MA 861 : Combinatorics I

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Contents

§0. Notation and Prerequisites

Given $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots, n\}$.

§1. Introduction

Exercise 1.1. Recall that the number of k-subsets of $[n]$ is $\binom{n}{k}$. Given a k-subset $S = \{x_1, \ldots, x_k\}$ of $[n]$, we write $S_< = \{x_1, \ldots, x_k\}_<$ to denote that $x_1 < x_2 < \cdots < x_k$. Determine the number of k-subsets $\{x_1, \ldots, x_k\}_<$ of $[n]$ such that $x_i \cong i \mod 2$.

For example, for $n = 6$ and $k = 3$, we have the subsets $\{1, 4, 5\}, \{1, 2, 3\}, \{1, 2, 5\}, \{3, 4, 5\}$.
Broadly, there are three types of "answers": a formula, a recurrence, and a generating function. A great example of the second and third is the following.
$p(n)$, the number of number partitions of n, is given by the generating function

$$\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1} \frac{1}{1-x^i}.$$

Using this, a recursion may be obtained as well. We do *not* plug in values for x in the above. We merely look at the coefficient of x^n in it. We want the coefficient to be a finite sum for all n. If it is an infinite sum, convergence issues may arise.

1.1. Counting in S_n

Recall that S_n is generated by transpositions. A transposition (i, j) is a permutation σ defined by

$$\sigma(k) = \begin{cases} j, & k = i, \\ i, & k = j, \\ k, & \text{otherwise.} \end{cases}$$

In fact, S_n is generated by just "adjacent transpositions" $S_i = (i, i + 1)$ for $1 \leq i < n$. We have

$$\begin{aligned} S_i^2 &= \mathrm{Id} \\ S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1} \\ S_i S_j &= S_j S_i \text{ if } |i-j| > 2. \end{aligned}$$

What is the minimum number of adjacent transpositions required to generate a given transposition? This number is referred to as the *length* $\ell(\pi)$ of the transposition π. What is $\sum_{\pi\in S(n)} q^{\ell(\pi)}$?

§2. Problem Sheets

2.1. Problem Sheet 1

Problem 2.1.1. Let $S(n, k)$ and $s(n, k)$ be Stirling numbers of the second and first kind respectively. Show that for all n, k, we have $s(n, k) \geq S(n, k)$.

Solution

Let $X_{S(n,k)}$ be the set of partitions of $[n]$ into exactly k parts and $X_{s(n,k)}$ the number of permutations of $[n]$ with exactly k cycles. Recall that by definition, $|X_{S(n,k)}| = S(n, k)$ and $|X_{s(n,k)}| = s(n, k)$. It suffices to demonstrate an injection f from $X_{S(n,k)}$ to $x_{s(n,k)}$. We do so as follows. Let $\{\{x_{1,1}, \ldots, x_{1,n_1}\}, \ldots, \{x_{k,1}, \ldots, x_{1,n_k}\}\}$ be a partition of $[n]$ into exactly k parts, where $x_{i,j_1} < x_{i,j_2}$ for $j_1 < j_2$. Then, we have a corresponding permutation of $[n]$ with exactly k cycles given by $(x_{1,1}, \ldots, x_{1,n_1}) \cdots (x_{k,1}, \ldots, x_{k,n_k})$. This map is clearly an injection, so we are done.

Problem 2.1.2. Show that

$$S(n, k) = \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{r!(k-r)!}.$$

Solution

$S(n,k)$ is merely the number of surjective functions from $[n]$ to $[k]$. The set of functions that are *not* surjective is

$$\bigcup_{i \in [n]} \{f \in [k]^{[n]} : i \notin \Im(f)\}.$$

The size of the above is quite easily determined by the inclusion-exclusion principle to get

$$k^n - S(n,k) = \sum_{r=1}^{k-1} (-1)^{r+1} \cdot \underbrace{\binom{k}{r}}_{\text{choosing } r \text{ elements in } [k] \text{ to ``avoid''}} \cdot \underbrace{(k-r)^n}_{\text{counting functions that avoid the chosen}},$$

and the desideratum immediately follows.

Problem 2.1.3. Let $A_n(y) = \sum_k S(n,k) y^k$. Show that $A_n(y) = (y + yD)^n 1$ where $D = \frac{\mathrm{d}}{\mathrm{d}y}$ is the derivative operator.

Solution

Problem 2.1.4. Let D_n be the number of derangements in $\mathfrak{S}_n$ and let $D(x) = \sum_{n \geq 0} D_n x^n / n!$ be its egf. Determine $D(x)$.

Solution

A permutation $\pi \in \mathfrak{S}_n$ is a derangement iff it has no cycles of length 1. Define $f : \mathbb{N} \to \mathbb{N}_0$ by

$$f(k) = \begin{cases} 0, & k = 1, \\ 1, & \text{otherwise.} \end{cases}$$

By the earlier observation, $\pi \in \mathfrak{S}_n$ is a derangement iff $f(|C_1|) \cdots f(|C_k|) = 1$ where $C_1, \ldots, C_k$ are the cycles of π. Using Corollary 5.1.9 in [SF99], we get that

$$D(x) = \exp\left(\sum_{n \geq 2} \frac{x^n}{n} \right) = \exp\big(-x - \log(1-x)\big) = \frac{e^{-x}}{1-x}.$$

References

[SF99] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.