Graphs

Def. A simple graph is a pair of sets $G = (V, E)$ where $E \subseteq \{\{a,b\} : a, b \in V,\ a \neq b\}$.

Recall how we used graphs for relations. (We did use directed graphs there though — basically (a,b) instead of $\{a,b\}$).

A simple graph is basically a symmetric irreflexive relation.
($\{a,b\}$ is modelled as (a,b) and (b,a))

In a non-simple graph, we allow more than one edge between a pair of nodes (multigraph) or more generally, we label edges with weights.

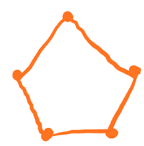
The complete graph K_n is the graph with n nodes and all possible edges between them.

$$(E = \{\{a,b\} : a, b \in V \text{ and } a \neq b\})$$

K_5

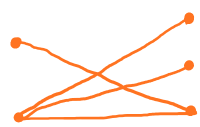
It is also known as a clique of size n.

A cycle C_n is a graph with $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{\{v_i, v_j\} : j = i+1 \text{ or } i = n \text{ and } j = 1\}$



C_5

A graph is said to be bipartite if there exist non-empty disjoint sets V_1 and V_2 such that $V = V_1 \dot\cup V_2$ and $E \subseteq \{\{a,b\} : a \in V_1 \text{ and } b \in V_2\}$. (there are no edges within a part)



For even n, C_n is bipartite

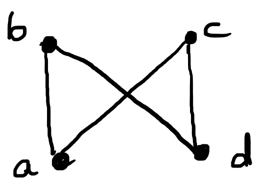
A complete bipartite graph K_{n_1, n_2} is a bipartite graph with $|V_1| = n_1$, $|V_2| = n_2$, and $E = \{\{a,b\} : a \in V_1, b \in V_2\}$.

$$(|E| = n_1 n_2)$$

Def. Graphs $G_1 = (E_1, V_1)$ and $G_2 = (E_2, V_2)$ are isomorphic if there is a bijection $f: V_1 \to V_2$ such that $\{u,v\} \in E_1$ iff $\{f(u), f(v)\} \in E_2$.

(They have the same "structure")

We can also describe graphs by adjacency matrices:



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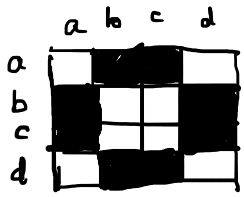


Then two graphs are isomorphic if we can permute the rows and columns (in the same sense) of one matrix to get the other.

A computational problem is to determine if two graphs are isomorphic from their adjacency matrices.
There is no general efficient algorithm known for the above for large graphs.

Def. A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$.

To get a subgraph,

1. Remove zero or more vertices along with the edges incident on them.
2. Further remove zero or more edges.

We get an induced subgraph by omitting the latter step.

Walks and Paths

Def. A walk of length $k \geq 0$ from node a to node b is a sequence of nodes $(a = v_0, v_1, \ldots, v_k = b)$ such that for all $0 \leq i \leq k-1$, $\{v_i, v_{i+1}\} \in E$.

If a walk has no repeating nodes, it is a path.

If a walk of length $k \geq 3$ has $v_0 = v_k$ and has no other repeating nodes, it is called a cycle.
A graph is acyclic if it has no cycles (there is no subgraph isomorphic to C_k)

Def. Nodes u and v are said to be connected if there exists a path from u to v.
Equivalently, they are connected if there is a walk between them. (Why?)

The connectedness relation is an equivalence relation.
The equivalence classes of this relation are called the connected components of G.

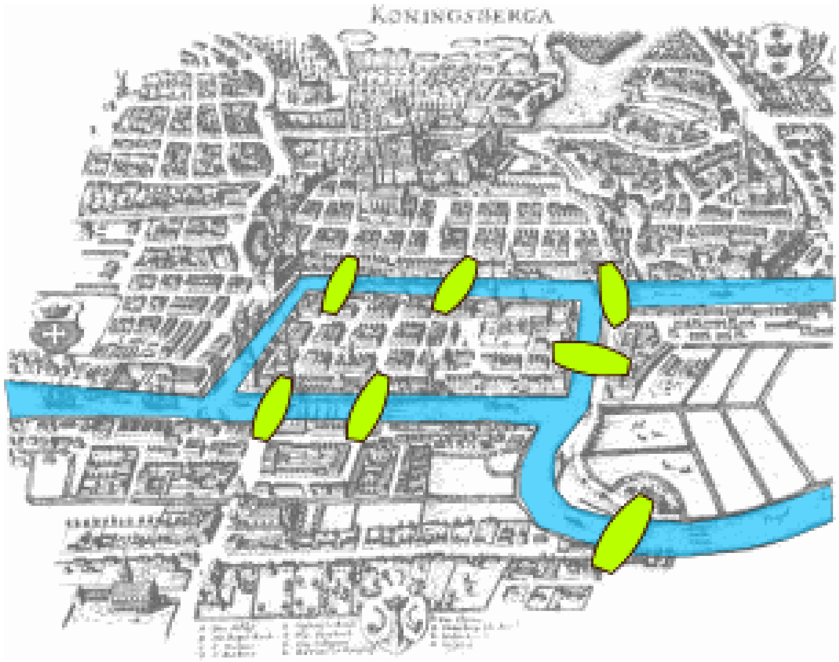
Given a simple graph $G = (V, E)$, the degree of $v \in V$ is the number of edges incident on v. That is,

$$\deg(v) = |\{u : \{u, v\} \in E\}|$$

Note that $2|E| = \sum_{v \in V} \deg(v)$ (each edge is counted twice)

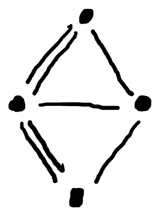
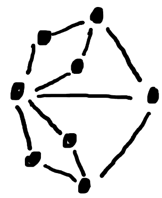
The degree sequence of a graph is a sorted list of degrees.
It is invariant under isomorphism.

We write $\Delta(G) = \max_{v \in V} \deg(v)$.



A famous question known as the "Seven Bridges of Königsberg" asks whether it is possible to walk through the city crossing each bridge exactly once.

We can model this as a graph:

 or equivalently, as a simple graph: 

With this motivation, define an Eulerian trail as a walk visiting each edge exactly once.

The question then asks if an Eulerian trail exists for the latter graph.

Note that if an Eulerian trail exists, there must be at most 2 odd degree nodes.

Indeed, define

$$\text{Enter}(v) = \{\{v_{i-1}, v_i\} : v_i = v\}$$

$$\text{Exit}(v) = \{\{v_i, v_{i+1}\} : v_i = v\}$$

that partitions all edges incident on v. Further, $|\text{Enter}(v)| = |\text{Exit}(v)|$ for all v except the start and end nodes.

$\Rightarrow$ There can be at most two odd degree nodes.

(Namely, the start and end)

As the graph drawn above has 3 odd degree nodes no Eulerian walk exists.

An Eulerian circuit is a closed walk ($v_0 = v_k$) visiting each edge exactly once.
If an Eulerian circuit exists, there are no odd degree nodes.

Further, if there are no odd degree nodes and all edges appear in a single connected component, there must exist an Eulerian circuit!
(Try splitting it into several cycles and "stitching" them together)
This also gives an efficient algorithm to find an Eulerian circuit if it exists.

A Hamiltonian cycle is a cycle that contains all nodes in the graph.

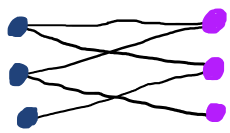
There is no efficient algorithm known to check if a graph has a Hamiltonian cycle.
Indeed, this is an "NP-hard" problem.

Given connected nodes u and v, the distance between them is the length of a shortest walk between them. (and ∞ if no walk exists)
this shortest walk must be a path.

We also define the diameter of a graph as the largest distance between two nodes in a graph.
(This is ∞ if the graph is not connected)

Graph Coloring

We can "color" bipartite graphs using two colors such that there is no edge between two nodes of the same color.



More generally, a coloring using k colors is proper if there is no edge between nodes of the same color.
A function $c: V \to [k]$ such that
$$\forall x,y \in V \quad \{x,y\} \in E \to c(x) \neq c(y)$$
The least number of colours possible in a proper colouring of G is called the chromatic number of G, denoted $\chi(G)$.

Suppose H is a subgraph of G. Then
G has a k-coloring $\Rightarrow H$ has a k-coloring.
That is, $\chi(H) \leq \chi(G)$.
In particular, if K_n is a subgraph of G, $\chi(G) \geq n$
If C_n for odd n is a subgraph, $\chi(G) > 2$.

Ex. Prove that isomorphic graphs have equal chromatic number.

There is no efficient algorithm known to calculate chromatic number.
(It is NP-hard)

Let us consider bipartite graphs.

For all integer $n \geq 1$, C_{2n+1} is not bipartite.
(Left as an exercise, try induction)

This extends to the following theorem.

Theo. A graph G (with $|V|>1$) is bipartite iff it has no odd cycle.

Proof. Let G be not bipartite.

Then G must contain a connected component that is bipartite (with >1 nodes) (why?)

Fix some v in this component and let

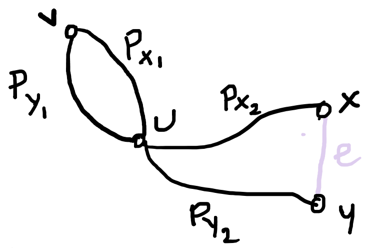
$A = \{x : \text{dist}(x,v) \text{ is even}\}$

$B = \{x : \text{dist}(x,v) \text{ is odd}\}$

As it is not bipartite, there exists edge $e=\{x,y\}$ where $x,y \in A$ or $x,y \in B$.

Let P_x and P_y be the shortest paths from v to x and v to y.

Suppose P_x and P_y both pass through node $u \neq v$.



Observe that $|P_{x_1}| = |P_{y_1}|$ due to the minimality of $|P_x|$ and $|P_y|$.

As $|P_x|$ and $|P_y|$ are either both even or both odd, $|P_{x_2}|$ and $|P_{y_2}|$ are either both even or both odd as well.

Using this, we can assume that P_x and P_y intersect only at v. (Otherwise, we can do this reduction repeatedly)

Then $P_x \cup P_y \cup \{e\}$ is a cycle of odd length!

(P_x, then e, then P_y^{rev})

For the converse, when G has no odd cycle, the coloring corresponding to

$A = \{x : \text{dist}(x,v) \text{ is even}\}$

$B = \{x : \text{dist}(x,v) \text{ is odd}\}$

is a valid one.

Let G have n nodes. $\chi(G)=n \longleftrightarrow G$ is isomorphic to K_n.

(If G is not isomorphic to K_n, there exist u,v s.t. $\{u,v\}\notin E$
Color u and v in the same color and every other vertex
differently $\Rightarrow \chi(G)\le n-1$)

For a graph G, the clique number $\omega(G)$ is the largest k such that G has a subgraph isomorphic to K_k.

$$\chi(G)\ge\omega(G)$$

The independence number $\alpha(G)$ is the largest k such that G has a set of k nodes with no edges among them.

(Such a subgraph with no edges is an anticlique)

In a colouring, the nodes of each colour must correspond to an anticlique.

Consider a coloring of G with $\chi(G)$ nodes.

$$n=\sum_c |\text{nodes with colour } c| \le \chi(G)\,\alpha(G)$$

$$\chi(G)\ge\frac{n}{\alpha(G)}$$

Theo. For any G, $\chi(G)\le\Delta(G)+1$

Proof. Try induction on the number of nodes.

Suppose for all $G=(V,E)$ with $|V|=k$, $\chi(G)\le\Delta(G)+1$

Let $G=(V,E)$ with $|V|=k+1$.

Let $G=(V',E')$ be obtained from G by removing some $v\in V$.
(and all incident edges)

$$\chi(G')\le\Delta(G')+1\le\Delta(G)+1.$$

Colour G' with $\Delta(G)+1$ colours.

However, $\deg(v)\le\Delta(G)$. Just colour v with a colour in $\{1,\dots,\Delta(G)+1\}$ that does not appear in its neighbourhood.

$\Rightarrow \chi(G)\le\Delta(G)+1$

Note that this describes a (recursive) algorithm to colour a graph G in $\Delta(G)+1$ colours.

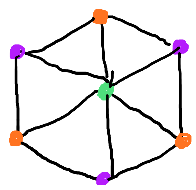
Note that equality holds in the above in K_n and C_{2n+1}.
Turns out that these are the only cases where equality holds.

A graph with no cycles is known as a forest.

The path graph P_n has
$V = \{1, 2, \cdots, n\}$ and $E = \{\{i, i+1\} : i \in [n-1]\}$

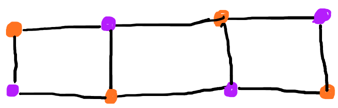
(P_7)

The wheel graph W_n $(n \geq 3)$ has
$V = \{hub\} \cup \mathbb{Z}_n$ (hub is some specific node)
$E = \{\{x, hub\} : x \in \mathbb{Z}_n\} \cup \{\{x, x+1\} : x \in \mathbb{Z}_n\}$



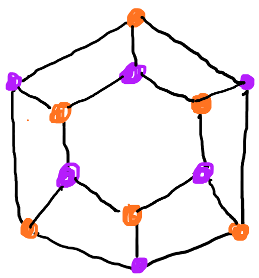
(W_6) Note that C_n is a subgraph of W_n.

The ladder graph L_n has
$V = \{0,1\} \times [n]$
$E = \{\{(0,i), (1,i)\} : i \in [n]\} \cup \{\{(b,i), (b,i+1)\} : b \in \{0,1\}, i \in [n-1]\}$



(L_4) Note that P_n is a subgraph of L_n (up to isomorphism)

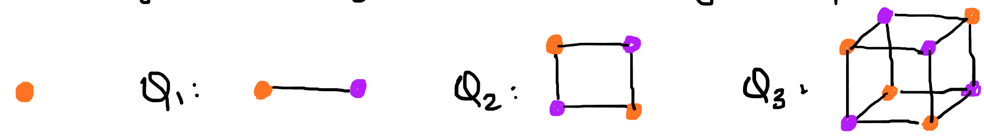
The circular ladder graph CL_n is the same as L_n but with two additional edges: $\{(b,n), (b,1)\}$ for $b \in \{0,1\}$.



(CL_6)

The hypercube graph Q_n has
V = set of all n-bit strings.
$E = \{\{x,y\} : x \text{ and } y \text{ differ at exactly one position}\}$

Q_0 : Q_1 : Q_2 : Q_3 :

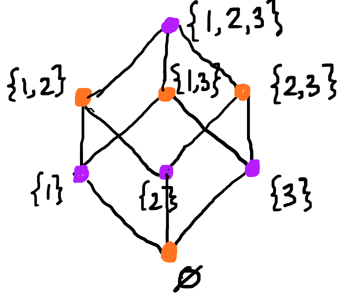


Q_n has 2^n nodes but the diameter $\left(\max\limits_{x,y \in V} \operatorname{dist}(x,y)\right)$ is only n.

It is n-regular and bipartite.
↳ label nodes with even no. of 1s in one colour.
odd no. of 1s in another.

Q_{n-1} is a subgraph of Q_n (up to isomorphism)

Now consider the graph isomorphic to Q_n where nodes correspond to subsets of $[n]$ and edges exist between subsets that differ by a single element.



Consider the graph $\overline{KG}_{n,k}$ which has the nodes at the k^{th} level of this, namely the subsets of size k. Insert an edge between any two nodes that intersect.
A clique in $\overline{KG}_{n,k}$ is a set of subsets which intersect pairwise.
For example, $\{\{n\} \cup S : S \subseteq [n-1], |S| = k-1\}$ is a clique with $C(n-1, k-1)$ nodes.

The Erdös-Ko-Rado Theorem states that if $k \leq n/2$, then there are no larger cliques than this.

The **Kneser Graph** $KG_{n,k}$ is the complement of this graph, that is, there is an edge between disjoint subsets.

So in $KG_{n,k}$, if $k \le n/2$, the largest anticlique is of size $C(n-1, k-1)$.

Let us define the "complement" we used above more concretely. The **complement** of $G=(V,E)$ is $\overline{G} = (V, \overline{E})$, where
$\overline{E} = \underbrace{\{\{a,b\} : a,b \in V, a \neq b\}}_{\text{this set is sometimes denoted } C(V,2)} \setminus E$.

We can similarly define the union, intersection, difference and symmetric difference of two graphs with the same vertex set.

We can further extend this to the union/intersection of $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ by taking the union intersection of the vertex sets and edge sets separately.

Given $G=(V,E)$, $G^2=(V,E')$ where
$$E' = E \cup \{\{x,y\} : \exists w\ \{x,w\},\{w,y\} \in E\}$$

More generally, G^k has an edge $\{x,y\}$ iff $\text{dist}(x,y) \le k$.

Given $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, their **cross product** or **tensor product** is $G_1 \times G_2 = (V_1 \times V_2, E)$ where
$$\{(u_1,v_1),(u_2,v_2)\} \in E \text{ iff } \{u_1,u_2\} \in E_1 \text{ and } \{v_1,v_2\} \in E_2.$$

For any graph G, $G \times K_2$ is a bipartite graph.

Given $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, their **box product** is $G_1 \square G_2 = (V_1 \times V_2, E)$ where
$$\{(u_1,v_1),(u_2,v_2)\} \in E \text{ iff } (\{u_1,u_2\} \in E_1 \text{ and } v_1=v_2) \text{ or } (\{v_1,v_2\} \in E_2 \text{ and } u_1=u_2)$$

For example, $Q_m \square Q_n = Q_{m+n}$ (isomorphically).

The Hamming graph $H_{n,q}$ is

$$K_q \square K_q \square \cdots \square K_q \quad \text{(n copies)}$$

Vertex set is $[q]^n$. There is an edge between two nodes if they are the same for all but one co-ordinate.