Representation Theory of Finite Groups

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Contents

§-1. Notation

Given a set X, S_X is the set of all bijections from X to itself.

§0. Preliminaries

We define precisely those things that the author of these notes did not know at the time of reading. If there are things in these notes you do not know that are not defined anywhere within the notes, too bad!

0.1. Algebra

Definition 0.1. Given a group G, $\hat{G}$ denotes the set of all group homomorphisms from $G \to \mathbb{C}^*$. This is a group under point-wise operations and is called the *dual group* of G.

Proposition 0.1. If G_1, G_2 are groups and $G = G_1 \times G_2$, then $\hat{G} \cong \hat{G}_1 \times \hat{G}_2$.

Proof. Let $\varphi \in \hat{G}$. Define the homomorphism $\Phi : \hat{G} \to \hat{G}_1 \times \hat{G}_2$ by $\varphi \mapsto (\varphi_1, \varphi_2)$, where

$$\varphi_1(g_1) = \varphi(g_1, 1) \text{ and } \varphi_2(g_2) = \varphi(2, g_2).$$

Let us show that Φ is injective by showing that precisely one $\varphi \in \hat{G}$ maps to $(1, 1)$. Indeed, this would imply that $\varphi(g_1, 1) = \varphi(1, g_2) = 1$ for all $g_1 \in G_2, g_2 \in G_2$. As φ is a homomorphism, $\varphi(g_1, g_2) = 1$ as well.
For surjectivity, let $(\rho_1, \rho_2) \in \hat{G}_1 \times \hat{G}_2$. Consider $\varphi : G \to \mathbb{C}^*$ defined by $\varphi(g_1, g_2) = \rho_1(g_1)\rho_2(g_2)$. It is not difficult to show that $\varphi \in \hat{G}$ since ρ_1, ρ_2 are homomorphisms ∎

Proposition 0.2. If $G = \mathbb{Z}/n\mathbb{Z}$, then $\hat{G} \sim G$.

Proof. Let $\varphi : G \to \mathbb{C}^*$ be a homomorphism. Observe that φ is determined by $\varphi(\overline{1})$. Further, since $n\overline{m} = 0$ for any $\overline{m} \in G$, $\varphi(\overline{m})^n = \varphi(0)$, so every $\varphi(g)$ is an nth root of unity. As a result, there are at most n elements in $\hat{G}$ corresponding to which of the n roots of unity $\overline{1}$ is mapped to – each such homomorphism $\varphi^{(k)}$ is of the form $\varphi(\overline{m}) = \omega_n^{km}$ for some k. It is easily checked that all of these do in fact correspond to homomorphisms.
Further, this group is cyclic because $\varphi^{(k)} = (\varphi^{(1)})^k$. ∎

Theorem 0.3. For any finite abelian group G, $\hat{G} \sim G$.

This follows directly using the classification theorem of finite groups and the previous two propositions.

0.2. Linear Algebra

Proposition 0.4. Let $W \leq V$ be vector spaces and $T \in \mathrm{GL}(V)$. Then, T is W-invariant iff $T(W) = W$.

Definition 0.2. Let V be a finite dimensional inner product space and $T \in \mathrm{End}(V)$. The *adjoint* of T is the unique linear operator T^* such that for any $v, w \in V$,

$$\langle Tv, w \rangle = \langle v, T^*w \rangle.$$

Proposition 0.5. Let V be an inner product space and $T \in \mathrm{End}(V)$. Suppose that $W \leq V$ be T-invariant. Then, $W^\perp$ is T^*-invariant.

Proof. Let $w \in W^\perp$. Then for any $v \in W$, we have

$$0 = \langle Tv, w \rangle = \langle v, T^*w \rangle,$$

so $T^*w \in W^\perp$ and the desideratum follows. ∎

Definition 0.3. Let V be an inner product space and $U \in \mathrm{GL}(V)$. U is said to be *unitary* if

$$\langle v, w \rangle = \langle Uv, Uw \rangle$$

for all $v, w \in V$.

In other words,
$$\langle v, U^*Uw \rangle = \langle v, w \rangle.$$
However, the identity map is its own adjoint, so $U^* = U^{-1}$.
The subset $U(V) \subseteq \mathrm{GL}(V)$ of all unitary operators forms a subgroup.

Definition 0.4. A matrix $U \in \mathrm{GL}_n(\mathbb{C})$ is said to be *unitary* if $U^*U = I_n$.

The set of all unitary matrices is denoted $U_n(\mathbb{C})$ and is a subgroup of $\mathrm{GL}_n(\mathbb{C})$.

Proposition 0.6. Let V be an inner product space and $T \in U(V)$. Suppose that $W \leq V$ is T-invariant. Then, $W^\perp$ is also T-invariant.

Proof. Recalling Proposition 0.5, we have that $W^\perp$ is T^{-1}-invariant. It then follows by Proposition 0.4 that $W^\perp$ is T-invariant as well $(T^{-1}(W^\perp) = W^\perp$ so $T(W^\perp) = W^\perp)$. ∎

Definition 0.5 (Minimal Polynomial). Let $T \in \mathrm{End}(V)$. The *minimal polynomial* of T is the unique monic polynomial $m(X) \in \mathbb{C}[X]$ of minimal degree such that $m(T)$ is the zero operator.

Definition 0.6 (Diagonalisable). Let $T \in \mathrm{End}(V)$. T is said to be *diagonalisable* if there exists a basis B of V consisting of eigenvectors of T.

For the rest of this section, we use T to denote an element of $\mathrm{End}(V)$ and $m(X)$ its minimal polynomial.

Proposition 0.7. Let $T \in \text{End}(V)$, $m(X)$ be its minimal polynomial, and $p(X) \in \mathbb{C}[X]$ be any polynomial such that $p(T) = 0$. Then $p(\lambda) = 0$ for any eigenvalue $\lambda \in \mathbb{C}$ of T. In particular, all the eigenvalues of T (in $\mathbb{C}$) are roots of the minimal polynomial.

Proof. Suppose
$$p(X) = a_0 + a_1 X + \cdots + a_r X^r.$$
If λ is an eigenvalue of T and v ($\neq 0$) a corresponding eigenvector, then $T^k v = \lambda^k v$. So,
$$0 = p(T)v = (a_0 + a_1 T + \cdots + a_r T^r) v$$
$$= \left(a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_r \lambda^r \right) v = p(\lambda)v.$$
As $v \neq 0$, $p(\lambda) = 0$. ■

As the minimal polynomial divides the characteristic polynomial, this says that the minimal polynomial and characteristic polynomial have precisely the same roots.

Theorem 0.8. Let $T \in \text{End}(V)$ and $m(X)$ be its minimal polynomial. T is diagonalisable if and only if $m(X)$ has distinct roots.

Proof. First, suppose that T is diagonalisable.
Let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of T. Consider
$$p(X) = (X - \lambda_1)(X - \lambda_2) \cdots (X - \lambda_r).$$
It is clear from the previous proposition that $p(X) \mid m(X)$. If we manage to show that $m(X) \mid p(X)$, then $m(X) = p(X)$ and we are done. To show this, it suffices to show that $p(T) = 0$, and to show this it suffices to show that $p(T)$ annihilates some basis of V. Towards this, let B be an eigenbasis of V with respect to T (since T is diagonalisable, this exists). Any $v \in B$ is annihilated by some $T - \lambda_i$. Since all the $(T - \lambda_i)$ commute, it follows that $p(T)$ vanishes on B, and thus V.

Now, suppose that $m(X)$ has distinct roots, and let it be equal to $(X - \lambda_1) \cdots (X - \lambda_r)$ for distinct $\lambda_i \in \mathbb{C}$. Let E_λ denote the eigenspace of λ. We wish to show that $V = \bigoplus_{i=1}^{r} E_{\lambda_i}$. Recall that eigenvectors corresponding to different eigenspaces are linearly independent. As a result, it suffices to show that $V = +_{i=1}^{r} E_{\lambda_i}$. For each $i \in [r]$, let
$$g_i(X) = \prod_{j \neq i} (X - \lambda_j)$$
and
$$f_i(X) = \frac{g_i(X)}{g_i(\lambda_i)}.$$
Observe that
$$1 = \sum_{i=1}^{r} f_i(X).$$
Indeed, they are polynomials of degree at most $r - 1$ that agree at the r points $(\lambda_i)_{i=1}^{r}$. As a result, for any $v \in V$,
$$v = \sum_{i=1}^{r} f_i(T)v.$$
If we manage to show that $f_i(T)v \in E_{\lambda_i}$, we are done. This is not too difficult to see as
$$(T - \lambda_i) f_i(T)v = \frac{1}{g_i(\lambda_i)} m(T)v = 0.$$ ■

§1. Introduction

1.1. Basic Definitions

1.1.1. Representations

Definition 1.1 (Representation). A *representation* of a group G is a homomorphism $\varphi : G \to \mathrm{GL}(V)$ for some finite-dimensional vector space V over $\mathbb{C}$. The *degree* of φ is the dimension of V.

Henceforth, V (or any other symbol for a vector space) is used to denote a non-trivial vector space over $\mathbb{C}$.
The map $\varphi(g)$ is typically denoted φ_g, and $\varphi_g(v)$ as $\varphi_g v$.
Note that since φ is a homomorphism, it is determined by its values on any generating set of G.
Recall that given a group G and set X, a group action of G on X is merely a function $G \to S_X$. Representations can thus be pictured as a special case of a group action where the image of any element is not just a bijection, it is linear.

Example. Let X be a set X and consider the *linearisation* $\mathbb{C}X$ of X, defined as the vector space with elements of the form $\sum_{x \in X} c_x x$, where each $c_x \in \mathbb{C}$, addition defined by

$$\sum_{x \in X} c_x x + \sum_{x \in X} d_x x = \sum_{x \in X} (c_x + d_x) x,$$

and scalar multiplication by

$$c \sum_{x \in X} c_x x = \sum_{x \in X} (c c_x) x.$$

It is clear that $\mathbb{C}X$ has basis X.
Observe that any group action of a group G on X extends to an action on $\mathbb{C}X$ as a representation! Indeed, we define

$$g \cdot \left(\sum_{x \in X} c_x x \right) = \sum_{x \in X} c_x (g \cdot x).$$

If V is a 1-dimensional vector space over $\mathbb{C}$, then $\mathrm{GL}(V) \cong \mathbb{C}$. In such a case, to stay sane, we usually write z instead of φ to denote a representation[1], so each z_g is a complex.
The trivial representation of a group G is the homomorphism $z : G \to \mathbb{C}^*$ given by $z_g = 1$ for all $g \in G$.

Example. Let $n \in \mathbb{N}$. Recall that $\mathrm{GL}(\mathbb{C}^*) \cong \mathrm{GL}(\mathbb{C})$, so any degree-one representation may be considered as a function $z : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$. However, recall from Proposition 0.2 that any representation is of the form defined by $z(\overline{m}) = \omega_n^{km}$ for some fixed $k \in \mathbb{N}$. It turns out that these are the "only" representations of a finite cyclic group (where "only" is defined in an appropriate sense, as we shall see later).

Observe that the above mentioned representations are incredibly restrictive, and we are barely using the fact that the representation is to $\mathbb{C}^*$.

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[1] one could say that in such a scenario, z is used to *represent a representation*.

Example. Now, consider the degree-one representations $z : G \to \mathbb{C}^*$ of a non-Abelian group G. Recall the (Abelian) *commutator subgroup* $[G, G]$ of G consisting of elements of the form $xy - yx$. Because $\mathbb{C}^*$ is Abelian, $[G, G] \subseteq \ker z$. Consequently, z factors through the quotient as

$$
\begin{array}{ccc}
G & \xrightarrow{\ z\ } & \mathbb{C}^* \\
\downarrow & \nearrow{\widetilde{z}} & \\
G/[G, G] & &
\end{array}
$$

As a result, when studying degree-one representations, it suffices to assume that the group is non-Abelian.

Why are representations useful? Consider the problem of, given a group G and $x, y \in G$, finding $x \in G$ such that $gxg^{-1} = y$. It might then be easier to find M such that $M\varphi_x M^{-1} = \varphi_y$ since testing matrix similarity is a well-studied problem. A g in the preimage of M may then be a good candidate for the required.

1.1.2. Equivalence

Let $\varphi : G \to \mathrm{GL}(V)$ be a degree-n representation. Let B, B' be two bases of V, and consider the two corresponding isomorphisms $T, T' : V \to \mathbb{C}^n$ mapping the basis elements of B, B' to the standard basis vectors of $\mathbb{C}^n$. We would then like that the two representations $\psi, \psi' : G \to \mathbb{C}^n$ defined by

$$
\psi_g = T\varphi_g T^{-1} \text{ and } \psi'_g T' \varphi_g (T')^{-1}
$$

are the same in some sense. Towards this, we define the following.

Definition 1.2. Two representations $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ are said to be *equivalent* if there exists an isomorphism (an *equivalence*) $T : V \to W$ such that $\psi_g = T\varphi_g T^{-1}$ for all $g \in G$. If this is the case, we write $\varphi \sim \psi$.

Note that T must be independent of g!
The above definition can be represented as saying that there exists an equivalence $T : V \to W$ such that the following commutes for all $g \in G$.

$$
\begin{array}{ccc}
V & \xrightarrow{\ \varphi_g\ } & V \\
\downarrow{T} & & \downarrow{T} \\
W & \xrightarrow{\ \psi_g\ } & W
\end{array}
$$

Observe that V, W must be isomorphic, that is, φ, ψ are of the same degree.

Proposition 1.1. Let G be a group and $z, z^* : G \to \mathbb{C}^*$ be degree-one representations. Then, $z \sim z^*$ iff $z = z^*$.

Proof. Let $z \sim z^*$ and $T : \mathbb{C}^* \to \mathbb{C}^*$ be an equivalence. Then, for any $g \in G$,

$$
\begin{aligned}
z_g^* v &= T z_g T^{-1} v \\
&= z_g T T^{-1} v && (T \text{ is linear}) \\
&= z_g v,
\end{aligned}
$$

so $z = z'$.   ■

From Theorem 0.3 and Section 1.1.1, we get the following.

Corollary 1.2. Any finite group G has exactly $|G/[G, G]|$ inequivalent degree-one representations.

Recall Definition 0.1.

Corollary 1.3. If G is an abelian group, its $|G|$ degree-one representations (up to equivalence) are just the elements of $\hat{G}$.

1.1.3. Irreducibility

Definition 1.3 (Invariant subspace). Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. A subspace $W \leq V$ is said to be G-*invariant* with respect to φ if for all $g \in G$ and $w \in W$, $\varphi_g(W) = W$.

Observe that if $W \leq V$ is G-invariant with respect to φ, then $\varphi|_W : G \to \mathrm{GL}(W)$ defined by $(\varphi|_W)_g(w) = \varphi_g(w)$ is a representation. In such a case, $\varphi|_W$ is said to be a *subrepresentation* of W.
Based on the direct sum of vector spaces, one can similarly define the direct sum of representations.

Definition 1.4 (Direct sum). Let $\varphi^{(1)} : G \to \mathrm{GL}(V_1)$ and $\varphi^{(2)} : G \to \mathrm{GL}(V_2)$ be representations. Then, their (external) *direct sum* is the representation $\varphi^{(1)} \oplus \varphi^{(2)} : G \to \mathrm{GL}(V_1 \oplus V_2)$ defined by

$$\left(\varphi^{(1)} \oplus \varphi^{(2)}\right)_g (v_1, v_2) = (\varphi_g^{(1)}(v_1), \varphi_g^{(1)}(v_2))$$

for all $g \in G$ and $(v_1, v_2) \in V_1 \oplus V_2$.

The above is more natural to picture using matrices.
If $V_1 = \mathrm{GL}_m(\mathbb{C})$ and $V_2 = \mathrm{GL}_n(\mathbb{C})$ above, then each $\varphi_g^{(i)}$ can be expressed as a matrix. The matrix in $\mathrm{GL}_{m+n}(\mathbb{C})$ corresponding to their direct sum is then given by

$$\left(\varphi^{(1)} \oplus \varphi^{(2)}\right)_g = \begin{pmatrix} \varphi_g^{(1)} & \\ & \varphi_g^{(2)} \end{pmatrix},$$

where the empty cells are appropriately sized 0 matrices.

Recall the trivial representation of a group G. Observe then that the representation $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ given by $\rho_g = I_n$ for all $g \in G$ is equivalent to the direct sum of n copies of the trivial representation.

Proposition 1.4. If $V_1, V_2 \leq V$ are G-invariant subspaces with respect to φ and $V = V_1 \oplus V_2$, then φ is equivalent to $\varphi|_{V_1} \oplus \varphi|_{V_2}$.

Proof. Consider the map $T : V \to V_1 \oplus V_2$ defined by $T(v_1 + v_2) = (v_1, v_2)$ ($V_1 \oplus V_2$ here is the external direct sum).

Let $\psi = \varphi|_{V_1} \oplus \varphi|_{V_2}$. Then,

$$
\begin{aligned}
\psi_g(v_1, v_2) &= \left(\left(\varphi|_{V_1} \right)_g (v_1), \left(\varphi|_{V_2} \right)_g (v_2) \right) \\
&= (\varphi_g(v_1), \varphi_g(v_2)) \\
&= T(\varphi_g(v_1) + \varphi_g(v_2)) \\
&= T\varphi_g(v_1 + v_2) \\
&= T\varphi_g T^{-1}(v_1, v_2).
\end{aligned}
$$
∎

Above, let B_i be a basis of V_i for $i = 1, 2$. Because each V_i is G-invariant, $\varphi_g(B_i) \subseteq \mathbb{C}B_i$. The matrix representation of $\varphi \cong \varphi^{(1)} \oplus \varphi^{(2)}$ is then

$$
[\varphi_g]_B = \begin{bmatrix} \left[\varphi_g^{(1)} \right]_{B_1} & \\ & \left[\varphi_g^{(2)} \right]_{B_2} \end{bmatrix}.
$$

Thus, it is seen that representations may be broken down into smaller representations which operate on invariant subspaces. The following definition arises naturally.

Definition 1.5 (Irreducible representation). A non-zero representation $\varphi : G \to \mathrm{GL}(V)$ is said to be *irreducible* if the only G-invariant subspaces of V are 0 and V.

Note however that if a represenation is reducible, it need not actually have a decomposition of the form described in Proposition 1.4. We shall see an example of this later in

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Example. Let G be a finite group with generators a and b, and suppose every element can be written as $a^i b^j$ for (non-negative) integers i, j. Since the inverse of any group element can be written as $a^i b^j$, it is seen that any group element can also be written as $b^{j'} a^{i'}$ for non-negative i', j' (let $i' = (|a| - 1)i$ and $j' = (|b| - 1)j$). So, assume that $n := |a| \leq |b|$.

We claim that any irreducible representation φ of G is of degree at most n. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. Let v be an eigenvector of φ_b and consider

$$
W = \langle v, \varphi_a v, \varphi_{a^2} v, \ldots, \varphi_{a^{n-1}} v \rangle.
$$

Clearly, $0 < \dim W \leq n$. If we manage to show that W is G-invariant, we are done since φ is irreducible so $V = W$ (in particular, $\dim V \leq n$).

Let $0 \leq k < n$ and consider some arbitrary $g = a^i b^j \in G$. Let $a^i b^j a^k = a^p b^q$. Then,

$$
\begin{aligned}
\varphi_{a^i b^j}(\varphi_{a^k} v) &= \varphi_{a^i b^j a^k} v \\
&= \varphi_{a^p} \varphi_{b^q} v \\
&= \varphi_{a^p} \lambda v && (v \text{ is an eigenvector of } \varphi_b \text{ and so } \varphi_{b^q}) \\
&= \lambda \varphi_{a^p} v \in W.
\end{aligned}
$$

In particular, any irreducible representation of the dihedral group D_n has degree at most two.

Proposition 1.5. Let $\rho : H \to \mathrm{GL}(V)$ be an irreducible representation and $\psi : G \to H$ be a surjective group homomorphism. Then, $\rho \circ \psi$ is an irreducible representation of G.

Proof. Let $\varphi = \rho \circ \psi$. Let W be a G-invariant subspace wrt φ. We shall show that it is also H-invariant wrt ρ to complete the proof. Indeed, for any $w \in W$ and $h \in H$, we have $h = \psi(g)$ for some $g \in G$. As a result,

$$\rho_h(w) = \rho_{\psi(g)}(w) = (\rho \circ \psi)_g(w) = \varphi_g(w) \in W.$$ ∎

Proposition 1.6. If $\varphi : G \to \mathrm{GL}(V)$ is a degree two representation, φ is irreducible iff there is no common eigenvector v to all φ_g with $g \in G$.

Proof. If v is an eigenvector of all φ_g, then $\mathbb{C}v$ is a G-invariant subspace, so the forward direction is done.
Now, suppose that there is no common eigenvector v to all of the φ_g but there is a non-trivial G-invariant subspace W of V. Because it is a degree-two representation, $W = \mathbb{C}v$ for some $v \in V$. It follows that for each $g \in G$, $\varphi_g v = \lambda_g v$ (for some $\lambda_g \in \mathbb{C}$), so the desideratum follows. ∎

Remark. The above result almost directly generalizes to degree three representations for *finite* groups as well. The key point to note is that if the representation of a finite group is reducible, then we can write $V = W \oplus W'$ for some non-zero G-invariant subspaces W, W'.
We shall prove this later in Proposition 1.15.

1.1.4. Decomposability

Definition 1.6 (Complete Reducibility). Let G be a group. A representation $\varphi : G \to \mathrm{GL}(V)$ is said to be *completely reducible* if $V = V_1 \oplus \cdots \oplus V_n$ where each V_i is G-invariant and $\varphi|_{V_i}$ is irreducible for each i.

Equivalently, by Proposition 1.4, the above is equivalent to saying that $\varphi = \varphi^{(1)} \oplus \cdots \varphi^{(n)}$ for some irreducible representations $\varphi^{(i)}$.

Remark. Note that any irreducible representation is completely reducible. Indeed, the V_i need not be proper subspaces of V.

In some sense, complete reducibility says that we do not run into the weird situation wherein the representation is not irreducible yet we cannot "reduce" it to a direct sum of 'smaller' representations.
The main result of this section is showing that any representation of a finite group is completely reducible.
Based on the above remark, we can further define the more logical thing to consider as follows.

Definition 1.7 (Decomposability). A non-zero representation φ is said to be *decomposable* if $V = V_1 \oplus V_2$ for some non-zero G-invariant subspaces $V_1, V_2 \leq V$. Otherwise, φ is said to be *indecomposable*.

First, let us show that irreducibility, complete reducibility, and decomposability are preserved under equivalence.

Lemma 1.7. Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be equivalent representations with $T : V \to W$ being a corresponding equivalence. If $V_1 \leq V$ is G-invariant, so is $W_1 = T(V_1)$.

Proof. Let $w \in W_1$ and $g \in G$. We have by definition that $\psi w = T\varphi T^{-1}w$. We have that $T^{-1}w \in V_1$, so since V_1 is G-invariant $\varphi T^{-1}w \in V_1$, so $T\varphi T^{-1}w \in W_1$ by definition of W_1. ∎

Lemma 1.8. Let $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ be equivalent representations. Then,

1. If φ is reducible, so is ψ.

2. If φ is decomposable, so is ψ.

3. If φ is completely reducible, so is ψ.

Proof. Let $T : V \to W$ be a corresponding equivalence.

1. Let $V_1 \leq V$ be a proper non-zero G-invariant subspace. Because T is an isomorphism, W_1 is also non-zero and proper. By the previous lemma, this is also G-invariant and we are done.

2. If $V = V_1 \oplus V_2$ for non-zero V_1, V_2, then $W = T(V_1) \oplus T(V_2)$ since T is an isomorphism. If V_1, V_2 are G-invariant, so are $T(V_1)$ and $T(V_2)$ by the previous lemma so we are done.

3. Again, if $V = V_1 \oplus \cdots \oplus V_n$, then $W = W_1 \oplus \cdots \oplus W_n$ where $W_i = T(V_i)$ and each V_i or W_i is G-invariant (with respect to the appropriate representation).
   We must check that if $\varphi|_{V_i}$ is irreducible, so is $\psi|_{W_i}$. However, this is direct as the following diagram commutes for all $g \in G$.

$$\begin{array}{ccc} V_i & \xrightarrow{\left(\varphi|_{V_i}\right)_g} & V_i \\ \Big\downarrow{T|_{V_i}} & & \Big\downarrow{T|_{V_i}} \\ W_i & \xrightarrow{\left(\psi|_{W_i}\right)_g} & W_i \end{array}$$

It is easily seen that $T|_{V_i}$ is an isomorphism from V_i to W_i. ∎

Proposition 1.9 (Irreducible representations of finite cyclic groups)**.** Let G be a finite cyclic group. Then all irreducible representations of G are of degree one.

Proof. We may assume that $G = \mathbb{Z}/n\mathbb{Z}$ by Lemma 1.8. Let $\varphi : G \to \mathrm{GL}_m(\mathbb{C})$ be a representation with $m \geq 2$.
Note that $\varphi_{\overline{1}}^n = I_n$. Recall Definition 0.5. It follows that the minimal polynomial of $\varphi_{\overline{1}}$ is a factor of $X^n - 1$, and in particular, has distinct roots. It follows from Theorem 0.8 that $\varphi_{\overline{1}}$ is diagonalisable.
Let D be a diagonal matrix and $T \in \mathrm{GL}_m(\mathbb{C})$ such that $T\varphi_{\overline{1}}T^{-1} = D$. Then $T\varphi_{\overline{k}}T^{-1} = D^k$ for any $1 \leq k \leq n$. Therefore, consider the equivalent representation $\psi : G \to \mathrm{GL}_m(\mathbb{C})$ defined by $\psi_g = T\varphi_g T^{-1}$. ψ_g is diagonal for all $g \in G$. Clearly, ψ is decomposable into m non-zero proper representations, contradicting irreducibility. ∎

1.2. Maschke's Theorem and Complete Reducibility

The aim of this section is to show that any representation of a **finite** group is completely reducible in **??**.
To begin, we shall show that a representation of a **finite** group is decomposable iff it is reducible in Theorem 1.14.
We first prove this for a specific type of representation in Lemma 1.10.

Definition 1.8 (Unitary)**.** Let V be an inner product space. A representation $\varphi : G \to \mathrm{GL}(V)$ is said to be *unitary* if φ_g is unitary for every $g \in G$.

That is, φ is a map from G to $U(V)$. Observe that unitarity depends on the inner product we place on the latent space! The usefulness arises when one observes certain properties of unitary representations (independent of the inner product), as we shall shortly see in Lemma 1.13.

Identifying $\mathrm{GL}_1(\mathbb{C})$ with $\mathbb{C}^*$, one sees that $U_1(\mathbb{C})$ ends up becoming S_1. Therefore, a degree-one unitary representation is a homomorphism $\varphi : G \to S^1$.

Recall that any degree-one representation of a finite group maps into S^1. Indeed, we have that $\varphi_g^{|G|} = 1$. Therefore, any such representation is unitary.

> **Example.** $\varphi : \mathbb{R} \to S^1$ defined by $t \mapsto \exp(2\pi\iota t)$ is a degree-one unitary representation of $(\mathbb{R}, +)$.

Recall that decomposability implies reducibility, but the converse need not hold for a general representation.

Lemma 1.10. Let $\varphi : G \to \mathrm{GL}(V)$ be a unitary representation. Then, φ is decomposable iff it is not irreducible.

Proof. Suppose that φ is not irreducible. We shall show that it is decomposable. Let $W \leq V$ be a non-zero proper G-invariant subspace. We are done if we show that $W^\top$ is G-invariant as well. Let $g \in G$. We know that φ_g is unitary and W is φ_g-invariant. Recalling Proposition 0.6, $W^\perp$ is φ_g-invariant. Since g was arbitrary, $W^\perp$ is G-invariant with respect to φ, completing the proof. ∎

As usual, we denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{C}^n$.

Over the next three lemmas, we define a new inner product and show that any representation is equivalent to a unitary representation using this inner product.

Lemma 1.11. Let G be a finite group and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ a representation. Consider the product $(\cdot, \cdot)$ on $\mathbb{C}^n$ defined by
$$(v, w) = \sum_{g \in G} \langle \rho_g v, \rho_g w \rangle.$$

$(\cdot, \cdot)$ is an inner product.

Note that the sum is well-defined because G is finite.

Proof. Let $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, v, w \in \mathbb{C}$. Then,

$$
\begin{aligned}
(c_1 v_1 + c_2 v_2, w) &= \sum_{g \in G} \langle \rho_g (c_1 v_1 + c_2 v_2), \rho_g w \rangle \\
&= \sum_{g \in G} \langle c_1 \rho_g v_1 + c_2 \rho_g v_2, \rho_g w \rangle \\
&= \sum_{g \in G} c_1 \langle \rho_g v_1, \rho_g w \rangle + c_2 \langle \rho_g v_2, w \rangle \\
&= c_1(v_1, w) + c_2(v_2, w).
\end{aligned}
$$

Next,

$$
\begin{aligned}
(w, v) &= \sum_{g \in G} \langle \rho_g w, \rho_g v \rangle \\
&= \sum_{g \in G} \overline{\langle \rho_g v, \rho_g w \rangle} \\
&= \overline{\sum_{g \in G} \langle \rho_g v, \rho_g w \rangle} = \overline{(v, w)}.
\end{aligned}
$$

Finally,

$$(v, v) = \sum_{g \in G} (\rho_g v, \rho_g v) \geq 0.$$

with equality iff $\rho_g v = 0$ for every $g \in G$. In particular, $v = \rho_1 v = 0$. $\blacksquare$

Lemma 1.12. With the same notation as in the previous lemma, ρ is unitary with respect to the inner product $(\cdot, \cdot)$.

Proof. Let v, w and $g \in G$. We would like to show that $(\rho_g v, \rho_g w) = (v, w)$. Indeed,

$$\begin{aligned}
(\rho_g v, \rho_g w) &= \sum_{g' \in G} \langle \rho_{g'} \rho_g v, \rho_{g'} \rho_g w \rangle \\
&= \sum_{g' \in G} \langle \rho_{g'g} v, \rho_{g'g} w \rangle \\
&= \sum_{g' \in G} \langle \rho_{g'} v, \rho_{g'} w \rangle && (g' \mapsto g'g \text{ is a bijection}) \\
&= (v, w). && \blacksquare
\end{aligned}$$

Lemma 1.13. Every representation of a finite group G is equivalent to a unitary representation.

Proof. Let $\varphi : G \to \mathrm{GL}(V)$ be a representation and $n = \dim V$. Fix an isomorphism $T : V \to \mathbb{C}^n$ and set $\rho_g = T\varphi_g T^{-1}$ for each $g \in G$. Clearly, ρ is a representation $G \to \mathrm{GL}_n(\mathbb{C})$ that is equivalent to φ.
By Lemma 1.12, ρ is a unitary representation with respect to the inner product defined in Lemma 1.11 and we are done. $\blacksquare$

Theorem 1.14. Let $\varphi : G \to \mathrm{GL}(V)$ be a non-zero representation of a finite group. Then, φ is reducible iff it is decomposable.

Proof. The desideratum follows directly from lemmas 1.8, 1.10 and 1.13. $\blacksquare$

The above further shows that if $\varphi : G \to \mathrm{GL}(V)$ is a representation of a finite group G and V_1 is a non-zero proper G-invariant subspace, then we can decompose $V = V_1 \oplus V_2$, where V_2 is the subspace orthogonal to V_1 (for an appropriate inner product structure) and is also G-invariant (and non-zero and proper).

Proposition 1.15. Let $\varphi : G \to \mathrm{GL}(V)$ be a degree 3 representation of a finite group. φ is reducible iff there is a common vector v to all the φ_g for $g \in G$.

The proof of the above is exactly as described in the remark after Proposition 1.6. Reducibility implies decomposability, so we get a one-dimensional invariant subspace.
Now, let us give an example of an infinite group that is reducible but not decomposable.

Example. Let $\varphi : \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ be the representation defined by

$$\varphi_n = \begin{bmatrix} 1 & n \\ & 1 \end{bmatrix}.$$

φ is reducible because $\mathbb{C}e_1$ is a $\mathbb{Z}$-invariant subspace. However, there is no other eigenvector to all the φ_n, so there is no other $\mathbb{Z}$-invariant subspace. As a result, φ is not decomposable.
It is worth noting that Lemma 1.10 is true even for an infinite group, and Lemma 1.11 is where it breaks.

Further, the above cannot be strengthened to degree 4 representations since we need not have a 1-dimensional invariant subspace.

Example. Let $\psi : D_4 \to \mathrm{GL}_4(\mathbb{C})$ be defined by

$$\psi_r = \begin{bmatrix} \iota & & & \\ & -\iota & & \\ & & \iota & \\ & & & -\iota \end{bmatrix} \text{ and } \psi_s = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}.$$

It may be checked that ψ_r and ψ_s have no common eigenvector.

Now, we arrive at the main result of this section.

Theorem 1.16 (Maschke's Theorem). Every representation of a finite group is completely reducible.

Proof. We prove this by induction on the degree of the representation $\varphi : G \to \mathrm{GL}(V)$.

If $\dim V = 1$, then φ is irreducible (so completely reducible) and we are done.

Let $n \geq 1$ and suppose that the statement is true for representations of degree $\leq n$. Let $\dim V = n + 1$. If φ is irreducible, we are done. Otherwise, by Theorem 1.14, $V = U \oplus W$ for non-zero G-invariant subspaces U, W. We may then apply the inductive hypothesis on U, W to write

$$U = U_1 \oplus \cdots \oplus U_n \text{ and } W = W_1 \oplus \cdots \oplus W_m$$

for non-zero G-invariant subspaces U_i and W_i such that $\varphi|_{U_i}$ and $\varphi|_{W_j}$ are irreducible for $1 \leq i \leq n$, $1 \leq j \leq m$. Consequently,

$$V = U_1 \oplus \cdots \oplus U_n \oplus W_1 \oplus \cdots \oplus W_m$$

and we are done. ∎

§2. Character Theory and Orthogonality Relations

2.1. Morphisms

Definition 2.1 (Morphism). Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be representations. A *morphism* from φ to ρ is a linear map $T : V \to W$ such that the following diagram commutes for all $g \in G$.

$$
\begin{array}{ccc}
V & \xrightarrow{\ \varphi_g\ } & V \\
\downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\
W & \xrightarrow{\ \rho_g\ } & W
\end{array}
$$

The set of all morphisms from φ to ρ is denoted $\mathrm{Hom}_G(\varphi, \rho)$.

By definition, $\mathrm{Hom}_G(\varphi, \rho) \subseteq \mathrm{Hom}(V, W)$.

Recall that any representation is just a special group action of G on the vector space of interest. Based off this, writing gv instead of $\varphi_g v$, the definition of a morphism can be alternatively written as saying that $Tgv = gTv$ for all $g \in G, v \in V$.[2].

Also observe that if $T \in \mathrm{Hom}_G(\varphi, \rho)$ is an isomorphism, then $\varphi \sim \rho$.

Remark. $T \in \mathrm{Hom}(V, V)$ is in $\mathrm{Hom}_G(\varphi, \varphi)$ iff it commutes with every φ_g. In particular, the identity map is an element of $\mathrm{Hom}_G(\varphi, \varphi)$.

Proposition 2.1. Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be representations, and $T \in \mathrm{Hom}_G(\varphi, \rho)$. $\ker T$ and $\mathrm{im}\, T$ are G-invariant subspaces of V and W with respect to φ and ρ respectively.

Proof. Let $v \in \ker T$. Then, for $g \in G$,
$$T(\varphi_g v) = \rho_g T v = 0,$$
so $\varphi_g v \in \ker T$. Similarly, for $w \in \mathrm{im}\, T$, letting $v \in V$ such that $Tv = w$,

$$\rho_g w = \rho_g T v = T(\varphi_g v) \in \mathrm{im}\, T.$$

■

We had mentioned earlier that $\mathrm{Hom}_G(\varphi, \rho) \subseteq \mathrm{Hom}(V, W)$. In fact, the following stronger statement is true.

Proposition 2.2. Let G be a group and $\varphi : G \to \mathrm{GL}(V)$, $\rho : G \to \mathrm{GL}(W)$ be representations. Then $\mathrm{Hom}_G(\varphi, \rho)$ is a subspace of $\mathrm{Hom}(V, W)$.

Proof. Clearly, $0 \in \mathrm{Hom}_G(\varphi, \rho)$. If $S, T \in \mathrm{Hom}_G(\varphi, \rho)$ and $\alpha \in \mathbb{C}$, then for any $g \in G$ and $v \in V$,

$$
\begin{aligned}
(S + \alpha T)\varphi_g v &= S\varphi_g v + \alpha T \varphi_g v \\
&= \rho_g S v + \alpha \rho_g T v \\
&= \rho_g S v + \rho_g(\alpha T) v = \rho_g (S + \alpha T) v,
\end{aligned}
$$

so $S + \alpha T \in \mathrm{Hom}_G(\varphi, \rho)$.

■

---

[2]the first g is a φ_g and the second is a ρ_g

Another expected result is that the homomorphism subspaces of equivalent representations are isomorphic.

Proposition 2.3. Let G be a group and $\varphi_i : G \to \mathrm{GL}(V_i)$, $\rho_i : G \to \mathrm{GL}(W_i)$ be representations for $i = 1, 2$. If $\varphi^{(1)} \sim \varphi^{(2)}$ and $\rho^{(1)} \sim \rho^{(2)}$, then $\dim \mathrm{Hom}_G(\varphi^{(1)}, \rho^{(1)}) = \dim \mathrm{Hom}_G(\varphi^{(2)}, \rho^{(2)})$.

Proof. Let $P : V_1 \to V_2$ and $R : W_1 \to W_2$ be corresponding equivalences. Consider $\Phi : \mathrm{Hom}_G(\varphi^{(1)}, \rho^{(1)}) \to \mathrm{Hom}_G(\varphi^{(2)}, \rho^{(2)})$ defined by $\Phi(S) = R \circ S \circ P^{-1}$. We claim that Φ is an isomorphism between the subspaces. Let us first show that this does indeed map into $\mathrm{Hom}_G(\varphi^{(2)}, \rho^{(2)})$. We have that for any $g \in G$ and $v \in V_1$,

$$
\begin{aligned}
\Phi(S)(\varphi^{(2)})_g v &= RSP^{-1}(\varphi^{(2)})_g v \\
&= RS(\varphi^{(1)})_g P^{-1} v && (P^{-1} \text{ is an equivalence}) \\
&= R(\rho^{(1)})_g SP^{-1} v && (S \in \mathrm{Hom}_G(\varphi^{(1)}, \rho^{(1)})) \\
&= (\rho^{(2)})_g RSP^{-1} v && (R \text{ is an equivalence}) \\
&= (\rho^{(2)})_g \Phi(S) v.
\end{aligned}
$$

It is clear that Φ is linear, and further that Φ is a bijection as an inverse is easily constructed similarly. $\blacksquare$

Lemma 2.4 (Schur's Lemma). Let G be a group, $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be irreducible representations of G, and $T \in \mathrm{Hom}_G(\varphi, \rho)$. Then, either T is an equivalence or $T = 0$.

Proof. Suppose that $T \neq 0$. It suffices to show that T is a bijection. If $\ker T \neq 0$, then we have a nonzero proper subspace $\ker T$ that is G-invariant (with respect to φ), contradicting irreducibility (of φ). Therefore, T is injective. Similarly, $\mathrm{im}\, T \neq 0$ and if $\mathrm{im}\, T \neq W$, we have a nonzero proper subspace $\mathrm{im}\, T$ that is G-invariant (with respect to ρ), contradicting irreducibility (of ρ). Therefore, T is surjective, completing the first part of the proof. $\blacksquare$

Corollary 2.5. Let G be a group, $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be irreducible representations of G, and $T \in \mathrm{Hom}_G(\varphi, \rho)$.

(a) If $\varphi \not\sim \rho$, then $\mathrm{Hom}_G(\varphi, \rho) = 0$.

(b) If $\varphi = \rho$, $T = \lambda I$ for some $\lambda \in \mathbb{C}$. That is, $\mathrm{Hom}_G(\varphi, \varphi)$ is one-dimensional with basis $\{I\}$.

Proof. (a) is immediate from Schur's Lemma.
For (b), let λ be an eigenvalue of T (which exists since $\mathbb{C}$ is algebraically closed). Recall that $I \in \mathrm{Hom}_G(\varphi, \varphi)$. It follows from Proposition 2.2 that $T - \lambda I \in \mathrm{Hom}_G(\varphi, \varphi)$. By the definition of an eigenvalue, $T - \lambda I$ is not invertible. Therefore, $T - \lambda I = 0$, proving the required. $\blacksquare$

Next, let us show that a direct sum of representations corresponds to a direct sum of their Homs as well.

Proposition 2.6. Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho_i : G \to \mathrm{GL}(W_i)$ be representations for $i = 1, 2$. It is true that

$$
\mathrm{Hom}_G(\varphi, \rho^{(1)} \oplus \rho^{(2)}) \cong \mathrm{Hom}_G(\varphi, \rho^{(1)}) \oplus \mathrm{Hom}_G(\varphi, \rho^{(2)}).
$$

In particular,

$$
\dim \mathrm{Hom}_G(\varphi, \rho^{(1)} \oplus \rho^{(2)}) = \dim \mathrm{Hom}_G(\varphi, \rho^{(1)}) + \dim \mathrm{Hom}_G(\varphi, \rho^{(2)}).
$$

Proof. Let $T \in \mathrm{Hom}_G(\varphi, \rho^{(1)} \oplus \rho^{(2)}) \subseteq \mathrm{Hom}(V, W_1 \oplus W_2)$. Letting π_i denote the projection maps, $\pi_i \circ T : V \to W_i$ is linear for $i = 1, 2$. Further, $(\pi_i \circ T) \in \mathrm{Hom}_G(\varphi, \rho_i)$ because

$$(\pi_i \circ T)\varphi_g v = \pi_i(\rho^{(1)} \oplus \rho^{(2)})_g Tv = (\rho_i)_g Tv.$$

On the other hand, given morphisms $T_i \in \mathrm{Hom}_G(V, W_i)$ for $i = 1, 2$, $T : V \to W$ defined by $T(v) = (T_1(v), T_2(v))$ is also a morphism. As a result, the correspondence $(T_1, T_2) \mapsto T$ is bijective and $\mathbb{C}$-linear, so is an isomorphism. ■

Corollary 2.7. Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be pairwise inequivalent irreducible representations of G. Set

$$\varphi = \underbrace{\varphi^{(1)} \oplus \cdots \oplus \varphi^{(1)}}_{m_1} \oplus \cdots \oplus \underbrace{\varphi^{(s)} \oplus \cdots \oplus \varphi^{(s)}}_{m_s}.$$

Then,

$$\dim \mathrm{Hom}_G(\varphi^{(r)}, \varphi) = m_r$$

for $1 \leq r \leq s$.

Proof. We have

$$\dim \mathrm{Hom}_G(\varphi^{(r)}, \varphi) = \sum_{i=1}^{s} m_i \dim \mathrm{Hom}_G(\varphi^{(r)}, \varphi^{(i)}) \qquad \text{(by Proposition 2.6)}$$

$$= m_r \qquad \text{(by Corollary 2.5)} ■$$

The above says that if we know that a representation is completely reducible and we know the (pairwise inequivalent and irreducible) representations that occur in a decomposition, then the number of times each representation occurs is fixed as well.

Corollary 2.8. Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ and $\psi^{(1)}, \ldots, \psi^{(r)}$ be pairwise[3] inequivalent irreducible representations of G. Let φ be a representation of G such that

$$\varphi \cong \bigoplus_{i=1}^{s} (\varphi^{(i)})^{m_i} \cong \bigoplus_{j=1}^{r} (\psi^{(j)})^{n_j}$$

where $m_i, n_j > 0$. Then, $r = s$ and there is a permutation σ of $[r]$ such that $\varphi^{(i)} \sim \psi^{(\sigma_i)}$ and $m_i = n_{\sigma(i)}$.

Proof. It suffices to show that each φ_i is equivalent to some ψ_j. Indeed, pairwise inequivalence then implies that $r = s$, and the previous corollary shows that $m_i = n_j$. Suppose instead that $\varphi^{(1)}$ is not equivalent to any ψ_j. Then, denoting by $(psi^{(j)})^{n_j}$ the direct sum of n_j $\psi^{(j)}$s,

$$m_1 = \mathrm{Hom}(\varphi^{(1)}, \varphi)$$

$$= \mathrm{Hom}\left(\varphi^{(1)}, \bigoplus_{j=1}^{s} (\psi^{(j)})^{n_j}\right)$$

$$= \sum_{j=1}^{s} n_j \mathrm{Hom}(\varphi^1, \psi^{(j)}) = 0,$$

leading to a contradiction and completing the proof. ■

Compare this to Maschke's Theorem. There we had that any representation of a finite group is completely reducible. Here, we have shown that the decomposition of any completely reducible representation is "unique"!

Recall Proposition 1.9.

Theorem 2.9 (Irreducible representations of finite abelian groups)**.** Let G be an abelian group. Any irreducible representation of G has degree 1.

Proof. Let $\varphi : G \to \mathrm{GL}(V)$ be an irreducible representation.
For any $h \in G$, for all $g \in G$ $\varphi_h \varphi_g = \varphi_g \varphi_h$, so $\varphi_h \in \mathrm{Hom}_G(\varphi, \varphi)$. Corollary 2.5(b) then shows that $\varphi_h = \lambda_h I$ for some $\lambda_h \in \mathbb{C}$ (this uses that φ is irreducible). Fix any $v \in V$. Then, $\varphi_h v = \lambda_h I v = \lambda_h v \in \mathbb{C}v$, so $\mathbb{C}v$ is a G-invariant subspace. By irreducibility, $V = \mathbb{C}v$ and is thus one-dimensional. ∎

Further recall that we had characterized the degree one representations of an abelian group in Corollary 1.3.

Corollary 2.10. Let G be a finite abelian group and $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ a representation. Then, there exists invertible T such that $T^{-1}\varphi_g T$ is diagonal for all $g \in G$.

Note that T is independent of g.

Proof. Since G is finite and abelian, we can write using Theorems 1.16 and 2.9 that

$$\varphi = \varphi^{(1)} \oplus \cdots \oplus \varphi^{(n)}$$

where each $\varphi^{(i)}$ is degree 1. If T is an isomorphism giving the above equivalence, then

$$T^{-1}\varphi_g T = \mathrm{diag}(\varphi_g^{(1)}, \ldots, \varphi_g^{(n)}).$$

∎

Corollary 2.11. Let $A \in \mathrm{GL}_n(\mathbb{C})$ be of finite order. Then A is diagonalisable.

Proof. Consider the representation $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_n(\mathbb{C})$ defined by $\overline{k} \mapsto A^k$. Corollary 2.10 implies that $\varphi_{\overline{1}} = A$ is diagonalisable (in fact, $I, A, \ldots, A^{n-1}$ are simultaneously diagonalisable). ∎

2.2. The Orthogonality Relations

For the remainder of this report, assume that any group G is finite unless otherwise mentioned.

Definition 2.2. Let G be a group. Define the *group algebra* $L(G) \coloneqq \mathbb{C}^G$. $L(G)$ is a vector space over $\mathbb{C}$ in the natural sense. It is also an inner product space when equipped with the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

In particular, the *norm* of $f \in L(G)$ is $\|f\| = \sqrt{\langle f, f \rangle}$.

Note that the sum involved in $\langle f_1, f_2 \rangle$ makes sense because G is finite. Given a representation $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$, we get n^2 elements $\varphi_{ij} : G \to \mathbb{C}$ corresponding to the n^2 entries of the matrix. We shall study φ_{ij} when φ is irreducible and unitary.

Proposition 2.12. Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be representations. Define for any linear transformation $T : V \to W$

$$T^\sharp = \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \varphi_g \in \mathrm{Hom}_G(\varphi, \rho).$$

Then,

(a) $T^\sharp \in \mathrm{Hom}_G(\varphi, \rho)$,

(b) if $T \in \mathrm{Hom}_G(\varphi, \rho)$, then $T^\sharp = T$, and

(c) the map $P : \mathrm{Hom}_{\mathbb{C}}(V, W) \to \mathrm{Hom}_G(\varphi, \rho)$ defined by $T \mapsto T^\sharp$ is a surjective linear map.

Proof.

1. For any $h \in H$,

$$
\begin{aligned}
T^\sharp \varphi_h &= \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \varphi_g \varphi_h \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \varphi_{gh} \\
&= \frac{1}{|G|} \sum_{g' \in G} \rho_{hg'^{-1}} T \varphi_{g'} \qquad\qquad (g \mapsto gh \text{ defines a bijection } G \to G) \\
&= \rho_h \frac{1}{|G|} \sum_{g' \in G} \rho_{g'^{-1}} T \varphi_{g'} = \rho_h T^\sharp.
\end{aligned}
$$

2. If $T \in \mathrm{Hom}_G(\varphi, \rho)$, then

$$
\begin{aligned}
T^\sharp &= \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T \varphi_g \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} \rho_g T \\
&= \frac{1}{|G|} \sum_{g \in G} T = T.
\end{aligned}
$$

3. (b) shows that P is surjective. For linearity, we have that for any $T_1, T_2 \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ and $c \in \mathbb{C}$,

$$
\begin{aligned}
P(cT_1 + T_2) &= \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} (cT_1 + T_2) \varphi_g \\
&= c \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T_1 \varphi_g + \frac{1}{|G|} \sum_{g \in G} \rho_{g^{-1}} T_2 \varphi_g \\
&= cP(T_1) + P(T_2). \qquad\qquad\blacksquare
\end{aligned}
$$

Proposition 2.13. Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be irreducible representations and let $T : V \to W$ be a linear map. Then,

(a) if $\varphi \not\sim \rho$, $T^\sharp = 0$ and

(b) if $\varphi = \rho$, $T^\sharp = \frac{\mathrm{Tr}\, T}{\deg \varphi} I$.

Proof.

(a) This is straightforward on an application of Schur's Lemma since $T^\sharp \in \mathrm{Hom}_G(\varphi, \rho)$.

(b) Again, by Schur's Lemma, we have that $T^\sharp = \lambda I$ for some $\lambda I \in \mathbb{C}$. Now, note that

$$\mathrm{Tr}\, T^\sharp = \mathrm{Tr}(\lambda I) = \lambda \deg \varphi.$$

We also have

$$\begin{aligned}
\mathrm{Tr}\, T^\sharp &= \mathrm{Tr}\left(\frac{1}{|G|} \sum_{g \in G} \varphi_{g^{-1}} T \varphi_g \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}\left(\varphi_{g^{-1}} T \varphi_g \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}\left(\varphi_{g^{-1}} \varphi_g T \right) \qquad\qquad (\mathrm{Tr}(ABC) = \mathrm{Tr}(ACB)) \\
&= \mathrm{Tr}\, T,
\end{aligned}$$

so the required follows.

∎

Suppose that $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$. P is then a linear form from $\mathrm{GL}(V, W) = M_{m \times n}(\mathbb{C})$ to itself. A natural question to ask is: how do we represent P as a matrix with respect to the standard basis vectors E_{ij}? (Recall that E_{ij} is the $m \times n$ matrix with 1 in the (i, j)th entry and 0 elsewhere)
It is a straightforward computational task to check that if $A = (a_{ij}) \in M_{r \times m}(\mathbb{C})$, $E_{ki} \in M_{m \times n}(\mathbb{C})$, and $B = (b_{ij}) \in M_{n \times s}(\mathbb{C})$, then

$$(AE_{ki}B)_{lj} = a_{lk} b_{ij}. \tag{2.1}$$

Lemma 2.14. Let $\varphi : G \to U_n(\mathbb{C})$ and $\rho : G \to U_m(\mathbb{C})$ be unitary representations of G. Let $A = E_{ki} \in M_{m \times n}(\mathbb{C})$. Then,

$$A^\sharp_{lj} = \langle \varphi_{ij}, \rho_{kl} \rangle.$$

Proof. Let $g \in G$. Because ρ_g is unitary, $\rho_{g^{-1}} = \rho_g^*$. As a result, $(\rho_{g^{-1}})_{lk} = \overline{(\rho_g)_{kl}}$. Consequently,

$$\begin{aligned}
(A^\sharp)_{lj} &= \frac{1}{|G|} \sum_{g \in G} (\rho_{g^{-1}} A \varphi_g)_{lj} \\
&= \frac{1}{|G|} \sum_{g \in G} (\rho_{g^{-1}})_{lk} (\varphi_g)_{ij} \\
&= \frac{1}{|G|} \sum_{g \in G} \overline{(\rho_g)_{kl}} (\varphi_g)_{ij} \\
&= \langle \varphi_{ij}, \rho_{kl} \rangle.
\end{aligned} \qquad\qquad ∎$$

Theorem 2.15 (Schur's Orthogonality Relations). Let $\varphi : G \to U_n(\mathbb{C})$ and $\rho : G \to U_m(\mathbb{C})$ be inequivalent irreducible unitary representations of a group G. Then,

(a) $\langle \varphi_{ij}, \rho_{kl} \rangle = 0$.

(b) $\langle \varphi_{ij}, \varphi_{kl} \rangle = \begin{cases} 1/n, & (i,j) = (k,l), \\ 0, & \text{otherwise.} \end{cases}$

In particular, the set $\{\varphi_{ij} : 1 \leq i,j \leq n\} \cup \{\rho_{kl} : 1 \leq k,l \leq m\}$ is a linearly independent set.

Proof.

(a) Let $A = E_{ki} \in M_{m \times n}(\mathbb{C})$. By Proposition 2.13, $A^{\sharp} = 0$ because $\varphi \not\sim \rho$, so in particular, using Lemma 2.14, $\langle \varphi_{ij}, \rho_{kl} \rangle = (A^{\sharp})_{lj} = 0$.

(b) Let $A = E_{ki} \in M_{n \times n}(\mathbb{C})$. By Proposition 2.13, $A^{\sharp} = \frac{\operatorname{Tr} A}{n} I$. $\operatorname{Tr} A$ is 1 if $k = i$ and 0 otherwise. We also have $\langle \varphi_{ij}, \rho_{kl} \rangle = \left(\frac{\operatorname{Tr} A}{n} I \right)_{lj}$, which is zero if $l \neq j$. That is, the quantity of interest is equal to $1/n$ if $k = i$ and $l = j$ and 0 otherwise. ∎

Corollary 2.16. Let φ be an irreducible unitary representation of G of degree n. The set $\{\sqrt{n} \varphi_{ij} : 1 \leq i,j \leq n\}$ of functions forms an orthonormal set.

Proposition 2.17. Let G be a (finite) group. The following are true.

(a) There are finitely many equivalence classes of irreducible representations of G.

(b) Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a transversal of unitary representatives of irreducible representations of G. Set $d_i = \deg \varphi^{(i)}$. Then, the set of functions

$$\{\sqrt{d_k} \varphi_{ij}^{(k)} : 1 \leq k \leq s, 1 \leq i,j \leq d_k\}$$

is orthonormal.

Proof.

(a) By Lemma 1.13, any equivalence class of representations (any class of irreducible representations in particular) contains a unitary representation. As $\dim L(G) = |G|$, no linearly independent set of vectors in $L(G)$ can contain more than $|G|$ elements. Because orthonormal sets are linearly independent, Corollary 2.16 and theorem 2.15(a) show that there can only be finitely many classes of irreducible representations.

(b) This again directly follows from Corollary 2.16 and theorem 2.15(a). ∎

In particular, using the same notation as the above proposition, we have that

$$s \leq d_1^2 + d_2^2 + \cdots + d_s^2 \leq |G|.$$

Indeed, the lower bound is obvious as each $d_i \geq 1$. For the upper bound, each representation of degree d_k corresponds to d_k^2 many orthonormal functions, so the overall set of representations corresponds to $\sum d_i^2$ orthonormal functions, which can be at most $\dim L(G) = |G|$. This also says that the degree of any irreducible representation is at most $\sqrt{|G|}$.

In fact, we shall see later that it is *exactly* $|G|$.

2.3. Characters and Class Functions

Recall the remark after Corollary 2.8, which said that the decomposition given by Maschke's Theorem is unique. In this section, we shall prove a stronger version of the same (explicitly finding the number of irreducible representations), arriving at some interesting results along the way.

Given an endomorphism of a finite dimensional vector space, we can talk about its trace, which is just the trace of any matrix representation after fixing an ordered basis. It is not too difficult to see that this trace is basis-invariant. We extensively use this fact in this section, namely that $\mathrm{Tr}(ABC) = \mathrm{Tr}(ACB)$ so if $C = A^{-1}$ then $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$.

Definition 2.3 (Character). Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. The *character* $\chi_\varphi : G \to \mathbb{C}$ of φ is defined by $\chi_\varphi(g) = \mathrm{Tr}\, \varphi_g$. The character of an irreducible representation is called an *irreducible character*.

As mentioned, the character does not depend on the basis we choose, so we may assume that we are talking about matrix representations. If $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ is a representation given by $\varphi_g = ((\varphi_g)_{ij})$, $\chi_\varphi(g) = \sum_{i=1}^n (\varphi_g)_{ii}$.

Occasionally, we cut out the explicit writing of the representation and directly refer to characters of a group. The degree of this character is just the degree of the corresponding representation.

Remark. If $z : G \to \mathbb{C}^* \hookrightarrow \mathbb{C}$ is a degree 1 representation, then $\chi_z = z$. Henceforth, we treat degree 1 representations and their characters as the same.

Proposition 2.18. If $\varphi : G \to \mathrm{GL}(V)$ is a representation, $\chi_\varphi(1) = \deg \varphi = \dim V$.

Proof. Indeed, $\varphi_1 = \mathrm{Id}_V$ so $\chi_\varphi(1) = \mathrm{Tr}\, \varphi_1 = \mathrm{Tr}\, \mathrm{Id}_V = \dim V = \deg \varphi$. ∎

Proposition 2.19. If φ and ρ are equivalent representations, $\chi_\varphi = \chi_\rho$.

Proof. We may assume that $\varphi, \rho : G \to \mathrm{GL}_n(\mathbb{C})$. If $T \in \mathrm{GL}_n(\mathbb{C})$ is an invertible matrix such that $\varphi_g = T\rho_g T^{-1}$ for all $g \in G$, then
$$\chi_\varphi(g) = \mathrm{Tr}\, \varphi_g = \mathrm{Tr}(T\rho_g T^{-1}) = \mathrm{Tr}\, \rho_g = \chi_\rho(g).$$
∎

Corollary 2.20. Let G be a group of order n and χ a character of degree m of G. Then, $\chi(g)$ is a sum of m nth roots of unity for each $g \in G$.

Proof. Because characters are invariant under equivalence, let us assume that the representation is of the form $\varphi : G \to \mathrm{GL}_m(\mathbb{C})$. Fix $g \in G$. Then, $\varphi_g^n = I$ so φ_g is diagonalisable by Corollary 2.11. So, we may assume that φ_g is diagonal. It has eigenvalues $(\lambda_i)_{i=1}^m$, where each λ_i is an nth root of unity. The desideratum follows. ∎

A proof similar to Proposition 2.19 also shows the following.

Proposition 2.21. Let χ be a character of G. Then, χ is constant on conjugacy classes of G.

Proof. Let $g, h \in G$ and φ be a representation corresponding to χ. Then,
$$\begin{aligned}
\chi(g) &= \mathrm{Tr}\, \varphi_g \\
&= \mathrm{Tr}(\varphi_{h^{-1}} \varphi_g \varphi_h) \\
&= \mathrm{Tr}\, \varphi_{h^{-1}gh} = \chi(h^{-1}gh).
\end{aligned}$$
∎

Functions such as these have a name of their own.

Definition 2.4 (Class function). A function $f : G \to \mathbb{C}$ is said to be a *class function* if $f(g) = f(h^{-1}gh)$ for all $g, h \in G$. The set of all class functions is denoted $Z(L(G))$.

Given a conjugacy class $C \subseteq G$ and a class function f, we denote by $f(C)$ the constant value taken by f on C.

Proposition 2.22. $Z(L(G))$ is a subspace of $L(G)$.

We omit the proof of the above as it is very straightforward.

Definition 2.5. Given a group G, the set of conjugacy classes of G is denoted $\mathrm{Cl}(G)$. For $C \in \mathrm{Cl}(G)$, we define $\delta_C : G \to \mathbb{C}$ by

$$\delta_C(g) = \begin{cases} 1, & g \in C, \\ 0, & \text{otherwise.} \end{cases}$$

That is, δ_C is the indicator function of C.

Proposition 2.23. The set $B = \{\delta_C : C \in \mathrm{Cl}(G)\}$ is a basis of $Z(L(G))$. In particular, $\dim Z(L(G)) = |\mathrm{Cl}(G)|$.

Proof. It is clear that $\delta_C \in \mathrm{Cl}(G)$ for each $C \in \mathrm{Cl}(G)$.
To show that B spans $Z(L(G))$, note that for any $f \in Z(L(G))$,

$$f = \sum_{C \in \mathrm{Cl}(G)} f(C)\delta_C.$$

This is easily checked by computing both sides at an arbitrary $g \in G$.
To show linear independence on the other hand, we have for $C, C' \in \mathrm{Cl}(G)$

$$\langle \delta_C, \delta_{C'} \rangle = \sum_{g \in G} \delta_C(g)\overline{\delta_{C'}(g)} = \begin{cases} 0, & C \neq C', \\ \frac{|C|}{|G|}, & C = C' \end{cases}$$

and any set of orthogonal nonzero vectors is linearly independent.
The desideratum follows. $\blacksquare$

Theorem 2.24. Let φ, ρ be irreducible representations of G. Then

$$\langle \chi_\varphi, \chi_\rho \rangle = \begin{cases} 1, & \varphi \sim \rho, \\ 0, & \varphi \nsim \rho. \end{cases}$$

Thus, the set of irreducible characters of G forms an orthonormal set of class functions.

Proof. By Lemma 1.13 and proposition 2.19, we may assume that $\varphi : G \to U_n(\mathbb{C})$ and $\rho : G \to U_m(\mathbb{C})$. We have

$$\langle \chi_\varphi, \chi_\rho \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g)\overline{\chi_\rho(g)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{1 \leq i \leq n} \varphi_{ii}(g) \right) \overline{\left(\sum_{1 \leq j \leq m} \rho_{jj}(g) \right)}$$

$$= \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \frac{1}{|G|} \sum_{g \in G} \varphi_{ii}(g)\overline{\rho_{jj}(g)}$$

$$= \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \langle \varphi_{ii}, \rho_{jj} \rangle.$$

Recall Schur's Orthogonality Relations. If $\varphi \nsim \rho$, then it immediately follows that the above quantity of interest is 0. If $\varphi \sim \rho$ on the other hand, we may assume that $\varphi = \rho$ by Proposition 2.19. We then again have by Schur's orthogonality relations that the summand is nonzero only when $i = j$, and in this case it is just equal to $1/n$. The overall sum is then $n \cdot 1/n = 1$, completing the proof. ∎

Corollary 2.25. Given two inequivalent irreducible representations φ, ρ of G, $\chi_\varphi \neq \chi_\rho$.

Proof. We have $\langle \chi_\varphi, \chi_\rho \rangle = 0$, but if $\chi_\varphi = \chi_\rho$ we also have $\langle \chi_\varphi, \chi_\varphi \rangle = 1$. ∎

Corollary 2.26. Two irreducible representations are equivalent if and only if they have the same character.

Consequently, there are at most $|\operatorname{Cl}(G)|$ equivalence classes of irreducible representations.

Definition 2.6. If V is a vector, φ is a representation, and $m \in \mathbb{N}$, then

$$mV = \underbrace{V \oplus \cdots \oplus V}_{m} \text{ and } m\varphi = \underbrace{\varphi \oplus \cdots \oplus \varphi}_{m}.$$

$0V$ is the zero vector space and 0φ is the degree zero representation.