Countability and Separation Axioms

Def. A space X is said to have a countable basis at $x \in X$ if there is a countable collection $\mathcal{B}$ of neighbourhoods of x such that any neighbourhood of x contains an element of $\mathcal{B}$.
A space that has a countable basis at each of its points is said to be first countable.

First countable

For example, any metrizable space is first countable — consider
$$\{B_d(x, 1/n) : n \in \mathbb{N}\}$$

Theo. (4.1) Let X be a topological space.
a) Let $A \subseteq X$. If there is a sequence of points in A that converges to $x \in X$, then $x \in \bar{A}$. The converse holds if X is first countable.
b) Let $f\ X \to Y$ be continuous. If (x_n) be a sequence of points in X that converges to x, then $f(x_n) \to f(x)$. The converse holds if X is first countable.

Proof. Recall that we have proved the above for metrizable X by considering $B_d(x, 1/n)$.
Here, just consider
$$B_n = U_1 \cap \cdots \cap U_n \text{ instead,}$$
where $\{U_n\}$ forms a countable basis at x.
The proof is relatively straightforward — let $x_n \in B_n \cap A$ for each n.
For (b), show that $f(\bar{A}) \subseteq \overline{f(A)}$. $\square$

Def. A topological space X is said to be second countable if it has a countable basis (for the topology).

Prop. (4.2) Any second countable space is first countable.

The proof is direct.

For example, $\mathbb{R}$ is second countable — consider $\{(a,b) : a,b \in \mathbb{Q}\}$.
Similarly, $\mathbb{R}^n$ is second countable as well.
$\mathbb{R}^\omega$ under the product topology is second countable as well.

$$\left\{ \prod_{n\in\mathbb{N}} U_n : U_n = (a,b) \text{ for } a,b \in \mathbb{Q} \text{ for finitely many } n \text{ and } \mathbb{R} \text{ otherwise} \right\}$$

↳ Why is this set countable?

→ Show that $\mathbb{R}^\omega$ under the uniform topology is not second countable.

Hint: Show that the subspace containing sequences of 0s and 1s has the discrete topology so does not have a countable basis. Use this to prove the required.

Theo. (4.3) A subspace of a first (resp. second) countable space is first (second) countable and a countable product of first (resp second) countable spaces is first (second) countable.

Proof We prove the above for the second countable case. If X has a countable basis $\mathcal{B}$, then $\{B \cap A : B \in \mathcal{B}\}$ is a countable basis for $A \subseteq X$ as a subspace.
If X_i has countable basis $\mathcal{B}_i$, then

$$\{\prod U_i : U_i \in \mathcal{B}_i \text{ for finitely many } i \text{ and } U_i = X_i \text{ otherwise}\}$$

forms a countable basis of $\prod_i X_i$. $\square$

Recall that a subset A of a space X is dense in X iff $\overline{A} = X$.

Theo (4.4) Let X be a second countable space.

(a) Every open covering of X contains a countable subcover.

(b) X has a countable dense subset.

Proof. (a) Let $\mathcal{A}$ be an open covering of X. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis of X. For each n, if possible, choose an $A_n \supseteq B_n$. Let $\mathcal{A}'$ be the collection of these A_n. It is clearly countable. Further, it covers X. Given $x \in X$, let $A \in \mathcal{A}$ such that $x \in A$ and $B_n \supseteq A$ be a basis element. Then $x \in A_n$, proving the claim.

(b) For each B_n, let $x_n \in B_n$. Let $D = \{x_n : n \in \mathbb{N}\}$. Then D is a countable dense subset since any basis element of X (and so any open set) intersects D. $\square$

Def. A space for which every open cover contains a countable subcover is called a Lindelöf space.

Lindelöf space

A space having a countable dense subset is said to be separable.

Separable space

Lecture 23 - 24/03/21 Separation Axioms

Obviously, any compact set is Lindelöf.

→ Show that $\mathbb{R}_\ell$ satisfies all countability axioms except the second.

The product of two Lindelöf spaces need not be Lindelöf.

→ Show that $\mathbb{R}_\ell$ is Lindelöf but $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not.

Hint: Consider $\mathbb{R}_\ell^2 \setminus \{(x,-x) : x \in \mathbb{R}_\ell\}$.

A subspace of a Lindelöf space need not be Lindelöf either.

→ Consider the ordered square $I_o^2 = [0,1]^2$ (under the dictionary order). Show that I_o^2 is compact (and Lindelöf) but the subspace $A = I_o \times (0,1)$ is not Lindelöf.

Def. Suppose that one point sets are closed in X. Then X is said to be *regular* if for each $x \in X$ and closed B disjoint from x, there exist disjoint open sets containing x and B.

Regular

Normal

The space is said to be *normal* if for disjoint closed A, B, there exist disjoint open sets containing A and B.

Observe that any normal space is regular and any regular space is Hausdorff.

Lemma. (4.5) Let one-point sets be closed in X.

a) X is regular iff given $x \in X$ and a nbd. U of x, there exists a neighbourhood V of x such that $\overline{V} \subseteq U$.

b) X is normal iff given a closed $A \subseteq X$ and open $U \supseteq A$, there is an open set $V \supseteq A$ with $\overline{V} \subseteq U$.

Proof

a) (Forward) Let $B = X \setminus U$ be closed. There exist disjoint open V and W containing x and B. Then $\overline{V} \cap B = \emptyset$. Therefore, $\overline{V} \subset U$

(Backward) Let $x \in X$ and B disjoint from $\{x\}$ be closed. Let $U = X \setminus B$. Let V be a nbd. of x such that $\overline{V} \subseteq U$. Then V and $X \setminus \overline{V}$ are disjoint open sets containing x and B respectively. Therefore, X is regular.

The argument for (b) is nearly identical, taking A instead of $\{x\}$. $\square$

Theo. (4.6) A subspace of a regular space is regular. A product of regular spaces is regular.

Proof. Let X be regular and $Y \subseteq X$. One-point sets are closed in Y. Let $x \in Y$ and B a closed subset of Y disjoint from $\{x\}$. Then $\overline{B} \cap Y = B$. Therefore, $x \notin \overline{B}$. Using regularity, let U, V be disjoint open sets containing x and $\overline{B}$. Then $(U \cap Y)$ and $(V \cap Y)$ are disjoint open sets of Y containing x and B.

Let (X_α) be a family of regular spaces and $X = \prod_\alpha X_\alpha$.
X is Hausdorff, so singletons are closed in X. Let $x \in X$ and U be a neighbourhood of x. Let $\prod_\alpha U_\alpha$ be a basis element containing x and contained in U. For each α, let V_α be a nbd. of x_α such that $\overline{V_\alpha} \subseteq U_\alpha$. If $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$. Then $V = \prod_\alpha V_\alpha$ is a neighbourhood of x in X. Since $\overline{V} = \prod_\alpha \overline{V_\alpha}$, $x \in \overline{V} \subseteq U$, so X is regular. $\square$
(by Theo 2.9)

→ Show that $\mathbb{R}_K$ is Hausdorff but not regular

Hint: Consider O and K.

→ Show that $\mathbb{R}_\ell$ is normal.

Theo. (4.7) Any second countable regular space is normal

Proof Let X be regular with countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. Each $x \in A$ has a nbd. U disjoint from B. Choose a nbd. V of x such that $\overline{V} \subseteq U$. Then, choose an element of $\mathcal{B}$ containing x and contained in V.

This gives a countable covering (U_n) of A by open sets whose closures do not intersect B. Similarly, choose a countable covering (V_n) of B.

$\bigcup U_n$ and $\bigcup V_n$ are open sets containing A and B, but need not be disjoint.

For each n, let

$$U_n' = U_n \setminus \bigcup_{i=1}^{n} \overline{V_i} \quad \text{and} \quad V_n' = V_n \setminus \bigcup_{i=1}^{n} \overline{U_i}.$$

Each U_i' and V_i' is open. Also, (U_n') covers A because for any $x \in A$, $x \in U_n$ for some n but $x \notin \overline{V_i}$ for $1 \leq i \leq n$.

The open sets

$$U' = \bigcup U_n' \quad \text{and} \quad V' = \bigcup V_n'$$

are disjoint. It is easy to show that U' and V' are disjoint. $\square$

Theo. (4.8) Any metrizable space is normal.

Proof. Let X be metrizable with metric d. Let A, B be disjoint closed subsets of X. For each $a \in A$, choose ε_a such that $B_d(a, \varepsilon_a) \cap B = \emptyset$.

Choose ε_b similarly. Then, let

$$U = \bigcup_{a \in A} B_d(a, \varepsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B_d(b, \varepsilon_b/2).$$

It is easy to show that these are disjoint (they are clearly open and contain A, B), completing the proof. $\square$

Theo- (4.9) Any compact Hausdorff space is normal.

Left as exercise.

Lecture 24 - 26/03/21 Urysohn Lemma and Completely Regular Spaces

Theo. (4.10) [Urysohn Lemma]
Let A, B be disjoint closed subsets of X. If X is normal, then for a closed interval $[a,b]$ in the real line, there exists a continuous map $f: X \to [a,b]$ such that $f(x) = a$ for every $a \in A$ and $f(x) = b$ for every $b \in B$.

Proof Clearly, it suffices to take $[a,b] = [0,1]$.
Let $P = \mathbb{Q} \cap [0,1]$. For each $p \in P$, we define open U_p such that if $p < q$, $\overline{U_p} \subseteq U_q$.
Arrange P as an infinite sequence (p_n) and for convenience, let $p_1 = 0$ and $p_2 = 1$. Let $U_1 = X \setminus B$ Because A is closed and $A \subseteq U_1$, we may choose (by normality) an open U_0 such that $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$.
In general, let $P_n = \{p_k : 1 \leq k \leq n\}$ and suppose open U_p is defined for $p \in P_n$ $(n \geqslant 2)$ such that $p < q \Rightarrow \overline{U_p} \subset U_q$.

Let $r = p_{n+1}$. Since P_{n+1} is finite, it has a simple ordering $<$ (derived from the usual ordering).
Let p_i and p_j be the immediate predecessor and successor respectively in P_{n+1} (Why do these exist?)
Now, choose U_r as an open set such that $\overline{U_{p_i}} \subseteq U_r \subseteq \overline{U_r} \subseteq U_{p_j}$ — such a U_r exists by using normality on the sets $\overline{U_{p_i}}$ and $X \setminus U_{p_j}$.
$\rightarrow$ This defines U_p for $p \in P$ such that $p < q \Rightarrow \overline{U_p} \subseteq U_q$.
Extend this to define U_p for all $p \in \mathbb{Q}$ as $U_p = \varnothing$ if $p < 0$ and $U_p = X$ if $p > 1$

Given $x \in X$, let $\mathbb{Q}(x) = \{p \in \mathbb{Q} : x \in U_p\}$.

Observe that $\mathbb{Q}(x)$ is bounded below (by, say, -1)

$\to$ Let $f(x) = \inf \mathbb{Q}(x) = \inf \{p \in \mathbb{Q} : x \in U_p\}$.

We claim that f is the desired function.

Note that $f(x) \in [0,1]$ for any $x \in X$ (Why?). $\quad$ (*)

For any $x \in A \subseteq U_0$, $f(x) = 0$.

For any $x \in B$, $x \notin p$ for any $p \leq 1$ ($U_1 = X \setminus B$). By (*), $f(x) = 1$.

It remains to show that f is continuous.

Observe that

- if $x \in \overline{U}_r$, $f(x) \leq r$
- if $x \notin U_r$, $f(x) \geq r$

This follows from the denseness of rationals.

Let $x_0 \in X$ and (c,d) be in $\mathbb{R}$ containing $f(x_0)$.

Choose rationals p, q such that $c < p < f(x_0) < q < d$.

Let $U = U_q \setminus \overline{U}_p$.

Then,

$$\left.\begin{array}{l} \cdot\ f(x_0) < q \Rightarrow x_0 \in U_q \\ \cdot\ f(x_0) > p \Rightarrow x_0 \notin \overline{U}_p \end{array}\right\} \Rightarrow x_0 \in U$$

- Let $x \in U$. Then $f(x) \in U_q \subseteq \overline{U}_q \Rightarrow f(x) \leq q < d$

$\qquad f(x) \notin \overline{U}_p \supseteq U_p \Rightarrow f(x) \geq p > c$

$\Rightarrow f(x_0) \in \underset{\text{open}}{\underset{\downarrow}{U}} \subseteq (c,d)$, so f is continuous by Theo 2.1 (iv). $\square$

Observe that the converse holds too — we may take $U = f^{-1}([0,1/2))$ and $V = f^{-1}((1/2, 1])$

Def A space X is *completely regular* if one-point sets are closed in X and for each $x_0 \in X$ and closed $A \not\ni x_0$, there is a continuous function $f: X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

By the Urysohn Lemma, normality implies complete regularity

The axioms are labelled as

T_1: for any x, y, there are open U, V such that $x \in U \not\ni y$ and $y \in V \not\ni x$.

T_2: Hausdorff

T_3 : Regular

$T_{3½}$: Completely regular

T_4 : Normal

Theo. (4.11) A subspace of a completely regular space is completely regular.
A product of completely regular spaces is completely regular.

(Go to next page)

The Urysohn Metrization Theorem and Tychonoff's Theorem

Theo. (4.12) [Urysohn Metrization Theorem]
A regular second countable space X is metrizable

Proof. We shall embed X in a metrizable space Y
Let $Y = \mathbb{R}^\omega$ under the product topology. We have already seen that Y is metrizable under the metric

$$D(x,y) = \sup_i \left\{ \frac{\min\{|x_i - y_i|, 1\}}{i} \right\}.$$

(The proof can also be carried out by taking Y as $\mathbb{R}^\omega$ under the uniform topology)
We will in fact embed X in $[0,1]^\omega$.

→ Claim 1 There exists a countable collection of continuous functions $f_n : X \to [0,1]$ such that for any $x_0 \in X$ and nbd. U of x_0, there is some n such that $f_n(x_0) \neq 0$ and $f(x) = 0$ for $x \in X \setminus U$.

- Let (B_n) be a countable basis for X. For each pair n,m with $\overline{B_n} \subseteq B_m$, use Theo. 3.43 to get a continuous function $g_{n,m} : X \to [0,1]$ such that $g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(X \setminus B_m) = \{0\}$.
  Then given any $x_0 \in X$ and neighbourhood U of x_0, we can choose a basis element B_m with $x_0 \in B_m \subseteq U$. By regularity and Lemma 3.38, we can let B_n be a basis element with $x_0 \in B_n \subseteq \overline{B_n} \subseteq B_m$.
  $(g_{n,m})$ then satisfies our requirements.

→ For this (f_n), define $F : X \to Y$ by

$$F(x) = (f_1(x), f_2(x), f_3(x), \dots)$$

- Because Y has the product topology and each f_n is continuous, F is continuous.
- For $x \neq y$, there is some index n such that $f_n(x) \neq 0$ and $f_n(y) = 0$.
  So, $F(x) \neq F(y)$ and F is injective.
- We must show that F is a homeomorphism of X to $f(X)$. We have already shown that it is a continuous bijection. Let U be open in X.
  Let $z_0 \in F(U)$ and $x_0 \in X$ with $F(x_0) = z_0$. Let N be such that $f_N(x_0) \neq 0$ and $f_N(X \setminus U) = \{0\}$.
  Let $V = \pi_N^{-1}((0,\infty)) \subseteq \mathbb{R}^\omega$. Let $W = V \cap F(X)$ be open in $F(X)$. Now, $\pi_N(z_0) = f_N(x_0) > 0$, so $z_0 \in W$. Further, $W \cap F(X \setminus U) = \emptyset$, so $W \subseteq F(U)$.
  Therefore, $F(U)$ is open and F is a homeomorphism. $\square$

Theo. [Embedding Theorem]

(4.13) Let X be a space in which one-point sets are closed. Let $(f_\alpha)_{\alpha\in J}$ be a family of continuous function $X \to \mathbb{R}$ such that for any $x_0 \in X$ and nbd. U of x_0, there is $\alpha \in J$ such that $f_\alpha(x_0) > 0$ and $f_\alpha(X \setminus U) = \{0\}$. Then $F : X \to \mathbb{R}^J$ defined by $(F(x))_\alpha = f_\alpha(x)$ is an embedding of X in $\mathbb{R}^J$.

A family of continuous functions that satisfies the hypothesis of the above theorem is said to separate points from closed sets.
For a space in which one-point sets are closed, this is seen to be equivalent to X being completely regular.

Corollary. (4.14) A space X is completely regular iff it is homeomorphic to $[0,1]^J$ for some J.

Theo. [Tietze Extension Theorem]

Let X be normal and A be closed in X.

(a) Any continuous map $A \to [a,b] \subseteq \mathbb{R}$ may be extended to a continuous map $X \to [a,b]$.

(b) Any continuous map $A \to \mathbb{R}$ may be extended to a continuous $X \to \mathbb{R}$.

The Tietze Extension Theorem can be used to prove the Urysohn Lemma.
(but its proof uses the Urysohn Lemma)

Theo. (4.15) [Tychonoff's Theorem] An arbitrary product of compact spaces is compact.

Let $(X_\alpha)_{\alpha \in J}$ be compact and $X = \prod_\alpha X_\alpha$.
We first prove a couple of lemmas.

Claim 1. Let X be a set and $\mathcal{A}$ a collection of subsets having the finite intersection property. Then there is a $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{D} \subseteq 2^X$, $\mathcal{D}$ has the finite intersection property, and no $\mathcal{F}$ with $\mathcal{D} \subsetneq \mathcal{F} \subseteq 2^X$ has the finite intersection property.

Proof. We use Zorn's Lemma to prove this.

↳ Given a strictly partially ordered set A in which every simply ordered subset has an upper bound, A has a maximal element.

The strict poset we consider is a set of collections of subsets of X.
Let

$$\mathbb{C} = \{ \mathcal{B} \subseteq 2^X : \mathcal{A} \subseteq \mathcal{B} \text{ and } \mathcal{B} \text{ has the finite intersection property} \}$$

with the strict partial order strict inclusion $\subsetneq$.
We want to show that $\mathbb{C}$ has a maximal element $\mathcal{D}$.
Let $\mathbb{B} \subseteq \mathbb{C}$ be a simply ordered subset. It suffices to show that

$$\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B} \in \mathbb{C}$$

and is an upper bound of $\mathbb{B}$ (which is obvious)
It is clear that $\mathcal{A} \subseteq \mathcal{C}$. Let $C_1, C_2, \dots, C_n \in \mathcal{C}$. For each i, choose $\mathcal{B}_i \in \mathbb{B}$ such that $C_i \in \mathcal{B}_i$.
$\{\mathcal{B}_i : 1 \le i \le n\}$ is simply ordered by proper inclusion and is finite, so has a maximal element $\mathcal{B}_k$. Then $C_1, C_2, \dots, C_n \in \mathcal{B}_k$. Since $\mathcal{B}_k$ has the finite intersection property, $\bigcap_{1 \le i \le n} C_i \neq \emptyset$, so $\mathcal{C}$ has the finite intersection property.
Using Zorn's Lemma completes the proof. $\square$

Completing the proof of Tychonoff's Theorem

Claim 2. Let X be a set and $\mathcal{D} \subseteq 2^X$ be as defined in the previous claim.
a) If B is a finite intersection of elements of $\mathcal{D}$, $B \in \mathcal{D}$.
b)

Proof a) Let B equal the intersection of finitely many elements in $\mathcal{D}$ and $\mathcal{E} = \mathcal{D} \cup \{B\}$. We show that $\mathcal{E}$ has the finite intersection property, so $\mathcal{E} = \mathcal{D}$.

Take finitely many elements of $\mathcal{E}$.

- → If none of them is B, their intersection is clearly nonempty.
- → If B is one of them, we can expand B as a finite intersection to get that the overall intersection is non-empty.

b) Left as exercise (idea similar to a)

We now come to the main proof of Tychonoff's Theorem. Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property.
We show that $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$
By Claim 1, choose $\mathcal{D} \supseteq \mathcal{A}$ as defined.
It suffices to show that $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$.
Consider for each $\alpha \in J$

$$\mathcal{D}_\alpha = \{\pi_\alpha(D) \cdot D \in \mathcal{D}\} \subseteq 2^{X_\alpha}$$

Because $\mathcal{D}$ has the finite intersection property, so does $\mathcal{D}_\alpha$.
By compactness, we may choose for each α, $x_\alpha \in X_\alpha$ such that

$$x_\alpha \in \bigcap_{D_\alpha \in \mathcal{D}_\alpha} \overline{D_\alpha}$$

Let $x = (x_\alpha)_{\alpha \in J} \in X$. If we show that $x \in \overline{D}$ for any $D \in \mathcal{D}$, we are done.
Let $D \in \mathcal{D}$ and U_β be a nbd of $x_\beta \in X_\beta$ Since $x_\beta \in \overline{\pi_\beta(D)}$, we can choose $y \in D$ such that $\pi_\beta(y) \in U_\beta \cap \pi_\beta(D)$.
Then, $y \in \pi_\beta^{-1}(U_\beta) \cap D$.
From (b) of Claim 2, every subbasis element containing x belongs to $\mathcal{D}$.
By (a) of Claim 2, every basis element containing x belongs to $\mathcal{D}$ and intersects every element of $\mathcal{D}$. Therefore, $x \in \overline{D}$ for all $D \in \mathcal{D}$.