
CS 779 : TUTORIAL SOLUTIONS

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§1. Tutorial 1

Exercise 1.1. Prove that the maximum number of subsets of $[n]$ with pairwise non-empty intersection is 2^{n-1} .

Solution

2^{n-1} is clearly attainable by taking $\{S \subseteq [n] : 1 \in S\}$. Furthermore, this is an upper bound since if \mathcal{S} is a family of subsets with pairwise non-empty intersection, then $\mathcal{S}' = \{S^c : S \in \mathcal{S}\}$ has zero intersection with \mathcal{S} and is of the same size, so $2|\mathcal{S}| = |\mathcal{S}'| + |\mathcal{S}| \leq 2^n$.

Exercise 1.2. Suppose you have a set system with m sets $(A_i)_{i=1}^m$ such that $|A_i|$ is odd for each i and $|A_i \cap A_j|$ is even for any $i \neq j$. Prove that $m \leq n$.

Solution

Consider the $m \times n$ matrix M where M_{ij} is 1 if $j \in A_i$ and is 0 otherwise. Then,

$$(MM^T)_{ij} = \sum_{k \in [n]} M_{ik}M_{jk} = |A_i \cap A_j|.$$

In particular, all the diagonal entries of MM^T are odd and all off-diagonal entries are even. Using this, it is not too difficult to show that $\det(MM^T) \neq 0$ (for an easy solution* of this, note that modulo 2, MM^T is congruent to the identity, which has nonzero determinant). Therefore, $m = \text{rank}(MM^T) = \text{rank}(M)$, so $m \leq n$.

Exercise 1.3. Prove that for matrices A, B , $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

Solution

It suffices to show that any column of $A + B$ is present in the space spanned by the column of A and B . This is straightforward since any column of $A + B$ is just the sum of the two corresponding columns in A and B .

Exercise 1.4. Suppose you have $A + A^T = J - I$, where J is the all ones matrix. Prove that $\text{rank}(A) \geq n/2$.

Solution

Using the previous exercise, we have $n = \text{rank}(J - I) = \text{rank}(A + A^T) \leq \text{rank}(A) + \text{rank}(A^T) = 2 \text{rank}(A)$.

Exercise 1.5. Suppose you have $A + A^T = J - I$, where J is the all ones matrix. Show that if $\text{rank}(A) < n - 1$, there is a vector x such that $Ax = 0$, $x \neq 0$, and $\mathbf{1}^T x = 0$. Using this, prove that $\text{rank}(A) \geq n - 1$.

Solution

Suppose $\text{rank}(A) < n - 1$. Then, $\dim \ker A \geq 2$. We also have $\dim \mathbf{1}^\perp = n - 1$. Therefore, $\ker A$ and $\mathbf{1}^\perp$ have nonzero intersection, and say $x \neq 0$ is in both. x satisfies the conditions mentioned in the question. Now,

$$\begin{aligned} 0 &= x^T(Ax) + (x^T A^T)x \\ &= x^T(J - I)x \\ &= \left(\sum_i x_i\right)^2 - \left(\sum_i x_i^2\right) = -\sum_i x_i^2, \end{aligned}$$

so $x = 0$, a contradiction. Therefore, $\text{rank}(A) \geq n - 1$.

Exercise 1.6. Suppose B_1, \dots, B_m are complete bipartite graphs whose edge disjoint union yields the complete graph K_n . Show that $m \geq n - 1$.

Solution

Suppose that B_i corresponds to the complete bipartite graph between sets $X_i, Y_i \subseteq [n]$, where $X_i \cap Y_i = \emptyset$. As a graph on vertex set $[n]$, on setting $M_i = \mathbb{1}_X \mathbb{1}_Y^\top$, B_i has adjacency matrix $M_i + M_i^\top$. Note that $\text{rank}(M_i) = 1$ for all i , since $\mathbb{1}_Y \subseteq \ker M_i$. Because the edge disjoint union of the B_i is K_n , we have $(\sum_i M_i) + (\sum_i M_i)^\top = J - I$. Using the previous exercise, $\text{rank}(\sum_i M_i) \geq n - 1$. Using Exercise 1.3 and the observation that $\text{rank}(M_i) = 1$ for all i , this implies that $m = \sum_{i=1}^m \text{rank}(M_i) \geq n - 1$, completing the proof.

Exercise 1.7. Suppose you have a set system of m sets such that for every pair of sets, the intersection size is fixed as $\lambda \geq 1$. Prove that $m \leq n$.

Solution

Let the set system be $(A_i)_{i=1}^m$. The size of at most one set is equal to λ . Furthermore, if $|A_1| = \lambda$, then $A_i \setminus A_1$ are disjoint for distinct i , so $m - 1 \leq n - \lambda$. Thus, we may assume that the size of every set is greater than λ . Define the matrix M exactly as in Exercise 1.2. We have that the off-diagonal entries of M are equal to λ . Now, $MM^\top = \lambda J + D$, for some diagonal matrix D with all positive diagonal entries. We wish to show that $\text{rank}(\lambda J + D) = m$. Let $x \neq 0$ in \mathbb{R}^n , and let u, v be the components of x along and orthogonal to $\mathbf{1}$ respectively, such that $x = t\mathbf{1}$. Then,

$$\begin{aligned} (\lambda J + D)x &= (\lambda J + D)(u + v) \\ &= n\lambda u + D(u + v) \\ &= D(D^{-1}n\lambda u + u + v). \end{aligned}$$

When $t = 0$, this is clearly nonzero as $v \neq 0$. Otherwise, to conclude, note that

$$\sum_i (D^{-1}n\lambda u + u + v)_i = \sum_i (D_{ii}^{-1}n\lambda + 1)u_i + v_i = \sum_i t(D_{ii}^{-1}n\lambda + 1),$$

which is nonzero as $d_{ii}, \lambda > 0$ and $t \neq 0$.

§2. Tutorial 2

Exercise 2.1. Find the dimension of the space spanned by the following polynomials over the given field.

- (a) $x_1, x_2, x_1x_2, x_1^2x_2, 1, (x_1 + x_2)^2, x_1^2 + x_2^2$ over \mathbb{R} and over \mathbb{F}_2 .
 (b) $x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n}$, where $i_1 + \cdots + i_n = m$ over \mathbb{R} and over \mathbb{F}_2 .

Solution

- (a) Over \mathbb{R} , it is clear that $(x_1 + x_2)^2 = (x_1^2 + x_2^2) + 2(x_1x_2)$, and the collection formed by removing $(x_1 + x_2)^2$ is linearly independent, so the dimension of the space is 6. Since linear dependence requires that there exists a linear combination of the polynomials which is equal to the zero polynomial (in the sense that every coefficient is 0), and not merely a polynomial that evaluates to 0 everywhere, the dimension of this space is 6 as well.
- (b) Over \mathbb{R} , all these monomials are linearly independent, so the dimension is the number of ways of choosing n non-negative numbers that sum to m . This is a routine exercise in combinatorics, with the answer being $\binom{m+n-1}{m}$. As in the first part, the dimension over \mathbb{F}_2 is $\binom{m+n-1}{m}$ as well.

Exercise 2.2. Given m sets with sizes greater than d and pairwise intersection d , prove that $m \leq (n + 1)$.

Hint. Associate a polynomial to each set so that the polynomials are linearly independent. Give an upper bound on the space spanned by these polynomials.

Solution

Let A_1, \dots, A_m be sets of the above form. Associate to each set the polynomial

$$p_i(x) = \sum_{j \in A_i} x_j - d.$$

Let u_i be the indicator vector of A_i , equal to 1 at precisely those coordinates j in A_i . Note that $p_i(u_j) \neq 0$ iff $i = j$, so the p_i are linearly independent. Furthermore, the span of the p_i is of dimension at most $n + 1$, corresponding to 1 and the n x_j . It follows that $m \leq (n + 1)$.

Exercise 2.3.

- How do we define the distance between a pair of points in \mathbb{R}^n ?
- Construct as many points as you can so that the distance between a pair is one of two distances, either d_1 or d_2 . You may also choose d_1 and d_2 to maximize the number.
- Consider m points with exactly two pairwise distances. Associate polynomials $p_i(x)$ to each point such that the polynomials are linearly independent.
- Deduce an upper bound on the dimension of the span of your polynomials. What does this imply about the number of points with exactly two pairwise distances?

Solution

- Given $x, y \in \mathbb{R}^n$, the L^2 distance between them is given by $\|x - y\|_2 = (x - y)^\top (x - y) = \sum_{i=1}^n (x_i - y_i)^2$.
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- Let the points be u_1, \dots, u_m . To each point, associate the polynomial $p_i(x) = (\|x - u_i\|_2^2 - d_1^2)(\|x - u_i\|_2^2 - d_2^2)$. Note that $p_i(u_j) \neq 0$ iff $i = j$. It follows that the polynomials are linearly independent.
- We have

$$\|x - u\|^2 - d^2 = \sum x_k^2 + \sum x_k(-2u_k) + \sum u_k^2 - d^2,$$

a linear combination of $\sum x_k^2$, the x_k , and 1. It follows that each p_i is a linear combination of $(\sum_{k=1}^n x_k^2)^2$, x_j ($\sum_{k=1}^n x_k^2$), x_j^2 , $x_j x_t$, x_j , and 1, where j, k range over n with $j \neq k$. Therefore, the dimension of the span of p_i is at most

$$1 + n + n + \binom{n}{2} + n + 1 = \frac{n^2}{2} + \frac{5n}{2} + 2.$$

Since the polynomials are linearly independent, this implies that the number of points with exactly two pairwise distances is at most the above quantity.

Exercise 2.4. A polynomial is called multilinear if the degree of each variable is at most one. What is the dimension of the space of multilinear polynomials of degree at most d over n variables?

Solution

The solution to this is near-identical to the second part of Exercise 2.1(b), with the answer being $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{d}$.

Exercise 2.5. Consider m sets A_1, \dots, A_m such that $|A_i| \equiv k \pmod{p}$ for some prime p . Assume that $|A_i \cap A_j| \in L \pmod{p}$ for some set L , such that $k \notin L$ and $|L| = \ell$. Show that $m \leq \binom{n}{0} + \dots + \binom{n}{\ell}$.

Solution

For each set, associate the polynomial

$$q_i(x) = \prod_{u \in L} \left(-u + \sum_{j \in A_i} x_j \right)$$

over \mathbb{F}_p . Denoting by u_j the vector over \mathbb{F}_p that is 1 precisely at coordinates in A_j and 0 elsewhere, note that $q_i(u_j) \neq 0$ iff $i = j$. Now, consider the polynomial p_i obtained by opening up the product in the above definition, and replacing any occurrence of x_j^t by x_j for $t \geq 1$. Since any coordinate of the u_j is 0 or 1, $p_i(u_j) = q_i(u_j)$ for any j . In particular, $p_i(u_j) \neq 0$ iff $i = j$ and so the p_i are linearly independent. Furthermore, since the p_i are multilinear, the dimension of their span is at most $\binom{n}{0} + \dots + \binom{n}{\ell}$ as in the previous problem.

Exercise 2.6. For a prime power $q = p^t$, prove that $\binom{r-1}{q-1}$ is divisible by p iff r is not divisible by q .

Solution

Exercise 2.7. Let $q = p^t$ and $k \in \mathbb{Z}$. Let $(A_i)_{i=1}^m$ be subsets of $[n]$ such that $|A_i| \equiv k \pmod{q}$ for each i and $|A_i \cap A_j| \not\equiv k \pmod{q}$ for $i \neq j$. Then, show that $m \leq \binom{n}{q-1} + \binom{n}{q-3} + \dots$.

Solution