MA 862 : Combinatorics II

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§1. Introduction

1.1. The Delsarte bound

Denote by $\mathcal{M}_n(\mathbb{C})$ the $\mathbb{C}$-vector space of all $n \times n$ complex matrices.

Definition 1.1. A subspace $\mathcal{A} \subseteq \mathcal{M}_n(\mathbb{C})$ is said to be a $*$*-algebra of matrices* if

1. $\mathcal{A}$ is closed under multiplication, in that if $A, B \in \mathcal{A}$, then $AB \in \mathcal{A}$, and
2. $\mathcal{A}$ is closed under conjugate transposes, in that if $A = (a_{ij}) \in \mathcal{A}$, then $A^\dagger = (\overline{a_{ji}}) \in \mathcal{A}$.
3. $\mathrm{Id} \in \mathcal{A}$.

That is, it is a subalgebra that is closed under conjugate transposes.
Let q be a prime power. Denote by $B_q(n)$ the set of all subspaces of $\mathbb{F}_q^n$ and $B_q(n,k)$ the set of all k-dimensional subspaces of $\mathbb{F}_q^n$. It is not too difficult to show that

$$|B_q(n,k)| = \binom{n}{k}_q = \frac{(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-k+1})}{(q^k-1)(q^k-q)(q^k-q^2)\cdots(q^k-q^{k-1})}.$$

We had also considered this quantity $\binom{n}{k}_q$ in Section 1.4 of Combinatorics I. Recall the q-Pascal recurrence

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k\binom{n}{k}_q \tag{1.1}$$

for $n \geq 0, k \geq 1$ with $\binom{n}{0}_q = 1$ and $\binom{0}{k} = \delta_{0,k}$. Is there a way to see this recurrence more directly using the subspace perspective of the q-binomial coefficient? If we have a (size k) basis of a k-dimensional subspace of $\mathbb{F}_q^n$, and consider the $k \times n$ matrix with rows equal to the vectors in this basis, we may bring this matrix to a *unique* row-reduced echelon form (independent of the basis used) using row operations wherein

(i) all rows are nonzero,

(ii) the first non-zero entry in every row is a 1. Suppose this entry occurs in column C_i in row i,

(iii) $C_1 < C_2 < \cdots C_k$, and

(iv) the submatrix comprising the $\{C_1, \ldots, C_k\}$ rows is a $k \times k$ identity matrix.

So, we can count $k \times n$ matrices in RREF instead of subspaces. Equation (1.1) then follows immediately by considering whether the last column is pivotal or not.

Definition 1.2. Let A be Hermitian. Then, $\langle A \rangle$, the $*$-algebra generated by A, is $\mathrm{span}\{\mathrm{Id}, A, A^2, \ldots\}$.

Note that this algebra is abelian. Furthermore, by the spectral theorem, $\dim(\langle A \rangle)$ is the number of distinct eigenvalues of A.

For $A \in \mathcal{M}^n(\mathbb{C})$ similar to a Hermitian matrix, that is, PAP^{-1} is Hermitian for some P, $P\langle A \rangle P^{-1}$ is a $*$-algebra.

Example 1 ($*$-algebras on graphs)**.** Let $G = (V, E)$ be a graph and A its adjacency matrix. $\langle A \rangle$ is called the *adjacency algebra* of G.

More specifically, consider the n-cube graph C_n with vertex set $B(n) = 2^{[n]}$ and an edge between X, Y if $|X \triangle Y| = 1$. Although $\langle A \rangle$ is $*$-algebra of $2^n \times 2^n$ matrices, its dimension is only $n+1$. The fact that we only require $n+1$ parameters to describe an arbitrary element of $\langle A \rangle$ is key to the Delsarte bound on binary code size we shall study in this section.

Let $k \leq n/2$. The Johnson graph has vertex set $B(n,k) = \binom{[n]}{k}$ and an edge between X, Y if $|X \cap Y| = k-1$. The dimension of this graph's adjacency algebra turns out to be $k+1$.

The Grassmann graph $J_q(n,k)$ has vertex set $B_q(n,k)$ (see above the example for definition) with $X, Y \in B_q(n,k)$ adjacent iff $\dim(X \cap Y) = k-1$. It turns out that the dimension of this graph's adjacency algebra is $k+1$ as well. Interestingly, the proof for this ends up just being a "q-analogue" of the proof for the Johnson graph.

The q-analogue of the n-cube $C_q(n)$ has vertex set $B_q(n)$ with X, Y adjacent iff $|\dim X - \dim Y| = 1$. We do not know the dimension of this graph's adjacency algebra! The adjacency matrix seems difficult to study (and is perhaps not even the right object to study). We shall instead study a weighted adjacency matrix of $C_q(n)$.

All the above examples are commutative. Recall that a *unitary representation* of a group G is a group homomorphism $\varphi : G \to \mathcal{U}_n(\mathbb{C})$.

Theorem 1.3. Let f be a unitary representation as above. Then,

$$\mathcal{A} = \{A \in \mathcal{M}_n(\mathbb{C}) : A\varphi(g) = \varphi(g)A \text{ for all } g \in G\}$$

is a $*$-algebra called the *commutant* of φ.

Proof. It is obvious that $\mathcal{A}$ is a subspace that is closed under multiplication. We have for $A \in \mathcal{A}, g \in G$ that

$$\varphi(g^{-1}) = \varphi(g)^{-1} = \varphi(g)^\dagger,$$

so

$$A^\dagger \varphi(g) = (\varphi(g)^\dagger A)^\dagger = (\varphi(g^{-1})A)^\dagger = (A\varphi(g)^{-1})^\dagger = \varphi(g)A^\dagger,$$

which easily yields the desideratum. ∎

The above $*$-algebra may be possible be non-commutative. Suppose that G acts on a set S. For each g, we can denote the group action by an $S \times S$ permutation matrix $\rho(g)$, with $(\rho(g))_{gs,s} = 1$. This gives a *representation* $\rho : G \to \mathcal{U}_S(\mathbb{C})$ – any group action thus yields a $*$-algebra.
We would like to analyze the set of matrices which commute with all $\rho(g)$. Let G act on the sets S, T, and let $\rho : G \to \mathcal{U}_S(\mathbb{C}), \tau : G \to \mathcal{U}_T(\mathbb{C})$ be the corresponding maps. Consider

$$\mathcal{A} = \left\{ M \in \mathcal{M}_{S \times S}(\mathbb{C}) : M\rho(g) = \tau(g)M \text{ for all } g \in G \right\}.$$

Finally, we shall set $S = T$ so that it is a $*$-algebra.

Lemma 1.4. Let $M \in \mathcal{M}_{S \times S}(\mathbb{C})$. Defining $\mathcal{A}$ as above, $M \in \mathcal{A}$ iff $M_{t,s} = M_{gt,gs}$ for all $g \in G, t \in T, s \in S$.

Proof. The t,sth entry of $M\rho(g)$ is equal to $M_{t,gs}$, and that of $\tau(g)M$ is $M_{g^{-1}t,s}$. The required follows. ∎