CS 779 : Tutorial solutions

Amit Rajaraman

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§1. Tutorial 1

Exercise 1.1. Prove that the maximum number of subsets of $[n]$ with pairwise non-empty intersection is 2^{n-1}.

Solution

2^{n-1} is clearly attainable by taking $\{S \subseteq [n] : 1 \in S\}$. Furthermore, this is an upper bound since if $\mathcal{S}$ is a family of subsets with pairwise non-empty intersection, then $\mathcal{S}' = \{S^c : S \in \mathcal{S}\}$ has zero intersection with $\mathcal{S}$ and is of the same size, so $2|\mathcal{S}| = |\mathcal{S}'| + |\mathcal{S}| \leq 2^n$.

Exercise 1.2. Suppose you have a set system with m sets $(A_i)_{i=1}^m$ such that $|A_i|$ is odd for each i and $|A_i \cap A_j|$ is even for any $i \neq j$. Prove that $m \leq n$.

Solution

Consider the $m \times n$ matrix M where M_{ij} is 1 if $j \in A_i$ and is 0 otherwise. Then,

$$(MM^\top)_{ij} = \sum_{k \in [n]} M_{ik} M_{jk} = |A_i \cap A_j|.$$

In particular, all the diagonal entries of $MM^\top$ are odd and all off-diagonal entries are even. Using this, it is not too difficult to show that $\det(MM^\top) \neq 0$ (for an easy solution* of this, note that modulo 2, $MM^\top$ is congruent to the identity, which has nonzero determinant). Therefore, $m = \operatorname{rank}(MM^\top) = \operatorname{rank}(M)$, so $m \leq n$.

Exercise 1.3. Prove that for matrices A, B, $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

Solution

It suffices to show that any column of $A + B$ is present in the space spanned by the column of A and B. This is straightforward since any column of $A + B$ is just the sum of the two corresponding columns in A and B.

Exercise 1.4. Suppose you have $A + A^\top = J - I$, where J is the all ones matrix. Prove that $\operatorname{rank}(A) \geq n/2$.

Solution

Using the previous exercise, we have $n = \operatorname{rank}(J - I) = \operatorname{rank}(A + A^\top) \leq \operatorname{rank}(A) + \operatorname{rank}(A^\top) = 2\operatorname{rank}(A)$.

Exercise 1.5. Suppose you have $A + A^\top = J - I$, where J is the all ones matrix. Show that if $\operatorname{rank}(A) < n - 1$, there is a vector x such that $Ax = 0$, $x \neq 0$, and $\mathbf{1}^\top x = 0$. Using this, prove that $\operatorname{rank}(A) \geq n - 1$.

Solution

Suppose $\operatorname{rank}(A) < n - 1$. Then, $\dim \ker A \geq 2$. We also have $\dim \mathbf{1}^\perp = n - 1$. Therefore, $\ker A$ and $\mathbf{1}^\perp$ have nonzero intersection, and say $x \neq 0$ is in both. x satisfies the conditions mentioned in the question. Now,

$$\begin{aligned} 0 &= x^\top (Ax) + (x^\top A^\top) x \\ &= x^\top (J - I) x \\ &= \left(\sum_i x_i \right)^2 - \left(\sum_i x_i^2 \right) = -\sum_i x_i^2, \end{aligned}$$

so $x = 0$, a contradiction. Therefore, $\operatorname{rank}(A) \geq n - 1$.

Exercise 1.6. Suppose $B_1, \ldots, B_m$ are complete bipartite graphs whose edge disjoint union yields the complete graph K_n. Show that $m \geq n - 1$.

Solution

Suppose that B_i corresponds to the complete bipartite graph between sets $X_i, Y_i \subseteq [n]$, where $X_i \cap Y_i = \varnothing$. As a graph on vertex set $[n]$, on setting $M_i = \mathbb{1}_X \mathbb{1}_Y^\top$, B_i has adjacency matrix $M_i + M_i^\top$. Note that $\operatorname{rank}(M_i) = 1$ for all i, since $\mathbb{1}_Y^\perp \subseteq \ker M_i$. Because the edge disjoint union of the B_i is K_n, we have $(\sum_i M_i) + (\sum_i M_i)^\top = J - I$. Using the previous exercise, $\operatorname{rank}(\sum_i M_i) \geq n-1$. Using Exercise 1.3 and the observation that $\operatorname{rank}(M_i) = 1$ for all i, this implies that $m = \sum_{i=1}^m \operatorname{rank}(M_i) \geq n-1$, completing the proof.

Exercise 1.7. Suppose you have a set system of m sets such that for every pair of sets, the intersection size is fixed as $\lambda \geq 1$. Prove that $m \leq n$.

Solution

Let the set system be $(A_i)_{i=1}^m$. The size of at most one set is equal to λ. Furthermore, if $|A_1| = \lambda$, then $A_i \setminus A_1$ are disjoint for distinct i, so $m - 1 \leq n - \lambda$. Thus, we may assume that the size of every set is greater than λ.
Define the matrix M exactly as in Exercise 1.2. We have that the off-diagonal entries of M are equal to λ. Now, $MM^\top = \lambda J + D$, for some diagonal matrix D with all positive diagonal entries. We wish to show that $\operatorname{rank}(\lambda J + D) = m$. Let $x \neq 0$ in $\mathbb{R}^n$, and let u, v be the components of x along and orthogonal to $\mathbf{1}$ respectively, such that $u = t\mathbf{1}$. Then,

$$\begin{aligned}(\lambda J + D)x &= (\lambda J + D)(u + v) \\ &= n\lambda u + D(u + v) \\ &= D(D^{-1}n\lambda u + u + v).\end{aligned}$$

When $t = 0$, this is clearly nonzero as $v \neq 0$. Otherwise, to conclude, note that

$$\sum_i (D^{-1}n\lambda u + u + v)_i = \sum_i (D_{ii}^{-1}n\lambda + 1)u_i + v_i = \sum_i t(D_{ii}^{-1}n\lambda + 1),$$

which is nonzero as $d_{ii}, \lambda > 0$ and $t \neq 0$.