Coordinate Hit-and-run

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Representations

Definition 1 (Representation)

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Representations can be thought of as group actions $G \to S_X$, with the additional specification that the images are not just bijections, they are isomorphisms.

Equivalence of representations

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Definition 2 (Equivalence)

Two representations $\varphi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ are said to be *equivalent* if there exists an isomorphism (an *equivalence*) $T : V \to W$ such that $\psi_g = T\varphi_g T^{-1}$ for all $g \in G$. If this is the case, we write $\varphi \sim \psi$.

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$$\begin{array}{ccc} V & \xrightarrow{\varphi_g} & V \\ \downarrow T & & \downarrow T \\ W & \xrightarrow{\psi_g} & W \end{array}$$

Irreducible representations

Definition 3 (Invariant subspace)

Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. A subspace $W \leq V$ is said to be *G-invariant* with respect to φ if for all $g \in G$ and $w \in W$, $\varphi_g(W) = W$.

Note that if $W \leq V$ is a G-invariant subspace, then $\varphi|_W : G \to \mathrm{GL}(W)$ defined by $(\varphi|_W)_g(w) = \varphi_g(w)$ is a representation!

Definition 4 (Irreducible representation)

A non-zero representation $\varphi : G \to \mathrm{GL}(V)$ is said to be *irreducible* if the only G-invariant subspaces of V are 0 and V.

Direct sum

Definition 5 (Direct sum)

Let $\varphi^{(1)} : G \to \mathrm{GL}(V_1)$ and $\varphi^{(2)} : G \to \mathrm{GL}(V_2)$ be representations. Then, their (external) *direct sum* is the representation $\varphi^{(1)} \oplus \varphi^{(2)} : G \to \mathrm{GL}(V_1 \oplus V_2)$ defined by

$$\left(\varphi^{(1)} \oplus \varphi^{(2)}\right)_g (v_1, v_2) = (\varphi^{(1)}_g(v_1), \varphi^{(1)}_g(v_2))$$

for all $g \in G$ and $(v_1, v_2) \in V_1 \oplus V_2$.

The above is more natural to picture using matrices.
If $V_1 = \mathrm{GL}_m(\mathbb{C})$ and $V_2 = \mathrm{GL}_n(\mathbb{C})$ above, then each $\varphi^{(i)}_g$ can be expressed as a matrix. The matrix in $\mathrm{GL}_{m+n}(\mathbb{C})$ corresponding to their direct sum is then given by

$$\left(\varphi^{(1)} \oplus \varphi^{(2)}\right)_g = \begin{pmatrix} \varphi^{(1)}_g & \\ & \varphi^{(2)}_g \end{pmatrix},$$

where the empty cells are appropriately sized 0 matrices.

More on decomposing representations

Definition 6 (Complete Reducibility)

Let G be a group. A representation $\varphi : G \to \mathrm{GL}(V)$ is said to be *completely reducible* if $V = V_1 \oplus \cdots \oplus V_n$ where each V_i is G-invariant and $\varphi|_{V_i}$ is irreducible for each i.

Note that even an irreducible representation is completely reducible. All we desire is that it can be "decomposed" into the direct sum of irreducible representations.

Definition 7 (Decomposability)

A non-zero representation φ is said to be *decomposable* if $V = V_1 \oplus V_2$ for some non-zero G-invariant subspaces $V_1, V_2 \leq V$. Otherwise, φ is said to be *indecomposable*.

Recall that a matrix $U \in \mathrm{GL}_n(\mathbb{C})$ is said to be *unitary* if $U^* U = I_n$.

Definition 8 (Unitary)

Let V be an inner product space. A representation $\varphi : G \to \mathrm{GL}(V)$ is said to be *unitary* if φ_g is unitary for every $g \in G$.

Lemma 9

Any representation of a finite group G is equivalent to a unitary representation.

Maschke's Theorem

Theorem 10 (Maschke's Theorem)

Every representation of a finite group is completely reducible.

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Definition 11 (Morphism)

Let $\varphi : G \to \mathrm{GL}(V)$ and $\rho : G \to \mathrm{GL}(W)$ be representations. A *morphism* from φ to ρ is a linear map $T : V \to W$ such that the following diagram commutes for all $g \in G$.

$$\begin{array}{ccc} V & \xrightarrow{\varphi_g} & V \\ \downarrow T & & \downarrow T \\ W & \xrightarrow{\rho_g} & W \end{array}$$

The set of all morphisms from φ to ρ is denoted $\mathrm{Hom}_G(\varphi, \rho)$.

Moreover, note that if T is an isomorphism (between V and W) in the above, it is an equivalence.

Lemma 12

Let φ, ρ be irreducible representations of a group G.

(a) If $\varphi \not\sim \rho$, $\mathrm{Hom}_G(\varphi, \rho) = 0$.

(b) $\mathrm{Hom}_G(\varphi, \varphi) = \{\lambda I : \lambda \in \mathbb{C}\}$.

Corollary 13

Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be pairwise inequivalent irreducible representations of G. Set

$$\varphi = \underbrace{\varphi^{(1)} \oplus \cdots \oplus \varphi^{(1)}}_{m_1} \oplus \cdots \oplus \underbrace{\varphi^{(s)} \oplus \cdots \oplus \varphi^{(s)}}_{m_s}.$$

Then, $\dim \mathrm{Hom}_G(\varphi^{(r)}, \varphi) = m_r$ for each r.

The group algebra

Definition 14

Let G be a group. Define the *group algebra* $L(G) = \mathbb{C}^G$. We endow it with the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Theorem 15

Let $\varphi : G \to U_n(\mathbb{C})$ and $\rho : G \to U_m(\mathbb{C})$ be inequivalent irreducible unitary representations of a group G. Then,

(a) $\langle \varphi_{ij}, \rho_{kl} \rangle = 0$.

(b) $\langle \varphi_{ij}, \varphi_{kl} \rangle = \begin{cases} 1/n, & (i,j) = (k,l), \\ 0, & \text{otherwise.} \end{cases}$

In particular, the set $\{\varphi_{ij} : 1 \leq i,j \leq n\} \cup \{\rho_{kl} : 1 \leq k,l \leq m\}$ is a linearly independent set.

Theorem 16

Let G be a (finite) group.

(a) There are finitely many equivalence classes of irreducible representations of G.

(b) Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a transversal of unitary irreducible representations of G. Set $d_i = \deg \varphi^{(i)}$. Then, the set of functions

$$\{\sqrt{d_k}\varphi_{ij}^{(k)} : 1 \leq k \leq s, 1 \leq i, j \leq d_k\}$$

is orthonormal.

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