

Def. A set $E \subseteq X$ is called **open** if $\forall e \in E, \exists r > 0$ s.t. $B(e, r) \subseteq E$.

Def. A neighbourhood of a point $x \in E$ is a ball $B(x, r)$ for some $r > 0$.

Theo. An arbitrary union of open sets is open.

Proof. Let $\{E_\alpha\}_{\alpha \in A}$ be open.

$$\begin{aligned} \text{Then } x \in \bigcup_{\alpha \in A} E_\alpha &\Rightarrow \exists \alpha \in A \text{ s.t. } x \in E_\alpha \\ &\Rightarrow \exists r > 0 \text{ s.t. } B(x, r) \subseteq E_\alpha \\ &\Rightarrow B(x, r) \subseteq \bigcup_{\alpha \in A} E_\alpha \Rightarrow \bigcup_{\alpha \in A} E_\alpha \text{ is open. } \quad \square \end{aligned}$$

What about an arbitrary intersection? No!

Consider $\left\{ \left(-\frac{1}{n}, \frac{1}{n} \right) \right\}_{n \in \mathbb{N}}$. The intersection of these sets is $\{0\}$, which is not open.

Ex. Prove that a finite intersection of open sets is open.

Theo. Let E be an open set in \mathbb{R} . Then E is ^{at most} a countable union of disjoint open intervals in \mathbb{R} .

Proof. For each $x \in E$, consider

$$J_x = \bigcup_{\substack{x \in I \\ I \text{ is an open interval}}} I. \quad J_x \text{ is the maximal open interval that contains } x.$$

Now, for $x, y \in E$, either $J_x = J_y$ or $J_x \cap J_y = \emptyset$. That is, E is a union of disjoint open intervals. This implies that it is an at most countable union.
 (As there is a rational number in each interval, the number of intervals is $\leq |\mathbb{Q}| = |\mathbb{N}|$). \square

Does an analogous result hold for \mathbb{R}^2 ?

Def. Let (X, d) be a metric space and $E \subseteq X$. A point $x \in E$ is called a **limit point** of E if for all $r > 0$,

$$B(x, r) \cap E \setminus \{x\} \neq \emptyset.$$

That is, there exist points in the set arbitrarily close to x .
 $\forall r > 0, \exists y \in E \text{ s.t. } y \neq x, d(x, y) < r$

Example. Consider $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}$.

0 is a limit point of E . (Consequence of the Archimedean property).

Ex. Prove that 0 is the only limit point of the above set.

We denote the set of limit points of a set E by E' .

Ex. Prove that $\mathbb{Q}' = \mathbb{R}$.

Ex. Let $E \subseteq \mathbb{R}$ be a finite set. Prove that $E' = \emptyset$.

Def. Let (X, d) be a metric space and $E \subseteq X$. E is said to be **closed** if $E' \subseteq E$.

Note that \emptyset is neither open nor closed (in \mathbb{R})

Also, \emptyset and \mathbb{R} are both open and closed (in \mathbb{R}).

We expect some relation between open sets and closed sets.
 (if anything, from the names "open" and "closed")

Theo. Let (X, d) be a metric space and $E \subseteq X$. Then
 E is closed $\Leftrightarrow E^c$ is open.

Proof. Let E^c be closed. For $x \in E$, x is not a limit point of E^c .

$$\Rightarrow \exists r > 0 \text{ s.t. } E^c \cap B(x, r) \setminus \{x\} = \emptyset$$

$$\Rightarrow E^c \cap B(x, r) = \emptyset$$

$$\Rightarrow B(x, r) \subseteq E \Rightarrow E \text{ is open.}$$

Let E be open. Let x be a limit point of E^c st. $x \notin E^c$.

$$\text{Then } \forall r > 0 \quad B(x, r) \cap E^c \setminus \{x\} \neq \emptyset$$

$$\Rightarrow \exists r > 0 \quad B(x, r) \cap E^c \neq \emptyset.$$

However, $\exists r_0 > 0 \quad B(x, r_0) \subseteq E$ (as $x \in E$ and E is open)

Contradiction! $\rightarrow E^c$ is closed. ■

Corollary. 1) Arbitrary intersection of closed sets is closed.

2) Finite union of closed sets is closed.

\hookrightarrow Not arbitrary: example. $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 1 - \frac{1}{n}] = (0, 1)$

Def. Let (X, d) be a metric space and $E \subseteq X$. Then the closure of

the set E , denoted \bar{E} , is given by

$$\bar{E} = E \cup E'$$

For example, $\overline{\mathbb{Q}} = \mathbb{R}$.

Theo. Given any E ,

1) \bar{E} is closed.

2) $E = \bar{E} \Leftrightarrow E$ is closed.

3) \bar{E} is the smallest closed set that contains E .

\hookrightarrow the intersection of all closed sets that contain E .

Proof. 1) We shall show that \bar{E}^c is open. Let $y \in \bar{E}^c \Rightarrow y$ is neither in E

nor a limit point of

E .

$$\begin{aligned} \Rightarrow \exists r > 0 \quad B(y, r) \cap E \setminus \{y\} &= \emptyset \\ \Rightarrow B(y, r) \cap E &= \emptyset \quad \text{Also, } B(y, r) \cap E' = \emptyset \\ \Rightarrow B(y, r) \cap \bar{E} &= \emptyset \Rightarrow B(y, r) \subseteq \bar{E}^c \\ &\Rightarrow \bar{E} \text{ is closed.} \end{aligned}$$

2) $\bar{E} = E \cup E'$

$$E = \bar{E} \Leftrightarrow E' \subseteq E \Leftrightarrow E \text{ is closed.}$$

3) let $F \supseteq E$ be a closed set. This implies that $E' \subseteq F' \subseteq F$.

$$\text{As } E \subseteq F \text{ and } E' \subseteq F, \bar{E} \subseteq F.$$

$\Rightarrow \bar{E}$ is the smallest closed set that contains E . ■

Def. Let (X, d) be a metric space. E is said to be dense (in X)

$$\text{if } \bar{E} = X.$$

*Ex. Prove that E is dense in X
iff for any open $U \subseteq X, E \cap U \neq \emptyset$.*

For example, \mathbb{Q} and $\mathbb{R} \setminus \mathbb{Q}$ are dense in \mathbb{R} .

Def. Let (X, d) be a metric space and $E \subseteq X$. An **open cover** (of E) is a collection of open sets $\{G_\alpha\}_{\alpha \in A}$ such that

$$E \subseteq \bigcup_{\alpha \in A} G_\alpha$$

Def. Let (X, d) be a metric space and $E \subseteq X$. E is said to be **compact** if every open cover of E has a finite subcover.

i.e. if $E \subseteq \bigcup_{\alpha \in A} G_\alpha$ where each G_α is open, then $\exists \alpha_1, \alpha_2, \dots, \alpha_n \in A$
st. $E \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$.

Example. Any finite set is a compact set

(We can choose sets G_1, G_2, \dots, G_n such that)
 $x_i \in G_i$ where the finite set is $\{x_1, x_2, x_3, \dots, x_n\}$

\mathbb{R} is not compact.

(Consider the open cover $\{(n-1, n+1)\}_{n \in \mathbb{Z}}$)

$(0, \infty)$ is not compact.

(Consider the open cover $\{\left(\frac{1}{n}, 2\right)\}_{n \in \mathbb{N}}$)

Ex. Let $E \subseteq \mathbb{R}$ be not bounded. Can E be compact?

(Consider the open cover of \mathbb{R} we used earlier)

\Rightarrow Any compact set is bounded.

Theo. Let (X, d) be a metric space and $E \subseteq X$. If E is compact, then E is closed.

Proof. We shall equivalently prove that E^c is open.

Let $p \in E^c$ and $q \in E$. Then $d(p, q) > 0$.

For each such q , choose r_q st. $0 < r_q < \frac{d(p, q)}{2}$.

Consider $\{B(q, r_q)\}_{q \in E}$. This is clearly an open cover of E .

As E is compact, there is a finite subcover:

$$E \subseteq B(q_1, r_{q_1}) \cup B(q_2, r_{q_2}) \cup \dots \cup B(q_n, r_{q_n})$$

for some $q_1, q_2, \dots, q_n \in E$.

Let $r = \min \{r_{q_1}, r_{q_2}, \dots, r_{q_n}\}$.

We now claim that $E \cap B(p, r) = \emptyset$

Let us assume otherwise. If $x \in E \cap B(p, r)$.

Then $d(x, p) < r$ and also,

$$x \in \bigcup_{1 \leq i \leq n} B(q_i, r_{q_i}) \Rightarrow x \in B(q_j, r_{q_j})$$

for some j .

$$\begin{aligned} \Rightarrow d(p, q_j) &\leq d(x, p) + d(x, q_j) \\ &< r + r_{q_j} \leq 2r_{q_j} < d(p, q_j) \end{aligned}$$

Contradiction! As $B(p, r) \subseteq E^c$, E is closed. ■

Ex. Prove that $\{\emptyset\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.

We now go towards the characterization of compact sets in \mathbb{R}^n .

Lemma. Let (X, d) be a metric space and $E \subseteq X$ be compact.
If $F \subseteq E$ is closed, then F is compact.

Proof. Let $(V_\alpha)_{\alpha \in A}$ be an open cover of F . Then $(V_\alpha)_{\alpha \in A} \cup (X \setminus F)$ is an open cover of E . $\exists \alpha_1, \alpha_2, \dots, \alpha_n \in A$ such that

$$\begin{aligned} F \subseteq E &\subseteq V_{\alpha_1} \cup \dots \cup V_{\alpha_n} \cup (X \setminus F) \\ \Rightarrow F &\subseteq V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n} \\ \Rightarrow F &\text{ is compact.} \end{aligned}$$

open set

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Lemma. Let $(I_n)_{n \in \mathbb{N}}$ be a sequence of closed and bounded intervals in \mathbb{R} such that $I_1 \supseteq I_2 \supseteq \dots$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Let $I_n = [a_n, b_n]$ for each $n \in \mathbb{N}$.

Let $E = \{a_n : n \in \mathbb{N}\}$. $E \neq \emptyset$ and E is bounded above by b_1 .

$$\text{Let } x = \sup E.$$

We claim that $x \in \bigcap_{n=1}^{\infty} I_n$.

For $n, m \in \mathbb{N}$, $a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$

$\Rightarrow x \leq b_m \quad \forall m$. We also have $a_n \leq x \quad \forall n$.

$\Rightarrow x \in [a_m, b_m] \quad \forall m$

$\Rightarrow x \in \bigcap_{n=1}^{\infty} [a_n, b_n].$

(Where did we
use closedness and
boundedness?)

■

\downarrow

$\{(1, \frac{1}{n}) : n \in \mathbb{N}\}$

and $\{[n, \infty) : n \in \mathbb{N}\}$

Theo. [Characterization of compact sets in \mathbb{R}^n] Let $E \subseteq \mathbb{R}^n$. E is compact if and only if E is closed and bounded.

Proof. Claim 1. Any closed and bounded interval is compact.

Suppose $(G_\alpha)_{\alpha \in A}$ is an open cover of $[a, b]$ that does not have a finite subcover. $\hookrightarrow I$.

Then at least one of $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ does not have a finite subcover, let it be I_2 . We can similarly split I_2 to get I_3 . Continue this process to get a sequence of closed and bounded intervals $(I_n)_{n \in \mathbb{N}}$ s.t. $I_1 \supseteq I_2 \supseteq \dots$ and none of them has a finite subcover of $(G_\alpha)_{\alpha \in A}$.

Further, if $x, y \in I_n$, then $d(x, y) \leq \frac{b-a}{2^{n-1}}$.

Let $x \in \bigcap_{n=1}^{\infty} I_n \subseteq \bigcup_{\alpha \in A} G_\alpha \Rightarrow x \in G_{\alpha_0}$ for some $\alpha_0 \in A$.
 \hookrightarrow open set

$$\Rightarrow \exists r > 0 \quad B(x, r) \subseteq G_{\alpha_0}$$

There then exists n_0 s.t. $\frac{b-a}{2^{n_0-1}} < r \Rightarrow I_{n_0} \subseteq B(x, r) \subseteq G_{\alpha_0}$
 \downarrow finite subcover

Contradiction! \Rightarrow Any closed and bounded interval is compact.

Claim 2. Any closed and bounded set in \mathbb{R} is compact.

Let E be a closed and bounded set and $I \supseteq E$ be a closed bounded interval. Then as I is compact, E is compact.
 \hookrightarrow a closed subset of I .

We have already proved that a compact subset of \mathbb{R} is closed and bounded. This completes the proof.

(The proof is similar for \mathbb{R}^n) ■

let us now consider what sets have limit points.

Clearly not every set (finite sets, \mathbb{N} , etc.)

Theo.: Let (X, d) be a metric space and $E \subseteq X$ be compact. Then any infinite subset of E has a limit point in E .

Proof: Suppose no point of E is a limit point of our infinite subset $F \subseteq E$.

\Rightarrow For each $x \in E$, $\exists r_x > 0$ s.t. $B(x, r_x) \cap F \setminus \{x\} = \emptyset$.

$\Rightarrow B(x, r_x)$ contains at most one point from F .

Note that $\bigcup_{x \in E} B(x, r_x)$ is an open cover of E , a compact set.

Let $F \subseteq E \subseteq \bigcup_{i=1}^n B(x_i, r_{x_i}) \Rightarrow F$ contains at most n points.

Contradiction! (F is an infinite set)

\Rightarrow Some point of E is a limit point of F .

Corollary: Every infinite bounded set in \mathbb{R} has a limit point in \mathbb{R} .
(Bolzano-Weierstrass Theorem)

(The set is a subset of some closed and bounded interval)

The converse of the above theorem holds as well.

Theo.: Let (X, d) be a metric space and $E \subseteq X$ s.t. any infinite subset of E has a limit point in E . Then E is compact.

Proof: We shall prove the result for $E \subseteq \mathbb{R}$.

Suppose E is not bounded. For each $n \in \mathbb{N}$, let $x_n \in E$ s.t. $|x_n| > n$.

Then $\{x_1, x_2, \dots\} \subseteq E$ has no limit point in E . $\rightarrow E$ is bounded.

Suppose E is not closed. Then $\exists x_0 \in E^c$ which is a limit point of E .

$$\rightarrow \forall n \in \mathbb{N} \quad E \cap B(x_0, \frac{1}{n}) \setminus \{x_0\} \neq \emptyset.$$

Let $x_n = E \cap B(x_0, \frac{1}{n}) \quad \forall n$.

Then $K = \{x_1, x_2, \dots\} \subseteq E$. We claim $K' = \{x_0\}$. Clearly, $x_0 \in K'$.

Let $y \neq x_0$, $y \in E$. We shall prove that $y \notin K'$.

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Note that $|y - x_n| \geq |x_0 - x_n| + |y - x_0| \quad \forall n \in \mathbb{N}$

$$\geq |y - x_0| - \frac{1}{n}$$

$$y \neq x_0 \Rightarrow \exists n_0 \text{ s.t. } |y - x_0| > \frac{2}{n_0}$$

$$\Rightarrow |y - x_0| > \frac{2}{n} \quad \forall n > n_0$$

$$\Rightarrow |y - x_n| \geq \frac{|y - x_0|}{2} \quad \forall n \geq n_0 \Rightarrow y \notin K'$$

$\Rightarrow E$ is closed $\Rightarrow E$ is compact.

Try proving this for a general metric space (it's true)

Note that this gives another necessary and sufficient condition for compactness (every infinite subset must have a limit point in it)

Def. Let (X, d) be a metric space and $Y \subseteq X$. Then $U \subseteq Y$ is said to be open in Y if it is open in the metric space $(Y, d|_Y)$
or open relative to Y

\hookrightarrow metric restricted to Y .

Theo. Let (X, d) be a metric space and $Y \subseteq X$. $U \subseteq Y$ is open in Y iff there is an open set $G \subseteq X$ such that $U = G \cap Y$.

Proof

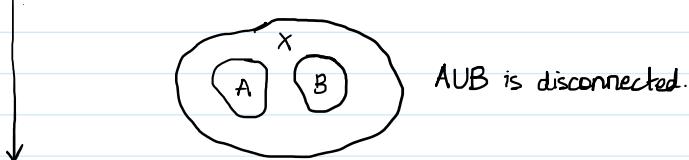
Therefore, we can define the **subspace topology** on $Y \subseteq X$ as $\{Y \cap G : G \subseteq X, G \text{ is open}\}$ which is the set of all open sets in Y considering the metric on X restricted to Y .

Def. Let (X, d) be a metric space. X is said to be **disconnected** if there exist non-empty $A, B \subseteq X$ such that $A \cup B = X$, $\overline{A} \cap B = \emptyset$, and $A \cap \overline{B} = \emptyset$.

X is said to be **connected** if it is not disconnected.

Let $E \subseteq X$. E is **disconnected** if there exist non-empty $A, B \subseteq E$ such that $A \cup B = E$, $\overline{A}^E \cap B = \emptyset$, and $A \cap \overline{B}^E = \emptyset$.

Ex. Let $U \subseteq Y \subseteq X$. Prove that $\overline{U}^Y = Y \cap \overline{U}^X$.



This implies that E is disconnected iff there exist non-empty $A, B \subseteq E$ such that $A \cup B = E$, $\overline{A}^E \cap B = \emptyset$, and $A \cap \overline{B}^E = \emptyset$.

Connectedness seems to correlate to our intuitive understanding of it.

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 $\left(\begin{array}{l} \text{something like "continuous"} \\ \text{sets — intervals?} \end{array} \right)$

Theo.

Let $E \subseteq \mathbb{R}$. If E is not an interval, E is disconnected.

Proof.

Suppose E is not an interval.

\rightarrow There exist $x, y \in E$ and $z \notin E$ with $x < z < y$. (If there is no such triplet, E is an interval by def.)

Let $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$.

$$\bar{A} \subseteq (-\infty, z] \Rightarrow \bar{A} \cap B = \emptyset$$

$$\bar{B} \subseteq [z, \infty) \Rightarrow A \cap \bar{B} = \emptyset$$

$$A \cup B = (\mathbb{R} \setminus \{z\}) \cap E = E$$

$\Rightarrow E$ is disconnected.

The contrapositive of the above just says:

If $E \subseteq \mathbb{R}$ is connected, E is an interval.

Theo.

If $E \subseteq \mathbb{R}$ is an interval, E is connected.

Suppose E is disconnected. Then $\exists A, B$ s.t. $A \cup B = E$, $\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$.

Let $x \in A$ and $y \in B$. Assume wlog that $x < y$.

Let $z = \sup(A \cap [x, y])$

$$\Rightarrow z \in \overline{A \cap [x, y]} \subseteq \overline{A \cap [x, y]} = \bar{A} \cap [x, y]$$

\hookrightarrow as $(\overline{A \cap [x, y]})$ is the smallest closed set containing $A \cap [x, y]$

$$\Rightarrow z \in \bar{A} \Rightarrow z \notin B \Rightarrow x \leq z < y.$$

\hookrightarrow as $y \in B$ and $z \in [x, y]$.

(a) If $z \notin A$, then $x < z < y$ and $z \notin E$. $\Rightarrow E$ is not an interval.

(b) If $z \in A$, then $z \notin \bar{B} = B \cup B'$ (as $A \cap \bar{B} = \emptyset$)

$$\Rightarrow \exists r > 0 \text{ s.t. } (z - r, z + r) \cap B = \emptyset.$$

$$\Rightarrow \forall \tilde{z} \text{ with } z < \tilde{z} < z + r, \tilde{z} \notin B.$$

$$\Rightarrow \exists z_1 \notin B, z_1 > z \text{ s.t. } z < z_1 < y$$

$$\text{Also, } z_1 \notin A. \quad \left(\begin{array}{l} z = \sup(A \cap [x, y]) \text{ and } x \leq z < y \\ \text{and } z_1 > z \end{array} \right)$$

$\rightarrow z \notin E$.

Therefore, E is not an interval.

$\therefore E \subseteq \mathbb{R}$ is connected if and only if E is an interval.

Note. This cannot be extended to \mathbb{R}^n .
(A circle is connected in \mathbb{R}^2 .)

Theo.

Let (X, d) be a metric space.

- 1. X is disconnected.
- 2. $X = C \cup D$ where C, D are non-empty, disjoint, and closed.
- 3. $X = C \cup D$ where C, D are non-empty, disjoint, and open.
- 4. There is a non-empty proper subset of X that is both open and closed.

} Equivalent.

Proof.

$1 \Rightarrow 2$:

Suppose $X = A \cup B$ with $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

We claim A is closed.

$$\overline{A} \subseteq X = A \cup B \text{ and } \overline{A} \cap B = \emptyset \Rightarrow \overline{A} \subseteq A$$

$\Rightarrow X = A \cup B$ where A, B are non-empty, disjoint, and closed.

$2 \Rightarrow 1$:

This follows directly.

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Def.

Let (X, d) be a metric space. X is path-connected if for any $x, y \in X$, there is a continuous function $f: [0, 1] \rightarrow X$ such that $f(0) = x$ and $f(1) = y$.

Theo.

Let (X, d) be a metric space.

X is path-connected $\Rightarrow X$ is connected.

$$\left(\begin{array}{l} \text{Converse is not true.} \\ \text{Consider } \{(x, \sin \frac{1}{x}) \in \mathbb{R}^2 : \\ x \in \mathbb{R} \setminus \{0\}\} \end{array} \right)$$

\cup open connected on $\mathbb{R}^n \Rightarrow$ path-connected