Topology

Lecture 1 - 06/01/21 Introduction and examples of topologies

Topology

Def. A topology on a set X is a collection $\mathcal{T}$ of subsets of X such that

i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

ii) If $U_i \in \mathcal{T}$ for all $i \in I$, where I is some indexing set, then
$$\bigcup_{i \in I} U_i \in \mathcal{T}.$$

iii) If $U_j \in \mathcal{T}$ for all $j \in J$, where J is some finite indexing set, then
$$\bigcap_{j \in J} U_j \in \mathcal{T}.$$

Equivalently, for $U_1, U_2 \in \mathcal{T}$, $U_1 \cap U_2 \in \mathcal{T}$.

Unless mentioned otherwise, assume $X \neq \emptyset$.

Recall the definition of a metric space and an open set. (from Real Analysis)

Since the set of open sets is closed under arbitrary unions and finite intersections, observe that the set of open subsets of a metric space (X,d) is a topology. That is,
$$\mathcal{T} = \{ U \subseteq X : U \text{ is open in } (X,d) \}$$
is a topology. ($\emptyset$ and X are trivially open)

Topologies essentially extend the idea of open sets. How?

Topological Space

Def. A topological space $(X, \mathcal{T})$ is a set X along with a topology $\mathcal{T}$ on X.

Open Set

For a topological space, we call the elements of $\mathcal{T}$ open.

$(X, \{\emptyset, X\})$ is a trivial topological space on a set X.

We now introduce the analogues of interior points, closed sets, etc. Since we don't have "balls" in topological spaces, we have to define everything in an alternate way that remains consistent.

Metric Topology

For a metric space (X,d), the topology
$$\mathcal{T} = \{ U \subseteq X : U \text{ is open}\}$$
is called the metric topology induced by the metric d.

Discrete Topology

For a set X, the topology $\mathcal{P}(X)$ is called the discrete topology on X.
Observe that this is the metric topology induced by the discrete metric. (for $x,y \in X$, $d(x,y) = 0$ if $x=y$ and 1 otherwise)

Indiscrete Topology

For a set X, the topology $\{\emptyset, X\}$ is called the indiscrete topology on X.

Finite Complement Topology

Let X be a set and
$$\mathcal{T}_f = \{\emptyset\} \cup \{U \subseteq X : X \setminus U \text{ is finite}\}.$$
$\mathcal{T}_f$ is a topology on X and is called the finite complement topology or the co-finite topology.

- Clearly, $\emptyset$ and X are in $\mathcal{T}_f$.
- For $(U_i)_{i\in I}$ in $\mathcal{T}_f$,
$$\left(\bigcup_{i\in I} U_i\right)^c = \bigcap_{i\in I} U_i^c \text{ is finite (since each } U_i^c \text{ is finite)}$$
- For $(U_i)_{i=1}^n$ in $\mathcal{T}_f$,
$$\left(\bigcap_{i=1}^n U_i\right)^c = \bigcup_{i=1}^n U_i^c \text{ is finite (a finite union of finite sets)}$$

We have seen that any metric defines a topology. Is the converse true?
No!
Topologies that are induced by a metric are said to be metrizable.
Consider the indiscrete topology $\{\emptyset, X\}$. (for $|X|>1$)
Use the fact that distinct points are separable by neighbourhoods.

If X is a finite set, the finite complement topology is the discrete topology.

Co-countable Topology

Similar to the co-finite topology $\mathcal{T}_f$, we can define $\mathcal{T}_c$, the co-countable topology.

$$\left(\{\emptyset\} \cup \{ U \subseteq X : X \setminus U \text{ is countable} \} \right)$$

Lecture 2 - 08/01/21 Bases of topologies

Def. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ and $\mathcal{T}$ is coarser than $\mathcal{T}'$. We can also define strictly finer and strictly coarser if there is a strict containment.

Finer
Coarser

$\mathcal{T}$ and $\mathcal{T}'$ are said to be comparable if $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq \mathcal{T}$.

(This is similar to the refinement of partitions in the Darboux integral)

Def. If X is a set, a basis (for a topology) on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

Basis

- $\forall x \in X$, $\exists B \in \mathcal{B}$ such that $x \in B$ (that is, $\bigcup_{B \in \mathcal{B}} B = X$)
- if $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ is a basis, the topology $\mathcal{T}$ generated by $\mathcal{B}$ is defined as

Generated Topology

$$\mathcal{T} = \left\{ U \subseteq X : U = \bigcup_{\substack{B \in \mathcal{B} \\ B \subseteq U}} B \right\} \qquad \left(\mathcal{T} = \left\{ U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U \right\} \right)$$

Alternatively, (why?)

$$\mathcal{T} = \left\{ U \subseteq X : U = \bigcup_{i \in I} B_i \text{ for some } (B_i)_{i \in I} \text{ in } \mathcal{B} \right\}$$

$\mathcal{B}$ is then said to be a basis of $\mathcal{T}$.

We take by convention that $\bigcup_{S \in \emptyset} S = \emptyset$.

Observe that

- we trivially have $\emptyset \in \mathcal{T}$
- the first condition implies that $X \in \mathcal{T}$.
- closure under (finite) intersections follows from the second condition. (Why?)
- closure under arbitrary unions follows from the way we define the topology.

Also note that $\mathcal{B} \subseteq \mathcal{T}$.

Note that bases here are extremely different from bases in linear algebra. A better analogue would be a spanning set.

How do we find a "smallest" basis though?

(an analogue of linear independence, perhaps?)

(for example, $\{(a,b) : a,b \in \mathbb{Q}\}$ generates the Euclidean metric topology of $\mathbb{R}^2$.)

Lecture 3 - 13/01/21 More about Bases and Topologies on $\mathbb{R}$

Also, how do we find a (non-trivial) basis for a topology?

Lemma. (1.1) Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open subsets of X such that for each open set U of X and each $x \in U$, there is $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis of $\mathcal{T}$.

Proof.
- Given $x \in X$, there is, by hypothesis, $C \in \mathcal{C}$ such that $x \in C \subseteq X$
- Next, let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open, so is $C_1 \cap C_2$. Therefore, $\exists C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subseteq C_1 \cap C_2$.

  $\Rightarrow \mathcal{C}$ is a basis on X.

Let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$.

- Let $U \in \mathcal{T}$. Then $\forall x \in U$, $\exists C \in \mathcal{C}$ s.t. $x \in C \subseteq U$.

$\Rightarrow \mathcal{T} \subseteq \mathcal{T}'$. (by the definition of a generated topology)

- Let $U \in \mathcal{T}'$. Then $\forall x \in U$, $\exists C_x \in \mathcal{C}$ s.t. $x \in C_x \subseteq U$.

  $\Rightarrow U = \bigcup_{x \in U} C_x$.

  However, each $C_x \in \mathcal{C} \subseteq \mathcal{T}$.

  $\Rightarrow U \in \mathcal{T}$. Therefore, $\mathcal{T}' \subseteq \mathcal{T}$

  so $\mathcal{T} = \mathcal{T}'$. $\square$

Lemma. (1.2) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. The following are equivalent.

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(ii) for each $x \in X$ and $B \in \mathcal{B}$ with $x \in B$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof. (ii) $\Rightarrow$ (i)

Let $U \in \mathcal{T}$ and $x \in U$. Let $B \in \mathcal{B}$ s.t. $x \in B \subseteq U$ (since $\mathcal{B}$ generates $\mathcal{T}$).

Let $B' \in \mathcal{B}'$ st. $x \in B' \subseteq B \subseteq U$.

$\Rightarrow U \in \mathcal{T}'$ by definition and therefore, $\mathcal{T} \subseteq \mathcal{T}'$.

(i) $\Rightarrow$ (ii)

Let $x \in X$ and $B \in \mathcal{B}$ with $x \in B$. By definition, $B \in \mathcal{T}$.

$\Rightarrow B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. $\square$

Example. The collection $\mathcal{B}$ of open circular regions in $\mathbb{R}^2$ generates the same topology as the collection $\mathcal{B}'$ of all open rectangular regions in $\mathbb{R}^2$. (Show that each is finer than the other using the above lemma)

Def. If $\mathcal{B}$ is the collection of all open intervals in the real line, then the topology generated by $\mathcal{B}$ is called the standard topology on $\mathbb{R}$. Unless mentioned otherwise, $\mathbb{R}$ is taken to have this topology.

Standard Topology

(This is the topology induced by the Euclidean metric)

If $\mathcal{B}'$ is the collection of all half-open intervals of the form $[a,b) : a,b \in \mathbb{R}$ where $a<b$, the topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$.

Lower Limit Topology

When $\mathbb{R}$ is given by this topology, it is denoted $\mathbb{R}_\ell$.

Let $K = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Let $\mathcal{B}''$ be the collection of all open intervals (a,b) along with sets of the form $(a,b) \setminus K$. The topology generated by $\mathcal{B}''$ is called the K-topology on $\mathbb{R}$.

K-Topology

When $\mathbb{R}$ is given by this topology, it is denoted $\mathbb{R}_K$.

(Do check that the above collections are bases)

Lemma (1.3). The topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with each other.

Proof- Let the topologies of $\mathbb{R}, \mathbb{R}_\ell, \mathbb{R}_K$ be $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ with bases $\mathcal{B}, \mathcal{B}', \mathcal{B}''$.

Given $(a,b) \in \mathcal{B}$ and $x \in (a,b)$, $x \in \underbrace{[x,b)}_{\in \mathcal{B}'} \subseteq (a,b)$

$\Rightarrow \mathcal{T} \subseteq \mathcal{T}'$.

Also, given $[x,d) \in \mathcal{B}'$, there are no (a,b) such that $x \in (a,b) \subseteq [x,d)$

$\Rightarrow \mathcal{T} \subsetneq \mathcal{T}'$

$\mathcal{T} \subseteq \mathcal{T}''$ is easily shown since $\mathcal{B} \subseteq \mathcal{B}''$.

To show that it is strictly finer, consider $B'' = (-1,1) \setminus K$ and $0 \in B$. There is no (a,b) such that $0 \in (a,b) \subseteq B''$.

$\Rightarrow \mathcal{T} \subsetneq \mathcal{T}''$.

To show that $\mathcal{T}'$ and $\mathcal{T}''$ are not comparable, consider $2 \in [2,3) \in \mathcal{B}'$ and $0 \in [-1,1] \setminus K \in \mathcal{B}''$. The details are left to the reader.

$\Rightarrow$ Neither $\mathcal{T}' \subseteq \mathcal{T}''$ nor $\mathcal{T}'' \subseteq \mathcal{T}'$.

$\square$

Def. A subbasis S for a topology $\mathcal{T}$ on X is a collection of subsets of X whose union is X.

Sub-basis

$$\left(S \subseteq \mathcal{P}(X) \text{ such that } \bigcup_{U \in S} U = X.\right)$$

The topology generated by S is the collection $\mathcal{T}$ of all unions of finite intersections of elements of S.

Let $\mathcal{B} = \left\{ \bigcap_{i=1}^{n} S_i : (S_i)_1^n \in S \right\}$.

The topology generated by S is just that generated by the basis $\mathcal{B}$.

↳ Why? This is easily checked using the definition (of a basis)

Lecture 4 - 15/01/21 Order Topologies

Def. We say that a relation C is a complete order on a set X if

Order

1. For $x, y \in X$, xCy or yCx.
2. For no $x \in X$, xCx.
3. If xCy and yCz, then xCz.

Def. Let A and B be two sets with orders $<_A$ and $<_B$. We say that A and B have the same order type if there is a bijection $f: A \to B$ such that for any $a_1, a_2 \in A$,

Same Order Type

$$a_1 <_A a_2 \iff f(a_1) <_B f(a_2)$$

Def. The dictionary order relation $<$ on $A \times B$ is defined by

Dictionary Order

$$a_1 \times b_1 < a_2 \times b_2 \iff a_1 <_A a_2 \text{ or } (a_1 = a_2 \text{ and } b_1 <_B b_2)$$

Two finite ordered sets of the same cardinality always have same order type. (why?)

Given an order $<$, we write $a \leq b$ if $a < b$ or $a = b$,
$a > b$ if $b < a$, and
$a \geqslant b$ if $a > b$ or $a = b$.

Let X be an ordered set with order $<$. Given $a, b \in X$, we define the four intervals

$$(a,b) = \{x \in X : a < x < b\}$$

$$[a,b) = \{x \in X : a \leq x < b\} \qquad (a,b] : \{x \in X : a < x \leq b\}$$

$$[a,b] = \{x \in X : a \leq x \leq b\}$$

with at least two elements

Lemma. (1.4) Let X be an ordered set. Let $\mathcal{B}$ be the collection of all sets of the types

1. (a,b), $a,b \in X$
2. $[a_0, b)$, where a_0 is the smallest element (if any) of X.
3. $(a, b_0]$, where b_0 is the largest element (if any) of X.

Order Topology

$\mathcal{B}$ is a basis for a topology on X, which is known as the order topology on X.

First of all, every $x \in X$ is contained in one of the above intervals.

$\rightarrow$ if $x \neq b_0$, $x \in [a_0, b_0) \in \mathcal{B}$

$\rightarrow$ if $x = b_0$, $x \in (a_0, b_0] \in \mathcal{B}$

Second, note that the intersection of two sets in $\mathcal{B}$ is either empty or in $\mathcal{B}$.

The result follows. $\square$

Observe that the standard topology on $\mathbb{R}$ is the order topology under the usual order.

Another example: the dictionary relation on $\mathbb{R} \times \mathbb{R}$ has

$$\mathcal{B} = \{ (a \times b, c \times d) : \underbrace{a < c}_{(a,c) \times \mathbb{R}} \text{ or } \underbrace{a = c \text{ and } b < d}_{a \times (b,d)} \}$$

Let $\mathcal{B}' = \{ a \times (b,d) : b < d \} \subseteq \mathcal{B}$.

Show that $\mathcal{B}'$ is a basis and that the topology generated by $\mathcal{B}'$ is the same as that generated by $\mathcal{B}$.

Hint. Use the theorem given last class to show the non-trivial containment

Lecture 5 - 20/01/21 Product and Subspace Topologies

If X is an ordered set and $a \in X$, we define the rays determined by a

$$\left.\begin{aligned}(a,\infty) &= \{x \in X : x > a\} \\ (-\infty, a) &= \{x \in X : x < a\}\end{aligned}\right\} \text{Open rays}$$

$$\left.\begin{aligned}[a,\infty) &= \{x \in X : x \geq a\} \\ (-\infty, a] &= \{x \in X : x \leq a\}\end{aligned}\right\} \text{Closed rays}$$

Note that any open ray is open in the order topology.

Def. Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology with basis

Product Topology

$$\mathcal{B} = \{ U \times V : U \text{ open in } X \text{ and } V \text{ open in } Y \}$$

Why is $\mathcal{B}$ a basis?

→ $X \times Y \in \mathcal{B}$

→ For $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$,

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$$

Also, $\mathcal{B}$ need not be a topology

Consider $\mathbb{R} \times \mathbb{R}$:

$$((1,3) \times (1,3)) \cup ((2,4) \times (2,4)) \notin \mathcal{B}.$$

Theo. (1.5) Let $\mathcal{B}$ and $\mathcal{C}$ bases for the topologies on X and Y respectively. Then

$$P = \{ B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C} \}$$

is a basis for the product topology on $X \times Y$.

Proof Let $U = U_1 \times U_2$ be open in $X \times Y$ and $(x_1, x_2) \in U$. By definition, there are $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x_1 \in B \subseteq U_1$ and $x_2 \in C \subseteq U_2$. Then $(x_1, x_2) \in B \times C \subseteq U_1 \times U_2$ so P is a basis of the product topology.
(by Lemma 1.1) $\blacksquare$

Def. Projection

Let $\pi_1 : X \times Y \to X$ be defined by $(x,y) \mapsto x$ and $\pi_2 : X \times Y \to Y$ by $(x,y) \mapsto y$. The maps π_1 and π_2 are called the projections of $X \times Y$ onto its first and second factors respectively.

Note that for $U \subseteq X$ and $V \subseteq Y$, $\pi^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$.

$\Rightarrow$ If U is open in X, $\pi^{-1}(U)$ is open in $X \times Y$.

If V is open in Y, $\pi^{-1}(V)$ is open in $X \times Y$

$\hookrightarrow$ This becomes relevant when we define continuous maps.

Theo. (1.6) The collection

$$S = \{ \pi^{-1}(U) : U \text{ open in } X \} \cup \{ \pi_2^{-1}(V) : V \text{ open in } Y \}$$

is a subbasis for the product topology on $X \times Y$.

Proof. S is obviously a subbasis as $(X \times Y) \in S$.

For any open $U \in X, V \in Y$,

$$U \times V = \pi^{-1}(U) \cap \pi_2^{-1}(V)$$

The required easily follows.

$\square$

Def. Subspace Topology

Let $(X, \mathcal{T})$ be a topological space. If $Y \subseteq X$,

$$\mathcal{T}_Y = \{ Y \cap U : U \in \mathcal{T} \}$$

is a topology on Y, called the subspace topology on Y and Y is then called a subspace of X.

(Proving that it is a topology is trivial)

Note that if Y is open, $\mathcal{T}_Y = \{ U \in \mathcal{T} : U \subseteq Y \}$.

Lemma (1.7) If $\mathcal{B}$ is a basis for the topology of X, the collection

$$\mathcal{B}_Y = \{ B \cap Y : B \in \mathcal{B} \}$$

is a basis for the subspace topology on Y.

(Proof left as exercise)

If Y is a subspace of X, we say that a set U is open in Y if it belongs to the topology of Y.
A set open in Y need not be open in X.
(but it is if Y is open)

Thm (1.8) If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is equal to the topology it inherits as a subspace of $X \times Y$.

Proof. Indeed, a basis of the former is

$$\{ (U \cap A) \times (V \cap B) : U \text{ open in } X, V \text{ open in } Y \}$$

and of the latter is

$$\{ (U \times V) \cap (A \times B) : U \text{ open in } X, V \text{ open in } Y \},$$

which are both equal. $\blacksquare$

Lecture 6 - 20/01/21

If X is an ordered set and $Y \subseteq X$, the order relation on Y need not be equal to that inherited as a subspace of X.
To see why,

1. consider $Y = [0,1] \subseteq \mathbb{R}$. For open $(a,b) \subseteq \mathbb{R}$,

$$(a,b) \cap Y = \begin{cases} (a,b), & a,b \in Y, \\ [0,b), & a \notin Y, b \in Y, \\ (a,1], & a \in Y, b \notin Y. \end{cases} \Big\} \text{ open in } Y \text{ but not in } \mathbb{R}.$$

2. consider $Y = [0,1] \cup \{2\} \subseteq \mathbb{R}$. Then $\{2\}$ is open in Y as a subspace of X, but not under the order topology on Y.

Observe that the order topology on Y is (always) coarser than that on Y as a subspace.
When are the two topologies equal?