CS 6001: Game Theory and Algorithmic Mechanism Design

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§1. Introduction

In typical linear programming, we have an objective that a *single* individual is trying to maximize (subject to some constraints). In game theory, we typically study systems where there are *multiple* individuals with objectives that they are trying to maximize, but each individual can only set some of the variables (the set of variables is shared). An optimal solution for one individual might not be the optimal solution for another. As a result, it is now better to look at *equilibria* instead of pure black-and-white optimality.
These equilibria are what we shall study.
The course is broadly split in two parts:

- *Game theory*, where we study games, interactions between agents who want to maximize their utilities. This is a predictive approach. We wish to find the most probably outcomes or responses of the agents/players.
- *Mechanism design*, where we try to design a game with desirable outcomes. This is a prescriptive approach. Given a reasonable outcome, we wish to build a game that yields this as a probable outcome.

The above is essentially analysis versus synthesis.

A well-known example of game theory is the *Prisoner's Dilemma*, which we slightly rephrase in the following.

Example 1 (Neighbouring kingdoms dilemma)**.** Two kingdoms can each decide to dedicate all their resources to either agriculture or warfare. If both kingdoms choose warfare, they each earn one unit of joy. If one chooses agriculture and the other chooses warfare, the former earns nothing while the latter earns six units of joy. Finally, if both choose agriculture, they each earn five units of joy.
This can be represented by the following matrix.

A, B	Agriculture	War
Agriculture	$5, 5$	$0, 6$
War	$6, 0$	$1, 1$

The above matrix is called a *game matrix*. Each of the two numbers in the cells are referred to as the *utilities* of the respective kingdoms.
Suppose that kingdom A decides to invest in agriculture. In this case, note that irrespective of what the other kingdom decides to choose, A can increase its payoff by switching to investing in warfare. Consequently, it makes more sense to invest in warfare (since the game is symmetric, this is true for both players). Even though both kingdoms heading to war is not the most profitable outcome, it appears to be the most likely outcome.

A *game* is a formal representation of the strategic interaction between multiple agents called *players*. The choices available to the players are called *actions*. A mapping from the state of the game to the set of actions is referred to as a *strategy*.
Depending on the context, games can be represented in different ways, normal form, extensive form, repeated form, stochastic form, etcetera.

Game theory is the formal study of the strategic interactions between players who are rational and intelligent.
A player is rational if they pick actions to maximize their payoff.
A player is intelligent if they know the rules of the game perfectly and picks actions assuming that the remaining players are also rational and intelligent. This also assumes that the player has sufficient computational ability to find the "optimal" action.

Let us now look at an example of mechanism design.

Example 2. Suppose we want to split a cake in two parts (for two children, say) in an "envy-free" fashion. That is, neither child would prefer the other piece of cake. We do not see the children's preferences, so we do not even know what a fair division might involve. This well-known problem has a well-known solution – make one child cut the cake and the other choose the piece. Why does this work? The first child splits it in a way that is exactly half from their perspective, and they are indifferent to the two pieces. The second child on the other hand gets a larger piece in their perspective.

While game theory allows us to study existing games, mechanism design helps us design games to attain specific goals.

The reader is no doubt familiar with the rules of chess. It has two players, *White* and *Black*, with 16 pieces each. Each piece has some legal moves (the players' actions are these moves). The game starts with White and players take alternating turns. White wins if they capture Black's king, and Black wins they capture White's king. There are a couple of conditions under which a draw can occur, which we do not detail.
There are numerous natural questions that arise. First and foremost, does White (or Black) have a winning strategy? A winning strategy is a plan of moves such that it wins irrespective of the moves performed by Black. Alternatively, is it possible to guarantee a draw? It is possible for none of these exist.

What is a strategy? Denote a board position by x_k. A *game situation* is a finite sequence $(x_0, \ldots, x_k)$, such that x_0 is the opening board position and $x_k \to x_{k+1}$ for even (resp. odd) k is created by a single action of White (resp. Black).
This set of game situations can be naturally represented by a tree, referred to as the *game tree*. The nodes are labelled with board positions (x_i), and arrows between nodes are labelled with actions. A *strategy* is a mapping from game situation to action, which describes what action to take at every vertex of this game tree. This is something of a contingency plan for every possible situation.
A strategy pair (s_W, s_B) which describes strategies for both players determines an outcome, also called a *play* of the game. This describes a path through the game tree.
All leaves in the game tree correspond to either White's victory, Black's victory, or a draw.

A *winning strategy* for White is a strategy s_W^* such that for any strategy s_B, (s_W^*, s_B) ends in a win for White. Similarly, we can define a *strategy guaranteeing at least a draw for White*, denoted s_W'.
It is not immediately obvious if such strategies exist.

Theorem 1.1 (von Neumann, 1928)**.** In chess, exactly one of the following statements is true.

(1) White has a winning strategy.

(2) Black has a winning strategy.

(3) Each player has a strategy guaranteeing a draw.

If any such strategy was known, the game would become boring.

Proof sketch. Each vertex of the game tree is a game situation. Denote by $\Gamma(x)$ the subtree rooted at x (including x itself) and by n_x the number of vertices in $\Gamma(x)$. $n_x = 1$ implies that x is a terminal vertex.
Using induction on n_x, we prove that one of the three statements must hold at any game situation. The theorem is clearly true for $n_x = 1$. Suppose x is a vertex with $n_x > 1$. By the inductive hypothesis, for all $y \in \Gamma(y) \setminus \{x\}$, the statement holds $(n_y < n_x)$. Let $C(x)$ be the vertices reachable from x in one step, and assume wlog that it is White's turn.

(a) If there exists $y_0 \in C(x)$ such that (1) is true, then (1) is true at x as well.

(b) If (2) is true for all $y \in C(x)$, then Black will win irrespective of White's move.

(c) Otherwise, because (a) does not hold, White does not have a winning strategy for any $y \in C(x)$. As a result, for every $y \in C(x)$, either Black has a winning strategy or both have a draw-guaranteeing strategy.
Because (b) does not hold, there is some $y \in C(x)$ where Black does not have a winning strategy. By the preceding argument, both players must have a draw-guaranteeing strategy at this node. ∎

Remark. The above proof is technically incorrect since the game tree for chess is infinite, so $\Gamma(x)$ need not be finite. This issue can be fixed through some clever manipulation of the "cycles" in the game tree to make the graph finite, but we do not go into more detail on this.

§2. Normal form games

2.1. Definitions

Normal form is a representation technique for games.

Definition 2.1 (Normal form game)**.** A *normal form game* is a 3-tuple $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where

1. $N = \{1, \dots, n\}$ is the set of players.
2. S_i is the set of *strategies* of player i. A particular strategy is denoted $s_i \in S_i$ and the set of *strategy profiles* is $S \coloneqq \bigtimes_{i\in N} S_i$, with specific elements $s = (s_1, \dots, s_n) \in S$. A strategy profile without i is $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.
3. $u_i : S \to \mathbb{R}$ is the *utility function* of player i.

If S_i is finite for all $i \in N$, the game is said to be a *finite* game.

As mentioned earlier, a player is rational if they pick actions that maximize their utility. A player is intelligent if they know the rules of the game perfectly, and pick actions assuming that all other players are rational and intelligent.

A fact is said to be *common knowledge* if

1. all players know the fact,
2. the fact that "all players know the fact" is also common knowledge.

Example 3 (Propagation of common knowledge)**.** There is an isolated island (with a hundred people, say) where all inhabitants have eye color either blue or black. There is no reflecting surface on the island (people cannot figure out their own eye color) and nobody can communicate with each other.
One day, a truth-telling god comes to the island and declares that all blue-eyed people are bad for the island and must leave as soon as possible. He also says that there is at least one blue-eyed person on the island. The inhabitants, being deeply devout, do listen to him and leave at the end of the day if they discover that their eyes are blue. In this setting, the fact that there is at least one blue-eyed person on the island is common knowledge.

If there was only one blue-eyed person, he would see that all other people have black eyes. Because the god said that there is a blue-eyed person, he infers that he must be the only blue-eyed person and leaves at the end of the first day.
If there were two, then on the second day everyone would notice that all people remain on the island, so they would infer that there are at least two blue-eyed people on the island. If one of the inhabitants sees that exactly one of the other four people is blue-eyed, then he, along with the other blue-eyed person, leaves on the second day.
This goes on, and it is seen that if there are exactly n blue-eyed people, then all of these n people leave at the end of the nth day.

Now, we discuss the concept of *domination* in NFGs. Consider the following game matrix:

$1,2$	L	M	R
U	$1,0$	$1,3$	$3,2$
D	$-1,6$	$0,5$	$3,3$

Observe that player 2 has no reason to ever play R. Indeed, no matter what player 1 chooses, they can increase their payoff by switching to M instead. In such a scenario, we say that R is dominated by M.

Definition 2.2. A strategy $s'_i \in S_i$ of player i is said to be *strictly dominated* if there exists another strategy $s_i \in S_i$ such that for every strategy profile $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}).$$

A strategy $s'_i \in S_i$ of player i is said to be *weakly dominated* if there exists another strategy $s_i \in S_i$ such that for every strategy profile $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

and in addition, there exists some $\tilde{s}_{-i} \in S_{-i}$ such that

$$u_i(s_i, \tilde{s}_{-i}) > u_i(s'_i, \tilde{s}_{-i}).$$

So, in the earlier example R is strictly dominated and D is weakly dominated.
There is no reason for a rational player to ever play a (weakly or strictly) dominated strategy.

Definition 2.3. A strategy $s_i \in S_i$ is *strictly (weakly) dominant* if it strictly (weakly) dominates all $s'_i \in S_i \setminus \{s_i\}$.

Recall Example 1. In this, warfare strictly dominates agriculture, which is precisely what we said there, albeit in more explicit terms.

Let us give another example of this.

Example 4 (Second price auctions)**.** Suppose there are two players having "values" v_1, v_2 respectively. Each player can choose a number in $[0, M]$, where $M \gg v_1, v_2$. The player who quotes the larger number wins the object (with ties being broken in favour of player 1, say), and pays the losing player's chosen number. The utility of the winning player is their value v_i minus their payment (the amount bid by the other player), and the utility of the losing player is 0.
As a NFG representation, $N = \{1, 2\}$, $S_1 = S_2 = [0, M]$, and

$$u_1(s_1, s_2) = \begin{cases} v_1 - s_2, & s_1 \geq s_2, \\ 0, & \text{otherwise,} \end{cases} \text{ and } u_1(s_1, s_2) = \begin{cases} v_2 - s_1, & s_1 < s_2, \\ 0, & \text{otherwise,} \end{cases}$$

It turns out that the strategy s_i where player i chooses v_i is a weakly dominant strategy! Let us check this for player 1.
Suppose that $s'_1 > v_1$ is in S_1 and $s_2 \in S_2$. We would like to show that $u_1(s_1, s_2) \geq u_1(s'_1, s_2)$. If $s_2 > s'_1$, both payoffs are zero. If $s_2 \leq s_1 < s'_1$, then both payoffs are equal to the same value. The remaining case is when $s_1 \leq s_2 \leq s'_1$. The payoff for s_1 is zero, but the payoff for s'_1 is non-positive since we are paying more than we value the item.

2.2. Some types of equilibria

Definition 2.4. A strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ is a *strictly (weakly) dominant strategy equilibrium* if each s_i^* is a strictly (weakly) dominant strategy.

We abbreviate the above as SDSE or WDSE.
No rational player would play dominated strategies, so we can eliminate dominated strategies one-by-one. A point of note here is that after eliminating a strategy, we get a reduced game with fewer strategies, and this game may have dominated strategies that were not there earlier.
For strictly dominated strategies, the order of elimination does not matter. For weakly dominated strategies however, some reasonable outcomes may be eliminated; this is because eliminating one player's weakly dominated strategies may introduce new dominated strategies for another.

Example 5 (Order of eliminating weakly dominated strategies matters)**.** Consider the following.

	L	C	R
T	$1,2$	$2,3$	$0,3$
M	$2,2$	$2,1$	$3,2$
B	$2,1$	$0,0$	$1,0$

Right off the bat, it is seen that T, B, and C are weakly dominated strategies. Suppose we start by imposing that player 1 does not play T. If we do this, then R becomes weakly dominated as well, so it makes sense to eliminate it. Similarly, we can go on to eliminate B and C. Finally, the payoff of (M, L) is $2, 2$.
On the other hand, if we eliminate strategies in the order of B, L, C, T, then the final remaining strategies are (M, R), which gives a payoff of $3, 2$, which is not the same as the previous $2, 2$!

Dominant strategies (and dominant strategy equilibria) need not exist in games in general.

Example 6 (Dominated strategies need not exist)**.** Consider the following game.

	L	R
L	$1,1$	$0,0$
R	$0,0$	$1,1$

As a result, dominance is not enough to give a reasonable outcome, so we must give a more refined notion.

Definition 2.5 (Nash Equilibrium)**.** A strategy profile (s_i^*, s_{-i}^*) is a *pure strategy Nash equilibrium* (PSNE) if for all $s_i \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

That is, fixing the remaining players' strategies, no player can increase their payoff by moving to another strategy. More succinctly, unilateral deviation is not beneficial.
In the above example, (L, L) and (R, R) are both PSNEs.
PSNEs need not exist either!

Example 7 (PSNEs need not exist)**.** The following game has no PSNE.

	L	R
L	$-1,1$	$1,-1$
R	$1,-1$	$1,-1$

Definition 2.6. A *best response* of player i against a strategy profile s_{-i} is a strategy that gives the maximum utility. That is,

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \text{ for all } s'_i \in S_i\}.$$

So, a PSNE is a strategy profile (s_i^*, s_{-i}^*) such that $s_i^* \in B_i(s_{-i}^*)$ for all $i \in N$.
A PSNE gives some sort of stability. Once there, no rational player has a reason to change their strategy.

One of our biggest assumptions thus far is that all players are rational and intelligent. There are, however, other types of rationality.
One is risk-aversion, where each player makes pessimistic estimates of others. This worst case optimal choice is called a max-min strategy.

Definition 2.7. A strategy s_i^{maxmin} is a *max-min strategy* for player i if

$$s_i^{\text{maxmin}} \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

The *max-min value* is defined by

$$\underline{v}_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

We have that for any $t_{-i} \in S_{-i}$,

$$u_i(s_i^{\text{maxmin}}, t_{-i}) = \underline{v}_i.$$

Theorem 2.8. Any dominant strategy is a maxmin strategy.

Proof. Let s_i^* be a dominant strategy for player i. We have that for any $s_{-i} \in S_{-i}$ and $s'_i \in S_i \setminus \{s_i^*\}$,

$$u_i(s_i^*, s_{-i}) \geq u_i(s'_i, s_{-i}).$$

In particular, when we set s_{-i} as any $s_{-i}^{\text{min}}(s_i^*) \in \arg\min_{s'_{-i} \in S_{-i}} u_i(s_i^*, s'_{-i})$, we get

$$\min_{s_{-i} \in S_{-i}} u_i(s_i^*, s_{-i}) = u_i(s_i^*, s_{-i}^{\text{min}}(s_i^*)) \geq u_i(s'_i, s_{-i}^{\text{min}}(s'_i)) \geq \min_{s_{-i} \in S_{-i}} u_i(s'_i, s_{-i})$$

and as a result,

$$s_i^* \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$ ■

Theorem 2.9. Every PSNE $s^* = (s_1^*, \ldots, s_n^*)$ of an NFG satisfies

$$u_i(s^*) \geq \underline{v}_i$$

for all $i \in N$.

Proof. We have

$$\begin{aligned} u_i(s_i^*, s_{-i}^*) &= \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) && \text{(by definition of PSNE)} \\ &\geq \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) = \underline{v}_i. \end{aligned}$$ ■

What happens to stability and security when games are eliminated? Recall that depending on the order in which dominant strategies are eliminated, the final value can change.

Example 8. The game matrix involved in this example is very similar to that in Example 5.

	L	C	R
T	$1, 2$	$2, 3$	$0, 3$
M	$2, 2$	$2, 1$	$3, 2$
B	$2, 0$	$0, 0$	$1, 0$

The initial maxmin values for the two players are 2 for player 1 and 0 for player 2, but after we eliminate B, the values go to 2 and 2.
It is not a coincidence that the maxmin value of 2 for player 1 is unchanged.

Theorem 2.10. Consider an NFG $G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$. Let $\hat{s}_j \in S_j$ be a dominated strategy, and $\hat{G}$ be the residual game after removing $\hat{s}_j$. The maxmin value of player j in $\hat{G}$ is equal to that in G.

The idea is that the eliminated strategy cannot be the unique maxmin strategy in G since it is dominated.

Proof. We are done if we show that there is a maxmin strategy in G in $S_j \setminus \{s_{-j}\}$ (Why?).
Let $\hat{s}_j$ be dominated by $t_j \in S_j \setminus \{\hat{s}_j\}$. Then, for all $s_{-j} \in S_{-j}$,

$$u_j(t_j, s_{-j}) \geq u_j(\hat{s}_j, s_{-j}).$$

We have

$$\min_{s_{-j}\in S_{-j}} u_j(t_j, s_{-j}) \geq \min_{s_{-j}\in S_{-j}} u_j(\hat{s}_j, s_{-j}),$$

so

$$\max_{s_j\in S_j\setminus\{\hat{s}_j\}} \min_{s_{-j}\in S_{-j}} u_j(s_j, s_{-j}) \geq \min_{s_{-j}\in S_{-j}} u_j(\hat{s}_j, s_{-j}),$$

completing the proof – there is another strategy whose value is at least that of $\hat{s}_j$. ∎

To summarize,

- eliminating strictly dominated strategies has no effect on the PSNEs,
- eliminating weakly dominated strategies may make the set of PSNEs smaller, but does not add new PSNEs, and
- the maxmin value is unaffected by eliminating (strictly or weakly) dominated strategies.

2.3. Matrix games

Definition 2.11. A *matrix game* or *two player zero-sum game* is a normal form game $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$ with $N = \{1, 2\}$ and $u_1 + u_2 \equiv 0$.

Example 9. Examples of matrix games are the following.

	L	C	R
T	$3, -3$	$-5, 5$	$-2, 2$
M	$1, -1$	$4, -4$	$1, -1$
B	$6, -6$	$-3, 3$	$-5, 5$

	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

A point of note here that we shall soon examine in more detail is that the first game has PSNEs while the second does not.

A matrix game can be represented by a single *utility matrix*, by considering the utilities of only one of the players, say player 1. Player 2's utilities are then just the negative of the matrix.
Given a utility matrix u, $u_1 \equiv u$ and $u_2 \equiv -u$.

What are the PSNEs, if any, of a matrix game?

Definition 2.12. A *saddle point* of a matrix A is an index A_{ij} that is the largest in the ith row and the smallest in the ith column.

Theorem 2.13. In a matrix game with utility matrix u, (s_1^*, s_2^*) is a PSNE if and only if it is a saddlepoint.

Proof. Indeed, (s_1^*, s_2^*) is a saddle point iff

$$u(s_1^*, s_2^*) \geq u(s_1, s_2*) \text{ for all } s_1 \in S_2 \text{ and}$$
$$u(s_1^*, s_2^*) \leq u(s_1^*, s_2) \text{ for all } s_2 \in S_2,$$

which is equivalent to being a PSNE since $u_1 \equiv u$ and $u_2 \equiv -u$. ■

Definition 2.14. Given a two player game with utility matrix u, define the *maxmin* value $\underline{v}$ by

$$\underline{v} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$$

and the *minmax* value $\overline{v}$ by

$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2).$$

The above can be slightly rephrased to say

$$\underline{v} = -\min_{s_1 \in S_1} \max_{s_2 \in S_2} u_2(s_1, s_2)$$
$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_2} u_1(s_1, s_2).$$

Lemma 2.15. For matrix games, $\overline{v} \geq \underline{v}$.

Proof. Let $\overline{v}$ and $\underline{v}$ be attained by (s_1, s_2) and (t_1, t_2) respectively. Then,

$$\overline{v} = u(s_1, s_2) \overset{(1)}{\geq} u(t_1, s_2) \overset{(2)}{\geq} u(t_1, t_2) = \underline{v},$$

where (1) is because (s_1, s_2) is a minmax strategy and (2) is because (t_1, t_2) is a maxmin strategy. ■

Going back to Example 9, check that in the first game, $\overline{v} = 1 = \underline{v}$, while in the second game, $\overline{v} = 1$ and $\underline{v} = -1$.

Theorem 2.16. A matrix game has a PSNE iff $\overline{v} = \underline{v}$.

Proof. We wish to show that a utility matrix u has a saddle point iff its maxmin and minmax values are equal.

Suppose that (s_1^*, s_2^*) is a saddle point. Then,

$$u(s_1^*, s_2^*) \geq \max_{s_1 \in S_1} u(s_1, s_2^*) \geq \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2) = \overline{v}.$$

An identical argument for player 2 (keeping in mind that $u_2 \equiv -u$!) yields that $\underline{v} \geq u(s_1^*, s_2^*)$.
Combining the two,

$$\underline{v} \geq u(s_1^*, s_2^*) \geq \overline{v}.$$

Recalling Lemma 2.15, we must have that $\overline{v} = \underline{v}$.

Now, suppose that $\overline{v} = \underline{v}$. There is a strategy (s_1^*, s_2^*) that is both a maxmin and minmax strategy (Why?).
For any $s_2 \in S_2$,

$$u(s_1^*, s_2) \geq \min_{t_2 \in S_2} u(s_1^*, t_2) = \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) = \underline{v}.$$

Similarly, for any $s_1 \in S_1$,

$$u(s_1, s_2^*) \leq \max_{t_1 \in S_1} u(t_1, s_2^*) = \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2) = \overline{v}.$$

The two equations above imply that (s_1^*, s_2^*) is a saddle point, so we are done. ■

2.4. Mixed strategies

So far, one issue is that PSNEs may not exist. We are also limiting ourselves to "pure" strategies, in the sense that there is a certain strategy that we definitely play.

Given a finite set A, define the set of probability distributions on A

$$\Delta A = \{p \in [0, 1]^A \sum_{a \in A} p_a = 1\}.$$

A *mixed strategy* of player i is some $\sigma_i \in \Delta(S_i)$.
Because we are looking at non-cooperative games, mixed strategies of distinct players are independent.
The utility of the ith player for a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_1 \in S_1} \cdots \sum_{s_n \in S_n} \left(\prod_{i \in [n]} \sigma_i(s_i) \right) u_i(s_1, \ldots, s_n).$$

That is, the utility of a mixed strategy is the expectation of the utility.

Example 10. Consider the following game.

	U	D
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

Suppose the mixed strategy σ_1 chooses L and R with probabilities $2/3$ and $1/3$ respectively, and σ_2 chooses U and D with probabilities $4/5$ and $1/5$ respectively. Then,

$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1) + \frac{2}{3}\frac{1}{5} \cdot 1 + \frac{1}{3}\frac{4}{5} \cdot 1 + \frac{1}{3}\frac{1}{5} \cdot (-1) = -1/5.$$

By linearity of expectation, we have

$$u_i(\lambda\sigma_i + (1-\lambda)\sigma_i', \sigma_{-i}) = \lambda u_i(\sigma_i, \sigma_{-i}) + (1-\lambda) u_i(\sigma_i', \sigma_{-i}). \tag{2.1}$$

This is referred to as mixing strategies.

Definition 2.17 (Mixed Strategy Nash Equilibrium)**.** A *mixed strategy nash equilibrium* (MSNE) is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ such that for all $i \in N$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*)$$

for any $\sigma_i' \in \Delta(S_i)$.

A PSNE is just a special case of a MSNE where all the mixed strategies are degenerate distributions.

Theorem 2.18. A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE iff

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*)$$

for all $s_i \in S_i$.

The forward implication is direct on setting the σ_i' appropriately, and the backward implication by an extension of Equation (2.1) to more than two strategies (How?).

Therefore, any PSNE is a MSNE. In any MSNE, there must be some amount of "balance" – in Example 10, the probability distributions that assign $1/2$ to each strategy together constitute a MSNE (check this!).
How do we make this notion of balance more formal?

Definition 2.19. Given a mixed strategy σ_i, the *support* of σ_i is

$$\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}.$$

We now characterize MSNEs.

Theorem 2.20. A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE iff

(a) $u_i(s_i, \sigma_{-i}^*)$ is equal for all $s_i \in \delta(\sigma_i^*)$, and

(b) $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$, $s_i' \notin \delta(\sigma_i^*)$.

We encourage the reader to use the above theorem to determine MSNEs in earlier examples.

Proof. We have

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) && \text{(similar to Theorem 2.18)} \\ &= \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \\ &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*). \end{aligned}$$

It is possible for the maximum and weighted average to be equal iff all the utilities are the same, so (a) is proved. Now, suppose that there is some $s_i \in \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*)$ such that

$$u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*).$$

Consider the mixed strategy that is identical to σ_i^* except that the weight that was on s_i is shifted to s_i'. This gives a strictly larger utility, contradicting the fact that we have an MSNE and proving (b).

Now, let us prove the backward direction. Let $u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ (using (a)). Using (b), we have $m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$. Then,

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \\ &= m_i(\sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*), \end{aligned}$$

so we are done by Theorem 2.18. ∎

We now try to convert the above theorem to an algorithm. Suppose we have an NFG $G = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$. The number of possible supports of $S_1 \times \cdots \times S_n$ is $K = (2^{|S_1|-1})(2^{|S_2|-1}) \cdots 2^{|S_n|-1}$.
For every possible support profile $X_1 \times \cdots \times X_n$, solve the following feasibility program.

$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}) \text{ for all } s_i \in X_i,\ i \in N \text{ and}$$

$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}) \text{ for all } s_i \in (S_i \setminus X_i),\ i \in N,$$

where for all $j \in N$, $\sigma_j(s_j) \geq 0$ for all $s_j \in S_j$ and $\sum_{s_j \in S_j} \sigma_j(s_j) = 1$.
This program is not linear unless there are only two players.
For games in general, there is no polynomial time algorithm known for this. In fact, the problem of finding an MSNE is *PPAD-complete*[1]. The interested reader may see [DGP09] for more details.

The previous algorithm may be applied to a smaller set of strategies by removing dominated strategies. Now, we can even talk about domination by a mixed strategy. As we saw earlier, removing a weak dominated strategy can upset the equilibrium.

Theorem 2.21. If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta(S_i)$, any MSNE of the game chooses s_i with probability zero.

[1]PPAD stands for "Polynomial Parity Argument on Directed graphs".

We can remove strictly dominated strategies without consequences.

Theorem 2.22 (Nash, [Nas51])**.** Any finite game has a mixed Nash equilibrium.

Above, "finite" means that the number of players and strategies are finite.

2.5. Correlated equilibria

So far, we have worked in the setting where each agent independently picks their own strategy. Now, we look at an alternative approach with a mediating agent or device. This is merely another version of rationality. It may lead to results where the utility of all players is improved, and is computationally tractable.

Example 11. Consider the following game, modelling the choices of cars at a crossroads.

	Wait	Go
Wait	$0, 0$	$1, 2$
Go	$2, 1$	$-10, -10$

It is clear that both Wait, Go and Go, Wait are PSNEs. It can also be seen however that a MSNE can assign some nonzero probability to the event where both players choose Go. In practice, a traffic light guides the players. This trusted third party is called the *mediator*. It randomizes over the strategy *profiles* (and not just strategies like earlier) and suggests the corresponding strategies to the players. If the startegies are enforceable, then it is an equilibrium.

Definition 2.23. A *correlated strategy* is a mapping $\pi \in \Delta S$.

In the setting of the previous example, an example of a sensible correlated strategy is that which chooses Wait, Go or Go, Wait with equal probability $1/2$.

Definition 2.24. A *correlated equilibrium* is a correlated strategy π such that

$$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i', s_{-i})$$

for all $s_i, s_i' \in S_i$ for all $i \in N$.

That is, no player benefits from (unilaterally) changing their strategy. This is largely similar to the definition of an MSNE, but the distribution is over strategy profiles and not strategies.

One massive advantage of correlated equilibria is that they can be computed efficiently! We merely need to find a solution to the set of constraints

$$\begin{aligned}
\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i', s_{-i}) & \text{ for all } s_i, s_i' \in S_i \text{ and } i \in N \\
\pi(s) \geq 0 & \text{ for all } s \in S \\
\sum_{s \in S} \pi(s) = 1. &
\end{aligned}$$

If $|S_i| = m$ for all i, this gives $O(nm^2)$ inequalities in the first part and $O(m^n)$ inequalities for the second.
All together, they represent a feasibility linear program.

In MSNEs, the total number of support profiles was $O(2^{mn})$. Here, we just have $O(m^n)$ inequalities, which is exponentially smaller.

Theorem 2.25. Given a MSNE σ^*, consider the strategy profile distribution π^* defined by

$$\pi^*(s_1, \ldots, s_n) = \prod_{i=1}^{n} \sigma_i^*(s_i).$$

Then, π^* is a correlated equilibrium.

We omit the proof of the above as it is straightforward.
Summarizing much of our discussion thus far, we have that the set of SDSEs is contained in the set of WDSEs, which is contained in the set of PSNEs, which is contained in the set of MSNEs, which is contained in the set of CEs.

§3. Long-form games

Thus far, we have only looked at "single-step" games in NFGs. Not all games can be represented by this however, with an obvious example that we have discussed being chess.

3.1. Perfect information extensive form games

Definition 3.1 (Perfect Information Extensive Form Game)**.** A perfect information extensive form game (PIEFG/EFG) is a 7-tuple $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N} \rangle$ where

- N is the set of players,
- A is the set of all possible actions (of all players)
- $\mathcal{H} \subseteq \bigcup_{k=0}^{\infty} A^k$ is the set of all sequences of actions (*histories*) satisfying
  - $\varnothing \in \mathcal{H}$ and
  - if $h = (a^{(0)}, a^{(1)}, \dots, a^{(\tau)}) \in \mathcal{H}$, any subsequence $h' = (a^{(0)}, a^{(1)}, \dots, a^{(t)})$ of h (for $t \leq \tau$) starting at the root is in $\mathcal{H}$,
- $Z \subseteq \mathcal{H}$ is called the set of all *terminal histories*, where a history $h = (a^{(0)}, a^{(1)}, \dots, a^{(\tau-1)})$ is terminal if there exists no $a^{(\tau)} \in A$ with $(a^{(0)}, a^{(1)}, \dots, a^{(\tau)}) \in \mathcal{H}$,
- $\mathcal{X} : \mathcal{H} \setminus Z \to 2^A$, called the *action set selection function*, gives the set of all valid actions given a non-terminal history,
- $P : \mathcal{H} \setminus Z \to N$ is the *player function* which gives the player who plays at a given non-terminal history, and
- $u_i : Z \to \mathbb{R}$ is the utility of player i.

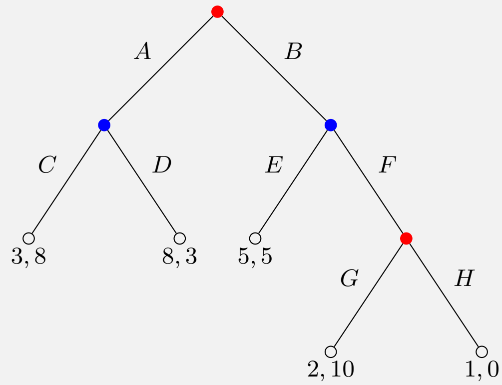
A history is essentially a path from the root in the game tree (recall our discussion of this from the first section).
A natural next question is: what is a strategy in a PIEFG?

Definition 3.2 (Strategy)**.** A *strategy* of a player in an EFG is a tuple of actions at every history where the player plays. That is, a strategy of a player i is an element of

$$S_i = \bigtimes_{h \in \mathcal{H} : P(h) = i} \mathcal{X}(h).$$

We can transform EFGs to NFGs! Indeed, the above definition explicitly describes the strategy set of each player, and associated to any tuple of strategies for all the players, we can determine the payoff.
This conversion has a massive explosion in size however, and equilibria in the converted NFG do not necessarily make sense in the context of the original EFG.

Example 12 (PSNEs in PIEFGs)**.** Consider the following game with two players, red and blue.



At the node BF, red has no reason to play H since choosing G results in a higher payoff. However, it is still seen that (AH, CF) (at each node, the player chooses whichever of these actions is available) is a PSNE! This is a consequence of the fact that this strategy profile never even reaches the node where red chooses G or H, so it never comes into play.

Henceforth, we talk about PSNEs of EFGs as PSNEs of their NFG conversion.

Definition 3.3. A *subgame* of a game is the restriction of the game to the descendants of a history.

The idea behind an equilibrium should be *subgame perfection*, where each player chooses the best possible action at each subgame where they play.

Definition 3.4 (Subgame Perfect Nash Equilibrium)**.** A *subgame perfect nash equilibrium* (SPNE) of an EFG G is a strategy profile $s \in S$ such that for any subgame G' of G, the restriction of s to G' is a PSNE of G'.

Similar to MSNEs, SPNEs are guaranteed to exist in finite PIEFGs. Observe that any SPNE is a PSNE. As we shall now see, the algorithm to find an SPNE is quite simple.

Algorithm 1: Backward induction to determine SPNEs

Input: An EFG G

Output: The utility and action to be taken by a given player at a certain history

backInd(*history* h)
  if $h \in Z$ **then**
    return $u(h), \varnothing$
  $\mathsf{bestUtil}_{P(h)} \leftarrow -\infty$
  foreach $a \in \mathcal{X}(h)$ **do**
    $\mathsf{utilAtChild}_{P(h)} \leftarrow$ backInd$((h, a))$
    if $\mathsf{utilAtChild}_{P(h)} > \mathsf{bestUtil}_{P(h)}$ **then**
      $\mathsf{bestUtil}_{P(h)} \leftarrow \mathsf{utilAtChild}_{P(h)}$
      $\mathsf{bestAction}_{P(h)} \leftarrow a$
  return $\mathsf{bestUtil}_{P(h)}, \mathsf{bestAction}_{P(h)}$

The idea of subgame perfection is intrinsically tied to the above algorithm.
The issue however is that we are essentially parsing the entire tree, so the algorithm is computationally very expensive. Further, some criticize the idea of SPNEs for assuming that the cognitive limit of the players is infinite (which is not realistic).
It is easy to find an SPNE in simple games such as tic-tac-toe.

Example 13 (Centipede game)**.** Players 1 and 2 alternate, and each can play a move from $\{\mathsf{take}, \mathsf{push}\}$, with a maximum of N rounds, say. The game terminates when take is played for the first time or the limit of N rounds if take is never played.
The game also fixes some quantities m_0, m_1 with $m_0 > m_1$. Suppose that the game ends on round $t \in \{0, \ldots, N-1\}$ with player p making the final move; let p' be the other player. Then, the payoffs for the two players are as follows:

- if p played take, then p, p' have payoffs of $2^t m_0$ and $2^t m_1$ respectively.
- if p played push, then p, p' have payoffs of $2^{t+1} m_1$ and $2^{t+1} m_0$ respectively.

That is, if a player plays push, they increase the size of the pot to be won, and if they play take, the game ends, with them getting a larger amount of money.
Most players except grandmasters play for a few rounds, with some of the reasons claimed for this being altruism, the difference in computational capacity of individuals and incentive difference.

There are some other criticisms of SPNEs as well, such as that it discusses what to do if the game reaches a certain history, but the equilibrium in earlier stages might show that we cannot actually reach this history.

3.2. Imperfect information extensive form games

EFGs are perfect information systems, where every player has perfect information about all developments in the game until that round. This is not useful in many practical scenarios, such as card games. EFGs also do not allow the representation of games with simultaneous moves. For example, the neighbouring kingdom dilemma we discussed in Example 1 depends significantly on the fact that both kingdoms move simultaneously.
This can be modeled as a sequential game, where the second player does not known for certain which history the game is in. This is called an *information set*. Such games where players deal with information sets instead of particular histories are called imperfect information extensive form games. When the information sets are singletons, we get back PIEFGs.

Definition 3.5. An *imperfect information extensive form games* (IIEFG) is an 7-tuple $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N}, (I_i)_{i \in N} \rangle$ where

- $\langle N, A, \mathcal{H}, \mathcal{X}, P, (u_i)_{i \in N} \rangle$ is a PIEFG,
- for every $i \in N$, $I_i = (I_i^1, I_i^2, \ldots, I_i^{k(i)})$ is a partition of $\{h \in \mathcal{H} \setminus Z : P(h) = i\}$ with the property that $\mathcal{X}(h) = \mathcal{X}(h')$ and $P(h) = P(h') = i$ whenever there exists some j such that $h, h' \in I_i^j$. The I_i^js are called the *information sets* of player i, and I_i the collection of information sets of i.

At a given information set, the player and all their available actions are the same. The player is not certain which history in the information set the game is at.
Since the actions now depend on the information sets and not the histories, we denote $\mathcal{X}(I_i^j) := \mathcal{X}(h)$ for any $h \in I_i^j$.
Strategies are also defined over information sets. The strategy set of player $i \in N$ is defined by

$$S_i = \bigtimes_{\tilde{I} \in I_i} \mathcal{X}(\tilde{I}).$$

With IIEFGs, NFGs can be represented using EFGs, though this is not very succinct. The representation used is typically chosen on the basis of the game we are working with.

3.2.1. Strategies

In NFGs, we had mixed strategies where we randomized over pure strategies. In EFGs, randomization can be done in several ways:

- randomize over the strategies defined at the very beginning of the game, and
- randomize over the actions at a given information set – we call such a strategy a behavioral strategy. This takes advantage of all the information a player has at a certain point of time.

Definition 3.6. A *behavioral strategy* of a player in an IIEFG is a function that maps each of their information sets to a probability distribution over the set of actions at that information set.

A couple of questions come to mind seeing this: can a player attain a higher payoff using one type of strategy? Is there any sort of equivalence between mixed and behavioral strategies.
Given a node x, mixed/behavioral strategy σ_i, and a mixed/behavioral strategy vector ξ_{-i}, denote by $p(x; \sigma_i, \chi_{-i})$ the probability of going to the node x in a game.

Definition 3.7 (Equivalence of mixed and behavioral strategies)**.** A mixed strategy σ_i and a behavioral strategy b_i of a player i in an IIEFG are *equivalent* if for every mixed/behavioral strategy vector ξ_{-i} of the other players and every vertex x in the game tree,

$$p(x; \sigma_i, \xi_{-i}) = p(x; b_i, \xi_{-i}).$$

Now, by definition, equivalence implies that the above equality holds at leaf nodes.
In fact, it suffices to check that it holds at leaf nodes! Given an arbitrary non-leaf nodes, the probability of reaching that node is equal to the sum of the probabilities of reaching the leaf nodes in the subtree, so equality at the leaves implies equality at non-leaves.
As a result, we get the following.

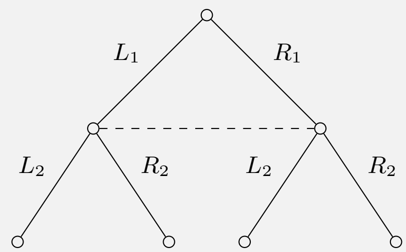
Theorem 3.8 (Utility Equivalence)**.** If σ_i, b_i are equivalent, then for every mixed/behavioral strategy vector of the other players ξ_{-i} and every $j \in N$,

$$u_j(\sigma_i, \xi_{-i}) = u_j(b_i, \xi_{-i}).$$

Corollary 3.9. Let σ and b be equivalent. Then, for all $i \in N$, $u_i(\sigma) = u_i(b)$.

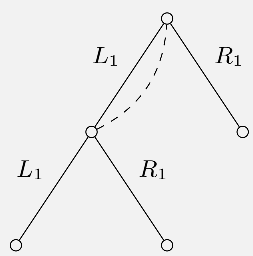
Behavioral strategies are more natural in IIEFGs. Players plan at a given stage (i.e. information set), and don't have a master plan from the very beginning. They are also far simpler computationally, since we only need to deal with fewer variables. For example, if a player has 4 information sets with 2 actions each, a mixed strategy would require $2^4 - 1$ variables whereas a behavioral strategy requires only 4.
It turns out that behavioral strategies and mixed strategies have incomparable power, as the following two examples demonstrate.

Example 14 (There exist mixed strategies without equivalent behavioral strategies)**.** Consider the following game tree with a single player.



Observe that there does not exist any behavioral strategy b for the player such that $p(b; L_1 R_2) = p(b; R_1 L_2) = 0$. Indeed, this would require $b_1(L_2) = b_1(R_2) = 0$.
On the other hand, there does exist a mixed strategy σ that sets $\sigma(L_1 R_2) = \sigma(R_1 L_2) = 0$. This is easily seen on realizing that $S_1 = \{L_1, L_2\} \times \{R_1, R_2\}$, so we have complete freedom in assigning probabilities to all four leaf ndoes.

Example 15 (There exist behavioral strategies without equivalent mixed strategies)**.** Consider the following game tree with a single player.



Observe that there exists no mixed strategy that goes to $L_1 R_1$ with nonzero probability. Indeed, here, $S_1 = \{L_1, R_1\}$, so the path $L_1 R_1$ is not even an option.
On the other hand, the behavioral strategy that picks L_1 or R_1 with equal probability $1/2$ does go to $L_1 R_1$ with probability $1/4$.

In the first example above, the player remembers that they made a move but does not remember which move they made. In the second, the player does not remember that they made a move.

So, when exactly does a behavioral strategy not have an equivalent mixed strategy?

Lemma 3.10. If there exists a path from the root to some vertex x that passes through the same information set twice, and if the action leading to x is not the same at each of these vertices, then the player of the information set has a behavioral strategy that has no equivalent strategy.

Theorem 3.11. Consider an IIEFG such that every vertex has at least two actions. Every behavioral strategy has an equivalent mixed strategy iff each information set of a player intersects every path emanating from the root at most once.

When does a mixed strategy not have an equivalent mixed strategy?
We need to figure out some way of formalizing the forgetfulness in Example 14.

Let $X = (x_0, x_1, \ldots, x_K)$ and $\hat{X} = (x_0, \hat{x}_1, \ldots, \hat{x}_L)$ be two paths in the game tree and I_i^j an information set of player i that intersects the two paths are precisely one vertex, say x_k and $\hat{x}_\ell$ respectively where $k < K$, $\ell < L$. Then, the two paths choose the same action at I_i^j if the two actions at these vertices are identical, that is,

$$a_i(x_k \to x_{k+1}) = a_i(\hat{x}_\ell \to \hat{x}_{\ell+1}).$$

In general, we also consider $a_i(x \to y)$ (where y is a descendant but not necessarily a child of x), defined by the action $a_i(x \to x')$, where x' is the child of x on the path from x to y.
Player i is said to have perfect recall if

1. any path from the root to a leaf intersects an information set of player i at most once, and
2. any two paths that end in the same information set of player i pass through the same information sets of i in the same order, and in every such information set, the two paths choose the same action.

Let us state this more formally.

Definition 3.12 (Perfect recall)**.** Player i is said to have *perfect recall* if for every I_i^j and pair of vertices $x, \hat{x} \in I_i^j$, if the decision vertices of i are $x_i^1, x_i^2, \ldots, x_i^L = x$ and $\hat{x}_i^1, \hat{x}_i^2, \ldots, \hat{x}_i^{\hat{L}} = \hat{x}$ respectively for the two paths from the root to x and $\hat{x}$, then

1. $L = \hat{L}$,
2. for any $1 \leq \ell \leq L$, $x_i^\ell, \hat{x}_i^\ell \in I_i^k$ for some k, and
3. for any $1 \leq \ell \leq L$, $a_i(x_i^\ell \to x_i^{\ell+1}) = a_i(\hat{x}_i^\ell \to \hat{x}_i^{\ell+1})$.

A game is said to be of perfect recall if every player has perfect recall.

Why is the game in Example 14 not of perfect recall?
Observe that perfect recall subsumes the condition in Theorem 3.11.
This effectively says that "every player remembers their choices, as well as whatever they remembered in the past".
Let us look at some consequences of perfect recall.
Given a node x, let $S_i^*(x)$ be the set of pure strategies of player i at which he chooses the actions leading to x. This is the intersections of members of S_i with the paths from the root to x.

Theorem 3.13. If i is a player with perfect recall and $x, \hat{x}$ are two vertices in the same information set of i, then $S_i^*(x) = S_i^*(\hat{x})$.

The above conclusion follows because the same information sets are seen and we take the same actions at these sets.

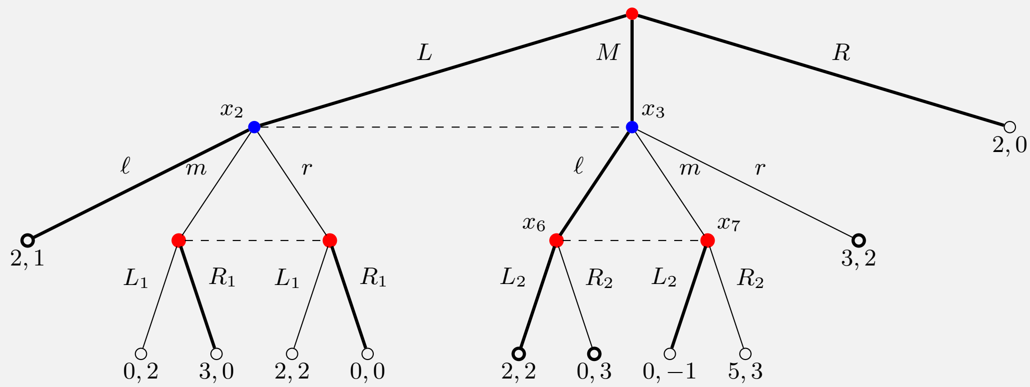
Theorem 3.14 (Kuhn)**.** In an IIEFG, if i is a player with perfect recall, then for any mixed strategy of i, there exists an equivalent behavioral strategy (and vice-versa).

The proof is constructive in nature, with the construction using perfect recall.
We remark that the converse of this theorem is *not* true, as seen in Example 15.

3.2.2. Bayesian equilibria

What are some notions of equilibria in IIEFGs? We could extend the subgame perfection in PIEFGs, but because the histories are uncertain, we are forced to use mixed strategies. Due to the information sets, a best response cannot be defined without the "belief" of each player.
Belief is a conditional probability distribution over the histories in an information set (where the conditioning is done over reaching the information set).

Example 16. Consider the following IIEFG, and the behavioral strategy described therein (by thick lines) – in the first round, player 1 plays L, M, R with probabilities $5/12, 4/12, 3/12$ respectively.



Now, suppose that player 2 believes that they are at x_6, x_7 with probabilities $p, 1-p$ respectively. If it is at x_6, it makes sense to choose L_2, and if at x_7, it makes sense to choose R_2. So, if the belief was $> 2/7$ $(1-p > 2/7)$, then the player chooses R_2. Such an action that maximizes expected utility at each information set is called *sequential rationality*.

Is the behavioral strategy described an equilibrium? Let us check this in the recursive method we did for usual SPNEs.
First of all, as mentioned, we require $p \geq 5/7$.
On the left subtree, the choice of R_1 is vacuously rational because we can never reach that subtree.
Now, let the belief of player 2 for the nodes x_2, x_3 be $q, 1-q$ respectively. If it plays ℓ, it gets a utility of 1 with probability q and 2 with probability $1-q$. If it chose m instead, it would get 0 with probability q and -1 with probability $1-q$. Finally, if it chose r, it would get 0 with probability q and 2 with probability $1-q$. It is seen that the expected utility for the choice of ℓ is always highest. In general, we need to compare it to the expected utilities for other choices. In this case, by nature of the behavioral strategy, the belief corresponds to $q = 5/9$.
Therefore, this behavioral strategy is indeed an equilibrium.
This is not a unique equilibrium however. As an exercise, check whether (MR_1R_2, m) is an equilibrium.

Definition 3.15. Let the information sets of player i be $I_i = \{I_i^k\}_{k=1}^{k(i)}$. The *belief* of player i is a probability distribution μ_i^j over I_i^j.
A belief $\mu_i = \{\mu_i^j\}_{j=1}^{k(i)}$ of player i is said to be a *Bayesian belief* wrt the behavorial strategy σ if it is derived from σ using Bayes' rule. That is, for each I_i^j such that there exists some $y \in I_i^j$ with $P_\sigma(y) \neq 0$,

$$\mu_i^j(x) = \frac{P_\sigma(x)}{\sum_{y \in I_i^j} P_\sigma(y)}$$

for all $x \in I_i^j$ and $j = 1, 2, \ldots, k(i)$.

A strategy σ_i of player i at information set I_i^j is said to be *sequentially rational* given σ_{-i} and partial beliefs μ_i^j if

$$\sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i, \sigma_{-i} \mid x) \geq \sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i', \sigma_{-i} \mid x).$$

for all actions σ_i' at I_i^j.
The tuple (σ, μ) is sequentially rational if it is sequentially rational for every player at every information set. Such a tuple (t, μ) is also called an *assessment*.

Sequential rationality is a refinement of Nash equilibria.

Proposition 3.16. In a PIEFG, a behavioral strategy profile σ is an SPNE iff the tuple $(\sigma, \hat{\mu})$ is sequentially rational.

Above, $\hat{\mu}$ assigns a degenerate distribution to every node.

Definition 3.17. An assessment (σ, μ) is a *perfect Bayesian equilibrium* (PBE) if for every player $i \in N$,

1. μ_i is Bayesian with respect to σ, and
2. σ_i is sequentially rational given σ_{-i} and μ_i.

Note that the belief μ_i is allowed to be arbitrary on information sets that are never reached. This is often represented with only σ, because μ is obtained from σ.

3.2.3. Example: peer-to-peer file sharing

In traditional file sharing models, download rate rapidly tapers off as the number of users increases. In peer-to-peer (P2P) models however, the download rate does not change significantly with an increase in users.
P2P models are very scalable and resilient to failures.

1. A *protocol* is the messages that can be sent and the actions that can be taken over the network.
2. A *client* is a particular process for sending messages and taking actions.
3. A *reference client* is a specific implementation of a client.
4. The *peers* are the ones using the protocol, and are the players in the game theoretic version of this problem which we shall discuss.

One of the earliest P2P systems, *Napster*, had a centralized database using which users could download music form one another.
Later, *Gnutella* became more popular. Each peer A can get a list of IP addresses of peers from a set of knwn peers (there is no server). To get a file, it broadcasts a query message to known peers, and the query response is sent by a peer B who has the desired file (this response is routed back to the original requester). Finally, A directly downloads the file from B.

Now, in Gnutella, each player can choose to either share, or "free ride" and gain the benefits of downloading without sharing anything. A corresponding game matrix looks something like

	Share	Free
Share	$2, 2$	$-1, 3$
Free	$3, -1$	$0, 0$

This is very similar to the neighbouring kingdoms dilemma we have seen. The issue with Gnutella is that it does not take these strategic thoughts into consideration. It does in fact turn out [AH00] that very few of the peers actually share anything.
In 2005, 85% (!) of the peers were free-riding. By 2013, Gnutella comprised less than 1% of worldwide P2P traffic. Many other P2P systems met the same fate.

We shall look at *BitTorrent*, which now comprises about 85% of P2P traffic in USA. It is used for both file sharing and software distribution (like Linux).
The key innovation is that it breaks files into pieces, and views it as a *repeated* game, with each piece shared separately.

1. First, the user goes to a searchable directory to find a link to a `.torrent` file corresponding to the desired content.
2. The `.torrent` file contains metadata about the content, in particular tracker URLs.
3. The tracker provides a list of peers participating in the swarm for the content (that is, their IP address and port).
4. The users BitTorrent client can now contact all these peers to download content.

The tracker is a centralized entity that controls traffic, tracks connection between peers and their speed of upload, download etc.
BitTorrent's algorithm is referred to as the *optimistic unchoking* algorithm. The reference client does as follows.

1. It sets a threshold r of upload speed.
2. If a peer j uploaded to i at a rate $\geq r$, unchoke j in the next period.
3. If a peer j uploaded to i at a rate $< r$, choke j in the next period.
4. Some peers who have just appeared may end up being choked because of this because they have not uploaded anything yet. So, every three time periods, optimistically unchoke a random peer from the neighbourhood who is currently choked, and leave them unchoked for three time periods.

This is forcing a repeated game by fragmenting files. The "leecher-seeder" (downloader-uploader) game is a repeated version of the neighbouring kingdoms dilemma, and the strategy of the seeder is just tit-for-tat – it cooperates as long as the leecher does as well.
The reader may find a simulation of the algorithm here.

There are various implementation-related questions and things to consider.

- How often should one contact the tracker?
- Which fragments of the file should be revealed?
- How many upload slots and which peers should be unchoked and at what speed?
- What data should others be allowed to download?
- Possible goals might be to minimize upload speed, maximize download speed, or perhaps some sort of balance.

There are numerous attacks on BitTorrent as well, such as

- BitThief [LMSW09]. The goal is to download files without uploading. It asks for peers from the tracker very frequently to grow its neighbourhood rapidly. It exploits optimistic unchoking, so never needs to upload. This can be fixed by modifying the tracker to block the IP address.
- Strategic Piece Revealer [LLSB08]. The reference client tells neighbours about new pieces, and requests the "rarest" piece first. It then tries to set up a monopoly while keeping others interested. Because we have established a monopoly over rare pieces, we can get any piece we desire.

§4. Bayesian Games

Thus far, we have only discussed non-cooperative games with complete information, where players only aim to maximize their own utilities (as opposed to cooperative games where players form coalitions and utilities are defined over coalitions, and incomplete information games where players do not deterministically know what game they are playing). There are various other types of games, such as repeated games, stochastic games etc.
We shall not discuss cooperative games in this course, but we shall now discuss incomplete information games. In such games, players receive private signals, known as *types*. We shall focus specifically on Bayesian games [Har67].

Example 17. Suppose we have a soccer game with two competing teams. Each can choose a game plan, where they either aim to win, or to draw. These two plans are the two possible types, which are private signals caused by external factories such as possibly weather, player injuries etc.
There are four possible type profiles in this example, WW, WD, DW, DD. Given this, we get four possible game matrices (DW is just the opposite of WD).

WW	Attack	Defend
Attack	$1, 1$	$2, 0$
Defend	$0, 2$	$0, 0$

WD	Attack	Defend
Attack	$2, 0$	$2, 1$
Defend	$0, 1$	$1, 0$

WW	Attack	Defend
Attack	$0, 0$	$1, 0$
Defend	$0, 1$	$-1, -1$

The probabilities of choosing each of the four profiles come from a common prior distribution, which is common knowledge.

Definition 4.1. A *Bayesian game* is represented by $\langle N, (\theta_i)_{i\in N}, P, (\Gamma_\theta)_{\theta\in \times_{i\in N}\Theta_i}\rangle$, where

1. N is the set of players,
2. Θ_i is the set of types of player i,
3. P is the common prior distribution over $\Theta \coloneqq \bigtimes_{i\in N}\Theta_i$, with the restriction that

$$\sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_i, \theta_{-i}) > 0$$

   for all $\theta_i \in \Theta_i$ for all $i \in N$. That is, the marginals for every type is positive (because otherwise we can eliminate θ_i and prune the type set), and
4. Γ_θ is an NFG $\langle N, (A_i(\theta))_{i\in N}, (u_i(\theta))_{i\in N}\rangle$ for the type profile $\theta \in \Theta$. We typically assume that $A_i(\theta) = A_i$ for all θ, and set $A \coloneqq \bigtimes_{i\in N} A_i$. As a result, u is a function $A \times \Theta \to \mathbb{R}$.

A Bayesian game is played as follows.

1. First, a profile θ is randomly drawn according to P.
2. Each player observes (only) their own type θ_i.
3. Player i picks an action $a_i \in A_i$ for all $i \in N$.
4. Player i gets payoff $u_i(a_i, a_{-i}; \theta)$.

Now, let us look at strategies in Bayesian games.

Definition 4.2. A (pure) *strategy* in a Bayesian game is a map $s_i : \Theta_i \to A_i$ for each i.

Similarly, a mixed strategy is a map $\sigma_i : \Theta_i \to \Delta A_i$.

Definition 4.3.

1. Let σ be a mixed strategy profile of a Bayesian game. The *ex-ante* utility of σ is the expected utility before observing one's own type. It is given by

$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta).$$

2. After observing the type θ_i, we can reevaluate the probabilities of the other players' profiles as

$$P(\theta_{-i} \mid \theta_i) = \frac{P(\theta_i, \theta_{-i})}{\sum_{\tilde{\theta}_{-i} \in \Theta_{-i}} P(\theta_i, \tilde{\theta}_{-i})}.$$

The *ex-interim* utility is the expected utility after observing one's own type θ_i is

$$u_i(\sigma \mid \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i} \mid \theta_i) u_i(\sigma(\theta); \theta).$$

More explicitly, we have

$$u_i(\sigma) = \sum_{\substack{\theta \in \Theta \\ (a_1, \ldots, a_n) \in A}} P(\theta) \left(\prod_{j \in N} \sigma_j(\theta_j)(a_j) \right) u_i(a_1, \ldots, a_n; \theta_1, \ldots, \theta_n).$$

Observe that when the types are independent, both utilities are the same.

Bayes' rule immediately yields that

$$u_i(\sigma) = \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma \mid \theta_i).$$

Example 18. Suppose we have a buyer and a seller trying to buy/sell an item. Let their type sets be $\Theta_1 = \Theta_2 = [100]$ and action sets $A_1 = A_2 = [100]$.
If the bid of the seller is less than or equal to that of the buyer, the trade goes through at the average of the two bids. Otherwise, the trade does not happen.
Suppose that type generation occurs independently and uniformly over Θ_1, Θ_2. Then, for all θ_1, θ_2,

$$P(\theta_2 \mid \theta_1) = P(\theta_1 \mid \theta_2) = \frac{1}{100}.$$

We also have

$$u_1(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \frac{a_1 + a_2}{2} - \theta_1, & a_2 \geq a_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$u_2(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \theta_2 - \frac{a_1 + a_2}{2} & a_1 \geq a_2 \\ 0, & \text{otherwise.} \end{cases}$$

Example 19 (Sealed bid auctions I: first price auctions)**.** Suppose we have two players, both willing to buy an object. Their values θ_i and bids b_i lie in $[0,1]$. The allocation functions that indicate the winner are

$$o_1(b_1,b_2) = \mathbb{1}_{b_1 \geq b_2} = \begin{cases} 1, & b_1 \geq b_2, \\ 0, & \text{otherwise} \end{cases} \text{ and } o_2(b_1,b_2) = 1 - o_1(b_1,b_2).$$

Again, suppose that the prior distribution is uniform over $[0,1]^2$. Then,

$$u_i(b_1,b_2;\theta_1,\theta_2) = o_i(b_1,b_2)(\theta_i - b_i).$$

Such a game where we pay the bid in the beginning is called a *first price auction*.

4.1. Equilibria

As discussed, there are two stages to the game, ex-ante before observing the type, and ex-interim after observing the type.
Of course, we have a Nash equilibrium in the ex-ante stage, where σ^* is an equilibrium if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*)$$

for all $\sigma_i', i \in N$.
In the ex-interim stage, we have the *Bayesian equilibrium*:

$$u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* \mid \theta_i) \geq u_i(\sigma_i'(\theta_i), \sigma_{-i}^* \mid \theta_i)$$

for all $\sigma_i', \theta_i \in \Theta_i, i \in N$.

For exactly the same reason as in Theorem 2.18, it suffices to check the right-hand-side for pure strategies a_i.

Theorem 4.4. In finite Bayesian games, a strategy profile is a Bayesian equilibrium iff it is a Nash equilibrium.

Proof.

$\Rightarrow$ Suppose that (σ^*, P) is a Bayesian equilibrium. Then,

$$\begin{aligned} u_i(\sigma_i', \sigma_{-i}^*) &= \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma_i'(\theta_i), \sigma_{-i}^* \mid \theta_i) \\ &\leq \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* \mid \theta_i) = u_i(\sigma_i^*, \sigma_{-i}^*). \end{aligned}$$

$\Leftarrow$ Suppose instead that (σ^*, P) is not a Bayesian equilibrium. That is, there exists some $i \in N, \theta_i \in \Theta_i, a_i \in A_i$ such that

$$u_i(a_i, \sigma_{-i}^* \mid \theta_i) > u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* \mid \theta_i).$$

Consider the strategy $\widehat{\sigma}_i$ defined by

- $\widehat{\sigma}_i(\theta_i') = \sigma_i^*(\theta_i')$ for $\theta_i' \in \Theta_i \setminus \{\theta_i\}$,
- $\widehat{\sigma}_i(\theta_i)(a_i) = 1$, and
- $\widehat{\sigma}_i(\theta_i)(b_i) = 0$ for $b_i \in A_i \setminus \{a_i\}$.

Then,

$$\begin{aligned}
u_i(\widehat{\sigma}_i, \sigma^*_{-i}) &= \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i) u_i(\widehat{\sigma}_i(\tilde{\theta}_i), \sigma^*_{-i} \mid \tilde{\theta}_i) \\
&= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) u_i(\widehat{\sigma}_i(\tilde{\theta}_i), \sigma^*_{-i} \mid \tilde{\theta}_i) + P(\theta_i) u_i(\widehat{\sigma}_i(\theta_i), \sigma^*_{-i} \mid \theta_i) \\
&= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) u_i(\sigma^*_i(\tilde{\theta}_i), \sigma^*_{-i} \mid \tilde{\theta}_i) + P(\theta_i) u_i(a_i, \sigma^*_{-i} \mid \theta_i) \\
&> \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) u_i(\sigma^*_i(\tilde{\theta}_i), \sigma^*_{-i} \mid \tilde{\theta}_i) + P(\theta_i) u_i(\sigma^*_i(\theta_i), \sigma^*_{-i} \mid \theta_i) \\
&= u_i(\sigma^*_i, \sigma^*_{-i}),
\end{aligned}$$

a contradiction. ■

Theorem 4.5. Every finite Bayesian game has a Bayesian equilibrium.

A finite Bayesian game is one where the set of players, action set, and type set are finite.
The idea of the proof is that we transform the game to a complete information game treating each type as a player, then we invoke Nash.

Example 20 (Sealed bid auctions II: equilibria in first price auctions)**.** This is a continuation of Example 19. In that example, we looked at *first price auctions*. We claim that the pure strategy profile defined by $b_1^* = \theta_1/2, b_2^* = \theta_2/2$ is a Bayesian equilibrium. For player 1, for a given θ_1, the utility under a unilateral deviation of b_1 is

$$\begin{aligned}
u_1(b_1, b_2^* \mid \theta_1) &= \int_0^1 f(\theta_2 \mid \theta_1)(\theta_1 - b_1) \mathbb{1}_{b_1 \geq \theta_2/2} \, \mathrm{d}\theta_2 \\
&= (\theta_1 - b_1) \int_0^1 \mathbb{1}_{\theta_2 \leq 2b_1} \, \mathrm{d}\theta_2. && (P \text{ is uniform})
\end{aligned}$$

If $b_1 \leq 1/2$, this is equal to $(\theta_1 - b_1)2b_1$, which is maximized for $b_1 = b_1^* = \theta_1/2$ as desired. When $b_1 > 1/2$, this is equal to $(\theta_1 - b_1) \leq (\theta_1 - 1/2) \leq \theta_1^2/2$.
Therefore, the payoff is indeed maximized for $b_1 = b_1^* = \theta_1/2$. Unilateral deviation for player 2 is similar, and the profile is a Bayesian equilibrium as claimed.

Example 21 (Sealed bid auctions III: second price auctions)**.** Next, let us look at *second price auctions*. The highest bidder wins as before, but the amount he pays is the second highest bid. Then,

$$u_1(b_1, b_2; \theta_1, \theta_2) = (\theta_1 - b_2)\mathbb{1}_{b_1 \geq b_2}$$
$$u_2(b_1, b_2; \theta_1, \theta_2) = (\theta_2 - b_1)\mathbb{1}_{b_1 < b_2}.$$

Here, we claim that $b_1^* = \theta_1, b_2^* = \theta_2$ is a Bayesian equilibrium.
Similar to earlier, player 1's payoff under unilateral deviation is

$$\int_0^1 (\theta_1 - \theta_2)\mathbb{1}_{b_1 \geq \theta_2} \, \mathrm{d}\theta_2 = \left(b_1\theta_1 - \frac{\theta_1^2}{2}\right).$$

This is maximized when $b_1 = b_1^* = \theta_1$. Unilateral deviation for player 2 is similar.

It turns out that even in the situation where the distributions of θ_1, θ_2 are arbitrary but still independent, $b_1^* = \theta_1$, $b_2^* = \theta_2$ is a Bayesian equilibrium. Indeed, for unilateral deviation of player 1,

$$\int_0^{b_1} f(\theta_2)(\theta_1 - \theta_2) = \theta_1 F\left(b_1\right) - \int_0^1 \theta_2 f(\theta_2).$$

Differentiating with respect to b_1 and setting the derivative to 0 to maximize it, we get

$$\theta_1 f\left(b_1\right) - b_1 f\left(b_1\right) = 0,$$

so the payoff is maximum when $b_1 = b_1^* = \theta_1$ (assuming the density is positive everywhere).
For any independent positive prior, bidding the true type is a Bayesian equilibrium of the induced Bayesian game in second price auctions.

§5. Social Choice

5.1. Introduction

Mechanism design is something of an inverse of game theory, where our task is to set the rules of the game given the desired objects/outcomes.
Some common examples of this are elections, resource allocation, matching students to universities, etc.
Let N be a set of players, X a set of outcomes, and Θ_i a set of private information of each agent i. Each element $\theta_i \in \Theta_i$ is called a *type*.

The type manifests in the preferences over the outcomes. We shall look at two ways this can happen.

1. *Ordinal*: θ_i defines an ordering over the outcomes, which describes a preference order. It does not describe how much something is preferred, however.
2. *Cardinal*: A utility function u_i maps an outcome/type pair to real numbers. In the *private value model*, u_i is a map $X \times \Theta_i \to \mathbb{R}$. In the *interdependent value model*, $(u_i)_{i \in N}$ is a map $X \times \Theta \to \mathbb{R}$.

Example 22. Let us look at the example of voting. Here, X is the set of candidates, and each θ_i is a ranking over the candidates. For example, (a, b, c) means that the voter prefers a over b over c.
Another example is single object allocation. Each outcome is $x = (a, p) \in X$, where $a = (a_1, \dots, a_n)$, where each $a_i \in \{0, 1\}$ and at most one of the a_is is 1, and $p = (p_1, \dots, p_n)$ where p_i is the payment charged to i. θ_i is the amount i values the object. The utility is given by $u_i(x, \theta_i) = a_i\theta_i - p_i$. We shall study this in more detail later in Section 5.4.3.

Now, the designer has a certain objective captured by a *social choice function* (SCF) $f : \Theta_1 \times \cdots \times \Theta_n \to X$. For example, in voting, if there is a candidate who beats everyone else in pairwise contests, we would like him to be chosen as the winner. In public project choice, where $\theta_i : X \to \mathbb{R}$ is the value of each project, we should pick $f(\theta) = \arg\max_{x \in X} \sum_{i \in N} \theta_i(x)$.
To create a game where $f(\theta)$ emerges as the outcome as an equilibrium, we require mechanism design.

Definition 5.1. A(n indirect) mechanism is a collection of message spaces and a decision rule $\langle M_1, \dots, M_n, g\rangle$, where M_i is the message space of agent i, and $g : M_1 \times \cdots \times M_n \to X$. A direct mechanism is the specific case where $M_i = \Theta_i$ and $g \equiv f$.

We shall see soon that it suffices to look at direct mechanisms.
All subsequent definitions assume cardinal preferences, but they can be replaced with ordinal preferences quite simply.

Definition 5.2. In a mechanism $\langle M_1, \dots, M_n, g\rangle$, a message m_i is *weakly dominant* for player i at θ_i if

$$u_i(g(m_i, \widetilde{m}_{-i}), \theta_i) \geq u_i(g(m_i', \widetilde{m}_{-i}), \theta_i) \tag{5.1}$$

for all $\widetilde{m}_{-i}, m_i'$.

Definition 5.3 (Dominant strategy implementability). An SCF $f : \Theta \to X$ is *implemented* in dominant strategies by $\langle M_1, \dots, M_n, g\rangle$ if

1. there exists a message mapping $s_i : \Theta_i \to M_i$ such that $s_i(\theta_i)$ is a dominant strategy for agent i at θ_i, and

2. $g(s_1(\theta_1), \dots, s_n(\theta_n)) = f(\theta)$ for all $\theta \in \Theta$.

We say that SCF f is *dominant strategy implementable* (DSI) by $\langle M_1, \dots, M_n, g\rangle$.

Definition 5.4 (Dominant strategy incentive compatibility)**.** A direct mechanism $\langle \Theta_1, \dots, \Theta_n, f\rangle$ is *dominant strategy incentive compatible* (DSIC) if

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\theta_i', \theta_{-i}), \theta_i) \tag{5.2}$$

for all $\theta_i, \theta_i', \widetilde{\theta}_{-i}, i$.

To determine if an SCF f is DSI, it seems that we must search over all possible indirect mechanisms.

Theorem 5.5 (Revelation principle)**.** If there exists an indirect mechanism that implements f in dominant strategies, then f is DSIC.

This means that we can focus on DSIC mechanisms without loss of generality.

Proof. Let f be implemented by $\langle M_1, \dots, M_n, g\rangle$, and fix $s_i : \Theta_i \to M_i$ as in the definition. Set $m_i' = s_i(\theta_i'), \widetilde{m}_{-i} = s_{-i}(\widetilde{\theta}_{-i})$ in Equation (5.1), to get

$$u_i(\underbrace{g(s_i(\theta_i), s_{-i}(\widetilde{\theta}_{-i}))}_{f(\theta_i, \widetilde{\theta}_{-i})}, \theta_i) \geq u_i(\underbrace{g(s_i(\theta_i'), s_{-i}(\widetilde{\theta}_{-i}))}_{f(\theta_i', \widetilde{\theta}_{-i})}, \theta_i),$$

so f is DSIC. ■

Now, let us look at a Bayesian extension, where agents may have probabilistic information about others' types. Recall bayesian games.

Definition 5.6 (Bayesian implementability)**.** A(n indirect) mechanism $\langle M_1, \dots, M_n, g\rangle$ implements an SCF f in *Bayesian equilibrium* if

1. there exists a message mapping profile $(s_1, \dots, s_n)$ such that $s_i(\theta_i)$ maximizes the ex-interim utility of agent i, that is,
$$\mathbb{E}_{-\theta_i}\left[u_i(g(s_i(\theta_i), s_{-i}(\theta_{-i})), \theta_i) \mid \theta_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i(g(m_i', s_{-i}(\theta_{-i})), \theta_i) \mid \theta_i\right]$$
for all $m_i', \theta_i, i \in N$, and
2. $g(s_i(\theta_i), s_{-i}(\theta_{-i})) = f(\theta_i, \theta_{-i})$ for all $i \in N$.

We say that f is Bayesian implementable via $\langle M_1, \dots, M_n, g\rangle$ under the prior P.

Proposition 5.7. If an SCF f is dominant strategy implementable, it is Bayesian implementable.

Similar to dominant strategy incentive compatibility, we have the following.

Definition 5.8 (Bayesian incentive compatibility)**.** A direct mechanism $\langle \Theta_1, \dots, \Theta_n, f \rangle$ is *Bayesian incentive compatible* (BIC) if

$$\mathbb{E}_{\theta_{-i}} \left[u_i(f(\theta_i, \widetilde{\theta}_{-i}), \theta_i) \mid \theta_i \right] \geq \mathbb{E}_{\theta_{-i}} \left[u_i(f(\theta_i', \widetilde{\theta}_{-i}), \theta_i) \mid \theta_i \right]. \tag{5.3}$$

for all $\theta_i, \theta_i', \widetilde{\theta}_{-i}, i$.

Theorem 5.9 (Revelation principle II)**.** If an SCF f is implementable in Bayesian equilibrium, it is BIC.

The proof is near-identical to that of the first revelation principle.

5.2. Arrow's Impossibility Theorem

Consider the setting where we have finite sets $A = \{a_1, \dots, a_m\}$ of alternatives and $N = \{1, \dots, n\}$ of players. Each player has a preference order R_i over a (a binary relation over A) such that

1. Completeness: for every $a, b \in A$, aR_ib or bR_ia or both,
2. Reflexivity: for all $a \in A$, aR_ia, and
3. Transitivity: for $a, b, c \in A$, if aR_ib and bR_ic, then aR_ic

The set of all such preference orders is denote $\mathcal{R}$. An ordering is linear if for $a, b \in A$, if aR_ib and bR_ia, then $a = b$. The set of all such linear orderings is denoted $\mathcal{P}$.

Given any ordering R_i, we can decompose it into an *asymmetric part* P_i and *symmetric part* I_i.

Definition 5.10 (Arrovian Social Welfare Function)**.** An Arrovian Social Welfare Function (ASWF) is a function $F : \mathcal{R}^n \to \mathcal{R}$.

That is, given the preference orders of multiple people, we try to consolidate them into a single preference order. Let $\hat{F}(R)$ and $\overline{F}(R)$ be the asymmetric and symmetric parts of $F(R)$ respectively.

Definition 5.11 (Pareto)**.** An ASWF F is said to be *weak Pareto* if for $a, b \in A$, if aP_ib for all $i \in N$, then $a\hat{F}(R)b$ for all $i \in N$.
An ASWF F is said to be *strong Pareto* if for $a, b \in A$, if aR_ib for all $i \in N$ and for some j, aP_jb, then $a\hat{F}(R)b$.

It is not too difficult to see that strong Pareto ASWFs are weak Pareto.
We say that $R_i, R_i' \in \mathcal{R}$ *agree* on $\{a, b\}$ if for agent i, aP_ib iff $aP_i'b$, bP_ia iff $bP_i'a$, and aI_ib iff bI_ia. We use the shorthand $R_i\big|_{a,b} = R_i'\big|_{a,b}$ to denote this. If it holds for every agent i, we write $R\big|_{a,b} = R'\big|_{a,b}$.

Definition 5.12 (Independence of irrelevant alternatives)**.** An ASWF F is said to satisfy *independence of irrelevant alternatives* (IIA) if for all $a, b \in A$, if $R\big|_{a,b} = R'\big|_{a,b}$, then $F(R)\big|_{a,b} = F(R')\big|_{a,b}$.

That is, if everyone prefers one option to another, then the final preference order also prefers this option to the other. Note that the dictatorship ASWF, defined by $F(R) = R_i$ for some fixed i, is both weak Pareto and IIA.

Theorem 5.13 (Arrow)**.** For $|A| \geq 3$, if an ASWF F is weak Pareto and IIA, then it is a dictator.

Over the rest of this section, we prove the above.
First, we need some notions of "decisiveness".

Definition 5.14. Let $F : \mathcal{R}^n \to \mathcal{R}$ be given, and let $G \subseteq N$ be non-empty.
G is said to be *almost decisive* over $\{a, b\}$, denoted $\overline{D}_G(a, b)$, if if aP_ib for all $i \in G$ and bP_ja for all $j \notin G$, then $a\hat{F}(R)b$.
G is *decisive* over $\{a, b\}$, denoted $D_G(a, b)$, if if aP_ib for all $i \in G$, then $a\hat{F}(R)b$.

If a group is (almost) decisive over all pairs of coordinates, we say that it is (almost) decisive.
The proof goes in two parts.

1. Field expansion: If a group is almost decisive over a pair of alternatives, then it is decisive over all pairs of alternatives.
2. Group contraction: If a group is decisive, a strict subset of that group is also decisive.

Lemma 5.15 (Field expansion)**.** Let F be WP and IIA. For all $a \neq b$ and $x \neq y$, and $G \subseteq N, G \neq \varnothing$, $\overline{D}_G(a, b)$ implies $D_G(x, y)$.

Proof. There are 6 cases to consider.

1. Case 1. $x = a$ and $y \neq a, b$. Pick arbitrary $R \in \mathcal{R}^n$ such that aP_iy for all $i \in G$. We wish to show that $a\hat{F}(R)y$. Construct another preference R' as follows. For $i \in G$, we have $aP_i'b$ and $bP_i'y$, and on $N \setminus G$, $bP_i'a$ and $bP_i'y$, and between a, y we retain the same ordering as in R_i.
   We ensure that $R_i'\big|_{a,y} = R_i|_{a,y}$ for all $i \in N$. Almost-decisiveness of F over $\{a, b\}$ for R' implies that $a\hat{F}(R')b$. Weak Pareto over b, y implies that $b\hat{F}(R')y$. So, by transitivity, $a\hat{F}(R')y$. Now, use IIA to conclude that $a\hat{F}(R)y$ as well, completing the proof.
2. Case 2. $y = b$ and $x \neq a, b$. The idea for this is nearly identical to that in case 1. R' is constructed such that for $i \in G$, $xP_i'a$ and $aP_i'b$, and on $N \setminus G$, $xP_i'a$ and $bP_i'a$, and the ordering of a, b is the same as that in R.
3. Case 3. $x \neq a, b$ and $y \neq a, b$. We have $\overline{D}_G(a, b)$, so $D_G(a, y)$ by case 1, so $\overline{D}_G(a, y)$ by definition, so $D_G(x, y)$ by case 2.
4. Case 4. $x \neq a, b$ and $y = a$. We have $\overline{D}_G(a, b)$, so $D_G(x, b)$ by case 2, so $\overline{D}_G(x, b)$ by definition, so $D_G(x, a)$ by case 1.
5. Case 5. $x = b$ and $y \neq a, b$. We have $\overline{D}_G(a, b)$, so $D_G(a, y)$ by case 1, so $\overline{D}_G(a, y)$ by definition, so $D_G(b, y)$ by case 2.
6. Case 6. $x = a$ and $y = b$. This similarly follows from previous parts.
7. Case 7. $x = b$ and $y = a$. This similarly follows from previous parts.

■

Lemma 5.16 (Group contraction)**.** Let F be WP and IIA. Let $G \subseteq N$ with $|G| \geq 2$ be decisive. Then, there exists non-empty $G' \subsetneq G$ which is decisive.

Proof. Arbitrarily choose $G_1 \subsetneq G$ and $G_2 = G \setminus G_1$. Let $a, b, c \in R$. Consider an ASWF R which orders aP_ibP_ic for $i \in G_1$, cP_iaP_ib for $i \in G_2$, and bP_icP_ia for $i \in N \setminus G$. Because aP_ib for all $i \in G$ and G is decisive, $a\hat{F}(R)b$.

1. Case 1. $a\hat{F}(R)c$. Observe that aP_ic for all $i \in G_1$ and cP_ia for $i \notin G_1$. Consider all R' where this condition holds. By IIA, $a\hat{F}(R')c$. Therefore, G_1 is almost decisive over $\{a, c\}$ and by field expansion, G_1 is decisive.

2. Case 2. $cF(R)a$. Recalling that $a\hat{F}(R)b$, we have $c\hat{F}(R)b$. Observe that cP_ib for all $i \in G_2$ and bP_ic for $i \notin G_2$. Considering all R' where this condition holds again, we get by IIA that $b\hat{F}(R')c$, so G_2 is almost decisive over $\{b, c\}$ and by field expansion, G_2 is decisive. ■

To complete the proof of Arrow, note that N is decisive, so by Lemma 5.16, there is a singleton decisive set and F is a dictator.

5.3. Social Choice

This means the Arrovian social welfare set up is too demanding. It says that achieving a social ordering in a democratic way is impossible. There are two suggestions to mitigate this:

1. consider a social choice setting, where we output an alternative instead of an ordering.

2. put restrictions on the agents' preferences.

Definition 5.17. A *social choice function* is a function $f : \mathcal{P}^n \to A$.

Note that we assume strict preferences.
Over the next few examples, we shall look at various voting rules.

Example 23 (Voting)**.**

1. Scoring rule. Let $s = (s_1, \ldots, s_m)$ be some common score vector. Each voter's kth preferred alternative is given a score of s_k. The scores are summed over all candidates, and the candidate with the highest score wins. There are various specific cases depending on s.

   - Plurality. $s = (1, 0, 0, \ldots, 0)$.
   - Veto. $s = (1, 1, \ldots, 1, 0)$.
   - Bonds. $s = (m - 1, m - 2, \ldots, 1, 0)$.
   - Harmonic. $s = (1, 1/2, \ldots, 1/m)$.
   - k-approval. $s = (1, \ldots, 1, 0, \ldots, 0)$ with exactly k 1s.

2. Plurality with runoff. This has two phases: in the first, the top 2 highest scoring candidates are retained and everyone else eliminated, and in the second everyone is asked to vote again. This is practiced in the French presidential election.

3. Maximin. This chooses the candidate with the largest margin of victory wins.

4. Copeland. Here we view the set of orderings as a collection of votes for pairwise elections, and choose the candidate who wins the most pairwise elections.

A *Condorcet* winner is a candidate that beats every other candidate in a pairwise election. It is not guaranteed to exist.
A voting rule that returns a Condorcet winner, if it exists, as the winner, is called *Condorcet consistent*. Clearly, Copeland is Condorcet consistent. It turns out that *no* scoring rule is Condorcet consistent (!), even seemingly logical strategies such as plurality.

Definition 5.18 (Pareto efficiency)**.** An alternative a is *Pareto dominated* by b if for all $i \in N$, bP_ia.
An SCF f is *Pareto efficient* if for all preference profiles P and $a \in A$, if a is Pareto dominated, then $f(P) \neq a$.

Definition 5.19 (Unanimity)**.** An SCF f is *unanimous* if for a preference profile P with $P_1(1) = P_2(1) = \cdots = P_n(1) = a$ (where $P_i(k)$ is the kth preferred alternative of agent i), $f(P) = a$.

Proposition 5.20. Any Pareto efficient SCF is unanimous, and the converse is not true.

Indeed, every alternative other than a is Pareto dominated by it. Further, this containment is strict, as can be seen by considering an SCF which picks a dominated alternative when not all the top alternatives are identical.

Definition 5.21 (Ontoness)**.** An SCF f is *onto* if for all $a \in A$, there exists $P^* \in \mathcal{P}^n$ such that $f(P^*) = a$.

Proposition 5.22. Any unanimous SCF is onto.

Indeed, a is chosen by any preference profile which has a at the top spot everywhere.

Definition 5.23 (Manipulability)**.** An SCF f is *manipulable* if there exists a profile P and $i \in N$ such that

$$f(P'_i, P_{-i})P_i f(P_i, P_{-i})$$

for some P'_i.

That is, false reporting can result in a more preferred outcome.
Non-manipulability means that false reporting can only result in a worse outcome.

Example 24 (Manipulability)**.** Consider a voting system with three candidates a, b, c (with ties broken in favour of $a > b > c$).
First, suppose we are in the the plurality setting, and the three voters have true preference orders abc, bac, cba.
If the third voter reports cba, then the winner is a. If he reports bca on the other hand, the winner is b, who is preferred to a.
Next, suppose we are in the Copeland setting, and the three voters have true preference orders abc, bca, cab.
If everyone reports their true preferences, a is the winner. If the second voter instead reports cba, then c becomes the Copeland winner.

Definition 5.24 (Strategyproofness)**.** An SCF is said to be *strategy-proof* (SP), *truthful*, or *incentive-compatible* if it is not manipulable by any agent at any profile.

Let us look at some properties of strategyproof SCFs.

Definition 5.25 (Dominated set)**.** For $a \in A$ and preference P_i, the *dominated set* of a at P_i is

$$D(a, P_i) = \{b \in A : aP_ib\}.$$

Definition 5.26 (Monotonicity)**.** An SCF f is *monotone* (MONO) if for any two profiles P, P' with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i)$ for all $i \in N$, then $f(P') = a$.

That is, if the relative position of a weakly improves from R to R', then a remains the winner.

Lemma 5.27. An SCF is strategyproof if and only if it is monotone.

Proof. Let us look at the forward direction first. Let f be SP and P, P' be two profiles with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i)$ for all $i \in N$. For $0 \leq k \leq n$, let $P^{(k)} = (P'_1, P'_2, \ldots, P'_k, P_{k+1}, \ldots, P_n)$ – note that $P^{(0)} = P$ and $P^{(n)} = P'$. We shall show inductively that $f(P^{(k)}) = a$ for each $0 \leq k \leq n$. The base case $k = 0$ is trivial.
Suppose that $f(P^{(k-1)}) = a$ but $f(P^{(k)}) = b \neq a$ for some $k \geq 1$. Now, note that it is not possible that bP_ka or aP'_kb as this would violate strategyproofness. This means that aP_kb and bP'_ka, but this contradicts $D(a, P_k) \subseteq D(a, P'_k)$, so we have a contradiction and are done.

Let us now look at the backward direction. Suppose instead that for some profile P, $i \in N$, and P'_i, $f(P'_i, P_{-i})P_if(P)$. Set $P' = (P'_i, P_{-i})$, $f(P) = a$, $f(P') = b$, so bP_ia. Construct a preference ordering P''_i with $P''_i(1) = b$ and $P''_i(2) = a$, and let $P'' = (P''_i, P_{-i})$.
We clearly have $D(a, P_j) \subseteq D(a, P''_j)$ for all $j \in N$. Monotonicity implies that $f(P'') = f(P) = a$. On the other hand, $D(b, P'_j) \subseteq D(b, P''_j)$ for all $j \in N$, so by monotonicity $f(P'') = f(P') = b$, resulting in a contradiction. ∎

We shall repeatedly see the above proof strategy.

Lemma 5.28. If an SCF is monotonic and onto, it is Pareto efficient.

Proof. Suppose instead that f is monotonic and onto, but not PE – let P be a preference profile and $a, b \in A$ such that bP_ia for all $i \in N$, but $f(P) = a$.
Using ontoness, let P' be a profile with $f(P') = b$, and construct a profile P'' with $P''_i(1) = b$ and $P''_i(2) = a$ for all $i \in N$. Clearly, $D(b, P') \subseteq D(b, P'')$, so $f(P'') = b$. We also have $D(a, P) \subseteq D(a, P'')$, so $f(P'') = a$, yielding a contradiction. ∎

Corollary 5.29. Let f be a strategyproof SCF. The following are equivalent.

1. f is Pareto efficient.
2. f is unanimous.
3. f is onto.

It turns out that even the above requirement is too strict, however.

Theorem 5.30 (Gibbard-Satterthwaite Theorem)**.** Suppose $|A| \geq 3$. f is strategyproof and onto if and only if it is dictatorial.

The above is untrue when $|A| = 2$ – plurality with some fixed tie-breaking rule is strategyproof, onto, and non-dictatorial. It may also break down if we place restrictions on the set of alternatives that a voter can report (we assume it to be $\mathcal{P}$).
As long as all ordinal ranks are feasible in the cardinal preference setting, this has an extension to that setting.
We shall only prove the result for $n = 2$ (so $N = \{1, 2\}$). It is possible to then apply induction on the number of agents to prove it for an arbitrary n, but we omit this part – interested readers may consult [Sen01] for more details.

Proof. First of all, we claim that for every preference profile P, $f(P) \in \{P_1(1), P_2(1)\}$.
If $P_1(1) = P_2(1)$, unanimity implies that $f(P)$ is this common value.
Suppose that $P_1(1) = a \neq b = P_2(1)$, and $f(P) = c \neq a, b$. Consider a preference P_1' with $P_1'(1) = a, P_1'(2) = b$, and a preference P_2' with $P_2'(1) = b, P_2'(2) = a$. Note that $f(P_1, P_2') \in \{a, b\}$, because every other alternative is Pareto dominated and f is Pareto efficient. Further, $f(P_1, P_2') \neq b$ because this would mean that f is manipulable (by player 2). Therefore, $f(P_1, P_2') = a$. Similarly, we get that $f(P_1', P_2) = b$. Now, due to monotonicity (because f is strategyproof), $D(a, P_1) \subseteq D(a, P_1')$, so $f(P_1', P_2') = a$, but similarly $D(b, P_2) \subseteq D(b, P_2')$ so $f(P_1, P_2') = b$, leading to a contradiction and proving the claim.

Now, let P, P' be two profiles with $P_1(1) = a \neq b = P_2(1)$ and $P_1'(1) = c, P_2'(1) = d$. It suffices to show that $f(P') = c$ if $f(P) = a$ and $f(P') = d$ if $f(P) = b$.
If $c = d$, this is trivial by unanimity, so assume $c \neq d$. We shall show that if $f(P) = a$, $f(P') = c$; the other side follows symmetrically.

1. First, consider the case where $(c, d) = (a, b)$. By the claim, $f(P') \in \{a, b\}$. Suppose instead that $f(P') = b$. Let profile $\hat{P}$ with $\hat{P}_1(1) = a, \hat{P}_1(2) = b, \hat{P}_2(1) = b, \hat{P}_2(2) = a$.
   By monotonicity using P, we get that $f(\hat{P}) = f(P) = a$, and by monotonicity using P', we get that $f(\hat{P}) = f(P') = b$, a contradiction.
2. Next, consider the case where $d = b$ but $c \neq a, b$. Again, we have that $f(P') \in \{c, b\}$, and suppose instead that $f(P') = b$. Let $\hat{P}_1$ be a preference order with $\hat{P}_1(1) = c, \hat{P}_1(2) = a$. By case 1, $f(\hat{P}_1, P_2) = f(P_1', P_2') = b$. Under true preferences $(\hat{P}_1, P_2)$, we see that player 1 misreporting their preference as P_1 would result in a more preferred alternative (a as opposed to b), contradicting strategyproofness.
3. Next, suppose that $c \neq a, b$ and $d \neq b$, and assume that $f(P') = d$. Let $\hat{P}$ be a profile such that $\hat{P}_1(1) = c, \hat{P}_2(1) = b$. By a symmetric version of case 2 applied for $P', \hat{P}$, $f(\hat{P}) = b$. By case 2 applied for $P, \hat{P}$, $f(\hat{P}) = c$, leading to a contradiction.

All remaining cases follow similar to case 3 using previously shown cases, so we omit the details. ■

A reason for a restrictive result like the GS theorem is that the domain of the SCF is too large, so a potential manipulator has far too many ways to manipulate.
What this means is that we must restrict the domain of our SCF.

5.4. Domain restriction

Consider the following different definition of strategyproofness.
An SCF f is strategyproof if for all $i \in \mathbb{N}, P_i, P_i' \in \mathcal{P}, P_{-i} \in \mathcal{P}^{n-1}$,

$$f(P_i, P_{-i}) P_i f(P_i', P_{-i}) \text{ or } f(P_i, P_{-i}) = f(P_i', P_{-i}).$$

If we reduce the set of preferences from $\mathcal{P}$ to $\mathcal{S} \subsetneq \mathcal{P}$, a function that is strategyproof on $\mathcal{P}$ continues to remain strategyproof on $\mathcal{S}$. However, we may get some more strategyproof functions.

There are various examples of domain restrictions which lead to interesting non-dictatorial strategyproof SCFs.

5.4.1. Single-peaked preferences

Consider the example of a room with several people in it, and each person has a preferred temperature t_i^* – anything above or below this is monotonically disliked more. The agent preferences are "single-peaked" in the sense that there is one unique temperature we like the most, and everything else is liked less in a monotonic fashion.
There is a common ordering, the temperature scale here, and each person has a unique preferred point, and on each side there is a monotone decrease.
This common ordering is denoted $<$, where the order is any total order (a binary relation is transitive and antisymmetric).
Why is this a domain restriction? If the ordering is over $\{a, b, c\}$ and the ordering is $a < b < c$, then preferences such as acb are ruled out.

Definition 5.31 (Single-peaked preference)**.** A preference order P_i (that is linear over A) of agent i is *single-peaked* with respect to the common order $<$ of the alternatives A if

1. for all $b, c \in A$ with $b < c \leq P_i(1)$, cP_ib and
2. for all $b, c \in A$ with $P_i(1) \leq b < c$, bP_ic.

Denote by $\mathcal{S}$ the set of single-peaked preferences.

So, under this domain restriction, our SCF is a function $\mathcal{S}^n \to A$.
We also typically take A to be a finite subset of $\mathbb{R}$.
This circumvents the Gibbards-Satterthwaite Theorem as seen by the strategyproof, onto, and non-dictatorial SCF which just outputs the leftmost peak. Clearly, the player with the leftmost peak has no reason to misreport. For any other player, misreporting only changes the outcome if they report something less than the leftmost peak, and they necessarily prefer this less than the current outcome.
A generalization of this is as follows.

Definition 5.32 (Median voter SCF)**.** An SCF $f : \mathcal{S}^n \to A$ is a *median voter SCF* if there exists a multiset $B = \{y_1, \ldots, y_{n-1}\}$ such that $f(P) = \text{median}(B, \text{peaks}(P))$ for all $P \in \mathcal{S}$.

The points in B are called the peaks of "phantom voters". Note that B is fixed for f and does not change with P.

Lemma 5.33 (Moulin 1980)**.** Every median voter SCF is strategyproof.

We do not prove this as it is relatively simple – the proof is similar to what we did when considering the leftmost peak SCF.

Proposition 5.34. f is Pareto efficient iff for each preference P, setting $p_{\min}, p_{\max}$ be the leftmost and rightmost peaks, $f(P) \in [p_{\min}, p_{\max}]$.

Proof. First, suppose that $f(P) \notin [p_{\min}, p_{\max}]$. Assume that $f(P) < p_{\min}$ without loss of generality. Then, every agent prefers $p_{\min}$ to $f(P)$, so $f(P)$ is dominated and f is not Pareto efficient.
On the other hand, if $f(P) \in [p_{\min}, p_{\max}]$, there does not exist any b such that $bP_if(P)$ for all $i \in N$. Indeed, if $b < f(P)$, it is less preferred by the player with peak $p_{\max}$, and if $b > f(P)$, it is less preferred by the player with peak $p_{\min}$. This vacuously implies Pareto efficiency. ∎

Although the proofs differ quite a bit, several results like in the unrestricted setting hold even here.

Theorem 5.35. Let $f : \mathcal{S}^n \to A$. If f is strategyproof, it is monotonic.

The proof of this is quite similar to that we did earlier in Lemma 5.27, although we need to argue why the construction in the reverse implication is valid.

Theorem 5.36. Let $f : \mathcal{S}^n \to A$ be strategyproof. The following are equivalent.

1. f is onto.
2. f is unanimous.
3. f is Pareto efficient.

Proof. We know that Pareto efficiency implies unanimity which implies ontoness. As a result, it suffices to show that ontoness implies Pareto efficiency. Suppose instead that f is strategyproof and onto but not Pareto efficient. Then, there exist $a, b \in A$ such that aP_ib for all $i \in N$ but $f(P) = b$. Because preferences are single-peaked, there is some alternative $c \in A$ which is a "neighbour" of b such that cP_ib for all $i \in N$ (possibly, $c = a$) – go "one step" in the direction of a from b.
Using ontoness, let P' such that $f(P') = c$. Consider P'' such that $P''_i(1) = c$ and $P''_i(2) = b$ for all $i \in N$. Due to monotonicity from P to P'', $f(P'') = b$, and due to monotonicity from P' to P'', $f(P'') = c$, leading to a contradiction. ■

Definition 5.37. Given a preference profile P and a permutation $\sigma : N \to N$, define a new preference profile P^σ by $(P \circ \sigma^{-1})$ – the preference of i under P is the preference of $\sigma(i)$ in the new profile.

Definition 5.38 (Anonymity)**.** An SCF $f : \mathcal{S}^n \to A$ is said to be *anonymous* if for every profile P and permutation σ of the agents, $f(P^\sigma) = f(P)$.

Any social choice function that specifically looks at the agent identities is not anonymous. In particular, dictatorial SCFs are not anonymous.
Anonymity seems a sensible property to desire from a good SCF.

Theorem 5.39 (Moulin 1980)**.** An SCF f is strategyproof, onto, and anonymous iff it is a median voter SCF.

Proof. The backward direction is trivial. We have already seen in Lemma 5.33 that f is strategyproof. It is also trivially anonymous because we only look at the set of peaks and do not assign importance to any particular player's peak. For ontoness, given an alternative a, we can consider the preference profile that places every player's peak at a. Clearly, the outcome is a.

Let $f : \mathcal{S}^n \to A$ be strategyproof, anonymous, and onto. Define P_i^0 to be agent i's preference with a peak at the leftmost position, and P_i^1 as that with a peak at the rightmost position. First, construct the phantom peaks

$$y_j = f(P_1^0, P_2^0, \ldots, P_{n-j}^0, P_{n-j+1}^1, \ldots, P_n^1)$$

for $j = 1, \ldots, n-1$.

First, we claim that the (y_j) is non-decreasing.
Note that the profiles we use to define y_j and y_{j+1} differ only at the $(n-j)$th coordinate. Consequently, due to strategyproofness, we must have $y_j P^0_{n-j} y_{j+1}$ or the two are equal. However, P^0_{n-j} is single-peaked with its peak at the leftmost position, so this just means that $y_j \leq y_{j+1}$.

Let P be an arbitrary profile, and p_i be the peaks of P_i. We claim that

$$f(P) = \operatorname{median}(p_1, \ldots, p_n, y_1, \ldots, y_n).$$

Due to anonymity, we may assume without loss of generality that $p_1 \leq \cdots \leq p_n$. Say a is the median we are interested in.

First, consider the case where $a = y_j$ for some j. By the previous claim and the definition of the median, we must have that $(j-1)$ phantom peaks and $(n-j)$ agent peaks are to the left, and $(n-1-j)$ phantom peaks and j agent peaks to the right of a. That is,

$$p_1 \leq \cdots \leq p_{n-j} \leq y_j = a \leq p_{n-j+1} \leq \cdots \leq p_n.$$

By definition,

$$y_j = f(P^0_1, P^0_2, \ldots, P^0_{n-j}, P^1_{n-j+1}, \ldots, P^1_n).$$

Let

$$b = f(P_1, , P^0_2, \ldots, P^0_{n-j}, P^1_{n-j+1}, \ldots, P^1_n).$$

Due to strategyproofness, $y_j P^0_1 b$, so $y_j \leq b$. By strategyproofness again, $b P_1 y_j$. Further, $p_1 \leq y_j$, so this means that $b \leq y_j$. Therefore, $b = y_j$! We can then repeat this argument for the first $(n-j)$ agents to get that

$$f(P_1, \ldots, P_{n-j}, P^1_{n-j+1}, \ldots, P^1_n) = y_j.$$

After this, we replace preferences from the right hand-side, to finally get that $y_j = f(P)$.

Next, suppose that $a = p_j$ for some j. We prove this in the case where there are two players, and the general case is a (slightly non-trivial) extension of this argument.
Let $a = P_1(1)$ and $b = P_2(1)$.
Let P' such that $P_i(1) = P'_i(1)$ for all $i \in N$. We claim that $f(P) = f(P')$. That is, as long as the peaks stay the same, the preferences stay the same.
Set $f(P) = x$ and $f(P'_1, P_2) = y$. Because f is strategyproof, $x P_1 y$ and $y P'_1 x$. If x and y are on the same side of the peak a, then they must be the same.
Assume without loss of generality that $x < a < y$ and $a < b$. Recall that f is Pareto efficient by Theorem 5.36, so by Proposition 5.34, $f(P) \in [a, b]$. However, we have $f(P) = x < a$. We repeat this same argument to go from (P'_1, P_2) to (P'_1, P'_2), thus proving the claim.

Now, set y_1 to be the phantom peak obtained from earlier. By our assumption, the median is an agent peak, so say it is equal to a. This means that $b < a < y_1$ or $y_1 < a < b$. Assume for contradiction that $f(P) = c \neq a$.
In the situation where $b < a < y_1$, we have by Pareto efficiency that $c < a$. Construct P'_1 such that $P'_1(1) = a$ and $y_1 P'_1 c$ (this is possible because c and y_1 are on different sides of the peak). Conider the profile (P^1_1, P_2). We have $b < y_1 \leq P^1_1(1)$ so the median of the three is y_1, a phantom peak, and we can use case 1 to conclude that $f(P^1_1, P_2) = y_1$. However, $y_1 P'_1 c$ by construction, contradicting strategyproofness since $f(P'_1, P_2) = c$.
The case where $y_1 < a < b$ is very similar, except that we consider P^0_1 instead of P^1_1. ■

For a complete proof of the above result (with n players), the reader may consult [Mou80].

5.4.2. Task allocation domain

The next domain restriction we shall look at is called task allocation domain. In this, we have a unit amount of task to be shared among n agents, and agent i gets a share $s_i \in [0,1]$ such that $\sum_{i\in N} s_i = 1$.
Similar to single-peaked preferences, every agent has a preferred shared of work.

> **Example 25.** Each task may have a reward, say a w amount of wages per unit time. If agent i works for time t_i, then they earn an amount of wt_i. This task also has costs (say physical tiredness) depending on the agent equal to $c_i t_i^2$. The net payoff is then $wt_i - c_i t_i^2$, which is maximized at $t_i^* = w/(2c_i)$.
> This is single-peaked over the *share of the task*, and not the alternatives.
> If the alternatives are $(0.2, 0.4, 0.4)$ and $(0.2, 0.6, 0.2)$, player 1 is indifferent. Note that if the preferences are single-peaked for all, we cannot have a single common order over the alternatives.

Denote the domain of task allocation by T (a single-peaked function over the task share), so the SCF is $f : T^n \to A$.
For a preference profile $P \in T^n$, we have $(f(P))_i \in [0,1]$ for each i and $\sum_{i\in N}(f(P))_i = 1$.
Suppose that player i has peak p_i over the share.

Definition 5.40. An SCF $f : T^n \to A$ is *Pareto efficient* if there exists no other share of the task that is weakly preferred by all agents and strictly preferred by at least one. That is, there exists no $a \in A$ such that $aR_i f(P)$ for all $i \in N$ and $aP_j f(P)$ for some j.

Lemma 5.41. An SCF $f : T^n \to A$ is Pareto efficient iff

1. for any preference profile P with $\sum_{i\in N} p_i = 1$, $f_i(P) = p_i$ for all $i \in N$.
2. for any preference profile P with $\sum_{i\in N} p_i > 1$, $f_i(P) \leq p_i$ for all $i \in N$.
3. for any preference profile P with $\sum_{i\in N} p_i = 1$, $f_i(P) \geq p_i$ for all $i \in N$.

Proof. First of all, if $\sum_{i\in N} p_i = 1$, then there is no problem and we must allocate tasks according to the peaks of the agents. This is the unique Pareto efficient SCF.

If $\sum_{i\in N} p_i > 1$, for any SCF f, there exists some $k \in N$ such that $f_k(P) < p_k$.
If f is Pareto efficient, can there exist an agent j such that $f_j(P) > p_j$? No, because if this were the case, we would increase k's share of the task and reduce j's share of the task, to make both players strictly better off.
Consequently, for all $j \in N$, $f_j(P) \leq p_j$.
Finally, an identical argument yields that if $\sum_{i\in N} p_i < 1$, $f_j(P) \geq P_j$ for all $j \in N$. ∎

Definition 5.42 (Anonymity)**.** For an anonymous SCF $f : T^n \to A$,

$$f_{\sigma(j)}(P^\sigma) = f_j(P).$$

Let us look at a couple of candidate SCFs.

1. Serial dictatorship. A predetermined sequence of agents is fixed. Each agent is given either their peak or a leftover share. If $\sum p_i < 1$, the last agent is given the leftover share. This is Pareto efficient and strategyproof but not anonymous. It is also quite unfair for later agents.

2. Proportional. Every player is assigned a share that is some constant multiplied by their peaks. That is, if $\sum_{i \in N} p_i = s$, we assign a share of p_i/s to each player.
   This is not strategyproof. For example, if the peaks are $0.2, 0.3, 0.1$, we have $s = 0.6$ and player 1 gets $1/3$. If they report $0.1, 0.3, 0.1$ instead however, player 1 gets 0.2.

We shall next describe a rule that is Pareto efficient, anonymous, and strategyproof, known as the *uniform rule*. Suppose that $\sum_{i \in N} p_i < 1$. We begin with everyone's allocation being 1. We keep reducing each person's allocation uniformly. Whenever we hit someone's peak, we stop reducing that agent's share. We stop reducing when $\sum f_i = 1$. When $\sum_{i \in N} p_i > 1$, we instead start with everyone's allocation being 0 and increase it gradually.

Definition 5.43. The uniform allocation rule f^u is described as follows.

1. If $\sum_i p_i = 1$, $f_i^u(P) = p_i$.
2. If $\sum_{i \in N} p_i < 1$, set $\mu(P)$ as the number that satisfies $\sum_{i \in N} \max\{p_i, \mu\} = 1$ and allocate $f_i^u(P) = \max\{p_i, \mu(P)\}$.
3. If $\sum_{i \in N} p_i > 1$, set $\lambda(P)$ as the number that satisfies $\sum_{i \in N} \min\{p_i, \lambda\} = 1$ and allocate $f_i^u(P) = \min\{p_i, \lambda(P)\}$.

Lemma 5.44. The uniform rule SCF is anonymous, Pareto efficient, and strategyproof.

Proof. It is quite easy to see anonymity since we only look at the agents' peaks and not the agents themselves. Pareto efficiency follows trivially by Lemma 5.41.
Strategyproofness is evident when $\sum p_i = 1$. Suppose now that $\sum_i p_i < 1$, so $f_i^u(P) \geq p_i$. The only agents who have any reason to manipulate are those with $f_i^u(P) > p_i$. That is, $\mu(P) > p_i$ and the allocation stopped before "reaching" p_i. The only way this allocation can change is by reporting $p_i' > \mu(P) > p_i$, but this is a worse allocation for i than $\mu(P)$. The proof when $\sum p_i > 1$ is similar. ∎

Theorem 5.45 (Sprumont)**.** An SCF $f : T^n \to A$ is strategyproof and Pareto efficient iff it is the uniform rule.

We do not describe the proof of the above, and refer the reader to [Spr91].

5.4.3. Mechanism design with transfers

In this case, the utility can be transferred and is thought of as something like money. A social choice function here is a function $F : \Theta \to X$, where an outcome $x \in X$ has two components – an allocation $a \in A$ and a payment vector $\pi \in \mathbb{R}^n$.

Example 26. One real-life example of this is a public decision of building a bridge or a park, so A is $\{\text{bridge}, \text{park}\}$.
We can think of the allocation of a divisible good (a "shared spectrum"), where $a \in [0,1]^n$ and $\sum a_i = 1$.
We can also have a single indivisible good, where $a \in \{0,1\}^n$ and $\sum a_i \leq 1$.
Alternatively, we can have multiple indivisible objects, wherein we have an overall set of objects S and

$$A = \{(A_1, \ldots, A_n) : A_i \subseteq S \text{ and } A_i \cap A_j = \varnothing \text{ for all distinct } i, j \in N\}.$$

Here, the type of an agent is $\theta_i \in \Theta_i$, the private information of i. An agent's benefit from an allocation is defined using a *valuation function* $v_i : A \times \Theta_i \to \mathbb{R}$.
In the previous example, if we look at the benefit to the environment for example, the valuation of a bridge would

be lower than that for a park.

The utility of a player i when their type is θ_i and the outcome is $x = (a, \pi)$ is given by

$$u_i((a, \pi), \theta_i) = v_i(a, \theta_i) - \pi_i.$$

Note that this is always linear in the payment, but can be non-linear in the allocation component. Due to this, we call this the *quasilinear domain*.
If we have two alternatives $(a, \pi^{(1)})$ and $(a, \pi^{(2)})$, agent i always prefers that with the lower π_i! This is the simple restriction intrinsic to this section, which opens up opportunity for numerous interesting SCFs.

We split the function F into the allocation part and the payment parts, $f : \Theta \to A$ and $p_i : \Theta \to \mathbb{R}$. Here, $\pi_i = p_i(\theta)$. The tuple of p_i is denoted p.

Example 27 (Allocation rules)**.** A couple of examples of allocation rules are as follows.

1. One uninteresting allocation rule is the *constant rule* where $f^c(\theta) = a$ for all $\theta \in \Theta$ for some fixed $a \in A$.
2. A slightly more inspired rule is the *dictatorial rule*, where

$$f^D(\theta) = \arg\max_{a \in A} v_d(a, \theta_d)$$

   for all $\theta \in \Theta$ for some fixed $d \in N$.
3. The *allocatively efficient* or *utilitarian* rule is

$$f^{\text{AE}}(\theta) = \arg\max_{a \in A} \sum_{i \in N} v_i(a, \theta_i)$$

   for all $\theta \in \Theta$, where we pick any arbitrary a in the maximizing subset of A.
4. The *affine maximizer* rule is defined by

$$f^{\text{AM}}(\theta) = \arg\max_{a \in A} \left(\sum_{i \in N} \lambda_i v_i(a, \theta_i) + \kappa(a) \right),$$

   where each λ_i is chosen to be non-negative and at least one is nonzero. Note that all three rules before this are specific cases of the affine maximizer rule.
5. The *max-min* or *egalitarian rule* has

$$f^{\text{MM}}(\theta) = \arg\max_{a \in A} \min_{i \in N} v_i(a, \theta_i).$$

Example 28 (Payment rules)**.** Three examples of rules that our payment function p might satisfy are

1. *No deficit*. Here, $\sum_{i \in N} p_i(\theta) \geq 0$ for all $\theta \in \Theta$.
2. *No subsidy*. Here, $p_i(\theta) \geq 0$ for all $\theta \in \Theta, i \in N$. Clearly, a no subsidy payment rule is a no deficit payment rule.
3. *Budget balanced*. Here, $\sum_i p_i(\theta) = 0$ for all $\theta \in \Theta$.

Recall incentive compatibility. This definition can be reformulated in the mechanism design with payment scenario as follows.

Definition 5.46 (Dominant strategy incentive compatibility)**.** A mechanism (f, p) is *dominant strategy incentive compatible* (DSIC) if for all $i \in N, \theta_{-i} \in \Theta_{-i}, \theta_i, \widetilde{\theta}_i \in \Theta_i$, if we set $\theta = (\theta_i, \theta_{-i})$ and $\widetilde{\theta} = (\widetilde{\theta}_i, \theta_{-i})$,

$$v_i(f(\theta), \theta_i) - p_i(\theta) \geq v_i(f(\widetilde{\theta}), \theta_i) - p_i(\widetilde{\theta}).$$

DSIC means that telling the truth is a weakly dominating strategy equilibrium.
Our goal is that the payment rule implements f in dominant strategies. In the quasilinear domain, we are often more interested in the allocation rule than the whole SCF (which also includes the payment).

Example 29. Suppose that there are two players $N = \{1, 2\}$ and $\Theta_1 = \Theta_2 = \{\theta^H, \theta^L\}$ and $f : \Theta_1 \times \Theta_2 \to A$. For (f, p) to be DSIC,

$$\begin{aligned} v_1(f(\theta^H, \theta_2), \theta^H) - p(\theta^H, \theta_2) &\geq v_1(f(\theta^L, \theta_2), \theta^H) - p_i(\theta^L, \theta_2) \\ v_1(f(\theta^L, \theta_2), \theta^L) - p(\theta^L, \theta_2) &\geq v_1(f(\theta^H, \theta_2), \theta^L) - p_i(\theta^H, \theta_2) \end{aligned}$$

and similar inequalities for the second player.

Now, suppose that (f, p) is incentive compatible. Consider another payment

$$q_i(\theta) = p_i(\theta) + h_i(\theta_{-i})$$

for all $\theta \in \Theta, i \in N$ for some function(s) h_i. Then, observe that (f, q) is incentive compatible as well!
Therefore, if we find a single payment that implements an allocation rule, we get an entire family of payments that implement the allocation rule. A converse question we can ask is: when do payments that implement f differ only by a factor of $h_i(\theta_{-i})$?

Example 30. Suppose the allocation is the same for two type profiles $\theta = (\theta_i, \theta_{-i})$ and $\widetilde{\theta} = (\widetilde{\theta}_i, \theta_{-i})$ and $f(\theta) = f(\widetilde{\theta}) = a$. if p implements f, then Definition 5.46 immediately yields that $p_i(\theta) = p_i(\widetilde{\theta})$.

Let us now move to Pareto efficiency in the quasilinear domain.

Definition 5.47 (Pareto optimality)**.** A mechanism (f, p) is said to be *Pareto optimal* if at every type profile $\theta \in \Theta$, there exists no allocation $b \neq f(\theta)$ and payments $(\pi_1, \ldots, \pi_n)$ with $\sum_i \pi_i \geq \sum_i p_i(\theta)$ such that

$$v_i(b, \theta_i) - \pi_i \geq v_i(f(\theta), \theta_i) - p_i(\theta)$$

for all $i \in N$, with the inequality being strict for some i.

The reason for having $\sum_i \pi_i \geq \sum_i p_i(\theta)$ is that otherwise, we can always put an excessive subsidy to every agent and make everyone better off.
Recall allocative efficiency from Example 27.

Theorem 5.48. A mechanism (f, p) is Pareto optimal if and only if it is allocatively efficient.

Proof.

- Suppose that (f, p) is not allocatively efficient. Then, there exists $b \neq f(\theta)$ such that

$$\sum_i v_i(b, \theta_i) > \sum_{i \in N} v_i(f(\theta), \theta_i)$$

for some θ. Let $\delta > 0$ be the difference between the two as above. Consider the payment

$$\pi_i = v_i(b, \theta_i) - v_i(f(\theta), \theta_i) + p_i(\theta) - (\delta/n).$$

Then,

$$\big(v_i(b, \theta_i) - \pi_i\big) - \big(v_i(f(\theta), \theta_i) - p_i(\theta)\big) = (\delta/n) > 0$$

and $\sum_i \pi_i = \sum_i p_i(\theta)$, showing that (f, p) is not Pareto optimal.

- Suppose that (f, p) is not Pareto optimal. Let $\theta, \pi, b \neq f(\theta)$ such that

$$\sum_i \pi_i \geq \sum_i p_i(\theta)$$

and

$$v_i(b, \theta_i) - \pi_i \geq v_i(f(\theta), \theta_i) - p_i(\theta)$$

for all $i \in N$, with the inequality being strict for some i.
Summing the second inequality over i,

$$\sum_i (v_i(b, \theta_i) - \pi_i) > \sum_i (v_i(f(\theta), \theta_i) - p_i(\theta))$$

$$\sum_i v_i(b, \theta_i) - \sum_i v_i(f(\theta), \theta_i) > \sum_i \pi_i - \sum_i p_i(\theta) \geq 0,$$

showing that (f, p) is not allocatively efficient.

■

So, for any Pareto optimal mechanism (f^{eff}, p), we have

$$f^{\text{eff}}(\theta) \in \arg\max_{a \in A} \sum_{i \in N} v_i(a, \theta_i).$$

This seems to suggest that any sensible mechanism has this (more or less) fixed allocation part. We must now come up with a sensible payment part to make it DSIC.

Definition 5.49 (Groves payment)**.** Given a Pareto optimal function f^{eff}, consider the *Groves payment* defined for each i by

$$p_i^G(\theta) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(f^{\text{eff}}(\theta), \theta_j)$$

where $h_i : \Theta_{-i} \to \mathbb{R}$ is any arbitrary function.
A *Groves mechanism* is a pair (f^{eff}, p^G), where f^{eff} is Pareto optimal and p^G is a Groves payment (for f^{eff}).

Perhaps surprisingly, this is a truthful mechanism!

Theorem 5.50. Any Groves mechanism is DSIC.

Proof. Arbitrarily choose player i, $\theta = (\theta_i, \theta_{-i})$, and $\widetilde{\theta} = (\widetilde{\theta}_i, \theta_{-i})$, and set $a = f^{\text{eff}}(\theta)$ and $\widetilde{a} = f^{\text{eff}}(\widetilde{\theta})$. By definition,

$$v_i(a, \theta_i) + \sum_{j \neq i} v_j(a, \theta_j) \geq v_i(\widetilde{a}, \theta_i) + \sum_{j \neq i} v_j(\widetilde{a}, \theta_j).$$

The utility of player i when they report θ_i is

$$\begin{aligned} v_i(a, \theta_i) - p_i(\theta) &= v_i(a, \theta_i) - h_i(\theta_{-i}) + \sum_{j \neq i} v_j(a, \theta_j) \\ &\geq v_i(\widetilde{a}, \theta_i) - h_i(\theta_{-i}) + \sum_{j \neq i} v_j(\widetilde{a}, \theta_j) \\ &= v_i(\widetilde{a}, \theta_i) - p_i(\widetilde{\theta}), \end{aligned}$$

proving the result. ■

Definition 5.51 (VCG mechanism)**.** Given a Pareto optimal function f^{eff}, the *pivotal mechanism* or *VCG mechanism* is the Groves mechanism $(f^{\text{eff}}, p^{\text{VCG}})$ with

$$h_i(\theta_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j).$$

That is,

$$p_i^{\text{VCG}}(\theta) = \max_{a \in A} \sum_{j \neq i} (v_j(a, \theta_j) - v_j(f^{\text{eff}}(\theta), \theta_j)).$$

Clearly, $p_i^{\text{VCG}}(\theta) \geq 0$ for all $\theta \in \Theta, i \in N$, that is, it is a no subsidy payment rule (and in particular, a no deficit payment rule). The first part is essentially the values of others in the absence of agent i. When agent i is introduced, the others get less than they could have in i's absence, and the expression above can be thought of as the "damages" caused by agent i which are compensated for by this payment. In other words,

$$v_i(f^{\text{eff}}(\theta), \theta_i) - p_i^{\text{VCG}}(\theta) = \underbrace{\sum_{j \in N} v_j(f^{\text{eff}}(\theta), \theta_j)}_{\text{social welfare in presence of } i} - \underbrace{\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)}_{\text{maximum social welfare in absence of } i},$$

is the marginal contribution of i to social welfare.
Let us try to understand this better with a couple of examples.

Example 31. Consider single object allocation where an agent's type θ_j is equal to the amount they value the object, and the value is equal to θ_j if they get the item and 0 otherwise. Suppose that there are 4 agents with valuations $10, 8, 9, 5$. Allocative efficiency means that the item is allocated to that agent with the highest valuation, namely 10. We have

$$p_i^{\mathrm{VCG}}(\theta) = \max_{a \in A} \sum_{j \neq i} (v_j(a, \theta_j) - v_j(f^{\mathrm{eff}}(\theta), \theta_j)).$$

For the agent with valuation 10, this is equal to the second highest bid 9. Indeed, the first summation is maximized when a assigns the item to the agent with the second highest bid 9, and the second summation is equal to 0. For all other agents, both summations are equal to the higest bid 10.
Therefore, under the VCG mechanism, the only agent who pays is the first agent, and he pays the second highest bid 9. This makes sense since the utility of agent i is equal to $10 - 9 = 1$, which is the amount they "profit" by paying less than the amount they value the item.
The VCG mechanism for single object allocation behaves is precisely a second price auction!
Let us look at another example, wherein a committee must decide to build one of a football stadium, a library, or a museum. The members of the committee A, B, C value each of these as follows.

	Football	Library	Museum
A	0	70	50
B	95	10	50
C	10	50	50

Due to allocative efficiency, the VCG mechanism decides to build the museum. If B is removed, it is instead decided to build the library which has corresponding value $70+50 = 120$. So, B pays an amount of $120-100 = 20$. Similarly, A pays $(95+10) - 100 = 5$, and C pays $(50+50) - 100 = 0$.
Note that agent C pays nothing! They can be thought of as a *non-pivotal* agent, and removing the player does not change the outcome of the game.

Agents whose presence changes the outcome are referred to as *pivotal agents*. These are precisely those players who have to pay some nonzero amount of money.
Over the next several examples, we more precisely look at combinatorial auctions from Example 26.

Example 32 (Combinatorial auctions I: An example)**.** Let us look at another example of selling multiple objects 1 and 2.

	$\varnothing$	$\{1\}$	$\{2\}$	$\{1,2\}$
θ_1	0	8	6	12
θ_2	0	9	4	14

The above table indicates each player value $v_i(a, \theta_i) =: \theta_i(a)$ under various actions. Note that the meaning of the table here is not as directly interpreted as that in the previous example – an allocation is a pair of non-intersecting subsets of $\{1, 2\}$. That is, we can decide to allocate $\{1\}$ to one agent and $\{2\}$ to another (but we cannot allocate item 1 to both agents).
In fact, this is what happens. Due to allocative efficiency, the VCG mechanism allocates $\{1\}$ to player 2 and $\{2\}$ to player 1. If player 1 is removed, we give the entire bundle to player 2 for value 14, so player 1 pays $14 - 9 = 5$ and similarly, player 2 pays $12 - 6 = 6$. The payoffs of the two players here are $6 - 5 = 1$ and $9 - 6 = 3$.

Example 33 (Combinatorial auctions II: Introduction). In a *combinatorial auction*, there is a fixed set M of objects, and $\Omega = 2^M$ is set of "bundles". The set of allocations A is comprised of all elements $a = (a_0, a_1, \ldots, a_n)$ of Ω^n such that $a_i \cap a_j = \varnothing$ for $i \neq j$ and $M = \bigcup_{i=0}^n a_i$. Here, a_0 is the set of unallocated objects.
Each player's type/value is given by a function $\theta_i : \Omega \to \mathbb{R}$. We assume that $\theta_i(S) \geq 0$ for all $S \in \Omega$, that is, objects are "goods" and not trash.
We further assume that $\theta_i(\varnothing) = 0$ and also that valuations are "selfish", that is, $\theta_i(a) = \theta_i(a_i)$ and agent i's valuation does not depend on what the others get.

We claim that in the allocation of goods, the VCG payment for an agent who gets no object in an efficient allocation is equal to 0.
Fix some $a^* \in \arg\max_{a \in A} \sum_{j \in N} \theta_j(a)$ with $a_i^* = 0$. Also let $a_{-i}^* = \arg\max_{a \in A} \sum_{j \in N \setminus \{i\}} \theta_j(a)$. Because a is the defined $\arg\max$, we have

$$p_i^{\text{VCG}}(\theta) = \sum_{j \in N \setminus \{i\}} \theta_j(a_{-i}^*) - \sum_{j \in N \setminus \{i\}} \theta_j(a^*) \leq 0,$$

so $p_i^{\text{VCG}}(\theta) = 0$ because it is non-negative.

Definition 5.52 (Individual rationality). A mechanism (f, p) is *individually rational* if for all $\theta \in \Theta, i \in N$,

$$v_i(f(\theta), \theta_i) - p_i(\theta) \geq 0.$$

Intuitively, what this says is that no player decides to participate if they are going to incur a negative payoff.

Example 34 (Combinatorial auctions III: Individual rationality). Next, we claim that in combinatorial auctions, the VCG mechanism is individually rational. Fix $i \in N$ and define a^*, a_{-i}^* as in the previous example. We have

$$\begin{aligned}
\theta_i(a^*) - p_i^{\text{VCG}}(\theta) &= \sum_j \theta_j(a^*) - \sum_{j \neq i} \theta_j(a_{-i}^*) \\
&= \sum_j \theta_j(a^*) - \sum_{j \neq i} \theta_j(a_{-i}^*) \\
&= \underbrace{\sum_j \theta_j(a^*) - \sum_j \theta_j(a_{-i}^*)}_{\geq 0 \text{ by definition of } a^*} + \underbrace{\theta_i(a_{-i}^*)}_{\geq 0} \geq 0.
\end{aligned}$$

Let us illustrate the concepts in the previous two examples using a more specific example, that of advertising on the internet.

Example 35 (Combinatorial auctions IV: Introduction to internet advertising). Internet advertising is very lucrative for numerous reasons:

1. it is quite easy to gather a lot of user data,
2. using this data it is possible to classify buyers into categories and measure their interests,
3. due to the low latency, we can have real-time automated bidding and make decisions on the fly.

Some examples of such advertisements are

1. Sponsored search ads: advertisers bid on the keywords entered by users.
2. Contextual ads: depending on the content of the page, we see a related message.
3. Display ads: this is akin to more traditional forms of advertising like banner advertisements in newspapers.

Modern internet advertising is handled via "ad exchanges", which act as an intermediary between the publisher and the client/advertiser. Small businesses customize their ads using these exchanges.
The decision of who gets ad positions on a page is done using *position auctions*.

1. In early position auctions, ads were ordered via bid-per-impression, charging agents based on the bids they place. Newspaper ads is an example of this, but puts all the risk on the advertiser since they are placing the bid without knowing for sure that anyone will see the ad.
2. Later, we switched to the pay-per-click model, where we rank it based on the number of clicks. This shares the risk with the publisher, since they are paid depending on the number of clicks. If ads are shown but not clicked, the publisher earns nothing.
3. Today, we rank ad spots based on the product on the probability of click (called click-through rate) and bid value. That is, we rank it based on the expected revenue.

Example 36 (Combinatorial auctions V: A deep-dive into internet advertising)**.** Suppose there are N advertisers $\{1, \ldots, n\}$ and m slots $\{1, 2, \ldots, m\}$ with $m \geq n$ and every ad is shown. Also assume that 1 is the "best" position and m the "worst" (where best and worst are defined more precisely a few paragraphs later).

Let us assume that clicks generate value for the advertisers, and all clicks are valued equally irrespective of where the ad is displayed. The position only affects the click-through rate for that ad. This allows us to decouple the value effect and the position effect.
The expected value of agent $i \in N$ when their ad is shown at position $j \in M$ is $v_{ij} = \mathrm{CTR}_{ij}\, v_i$, where CTR is the click-through rate for i at j and v_i is the click value for i. CTR_{ij} has two components: the quality component e_i and the position component p_j, so $v_{ij} = p_j \cdot (e_i v_i)$. We have assumed that the position effect is decreasing with position, that is, $p_1 = 1$ and $p_j > p_{j+1}$ for $j = 1, \ldots, m-1$.
Note that v_i is the only private information of the advertisers, and both p_j, e_i are measurable. Search engines estimate e_i as $\hat{e}_i$. If the bidders bid b_i, we rank the ads in decreasing order of $\hat{e}_i b_i$.

In this setting, we just have a combinatorial auction where the allocation determines which ad spots each client gets. We get that

$$v_i(a, \theta_i) = p_{a_i}(\hat{e}_i \theta_i)$$

for an allocation $a = (a_1, \ldots, a_n)$, where $p_{a_i} = \sum_{j \in a_i} p_j$ and the reported click value is θ_i. Because of allocative efficiency, we pick an allocation $a^* \in \arg\max_{a \in A} \sum_{i \in N} v_i(a, \theta_i)$.
We claim that an allocation of slots is efficient iff it is the rank-by-expected-revenue mechanism we have described. This is rather easily proved as the above is just that maximizing a weighted sum.
The slot allocation problem is merely a sorting problem, and is computationally tractable. Now, we must use payments to make it DSIC. A natural candidate is the VCG mechanism, which is (at its core) what is used in ad auctions on Facebook!
Given reported bids $(\theta_1, \ldots, \theta_n)$, assume that $\hat{e}_1\theta_1 \geq \hat{e}_2\theta_2 \geq \cdots \geq \hat{e}_n\theta_n$. Assume that the allocation a^* is such that $a_i^* = i$.
Consider $a_{-i}^* = \arg\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$. Removing agent i means that players after i get an earlier slot. Then,

$$\begin{aligned}
p_i^{\mathrm{VCG}}(\theta) &= \sum_{j \neq i} v_j(a_{-i}^*, \theta_j) - \sum_{j \neq i} v_j(a^*, \theta_j) \\
&= \sum_{j=i}^{n-1} p_j \cdot (\hat{e}_{j+1}\theta_{j+1}) - \sum_{j=i}^{n-1} p_j \cdot (\hat{e}_{j+1}\theta_{j+1}) \\
&= \sum_{j=i}^{n-1} (p_j - p_{j+1})\hat{e}_{j+1}\theta_{j+1}.
\end{aligned}$$

In particular, $p_n^{\mathrm{VCG}}(b) = 0$. This is the total expected payment. To convert this to the pay-per-click we are interested in, we look at $p_i^{\mathrm{VCG}}(b)/p_i\hat{e}_i$.

Now, let us look at a couple of advantages and disadvantages of the VCG mechanism. Some clear advantages are:

1. It is DSIC.
2. It is a no subsidy mechanism if items are goods.
3. It never charges a losing agent.
4. It is individually rational to participate, so nobody loses money (at least in the combinatorial auction setting).

On the other hand, some criticisms are:

1. It is too transparent and provides no privacy. As a consequence of being a truthful mechanism, it reveals true valuations. Competing companies might not like such information to be made public. Additionally, malicious

auctioneers can introduce fake bidders to extract a higher payment from the bidders – this is relevant in the case where the highest bid is far higher than the second-highest bid.

2. It is very susceptible to collusion. For example, consider the following public goods setting, with the payments indicated.

	F	M	payment
A	200	0	150
B	100	0	50
C	0	250	0

If the first two players collude, they might bid higher, pay much less, and increase their utility!

	F	M	payment
A	250	0	100
B	150	0	0
C	0	250	0

3. It is not frugal, and payments can be very large. The VCG is guaranteed to be no deficit, but it can charge a payment much higher than the cost.

4. Revenue does not weakly increase with the number of players. Consider the following public good settings, with valuations as indicated.

	F	M	payment
A	0	90	0
B	100	0	90

Now suppose we add a third player to make the game the following.

	F	M	payment
A	0	90	0
B	100	0	0
C	100	0	0

Nobody is pivotal here.

5. It is not budget balanced. While it is a no deficit mechanism, it nearly always keeps a surplus, which is often quite large. The issue here is that the money cannot be redistributed among the same players because this will change payoffs and violate DSIC. If the players are partitioned into two groups and the surplus of one group is redistributed over the other group, it is budget balanced but the overall efficiency is compromised.

The final point suggests that the surplus has to be taken away or eliminated somehow. To understand this tradeoff better, we refer the reader to [NS19]. Despite these limitations, the VCG mechanism is quite elegant and has many use cases.

Now, let us try to generalize the VCG mechanism to come up with a larger class of DSIC mechanisms in the quasi-linear domain. To do this, we use the affine maximizer rule defined in Example 27 as

$$f^{\text{AM}}(\theta) = \arg\max_{a \in A} \left(\sum_{i \in N} w_i \theta_i(a) + \kappa(a) \right),$$

where the $w_i \geq 0$ are not all zero, and $\kappa : A \to \mathbb{R}$ is any arbitrary function, known as the translation.

When $\kappa \equiv 0$ and all the w_i are equal, we get allocative efficiency again. When $\kappa \equiv 0$ and $w_d = 1$ for some d and $w_i = 0$ for $i \neq d$, we get a dictatorial allocation.
If all the w_i are not identical, then the mechanism is not anonymous. If κ is non-constant, then we are giving different importance to different allocations.
In this wider class of affine maximizers, one can try to characterize the mechanisms that satisfy some property.

Definition 5.53. An affine maximizer rule f^{AM} with weights $(w_i)_{i\in N}$ and translation κ is said to satisfy *independence of non-influential agents* (INA) if for all $i \in N$ with $w_i = 0$,

$$f^{\text{AM}}(\theta_i, \theta_{-i}) = f^{\text{AM}}(\theta'_i, \theta_{-i})$$

for all $\theta_i, \theta'_i, \theta_{-i}$.

This says that tie-breaking is not done on the basis of any weight zero agents.
If INA were not satisfied, the affine maximizer could be manipulated by a weight 0 agent.

Theorem 5.54. An affine maximizer rule satisfying INA is implementable in dominant strategy.

Proof. We need to construct a payment function p^{AM} such that $(f^{\text{AM}}, p^{\text{AM}})$ is DSIC. Define

$$p_i^{\text{AM}}(\theta_i, \theta_{-i}) = \begin{cases} \frac{1}{w_i}\left(h_i(\theta_{-i}) - \left(\sum_{j\neq i} w_j\theta_j(f^{\text{AM}}(\theta)) + \kappa(f^{\text{AM}}(\theta))\right)\right), & w_i > 0, \\ 0, & w_i = 0. \end{cases}$$

This is quite similar to the Groves payment. If $w_i > 0$, the payoff is

$$\begin{aligned} \theta_i(f^{\text{AM}}(\theta_i, \theta_{-i})) - p_i^{\text{AM}}(\theta_i, \theta_{-i}) &= \frac{1}{w_i}\left(\sum_{j\in N} w_j\theta_j(f^{\text{AM}}(\theta_i, \theta_{-i})) + \kappa(f^{\text{AM}}(\theta)) - h_i(\theta_{-i})\right) \\ &\geq \frac{1}{w_i}\left(\sum_{j\in N} w_j\theta_j(f^{\text{AM}}(\theta'_i, \theta_{-i})) + \kappa(f^{\text{AM}}(\theta'_i, \theta_{-i})) - h_i(\theta_{-i})\right) \\ &= \theta_i(f^{\text{AM}}(\theta'_i, \theta_{-i})) - p_i^{\text{AM}}(\theta'_i, \theta_{-i}), \end{aligned}$$

where the second step is by the definition of the affine maximizer rule. The case where $w_i = 0$ follows due to INA. ■

Now, consider the scenario where Θ_i has all valuation functions $\theta_i : A \to \mathbb{R}$ with no restriction on the functions. In this unrestricted space, we can characterize the class of DSIC mechanisms in the quasilinear domain.

Theorem 5.55 (Roberts)**.** Let A be finite with $|A| \geq 3$. If the type space is unrestricted, any onto and dominant strategy implementable allocation rule is an affine maximizer.

We omit the proof of the above, but the interested reader can find one at [LMN09].

5.4.4. Indivisible object allocation

In the single indivisible object allocation setting, recall that due to the truthfulness of VCG, Example 31 implies that the second-price auction mechanism is truthful.

The setup is as follows. Each agent i has a type set $T_i \subseteq \mathbb{R}$, and $t_i \in T_i$ denotes the value of agent i if they win the object. An allocation a is a vector that represents the *probability* of winning the object, so the set of allocations is

$$\Delta A = \{a \in [0,1]^n : \sum_{i=1}^{n} a_i = 1\}.$$

The allocation rule is some $f : T_1 \times T_2 \times \cdots \times T_n \to \Delta A$. The valuation is said to be in the *product form* if it is given by $v_i(a, t_i)$ equal to $a_i \cdot t_i$, namely the expected valuation. That is, $f_i(t_i, t_{-i})$ is the probability of winning the object for agent i when the type profile is (t_i, t_{-i}).

In the second-price auction, also called the *Vickrey* auction, the types are v_i, and defining $t_{-i}^{(2)} = \max_{j \neq i} v_j$, agent i wins if $v_i > t_{-i}^{(2)}$ and loses if $v_i < t_{-i}^{(2)}$. Some tiebreaking rule decides the equality case. Because the payment is $t_{-i}^{(2)}$ for the winner i, the utility is zero if there is a tie. That is,

$$u_i = \begin{cases} 0, & v_i \leq t_{-i}^{(2)}, \\ v_i - t_{-i}^{(2)}, & v_i > t_{-i}^{(2)}. \end{cases}$$

Observe the following.

1. As a function of v_i when all the other v_j are fixed, the utility is convex.
2. Wherever differentiable, the utility's derivative coincides with the allocation probability.

We now describe some known results from convex analysis.

Fact 1. Convex functions are continuous on the interior of their domain.

Definition 5.56. Let $g : I \to \mathbb{R}$ be convex. For any $x \in I$, x^* is a *subgradient* of g at x^* if

$$g(z) \geq g(x) + x^* \cdot (z - x)$$

for all $z \in I$.

Fact 2. Let $g : I \to \mathbb{R}$ be convex. For every $x \in I$, a subgradient of g exists. Further, if x is in the interior of I and g is differentiable at x, $g'(x)$ is the unique subgradient at x.

Fact 3. Let $I' \subseteq I$ be the set of points where g is differentiable. Then, $I \setminus I'$ is of measure 0.

"Measure zero" means that there is no interval contained in this set.

Fact 4. Further, the set of subgradients at a point forms a convex set. More precisely, it is equal to $[g'_-(x), g'_+(x)]$.

Denote the set of subgradients of g at $x \in I$ as $\partial g(x)$. Then, we have that $\partial g(x)$ is non-empty and convex at all $x \in I$, and if g is differentiable at x, it is a singleton $\{g'(x)\}$.
Similar to the non-decreasingness of the derivative of a differentiable convex function, we get the following.

Lemma 5.57. Let $g : I \to \mathbb{R}$ be a convex function. Let $\varphi : I \to \mathbb{R}$ a subgradient function, that is, $\varphi(z) \in \partial g(z)$ for all $z \in I$. Then, φ is non-decreasing, that is, for all $x, y \in I$ with $x > y$, $\varphi(x) \geq \varphi(y)$.

Lemma 5.58. Let $g : I \to \mathbb{R}$ be a convex function. Then, for any $x, y \in I$,

$$g(x) = g(y) + \int_y^x \varphi(z) \, \mathrm{d}z,$$

where $\varphi : I \to \mathbb{R}$ is a subgradient function of g.

Definition 5.59. An allocation rule is *non-decreasing* if for every agent $i \in N$ and $s_i, t_i \in T_i$ with $t_i > s_i$ and $t_{-i} \in T_{-i}$,

$$f_i(t_i, t_{-i}) \geq f_i(s_i, t_{-i}).$$

Theorem 5.60 (Myerson)**.** Suppose $T_i = [0, b_i]$ for all $i \in N$, and the valuations are in the product form. An allocation rule $f : T \to \Delta A$ and a payment rule $(p_1, \ldots, p_n)$ are DSIC iff

(a) f is non-decreasing and

(b) payments are given by
$$p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_{-i}) \, \mathrm{d}x$$
for all t_i, t_{-i}, i.

Proof. Let us first prove the forward direction. The utility of agent i is

$$u_i(t_i, t_{-i}) = t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}).$$

Due to DSIC-ness,

$$\begin{aligned} u_i(t_i, t_{-i}) &= t_i f(t_i, t_{-i}) - p_i(t_i, t_{-i}) \\ &\geq t_i f(s_i, t_{-i}) - p_i(s_i, t_{-i}) \\ &= s_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i}) + f_i(s_i, t_{-i})(t_i - s_i) \\ &= u_i(s_i, t_{-i}) + f_i(s_i, t_{-i})(t_i - s_i). \end{aligned}$$

Fixing t_{-i}, define $g(t_i) = u_i(t_i, t_{-i})$ and $\varphi(t_i) = f_i(t_i, t_{-i})$. Then, the above just says that

$$g(t_i) \geq g(s_i) + \varphi(s_i)(t_i - s_i),$$

so φ is a subgradient function of g. Next, let us show that g is convex. Pick $x_i, z_i \in T_i$, and let $y_i = \lambda x_i + (1 - \lambda) z_i$ for some $\lambda \in [0, 1]$. Due to DSIC-ness,

$$\begin{aligned} g(x_i) &\geq g(y_i) + \varphi(y_i)(x_i - y_i) \text{ and} \\ g(z_i) &\geq g(y_i) + \varphi(y_i)(z_i - y_i). \end{aligned}$$

Taking a convex combination of the two equations above,

$$\lambda g(x_i) + (1 - \lambda) g(z_i) \geq g(y_i) + \varphi(y_i) \underbrace{(\lambda x_i + (1 - \lambda z_i) - y_i)}_{0},$$

so g is convex. Lemma 5.57 proves the first part, and Lemma 5.58 proves the second.

The backward direction is straightforward. Assuming for simplicity that $p_i(0, t_{-i}) = 0$,

$$\big(t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i})\big) - \big(t_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i})\big) = (s_i - t_i) f_i(s_i, t_{-i}) + \int_{s_i}^{t_i} f_i(x, t_{-i}) \, \mathrm{d}x \geq 0.$$ ∎

Corollary 5.61. An allocation rule in the single object allocation setting is implementable in dominant strategy iff it is non-decreasing.

Example 37 (Single object allocation mechanisms)**.** Let us look at some truthful single object allocation mechanisms.

1. The constant allocation mechanism is non-decreasing and the payment is some constant.
2. The dictatorial allocation mechanism gives the object only to the dictator, which is non-decreasing, and the payment is some constant again.
3. In a second-price auction, $p_i(0, t_{-i})$ and $f_i(\cdot, t_{-i})$ is an appropriate step function with the step at $t_{-i}^{(2)}$.
4. Efficient allocation with a reserve price is also non-decreasing. This is a modification of the second-price auction such that if the highest value is below some reserve price r, nobody gets the object. Similar to the second-price auction, here, $f_i(\cdot, t_{-i})$ is a step function with the step at $\max\{t_{-i}^{(2)}, r\}$.
5. Let us look at an uncommon allocation rule that might be difficult to arrive at without Myerson's rule. Suppose $N = \{1, 2\}$ and $A = \{a_0, a_1, a_2\}$ (a_0 meaning unsold, and a_i for $i \neq 0$ meaning that the object is given to i). Given a type profile t, the seller computes $\max\{2, t_1^2, t_2^3\}$, and selects that a_i corresponding to the maximum, breaking ties as $0 > 1 > 2$. Player 1 gets the object if $t_1 > \sqrt{\max\{2, t_2^3\}}$ and player 2 gets the object if $t_2 > \sqrt[3]{\max\{2, t_1^2\}}$. Both of these are non-decreasing step functions.

Definition 5.62. A mechanism (f, p) is *ex-post individually rational* if

$$t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) \geq 0$$

for all $t \in T$ and $i \in N$.

That is, even after *all* agents have revealed their types, participation is weakly preferred.

Lemma 5.63. Let (f, p) be a DSIC mechanism in the single object allocation setting.

(a) (f, p) is individually rational iff $p_i(0, t_{-i}) \leq 0$ for all $t_{-i} \in T_{-i}$ and $i \in N$.

(b) (f, p) is individually rational and is no-subsidy iff $p_i(0, t_{-i}) = 0$ for all $t_{-i} \in T_{-i}$ and $i \in N$.

Proof.

(a) Using individual rationality in the $t_i = 0$ case gives the forward direction of the first part. For the converse, using DSIC-ness, the payoff of i is

$$\begin{aligned} t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) &= t_i f_i(t_i, t_{-i}) - p_i(0, t_{-i}) - t_i f_i(t_i, t_{-i}) + \int_0^{t_i} f_i(x, t_{-i})\, \mathrm{d}x \\ &= \int_0^{t_i} f_i(x, t_{-i}) - p_i(0, t_{-i}) \geq 0. \end{aligned}$$

(b) This is direct since we have $p_i(0, t_{-i}) \leq 0$ from the first part, and $p_i(0, t_{-i}) \geq 0$ due to no-subsidy.

■

Let us next describe some non-Vickrey auctions, with a focus on budget balanced mechanisms (recall what this means from Example 28) or something close to one.

Example 38.

1. The object goes to the highest bidder, and the payment is such that everyone is compensated some amount.
   The highest two bidders are compensated $1/n$ of the third highest bid, and everyone else receives $1/n$ of the second highest bid. Assuming without loss of generality that $t_1 > t_2 > \cdots > t_n$, agent 1 pays
   $$-\frac{1}{n}t_3 + t_1 - \int_0^{t_1} f_1(x, t_{-1})\,\mathrm{d}x = t_2 - \frac{1}{n}t_3,$$
   agent 2 pays $-t_3/n$, and everyone else pays $-t_2/n$. The total payment is
   $$t_2 - \frac{1}{n}t_3 - \frac{1}{n}t_3 - \frac{n-2}{n}t_2 = \frac{2}{n}(t_2 - t_3),$$
   which tends to 0 for large n.
2. Allocate the object with probability $\left(1 - \frac{1}{n}\right)$ to the highest bidder, and with probability $\frac{1}{n}$ to the second highest bidder. We also set $p_i(0, t_{-i}) = \frac{-1}{n}t_{-i}^{(2)}$. Then, agent 1 pays
   $$-\frac{1}{n}t_3 + \left(1 - \frac{1}{n}\right)t_1 - \underbrace{\frac{1}{n}(t_2 - t_3)}_{\int_{t_3}^{t_2} \frac{1}{n}\,\mathrm{d}x} - \underbrace{\left(1 - \frac{1}{n}\right)(t_1 - t_2)}_{\int_{t_2}^{t_1}\left(1 - \frac{1}{n}\right)\mathrm{d}x} = \left(1 - \frac{2}{n}\right)t_2,$$
   agent 2 pays
   $$-\frac{1}{n}t_3 + \frac{1}{n}t_2 - \frac{1}{n}(t_2 - t_3) = 0,$$
   and everyone else pays $-t_2/n$. The sum of all these is 0.

5.5. Optimal mechanism design

We now come to another question: how do we maximize the revenue of the auctioneer? We begin with the same quasilinear domain from the previous subsubsection.
Before getting to the details, we must set up some notation.

Definition 5.64 (Bayesian Incentive Compatibility)**.** Let $T_i = [0, b_i]$ for each i and G a common prior over T with density g. We let $G_{-i}(s_{-i} \mid s_i)$ to be the conditional distribution over T_{-i} given that i's type is s_i, having density $g_{-i}(s_{-i} \mid s_i)$.

g_{-i} is derived using Bayes' rule.
Any mechanism $(f, p_1, \ldots, p_n)$ induces an expected allocation and payment rule (α, π). Here,

$$\alpha_i(s_i \mid t_i) = \int_{s_{-i} \in T_{-i}} f_i(s_i, s_{-i}) g_{-i}(s_i \mid t_i)\,\mathrm{d}s_i,$$

where s_i is the reported type, t_i is the true type, f_i is the probabilistic allocation as before, and g_{-i} encodes the probabilistic knowledge of the types of the other agents in the common prior. The conditioning on t_i means that we are conditioning on the true type being t_i. The expected payment is

$$\pi_i(s_i \mid t_i) = \int_{s_{-i} \in T_{-i}} p_i(s_i, s_{-i}) g_{-i}(s_{-i} \mid t_i)\,\mathrm{d}s_{-i}.$$

The expected utility of agent i is

$$t_i\alpha_i(t_i \mid t_i) - \pi_i(t_i \mid t_i).$$

Definition 5.65. A mechanism (f, p) is *Bayesian incentive compatible* (BIC) if for all $i \in N$, $s_i, t_i \in T_i$,

$$t_i\alpha_i(t_i \mid t_i) - \pi_i(t_i \mid t_i) \geq t_i\alpha_i(s_i \mid t_i) - \pi_i(s_i \mid t_i).$$

Similarly, f is Bayesian implementable if there exists some p such that (f, p) is BIC.

Now, let us make the assumption that the priors are independent. That is, agent i's value is drawn from a distribution G_i with density g_i independently from the other agents, so

$$g(s_1, s_2, \ldots, s_n) = \prod_{i \in N} g_i(s_i)$$

and

$$g(s_{-i} \mid t_i) = \prod_{j \neq i} g_j(s_j).$$

We use the shorthand $\alpha(t_i) = \alpha(t_i \mid t_i)$.

Definition 5.66. An allocation rule is *non-decreasing in expectation* (NDE) if for all $i \in N, s_i, t_i \in T_i$ with $s_i < t_i$, $\alpha_i(s_i) \leq \alpha_i(t_i)$.

It is easily seen that non-decreasing rules are non-decreasing in expectation.
Similar to Myerson's characterization, we get the following Bayesian version.

Theorem 5.67. A mechanism (f, p) in the independent prior setting is BIC iff

1. f is non-decreasing in expectation and
2. p_i satisfies
$$\pi_i(t_i) = \pi_i(0) + t_i\alpha(t_i) - \int_0^{t_i} \alpha_i(x)\,\mathrm{d}x.$$

The proof is very similar to that of Myerson, and we omit it.

Example 39 (NDE rules need not be ND)**.** Let us look at one rule that is NDE but not non-decreasing. Suppose there are two players with 5 types each, and all 5 types are equally likely.

$$\begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 1 \end{pmatrix}$$

Here, the ijth entry depicts who wins if player 2 chooses type i and player 1 chooses type j. Suppose the true types are $t_1 = t_2 = 3$. For player 1, the allocation is indeed non-decreasing in expectation since $\alpha_1(t_1')$ is equal to $0, 0, 1/5, 2/5, 3/5$ for $t_1' = 1, 2, 3, 4, 5$. We can similarly check it for player 2. However, it is not non-decreasing because $f(4, 3) = 1$ and $f(5, 3) = 0$ although $5 > 4$.

Individual rationality now becomes the following.

Definition 5.68. A mechanism (f, p) is *interim individually rational* (IIR) if for every bidder $i \in N, t_i \in T_i$,

$$t_i \alpha_i(t_i) - \pi_i(t_i) \geq 0.$$

Similar to Lemma 5.63, we get the following.

Lemma 5.69. A BIC mechanism (f, p) is IIR iff for all $i \in N$,

$$\pi_i(0) \leq 0.$$

The proof is again quite similar.

Let us return to our question of revenue maximization. Suppose we have a single agent for now, with type set $T = [0, \beta]$. Our mechanism is (f, p), where $f : [0, \beta] \to [0, 1]$ and $p : [0, \beta] \to \mathbb{R}$. Incentive compatibility (BIC and DSIC are equivalent since there is only one agent) gives that

$$tf(t) - p(t) \geq tf(s) - p(s)$$

for all $t, s \in T$. Individual rationality gives that

$$tf(t) - p(t) \geq 0$$

for all $t \in T$. The expected revenue earned by a mechanism is

$$\pi^M = \int_0^\beta p(t)g(t) \,\mathrm{d}t.$$

We need to find a mechanism M^* in the class of incentive-compatible and individually rational mechanisms such that $\pi^{M^*} \geq \pi^M$ for all M. Call M^* the *optimal mechanism*. By the characterization lemmas Theorem 5.60 and lemma 5.63, we have

$$\begin{aligned} &p(t) = p(0) + tf(t) - \int_0^t f(x) \,\mathrm{d}x \text{ and} \\ &p(0) \leq 0. \end{aligned}$$

In order to maximize revenue, $p(0) = 0$. So, the payment formula is

$$p(t) = tf(t) - \int_0^t f(x) \,\mathrm{d}x.$$

The expected revenue is

$$\pi^f = \int_0^\beta \left(tf(t) - \int_0^t f(x) \,\mathrm{d}x \right) g(t) \,\mathrm{d}t,$$

which we want to maximize by choosing an appropriate f.

Now, we have

$$\begin{aligned}
\pi^f &= \int_0^\beta \left(tf(t) - \int_0^t f(x)\,\mathrm{d}x \right) g(t)\,\mathrm{d}t \\
&= \int_0^\beta tf(t)g(t)\,\mathrm{d}t - \int_0^\beta \left(\int_0^t f(x)\,\mathrm{d}x \right) g(t)\,\mathrm{d}t \\
&= \int_0^\beta tf(t)g(t)\,\mathrm{d}t - \int_0^\beta \left(\int_x^\beta g(t)\,\mathrm{d}t \right) f(x)\,\mathrm{d}x \\
&= \int_0^\beta tf(t)g(t)\,\mathrm{d}t - \int_0^\beta \left(\int_t^\beta g(x)\,\mathrm{d}x \right) f(t)\,\mathrm{d}t \\
&= \int_0^\beta \left(tf(t)g(t) - f(t) \left(\int_t^\beta g(x)\,\mathrm{d}x \right) \right) \mathrm{d}t \\
&= \int_0^\beta \left(tf(t)g(t) - f(t)(1 - G(t)) \right) \mathrm{d}t \\
\pi^f &= \int_0^\beta \left(t - \frac{1-G(t)}{g(t)} \right) g(t)f(t)\,\mathrm{d}t.
\end{aligned}$$

Now, we want to figure out

$$\max_{\text{non-decreasing } f} \int_0^\beta \left(t - \frac{1-G(t)}{g(t)} \right) g(t)f(t)\,\mathrm{d}t. \tag{5.4}$$

Let us assume that G satisfies the *monotone hazard rate* condition, which means that $\frac{g(x)}{1-G(x)}$ is non-decreasing in x. Standard distributions such as the uniform and exponential distributions satisfy this.

Fact 5. If G satisfies the MHR condition, there is a unique solution to

$$x = \frac{1-G(x)}{g(x)}.$$

Indeed, the right-hand side is non-increasing and the left is strictly increasing. Set

$$w(x) = x - \frac{1-G(x)}{g(x)},$$

and let $w(x^*) = 0$. Then, $w(x) > 0$ for $x > x^*$ and $w(x) < 0$ for $x < x^*$. The unrestricted solution to eq. (5.4) is therefore

$$f(t) = \begin{cases} 0, & t < x^*, \\ 1, & t > x^*, \\ \alpha, & t = x^*, \end{cases} \tag{5.5}$$

where $\alpha \in [0,1]$. Note that this is basically just a step function. This f is also non-decreasing and is therefore the optimal solution of eq. (5.4).

Theorem 5.70. In the single agent single object allocation setting, a mechanism (f, p) under the MHR condition on the prior is optimal iff

1. f is given by Equation (5.5).
2. For all $t \in T$,

$$p(t) = \begin{cases} x^*, & t \geq x^*, \\ 0, & \text{otherwise.} \end{cases}$$

Let us now shift to the multiple agent setting. Call a mechanism optimal if it is BIC, IIR, and maximizes revenue. Recalling Theorem 5.67 and lemma 5.69, the expected payment made by agent i is

$$\int_{T_i} \pi_i(t_i) g_i(t_i)\,\mathrm{d}t_i,$$

where $T_i = [0, b_i]$. Exactly as before, if we set

$$w_i(t_i) = t_i - \frac{1 - G_i(t_i)}{g_i(t_i)},$$

called the *virtual valuation* of player i, the expected payment simplifies to

$$\int_0^{b_i} w_i(t_i) g_i(t_i) \alpha_i(t_i),$$

where

$$\alpha_i(t_i) = \int_{T_{-i}} f_i(t_i, t_{-i}) g_{-i}(t_{-i})\,\mathrm{d}t_{-i}.$$

Substituting α_{-i}, this gives the payment as

$$\int_T w_i(t_i) f_i(t) g(t)\,\mathrm{d}t.$$

Therefore, the total revenue generated by all players is

$$\sum_{i \in N} \int_T w_i(t_i) f_i(t) g(t)\,\mathrm{d}t = \int_T \left(\sum_{i \in N} w_i(t_i) f_i(t) \right) g(t)\,\mathrm{d}t$$

which we wish to maximize, subject to f being NDE. As before, if we try to solve the unconstrained optimization problem, we get

$$f_i(t) = \begin{cases} 1, & w_i(t_i) \geq w_j(t_j) \text{ for all } j \text{ and } w_i(t_i) \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{5.6}$$

However, this can result in fs that are not NDE. See [Mye81] for details. So, like the MHR constraint earlier, we enforce something here as well.

Definition 5.71. A virtual valuation v_i is *regular* if for all $s_i, t_i \in T_i$ with $s_i < t_i$, $w_i(s_i) < w_i(t_i)$.

This is weaker than MHR.
Regularity says that $w_i(t_i) > w_i(s_i)$ for all $t_i > s_i$. As a result, the optimal allocation rule eq. (5.6) also satisfies

$$f_i(t_i, t_{-i}) \geq f_i(s_i, t_{-i}),$$

so f is non-decreasing, and NDE in particular.
Note that this f is deterministic (in the sense that it either allocates the whole object or nothing). Therefore, not only is it BIC and IIR, it is DSIC and IR.

Theorem 5.72. Suppose every agent's valuation is regular. Define the mechanism (f, p) as follows.

1. If $w_i(t_i) < 0$ for all $i \in N$, $f_i(t) = 0$ for all $i \in N$.

2. Otherwise,
$$f_i(t) = \begin{cases} 1, & w_i(t_i) \geq w_j(t_j) \text{ for all } j \\ 0, & \text{otherwise,} \end{cases}$$
with ties being broken arbitrarily.

3. Payments are given by
$$p_i(t) = \begin{cases} 0, & f_i(t) = 0, \\ \max\{w_i^{-1}(0), \kappa_i^*(t_{-i})\}, & f_i(t) = 1. \end{cases}$$

Then, (f, p) is an optimal mechanism.

Above, $w_i^{-1}(0)$ is the value of t_i such that $w_i(t_i) = 0$ and
$$\kappa_i^*(t_{-i}) = \inf\{t_i : f_i(t_i, t_{-i}) = 1\},$$
the smallest value of t_i where i starts to win.
This can be thought of as something like a second-price auction with a reserve price.

Example 40 (Uniform independent priors)**.** Suppose there are two buyers with $T_1 = [0, 12]$ and $T_2 = [0, 18]$, with a uniform independent prior. Then,
$$w_1(t_1) = t_1 - \frac{1 - G_1(t_1)}{g_1(t_1)} = t_1 - \frac{1 - \frac{t_1}{12}}{\frac{1}{12}} = 2t_1 - 12$$
and $w_2(t_2) = 2t_2 - 18$. Then, we can tabulate some data as follows. Note that gradual change of when each player's virtual valuation becomes non-negative, and also when they start winning.

t_1	t_2	allocation	p_1	p_2
4	8	unsold	0	0
2	12	2	0	9
6	6	1	6	0
9	9	1	6	0
8	15	2	0	11

Example 41 (Symmetric bidders)**.** Suppose the valuations are drawn from the same distribution, so $T_i = T$, the g_i are some common g, and all the common valuations w_i are equal to w. Here, $w(t_i) > w(t_j)$ iff $t_i > t_j$, so the object goes to the higher bidder, and it is not sold if $w^{-1}(0) > t_i$. This is precisely a second-price auction with a reserve price of $w^{-1}(0)$, and is efficient.

Finally, we discuss a few other optimal mechanisms.

Theorem 5.73 (Green-Laffont-Holmström)**.** If the type space is "sufficiently rich", every efficient DSIC mechanism is a Groves mechanism.

"Sufficient richness" means that everything is feasible in some sense.

Proof sketch. Suppose there are two alternatives $A = \{a, b\}$. Consider the welfares $\sum_i t_i(a)$ and $\sum_i t_i(b)$. We choose a if the former is greater than the latter. Fix the valuations of the other agents to t_{-i}, and fix the value of i at alternative

b at $t_i(b)$. Then, there exists some threshold $t_i^*(a)$ such that for all $t_i(a) \geq t_i^*(a)$, a is the outcome, and for $t_i(a) < t_i^*(a)$, b is the outcome.
Using DSIC-ness for $t_i^*(a) + \epsilon = t_i(a)$, we get

$$t_i^*(a) + \epsilon - p_{ia} \geq t_i(b) - p_{ib}$$

and similarly,

$$t_i(b) - p_{ib} \geq t_i^*(a) - \delta - p_{ia}.$$

Because ϵ, δ are arbitrary, $t_i^*(a) - p_{ia} = t_i(b) - p_{ib}$. Because $t_i^*(a)$ is the reshold of the efficient outcome,

$$t_i^*(a) + \sum_{j \neq i} t_j(a) = t_i(b) + \sum_{j \neq i} t_j(b).$$

Combining these two,

$$p_{ia} - p_{ib} == \sum_{j \neq i} t_j(b) - t_j(a).$$

Therefore, the payment is of the form $p_{ix} = h_i(t_{-i}) - \sum_{j \neq i} t_j(x)$. ■

Theorem 5.74 (Green-Laffont)**.** No Groves mechanism is budget-balanced.

Corollary 5.75. If the type space is sufficiently rich, no efficient mechanism is both DSIC and budget-balanced.

Given this, we try to weaken the notion of DSIC-ness. Allocation is still efficient, and payment is now defined via a prior as

$$\delta_i(t_i) = \mathbb{E}_{t_{-i}|t_i} \sum_{j \neq i} t_j(a^*(t)).$$

For example,

$$a^*(t) \in \arg\max_{a \in A} \sum_{i \in N} t_i(a)$$
$$p_i^{\text{dAGVA}} = \frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) - \delta_i(t_i),$$

named after d'Aspremont, Gerard-Varet, Arrow, and is also known as the "expected VCG" mechanism. It can be shown that the dAGVA mechanism is efficient, BIC, and budget-balanced. However, it is not IIR.

Theorem 5.76 (Myerson-Satterthwaite)**.** In a bilateral trade (that involves two types of agents, seller and buyer), no mechanism can be simultaneously BIC, efficient, IIR, and budget-balanced.

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