The KLS Conjecture

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§0. Notation

- We refer to measures by greek symbols such as μ and ν and their densities by lowercase alphabets beginning from p.
- B refers to the Euclidean ball of radius 1 in $\mathbb{R}^n$ (the value of n is usually understandable from context).
- Given a measure μ on $\mathbb{R}^n$ and a set X whose support is at most $(n-1)$-dimensional, $\mu^+(X)$ refers to the "surface area" of the set X, that is,
$$\mu^+(X) = \lim_{\varepsilon \to 0} \frac{\mu(X + \varepsilon B)}{2\varepsilon}.$$
Alternatively, if $X \subseteq \mathbb{R}^n$, then
$$\mu^+(\partial X) = \lim_{\varepsilon \to 0} \frac{\mu(X + \varepsilon B) - \mu(X)}{\varepsilon}.$$
- While needles in [KLS95] refer to one-dimensional segments with a polynomial factor (ℓ^{n-1} where ℓ is linear) in particular, we use them more generally to refer to log-concave measures with a one-dimensional support.

§1. Measure Disintegration

1.1. Introduction

To begin with, let us describe a needle decomposition procedure given in [LV18] to prove the KLS Conjecture. Suppose that we are given a log-concave measure μ with density p with compact convex support K. Let us also fix a subset $E \subseteq K$ of measure $1/2$. We would like to bound $\mu^+(\partial E)$ below (over all such E).
Now, suppose that we have some hyperplane H that divides space into two half-spaces H_1 and H_2. Let $K_i = K \cap H_i$ and further assume that $\mu(E \cap H_i) = \frac{1}{2}\mu(K_i)$ for each i. Consider the measures μ_1 and μ_2 with densities

$$p_i = p \mathbb{1}_{K_i} \frac{\mu(K)}{\mu(K_i)}.$$

Observe that

$$\begin{aligned} p &= p_1 \frac{\mu(K_1)}{\mu(K)} + p_2 \frac{\mu(K_2)}{\mu(K)} \\ \mu &= \mu_1 \frac{\mu(K_1)}{\mu(K)} + \mu_2 \frac{\mu(K_2)}{\mu(K)} \end{aligned} \tag{1.1}$$

More generally, suppose we have some space Ω with a probability measure ν on it such that

$$\mu = \int_\Omega \mu_\omega \, \mathrm{d}\nu(\omega), \tag{1.2}$$

where the (μ_ω) are log-concave measures on $\mathbb{R}^n$. In the above example, we can take $\Omega = \{1, 2\}$ and $\nu(\{i\}) = \mu(K_i)/\mu(K)$ for $i \in \Omega$.
Then, given any set E of measure $1/2$, we have

$$\begin{aligned}\mu^+(\partial E) &= \int_\Omega \mu_\omega^+(\partial E) \, \mathrm{d}\nu(\omega) \\ &\geq \int_\Omega \psi_\omega \mu_\omega(E)(1 - \mu_\omega(E)) \, \mathrm{d}\nu(\omega), \end{aligned} \tag{1.3}$$

where ψ_ω is the isoperimetric constant of μ_ω. If we manage to bound the expression on the right below by some constant independent of E, then the KLS conjecture follows. It is also worth noting that the decomposition we choose may be dependent on E itself, we only require that the lower bound constant does not depend on this choice of E.

1.2. A proof of the $n^{-1/2}$ bound using needle decompositions

"Needle decomposition" refers to the process of performing the step we used to obtain (1.1) until the bodies K_ω become one-dimensional. We repeatedly split the bodies in a way that the quantity $\mu_\omega(E)$ remains constant at $1/2$. Suppose that we do so and the final limiting set of needles is $(K_\omega)_{\omega \in \Omega}$. Then, we can use one-dimensional isoperimetry to get that for any ω, $\psi_\omega \gtrsim \|A_\omega\|_{\mathrm{op}}^{-1/2}$. We also have that $\mu_\omega(E) = 1/2$, so

$$\mu^+(\partial E) \gtrsim \int_\Omega \|A_\omega\|_{\mathrm{op}}^{-1/2} \, \mathrm{d}\nu(\omega). \tag{1.4}$$

We wish to bound the integral on the right below.

To do so, consider (1.2) (or rather, the similar expression for the density p). Then, we have that

$$\int_{\mathbb{R}^n} p(x) x x^\top \, \mathrm{d}x = \int_\Omega \int_{\mathbb{R}^n} p_\omega(x) x x^\top \, \mathrm{d}x \, \mathrm{d}\nu(\omega).$$

Thus,

$$A + bb^\top = \int_\Omega A_\omega + b_\omega b_\omega^\top \, \mathrm{d}\nu(\omega), \tag{1.5}$$

where A and b (resp. A_ω and b_ω) refer to the covariance matrix and barycenter of μ (resp. μ_ω) respectively.
Assume without loss of generality that $b = 0$. Taking the trace on either side of the above expression,

$$\begin{aligned}\operatorname{Tr}(A) &= \int_\Omega \operatorname{Tr}(A_\omega) + \|b_\omega\|^2 \, \mathrm{d}\nu(\omega) \\ &\geq \int_\Omega \|A_\omega\|_{\mathrm{op}} \, \mathrm{d}\nu(\omega), \end{aligned}$$

where the inequality follows from the fact that A_ω is a covariance matrix so is positive semi-definite. One can then use Hölder's inequality to get

$$\left(\int_\Omega \|A_\omega\|_{\mathrm{op}} \, \mathrm{d}\nu(\omega) \right) \left(\int_\Omega \|A_\omega\|_{\mathrm{op}}^{-1/2} \, \mathrm{d}\nu(\omega) \right)^2 \geq 1$$

and so,

$$\int_\Omega \|A_\omega\|_{\mathrm{op}}^{-1/2} \, \mathrm{d}\nu(\omega) \gtrsim \operatorname{Tr}(A)^{-1/2}.$$

Substituting this back in (1.4), we get $\psi_p \gtrsim \operatorname{Tr}(A)^{-1/2}$, that is, $\psi_n \gtrsim n^{-1/2}$.

1.3. An alternate way to look at stochastic localization

Let us return to (1.3). In the above method of needle decomposition, we attempted to exercise control over the quantity $\mu_\omega(E)(1-\mu_\omega(E))$ for all ω by fixing $\mu_\omega(E)$ at 1/2.

How does stochastic localization fit into this? Instead of controlling $\mu_\omega(E)$, we try to control ψ_ω by defining a martingale (p_t) whose isoperimetric constant is easily bounded. That is, $\mathbf{E}[p_t] = p$ (this is just an integral of the form of (1.1)) and further, the isoperimetric constant of μ_t is lower bounded by $t^{1/2}$. Then, the problem comes down to estimating

$$\int_\Omega \mu_t(E)(1-\mu_t(E))\,\mathrm{d}\nu(\omega),$$

which is exactly what papers such as [Che21] do.

1.4. What next?

Going back to needle decompositions again, we wish to show that there exists a needle decomposition conserving $\mu_\omega(E) = 1/2$ such that

$$\int_\Omega \frac{1}{\|A_\omega\|_{\mathrm{op}}^{1/2}}\,\mathrm{d}\nu(\omega) \gtrsim \|A\|_{\mathrm{op}}^{-1/2}.$$

(1.5) for $b = 0$ gives

$$\|A\|_{\mathrm{op}} = \left\| \int_\Omega A_\omega + b_\omega b_\omega^\top \,\mathrm{d}\nu(\omega) \right\|_{\mathrm{op}}.$$

Therefore, it would suffice to show that

$$\int_\Omega \frac{1}{\|A_\omega\|_{\mathrm{op}}^{1/2}}\,\mathrm{d}\nu(\omega) \gtrsim \left\| \int_\Omega A_\omega + b_\omega b_\omega^\top \,\mathrm{d}\nu(\omega) \right\|_{\mathrm{op}}^{-1/2}$$

for some needle decomposition that conserves $\mu_\omega(E)$.[1]

Using Hölder's inequality as we did in the proof of the $n^{-1/2}$ bound, it is seen that it suffices to show

$$\int_\Omega \|A_\omega\|_{\mathrm{op}}\,\mathrm{d}\nu(\omega) \lesssim \left\| \int_\Omega A_\omega + b_\omega b_\omega^\top \,\mathrm{d}\nu(\omega) \right\|_{\mathrm{op}}$$

for some needle decomposition preserving $\mu_\omega(E)$.
Neglecting the $b_\omega b_\omega^\top$ term, it suffices to show that

$$\int_\Omega \|A_\omega\|_{\mathrm{op}}\,\mathrm{d}\nu(\omega) \lesssim \left\| \int_\Omega A_\omega \,\mathrm{d}\nu(\omega) \right\|_{\mathrm{op}}.$$

The above inequality essentially asks if there exists a needle decomposition where the needles are "nearly aligned". Indeed, if the segments of the needles are perfectly aligned, then equality holds above. We are allowing a constant factor of leeway.

References

[Che21] Yuansi Chen. An almost constant lower bound of the isoperimetric coefficient in the KLS conjecture, 2021.

[KLS95] R. Kannan, L. Lovász, and M. Simonovits. Isoperimetric problems for convex bodies and a localization lemma. *Discrete & Computational Geometry*, 13(3):541–559, Jun 1995.

[LV18] Yin Tat Lee and Santosh S. Vempala. The Kannan-Lovász-Simonovits Conjecture, 2018.

[1] Is this inequality equivalent to the KLS Conjecture? Do there exist needle decompositions not obtained by the bisection method that conserve $\mu_\omega(E)$ and satisfy the above inequality?