

Probability Theory

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Summer 2020

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§0. Notation

\mathbb{N} represents the set $\{1, 2, \dots\}$.

\mathbb{N}_0 represents the set $\{0, 1, 2, \dots\}$.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\binom{x}{r} = \frac{x(x-1) \cdots (x-r+1)}{r!}$$

is the generalised binomial coefficient.

For $a \in \mathbb{R}^n$, we denote the i th coordinate of a by a_i for each $i = 1, 2, \dots, n$.

For $a, b \in \mathbb{R}^n$, we write $a < b$ if $a_i < b_i$ for each $i = 1, 2, \dots, n$.

§1. Measure Theory

Before beginning a rigorous study of probability theory, it is necessary to understand some parts of basic measure theory.

1.1. Classes of Sets

Let Ω be a non-empty set and $\mathcal{A} \subseteq 2^\Omega$, where 2^Ω is the power set of Ω . Then

Definition 1.1. \mathcal{A} is called

- \cap -closed (closed under intersections) or a π -system if $A \cap B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.
- σ - \cap -closed (closed under countable intersections) if $\cap_{i=1}^\infty A_i \in \mathcal{A}$ for any choice of countably many sets $A_1, A_2, \dots \in \mathcal{A}$.
- \cup -closed (closed under unions) if $A \cup B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.
- σ - \cup -closed (closed under countable unions) if $\cup_{i=1}^\infty A_i \in \mathcal{A}$ for any choice of countably many sets $A_1, A_2, \dots \in \mathcal{A}$.
- \setminus -closed (closed under differences) if $A \setminus B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.
- closed under complements if $A^c = \Omega \setminus A \in \mathcal{A}$ for all $A \in \mathcal{A}$.

Theorem 1.1. Let \mathcal{A} be closed under complements. Then \mathcal{A} is \cup -closed (σ - \cup -closed) if and only if \mathcal{A} is closed \cap -closed (σ - \cap -closed).

The above is relatively straightforward to prove using De Morgan's Laws.

Theorem 1.2. Let \mathcal{A} be \setminus -closed. Then

- (a) \mathcal{A} is \cap -closed,
- (b) if \mathcal{A} is σ - \cup -closed, then \mathcal{A} is σ - \cap -closed.
- (c) Any countable union of sets in \mathcal{A} can be expressed as a countable union of pairwise disjoint sets in \mathcal{A} .

Proof.

- (a) For $A, B \in \mathcal{A}$, $A \cap B = A \setminus (A \setminus B) \in \mathcal{A}$.
- (b) Let $A_1, A_2, \dots \in \mathcal{A}$. Then

$$\begin{aligned} \cap_{i=1}^\infty A_i &= \cap_{i=1}^\infty (A_1 \cap A_i) \\ &= \cap_{i=1}^\infty A_1 \setminus (A_1 \setminus A_i) \\ &= A_1 \setminus \cup_{i=1}^\infty (A_1 \setminus A_i). \end{aligned}$$

- (c) Let $A_1, A_2, \dots \in \mathcal{A}$. Representing the disjoint union of sets as \uplus , we have

$$\cup_{i=1}^\infty A_i = A_1 \uplus (A_2 \setminus A_1) \uplus ((A_3 \setminus A_2) \setminus A_1) \uplus \dots$$

The result follows. ■

Definition 1.2 (Algebra). A class of sets $\mathcal{A} \subseteq 2^\Omega$ is called an *algebra* if

- (i) $\Omega \in \mathcal{A}$,
- (ii) \mathcal{A} is \setminus -closed, and

(iii) \mathcal{A} is \cup -closed.

Definition 1.3 (σ -algebra). A class of sets $\mathcal{A} \subseteq 2^\Omega$ is called a σ -algebra if

- (i) $\Omega \in \mathcal{A}$,
- (ii) \mathcal{A} is closed under complements, and
- (iii) \mathcal{A} is σ - \cup -closed.

σ -algebras are also known as σ -fields.

Note that any σ -algebra is an algebra (but the converse is not true).

Theorem 1.3. A class of sets $\mathcal{A} \subseteq 2^\Omega$ is an algebra if and only if

- (a) $\Omega \in \mathcal{A}$,
- (b) \mathcal{A} is closed under complements, and
- (c) \mathcal{A} is \cap -closed.

The proof of the above is left as an exercise to the reader.

Definition 1.4 (Ring). A class of sets $\mathcal{A} \subseteq 2^\Omega$ is called a *ring* if

- (i) \mathcal{A} is \setminus -closed, and
- (ii) \mathcal{A} is \cup -closed.

Note that if \mathcal{A} is a non-empty ring with $X \in \mathcal{A}$, then $\emptyset = X \setminus X \in \mathcal{A}$.

Further, a ring is a σ -ring if it is σ - \cup -closed.

Definition 1.5 (Semiring). A class of sets $\mathcal{A} \subseteq 2^\Omega$ is called a *semiring* if

- (i) $\emptyset \in \mathcal{A}$,
- (ii) for any $A, B \in \mathcal{A}$, $A \setminus B$ is a finite union of mutually disjoint sets in \mathcal{A} , and
- (iii) \mathcal{A} is \cap -closed.

Definition 1.6 (λ -system). A class of sets $\mathcal{A} \subseteq 2^\Omega$ is called a λ -system (or *Dynkin's λ -system*) if

- (i) $\Omega \in \mathcal{A}$,
- (ii) for any $A, B \in \mathcal{A}$ with $B \subseteq A$, $A \setminus B \in \mathcal{A}$, and
- (iii) $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ for any choice of countably many pairwise disjoint sets $A_1, A_2, \dots \in \mathcal{A}$.

The above definitions are those that are required during our study of probability theory. σ -algebras in particular are extremely important as we shall use them when defining probabilities.

Theorem 1.4.

- (a) Every σ -algebra is also a λ -system, an algebra and a σ -ring.
- (b) Every σ -ring is a ring, and every ring is a semiring.
- (c) Every algebra is a ring. An algebra on a finite set Ω is a σ -algebra.

Proof.

- (a) Let \mathcal{A} be a σ -algebra. Then for any $A, B \in \mathcal{A}$, $A \setminus B = (A^c \cup B)^c \in \mathcal{A}$ and $A \cap B = (A^c \cup B^c)^c \in \mathcal{A}$, that is, \mathcal{A} is \setminus -closed and \cup -closed. The result follows.
- (b) Let \mathcal{A} be a ring. Then 1.1 implies that \mathcal{A} is \cap -closed. The result follows.
- (c) Let \mathcal{A} be an algebra. With proof similar to the first part of this theorem, it is seen that \mathcal{A} is \setminus -closed. We have $\emptyset = \Omega \setminus \Omega \in \mathcal{A}$ and thus, it is a ring. If Ω is finite, then \mathcal{A} is finite. Thus any countable union of sets is a finite union of sets and the result follows. ■

Definition 1.7. Let A_1, A_2, \dots be subsets of Ω . Then

$$\liminf_{n \rightarrow \infty} A_n := \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_j \text{ and } \limsup_{n \rightarrow \infty} A_n := \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j$$

are called the *limes inferior* and *limes superior*, respectively, of the sequence $(A_n)_{n \in \mathbb{N}}$.

The above may be rewritten as

$$A_* := \liminf_{n \rightarrow \infty} A_n = \{\omega \in \Omega \mid |n \in \mathbb{N} : \omega \notin A_n| < \infty\}$$

$$A^* := \limsup_{n \rightarrow \infty} A_n = \{\omega \in \Omega \mid |n \in \mathbb{N} : \omega \in A_n| = \infty\}$$

Definition 1.8 (Indicator function). Let A be a subset of Ω . The *indicator function on A* is defined by

$$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

With the above notation, it may be shown that

$$\mathbb{1}_{A_*} = \liminf_{n \rightarrow \infty} \mathbb{1}_{A_n} \text{ and } \mathbb{1}_{A^*} = \limsup_{n \rightarrow \infty} \mathbb{1}_{A_n}.$$

If $\mathcal{A} \subseteq 2^\Omega$ is a σ -algebra and if $A_n \in \mathcal{A}$ for every $n \in \mathbb{N}$, then $A_* \in \mathcal{A}$ and $A^* \in \mathcal{A}$.

Proving the above statements is left as an exercise to the reader.

Theorem 1.5. Let I be some index set and \mathcal{A}_i be a σ -algebra for each $i \in I$. Then the intersection $A_I \cap_{i \in I} A_i$ is also a σ -algebra.

Proof. We can prove this by using the three conditions in the definition of a σ -algebra.

- (i) Since $\Omega \in \mathcal{A}_i$ for every $i \in I$, $\Omega \in A_I$.
- (ii) Let $A \in A_I$. Then $A \in \mathcal{A}_i$ for each $i \in I$ and thus $A^c \in \mathcal{A}_i$ for each $i \in I$. Therefore, $A^c \in A_I$.
- (iii) Let $A_1, A_2, \dots \in A_I$. Then $A_n \in \mathcal{A}_i$ for each $n \in \mathbb{N}$ and $i \in I$. Thus $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_i$ for each i as well. The result follows. ■

Theorem 1.6. Let $\mathcal{E} \subseteq 2^\Omega$. Then there exists a smallest σ -algebra $\sigma(\mathcal{E})$ with $\mathcal{E} \subseteq \sigma(\mathcal{E})$:

$$\sigma(\mathcal{E}) = \bigcap_{\substack{\mathcal{A} \subseteq 2^\Omega \\ \mathcal{E} \subseteq \mathcal{A} \\ \mathcal{A} \text{ is a } \sigma\text{-algebra}}} \mathcal{A}.$$

$\sigma(\mathcal{E})$ is called the σ -algebra generated by \mathcal{E} and \mathcal{E} is called a *generator* of $\sigma(\mathcal{E})$.

Proof. 2^Ω is a σ -algebra that contains \mathcal{E} so the intersection is non-empty. By 1.5, $\sigma(\mathcal{E})$ is a σ -algebra. ■

Similar to the above, $\delta(\mathcal{E})$ is defined as the λ -system generated by \mathcal{E} .

It may be noted that we always have the following:

1. $\mathcal{E} \subseteq \sigma(\mathcal{E})$.
2. If $\mathcal{E}_1 \subseteq \mathcal{E}_2$, then $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$.
3. \mathcal{A} is a σ -algebra if and only if $\sigma(\mathcal{A}) = \mathcal{A}$.

Similar statements hold for λ -systems. Further, $\delta(\mathcal{E}) \subseteq \sigma(\mathcal{E})$.

Theorem 1.7 (\cap -closed λ -system). Let $\mathcal{D} \subseteq 2^\Omega$ be a λ -system. Then \mathcal{D} is a π -system if and only if \mathcal{D} is a σ -algebra.

Proof. If \mathcal{D} is a σ -algebra, then it is obviously a π -system. Let \mathcal{D} be a π -system. Then

- (a) As \mathcal{D} is a λ -system, $\Omega \in \mathcal{D}$.
- (b) Let $A \in \mathcal{D}$. Since $\Omega \in \mathcal{D}$ and \mathcal{D} is a λ -system, $A^c = \Omega \setminus A \in \mathcal{D}$.
- (c) Let $A, B \in \mathcal{D}$. We have $A \cap B \in \mathcal{D}$. We now have $A \setminus B = A \setminus (A \cap B) \in \mathcal{D}$, that is, \mathcal{D} is \setminus -closed.
Let $A_1, A_2, \dots \in \mathcal{D}$. Then by 1.2, there exist $B_1, B_2, \dots \in \mathcal{D}$ such that

$$\bigcup_{i=1}^{\infty} A_i = \biguplus_{i=1}^{\infty} B_i \in \mathcal{D}.$$

This completes the proof. ■

Theorem 1.8 (Dynkin's π - λ theorem). If $\mathcal{E} \subseteq 2^\Omega$ is a π -system, then $\delta(\mathcal{E}) = \sigma(\mathcal{E})$.

Proof. We already have $\delta(\mathcal{E}) \subseteq \sigma(\mathcal{E})$. We must now prove the reverse inclusion. We shall show that $\delta(\mathcal{E})$ is a π -system.

For each $E \in \delta(\mathcal{E})$, let

$$\mathcal{D}_E = \{A \in \delta(\mathcal{E}) : A \cap E \in \delta(\mathcal{E})\}.$$

To show that $\delta(\mathcal{E})$ is a π -system, it suffices to show that $\delta(\mathcal{E}) \subseteq \mathcal{D}_E$ for all $E \in \delta(\mathcal{E})$. We shall first show that \mathcal{D}_E is a λ -system for each $E \in \mathcal{E}$ by checking each of the conditions in 1.6.

- (a) We clearly have $\Omega \in \mathcal{D}_E$ as $\Omega \cap E = E$.
- (b) For any $A, B \in \mathcal{D}_E$ with $A \subseteq B$,

$$(B \setminus A) \cap E = (B \cap E) \setminus (A \cap E) \in \delta(\mathcal{E}).$$

- (c) Let $A_1, A_2, \dots \in \mathcal{D}_E$ be mutually disjoint sets. Then

$$\left(\biguplus_{i=1}^{\infty} A_i \right) \cap E = \biguplus_{i=1}^{\infty} (A_i \cap E) \in \delta(\mathcal{E}).$$

Now since \mathcal{D}_E is a λ -system and $\mathcal{E} \subseteq \mathcal{D}_E$ (Why?), $\delta(\mathcal{E}) \subseteq \mathcal{D}_E$.

Now that we have shown that $\delta(\mathcal{E})$ is a π -system, the result follows by 1.7. ■

Definition 1.9 (Topology). Let $\Omega \neq \emptyset$ be an arbitrary set. A class of sets $\tau \subseteq 2^\Omega$ is called a *topology* on 2^Ω if

- (i) $\emptyset, \Omega \in \tau$,
- (ii) τ is \cap -closed, and
- (iii) for any $\mathcal{F} \subseteq \tau$, $\bigcup_{A \in \mathcal{F}} A \in \tau$.

In the above case, the pair (Ω, τ) is called a *topological space*. The sets $A \in \tau$ are called *open* and the sets $A \subseteq \Omega$ with $A^c \in \tau$ are called *closed*.

Note that in contrast with σ -algebras, topologies are closed under only finite intersections but are also closed under arbitrary unions.

For example, consider the natural topology on \mathbb{R} which consists of all open intervals in \mathbb{R} and any arbitrary union of them.

Definition 1.10 (Borel σ -algebra). Let (Ω, τ) be a topological space. The σ -algebra

$$\mathcal{B}(\Omega) = \mathcal{B}(\Omega, \tau) = \sigma(\tau)$$

that is generated by the open sets is called the *Borel σ -algebra on Ω* . The elements $A \in \mathcal{B}(\Omega, \tau)$ are called *Borel sets* or *Borel measurable sets*.

A Borel σ -algebra that we shall often encounter is $\mathcal{B}(\mathbb{R}^n)$ for $n \in \mathbb{N}$. Consider the following classes of sets:

$$\begin{aligned} \mathcal{A}_1 &= \{A \subseteq \mathbb{R}^n : A \text{ is open}\} \\ \mathcal{A}_2 &= \{A \subseteq \mathbb{R}^n : A \text{ is closed}\} \\ \mathcal{A}_3 &= \{A \subseteq \mathbb{R}^n : A \text{ is compact}\} \\ \mathcal{A}_4 &= \{(a, b) : a, b \in \mathbb{Q}^n \text{ and } a < b\} \\ \mathcal{A}_5 &= \{(a, b] : a, b \in \mathbb{Q}^n \text{ and } a < b\} \\ \mathcal{A}_6 &= \{[a, b) : a, b \in \mathbb{Q}^n \text{ and } a < b\} \\ \mathcal{A}_7 &= \{[a, b] : a, b \in \mathbb{Q}^n \text{ and } a < b\} \\ \mathcal{A}_8 &= \{(-\infty, b) : b \in \mathbb{Q}^n\} \\ \mathcal{A}_9 &= \{(-\infty, b] : b \in \mathbb{Q}^n\} \\ \mathcal{A}_{10} &= \{(a, \infty) : a \in \mathbb{Q}^n\} \\ \mathcal{A}_{11} &= \{[a, \infty) : a \in \mathbb{Q}^n\} \end{aligned}$$

It may be proved that $\mathcal{B}(\mathbb{R}^n)$ is generated by any of the classes of sets $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_{11}$.

For a measurable $A \in \mathcal{B}(\mathbb{R})$, we represent by $\mathcal{B}(\mathbb{R})|_A$ the restriction of $\mathcal{B}(\mathbb{R})$ to A . It may be proved that this is equal to $\mathcal{B}(A)$, the σ -algebra generated by the open subsets of A .

1.2. Measure

Definition 1.11. Let $\mathcal{A} \subseteq 2^\Omega$ and let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a set function. We say that μ is

- (i) *monotone* if for any $A, B \in \mathcal{A}$, $A \subseteq B$ implies that $\mu(A) \leq \mu(B)$,
- (ii) *additive* if for any choice of finitely many mutually disjoint sets $A_1, \dots, A_n \in \mathcal{A}$ with $\biguplus_{i=1}^n A_i \in \mathcal{A}$,

$$\mu\left(\biguplus_{i=1}^n A_i\right) = \sum_{i=1}^n \mu(A_i),$$

- (iii) σ -additive if for any choice of countably many mutually disjoint sets $A_1, A_2, \dots \in \mathcal{A}$ with $\biguplus_{i=1}^\infty A_i \in \mathcal{A}$,

$$\mu\left(\biguplus_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i),$$

- (iv) subadditive if for any choice of finitely many sets $A, A_1, A_2, \dots, A_n \in \mathcal{A}$ with $A \subseteq \bigcup_{i=1}^n A_i$, we have

$$\mu(A) \leq \sum_{i=1}^n \mu(A_i), \text{ and}$$

(v) σ -subadditive if for any choice of countably many sets $A, A_1, A_2, \dots \in \mathcal{A}$ with $A \subseteq \bigcup_{i=1}^{\infty} A_i$, we have

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Definition 1.12. Let \mathcal{A} be a semiring and $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a set function with $\mu(\emptyset) = 0$. μ is called a

- (i) *content* if μ is additive,
- (ii) *premeasure* if μ is σ -additive, and
- (iii) *measure* if μ is σ -additive and \mathcal{A} is a σ -algebra.

Definition 1.13 (Finite content). Let \mathcal{A} be a semiring. A content μ on \mathcal{A} is called

- (i) *finite* if $\mu(A) < \infty$ for all $A \in \mathcal{A}$ and
- (ii) σ -*finite* if there exists a sequence of sets $\Omega_1, \Omega_2, \dots \in \mathcal{A}$ such that $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ and $\mu(\Omega_i) < \infty$ for every $i \in \mathbb{N}$.

Theorem 1.9 (Properties of contents). Let \mathcal{A} be a semiring and μ be a content on \mathcal{A} . Then

- (a) If \mathcal{A} is a ring, then $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$.
- (b) μ is monotone. If \mathcal{A} is a ring, then $\mu(B) = \mu(A) + \mu(B \setminus A)$ for any $A, B \in \mathcal{A}$ with $A \subseteq B$.
- (c) μ is subadditive. If μ is σ -additive, then it is also σ -subadditive
- (d) If \mathcal{A} is a ring, then

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$

for any choice of countably many mutually disjoint sets $A_1, A_2, \dots \in \mathcal{A}$ with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Proof.

- (a) Note that $A \cup B = A \uplus (B \setminus A)$ and $B = (A \cap B) \uplus (B \setminus A)$. As μ is additive,

$$\mu(A \cup B) = \mu(A) + \mu(B \setminus A) \text{ and } \mu(B) = \mu(A \cap B) + \mu(B \setminus A).$$

The result follows.

- (b) Let $A \subseteq B$. If $B \setminus A \in \mathcal{A}$ (which is true in the case of a ring), we have $B = A \uplus (B \setminus A)$ and thus

$$\mu(B) = \mu(A) + \mu(B \setminus A).$$

If \mathcal{A} is just a semiring, then there exist $n \in \mathbb{N}$ and mutually disjoint sets $C_1, C_2, \dots, C_n \in \mathcal{A}$ such that

$$B \setminus A = \bigsqcup_{i=1}^n C_i.$$

In either case, we have $\mu(A) \leq \mu(B)$.

- (c) Let $A, A_1, A_2, \dots, A_n \in \mathcal{A}$ such that $A \subseteq \bigcup_{i=1}^n A_i$. Let $B_1 = A_1$ and for each $k = 2, 3, \dots, n$, let

$$B_k = A_k \setminus \left(\bigcup_{i=1}^{k-1} A_i \right).$$

Note that any two B_i s are disjoint. As μ is additive and monotone, we have

$$\begin{aligned}\mu(A) &\leq \mu\left(\bigcup_{i=1}^n A_i\right) \\ &= \mu\left(\biguplus_{i=1}^n B_i\right) \\ &= \sum_{i=1}^n \mu(B_i) \leq \sum_{i=1}^n \mu(A_i).\end{aligned}$$

We can similarly prove that if μ is σ -additive, then it is σ -subadditive.

(d) Let $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Since μ is monotone,

$$\sum_{i=1}^m \mu(A_i) = \mu\left(\biguplus_{i=1}^m A_i\right) \leq \mu(A) \text{ for any } m \in \mathbb{N}.$$

The result follows. ■

Definition 1.14. Let A, A_1, A_2, \dots be sets. We write

- (i) $A_n \uparrow A$ if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and $\bigcup_{i=1}^{\infty} A_i = A$. In this case, we say that A_n increases to A .
- (ii) $A_n \downarrow A$ if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ and $\bigcap_{i=1}^{\infty} A_i = A$. In this case, we say that A_n decreases to A .

Definition 1.15 (Continuity of contents). Let μ be a content on the ring \mathcal{A} . μ is called

- (i) *lower semicontinuous* if $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A)$ for any $A \in \mathcal{A}$ and sequence $(A_n)_{n \in \mathbb{N}}$ with $A_n \uparrow A$,
- (ii) *upper semicontinuous* if $\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A)$ for any $A \in \mathcal{A}$ and sequence $(A_n)_{n \in \mathbb{N}}$ with $\mu(A_n) < \infty$ for some n (this implies that it holds for all $n \in \mathbb{N}$) and $A_n \downarrow A$,
- (iii) *\emptyset -continuous* if (ii) holds for $A = \emptyset$.

Theorem 1.10. Let μ be a content on the ring \mathcal{A} . The following properties are equivalent:

- (a) μ is σ -additive (and hence a premeasure).
- (b) μ is σ -subadditive.
- (c) μ is lower semicontinuous.
- (d) μ is \emptyset -continuous.
- (e) μ is upper semicontinuous.

Then (a) \iff (b) \iff (c) \implies (d) \iff (e). If μ is finite, then all four statements are equivalent.

Proof.

- (a) \implies (b).

The proof of this is similar to that involved in 1.9(c)

- (b) \implies (a).

This follows from 1.9(c).

- (a) \implies (c).

Let μ be a premeasure and $A \in \mathcal{A}$. Let $A_1, A_2, \dots \in \mathcal{A}$ such that $A_n \uparrow A$ and let $A_0 = \emptyset$. Then

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i \setminus A_{i-1}) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A_i \setminus A_{i-1}) = \lim_{n \rightarrow \infty} \mu(A_n).$$

- (c) \implies (a).

Let $B_1, B_2, \dots \in \mathcal{A}$ be mutually disjoint and let $B = \biguplus_{n=1}^{\infty} B_n \in \mathcal{A}$. Let $A_n = \bigcup_{i=1}^n B_i$ for all $n \in \mathbb{N}$. Then

$$\mu(B) = \lim_{n \rightarrow \infty} \mu(A_n) = \sum_{i=1}^{\infty} \mu(B_i).$$

Thus μ is σ -additive.

- (d) \implies (e)

Let $A, A_1, A_2, \dots \in \mathcal{A}$ with $A_n \downarrow A$ and $\mu(A_1) < \infty$. Define $B_n = A_n \setminus A \in \mathcal{A}$ for valid n . Then $B_n \downarrow \emptyset$. Thus

$$\lim_{n \rightarrow \infty} \mu(A_n) - \mu(A) = \lim_{n \rightarrow \infty} \mu(B_n) = 0$$

and the result is proved.

- (e) \implies (d)

This is obvious.

- (c) \implies (d)

Let $A_1, A_2, \dots \in \mathcal{A}$ with $A_n \downarrow \emptyset$ and $\mu(A_1) < \infty$. Then $A_1 \setminus A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ and $A_1 \setminus A_n \uparrow A_1$. Thus

$$\mu(A_1) = \lim_{n \rightarrow \infty} \mu(A_1) - \mu(A_n).$$

Since $\mu(A_1) < \infty$, $\lim_{n \rightarrow \infty} \mu(A_n) = 0$ and the result is proved.

- (d) \implies (c) if μ is finite.

Let $A, A_1, A_2, \dots \in \mathcal{A}$ with $A_n \uparrow A$. Then $A \setminus A_n \downarrow \emptyset$ and

$$\lim_{n \rightarrow \infty} \mu(A) - \mu(A_n) = \lim_{n \rightarrow \infty} \mu(A \setminus A_n) = 0.$$

The result follows. ■

Definition 1.16 (Measurable spaces).

- A pair (Ω, \mathcal{A}) consisting of a nonempty set Ω and a σ -algebra $\mathcal{A} \subseteq 2^\Omega$ is called a *measurable space*. The sets $A \in \mathcal{A}$ are called *measurable sets*. If Ω is countable and $\mathcal{A} = 2^\Omega$, then the space $(\Omega, 2^\Omega)$ is called *discrete*.
- A triple $(\Omega, \mathcal{A}, \mu)$ is called a *measure space* if (Ω, \mathcal{A}) is a measurable space and μ is a measure on \mathcal{A} .

Definition 1.17 (Measurable map). Let (Ω, \mathcal{A}) and (Ω', \mathcal{A}') be measurable spaces. A map $X : \Omega \rightarrow \Omega'$ is called $\mathcal{A} - \mathcal{A}'$ -*measurable* (or just measurable) if

$$X^{-1}(A') \in \mathcal{A} \text{ for any } A' \in \mathcal{A}'.$$

In this case, we write $X : (\Omega, \mathcal{A}) \rightarrow (\Omega', \mathcal{A}')$.

Note that if X is measurable, then X^{-1} , which maps elements to their preimage, is a function from Ω' to \mathcal{A} .

1.3. The Measure Extension Theorem

Lemma 1.11. Let $(\Omega, \mathcal{A}, \mu)$ be a σ -finite measure space and $\mathcal{E} \subseteq \mathcal{A}$ be a π -system that generates \mathcal{A} . Assume there exists sequence $\Omega_1, \Omega_2, \dots \in \mathcal{E}$ such that $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$ and $\mu(\Omega_i) < \infty$ for all $i \in \mathbb{N}$. Then μ is uniquely determined by the values $\mu(E)$, $E \in \mathcal{E}$.

If $\Omega \in \mathcal{A}$ and $\mu(\Omega) = 1$, then the existence of the sequence $(\Omega_n)_{n \in \mathbb{N}}$ is not required.

Proof. Let ν be a σ -finite measure on (Ω, \mathcal{A}) such that $\mu(E) = \nu(E)$ for all $E \in \mathcal{E}$.

Let $E \in \mathcal{E}$ with $\mu(E) < \infty$. Consider

$$\mathcal{D}_E = \{A \in \mathcal{A} \mid \mu(A \cap E) = \nu(A \cap E)\}.$$

We claim that \mathcal{D}_E is a λ -system. We shall prove this by checking each of the conditions of 1.6.

(a) Clearly, $\Omega \in \mathcal{D}_E$.

(b) Let $A, B \in \mathcal{D}_E$ with $B \subseteq A$. Then

$$\begin{aligned} \mu((A \setminus B) \cap E) &= \mu(A \cap E) - \mu(B \cap E) \quad (\text{using 1.9}) \\ &= \nu(A \cap E) - \nu(B \cap E) \\ &= \nu((A \setminus B) \cap E). \end{aligned}$$

That is, $(A \setminus B) \in \mathcal{D}_E$.

(c) Let $A_1, A_2, \dots \in \mathcal{D}_E$ be mutually disjoint sets. Then

$$\begin{aligned} \mu\left(\left(\biguplus_{i=1}^{\infty} A_i\right) \cap E\right) &= \sum_{i=1}^{\infty} \mu(A_i \cap E) \\ &= \sum_{i=1}^{\infty} \nu(A_i \cap E) \\ &= \nu\left(\left(\biguplus_{i=1}^{\infty} A_i\right) \cap E\right). \end{aligned}$$

Therefore, $\biguplus_{i=1}^{\infty} A_i \in \mathcal{D}_E$ and \mathcal{D}_E is a λ -system.

As $\mathcal{E} \subseteq \mathcal{D}_E$ (Why?), $\delta(\mathcal{E}) \subseteq \mathcal{D}_E$. Since \mathcal{E} is a π -system, 1.8 implies that

$$\mathcal{A} \supseteq \mathcal{D}_E \supseteq \delta(\mathcal{E}) = \sigma(\mathcal{E}) = \mathcal{A}.$$

Hence $\mathcal{D}_E = \mathcal{A}$.

Therefore, $\mu(A \cap E) = \nu(A \cap E)$ for any $A \in \mathcal{A}$ and $E \in \mathcal{E}$ with $\mu(E) < \infty$.

Now, let $\Omega_1, \Omega_2, \dots \in \mathcal{E}$ be a sequence such that $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$ and $\mu(\Omega_i) < \infty$ for all $i \in \mathbb{N}$. Let $E_0 = \emptyset$ and $E_n = \biguplus_{i=1}^n \Omega_i$ for each $n \in \mathbb{N}$. Note that

$$E_n = \biguplus_{i=1}^n (E_{i-1}^c \cap \Omega_i).$$

Therefore for any $A \in \mathcal{A}$ and $n \in \mathbb{N}$,

$$\begin{aligned} \mu(A \cap E_n) &= \sum_{i=1}^n \mu((A \cap E_{i-1}^c) \cap \Omega_i) \\ &= \sum_{i=1}^n \nu((A \cap E_{i-1}^c) \cap \Omega_i) = \nu(A \cap E_n). \end{aligned}$$

Now, since $E_n \uparrow \Omega$ and μ, ν are lower semicontinuous (by 1.10),

$$\begin{aligned}\mu(A) &= \lim_{n \rightarrow \infty} \mu(A \cap E_n) \\ &= \lim_{n \rightarrow \infty} \nu(A \cap E_n) = \nu(A)\end{aligned}$$

This proves the result.

The second part of the theorem is trivial as $\mathcal{E} \cup \{\Omega\}$ is a π -system that generates \mathcal{A} . Hence one can choose the constant sequence $E_n = \Omega, n \in \mathbb{N}$. ■

Definition 1.18 (Outer Measure). A function $\mu^* : 2^\Omega \rightarrow [0, \infty]$ is called an *outer measure* if

- (i) $\mu^*(\emptyset) = 0$,
- (ii) μ^* is monotone, and
- (iii) μ^* is σ -subadditive.

Lemma 1.12. Let $\mathcal{A} \subseteq 2^\Omega$ be an arbitrary class of sets with $\emptyset \in \mathcal{A}$ and let μ be a nonnegative set function on \mathcal{A} with $\mu(\emptyset) = 0$. For $A \subseteq \Omega$, define the set of countable coverings \mathcal{F} with sets $F \in \mathcal{A}$

$$\mathcal{U}(A) = \left\{ \mathcal{F} \subseteq \mathcal{A} : \mathcal{F} \text{ is countable and } A \subseteq \bigcup_{F \in \mathcal{F}} F \right\}.$$

Define

$$\mu^*(A) = \inf \left\{ \sum_{F \in \mathcal{F}} \mu(F) : \mathcal{F} \in \mathcal{U}(A) \right\}$$

where $\inf \emptyset = \infty$. Then μ^* is an outer measure. If μ is σ -subadditive then $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Proof. Let us check each of the three conditions in the definition of an outer measure.

- (a) Since $\emptyset \in \mathcal{A}$, we have $\{\emptyset\} \in \mathcal{U}(\emptyset)$ and hence $\mu(\emptyset) = 0$.
- (b) If $A \subseteq B$, then $\mathcal{U}(A) \subseteq \mathcal{U}(B)$, and hence $\mu^*(A) \leq \mu^*(B)$.
- (c) Let $A, A_1, A_2, \dots \subseteq \Omega$ such that $A \subseteq \bigcup_{i=1}^\infty A_i$. We claim that $\mu^*(A) \leq \sum_{i=1}^\infty \mu^*(A_i)$.

Without loss of generality, assume that $\mu^*(A_i) < \infty$ and hence $\mathcal{U}(A_i) \neq \emptyset$ for all $i \in \mathbb{N}$. Fix some $\varepsilon > 0$. Now, for every $n \in \mathbb{N}$, we may choose a covering $\mathcal{F}_n \in \mathcal{U}(A_n)$ such that

$$\sum_{F \in \mathcal{F}_n} \mu(F) \leq \mu^*(A_n) + \varepsilon 2^{-n}.$$

Then let $\mathcal{F} = \bigcup_{n=1}^\infty \mathcal{F}_n \in \mathcal{U}(A)$.

$$\mu^*(A) \leq \sum_{F \in \mathcal{F}} \mu(F) \leq \sum_{n=1}^\infty \sum_{F \in \mathcal{F}_n} \mu(F) \leq \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

This proves the first part of the result.

To prove the next part of the result, first note that since $\{A\} \in \mathcal{U}(A)$, we have $\mu^*(A) \leq \mu(A)$. If μ is σ -subadditive, then for any $\mathcal{F} \in \mathcal{U}(A)$,

$$\sum_{F \in \mathcal{F}} \mu(F) \geq \mu(A).$$

It follows that $\mu^*(A) \geq \mu(A)$. ■

Definition 1.19 (μ^* -measurable sets). Let μ^* be an outer measure. A set $A \in 2^\Omega$ is called μ^* -measurable if

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E) \text{ for any } E \in 2^\Omega.$$

We write $\mathcal{M}(\mu^*) = \{A \subseteq \Omega : A \text{ is } \mu^*\text{-measurable}\}$.

Lemma 1.13. $A \in \mathcal{M}(\mu^*)$ if and only if

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \leq \mu^*(E) \text{ for any } E \in 2^\Omega.$$

Proof. As μ^* is subadditive, we trivially have

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \geq \mu^*(E) \text{ for any } E \in 2^\Omega.$$

The result follows. ■

Lemma 1.14. $\mathcal{M}(\mu^*)$ is an algebra.

Proof. We shall check the conditions given in the definition of an algebra 1.2.

- (a) We trivially have $\Omega \in \mathcal{M}(\mu^*)$.
- (b) By definition, $\mathcal{M}(\mu^*)$ is closed under complements.
- (c) We must check that $\mathcal{M}(\mu^*)$ is closed under intersections. Let $A, B \in \mathcal{M}(\mu^*)$ and $E \subseteq \Omega$. Then

$$\begin{aligned} \mu^*((A \cap B) \cap E) + \mu^*((A \cap B)^c \cap E) &= \mu^*((A \cap B) \cap E) \\ &\quad + \mu^*((A \cap B^c \cap E) \cup (A^c \cap B \cap E) \cup (A^c \cap B^c \cap E)) \\ &\leq \mu^*(A \cap (B \cap E)) + \mu^*(A \cap (B^c \cap E)) \\ &\quad + \mu^*(A^c \cap (B \cap E)) + \mu^*(A^c \cap (B^c \cap E)) \\ &= \mu^*(B \cap E) + \mu^*(B^c \cap E) \quad (\text{since } A \in \mathcal{M}(\mu^*)) \\ &= \mu^*(E). \quad (\text{since } B \in \mathcal{M}(\mu^*)) \end{aligned}$$

This proves the result. ■

Lemma 1.15. An outer measure μ^* is σ -additive on $\mathcal{M}(\mu^*)$.

Proof. Let $A, B \in \mathcal{M}(\mu^*)$ with $A \cap B \neq \emptyset$. Then

$$\begin{aligned} \mu^*(A \cup B) &= \mu^*(A \cap (A \cup B)) + \mu^*(A^c \cap (A \cup B)) \\ &= \mu^*(A) + \mu^*(B). \end{aligned}$$

That is, μ^* is additive (and is thus a content). Since μ^* is σ -subadditive, 1.10 gives the required result. ■

Lemma 1.16. If μ^* is an outer measure, $\mathcal{M}(\mu^*)$ is a σ -algebra.

Proof. We have already shown that $\mathcal{M}(\mu^*)$ is an algebra (and thus a π -system). Using 1.7, it is sufficient to show that $\mathcal{M}(\mu^*)$ is a λ -system.

Let $A_1, A_2, \dots \in \mathcal{M}(\mu^*)$ be mutually disjoint sets and let $A = \biguplus_{i=1}^\infty A_i$. Further, for each $n \in \mathbb{N}$, let $B_n = \bigcup_{i=1}^n A_i$. For any E and valid $n \in \mathbb{N}$, we have

$$\begin{aligned} \mu^*(E \cap B_{n+1}) &= \mu^*((E \cap B_{n+1}) \cap B_n) + \mu^*((E \cap B_{n+1}) \cap B_n^c) \\ &= \mu^*(E \cap B_n) + \mu^*(E \cap A_{n+1}). \end{aligned}$$

By a simple induction, it follows that

$$\mu(E \cap B_n) = \sum_{i=1}^n \mu^*(E \cap A_i).$$

Since μ^* is monotonic, we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\ &\geq \mu^*(E \cap B_n) + \mu^*(E \cap A^c) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap A^c).\end{aligned}$$

Letting $n \rightarrow \infty$ and using the fact that μ^* is σ -subadditive, we have

$$\begin{aligned}\mu^*(E) &\geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap A^c) \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c)\end{aligned}$$

■

Theorem 1.17 (Measure Extension Theorem). Let \mathcal{A} be a semiring and let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be an additive, σ -subadditive and σ -finite set function with $\mu(\emptyset) = 0$. Then there is a unique σ -finite measure $\tilde{\mu} : \sigma(\mathcal{A}) \rightarrow [0, \infty]$ such that $\tilde{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Proof. Since \mathcal{A} is a π -system, if such a $\tilde{\mu}$ exists, it is uniquely defined due to 1.11.

We shall explicitly construct a function that satisfies the given conditions. In order to do so, define as in 1.12

$$\mu^*(A) = \inf \left\{ \sum_{F \in \mathcal{F}} \mu(F) : \mathcal{F} \in \mathcal{U}(A) \right\} \text{ for any } A \subseteq \Omega.$$

By 1.12, μ^* is an outer measure and $\mu^*(A) = \mu(A)$ for any $A \in \mathcal{A}$.

We first claim that $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$.

To prove this, let $A \in \mathcal{A}$ and $E \subseteq \Omega$ with $\mu^*(E) < \infty$. Fix some $\varepsilon > 0$. Then by the definition of μ^* , there exists a sequence $E_1, E_2, \dots \in \mathcal{A}$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} E_i \text{ and } \sum_{i=1}^{\infty} \mu(E_i) \leq \mu^*(E) + \varepsilon.$$

For each n , define $B_n = E_n \cap A$. Since \mathcal{A} is a semiring, there exists for each n some $m_n \in \mathbb{N}$ and mutually disjoint sets $C_{n,1}, C_{n,2}, \dots, C_{n,m_n}$ such that

$$E_n \setminus A = E_n \setminus B_n = \biguplus_{i=1}^{m_n} C_{n,i}$$

Then we have that

$$\begin{aligned}E \cap A &\subseteq \bigcup_{n=1}^{\infty} B_n, \\ E \cap A^c &\subseteq \bigcup_{n=1}^{\infty} \biguplus_{i=1}^{m_n} C_{n,i}, \text{ and} \\ E_n &= B_n \uplus \biguplus_{i=1}^{m_n} C_{n,i}.\end{aligned}$$

This implies that

$$\begin{aligned}
\mu^*(E \cap A) + \mu^*(E \cap A^c) &\leq \sum_{n=1}^{\infty} \mu(B_n) + \sum_{n=1}^{\infty} \sum_{i=1}^{m_n} \mu(C_{n,i}) \quad (\text{since } \mu \text{ is } \sigma\text{-subadditive}) \\
&= \sum_{n=1}^{\infty} \left(\mu(B_n) + \sum_{i=1}^{m_n} \mu(C_{n,i}) \right) \\
&= \sum_{n=1}^{\infty} \mu(E_n) \quad (\text{since } \mu \text{ is additive}) \\
&\leq \mu^*(E) + \varepsilon.
\end{aligned}$$

1.13 implies that $A \in \mathcal{M}(\mu^*)$, that is, $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$. This in turn implies that $\sigma(\mathcal{A}) \subseteq \mathcal{M}(\mu^*)$. Define the required function by $\tilde{\mu} : \sigma(\mathcal{A}) \rightarrow [0, \infty]$, $A \mapsto \mu^*(A)$. By **1.15**, $\tilde{\mu}$ is σ -additive. Since μ is σ -finite, $\tilde{\mu}$ is σ -finite as well. This proves the result. \blacksquare

Now that we have the Measure Extension Theorem, we may introduce the Lebesgue-Stieltjes measure, which is given as follows.

Definition 1.20 (Lebesgue-Stieltjes Measure). Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be monotone increasing and right continuous. The measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$\mu_F((a, b]) = F(b) - F(a) \text{ for all } a, b \in \mathbb{R} \text{ such that } a < b$$

is called the *Lebesgue-Stieltjes measure* with distribution function F .

The Lebesgue-Stieltjes measure is well-defined due to the Measure Extension Theorem **1.17**.

To see this more clearly, let $\mathcal{A} = \{(a, b] : a, b \in \mathbb{R} \text{ and } a \leq b\}$. It may be checked that \mathcal{A} is a semiring. Further, $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$. Now, define the function $\tilde{\mu}_F : \mathcal{A} \rightarrow [0, \infty]$ by $(a, b] \mapsto F(b) - F(a)$. Clearly $\tilde{\mu}_F(\emptyset) = 0$ and the function is additive. It remains to check that $\tilde{\mu}_F$ is σ -subadditive.

Let $(a, b], (a_1, b_1], (a_2, b_2], \dots \in \mathcal{A}$ such that $(a, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$. Fix some $\varepsilon > 0$ and choose $a_\varepsilon \in (a, b)$ such that

$$F(a_\varepsilon) - F(a) < \varepsilon/2 \implies \tilde{\mu}_F((a, b]) - \tilde{\mu}_F((a_\varepsilon, b]) < \varepsilon/2.$$

It is possible to choose such an ε due to the right continuity of F . Also, for any $k \in \mathbb{N}$, choose $b_{k,\varepsilon}$ such that

$$F(b_{k,\varepsilon}) - F(b_k) < \varepsilon 2^{-k-1} \implies \tilde{\mu}_F((a_k, b_{k,\varepsilon}]) - \tilde{\mu}_F((a_k, b_k]) < \varepsilon 2^{-k-1}.$$

We now have

$$[a_\varepsilon, b] \subseteq (a, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \subseteq \bigcup_{k=1}^{\infty} (a_k, b_{k,\varepsilon}]$$

Due to the compactness of $[a_\varepsilon, b]$, there then exists some $k_0 \in \mathbb{N}$ such that

$$(a_\varepsilon, b] \subseteq \bigcup_{k=1}^{k_0} (a_k, b_{k,\varepsilon}].$$

This implies that

$$\begin{aligned}
\tilde{\mu}_F((a, b]) &\leq \frac{\varepsilon}{2} + \tilde{\mu}_F((a_\varepsilon, b]) \\
&\leq \frac{\varepsilon}{2} + \sum_{k=1}^{k_0} \tilde{\mu}_F((a_k, b_{k,\varepsilon}]) \\
&\leq \frac{\varepsilon}{2} + \sum_{k=1}^{k_0} (\tilde{\mu}_F((a_k, b_k]) + \varepsilon 2^{-k-1}) \\
&\leq \varepsilon + \sum_{k=1}^{\infty} \tilde{\mu}_F((a_k, b_k])
\end{aligned}$$

As this is true for any choice of ε , $\tilde{\mu}_F$ is σ -subadditive.

Then the extension of $\tilde{\mu}_F$ uniquely to a σ -finite measure is guaranteed by 1.17. This measure is known as the Lebesgue-Stieltjes measure.

The measure that results when the function F is equal to the identity function is referred to the *Lebesgue measure* on \mathbb{R}^1 . Similar to this, we can define the Lebesgue measure in general as follows.

Definition 1.21 (Lebesgue Measure). There exists a unique measure λ^n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that for all $a, b \in \mathbb{R}^n$ with $a < b$,

$$\lambda^n((a, b]) = \prod_{i=1}^n (b_i - a_i).$$

λ^n is called the *Lebesgue measure* on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ or the *Lebesgue-Borel measure*.

1.4. Miscellaneous

Let E be a finite nonempty set and $\Omega = E^{\mathbb{N}}$. If $\omega_1, \omega_2, \dots, \omega_n \in E$, we define the following.

$$[\omega_1, \omega_2, \dots, \omega_n] = \{\omega' \in \Omega : \omega'_i = \omega_i \text{ for } i = 1, 2, \dots, n\}.$$

This represents the set of all sequences whose first n elements are $\omega_1, \omega_2, \dots, \omega_n$.

Theorem 1.18 (Finite Products of Measures). Let $n \in \mathbb{N}$ and $\mu_1, \mu_2, \dots, \mu_n$ be Lebesgue-Stieltjes measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then there exists a unique σ -finite measure μ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that for all $a, b \in \mathbb{R}^n$ with $a < b$,

$$\mu((a, b]) = \prod_{i=1}^n \mu_i((a_i, b_i])$$

We call μ the *product measure* of $\mu_1, \mu_2, \dots, \mu_n$ and denote it by $\bigotimes_{i=1}^n \mu_i$.

The proof of the above is similar to that of 1.17. We choose intervals $(a, b_\varepsilon]$ and so on such that $\mu((a, b_\varepsilon]) < \mu((a, b]) + \varepsilon$. Such b_ε exists due to the right continuity of each of the F_i s corresponding to each of the μ_i s.

§2. Introduction to Probability

2.1. Basic Definitions

Definition 2.1 (Probability Measure). Let \mathcal{A} be a σ -algebra and μ be a measure on \mathcal{A} . μ is called a *probability measure* if $\mu(\Omega) = 1$.

Let Ω be a countable non-empty set and $\mathcal{A} = 2^\Omega$. Further let $(p_\omega)_{\omega \in \Omega}$ be non-negative numbers. The map given by $A \mapsto \mu(A) = \sum_{\omega \in A} p_\omega$ defines a σ -finite measure on 2^Ω . $p = (p_\omega)_{\omega \in \Omega}$ is called the *weight function* of μ . p_ω is called the *weight of μ at point ω* .

In the case where $\sum_{\omega \in \Omega} p_\omega = 1$, μ is a probability measure. Then the vector $(p_\omega)_{\omega \in \Omega}$ is called a *probability vector*.

Definition 2.2 (Probability Distribution Function). A right continuous monotonically increasing function $F : \mathbb{R} \rightarrow [0, 1]$ such that $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$ is called a (*proper*) *probability distribution function*, often abbreviated as *p.d.f.* If we instead have $\lim_{x \rightarrow \infty} F(x) \leq 1$, F is called a (*possibly*) *defective p.d.f.* If μ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then the function F_μ given by $x \mapsto \mu((-\infty, x])$ is called the *distribution function* of μ .

Note that a probability measure is uniquely determined by its distribution function.

Definition 2.3 (Probability Space). Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If in addition $\mu(\Omega) = 1$, then $(\Omega, \mathcal{A}, \mu)$ is called a *probability space*.

In the above definition, Ω is called the *sample space*, \mathcal{A} is called the *event space* (and its elements are called *events*), and μ is called the *probability function*.

Let Ω be a finite nonempty set. Let $\mathcal{A} = 2^\Omega$ and consider the function $\mu : \mathcal{A} \rightarrow [0, 1]$ given by

$$\mu(A) = \frac{|A|}{|\Omega|} \text{ for each } A \subseteq \Omega.$$

This defines a probability measure on \mathcal{A} . This function μ is called the *uniform distribution on Ω* and is denoted \mathcal{U}_Ω . The resulting probability space $(\Omega, \mathcal{A}, \mathcal{U}_\Omega)$ is called a *Laplace space*.

Another example is as follows. Let $\omega \in \Omega$ and $\delta_\omega(A) = \mathbb{1}(\{\omega\})$. Then δ_ω is a probability measure on any σ -algebra $\mathcal{A} \subseteq 2^\Omega$. δ_ω is called the *Dirac measure* for the point ω .

The Dirac measure is useful in constructing discrete probability distributions.

Consider the example of a coin toss. The sample space Ω has two elements, H (for heads) and T (for tails). The event space \mathcal{A} then has four elements \emptyset , $\{H\}$, $\{T\}$, and $\{H, T\}$. Each of these events have associated probabilities $0, \frac{1}{2}, \frac{1}{2}$, and 1 respectively. Note that $\{H, T\}$ represents the event that either a heads or a tails occurs.⁴

Definition 2.4 (Random Variable). Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, (Ω', \mathcal{A}') a measurable space, and $X : \Omega \rightarrow \Omega'$ be measurable. Then

- (i) X is called a *random variable* with values in (Ω', \mathcal{A}') . If $(\Omega', \mathcal{A}') = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then X is called a *real random variable*.
- (ii) For $A' \in \mathcal{A}'$, we often denote

$$\mathbf{P}[X^{-1}(A')] \text{ as } \mathbf{P}[X \in A'] \text{ and } X^{-1}(A') \text{ as } \{X \in A'\}.$$

In particular, we let $\{X \geq 0\} = X^{-1}([0, \infty))$ and define $\{X \leq b\}$ and other terms similarly.

As we shall primarily deal with real random variables in our study of probability, we often drop the “real” and refer to them as just random variables.

Definition 2.5. Let X be a random variable with underlying probability space $(\Omega, \mathcal{A}, \mathbf{P})$.

- (i) The probability measure $\mathbf{P}_X = \mathbf{P} \circ X^{-1}$ is called the *distribution* of X .
- (ii) For a real random variable X , the map F_X given by $x \mapsto \mathbf{P}[X \leq x]$ is called the *distribution function* of P_X (or X). If $\mu = \mathbf{P}_X$, we write $X \sim \mu$ and say that X has distribution μ .

- (iii) A family $(X_i)_{i \in I}$ of random variables is called *identically distributed* if $\mathbf{P}_{X_i} = \mathbf{P}_{X_j}$ for all $i, j \in I$. We write $X \stackrel{\mathcal{D}}{=} Y$ if $\mathbf{P}_X = \mathbf{P}_Y$ (\mathcal{D} for *distribution*).

Theorem 2.1. For any p.d.f. F , there exists a real random variable X with $F_X = F$.

Proof. We shall explicitly construct a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and random variable $X : \Omega \rightarrow \mathbb{R}$ such that $F_X = F$. One choice that might come to mind is to take $(\Omega, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $X : \mathbb{R} \rightarrow \mathbb{R}$ as the identity function, and P the Lebesgue-Stieltjes measure with distribution function F .

While this choice of ours works, let us attempt to construct another more “standard” choice that is perhaps more enlightening. Let $\Omega = (0, 1)$, $\mathcal{A} = \mathcal{B}(\Omega)|_\Omega$ and \mathbf{P} be the Lebesgue measure on (Ω, \mathcal{A}) . This is standard in the sense that given any F , we construct a random variable over the same probability space. Define the left continuous inverse of F as

$$F^{-1}(t) = \inf\{x \in \mathbb{R} : F(x) \geq t\} \text{ for } t \in (0, 1).$$

Note that $F^{-1}(t) \leq x$ if and only if $F(x) \geq t$. In particular,

$$\{t : F^{-1}(t) \leq x\} = (0, F(x)] \cap (0, 1)$$

and so $F^{-1} : (\Omega, \mathcal{A}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. Thus

$$\mathbf{P}[\{t : F^{-1}(t) \leq x\}] = F(x).$$

This implies that F^{-1} is the random variable we wish to construct. ■

Note that the above implies that there is a bijection between probability distribution functions and distribution functions corresponding to random variables.

Definition 2.6. If a distribution $F : \mathbb{R}^n \rightarrow [0, 1]$ is of the form

$$F(x) = \int_{-\infty}^{x_1} dt_1 \int_{-\infty}^{x_2} dt_2 \cdots \int_{-\infty}^{x_n} dt_n f(t_1, t_2, \dots, t_n) \text{ for } (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

for some integrable function $f : \mathbb{R}^n \rightarrow [0, \infty)$, then f is called the *density of the distribution*.

2.2. Important Examples of Random Variables

We now give several important examples of random variables that we shall encounter several times in our study of probability.

1. Bernoulli Distribution.

Let $p \in [0, 1]$ and $\mathbf{P}[X = 1] = p$, $\mathbf{P}[X = 0] = 1 - p$. Then \mathbf{P}_X is called the *Bernoulli distribution with parameter p* and is denoted Ber_p . More formally,

$$\text{Ber}_p = (1 - p)\delta_0 + p\delta_1.$$

Its distribution function is

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - p, & x \in [0, 1) \\ 1, & x \geq 1 \end{cases}$$

Note that the above can be likened to the outcome of a weighted coin, with heads and tails corresponding to 0 and 1.

The distribution \mathbf{P}_Y of $Y = 2X - 1$ is called the *Rademacher distribution with parameter p* . More formally,

$$\text{Rad}_p = (1 - p)\delta_{-1} + p\delta_1.$$

$\text{Rad}_{1/2}$ is simply called the Rademacher distribution.

2. Binomial Distribution.

Let $p \in [0, 1]$ and $n \in \mathbb{N}$. Let $X : \Omega \rightarrow \{0, 1, 2, \dots, n\}$ be such that for each valid k ,

$$\mathbf{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}.$$

Then \mathbf{P}_X is called the *binomial distribution with parameters n and p* and is denoted $b_{n,p}$. More formally,

$$b_{n,p} = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} \delta_k.$$

3. Geometric Distribution.

Let $p \in (0, 1]$ and $X : \Omega \rightarrow \mathbb{N}_0$ be such that for each $n \in \mathbb{N}_0$,

$$\mathbf{P}[X = n] = p(1-p)^n.$$

Then \mathbf{P}_X is called the *geometric distribution with parameter p* and is denoted γ_p or $b_{1,p}^-$. More formally,

$$\gamma_p = \sum_{n=0}^{\infty} p(1-p)^n \delta_n.$$

4. Negative Binomial Distribution.

Let $r > 0$ and $p \in (0, 1]$. We denote by

$$b_{r,p}^- = \sum_{k=0}^{\infty} \binom{-r}{k} (-1)^k p^r (1-p)^k \delta_k$$

the *negative binomial distribution* or *Pascal distribution* with parameters r and p . Note that r need not be an integer.

5. Poisson Distribution.

Let $\lambda \in [0, \infty)$ and $X : \Omega \rightarrow \mathbb{N}_0$ be such that for each $n \in \mathbb{N}_0$,

$$\mathbf{P}[X = n] = e^{-\lambda} \frac{\lambda^n}{n!}.$$

Then $\mathbf{P}_x = \text{Poi}_\lambda$ is called the *Poisson distribution with parameter λ* .

6. Hypergeometric Distribution.

Consider a basket with $B \in \mathbb{N}$ black balls and $W \in \mathbb{N}$ white balls. If we draw $n \in \mathbb{N}$ balls from the basket, some simple combinatorics shows that the probability of drawing (exactly) $b \in \{0, 1, 2, \dots, n\}$ black balls is given by the *hypergeometric distribution with parameters B, W, n* :

$$\text{Hyp}_{B,W;n}(\{b\}) = \frac{\binom{B}{b} \binom{W}{n-b}}{\binom{B+W}{n}}.$$

In general, if we have k colors with B_i balls of colour i for each i , the probability of drawing exactly b_i balls of colour i for each i is given by the *generalised hypergeometric distribution*:

$$\text{Hyp}_{B_1, B_2, \dots, B_k; n}(\{(b_1, b_2, \dots, b_k)\}) = \frac{\binom{B_1}{b_1} \binom{B_2}{b_2} \dots \binom{B_k}{b_k}}{\binom{B_1 + B_2 + \dots + B_k}{n}}$$

where $n = b_1 + b_2 + \dots + b_k$.

7. Gaussian Normal Distribution.

Let $\mu \in \mathbb{R}, \sigma^2 > 0$. Let X be a real random variable such that for $x \in \mathbb{R}$,

$$\mathbf{P}[X \leq x] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^x \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) dt$$

Then \mathbf{P}_X is called the *Gaussian normal distribution* (or just *normal distribution*) with parameters μ and σ^2 and is denoted $\mathcal{N}_{\mu, \sigma^2}$. In particular, $\mathcal{N}_{0,1}$ is the standard normal distribution.

8. Exponential Distribution.

Let $\theta > 0$ and X be a nonnegative random variable such that for each $x \geq 0$,

$$\mathbf{P}[X \leq x] = \mathbf{P}[X \in [0, x]] = \int_0^x \theta e^{-\theta t} dt.$$

Then \mathbf{P}_X is called the *exponential distribution with parameter θ* and is denoted \exp_θ .

2.3. The Product Measure

Let E be a finite set and $\Omega = E^{\mathbb{N}}$. Let $(p_e)_{e \in E}$ be a probability vector. Define

$$\mathcal{A} = \{[\omega_1, \dots, \omega_n] : \omega_1, \dots, \omega_n \text{ and } n \in \mathbb{N}\}$$

and a content μ on \mathcal{A} by

$$\mu([\omega_1, \omega_2, \dots, \omega_n]) = \prod_{i=1}^n p_{\omega_i}$$

We wish to extend μ to a measure on $\sigma(\mathcal{A})$. Similar to how we proved the existence of the Lebesgue-Stieltjes measure 1.20, we use a compactness argument to show that μ is σ -subadditive.

Let $A, A_1, A_2, \dots \in \mathcal{A}$ such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$. We claim that there exists $n \in \mathbb{N}$ such that $A \subseteq \bigcup_{i=1}^n A_i$.

For each $n \in \mathbb{N}$, let $B_n = A \setminus \bigcup_{i=1}^n A_i$. We assume that $B_n \neq \emptyset$ for all $n \in \mathbb{N}$ and prove the required by contradiction. Due to the pigeonhole principle, there exists some $\omega_1 \in E$ such that $[\omega_1] \cap B_n \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. Since $B_1 \supseteq B_2 \supseteq \dots$, we have that

$$[\omega_1] \cap B_n \neq \emptyset \text{ for all } n \in \mathbb{N}.$$

Similarly, there exist $\omega_2, \omega_3, \dots \in E$ such that

$$[\omega_1, \dots, \omega_k] \cap B_n \neq \emptyset \text{ for all } k, n \in \mathbb{N}.$$

Each B_n is a disjoint union of sets $C_{n,1}, \dots, C_{n,m_n} \in \mathcal{A}$. Thus for each $n \in \mathbb{N}$, there is some $i_n \in \{1, 2, \dots, m_n\}$ such that

$$[\omega_1, \omega_2, \dots, \omega_k] \cap C_{n,i_n} \neq \emptyset \text{ for infinitely many } k \in \mathbb{N}.$$

As $[\omega_1] \supseteq [\omega_1, \omega_2] \supseteq \dots$, this implies that

$$[\omega_1, \omega_2, \dots, \omega_k] \cap C_{n,i_n} \neq \emptyset \text{ for all } k \in \mathbb{N}$$

As $C_{n,i_n} \in \mathcal{A}$, for fixed n and large k ($k \geq m_n$), we have

$$[\omega_1, \omega_2, \dots, \omega_k] \subseteq C_{n,i_n}.$$

This implies that $\omega = (\omega_1, \omega_2, \dots) \in C_{n,i_n} \subseteq B_n$. This in turn implies that $\bigcap_{i=0}^{\infty} B_i \neq \emptyset$, which yields a contradiction. Therefore, $A \subseteq \bigcup_{i=1}^n A_i$ for some $n \in \mathbb{N}$. Since μ is known to be (finite) subadditive, we have

$$\mu(A) \leq \sum_{i=1}^n \mu(A_i) \leq \sum_{i=1}^{\infty} \mu(A_i),$$

which is the required result.

Definition 2.7 (Product Measure). Let E be a finite nonempty set and $\Omega = E^{\mathbb{N}}$. Let $(p_e)_{e \in E}$ be a probability vector. There then is a unique probability measure μ on $\sigma(\mathcal{A}) = \mathcal{B}(\Omega)$ (where \mathcal{A} is defined as above) such that

$$\mu([\omega_1, \omega_2, \dots, \omega_n]) = \prod_{i=1}^n p_{\omega_i} \text{ for all } \omega_i \in E \text{ and } n \in \mathbb{N}.$$

μ is called the *product measure* or *Bernoulli measure* on Ω with weights $(p_e)_{e \in E}$ and is denoted by $(\sum_{e \in E} p_e \delta_e)^{\otimes \mathbb{N}}$. The σ -algebra $\sigma(\mathcal{A})$ is called the *product σ -algebra* on Ω and is denoted by $(2^E)^{\otimes \mathbb{N}}$.

We explain the above more intuitively in the following subsection.

2.4. Independent Events

In the following, let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and the sets $A \in \mathcal{A}$ be events. The \Pr or \mathbf{P} symbol will denote the universal object of a probability measure (we used \mathbf{P} to denote this until now), and the probabilities $\Pr[\cdot]$ are always written in (square) brackets.

Definition 2.8. Two events A and B are said to be *independent* if

$$\Pr[A \cap B] = \Pr[A] \Pr[B].$$

An common example of independent events is that of rolling a die twice, where the outcome of the first roll is independent of the second.

Here, $\Omega = \{1, 2, \dots, 6\}^2$, $\mathcal{A} = 2^\Omega$ and the probability distribution is $\mathbf{P} = \mathcal{U}_\Omega$. Our claim may be verified as follows. Let $\tilde{A}, \tilde{B} \subseteq \Omega$, $A = \tilde{A} \times \Omega$ and $B = \Omega \times \tilde{B}$. We must show that $\Pr[A] \Pr[B] = \Pr[A \cap B]$. This is obvious as follows:

$$\begin{aligned} \Pr[A] &= \frac{|\tilde{A}|}{36} = \frac{|\tilde{A}|}{6} \\ \Pr[B] &= \frac{|\tilde{B}|}{36} = \frac{|\tilde{B}|}{6} \\ \Pr[A \cap B] &= \frac{|\tilde{A} \cap \tilde{B}|}{36} = \frac{|\tilde{A}| |\tilde{B}|}{36} = \Pr[A] \Pr[B]. \end{aligned}$$

While in the above example it is intuitively clear that the two events must be independent, we can have less obvious examples as well. For example, the event that the sum of the two rolls is odd and the event that the first roll gives at most a three are independent. We leave it to the reader to verify this claim.

We extend this definition of two independent events to any number of independent events as follows.

Definition 2.9 (Independence of Events). Let I be an index set and $(A_i)_{i \in I}$ be a family of events. The family $(A_i)_{i \in I}$ is called *independent* if for any finite subset $J \subseteq I$, the following holds:

$$\Pr \left[\bigcap_{j \in J} A_j \right] = \prod_{j \in J} \Pr[A_j].$$

Let us now return to the product measure discussed in 2.7, which can be understood intuitively as follows. If E is a finite set of outcomes, consider the probability space comprising $\Omega = E^{\mathbb{N}}$, the σ -algebra

$$\mathcal{A} = \sigma([\omega_1, \dots, \omega_n] : \omega_1, \dots, \omega_n \in E \text{ and } n \in \mathbb{N})$$

and the product measure $\mathbf{P} = (\sum_{e \in E} p_e \delta_e)^{\otimes \mathbb{N}}$. This basically represents that we repeatedly conduct the experiment of choosing an outcome from E . Let $\tilde{A}_i \subseteq E$ for any $i \in \mathbb{N}$ and let A_i be the event such that \tilde{A}_i occurs in the i th experiment, given by

$$A_i = \{\omega \in \Omega : \omega_i \in \tilde{A}_i\} = \bigcup_{(\omega_1, \dots, \omega_i) \in E^{i-1} \times \tilde{A}_i} [\omega_1, \dots, \omega_i]$$

Intuitively, the family $(A_i)_{i \in \mathbb{N}}$ should be independent, since the outcome of one of the conducted experiments does not depend on the outcomes of the other experiments.

Let us check this. Let $J \subseteq \mathbb{N}$. For $j \in J$, let $B_j = A_j$ and $\tilde{B}_j = \tilde{A}_j$ and for $j \in \{1, 2, \dots, n\} \setminus J$, let $B_j = \Omega$ and $\tilde{B}_j = E$. Then

$$\begin{aligned} \Pr \left[\bigcap_{j \in J} A_j \right] &= \Pr \left[\bigcap_{j=1}^n B_j \right] \\ &= \Pr \left[\left\{ \omega \in \Omega : \omega_j \in \tilde{B}_j \text{ for each } j \in \{1, 2, \dots, n\} \right\} \right] \\ &= \sum_{e_1 \in \tilde{B}_1} \cdots \sum_{e_n \in \tilde{B}_n} \prod_{j=1}^n p_{e_j} \\ &= \prod_{j=1}^n \left(\sum_{e \in \tilde{B}_j} p_e \right) \\ &= \prod_{j \in J} \left(\sum_{e \in \tilde{A}_j} p_e \right) \end{aligned}$$

As this is true in particular for $|J| = 1$, we have for some fixed $i \in \{1, 2, \dots, n\}$,

$$\Pr[A_i] = \left(\sum_{e \in \tilde{A}_i} p_e \right).$$

Substituting the above, we have

$$\Pr \left[\bigcap_{j \in J} A_j \right] = \prod_{j \in J} \left(\sum_{e \in \tilde{A}_j} p_e \right) = \prod_{j \in J} \Pr[A_j].$$

This proves the result.

Note that if events A and B are independent, then the events A^c and B are independent as well. This can be described more precisely as follows.

Theorem 2.2. Let I be an index set and $(A_i)_{i \in I}$ be a family of events. Define $B_i^0 = A_i$ and $B_i^1 = A_i^c$ for each $i \in I$. Then the following statements are equivalent.

- (a) The family $(A_i)_{i \in I}$ is independent.
- (b) There is some $\alpha \in \{0, 1\}^I$ such that the family $(B_i^{\alpha_i})_{i \in I}$ is independent.
- (c) For all $\alpha \in \{0, 1\}^I$, the family $(B_i^{\alpha_i})_{i \in I}$ is independent.

We leave the proof of the above to the reader.

Now, recall the limes superior, defined in 1.7. The limes superior represents the event that a particular event occurs an infinite amount of times. For example, if we roll a die an infinite number of times, we could consider the event that we roll a four an infinite number of times. This is formalized in the following.

Theorem 2.3 (Borel-Cantelli Lemma). Let A_1, A_2, \dots be events and define $A^* = \limsup_{n \rightarrow \infty} A_n$. Then

- (a) If $\sum_{n=1}^{\infty} \Pr[A_n] < \infty$, $\Pr[A^*] = 0$.
- (b) If $(A_n)_{n \in \mathbb{N}}$ is independent and $\sum_{n=1}^{\infty} \Pr[A_n] = \infty$, then $\Pr[A^*] = 1$.

Proof. By 1.10, \mathbf{P} is upper semicontinuous, lower semicontinuous and σ -subadditive.

1. As \mathbf{P} is upper semicontinuous and σ -subadditive,

$$\begin{aligned}\Pr[A^*] &= \lim_{n \rightarrow \infty} \Pr \left[\bigcup_{m=n}^{\infty} A_m \right] \\ &\leq \lim_{n \rightarrow \infty} \Pr[A_n] = 0\end{aligned}$$

The result follows.

2. As \mathbf{P} is lower semicontinuous and the family $(A_n)_{n \in \mathbb{N}}$ is independent, we have

$$\begin{aligned}\Pr[(A^*)^c] &= \Pr \left[\bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c \right] \\ &= \lim_{n \rightarrow \infty} \Pr \left[\bigcap_{m=n}^{\infty} A_m^c \right] \\ &= \lim_{n \rightarrow \infty} \prod_{m=n}^{\infty} (1 - \Pr[A_m])\end{aligned}$$

Now for any $n \in \mathbb{N}$, as $\log(1 - x) \leq -x$

$$\begin{aligned}\prod_{m=n}^{\infty} (1 - \Pr[A_m]) &= \exp \left(\sum_{m=n}^{\infty} \log(1 - \Pr[A_m]) \right) \\ &\leq \exp \left(- \sum_{m=n}^{\infty} \Pr[A_m] \right) = 0\end{aligned}$$

The result follows. ■

The reader may verify using the Borel-Cantelli Lemma that if we roll a die an infinite number of times, the probability of rolling a four an infinite number of times is 1 by considering the events $A_n = \{\omega \in \Omega : \omega_n = 6\}$ for each $n \in \mathbb{N}$ where $\Omega = \{1, \dots, 6\}^{\mathbb{N}}$.

We now extend the definition of independence of a family of events as follows.

Definition 2.10 (Independence of classes of events). Let I be an index set and $\mathcal{E}_i \subseteq \mathcal{A}$ for all $i \in I$. The family $(\mathcal{E}_i)_{i \in I}$ is called *independent* if for any finite $J \subseteq I$ and any choice of $E_j \in \mathcal{E}_j$ and $j \in J$, the family $(E_j)_{j \in J}$ is independent.

For example, if we roll a die an infinite number of times, for each $i \in \mathbb{N}$, consider the class of events given by $\mathcal{E}_i = \{\{\omega \in \Omega : \omega_i \in A\} : A \subseteq \{1, \dots, 6\}\}$ where $\Omega = \{1, \dots, 6\}^{\mathbb{N}}$. Then the family $(\mathcal{E}_i)_{i \in I}$ is independent.

Theorem 2.4. Let I be an index set and for each $i \in I$, let $\mathcal{E}_i \subseteq \mathcal{A}$. Then

- (a) Let I be finite. If $\Omega \in \mathcal{E}_i$ for each i , then $(\mathcal{E}_i)_{i \in I}$ is independent if and only if $(E_i)_{i \in I}$ is independent for any choice of $E_i \in \mathcal{E}_i, i \in I$.
- (b) If $(\mathcal{E}_i \cup \{\emptyset\})$ is \cap -closed for each i , then $(\mathcal{E}_i)_{i \in I}$ is independent if and only if $(\sigma(\mathcal{E}_i))_{i \in I}$ is independent.

Proof.

- (a) The forward implication is obvious from the definition. To prove the backward implication, for $J \subseteq I$ and $j \in I \setminus J$, choose $E_j = \Omega$.

(b) The backward implication is obvious. Let us now prove the forward implication.

First, we claim that for any $J \subseteq J' \subseteq I$ where J is finite,

$$\Pr \left[\bigcap_{i \in J'} E_i \right] = \prod_{i \in J'} \Pr[E_i]$$

for any choice of $E_i \in \sigma(\mathcal{E}_i)$ if $i \in J$ and $E_i \in \mathcal{E}_i$ if $i \in J' \setminus J$.

We shall prove the above claim by induction on $|J|$. If $|J| = 0$, then the claim is true as $(\mathcal{E}_i)_{i \in I}$ is independent. Now assume that the claim is true for all $J \subseteq I$ with $|J| = n$ and all finite $J' \supseteq J$. Fix such a J . Let $j \in I \setminus J$. Define $\tilde{J} = J \cup \{j\}$ and choose some $J' \supseteq \tilde{J}$. We shall show that the claim is true if we replace J with \tilde{J} , thus proving the inductive step.

Fix $E_i \in \sigma(\mathcal{E}_i)$ for each $i \in J$ and $E_i \in \mathcal{E}_i$ for each $i \in J' \setminus \tilde{J}$. Consider measures μ, ν on (Ω, \mathcal{A}) such that

$$\begin{aligned} \mu : E_j &\mapsto \Pr \left[\bigcap_{i \in J'} E_i \right] \\ \nu : E_j &\mapsto \prod_{i \in J'} \Pr[E_i] \end{aligned}$$

By the induction hypothesis, $\mu(E_j) = \nu(E_j)$ for all $E_j \in \mathcal{E}_j \cup \{\emptyset, \Omega\}$. As $\mathcal{E}_j \cup \{\emptyset\}$ is \cap -closed, 1.11 implies that $\mu(E_j) = \nu(E_j)$ for all $E_j \in \sigma(\mathcal{E}_j)$.

This proves our claim. Setting $J = J'$ yields the required result. ■