Topology

Lecture 1 - 06/01/21 Introduction and examples of topologies

Topology

Def. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that

i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

ii) If $U_i \in \mathcal{T}$ for all $i \in I$, where I is some indexing set, then
$$\bigcup_{i \in I} U_i \in \mathcal{T}.$$

iii) If $U_j \in \mathcal{T}$ for all $j \in J$, where J is some finite indexing set, then
$$\bigcap_{j \in J} U_j \in \mathcal{T}.$$

Equivalently, for $U_1, U_2 \in \mathcal{T}$, $U_1 \cap U_2 \in \mathcal{T}$. ←

Unless mentioned otherwise, assume $X \neq \emptyset$.

Recall the definition of a metric space and an open set. (from Real Analysis)

Since the set of open sets is closed under arbitrary unions and finite intersections, observe that the set of open subsets of a metric space (X,d) is a topology. That is,
$$\mathcal{T} = \{ U \subseteq X : U \text{ is open in } (X,d) \}$$
is a topology. ($\emptyset$ and X are trivially open)

Topologies essentially extend the idea of open sets. How?

Topological Space

Def. A *topological space* $(X, \mathcal{T})$ is a set X along with a topology $\mathcal{T}$ on X.

Open Set

For a topological space, we call the elements of $\mathcal{T}$ *open*.

$(X, \{\emptyset, X\})$ is a trivial topological space on a set X.

We now introduce the analogues of interior points, closed sets, etc. Since we don't have "balls" in topological spaces, we have to define everything in an alternate way that remains consistent.

Metric Topology

For a metric space (X,d), the topology
$$\mathcal{T} = \{U \subseteq X : U \text{ is open}\}$$
is called the metric topology induced by the metric d.

Discrete Topology

For a set X, the topology $\mathcal{P}(X)$ is called the discrete topology on X.
Observe that this is the metric topology induced by the discrete metric. (for $x,y \in X$, $d(x,y) = 0$ if $x=y$ and 1 otherwise)

Indiscrete Topology

For a set X, the topology $\{\emptyset, X\}$ is called the indiscrete topology on X.

Finite Complement Topology

Let X be a set and
$$\mathcal{T}_f = \{\emptyset\} \cup \{U \subseteq X : X \setminus U \text{ is finite}\}.$$
$\mathcal{T}_f$ is a topology on X and is called the finite complement topology or the co-finite topology.

- Clearly, $\emptyset$ and X are in $\mathcal{T}_f$.
- For $(U_i)_{i \in I}$ in $\mathcal{T}_f$,
$$\left(\bigcup_{i\in I} U_i\right)^c = \bigcap_{i \in I} U_i^c \text{ is finite} \quad (\text{since each } U_i^c \text{ is finite})$$
- For $(U_i)_{i=1}^n$ in $\mathcal{T}_f$,
$$\left(\bigcap_{i=1}^n U_i\right)^c = \bigcup_{i=1}^n U_i^c \text{ is finite} \quad (\text{a finite union of finite sets})$$

We have seen that any metric defines a topology. Is the converse true?

No!

Topologies that are induced by a metric are said to be metrizable.

Consider the indiscrete topology $\{\emptyset, X\}$. (for $|X|>1$)
Use the fact that distinct points are separable by neighbourhoods.

If X is a finite set, the finite complement topology is the discrete topology.

Co-countable Topology

Similar to the co-finite topology τ_f, we can define τ_c, the co-countable topology.

$$\left(\{\emptyset\} \cup \{ U \subseteq X : X \setminus U \text{ is countable} \} \right)$$

Lecture 2 - 08/01/21

Bases of topologies

Def. Suppose τ and τ' are two topologies on a set X. If $\tau' \supseteq \tau$, we say that τ' is finer than τ and τ is coarser than τ'. We can also define strictly finer and strictly coarser if there is a strict containment.

Finer

Coarser

τ and τ' are said to be comparable if $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$.

(This is similar to the refinement of partitions in the Darboux integral)

Def. If X is a set, a basis (for a topology) on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

Basis

- $\forall x \in X$, $\exists B \in \mathcal{B}$ such that $x \in B$ (that is, $\bigcup_{B \in \mathcal{B}} B = X$)
- if $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

If $\mathcal{B}$ is a basis, the topology τ generated by $\mathcal{B}$ is defined as

$$\tau = \left\{ U \subseteq X : U = \bigcup_{\substack{B \in \mathcal{B} \\ B \subseteq U}} B \right\}$$

Alternatively, (Why?)

$$\tau = \left\{ U \subseteq X : U = \bigcup_{i \in I} B_i \text{ for some } (B_i)_{i \in I} \text{ in } \mathcal{B} \right\}$$

$\mathcal{B}$ is then said to be a basis of τ.

We take by convention that $\bigcup_{S \in \emptyset} S = \emptyset$.

Observe that

- we trivially have $\emptyset \in \mathcal{T}$
- the first condition implies that $X \in \mathcal{T}$.
- closure under (finite) intersections follows from the second condition. (Why?)
- closure under arbitrary unions follows from the way we define the topology.

Also note that $\mathcal{B} \subseteq \mathcal{T}$.

Note that bases here are extremely different from bases in linear algebra. A better analogue would be a spanning set.

How do we find a smallest basis though?

(an analogue of linear independence, perhaps?)

Lemma. Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis of $\mathcal{T}$.