MA 412: Complex Analysis

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§1. Introduction

1.1. Some basic definitions

Consider the equation $X^2 + 1 = 0$. Clearly, this equation has no roots over $\mathbb{R}$. Consider the set

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} = \mathbb{R}^2,$$

and define addition and subtraction over $\mathbb{C}$ as

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b) \cdot (c, d) &= (ac - bd, ad + bc).\end{aligned}$$

It is easy to show that $(\mathbb{C}, +, \cdot)$ is a field with additive identity $(0, 0)$ and multiplicative identity $(1, 0)$. Further observe that $\mathbb{R}$ is a subfield of $\mathbb{C}$ – consider the field homomorphism $\mathbb{R} \to \mathbb{C}$ defined by $a \mapsto (a, 0)$.
Now, we denote $\iota = (0, 1)$, and write (a, b) as $a + b\iota$.

Observe that the equation $X^2 + 1 = 0$ *does* have roots over $\mathbb{C}$ since it can be written as $(X + \iota)(X - \iota)$. For the sake of completeness, we also note that the multiplicative identity of $a + \iota b$ is

$$\frac{1}{a + \iota b} = \frac{a - \iota b}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\iota.$$

When writing $z = a + b\iota$ where $a, b \in \mathbb{R}$, we write $a = \Re z$ (the real part of z) and $b = \Im z$ (the imaginary part of z). We also define the absolute value $|z| = (a^2 + b^2)^{1/2}$ of z, and the *conjugate* $\overline{z} = a - \iota b$ of z. We clearly have

$$\begin{aligned}z\overline{z} &= |z|^2 \\ \Re z &= \frac{z + \overline{z}}{2} \\ \Im z &= \frac{z - \overline{z}}{2\iota}.\end{aligned}$$

It is easy to check that

$$\overline{z + w} = \overline{z} + \overline{w} \text{ and } \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

We also have

$$\begin{aligned}\left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \\ |\overline{z}| &= |z|.\end{aligned}$$

Exercise 1.1. Check that the set

$$M = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R}$$

with matrix addition and multiplication is a field isomorphic to $\mathbb{C}$.

To close out the tedious part of things, we have

$$\begin{aligned}|z + w|^2 &= |z|^2 + |w|^2 + 2\Re(z\overline{w}) \\ |z + w| &\leq |z| + |w| \qquad (1.1)\end{aligned}$$

Equation (1.1) is referred to as the *triangle inequality*.

1.2. Polar representations and roots

Consider $z = x + \iota y \in \mathbb{C}$. We may then define

$$x = r\cos\theta \qquad y = r\sin\theta,$$

where $|z| = r$ and the angle θ is called the *argument* of z as is denoted $\theta = \arg z$. We typically restrict θ to $(-\pi, \pi]$. We denote $\operatorname{cis}\theta = \cos\theta + \iota\sin\theta$. Therefore, we have

$$z = |z|\operatorname{cis}(\arg z).$$

Observe that rather conveniently,

$$\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2).$$

Therefore, inductively,

$$z_1 z_2 \cdots z_n = \left(\prod_i |z_i|\right)\operatorname{cis}\left(\sum_i \arg z_i\right).$$

In particular,

$$z^n = r^n \operatorname{cis}(n\theta)$$

for any $n > 0$. If $z \neq 0$ (equivalently, $r \neq 0$), the above holds for all $n \in \mathbb{Z}$.
In the case where $r = 1$, we have

$$(\cos\theta + \iota\sin\theta)^n = \cos(n\theta) + \iota\sin(n\theta) \tag{1.2}$$

Equation (1.2) is referred to as *de Moivre's Formula.*

Let us consider the equation $z^n = a$. This equation has n roots of the form

$$z = |a|^{1/n}\operatorname{cis}\left(\frac{2k\pi + \arg a}{n}\right)$$

for $k = 0, 1, \ldots, n-1$.
A *line* in the complex plane is a set of the form

$$L = \{z = a + tb : t \in \mathbb{R}\},$$

for some fixed $a, b \in \mathbb{C}$, where b is a *directional* vector whose absolute value may be assumed to be 1. Since $b \neq 0$, we equivalently have

$$L = \left\{z : \Im\left(\frac{z-a}{b}\right) = 0\right\}.$$

We can also define the half-planes

$$H_a = \left\{z : \Im\left(\frac{z-a}{b}\right) > 0\right\}$$
$$K_a = \left\{z : \Im\left(\frac{z-a}{b}\right) < 0\right\}.$$

Note that $H_a = a + H_0$, where the addition is Minkowski addition:

$$H_a = \{a + z : z \in H_0\}.$$

1.3. The extended plane

Define $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and let $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. We shall show a bijection from $\mathbb{C}_\infty$ to S.
Let $N = (0, 0, 1)$ be the 'north pole' of S, and orient $\mathbb{C}$ (as $\mathbb{R}^2$) in the horizontal plane in a manner such that $\mathbb{C}$ cuts S along the equator. For $z = x + \iota y \in \mathbb{C}$, let us define the corresponding point $Z = (x_1, x_2, x_3) \in S$. We shall draw a line connecting z to N, and let Z be the point of intersection (other than N) of this line with S. Finally, we shall map ∞ to N.
Let us define this more explicitly. The line through N and z is

$$L = \{tN + (1 - t)z : t \in \mathbb{R}\}.$$

Then, letting $z = (x, y, 0)$, we have

$$t^2 + (1 - t)^2 |z|^2 = 1.$$

So,

$$|z|^2 = \frac{1 - t^2}{(1 - t)^2} = \frac{1 + t}{1 - t}$$

and

$$t = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Therefore, we map z to

$$Z = \left(\frac{2\Re z}{|z|^2 + 1}, \frac{2\Im z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}. \right) \in S.$$

Based on this, we can define a distance metric between points in $\mathbb{C}_\infty$. For $z, z' \in \mathbb{C}_\infty$ mapping to $Z, Z' \in S$, we let $d(z, z')$ be the Euclidean distance between Z, Z' in $\mathbb{R}^3$. More explicitly,

$$\begin{aligned} d(z, z')^2 &= (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 \\ &= 2 - 2(x_1 x_1' + x_2 x_2' + x_3 x_3') \\ &= \frac{2|z - z'|}{\left((|z|^2 + 1)(|z'|^2 + 1)\right)^{1/2}} \end{aligned}$$

when $z, z' \in \mathbb{C}$ and if $z' = \infty$ (so $Z' = (0, 0, 1)$), we have

$$d(z, z') = \frac{4}{|z|^2 + 1}$$

This correspondence between points of S and $\mathbb{C}_\infty$ is called the *stereographic projection*.

Exercise 1.2. If P is a plane in $\mathbb{R}^3$ and $\Lambda = P \cap S$ is a circle on S, show that the projection of Λ on $\mathbb{C}$ under the stereographic projection is a circle as well (possibly a circle of infinite radius, namely a line).

1.4. Power series

In this section, we begin discussing convergence of series in $\mathbb{C}$ and related properties.

Definition 1.1. If $a_n \in \mathbb{C}$ for every $n \geq 0$, the series $\sum_{n=0}^\infty a_n$ is said to *converge* to z iff for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{n=0}^{m} a_n - z \right| < \epsilon$$

for all $m \geq N$.
The series $\sum_{n=0}^\infty a_n$ is said to converge *absolutely* if $\sum_{n=0}^\infty |a_n|$ converges.

Theorem 1.1. $\mathbb{C}$ is complete. That is, every Cauchy sequence in $\mathbb{C}$ is convergent.

Proof. Suppose $\{x_n + \iota y_n\}$ is a Cauchy sequence in $\mathbb{C}$, where $x_n, y_n \in \mathbb{R}$ for each n. We then have the existence of $N \in \mathbb{N}$ such that for all $m, k > N$, $|(x_m - x_k) + \iota(y_m - y_k)| < \epsilon$. Consequently, $|x_m - x_k| < \epsilon$ and $|y_m - y_k| < \epsilon$. However, since $\mathbb{R}$ is complete, this implies that (x_n) and (y_n) are convergent, completing the proof. ∎

Theorem 1.2. If $\sum a_n$ converges absolutely, $\sum a_n$ converges.

Proof. Let $\epsilon > 0$, $z_n = \sum_{i=0}^n a_i$, and $S_n = \sum_{i=0}^n |a_i|$. Because $\mathbb{C}$ is complete, it suffices to show that (z_n) is Cauchy. Since $\sum |a_n|$ is convergent, there exists $N \in \mathbb{N}$ such that $|S_m - S_k| < \epsilon$ for all $m, k > N$. Supposing $m > k$, we have

$$S_m - S_k = \sum_{i=k+1}^{m} |a_i|.$$

So,

$$\begin{aligned} |z_m - z_k| &= \left| \sum_{i=k+1}^{m} a_i \right| \\ &\leq \sum_{i=k+1}^{m} |a_i| < \epsilon, \end{aligned}$$

completing the proof. ∎

Exercise 1.3. Show that $\sum_{n=0}^{\infty} z^n$ converges iff $|z| < 1$.

Theorem 1.3. For a given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, define the number R $(0 \leq R \leq \infty)$ by

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n}.$$

Then,

(a) If $|z - a| < R$, the series converges absolutely.

(b) If $|z - a| > R$, the terms of the series become unbounded and the series diverges.

(b) If $0 < r < R$, the series converges uniformly on the set $\{z : |z - a| \leq r\}$.

This R is referred to as the *radius of convergence* of the power series.

Proof.

(a) We assume without loss of generality that $a = 0$. If $|z| < R$, there exists r with $|z| < r < R$. By the definition of R, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{R} - \epsilon < \sup_{k \geq n} |a_k|^{1/k} < \frac{1}{R} + \epsilon$$

for all $n > N$. If we take $\epsilon = 1/r - 1/R$, it follows that $|a_n|^{1/n} < 1/r$ for all $n > N$. That is, for all $n > N$, $|a_n| < 1/r^n$ and so

$$|a_n z^n| < \left(\frac{|z|}{r} \right)^n.$$

Therefore, $\sum_{n=N}^{\infty} a_n z^n$ is dominated by $\sum_{n=N}^{\infty} (|z|/r)^n$. Now however, we can just use the result of Exercise 1.3 to conclude absolute convergence since $|z|/r < 1$.

(b) Let $|z| > R$ and choose r with $|z| > r > R$. For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{R} - \epsilon < \sup_{k \geq n} |a_k|^{1/k} \text{ for all } n > N.$$

Choosing $\epsilon = 1/R - 1/r$,

$$|a_n|^{1/n} > 1/r$$

for infinitely many $n \in \mathbb{N}$. It follows that $|a_n z^n| > (|z|/r)^n$ for infinitely many $n \in \mathbb{N}$. Since $|z|/r > 1$, these terms become unbounded and therefore the series diverges.

(c) Now, suppose $r < R$ and choose ρ such that $r < \rho < R$. Similar to the argument in (a), we get that

$$|a_n| < \frac{1}{\rho^n} \text{ for all } n \geq N.$$

If $|z| \leq r$, $|a_n z^n| \leq (r/\rho)^n$ and $r/\rho < 1$. The Weierstrass M-test then gives that the power series converges uniformly on $\{z : |z| \leq r\}$. ■

It should be noted that we cannot conclude anything when $|z - a| = R$.

Theorem 1.4. If $\sum a_n (z-a)^n$ is a power series with radius of convergence R, then if it exists,

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R.$$

Proof. Again, assume that $a = 0$ and let $\alpha = \lim |a_n/a_{n+1}|$, which we assume exists. Suppose that $|z| < \alpha$ and take $r \in \mathbb{R}$ such that $|z| < r < \alpha$. For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\alpha - \epsilon < \left| \frac{a_n}{a_{n+1}} \right| < \alpha + \epsilon.$$

Taking $\epsilon = \alpha - r$, $|a_n/a_{n+1}| > r$ for all $n \geq N$. Let $B = |a_N| r^N$. Then,

$$a_{N+1} r^{N+1} = |a_{N+1}| r \cdot r^N < |a_N| r^N = B.$$

Similarly, we get that $|a_n| r^n < B$ for all $n \geq N$. Therefore,

$$|a_n z^n| < B \left(\frac{|z|}{r} \right)^n$$

for all $n \geq N$. Thus, the sequence converges absolutely since $|z| < r$.
Since $r < \alpha$ was arbitrary, this implies that $\alpha \leq R$.

On the other hand, if $|z| > \alpha$, take $r \in \mathbb{R}$ such that $|z| > r > \alpha$. Taking $\epsilon = r - \alpha$, we get $N \in \mathbb{N}$ such that

$$\left| \frac{a_n}{a_{n+1}} \right| < r$$

for all $n \geq N$. Letting $B = |a_N| r^N$ again, we once more obtain that $|a_n| r^n > B$ for all $n \geq N$. This gives that

$$|a_n z^n| > B \left(\frac{|z|}{r} \right)^n$$

for all $n \geq N$, and since $|z| > r$, the sequence diverges (we may assume that $B \neq 0$ by making N larger if required to ensure that $a_N \neq 0$ – if this is not possible, the problem is trivial since it means that (a_n) is eventually 0). Since the choice of r was arbitrary, this implies that $R \leq \alpha$, completing the proof. ■

Now, consider the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The radius of convergence of this series is ∞. So, it converges for any complex number z, and convergence is uniform on every compact subset of $\mathbb{C}$.
The above defines a function $\exp : \mathbb{C} \to \mathbb{C}$.
We also denote $e^z = \exp(z)$.

Definition 1.2 (Differentiability)**.** If G is an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$, then f is said to be *differentiable* at a point $a \in G$ if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If it exists, the value of this limit is denoted $f'(a)$ and is called the *derivative* of f at a.

If f is differentiable at each point of G, we say that f is differentiable on G. Note that if f is differentiable on G, then $f' : G \to \mathbb{C}$ is a function. If f' is continuous, f is said to be *continuously differentiable.*

Theorem 1.5. If $f : G \to \mathbb{C}$ is differentiable at a point $a \in G$, f is continuous at a.

Proof. The proof of this is direct:

$$\begin{aligned}\lim_{z \to a} |f(z) - f(a)| &= \left(\lim_{z \to a} \frac{|f(z) - f(a)|}{|z-a|} \right) \cdot \lim_{z \to a} |z - a| \\ &= f'(a) \cdot 0 = 0.\end{aligned}$$

■

Definition 1.3. A function $f : G \to \mathbb{C}$ is said to be *analytic* if f is continuously differentiable on G.

Let f, g be analytic on G and Ω respectively, and suppose that $f(G) \subseteq \Omega$. Then, $g \circ f$ is analytic on G and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$$

for all $z \in G$. This is called the *chain rule.*

We shall show later that if f is differentiable then its derivative is continuous, and so f is analytic.

Theorem 1.6. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$. Then

(a) For each $k \geq 1$, the series

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z-a)^{n-k}$$

has radius of convergence R.

(b) The function f is infinitely differentiable on $B(a, R)$ (the open ball of radius R centered at a), and further, $f^{(k)}(z)$ is given by the series in (a) for all $k \geq 1$ and $|z - a| < R$.

(c) For $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

Proof. Assume that $a = 0$.

(a) Note that it suffices to prove the result for $k = 1$ (Why?). To show this, it is enough to show that

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |na_n|^{1/(n-1)}$$

First, it is not difficult to show that $\lim_{n\to\infty} n^{1/(n-1)} = 1$. It may be shown that for any sequences $(c_n), (d_n)$ in $\mathbb{R}$ where $c_n \geq 0$, if $\lim c_n = c$ and $\limsup d_n = d$, then $\limsup c_n d_n = cd$. Therefore, we are done if we show that $\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/(n-1)}$.

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + z \sum_{n=0}^{\infty} a_{n+1} z^n.$$

Let R' be the radius of convergence of $\sum_{n=0}^{\infty} a_{n+1} z^n$. We want to show that $R' = R$.
If $|z| < R'$, then

$$\sum |a_n z^n| \leq |a_0| + |z| \sum_{n=0}^{\infty} |a_{n+1} z^n| < \infty,$$

so $R' \leq R$. On the other hand, if $|z| < R$ and $z \neq 0$,

$$\sum |a_{n+1} z^n| < \frac{1}{|z|} \left(\sum |a_n z^n| + |a_0| \right) < \infty,$$

so $R \leq R'$ and we are done.

(b) Once again, it suffices to prove the result for $k = 0$. For $|z| < R$ and $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$,

$$s_n(z) = \sum_{k=0}^{n} a_k z^k \text{ and } R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k,$$

fix a point $w \in B(0, R)$ and r such that $|w| < r < R$. We wish to show that $f'(w)$ exists and is equal to $g(w)$. Let $\delta > 0$ be arbitrary with $\overline{B(w, \delta)} \subseteq B(0, r)$. Letting $z \in B(w, \delta)$, we have

$$\frac{f(z) - f(w)}{z - w} - g(w) = \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) + s_n'(w) - g(w) + \frac{R_n(z) - R_n(w)}{z - w}.$$

We have

$$|z^k - w^k| = |z - w||z^{k-1} + z^{k-2} w + \cdots + w^{k-1}| \leq |z - w| k r^{k-1}.$$

Therefore,

$$\left| \frac{R_n(z) - R_n(w)}{z - w} \right| = \left| \sum_{k=n+1}^{\infty} a_k \frac{z^k - w^k}{z - w} \right| \leq \sum_{k=n+1}^{\infty} |a_k| k r^{k-1}.$$

Since $r < R$, $\sum_{k=1}^{\infty} |a_k| k r^{k-1}$ converges and so for any $\epsilon > 0$, there exists $N_1 \in N$ such that for $n \geq N_1$,

$$\left| \frac{R_n(z) - R_n(w)}{z - w} \right| < \epsilon/3.$$

Since $\lim s_n'(w) = g(w)$, there exists $N_2 \in \mathbb{N}$ such that

$$|s_n'(w) - g(w)| < \epsilon/3$$

for $n \geq N_2$. Choose $n \geq \max(N_1, N_2)$. Then, there exists $\delta > 0$ such that whenever $0 < |z - w| < \delta$,

$$\left| \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) \right| < \epsilon/3.$$

Putting all these together, we get the desideratum.

(c) This is straightforward using the explicit expression for $f^{(k)}(a)$. ■

If the series $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence $R > 0$, then f is analytic on $B(a, R)$. Therefore, exp is analytic on $\mathbb{C}$.
Further, letting $g = \exp$,

$$g'(z) = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = g(z).$$

Define the functions cos and sin using power series as

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + (-1)^k \frac{z^{2k}}{(2k)!} + \cdots$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^k \frac{z^{2k+1}}{(2k+1)!} + \cdots$$

Note that

$$\cos z = \frac{e^{\iota z} + e^{-\iota z}}{2} \text{ and } \sin z = \frac{e^{\iota z} - e^{-\iota z}}{2\iota}.$$

Therefore,

$$e^{\iota z} = \cos z + \iota \sin z.$$

In particular, if $z = \theta \in \mathbb{R}$,

$$e^{\iota\theta} = \cos\theta + \iota \sin\theta.$$

It is direct to show next that $\cos^2 z + \sin^2 z = 1$ for $z \in \mathbb{C}$.

Definition 1.4. A function f is said to be *periodic* with period c if $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

e^z is periodic with period $2\pi\iota$.

Similar to cos and sin, one can define the function log as

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

$\log z$ is defined only when $|z-1| < 1$. Further note that we cannot define log as the inverse of exp (as we do over the reals) since exp is not injective here.
We would like to define log such that $w = \exp z$ when $z = \log w$. Since exp is non-zero, also suppose that $w \neq 0$. If $z = x + \iota y$, then $|w| = e^x$ and $\arg w = y + 2\pi k\iota$ for some $k \in \mathbb{Z}$. Therefore, the solution set for $e^z = w$ is

$$\{\log|w| + \iota(\arg w + 2\pi k) : k \in \mathbb{Z}\}.$$

Definition 1.5. If G is an open connected set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ is a continuous function such that $z = \exp(f(z))$ for all $z \in G$, then f is a *branch of the logarithm.*

Lemma 1.7. If $G \subseteq \mathbb{C}$ is open and connected and f is a branch of the logarithm on G, then the totality of the branches of $\log z$ are the functions $f(z) + 2\pi k\iota$ for $k \in \mathbb{Z}$.

Proof. If $g(z) = f(z) + 2\pi k\iota$ for some $k \in \mathbb{Z}$, then $\exp(g(z)) = \exp(f(z)) = z$, so g is also a branch of the logarithm. On the other hand, suppose that g is a branch of the logarithm. For $z \in G$, $\exp(f(z)) = \exp(g(z)) = z$, so $g(z) = f(z) + 2\pi k\iota$. However, note that this k depends on z. We must show that the same k works for all z. Indeed, $h(z) = (g(z) - f(z))/2\pi\iota$ is continuous on G and $h(G) \subseteq \mathbb{Z}$, so the required follows. ■

Now, let $G = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Clearly, G is connected and each $z \in G$ can be uniquely denoted by $|z|e^{\iota\theta}$, where $-\pi < \theta < \pi$. For θ in this range, define

$$f(re^{\iota\theta}) = \log r + \iota\theta.$$

This is a branch of the logarithm on G, and is commonly referred to as the *principal branch.*

Theorem 1.8. Let G, Ω be open subsets of $\mathbb{C}$. Suppose that $f : G \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are continuous such that $g(f(z)) = z$ for all $z \in G$. If G is differentiable and $g'(z) \neq 0$, f is differentiable and

$$f'(z) = \frac{1}{g'(f(z))}.$$

If g is analytic, so is f.

Proof. Fix $a \in G$ and let $h \in \mathbb{C} \setminus \{0\}$ with $a + h \in G$. Since $g(f(a)) = a \neq a + h = g(f(a + h))$, $f(a) \neq f(a + h)$. Also,

$$1 = \frac{g(f(a+h)) - g(f(a))}{h} = \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \cdot \frac{f(a+h) - f(a)}{h}.$$

Take the limit of either side as $h \to 0$. The first fraction is equal to $g'(f(a))$ since $\lim_{h\to 0}(f(a + h) - f(a)) = 0$, and therefore $\lim_{h\to 0}(f(a + h) - f(a))/h = f'(a)$ exists, and $1 = g'(f(a)) \cdot f'(a)$. The required follows.
If g is analytic, then g' is continuous so f is analytic. ■

Corollary 1.9. Any branch of the logarithm function is analytic and has derivative $z \mapsto 1/z$.

Given a branch of the logarithm f on an open connected set G and fixed $b \in \mathbb{C}$, define $g(z) = \exp(bf(z))$. If $b \in \mathbb{Z}$, $g(z) = z^b$. In general, this defines a branch of z^b ($b \in \mathbb{C}$) for any open connected set on which there is a branch of $\log z$.
If we write z^b as a function, it is implicitly understood that the f in $\exp(bf(z))$ is the principal branch of the logarithm. Since log is analytic, so is $z \mapsto z^b$.

1.5. Cauchy-Riemann Equations

Let $f : G \to \mathbb{C}$ be analytic and let

$$u(x, y) = \Re(f(x + \iota y)), v(x, y) = \Im(f(x + \iota y))$$

for $x + \iota y \in G$. Let us evaluate the limit

$$f'(z) = \lim_{h\to 0} \frac{f(z + h) - f(z)}{h}.$$

in two different ways.
First, if we let $h \to 0$ through real values, we get

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + \iota \frac{\partial v}{\partial x}(x, y).$$

Along the imaginary axis, we get

$$f'(z) = -\iota \frac{\partial u}{\partial y}(x, y) + \frac{\partial v}{\partial y}(x, y).$$

Therefore,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Supposing that u and v have continuous second derivative (we shall later show that they are infinitely differentiable), we have that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Therefore, since the second derivatives are continuous,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.3}$$

A function u with continuous second partial derivatives satisfying Equation (1.3) is said to be *harmonic*. Similarly, v is also harmonic.

Theorem 1.10. Let u, v be real-valued functions defined on an open connected set (a *region*) G and suppose that they have continuous second partial derivatives. Then, $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + \iota v(z)$ is analytic iff u and v satisfy the Cauchy-Riemann equations.

Proof. We have already shown the forward direction.
For the other direction, let $z = x + \iota y \in G$ and $B(z,r) \subseteq G$. Let $h = s + \iota t \in B(0,r)$. Our goal is to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(z+h) - f(z) - f'(z)h}{h} \right| < \epsilon$$

for all $h \in B(0,\delta)$ for some $f'(z) \in \mathbb{C}$. Note that

$$u(x+s, y+t) - u(x,y) = \big(u(x+s,y+t) - u(x,y+t)\big) + \big(u(x,y+t) - u(x,y)\big).$$

Now, for fixed $t \in (-r,r)$, $s \mapsto u(x+s,y+t)$ is a differentiable function on $(-r,r)$. We apply the mean value theorem to conclude that there exist $s_1, t_1 \in (-r,r)$ for each $s + \iota t \in B(0,r)$ such that $|s_1| < |s|$, $|t_1| < |t|$, and

$$\begin{aligned} u(x+s,y+t) - u(x,y+t) &= u_x(x+s_1,y+t)s \\ u(x,y+t) - u(x,y) &= u_y(x,y+t_1)t. \end{aligned}$$

Now, let

$$\varphi(s,t) = \big(u(x+s,y+t) - u(x,y)\big) - \big(u_x(x,y)s + u_y(x,y)t\big).$$

We get that

$$\varphi(s,t) = \big(su_x(x+s_1,y+t) - su_x(x,y)\big) + \big(tu_y(x,y+t_1) - tu_y(x,y)\big).$$

So,

$$\frac{\varphi(s,t)}{s+\iota t} = \frac{s}{s+\iota t}\big(u_x(x+s_1,y+t) - u_x(x,y)\big) + \frac{t}{s+\iota t}\big(u_y(x,y+t_1) - u_y(x,y)\big)$$

and on taking the limit of both sides as $s + \iota t \to 0$, we can use the fact that $|s| \le |s+\iota t|$, $|t| \le |s+\iota t|$, $|s_1| < |s|$, $|t_1| < t$, and the continuity of u_x, u_y, to conclude that

$$\lim_{s+\iota t \to 0} \frac{\varphi(s,t)}{s+\iota t} = 0.$$

Therefore,

$$u(x+s,y+t) - u(x,y) = u_x(x,y)s + u_y(x,y)t + \varphi(s,t).$$

We get a similar equation for v as well, with a function ψ (in place of φ). Combining the two,

$$\begin{aligned} \frac{f(z+s+\iota t) - f(z)}{s+\iota t} &= \frac{u(x+s,y+t) - u(x,y)}{s+\iota t} + \iota\frac{v(x+s,y+t) - v(x,y)}{s+\iota t} \\ &= \frac{su_x(x,y) + tu_y(x,y) + \varphi(s,t) + \iota\left(sv_x(x,y) + tv_y(x,y) + \psi(s,t)\right)}{s+\iota t} \\ &= \frac{u_x(x,y)(s+\iota t) + \iota v_x(x,y)(s+\iota t) + \varphi(s,t) + \iota\psi(s,t)}{s+\iota t}, \end{aligned}$$

where we used Cauchy-Riemann equations in the final step and thus,

$$\lim_{s+\iota t \to 0} \frac{f(z+s+\iota t) - f(z)}{s+\iota t} = u_x(x,y) + \iota v_x(x,y),$$

completing the proof. Since u_x and v_x are continuous, f' is continuous and f is analytic. ∎

A next question is: given some u such that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
when does there exist harmonic v such that $u + \iota v$ is analytic? Such a v is referred to as a *harmonic conjugate* of u. It turns out that the answer is not always. Indeed, $u(x, y) = \log((x^2 + y^2)^{1/2})$ on $\mathbb{C} \setminus \{0\}$, despite being harmonic, does not have a harmonic conjugate.

Theorem 1.11. Let G be either the entirety of $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is a harmonic function, then u has a harmonic conjugate.

Proof. Let $G = B(0, R)$ for some $0 < R \leq \infty$ and let $u : G \to \mathbb{R}$ be analytic. Define
$$v(x, y) = \int_0^y u_x(x, t)\,\mathrm{d}t + \varphi(x)$$
so that $u_x = v_y$. We shall determine φ such that $v_x = -u_y$. Differentiating with respect to x, we get
$$\begin{aligned} v_x(x, y) &= \int_0^y u_{xx}(x, t)\,\mathrm{d}t + \varphi'(x) \\ &= -\int_0^y u_{yy}(x, t)\,\mathrm{d}t + \varphi'(x) \\ &= -u_y(x, y) + u_y(x, 0) + \varphi'(x). \end{aligned}$$
Therefore, $\varphi'(x) = -u_y(x, 0)$, and the function
$$v(x, y) = \int_0^y u_x(x, t)\,\mathrm{d}t - \int_0^x u_y(s, 0)\,\mathrm{d}s$$
is a harmonic conjugate of u. ■

The above proof requires that the entire segments $[(0, 0), (x, 0)]$ $[(x, 0), (x, y)]$ are contained in G, which is true when we are on a disk.

1.6. Transformations

Consider the two hyperbolas defined by
$$\begin{aligned} x^2 - y^2 &= c \\ 2xy &= d, \end{aligned}$$
where $c, d \neq 0$.
This gives
$$y = \pm\sqrt{\frac{-c \pm \sqrt{d^2 + c^2}}{2}}.$$
Consider the functions
$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy. \end{aligned}$$
The two hyperbolas above are mapped by this $f = u + \iota v$ to the straight lines $u = c$ and $v = d$.

Definition 1.6. A *path* in a region $G \subseteq \mathbb{C}$ is a continuous function $\gamma : [a, b] \to G$ for some interval $[a, b]$ in $\mathbb{R}$. If $\gamma'(t)$ exissts for each $t \in [a, b]$ and $\gamma' : [a, b] \to \mathbb{C}$ is continuous, then γ is said to be *smooth.* γ is said to be *piecewise smooth* if there is a partition $a = t_0 < t_1 < \cdots t_{n-1} < t_n = b$ of $[a, b]$ such that γ is smooth on each subinterval $[t_{i-1}, t_i]$ for $1 \leq i \leq n$.
For a path $\gamma : [a, b] \to \mathbb{C}$, $\gamma([a, b])$ is sometimes referred to as the *trace* of γ and denoted $\{\gamma\}$.

By the existence of γ', we mean that the two-sided limit

$$\lim_{h \to 0} \frac{\gamma(t + h) - \gamma(t)}{h}$$

exists for $t \in (a, b)$ and the right and left sided limits exist for $t = a, b$ respectively. This is equivalent to saying that $\Re\gamma$ and $\Im\gamma$ have derivatives.
Suppose $\gamma : [a, b] \to G$ is a smooth path and for some $t_0 \in (a, b)$, $\gamma'(t_0) \neq 0$. Then, γ has a *tangent line* at the point $z_0 = \gamma(t_0)$. This line goes through the point z_0 in the direction of the vector $\gamma'(t_0)$, that is, the slope of the line is $\tan(\arg \gamma'(t_0))$.

If γ_1 and γ_2 are two smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1), \gamma_2'(t_2) \neq 0$, then define the *angle* between the paths γ_1, γ_2 at z_0 to be $\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1))$.

Suppose γ is a smooth path in G and $f : G \to \mathbb{C}$ is analytic. Then, $\sigma = f \circ \gamma$ is also a smooth path and $\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t)$. Further, if z_0 is a fixed point of f with $\gamma(t_0) = z_0$,

$$\arg(\sigma'(t_0)) - \arg(\gamma'(t_0)) = \arg(f'(z_0)).$$

Let γ_1, γ_2 be smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ with non-zero derivatives at t_1, t_2 respectively, and let $\sigma_1 = f \circ \gamma_1, \sigma_2 = f \circ \gamma_2$. Further suppose that the two paths γ_1, γ_2 are not tangent to each other at z_0. Then,

$$\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1)) = \arg(\sigma_2'(t_2)) - \arg(\sigma_1'(t_1)).$$

This says that the angle between two paths are preserved after applying an analytic function to both. A function f satisfying this is said to have the *angle-preserving property.*

Definition 1.7. A function $f : G \to \mathbb{C}$ which has the angle-preserving property and also has

$$\lim_{z \to a} \left| \frac{f(z) - f(a)}{z - a} \right|$$

existing for all $a \in G$ is called a *conformal map.*

It turns out that a function f is a conformal map if and only if it is analytic and $f'(z) \neq 0$ for all z (How?).

Definition 1.8. A mapping of the form

$$S(z) = \frac{az + b}{cz + d}$$

is called a *linear fractional transformation.* If we further have that $ad - bc \neq 0$, then $S(z)$ is called a *Möbius transformation.*

We have

$$S'(z) = \frac{ad - bc}{(cz + d)^2}.$$

If $w = S(z)$, it is relatively simple to show that

$$z = S^{-1}(w) = \frac{dw - b}{-cw + a}.$$

Therefore, the inverse of a Möbius transformation is a Möbius transformation. The composition of two Möbius transformations is a Möbius transformation as well.
Also observe that the coefficients a, b, c, d for a given Möbius transformation are not unique since we can multiply them by a constant. We may also extend S to $\mathbb{C}_\infty$ with $S(\infty) = a/c$ and $S(-d/c) = \infty$.

$S(z) = z + a$ is called a *translation*, $S(z) = az$ with $a \neq 0$ is called a *dilation*, $S(z) = e^{\iota\theta} z$ is called a *rotation*, and $S(z) = 1/z$ is called the *inversion*. We shall see later that any Möbius transformation is a composition of these five types of transformations.
What are the fixed points of a Möbius transformation S? $S(z) = z$ gives

$$cz^2 + (a - d)z + b = 0.$$

Therefore, a Möbius transformation has at most two fixed points unless $S(z) = z$ for all $z \in \mathbb{C}_\infty$.

Let $a, b, c \in \mathbb{C}_\infty$ be distinct with $S(a) = \alpha$, $S(b) = \beta$, $S(c) = \gamma$. Let T be another Möbius transformation with $T(a) = \alpha$, $T(b) = \beta$, $T(c) = \gamma$. Then $T^{-1} \circ S$ has three (distinct) fixed points, and therefore $S = T$.
Therefore, any Möbius transformation is uniquely determined by its value at any three distinct points.

Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Define $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$S(z) = \begin{cases} \frac{(z - z_3)/(z - z_4)}{(z_2 - z_3)/(z_2 - z_4)}, & z_2, z_3, z_4 \in \mathbb{C}, \\ \frac{z_2 - z_4}{z - z_4}, & z_3 = \infty, \\ \frac{z - z_3}{z_2 - z_3}, & z_4 = \infty. \end{cases}$$

In any case, $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$, and S is the only transformation having this property.

Definition 1.9. If $z_1 \in \mathbb{C}_\infty$, then (z_1, z_2, z_3, z_4) is referred to as the *cross-ratio* of z_1, z_2, z_3, z_4 and is the image of z_1 under the Möbius transformation described above, which is the unique Möbius transformation taking z_2 to 1, z_3 to 0, and z_4 to ∞.

For example, $(z_2, z_2, z_3, z_4) = 1$ and $(z, 1, 0, \infty) = z$.
If M is any Möbius transformation with $M(w_2) = 1$, $M(w_3) = 0$, $M(w_4) = \infty$, then $M(z) = (z, w_2, w_3, w_4)$ for all $z \in \mathbb{C}_\infty$.

Theorem 1.12. If z_2, z_3, z_4 are distinct points and T is any Möbius transformation, then

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4).$$

Proof. Let $S(z) = (z, z_2, z_3, z_4)$. If $M = ST^{-1}$, then

$$M(T(z_2)) = 1, \quad M(T(z_3)) = 0, \quad M(T(z_4)) = \infty.$$

Therefore, $M = (z, Tz_2, Tz_3, Tz_4)$. That is,

$$ST^{-1}z = (z, Tz_2, Tz_3, Tz_4)$$

for all $z \in \mathbb{C}_\infty$. Setting $z = Tz_1$ yields the required. ∎

Lemma 1.13. If $\{z_2, z_3, z_4\}, \{w_2, w_3, w_4\} \subseteq \mathbb{C}_\infty$, then there exists a unique Möbius transformation S with $Sz_i = w_i$ for each i.

We omit the proof of the above.

Lemma 1.14. Let $\{z_1, z_2, z_3, z_4\} \subseteq \mathbb{C}_\infty$. Then, (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle.

Proof. Define $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $Sz = (z, z_2, z_3, z_4)$. We are done if we show that $S^{-1}(\mathbb{R}_\infty)$ is a circle (since a circle is uniquely determined by three distinct points on it).
Let $S(z) = (az + b)/(cz + d)$.

First, let us show that $S^{-1}(\mathbb{R}_\infty) \subseteq \Gamma$ for a circle Γ in $\mathbb{C}_\infty$. Let $w \in S^{-1}(\mathbb{R}_\infty)$. Then, $Sw = \overline{Sw}$ so

$$\frac{aw + b}{cw + d} = \frac{\overline{aw} + \overline{b}}{\overline{cw} + \overline{d}}.$$

This gives that

$$(a\overline{c} - \overline{a}c)|w|^2 + (a\overline{d} - \overline{b}c)w + (b\overline{c} - d\overline{a})\overline{w} + (b\overline{d} - \overline{b}d) = 0. \tag{$*$}$$

If $a\overline{c}$ is real, we get that

$$\Im\left((a\overline{d} - \overline{b}c)w + b\overline{d}\right) = 0,$$

which is a circle through ∞ (a line).
If on the other hand $a\overline{c}$ is not real, then ($*$) becomes

$$2\iota \underbrace{\Im(a\overline{c})}_{\alpha \neq 0} |w|^2 + (a\overline{d} - b\overline{c})w + (b\overline{c} - \overline{a}d)\overline{w} + (b\overline{d} - \overline{b}d) = 0.$$

Dividing by $2\iota\alpha$,

$$|w|^2 + \frac{(a\overline{d} - b\overline{c})w}{2\iota\alpha} + \frac{(b\overline{c} - \overline{a}d)\overline{w}}{2\iota\alpha} + \frac{(b\overline{d} - \overline{b}d)}{2\iota\alpha} = 0.$$

Since α is real,

$$\overline{\frac{(b\overline{c} - \overline{a}d)\overline{w}}{2\iota\alpha}} = \frac{(a\overline{d} - b\overline{c})w}{2\iota\alpha}$$

and

$$\frac{(b\overline{d} - \overline{b}d)}{2\iota\alpha}$$

is real. This gives

$$|w|^2 + \overline{\gamma}w + \gamma\overline{w} - \delta = 0$$

for some $\gamma \in \mathbb{C}, \delta \in \mathbb{R}$. This is equivalent to $|w + \gamma| = (|\gamma|^2 + \delta)^{1/2}$, which is the equation of a circle[1].

Letting $T = S^{-1}$ and Γ be the circle obtained in the previous part of the proof, we must now show that $T(\mathbb{R}_\infty) = \Gamma$. Since $\mathbb{R}_\infty$ is connected and compact and T is a homeomorphism, $T(\mathbb{R}_\infty)$ is a closed arc, say Γ_1, of Γ. If $\Gamma_1 \neq \Gamma$, let z_1, z_2 be the endpoints of this arc. If $T(\infty) = z_3$ which is distinct from z_1, z_2, then $\mathbb{R}_\infty \setminus \{\infty\}$ is connected but $\Gamma_1 \setminus \{z_1\}$ is disconnected, which is a contradiction. So, suppose $T(\infty) = z_1$. Then, $\mathbb{R}_\infty \setminus \{\infty, T^{-1}(z_2)\}$ is disconnected but $\Gamma_1 \setminus \{z_1, z_2\}$ is connected, yielding a contradiction once more and completing the proof. ■

Next, we give a more general version of the above.

Theorem 1.15. A Möbius transformation takes circles to circles.

Note that Lemma 1.14 follows from this since $\mathbb{R}_\infty$ is a circle (of infinite radius) in $\mathbb{C}_\infty$.

[1] it may be checked that $|\gamma|^2 + \delta$ is a positive real by substituting their values.

Proof. Let Γ be a circle in $\mathbb{C}_\infty$ and S a Möbius transformation. Let z_2, z_3, z_4 be three distinct points on Γ, and set $w_j = Sz_j$ for each j. We claim that $S(\Gamma)$ is the circle Γ' determined by w_2, w_3, w_4. Indeed,

$$(z, z_2, z_3, z_4) = (Sz, w_2, w_3, w_4)$$

for any z, and if $z \in \Gamma$, the LHS is real by Lemma 1.14, and using the same theorem on the RHS completes the proof. ■

Definition 1.10. Let Γ be a circle through z_2, z_3, z_4. The points $z, z^* \in \mathbb{C}_\infty$ are said to be *symmetric* with respect to Γ if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

Remark. The above definition only depends on Γ, not the choice of z_2, z_3, z_4.

Observe that z is symmetric with respect to itself with respect to Γ if and only if $z \in \Gamma$. Indeed, it implies that (z, z_2, z_3, z_4) is real, which by Lemma 1.14 implies that $z \in \Gamma$.

What does it mean for z, z^* to be symmetric?
If Γ is a straight line, z, z^* are symmetric with respect to Γ iff their perpendicular bisector is equal to Γ. That is, the line joining z, z^* is perpendicular to Γ and they are the same distance from Γ (but on opposite sides). Indeed, choosing $z_4 = \infty$, we get that

$$\frac{z^* - z_3}{z_2 - z_3} = \frac{\overline{z} - \overline{z_3}}{\overline{z_2} - \overline{z_3}},$$

so

$$|z - z_3| = |z^* - z_3|$$

for all $z_3 \in \Gamma$.
Now, suppose that $\Gamma = \{z : |z - a| = R\}$ for some $0 < R < \infty$. We extensively use Theorem 1.12 and the five types of Möbius translations in the following sequence of equations. Then,

$$\begin{aligned}
(z^*, z_2, z_3, z_4) &= \overline{(z, z_2, z_3, z_4)} \\
&= \overline{(z - a, z_2 - a, z_3 - a, z_4 - a)} \\
&= \left(\overline{z} - \overline{a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}, \frac{R^2}{z_4 - a}\right) \\
&= \left(\frac{R^2}{\overline{z} - \overline{a}}, z_2 - a, z_3 - a, z_4 - a\right) \\
&= \left(\frac{R^2}{\overline{z} - \overline{a}} + a, z_2, z_3, z_4\right).
\end{aligned}$$

Therefore, $z^* = a + \frac{R^2}{\overline{z} - \overline{a}}$, that is,

$$(z^* - a)(\overline{z} - \overline{a}) = R^2.$$

Since

$$\frac{z^* - a}{z - a} = \frac{R^2}{|z - a|^2} > 0$$

is real, it follows that z^* is on the ray $\{a + t(z - a) : 0 < t < \infty\}$. We also have that

$$|z^* - a||z - a| = R^2,$$

so one can easily obtain z^* from z or vice-versa.

Lemma 1.16 (Symmetry Principle)**.** If a Möbius transformation takes a circle Γ_1 to the circle Γ_2, then any pair of points symmetric with respect to Γ_1 is mapped to a pair of points symmetric with respect to Γ_2.

Proof. The proof of this is near-direct.

$$\begin{aligned}(Tz, Tz_2, Tz_3, Tz_4) &= (z^*, z_2, z_3, z_4) \\ &= \overline{(z, z_2, z_3, z_4)} \\ &= \overline{(Tz, Tz_2, Tz_3, Tz_4)}.\end{aligned}$$

■

Definition 1.11. If Γ is a circle, then an *orientation* for Γ is an ordered triple (z_1, z_2, z_3) of points in Γ.

An orientation is used to represent a "direction" of the circle, where we "go" from z_1 to z_2 to z_3.

Let $\Gamma = \mathbb{R}$ and $z_1, z_2, z_3 \in \mathbb{R}$. Also put $Tz = (z, z_1, z_2, z_3)$. Since $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$, a, b, c, d can be chosen to be reals. Then,

$$\begin{aligned}Tz &= \frac{az+b}{cz+d} \\ &= \frac{az+b}{|cz+d|^2}(c\overline{z}+d) \\ &= \frac{1}{|cz+d|^2}\left(ac|z|^2 + bd + bc\overline{z} + adz\right).\end{aligned}$$

So,

$$\Im(z, z_1, z_2, z_3) = \frac{ad-bc}{|cz+d|^2}\Im z$$

and thus, $\{z : \Im(z, z_1, z_2, z_3) > 0\}$ is either the upper or lower half-plane depending on whether $ad - bc$ is negative or positive. Note that $ad - bc$ is the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let Γ be an arbitrary circle and suppose that $z_1, z_2, z_3 \in \Gamma$. Then, for any Möbius transformation S,

$$\begin{aligned}\{z : \Im(z, z_1, z_2, z_3) > 0\} &= \{z : \Im(Sz, Sz_1, Sz_2, Sz_3) > 0\} \\ &= S^{-1}\{z : \Im(z, Sz_1, Sz_2, Sz_3) > 0\}.\end{aligned}$$

So, if S is chosen to map Γ to $\mathbb{R}_\infty$, then the above set is equal to S^{-1} of either the upper or lower halfspace.

Definition 1.12. If z_1, z_2, z_3 is an orientation of Γ, we denote the *right side* and *left side* of Γ (with respect to (z_1, z_2, z_3)) to be

$$\{z : \Im(z, z_1, z_2, z_3) > 0\} \text{ and } \{z : \Im(z, z_1, z_2, z_3) < 0\}$$

respectively.

Theorem 1.17 (Orientation Principle)**.** Let Γ_1, Γ_2 be circles in $\mathbb{C}_\infty$ such that $T\Gamma_1 = \Gamma_2$ for some Möbius transformation T. Let (z_1, z_2, z_3) be an orientation of Γ_1. Then, T takes the right side (resp. left side) of Γ_1 with respect to the orientation (z_1, z_2, z_3) to the right side (resp. left side) of Γ_2 with respect to the orientation (Tz_1, Tz_2, Tz_3).

The proof of the above is left as an exercise to the reader.

Since $(z, 1, 0, \infty) = z$ by definition, the right side of $\mathbb{R}_\infty$ with respect to the orientation $(1, 0, \infty)$ is the upper half-plane.

Exercise 1.4. Find an analytic function $f : G \to \mathbb{C}$ where $G = \{z : \Re z > 0\}$, such that $f(G) = \{z : |z| < 1\}$.

Similar to the above exercise, one may show that

$$g(z) = \frac{e^z - 1}{e^z + 1}$$

maps the infinite strip $\{z : |\Im z| < \pi/2\}$ to the open unit disk D.

§2. Integration

2.1. Basic definitions

2.1.1. Integrals of real functions

First, let us recall the definition of the Riemann integral[2] of functions on $\mathbb{R}$.

Definition 2.1 (Riemann Integral)**.** Let $[a, b]$ be a given interval. A *partition* $\mathcal{P}$ of $[a, b]$ is a finite set of points $x_0, x_1, \dots, x_n$ where

$$a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b.$$

We also write $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, \dots, n$.
For a bounded real function f on $[a, b]$ and each partition $\mathcal{P}$ of $[a, b]$, we set

$$M_i = \sup_{x_{i-1} \le x \le x_i} f(x), \qquad m_i = \inf_{x_{i-1} \le x \le x_i} f(x).$$

Further, set

$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i, \qquad L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

as the upper and lower Riemann sum respectively, and finally,

$$\overline{\int_a^b} f \,\mathrm{d}x = \inf_{\mathcal{P}} U(\mathcal{P}, f), \qquad \underline{\int_a^b} f \,\mathrm{d}x = \sup_{\mathcal{P}} L(\mathcal{P}, f)$$

as the upper and lower Riemann integrals of f.

Next, we define the slightly more general Riemann-Stieltjes integral. Note that this is the same as the usual Riemann integral when α is the identity function.

Definition 2.2 (Riemann-Stieltjes Integral)**.** Let $\alpha : [a, b] \to \mathbb{R}$ be a monotonically increasing function on $[a, b]$. Corresponding to each partition $\mathcal{P}$ of $[a, b]$, write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Clearly, $\Delta\alpha_i \ge 0$ for each i.
For any real function f which is bounded on $[a, b]$, we put

$$U(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i, \qquad L(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i,$$

where M_i, m_i are defined as in the definition of the Riemann integral. We then define

$$\overline{\int_a^b} f \,\mathrm{d}\alpha = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha), \qquad \underline{\int_a^b} f \,\mathrm{d}\alpha = \sup_{\mathcal{P}} L(\mathcal{P}, f, \alpha).$$

If these two are equal, we say that f is *Riemann-Stieltjes integrable* with respect to α on $[a, b]$ and denote the common value as $\int_a^b f \,\mathrm{d}\alpha$.

[2] technically the Darboux integral?

We also remark that
$$\int_a^b f \, \mathrm{d}\alpha = \lim_{\max \Delta\alpha_k \to 0} \sum_{k=1}^{n} f(\tau_k)\Delta\alpha_k,$$
where $x_{k-1} \leq \tau_k \leq x_k$ for each k.
More generally, we define the *mesh* of $\mathcal{P}$ with respect to α as
$$\|\mathcal{P}\| = \max\{\Delta\alpha_i : 1 \leq i \leq n\}.$$
So for all $\epsilon > 0$, there exists $\delta > 0$ such that for any partition $\mathcal{P}$ of $[a, b]$ with $\|P\| < \delta$, then
$$\left| \sum_{k=1}^{n} f(\tau_k)\Delta\alpha_k - \int_a^b f \, \mathrm{d}\alpha \right| < \epsilon$$
for any choice of points $x_{k-1} \leq \tau_k \leq x_k$.

2.1.2. Riemann-Stieltjes integrals of complex-valued functions

Definition 2.3. A function $\gamma : [a, b] \to \mathbb{C}$ for $[a, b] \subseteq \mathbb{R}$ is said to be of *bounded variation* if there exists $M > 0$ such that for any partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b\}$ of $[a, b]$,
$$v(\gamma; \mathcal{P}) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \leq M.$$
The *total variation* $V(\gamma)$ of γ is defined by
$$V(\gamma) = \sup\{v(\gamma; \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}.$$
Clearly, $V(\gamma) \leq M < \infty$.

Lemma 2.1. Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation. Then,

1. If $\mathcal{P}, \mathcal{Q}$ are partitions of $[a, b]$ with $\mathcal{P} \subseteq \mathcal{Q}$, then $v(\gamma; \mathcal{P}) \leq v(\gamma; \mathcal{Q})$.
2. If $\sigma : [a, b] \to \mathbb{C}$ is also of bounded variation and $\alpha, \beta \in \mathbb{C}$, then $\alpha\gamma + \beta\sigma$ is of bounded variation and
$$V(\alpha\gamma + \beta\sigma) \leq |\alpha|V(\gamma) + |\beta|V(\sigma).$$

We omit the proof of the above, which is direct on using the triangle inequality on the definition of $v(\gamma; \mathcal{P})$.

Lemma 2.2. If $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth, γ is of bounded variation and
$$V(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}t.$$

Proof. It suffices to show the required in the case where γ is smooth, since in general we can consider the refinement of any partition that splits along the pieces along which γ is smooth.
The right hand side is well-defined since γ' is continuous. Let $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b\}$. By definition,
$$\begin{aligned} v(\gamma, \mathcal{P}) &= \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} \gamma'(t) \, \mathrm{d}t \right| \\ &\leq \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} |\gamma'(t)| \, \mathrm{d}t = \int_a^b |\gamma'(t)| \, \mathrm{d}t. \end{aligned}$$

Therefore, $V(\gamma) \leq \int_a^b |\gamma'(t)|\, \mathrm{d}t$, so γ is of bounded variation.
Since γ' is continuous, it is uniformly continuous. So, if $\epsilon > 0$, we may choose $\delta_1 > 0$ such that

$$|s - t| < \delta_1 \implies |\gamma'(s) - \gamma'(t)| < \epsilon.$$

Also, let $\delta_2 > 0$ such that if $\|P\| < \delta_2$, then

$$\left| \int_a^b |\gamma'(t)|\, \mathrm{d}t - \sum_{k=1}^{m} |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \epsilon,$$

where τ_k is any point in $[t_{k-1}, t_k]$. Therefore,

$$\begin{aligned} \int_a^b |\gamma'(t)|\, \mathrm{d}t &\leq \epsilon + \sum_{k=1}^{m} |\gamma'(t_k)|(t_k - t_{k-1}) \\ &= \epsilon + \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k)\, \mathrm{d}t \right| \\ &\leq \epsilon + \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} (\gamma'(\tau_k) - \gamma'(t))\, \mathrm{d}t \right| + \sum_{k=1}^{m} \left| \int_{t_{k-1}}^{t_k} \gamma'(t)\, \mathrm{d}t \right|. \end{aligned}$$

If $\|P\| < \delta = \min(\delta_1, \delta_2)$, then $|\gamma'(\tau_k) - \gamma'(t)| < \epsilon$ for all $t \in [t_{k-1}, t_k]$ and

$$\begin{aligned} \int_a^b |\gamma'(t)\, \mathrm{d}t| &\leq \epsilon + \epsilon(b - a) + \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \epsilon(1 + b - a) + V(\gamma; P) \leq \epsilon(1 + b - a) + V(\gamma), \end{aligned}$$

so we are done since $1 + b - a > 0$ is finite and ϵ can be made arbitrarily small. ∎

Theorem 2.3. Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation and suppose that $f : [a, b] \to \mathbb{C}$ is continuous. Then, there exists a (unique) complex number $\mathcal{I}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that when $\mathcal{P} = \{t_0 < t_1 < \cdots < t_m\}$ is a partition of $[a, b]$ with $\|P\| = \max_{1 \leq k \leq m}(t_k - t_{k-1}) < \delta$,

$$\left| \mathcal{I} - \sum_{k=1}^{m} f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \right| < \epsilon$$

for any choice of points τ_k with $t_{k-1} \leq \tau_k \leq t_k$.
This $\mathcal{I}$ is called the integral of f with respect to γ over $[a, b]$ and is denoted by

$$\mathcal{I} = \int_a^b f\, \mathrm{d}\gamma = \int_a^b f(t)\, \mathrm{d}\gamma(t).$$

Proof. First of all, note that it suffices to consider the case where γ is real-valued, since we can write $\gamma = \gamma_1 + \iota\gamma_2$, where γ_1, γ_2 are real-valued, to get two integrals $\mathcal{I}_1, \mathcal{I}_2$ (for γ_1, γ_2 respectively), and finally use the triangle inequality to get $\mathcal{I} = \mathcal{I}_1 + \iota\mathcal{I}_2$.
Since f is continuous, it is uniformly continuous. We can (inductively) find positive numbers $\delta_1 > \delta_2 > \cdots$ such that if $|s - t| < \delta_m$, $|f(s) - f(t)| < 1/m$. For each $M \geq 1$, let $\mathcal{P}_m$ be the collection of all partitions P of $[a, b]$ with $\|P\| \leq \delta_m$, so $\mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \cdots \supseteq \mathcal{P}_m \supseteq \cdots$. Finally, define F_m to be the closure of the set

$$\left\{ \sum_{k=1}^{n} f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) : P \in \mathcal{P}_m \text{ and } t_{k-1} \leq \tau_k \leq t_k \right\}.$$

Because $\mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \cdots$, it follows trivially that

$$F_1 \supseteq F_2 \supseteq \cdots.$$

We claim that

$$\operatorname{diam} F_m \leq \frac{2}{m} V(\gamma). \tag{2.1}$$

If we do this, then Cantor's Theorem (since $\mathbb{C}$ is complete) implies that there is precisely one complex number $\mathcal{I}$ such that $\mathcal{I} \in F_m$ for all $m \geq 1$. Then, for any $\epsilon > 0$, we may let $m > (2/\epsilon)V(\gamma)$ so $\epsilon > (2/m)V(\gamma) \geq \operatorname{diam} F_m$. Since $\mathcal{I} \in F_m$, $F_m \subseteq B(\mathcal{I}, \epsilon)$. Therefore, $\delta = \delta_m$ gets the job done.
So, we must show that

$$\operatorname{diam} \left\{ f(\tau_k) \left(\gamma(t_k) - \gamma(t_{k-1}) \right) : P \in \mathcal{P}_m \text{ and } t_{k-1} \leq \tau_k \leq t_k \right\} \leq \frac{2}{m} V(\gamma).$$

To do this, if $P = \{t_0 < \cdots < t_n\}$ is a partition, denote by $S(P)$ a sum of the form $\sum f(\tau_k) \left(\gamma(t_k) - \gamma(t_{k-1}) \right)$ where $t_{k-1} \leq \tau_k \leq t_k$ for each k. Fixing $m \geq 1$, let $P \in \mathcal{P}_m$. If $P \subseteq Q$ (so $Q \in \mathcal{P}_m$ as well), then

$$|S(P) - S(Q)| < \frac{1}{m} V(\gamma).$$

We only show this in the case where Q is obtained from P by adding a single extra partition point (the general case follows similarly). Let $Q = \{t_0 < t_1 < \cdots < t_{p-1} < t^* < t_p < \cdots t_n\}$. If $t_{p-1} \leq \sigma \leq t^*$ and $t^* \leq \sigma' \leq t_p$. Then,

$$S(Q) = \sum_{k \neq p} f(\sigma_k)(\gamma(t_k) - \gamma(t_{k-1})) + f(\sigma) \left(\gamma(t^*) - \gamma(t_{p-1}) \right) + f(\sigma') \left(\gamma(t_p) - \gamma(t^*) \right).$$

Then, using the definition of δ_m,

$$\begin{aligned}
\left| S(P) - S(Q) \right| &= \Bigg| \sum_{k \neq p} \left(f(\tau_k) - f(\sigma_k) \right) \left(\gamma(t_k) - \gamma(t_{k-1}) \right) \\
&\quad + f(\tau_p)(\gamma(t_p) - \gamma(t_{p-1})) - f(\sigma)(\gamma(t^*) - \gamma(t_{p-1})) - f(\sigma')(\gamma(t_p) - \gamma(t^*)) \Bigg| \\
&\leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k-1}| + \left| \left(f(\tau_p) - f(\sigma) \right) \left(\gamma(t^*) - \gamma(t_{p-1}) \right) + \left(f(\tau_p) - f(\sigma') \right) \left(\gamma(t_p) - \gamma(t^*) \right) \right| \\
&\leq \frac{1}{m} \sum_{k \neq p} \left| \gamma(t_k) - \gamma(t_{k-1}) \right| + \frac{1}{m} \left| \gamma(t^*) - \gamma(t_{p-1}) \right| + \frac{1}{m} \left| \gamma(t_p) - \gamma(t^*) \right| \\
&\leq \frac{1}{m} V(\gamma).
\end{aligned}$$

Next, let P, R be any two partitions in $\mathcal{P}_m$, and $Q = P \cup R$ a partition that contains P and R. Using the first part,

$$|S(P) - S(Q)| \leq |S(P) - S(Q)| + |S(Q) - S(R)| \leq \frac{2}{m} V(\gamma).$$

It follows that the diameter of the set of interest is at most $(2/m)V(\gamma)$, completing the proof. ■

Theorem 2.4. Let f, g be continuous functions on $[a, b]$ and let γ, σ be functions of bounded variation on $[a, b]$. Then for any scalars α, β,

$$\begin{aligned}
\int_a^b (\alpha f + \beta g) \, \mathrm{d}\gamma &= \alpha \int_a^b f \, \mathrm{d}\gamma + \beta \int_a^b g \, \mathrm{d}\gamma \\
\int_a^b f \, \mathrm{d}(\alpha\gamma + \beta\sigma) &= \alpha \int_a^b f \, \mathrm{d}\gamma + \beta \int_a^b f \, \mathrm{d}\sigma.
\end{aligned}$$

Proposition 2.5. Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and let $f : [a,b] \to \mathbb{C}$ be continuous. If $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, then

$$\int_a^b f \,\mathrm{d}\gamma = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f \,\mathrm{d}\gamma.$$

We omit the proofs of the above.

Theorem 2.6. If γ is piecewise smooth and $f : [a,b] \to \mathbb{C}$ is continuous, then $\int_a^b f \,\mathrm{d}\gamma = \int_a^b f(t)\gamma'(t) \,\mathrm{d}t$.

Proof. It suffices to consider the case where γ is smooth by Proposition 2.5. Also, by looking at the real and imaginary parts of γ separately, it suffices to consider the case where γ is real-valued on $[a,b]$. Let $\epsilon > 0$ and choose $\delta > 0$ such that if $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ has $\|P\| < \delta$, then

$$\left| \int_a^b f \,\mathrm{d}\gamma - \sum_{k=1}^{n} f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \right| < \epsilon/2$$

and

$$\left| \int_a^b f(t)\gamma'(t) \,\mathrm{d}t - \sum_{k=1}^{n} f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \epsilon/2$$

for any $t_{k-1} \leq \tau_k \leq t_k$ for each k.
Applying the mean value theorem on γ (this requires that γ be real-valued), one gets that there exists $\tau_k \in [t_{k-1}, t_k]$ for each k such that

$$\gamma'(\tau_k) = \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}}.$$

Using these τ_k specifically,

$$\left| \int_a^b f \,\mathrm{d}\gamma - \sum_{k=1}^{n} f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \epsilon/2,$$

so

$$\left| \int_a^b f \,\mathrm{d}\gamma - \int_a^b f(t)\gamma'(t) \,\mathrm{d}t \right| < \epsilon,$$

completing the proof. ■

2.2. Integrals On Curves

Definition 2.4. $\gamma : [a,b] \to \mathbb{C}$ is called a *rectifiable path* if it is continuous and of bounded variation. Note that if γ is piecewise smooth, then it is rectifiable and its length is

$$\int_a^b |\gamma'(t)| \,\mathrm{d}t = V(\gamma).$$

Definition 2.5. If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and f is a function continuous on $\{\gamma\}$, then the *(line) integral* of f along γ is

$$\int_a^b f(\gamma(t))\,\mathrm{d}\gamma(t).$$

This line integral is also denoted as

$$\int_\gamma f = \int_\gamma f(z)\,\mathrm{d}z.$$

For example, if $\gamma : [0, 2\pi] \to \mathbb{C}$ as $\gamma(t) = e^{\iota t}$,

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} e^{-\iota t}(\iota e^{\iota t})\,\mathrm{d}t = 2\pi\iota.$$

and

$$\int_\gamma z^m\,\mathrm{d}z = \int_0^{2\pi} e^{\iota m t}(\iota e^{\iota t})\,\mathrm{d}t = \iota\int_0^{2\pi} \cos((m+1)t)\,\mathrm{d}t - \int_0^{2\pi} \sin((m+1)t)\,\mathrm{d}t = 0.$$

Theorem 2.7. If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path and $\varphi : [c, d] \to [a, b]$ is a continuous non-decreasing function with $\varphi(c) = a, \varphi(d) = b$, then for any function f continuous on γ,

$$\int_\gamma f = \int_{\gamma\circ\varphi} f.$$

Remark. The above uses the fact that $\gamma \circ \varphi$ is also rectifiable (Why is this true?).

Proof. Let $\epsilon > 0$ and choose $\delta_1 > 0$ such that for a partition $\{s_0 < s_1 < \cdots < s_n\}$ of $[c, d]$ with $(s_k - s_{k-1}) < \delta_1$ and any $s_{k-1} \le \sigma_k \le s_k$,

$$\left| \int_{\gamma\circ\varphi} f - \sum_{k=1}^{n} f((\gamma\circ\varphi)(s_k)) - f((\gamma\circ\varphi)(s_{k-1})) \right| < \epsilon/2.$$

Similarly, choose $\delta_2 > 0$ such that for a partition $\{t_0 < t_1 < \cdots < t_m\}$ of $[a, b]$ with $(t_k - t_{k-1}) < \delta_2$ and $t_{k-1} \le \tau_k \le t_k$,

$$\left| \int_\gamma f - \sum_{k=1}^{m} f(\gamma(t_k)) - f(\gamma(t_{k-1})) \right| < \epsilon/2.$$

Since φ is uniformly continuous on $[c, d]$, there exists $\delta > 0$ less than δ_1 such that $|\varphi(s) - \varphi(t)| < \delta_2$ whenever $|s - t| < \delta$. So, if $\{s_0 < s_1 < \cdots < s_n\}$ is a partition of $[c, d]$ with $(s_k - s_{k-1}) < \delta < \delta_1$ and $t_k = \varphi(s_k)$, then $\{t_0 < t_1 < \cdots < t_n\}$ is a partition of $[a, b]$ with $(t_k - t_{k-1}) < \delta_2$. If $s_{k-1} \le \sigma_k \le s_k$ and $\tau_k = \varphi(\sigma_k)$, then we can use the two earlier inequalities to conclude that

$$\left| \int_\gamma f - \int_{\gamma\circ\varphi} f \right| < \epsilon,$$

completing the proof. ■

Definition 2.6. Let $\gamma : [a, b] \to \mathbb{C}$ be a rectifiable path, and for $a \le t \le b$, set $|\gamma|(t) = V(\gamma; [a, t])$. That is,

$$|\gamma|(t) = \sup\left\{ \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| : \{t_0 < t_1 < \cdots < t_n\} \text{ is a partition of } [a, t] \right\}.$$

Clearly, $|\gamma|$ is increasing on $[a, b]$ and of bounded variation. In fact, $V(|\gamma|; [a, b]) = |\gamma|(b) - |\gamma|(a)$. If f is continuous on $[a, b]$, define

$$\int f|\,\mathrm{d}z| = \int_a^b f(\gamma(t))\,\mathrm{d}|\gamma|(t).$$

Theorem 2.8. Let $\gamma : [a, b] \to \mathbb{C}$ be a rectifiable curve and suppose that f is a function continuous on $\{\gamma\}$. Then,

$$\int_\gamma f = -\int_{-\gamma} f \tag{2.2}$$

where $(-\gamma)(t) = \gamma(a + b - t)$,

$$\left|\int_\gamma f\right| \le \int_\gamma |f|\,|\,\mathrm{d}z| \le V(\gamma)\sup\{|f(z)| : z \in \{\gamma\}\}, \tag{2.3}$$

and for $c \in \mathbb{C}$,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma+c} f(z-c)\,\mathrm{d}z. \tag{2.4}$$

Proof. Equations (2.2) and (2.4) follow near-directly from the definition, so we prove only Equation (2.3). Let $\epsilon > 0$. Then, there exists $\delta > 0$ such that if $P = \{t_0 < t_1 < \cdots t_n\}$ is a partition of $[a, b]$ with $\|P\| < \delta$, then

$$\left|\left|\int_\gamma f(z)\,\mathrm{d}z\right| - \left|\sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1}))\right|\right| \le \left|\int_\gamma f(z)\,\mathrm{d}z - \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1}))\right| < \epsilon/2$$

for any $t_{k-1} \le \tau_k \le t_k$. That is,

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &< \left|\sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1}))\right| + \epsilon/2 \\ &\le \sum_{k=1}^n \big|f(\gamma(\tau_k))\big|\,\big|\gamma(t_k) - \gamma(t_{k-1})\big| + \epsilon/2.\end{aligned}$$

We may also assume that for this same δ,

$$\sum_{k=1}^n |f(\gamma(t_k))|(|\gamma|(t_k) - |\gamma|(t_{k-1})) < \int_\gamma |f(z)|\,|\,\mathrm{d}z| + \epsilon/2.$$

Recall that $|\gamma|(t)$ is an increasing function. So,

$$|\gamma|(t_k) - |\gamma|(t_{k-1}) \ge |\gamma(t_k) - \gamma(t_{k-1})|$$

Therefore,

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &< \sum_{k=1}^n |f(\gamma(\tau_k))|\,\big(|\gamma|(t_k) - |\gamma|(t_{k-1})\big) + \epsilon/2 \\ &< \int_\gamma |f(z)|\,|\,\mathrm{d}z| + \epsilon.\end{aligned}$$

It follows that

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \int_\gamma |f(z)|\,|\,\mathrm{d}z|.$$

To conclude the proof, note that

$$\int_\gamma |\,\mathrm{d}z| = |\gamma|(b) - |\gamma|(a) = |\gamma|(b) = V(\gamma),$$

so

$$\int_\gamma |f(z)||\,\mathrm{d}z| \le V(\gamma) \sup_{z\in\{\gamma\}} |f(z)|.$$

■

Lemma 2.9. If G is an open set in $\mathbb{C}$, $\gamma : [a,b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous, then for every $\epsilon > 0$ there exists a polygonal path Γ in G such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and

$$\left|\int_\gamma f - \int_\Gamma f\right| < \epsilon$$

Proof. We prove the result in the case where G is an open disk. In the general case where G need not be a disk, since $\{\gamma\}$ is compact, there exists a number r with $0 < r < d(\{\gamma\}, \partial G)$. Choose $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ when $|s-t| < \delta$. The idea is that we shall take several smaller disks and stitch together the polygonal paths on each of these sections.
If $P = \{t_0 < t_1 < \cdots < t_n\}$ is a partition of $[a,b]$ with $\|P\| < \delta$, then $|\gamma(t_k) - \gamma(t_{k-1})| < r$ for $t_{k-1} \le t \le t_k$. That is, if $\gamma_k : [t_{k-1}, t_k] \to G$ is defined by $\gamma_k(t) = \gamma(t)$, then $\{\gamma_k\} \subseteq B(\gamma(t_{k-1}), r)$ for $1 \le k \le n$. Getting a polygonal path Γ_k for each k such that

$$\left|\int_{\gamma_k} f - \int_{\Gamma_k} f\right| < \epsilon/n,$$

defining $\Gamma(t) = \Gamma_k(t)$ on $[t_{k-1}, t_k]$ does the job.

Now, let us prove the result in the disk case.
Because $\{\gamma\}$ is a compact set, $d = d(\{\gamma\}, \partial G) > 0$. It follows that if $G = B(c,r)$, then $\{\gamma\} \subseteq B(c,\rho)$ where $\rho = r - d/2$.
Now, observe that f is uniformly continuous on $\overline{B}(c,\rho) \subseteq G$. Thus, we may assume without loss of generality that f is uniformly continuous on G. Now, choose $\delta > 0$ such that if $|z-w| < \delta$, then $|f(z) - f(w)| < \epsilon$. If $\gamma : [a,b] \to G$, then γ is uniformly continuous so there is a partition $P = \{t_0 < t_1 < \cdots < t_n\}$ of $[a,b]$ such that if $t_{k-1} \le s, t \le t_k$, $|\gamma(s) - \gamma(t)| < \delta$, and such that for $t_{k-1} \le \tau_k \le t_k$,

$$\left|\int_\gamma f - \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1}))\right| < \epsilon.$$

Now, define $\Gamma : [a,b] \to G$ by

$$\Gamma(t) = \frac{(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k)}{t_k - t_{k-1}}$$

if $t_{k-1} \le t \le t_k$. This is the polygonal path we shall consider. Indeed,

$$\Gamma(t) - \gamma(\tau_k) = \frac{t_k - t}{t_k - t_{k-1}}(\gamma(t_{k-1}) - \gamma(\tau_k)) + \frac{t - t_{k-1}}{t_k - t_{k-1}}(\gamma(t_k) - \gamma(\tau_k)),$$

so

$$\begin{aligned}|\Gamma(t) - \gamma(\tau_k)| &\le \left|\frac{t_k - t}{t_k - t_{k-1}}\right| \left|\gamma(t_{k-1}) - \gamma(\tau_k)\right| + \left|\frac{t - t_{k-1}}{t_k - t_{k-1}}\right| \left|\gamma(t_k) - \gamma(\tau_k)\right| \\ &\le \left|\gamma(t_{k-1}) - \gamma(\tau_k)\right| + \left|\gamma(t_k) - \gamma(\tau_k)\right| < 2\delta.\end{aligned}$$

Thus,

$$\begin{aligned}\int_\Gamma f &= \int_a^b f(\Gamma(t))\Gamma'(t)\,\mathrm{d}t \\ &= \sum_{k-1}^{n} \frac{\gamma(t_k)-\gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t))\,\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}\left|\int_\gamma f - \int_\Gamma f\right| &= \left|\int_\gamma f - \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k)-\gamma(t_{k-1}))\right| + \left|\sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k)-\gamma(t_{k-1})) - \int_\Gamma f\right| \\ &\le \epsilon + \left|\sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k)-\gamma(t_{k-1})) - \int_\Gamma f\right| \\ &\le \epsilon + \sum_{k=1}^n \frac{|\gamma(t_k)-\gamma(t_{k-1})|}{t_k-t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\Gamma(t)) - f(\gamma(\tau_k))|\,\mathrm{d}t \\ &\le \epsilon + \epsilon\sum_{k=1}^n |\gamma(t_k)-\gamma(t_{k-1})| \\ &\le \epsilon(1+V(\gamma)),\end{aligned}$$

which can be made arbitrarily small, thus completing the proof. ■

The following can be thought of as an analogue of the Fundamental Theorem of Calculus for complex functions.

Theorem 2.10. Let G be open in $\mathbb{C}$ and γ be a rectifiable path in G with initial and end points α, β respectively. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$ (F is differentiable and $F' = f$), then

$$\int_\gamma f = F(\beta) - F(\alpha).$$

Proof. When $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth,

$$\begin{aligned}\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b F'(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b (F\circ\gamma)'(t)\,\mathrm{d}t \\ &= (F\circ\gamma)(b) - (F\circ\gamma)(a) && \text{(by the Fundamental Theorem of Calculus)} \\ &= F(\beta) - F(\alpha).\end{aligned}$$

In general, we may use Lemma 2.9. For $\epsilon > 0$, let Γ be a polygonal path of the described form. Since Γ is piecewise smooth, $\int_\Gamma f = F(\beta) - F(\alpha)$, so

$$\left|\int_\gamma f - (F(\beta) - F(\alpha))\right| < \epsilon.$$

Since ϵ was chosen arbitrarily, the desideratum follows. ■

The fundamental theorem of calculus says that each continuous function has a primitive. However, this is not true for functions of complex variables. For example, letting $f(z) = |z|^2$, if F is a primitive of f, then F is analytic. So, if $F = U + \iota V$, $x^2 + y^2 = F'(x + \iota y)$. Consequently,

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} = x^2 + y^2 \\ \frac{\partial U}{\partial y} &= \frac{\partial V}{\partial x} = 0.\end{aligned}$$

However, $\frac{\partial U}{\partial y} = 0$ implies that $U(x, y) = u(x)$ for some function u, which implies that $u'(x) = x^2 + y^2$, a contradiction.

2.3. Power series representation of analytic functions

Recall the following result which we had used in the proof of Theorem 1.11.

Theorem 2.11. Let $\varphi : [a, b] \times [c, d] \to \mathbb{C}$ be a continuous function and defined $g : [c, d] \to \mathbb{C}$ by

$$g(t) = \int_a^b \varphi(s, t)\,\mathrm{d}s.$$

Then g is continuous. Moreover, if $\frac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a, b] \times [c, d]$, then g is continuously differentiable and

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t)\,\mathrm{d}s.$$

This is referred to as the Leibniz rule.

For example, this may be used to prove that if $|z| < 1$,

$$\int_0^{2\pi} \frac{e^{\iota s}}{e^{\iota s} - z} = 2\pi.$$

To do so, let $\varphi(s, t) = e^{\iota s}/(e^{\iota s} - tz)$ for $0 \le t \le 1$ and $0 \le s \le 2\pi$. Observe that φ is continuously differentiable since $|z| < 1$. Thus,

$$g(t) = \int_0^{2\pi} \varphi(s, t)\,\mathrm{d}s$$

is continuously differentiable. Since $\varphi(s, 0) = 1$, $g(0) = 2\pi$. Now,

$$\begin{aligned}g'(t) &= \int_0^{2\pi} \frac{\partial \varphi}{\partial t}(s, t)\,\mathrm{d}s \\ &= \int_0^{2\pi} \frac{z e^{\iota s}}{(e^{\iota s} - tz)^2}\,\mathrm{d}s.\end{aligned}$$

For fixed t, $\Phi(s) = z\iota/(e^{\iota s} - tz)$ satisfies

$$\Phi'(s) = -\frac{\iota z}{(e^{\iota s} - tz)^2} \cdot \iota e^{\iota s} = \frac{z e^{\iota s}}{(e^{\iota s} - tz)^2}.$$

Therefore, $g'(t) = \Phi(2\pi) - \Phi(0) = 0$, so g is a constant and $g(t) = g(0) = 2\pi$ for any t, 1 in particular.

Theorem 2.12. Let $f : G \to \mathbb{C}$ be analytic and suppose that $\overline{B(a,r)} \subseteq G$ for some $r > 0$. If $\gamma(t) = a + re^{\iota t}$ for $0 \le t \le 2\pi$, then

$$f(z) = \frac{1}{2\pi\iota} \int_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w$$

for $|z - a| < r$.

Proof. Defining $G_1 = \{(z-a)/r : z \in G\}$ and $g(z) = f(a + rz)$, it suffices to consider the case where $a = 0$ and $r = 1$.
Fix z with $|z| < 1$. It must be shown that

$$f(z) = \int_{2\pi\iota} \int_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{\iota s}) e^{\iota s}}{e^{\iota s} - z} \, \mathrm{d}s.$$

That is, we want to show that

$$\begin{aligned} 0 &= \int_0^{2\pi} \frac{f(e^{\iota s}) e^{\iota s}}{e^{\iota s} - z} \, \mathrm{d}s - 2\pi f(z) \\ &= \int_0^{2\pi} \left(\frac{f(e^{\iota s}) e^{\iota s}}{e^{\iota s} - z} - f(z) \right) \mathrm{d}s. \end{aligned}$$

For this, let

$$\varphi(s,t) = \frac{f(z + t(e^{\iota s} - z)) e^{\iota s}}{e^{\iota s} - z} - f(z)$$

for $0 \le t \le 1$ and $0 \le s \le 2\pi$, and

$$g(t) = \int_0^{2\pi} \varphi(s,t) \, \mathrm{d}s.$$

We wish to show that $g(1) = 0$. Observe that

$$g(0) = \int_0^{2\pi} \frac{f(z) e^{\iota s}}{e^{\iota s} - z} - f(z) \, \mathrm{d}s = f(z) \int_0^{2\pi} \frac{e^{\iota s}}{e^{\iota s} - z} \, \mathrm{d}s - 2\pi f(z) = 0.$$

Also,

$$\begin{aligned} g'(t) &= \int_0^{2\pi} \frac{\partial \varphi}{\partial t} (s,t) \, \mathrm{d}s \\ &= \int_0^{2\pi} \frac{e^{\iota s}}{e^{\iota s} - z} f'(z + t(e^{\iota s} - z))(e^{\iota s} - z) \, \mathrm{d}s \\ &= \int_0^{2\pi} e^{\iota s} f'(z + t(e^{\iota s - z})) \, \mathrm{d}s \\ &= \frac{1}{t} f(z + t(e^{\iota s} - z)) \Big|_{s=0}^{s=2\pi} \\ &= 0, \end{aligned}$$

completing the proof. ∎

If $|z - a| < r$ and w is such that $|w - a| = r$, then

$$\frac{1}{w - z} = \frac{1}{w - a} \cdot \frac{1}{1 - \frac{z-a}{w-a}} = \frac{1}{w - a} \sum_{i=0}^{\infty} \left(\frac{z - a}{w - a} \right)^i.$$

since $|z-a| < |w-a|$.
Now, multiplying by $f(w)/2\pi\iota$ and integrating around the circle γ defined by $|w-a| = r$, we get that

$$f(z) = \int_\gamma \frac{f(w)}{2\pi\iota} \sum_{i=0}^{\infty} \frac{(z-a)^i}{(w-a)^{i+1}} \, \mathrm{d}w.$$

But how do we simplify the right hand side? We do not know (*yet*) that the integral and summation may be switched. So, let us get to showing this.

Lemma 2.13. Let γ be a rectifiable curve in $\mathbb{C}$ and suppose that F_n and F are continuous functions on $\{\gamma\}$. If (F_n) uniformly converges to F on $\{\gamma\}$, then

$$\int_\gamma = \lim_{n\to\infty} \int_\gamma F_n.$$

Proof. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ such that

$$|F_n(w) - F(w)| < \frac{\epsilon}{V(\gamma)}$$

for $n \geq N$. This implies that

$$\left| \int_\gamma F_n - \int_\gamma F \right| \leq V(\gamma) \sup_w |F_n(w) - F(w)| \leq \epsilon$$

for $n \geq N$, completing the proof. ■

Theorem 2.14. Let f be analytic on $B(a,R)$. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for all $|z-a| < R$, where $a_n = f^{(n)}(a)/n!$ and this series has radius of convergence at least R.

Proof. Let $0 < r < R$ such that $\overline{B(a,r)} \subseteq B(a,R)$. Let $\gamma(t) = a + re^{\iota t}$ $(0 \leq t \leq 2\pi)$. Since $|z-a| < r$, if $M = \max\{|f(w)| : |w-a| = r\}$,

$$\frac{|f(w)||z-a|^n}{|w-a|^{n+1}} \leq \frac{M}{r}\left(\frac{|z-a|}{r}\right)^n.$$

Since $|z-a| < r$,

$$\sum_{n=0}^{\infty} f(w) \frac{(z-a)^n}{(w-a)^{n+1}}$$

converges uniformly for w on $\{\gamma\}$. By the discussion before the previous lemma together with the lemma itself,

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi\iota} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \right) (z-a)^n. \tag{$*$}$$

Since

$$a_n = \frac{1}{2\pi\iota} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}.$$

is independent of z, $(*)$ converges for $|z-a| < R$. However, we now know from Theorem 1.6(c) that $a_n = f^{(n)}(a)/n!$, completing the proof. ■

Corollary 2.15. If f is analytic,

$$f^{(n)}(a) = \frac{1}{2\pi\iota} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, \mathrm{d}w$$

where $\gamma = a + re^{\iota t}$ and $r < R$, the radius of convergence of the series.

Corollary 2.16. If $f : G \to \mathbb{C}$ is analytic, then f is infinitely differentiable.

Indeed, this follows directly from the fact that

$$f^{(n)}(a) = \frac{n!}{2\pi\iota} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, \mathrm{d}w$$

where $\gamma(t) = a + re^{\iota t}$ for $0 \leq t \leq 2\pi$.

Corollary 2.17 (Cauchy's Estimate)**.** Let f be analytic on $B(a, R)$ and suppose $|f(z)| \leq M$ for all $z \in B(a, R)$. Then

$$|f^{(n)}(a)| \leq \frac{n! M}{R^n}.$$

Indeed, the above applies with $r < R$ so we get that

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi} \int_\gamma \frac{|f(w)|}{|w-a|^{n+1}} |\,\mathrm{d}w| \leq \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n! M}{r^n}.$$

Since $r < R$ is arbitrary, we may let $r \to R^-$.

Proposition 2.18. Let f be analytic on the disk $B(a, R)$ and suppose that γ is a closed rectifiable curve in $B(a, R)$. Then $\int_\gamma f = 0$.

Proof. Due to Theorem 2.10, it suffices to show that f has a primitive. We know that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for $|z - a| < R$, where $a_n = f^{(n)}(a)/n!$. Consider the function

$$F(z) = (z-a) \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-a)^n.$$

Since $\lim_{n\to\infty} (n+1)^{1/n} = 1$, this power series has the same radius of convergence as $\sum a_n (z-a)^n$. Therefore, F is defined on $B(a, R)$. Moreover, $F'(z) = f(z)$ for $|z - a| < R$ by Theorem 1.6(b), completing the proof. ■

Definition 2.7. An *entire* function is a function which is defined and analyitc on the whole complex plane $\mathbb{C}$.

Proposition 2.19. If f is entire, then it has a power series expansion with infinite radius of convergence.

Therefore, entire functions may be considered as polynomials of "infinite degree". Polynomials of finite non-zero degree are typically unbounded. These two insights lead to the following result.

Theorem 2.20 (Liouville's Theorem)**.** If f is a bounded entire function, then f is constant.

Proof. Suppose that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We shall show that $f'(z) = 0$ for all $z \in \mathbb{C}$. By Cauchy's Estimate, since f is analytic on any disk $B(z, R)$, $|f'(z)| \leq M/R$. However, R is arbitrary so $f'(z) = 0$ for any $z \in \mathbb{C}$. ■

Theorem 2.21 (Fundamental Theorem of Algebra)**.** If p is a non-constant polynomial with coefficients in $\mathbb{C}$, then there exists $a \in \mathbb{C}$ with $p(a) = 0$.

Proof. Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Consider the entire function $f(z) = 1/p(z)$. This function is then bounded as $p(z)$ goes to ∞ as z goes to infinity. By Liouville's Theorem, f (and thus p) is constant, which is a contradiction. ■

Due to the above, $\mathbb{C}$ is an algebraically closed field.

Corollary 2.22. If $p(z)$ is a polynomial and its roots are (p_j) with multiplicity k_j (for $1 \leq j \leq m$), then $p(z) = C(z-a_1)^{k_1}(z-a_2)^{k_2}\cdots(z-a_m)^{k_m}$ for some constant C, where $\sum k_j$ is the degree of p.

It is not too difficult to show that if $p(z)$ is a non-constant polynomial, then p is a surjective analytic function on $\mathbb{C}$. However, we know that the map $z \mapsto e^z$ is an entire function but there is no $b \in C$ such that $e^b = 0$. So, power series ("polynomials of infinite degree") cannot be thought of in the same way as ordinary polynomials (of finite degree). However, we shall see later that given a non-constant entire function f, there exists at most one $a \in \mathbb{C}$ that is not in the image of f. This is referred to as Little Picard's Theorem.

Theorem 2.23. Let G be a connected open set and $f : G \to \mathbb{C}$ be analytic. Then, the following are equivalent statements.

(a) f is identically zero.

(b) There exists $a \in \mathbb{C}$ such that for all $n \geq 0$, $f^{(n)}(a) = 0$.

(c) $\{z \in G : f(z) = 0\}$ has a limit point in G.

Proof. Clearly, (a) implies (b) and (c).
Next, let us show that (c) implies (b). Let $a \in G$ be a limit point of the zero set of f. Let $R > 0$ such that $B(a, R) \subseteq G$. Since a is a limit point of z and f is continuous, $f(a) = 0$. Let $n \geq 1$ such that $f^{(k)}(a) = 0$ for $k < n$ and $f^{(n)}(a) \neq 0$. Expanding f as a power series about a gives that

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k$$

for $|z - a| < R$ and $a_n \neq 0$. Let

$$g(z) = \sum_{k=n}^{\infty} a_k (z-a)^{k-n}.$$

Since g is continuous in $B(a, R)$ and $g(a) \neq 0$, let $r < R$ such that $g(z) \neq 0$ when $|z - a| < r$. Since a is a limit point of z, there exists b with $f(b) = 0$ and $0 < |a - b| < r$. This gives $0 = (b-a)^n g(b)$, so $g(b) = 0$, a contradiction. Therefore, no such n can be found and (b) is true.
Finally, let us show that (b) implies (a). Let

$$A = \{z \in G : f^{(n)}(z) = 0 \text{ for all } n \geq 0\}.$$

By the definition of (b), $A \neq \ emptyset$. We shall show that A is both open and closed in G, and by the connectedness of G is follows that A is the entirety of G. Showing that A is closed is direct – if $z \in \overline{A}$ and (z_k) a sequence such that $z_k \to z$, then since each $f^{(k)}$ is continuous, $f^{(n)}(z) = \lim f^{(n)}(z_k) = 0$ for all $n \geq 0$, and so $z \in A$. On the other hand, if $a \in A$, we can write $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = 0$ on $B(a, R)$ (for some $R > 0$), so $B(a, R) \subseteq A$ and A is open, completing the proof. ■

Corollary 2.24. If f, g are analytic on a region G, then $f \equiv g$ iff $\{z \in G : f(z) = g(z)\}$ has a limit point in G.

Corollary 2.25. If f is non-trivial and analytic on an open connected set G, then each zero of f has finite multiplicity. More explicitly, for each $a \in G$ with $f(a) = 0$, there is an integer $n \geq 1$ and an analytic function $g : G \to \mathbb{C}$ such that $g(a) \neq 0$ and $f(z) = (z-a)^n g(z)$ for all $z \in G$.

Proof. It is clear that there exists a largest $n \geq 1$ such that $f^{(k)}(a) = 0$ for all $k \leq n-1$. ∎

Corollary 2.26. If $f : G \to \mathbb{C}$ is non-trivial and analytic, and $a \in G$ with $f(a) = 0$, then there exists $R > 0$ such that $B(a, R) \subseteq G$, and $f(z) \neq 0$ for all $0 < |z-a| < R$.

The above follows from the fact that the zeros of f are isolated.

Theorem 2.27 (Maximum Modulus Theorem)**.** If G is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

That is, if $|f|$ attains its maximum, f is constant.

Proof. Let $\overline{B(a,r)} \subseteq G$ and $\gamma(t) = a + re^{\iota t}$ for $0 \leq t \leq 2\pi$. Then,

$$\begin{aligned} f(a) &= \frac{1}{2\pi\iota} \int_\gamma \frac{f(w)}{w-a} \, \mathrm{d}w \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{\iota t}) \, \mathrm{d}t. \end{aligned}$$

Therefore,

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{\iota t})| \, \mathrm{d}t \leq |f(a)|.$$

Therefore,

$$0 = \int_0^{2\pi} \left(|f(a)| - |f(a + re^{\iota t})| \right) \mathrm{d}t.$$

Since the integrand is continuous is non-negative, $|f(a)| = |f(a + re^{\iota t})|$ for all $t \in [0, 2\pi]$. If $f(a) = 0$, we are clearly done. Otherwise, since r was arbitrary, f maps any disk $B(a,R)$ to the circle $|z| = |f(a)|$. It may then be shown using the Cauchy-Riemann equations that f is constant on $B(a,R)$ and is equal to $f(a)$ for all $|z-a| < R$. Therefore, $f(z) = f(a)$ for all $z \in G$ since the zeros of $f - f(a)$ are not isolated. ∎

Recall that

$$\int_\gamma \frac{1}{z-a} \, \mathrm{d}z = 2\pi\iota n$$

if $\gamma(t) = a + e^{\iota n t}$ for $t \in [0, 2\pi]$. However, this property is not peculiar to the path γ, as shown by the following result.

Theorem 2.28. If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$, then

$$\frac{1}{2\pi\iota} \int_\gamma \frac{1}{z-a} \, \mathrm{d}z$$

is an integer.

Proof. Using Lemma 2.9, we may assume that γ is piecewise smooth (Why?).
Let us assume that γ is smooth. Define $g : [0,1] \to \mathbb{C}$ by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a}\,\mathrm{d}s.$$

Then, $g(0) = 0$ and $g(1) = \int_\gamma 1/(z-a)\,\mathrm{d}z$. We also have that

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - a}$$

for $0 \leq t \leq 1$. This gives that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-g(t)}(\gamma(t) - a)\right) = e^{-g(t)}\gamma'(t) - g'(t)e^{-g(t)}(\gamma(t) - a) = 0.$$

Therefore,

$$e^{-g(0)}(\gamma(0) - a) = e^{-g(1)}(\gamma(1) - a).$$

Because $\gamma(0) = \gamma(1)$ (the curve is closed) and $g(0) = 0$, $g(1) = 2\pi\iota n$ for some integer n. In the case where γ is piecewise-smooth, we can define g by integrating over each of the smooth intervals and the result follows near-identically. ∎

Definition 2.8. If γ is a closed rectifiable curve in $\mathbb{C}$ then for $a \notin \{\gamma\}$,

$$n(\gamma; a) = \frac{1}{2\pi\iota}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z$$

is called the *index* of γ with respect to the point a. It is also sometimes referred to as the *winding number* of γ around a.

Recall the definition of $(-\gamma)$ from (2.2), also denoted γ^{-1}. If γ and σ are curves on $[0,1]$ with $\gamma(1) = \sigma(0)$, $\gamma + \sigma$ is the curve

$$(\gamma + \sigma)(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq 1/2, \\ \sigma(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Proposition 2.29. If σ, γ are closed rectifiable curves with the same initial (and final) points, then

$$n(\gamma; a) = -n(-\gamma; a) \tag{2.5}$$

for all $a \notin \{\gamma\}$ and

$$n(\gamma + \sigma; a) = n(\gamma; a) + n(\sigma; a) \tag{2.6}$$

for all $a \notin \{\sigma\} \cup \{\gamma\}$.

We omit the proof of the above.

The reason for $n(\cdot;\cdot)$ being called the winding number is clear from what happens in the case of a circle. For $a + e^{2\pi\iota n t}$, then $n(\gamma; a) = n$ is the number of times this curve "winds" or "wraps" around a. In fact, if $|b - a| < 1$, $n(\gamma; b) = n$ and if $|b - a| > 1$, $n(\gamma; b) = 0$.

Theorem 2.30. Let γ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma; a)$ is constant for a belonging to a component of $G = \mathbb{C} \setminus \{\gamma\}$. Also, $n(\gamma; a) = 0$ for a belonging to the unbounded component of G.

Remark. By components, we mean maximal connected subsets of G.
Since $\{\gamma\}$ is compact, $\{z : |z| > R\} \subseteq G$ for sufficiently large R, so γ has precisely one unbounded component.

Proof. Define $f : G \to \mathbb{C}$ by $f(a) = n(\gamma; a)$. If we manage to show that f is continuous on G, we are done since the image of this map is a subset of the integers and each component is connected by definition.
Recall that components of G are open. Fix $a \in G$ and let $r = d(a; \{\gamma\}) > 0$. If $|a - b| < \delta < r/2$, then

$$\begin{aligned}
|f(a) - f(b)| &= \frac{1}{2\pi}\left|\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right) \mathrm{d}z\right| \\
&= \frac{1}{2\pi}\left|\int_\gamma \frac{a-b}{(z-a)(z-b)} \mathrm{d}z\right| \\
&\leq \frac{|a-b|}{2\pi} \int_\gamma \frac{1}{|z-a||z-b|} |\,\mathrm{d}z|.
\end{aligned}$$

■