Geometric Topology

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Last updated March 12, 2021

Contents

§0. Pre-Requisites

We assume that the reader has a decent knowledge of General Topology, Analysis, and some parts of Basic Algebra. Nevertheless, we restate some important topological definitions and results.

Definition 0.1. A topological space X is said to be a *Hausdorff space* if for any distinct $x_1, x_2 \in X$, there exist disjoint open subsets U_1 and U_2 such that $x_1 \in U_1$ and $x_2 \in U_2$.

Definition 0.2. If X is a topological space and $x \in X$, a collection $\mathcal{B}_x$ of neighbourhoods of x is called a *neighbourhood basis for X at x* if any neighbourhood of x contains some $B \in \mathcal{B}_x$.
X is said to be *first countable* if there is a countable neighbourhood basis at each point.

It is often more useful to have a nested neighbourhood basis, defined as follows.

Definition 0.3. If X is a topological space and $x \in X$, a sequence (U_n) of neighbourhoods of x is said to be a *nested neighbourhood basis at x* if $U_{n+1} \subseteq U_n$ for each n and any neighbourhood of x contains some U_n.

The following lemma makes first countable spaces slightly easier to handle.

Lemma 0.1 (Nested Neighbourhood Basis Lemma)**.** A space X is first countable iff every $x \in X$ has a nested neighbourhood basis.

Theorem 0.2. Let X be a first countable space, $A \subseteq X$, and $x \in X$. Then

(a) $x \in \overline{A}$ iff x is a limit of a sequence of points in A.

(b) $x \in A^\circ$ iff every sequence in X converging to x is eventually in A.

(c) A is closed iff it contains every limit of every convergent sequence of points in A.

(d) A is open iff every sequence of points converging to a point in A is eventually in A.

Definition 0.4. A topological space X is said to be *second countable* if it has a countable basis.

Theorem 0.3. Let X be a second countable space. Then

(a) X is first countable.

(b) X contains a countable dense subset (it is separable).

(c) Every open cover of X contains a countable subcover.

§1. Introduction

Manifolds are essentially like curves or surfaces, except that they might be of higher dimension. Loosely speaking, an n-dimensional manifold is something that, at every point, is locally like $\mathbb{R}^n$.
So for example, manifolds of dimension 1 would be lines and curves. Manifolds of dimension 2 are surfaces, such as the unit sphere in $\mathbb{R}^3$, torii, paraboloids, etc.
To help us formalize this notion, let us make concrete what "locally like $\mathbb{R}^n$" means.

Definition 1.1. A topological space M is said to be *locally Euclidean of dimension* n if every point in M has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$.

Further, we see that since any open subset of $\mathbb{R}^n$ contains (an affine shift) of the Euclidean ball, the following follows as well.

Lemma 1.1. Let M be a topological space. The following are equivalent.

(a) M is locally Euclidean of dimension n.

(b) Every point of M has a neighbourhood homeomorphic to the open ball in $\mathbb{R}^n$.

(c) Every point of M has a neighbourhood homeomorphic to $\mathbb{R}^n$.

We represent the open ball in $\mathbb{R}^n$ by $\mathbb{B}^n$.
The proof of the above is not too difficult – we can use the fact that in $\mathbb{R}^n$, *translations* ($x \mapsto x + x_0$), *dilations* ($x \mapsto cx$), and *boundedness* are not topological properties (that is, they need not be preserved under homeomorphism). In particular, $\mathbb{B}^n$ is homeomorphic to $\mathbb{R}^n$.

Suppose U is locally Euclidean of dimension n and $U \subseteq M$ is open and homeomorphic to some open subset of $\mathbb{R}^n$. In this context, U is called a *coordinate domain* and any homeomorphism φ from U to an open subset of $\mathbb{R}^n$ is called a *coordinate map*. (U, φ) is then called a *coordinate chart*. If U is homeomorphic to $\mathbb{B}^n$, it is called a *coordinate ball* (or sometimes, a *coordinate disk* in dimension 2).
If $p \in M$ and $U \ni p$ is a coordinate domain, it is called a *coordinate neighbourhood* or *Euclidean neighbourhood of* p.
Observe that the definition of being locally Euclidean even extends to dimension 0 – this is equivalent to saying that the space is discrete.

Definition 1.2 (Manifold)**.** An *n-dimensional topological manifold* is a second countable Hausdorff space that is locally Euclidean of dimension n.

We often refer to them as n-dimensional manifolds, n-manifolds, or even just manifolds if the dimension is known or unimportant.
First of all, note that any manifold has *a* well-defined dimension. However, this begs the question – does a manifold have a *unique* dimension? This might seem quite obvious, but proving it is quite non-trivial. We state the result here, and return to it much later.

Theorem 1.2. A non-empty topological space cannot be both an m-manifold an an n-manifold for some $m \neq n$.

The proof of the above is quite obvious in the case where one of m and n is 0. Indeed, this would mean that the space is discrete, but a singleton set (that is open), does not contain any open subset homeomorphic to $\mathbb{B}^n$.

At the beginning of this section, we described how manifolds can be thought of as curves or surfaces, but of a higher dimension. Is this true in general? It turns out that it is, and any n-manifold is homeomorphic to some subset of a Euclidean space.

In the definition of a manifold, some authors require separability instead of second countability. This is in fact equivalent to our definition, as is not too difficult to show (one direction is obvious from Theorem 0.3).