MA 861 : COMBINATORICS I

Amit Rajaraman

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Contents

§0. Notation and Prerequisites

Given $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots, n\}$ and $[n]_0$ denotes the set $[n] \cup \{0\}$.
$S(n, k)$, a Stirling number of the second kind, is the number of partitions of $[n]$ into exactly k parts. $s(n, k)$, a Stirling number of the first kind, is the number of permutations of $[n]$ with exactly k cycles.

§1. Introduction

Exercise 1.1. Recall that the number of k-subsets of $[n]$ is $\binom{n}{k}$. Given a k-subset $S = \{x_1, \dots, x_k\}$ of $[n]$, we write $S_< = \{x_1, \dots, x_k\}_<$ to denote that $x_1 < x_2 < \cdots < x_k$. Determine the number of k-subsets $\{x_1, \dots, x_k\}_<$ of $[n]$ such that $x_i \cong i \mod 2$.

For example, for $n = 6$ and $k = 3$, we have the subsets $\{1, 4, 5\}, \{1, 2, 3\}, \{1, 2, 5\}, \{3, 4, 5\}$.
Broadly, there are three types of "answers": a formula, a recurrence, and a generating function. A great example of the second and third is the following.
$p(n)$, the number of number partitions of n, is given by the generating function

$$\sum_{n \geq 0} p(n) x^n = \prod_{i \geq 1} \frac{1}{1 - x^i}.$$

Using this, a recursion may be obtained as well. We do *not* plug in values for x in the above. We merely look at the coefficient of x^n in it. We want the coefficient to be a finite sum for all n. If it is an infinite sum, convergence issues may arise.

1.1. Counting in $\mathfrak{S}_n$

Recall that $\mathfrak{S}_n$ is generated by transpositions. A transposition (i, j) is a permutation σ defined by

$$\sigma(k) = \begin{cases} j, & k = i, \\ i, & k = j, \\ k, & \text{otherwise.} \end{cases}$$

In fact, $\mathfrak{S}_n$ is generated by the set of just "adjacent transpositions" $S_i = (i, i+1)$ for $1 \leq i < n$. We have

$$\begin{aligned} S_i^2 &= \mathrm{Id} \\ S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1} \\ S_i S_j &= S_j S_i \text{ if } |i - j| > 2. \end{aligned}$$

Definition 1.1. Given a permutation $\pi \in \mathfrak{S}_n$, define the *length* $\ell(\pi)$ of π to be the smallest k such that there exist adjacent transpositions $\sigma_1, \cdots, \sigma_k$ such that $\pi = \sigma_1 \cdots \sigma_k$.

Proposition 1.1. Consider the *inversion number* $\mathrm{inv}(\pi)$ of a permutation, defined by

$$\mathrm{inv}(\pi) = \left| \{1 \leq i \leq j \leq n : \pi_i > \pi_j\} \right|.$$

Then, $\ell(\pi) = \mathrm{inv}(\pi)$.

Definition 1.2. The *sign* of a permutation π is defined by $\mathrm{sign}(\pi) = (-1)^{\mathrm{inv}(\pi)}$. Equivalently,

$$\mathrm{sign}(\pi) = \frac{\prod_{1 \leq i < j \leq n} (x_{pi_i} - x_{\pi_j})}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}.$$

It is straightforward to see that for all $\pi \in \mathfrak{S}_n$, $0 \leq \operatorname{inv}(\pi) \leq \binom{n}{2}$.

Proposition 1.2. Consider $\operatorname{inv}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{inv}(\pi)}$. Then,

$$\operatorname{inv}_n(q) = \prod_{1 \leq m \leq n} [n]_q,$$

where

$$[m]_q = \begin{cases} 1 + q + \cdots + q^{m-1}, & m \geq 1, \\ 0, & m = 0. \end{cases}$$

This quantity $[m]_q$ is called the q-analogue of m, and similarly, the q-analogue of $n!$ is $\prod_{i=1}^{n} [m]_q$ (this is slightly vague). Note in particular that $n! = \operatorname{inv}_n(1)$.

Proof. We prove this by induction. It is easily verified for $n = 2$.
Take $\sigma \in \mathfrak{S}_{n-1}$. There are n "gaps" where n can be "placed" in σ to get a permutation in $\mathfrak{S}_n$. If we place it in the ith position from the end (for $0 \leq i \leq n-1$), the inversion number of the newly obtained permutation is i more than the inversion number of σ.
As a result,

$$\operatorname{inv}_n(q) = \operatorname{inv}_{n-1}(q) + q \operatorname{inv}_{n-1}(q) + q^2 \operatorname{inv}_{n-1}(q) + \cdots + q^{n-1} \operatorname{inv}_{n-1}(q) = [n]_q \operatorname{inv}_{n-1}(q),$$

where the $q^i \operatorname{inv}_{n-1}(q)$ term corresponds to the case where n is placed in the ith position from the end. The required follows by the inductive hypothesis. ■

Definition 1.3 (Descent)**.** For $\pi \in \mathfrak{S}_n$, define the *descents* $\operatorname{DES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$, $\operatorname{des}(\pi) = |\operatorname{DES}(\pi)|$, and $\operatorname{maj}(\pi) = \sum_{i \in \operatorname{DES}(\pi)} i$.

There are central limit theorems for many of these parameters, which we shall not study.
A permutation π has $\operatorname{des}(\pi) + 1$ many "increasing runs".
For example, for the permutation $\pi = (1 \mapsto 5, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 3) \in \mathfrak{S}_6$, $\operatorname{DES}(\pi) = \{1, 4, 5\}$, $\operatorname{des}(\pi) = 3$, and $\operatorname{maj}(\pi) = 10$.

Proposition 1.3. The distribution of $\operatorname{maj}(\pi)$ over $\mathfrak{S}_n$ is the same as that of $\operatorname{inv}(\pi)$. Equivalently,

$$\operatorname{maj}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{maj}(\pi)} = \prod_{m=1}^{n} [m]_q = \operatorname{inv}_n(q).$$

This result took nearly 50 years to prove!

Proof. The strategy is similar to that of Proposition 1.2. Let $\pi \in \mathfrak{S}_{n-1}$. As before, there are n positions to insert n.

- Label the positions of descents of σ and the last position from right ot left as $0, 1, \ldots, \operatorname{des}(\pi)$.
- Label the remaining positions from left to right as $\operatorname{des}(\pi) + 1, \ldots, n-1$.

We claim that inserting n at a position increases maj by the labelled amount.
If inserted anywhere, all the descent positions starting from there increase by 1. This explains why the increase is equal to the labelled quantity for positions that are descents, since no new descents are introduced. In the case where we insert it in a position of non-descent, we further introduce a new descent at the position of insertion of n, which explains why the increase is equal tot he labelled quantity for positions that are not descents.
The remainder of the proof is identical to that of Proposition 1.2, since the increases are in bijection with $[n-1]_0$. ■

Definition 1.4. A parameter $f : \mathfrak{S}_n \to \mathbb{R}$ of permutations such that

$$\sum_{\pi \in \mathfrak{S}_n} q^{f(\pi)} = \prod_{1 \leq m \leq n} [m]_q$$

is said to be *Mahonian*.

As we saw in **??** 1.2**??** 1.3, both inv and maj are Mahonian.

1.2. Counting spanning trees

Problem. Count the number of spanning trees in an arbitrary (finite) graph G.

This was solved by Kirchhoff using the Matrix Tree Theorem.

Theorem 1.4 (Matrix Tree Theorem)**.** Consider the *Laplacian* $L = D - A$ of a graph G, where A is its adjacency matrix and D is a diagonal matrix with the diagonal entries being the degrees of the vertices. The determinant of any $(n-1) \times (n-1)$ submatrix of L obtained by omitting any arbirary row and column is equal to the number of spanning trees of G.

In particular, when $G = K_n$, we end up getting the following.

Theorem 1.5 (Cayley's Theorem)**.** The number of spanningtrees in K_n is n^{n-2}.

One proof by Prüfer gives an explicit bijection between spanning trees and sequences $(v_1, \ldots, v_{n-2})$ of vertices in G. Another proof is of course using the matrix tree theorem, which reduces it to a simple determinant calculation. Joyal gave another bijection between elements of the form (T, u, v) where T is a spanning tree and u, v are vertices in G, and functions from $[n] \to [n]$.

The proof we give uses exponential generating functions. Recall the following result, which we give without proof. Interested readers may consult Corollary 5.1.6 of [SF99] for further details.

Theorem 1.6 (Exponential Formula)**.** Let $\{f_n\}$ be a sequence with exponential generating function

$$F(x) = \sum_{n \geq 1} f_n \frac{x^n}{n!}.$$

Define the sequence h_n by

$$h_n = \sum_{\substack{\pi \in \mathrm{SetPartn}([n]) \\ \pi = \{S_1, \ldots, S_k\}}} f_{|S_1|} f_{|S_2|} \cdots f_{|S_k|}$$

and $h_0 = 1$, and let

$$H(x) = \sum_{n \geq 0} h_n \frac{x^n}{n!}.$$

Then,

$$H(x) = \exp(F(x))$$

Note that the summation of F is for $n \geq 1$, because we may assume that $f_0 = 0$ since f_0 does not appear in the expression of any h_n.

Definition 1.5 (Compositional inverse)**.** Generating functions F and G are said to be *compositional inverses* (of each other) if $F(G(x)) = G(F(x)) = x$.

Let

$$F(x) = \sum_{n \geq 0} f_n x^n \text{ and } G(x) = \sum_{n \geq 0} g_n x^n$$

be compositional inverses of each other. It is reasonably straightforward to show that $f_0 = g_0 = 0$ and $f_1, g_1 \neq 0$. The first condition implies that the coefficient of any x^n in $F \circ G$ (or $G \circ F$) is finite.

Theorem 1.7 (Lagrange Inversion Theorem)**.** Let

$$F(x) = \sum_{n \geq 0} f_n x^n \text{ and } G(x) = \sum_{n \geq 0} g_n x^n$$

be compositional inverses of each other. Then, ng_n is the coefficient of $1/x$ in $(1/F(x))^n$.

Equivalently, ng_n is the coefficient of x^{n-1} in $(x/F(x))^n$.

Proof. We have

$$x = G(F(x)) = \sum_{i \geq 0} g_i F(x)^i.$$

Differentiating,

$$1 = \sum_{i \geq 0} g_i i F(x)^{i-1} F'(x).$$

As a result,

$$\left(\frac{1}{F(x)}\right)^n = \sum_{i \geq 0} g_i i F(x)^{i-1-n} F'(x).$$

Whenever $i \neq n$, the coefficient of $1/x$ in $F(x)^{i-1-n} F'(x) = \left(F(x)^{i-n}/(i-n)\right)'$ is zero. Indeed, recall that the coefficient of $1/x$ in the derivative of any power series with possibly negative exponents is zero.
As a result, the coefficient of $1/x$ in $(1/F(x))^n$ is equal to the coefficient of $1/x$ in $g_n n F'(x)/F(x)$. We have

$$\frac{F'(x)}{F(x)} = \frac{f_1 + 2f_2 x + \cdots}{f_1 x + f_2 x^2 + \cdots}.$$

The constant term in this is $f_1/f_1 = 1$, and the desideratum follows. ■

At long last, let us return to Cayley's Theorem.

Proof of Cayley's Theorem. Instead of looking at the number T_n of spanning trees, we shall look at RT_n, the number of *rooted* spanning trees. Clearly, $\mathrm{RT}_n = n\mathrm{T}_n$.
Define RF_n to be the number of rooted forests on $[n]$ and let

$$\mathrm{RF}(x) = \sum_{n \geq 0} \mathrm{RF}_n \frac{x^n}{n!}$$

$$\mathrm{RT}(x) = \sum_{n \geq 0} \mathrm{RT}_n \frac{x^n}{n!}.$$

Using Theorem 1.6, it is not too difficult to see that

$$\mathrm{RF}(x) = \exp(\mathrm{RT}(x)). \tag{1.1}$$

Claim (Polya). $\mathrm{RT}_{n+1} = (n+1)\mathrm{RF}_n$.
Indeed, any rooted tree on K_{n+1} may be obtained from a rooted forest F on K_n by adding a new vertex v, adding the edge between each root in F and v to the spanning tree, removing the "root status" from all vertices except v. v can be labelled in $n+1$ ways, so we are done.

As a result,

$$\mathrm{RF}(x) = \sum_{n \geq 0} \frac{\mathrm{RT}_{n+1}}{n+1} \cdot \frac{x^n}{n!} = \frac{1}{x}\mathrm{RT}(x). \tag{1.2}$$

Combining Equations (1.1) and (1.2),

$$\mathrm{RT}(x) = x\exp(\mathrm{RT}(x)).$$

That is, RT is the compositional inverse of $x \mapsto xe^{-x}$. Now, we use the Lagrange Inversion Theorem to get that $n\mathrm{RT}_n/n!$ is equal to the coefficient of x^{n-1} in $(x/xe^{-x})^n = e^{nx}$, which is $n^{n-1}/(n-1)!$. Therefore, $\mathrm{T}_n = \mathrm{RT}_n/n = n^{n-2}$ and we are done. ∎

1.3. Chebyshev polynomials

We would like a polynomial $T_n(x)$ such that $T_n(\cos\theta) = \cos(n\theta)$. Why does such a polynomial even exist? Recall that

$$(\cos\theta + \iota\sin\theta)^n = \cos n\theta + \iota\sin n\theta.$$

Since the real part of the left only has even powers of $\sin$, we can convert it to a polynomial of $\cos\theta$s alone.
For example,

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1. \end{aligned}$$

Proposition 1.8.

$$T_n(x) = \begin{cases} 1, & n = 0, \\ x, & n = 1, \\ 2xT_{n-1}(x) - T_{n-2}(x), & n \geq 2. \end{cases}$$

We give the proof of the above in the solution to Proposition 1.9.

Proposition 1.9. $T_0(x) = 1$, $T_1(x) = x$, and for $n \geq 2$,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

Proof. Let $\cos\theta = x$. We have

$$\begin{aligned} T_n(x) = \cos n\theta &= \cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta \\ &= xT_{n-1}(x) - (\sin(n-2)\theta\cos\theta + \cos(n-2)\theta\sin\theta)\sin\theta \\ &= xT_{n-1}(x) - T_{n-2}(x)(1-x^2) - x(\sin\theta\sin(n-2)\theta) \\ &= xT_{n-1}(x) + x^2T_{n-2}(x) - T_{n-2}(x) - x(\cos\theta\cos(n-2)\theta - \cos(n-1)\theta) \\ &= 2xT_{n-1}(x) - T_{n-2}(x). \end{aligned}$$

∎

Definition 1.6 (Chebyshev polynomials)**.** The nth *Chebyshev polynomial of the first kind* T_n is defined as above. The nth *Chebyshev polynomial of the second kind* U_n is defined by

$$U_n(x) = \begin{cases} 1, & n = 0, \\ 2x, & n = 1, \\ 2xU_{n-1}(x) - U_{n-2}(x), & n \geq 2. \end{cases}$$

Consider the number of tilings of a $1 \times n$ board B_n using squares (1×1 pieces) and dimers (1×2 pieces). It is not too difficult to show that this corresponds to the Fibonacci numbers.

Now, instead consider a *weighted* version of this problem, where we give squares a weight of $2x$ and dimers a weight of -1. The weight $\mathrm{wt}(T)$ of a given tiling T is equal to the product of the weights of the pieces used. Then, the Chebyshev polynomial U_n is just the sum of the weights of all tilings of B_n!

$$U_n(x) = \sum_{\text{tilings } T \text{ of } B_n} \mathrm{wt}(T).$$

Similar to this, we can get a combinatorial model for T_n as well, with the only difference being that a square piece has weight x if it is at the lefmost $(1,1)$ position.

Given a tiling T, let $S(T)$ and $D(T)$ be the number of squares and dimers in the tiling respectively. In general, define

$$F_n(s,t) = \sum_{\text{tilings } T \text{ of } B_n} s^{S(T)} t^{D(T)}.$$

Then,

$$\begin{aligned} F_0(s,t) &= 1, \\ F_1(s,t) &= s, \\ F_n(s,t) &= sF_{n-1}(s,t) + tF_{n-2}(s,t). \end{aligned}$$

1.4. More on q-analogues

Recall the definition of $[n]_q! = \prod_{i=1}^n [i]_q$. Inspired by this, define

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q[n-k]_q!}.$$

This is clearly a rational function of q. It turns out that this is a polynomial in q! For example,

$$\binom{5}{2}_q = \frac{[5]_q[4]_q}{[2]_q[1]_q} = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

Recall that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Exercise 1.2. Show that

$$\binom{n}{q}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q.$$

These are called the q-Pascal's recurrences.

Corollary 1.10. $\binom{n}{k}_q$ is a polynomial. in q with non-negative coefficients.

It turns out that the coefficients of the polynomial are unimodal and symmetric! We do not prove this, the reader can see _ for more details.
A natural question to ask then is: what do the coefficients of this polynomial count?
Let $\binom{n}{k}_q = f_{n,k}(q) = \sum_{r \geq 0} a_{n,k}^{(r)} q^r$. Can we have

$$\binom{n}{k}_q = \sum_{T \in \binom{[n]}{k}} q^{\text{parameter}(T)}?$$

$a_{n,k}^{(r)}$ then just counts the number of T with the given parameter value.
Recall that $\binom{n}{k}$ is the number of paths from $(0,0)$ to $(n-k,k)$ if only upwards and rightwards movements on the integer lattice $\mathbb{Z}^2$ are allowed. Let P be such a path.
Consider the portion of the box above P. This can be viewed as the Ferrer diagram of some number partition $\lambda(P)$. $\lambda(P)$ has at most k parts, and no part is of size more than $n-k$. In fact, *all* such partitions correspond to some path! What number is $\lambda(P)$ a number partition of? Denote this number as $|\lambda(P)|$. Let $S_{n,k}$ be the set of all paths of the mentioned form.

Theorem 1.11.

$$\sum_{P \in S_{n,k}} q^{|\lambda(P)|} = \binom{n}{k}_q.$$

Surprisingly, the proof of the above is near-straightforward using the q-Pascal recurrence – merely consider two cases depending on whether the first step of the path is right or upwards.

1.5. Derivative polynomials

We begin this section by recalling the following rather interesting result.
Define the *Bell polynomial* $B_{n,k}$ by

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{j_1!(1!)^{j_1} j_2!(2!)^{j_2} \cdots j_{n-k+1}!((n-k+1)!)^{n-k+1}} \cdot x_1^{j_1} x_2^{j_2} \cdots x_{n-k+1}^{j_{n-k+1}},$$

where the summation is taken over all indices $j_1, \ldots, j_{n-k+1}$ of non-negative integers such that

$$k = j_1 + j_2 + \cdots + j_{n-k+1} \text{ and}$$
$$n = j_1 + 2j_2 + 3j_3 + \cdots + (n-k+1)j_{n-k+1}.$$

This has a natural correspondence to the Stirling numbers of the second kind, with j_i representing the number of partitions of size i. In particular, the sum of coefficients of $B_{n,k}$ is $S_{n,k}$.

Proposition 1.12 (Faà di Bruno's Formula,[dB55]).

$$D^n f(g(x)) = \sum f^{(k)}(g(x)) \cdot B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)).$$

To illustrate this better, let us look at the first few derivatives explicitly. Dropping the (x) on the right to make the notation more succinct, we have

$$\begin{aligned} Df(g(x)) &= f'(g)g' \\ D^2f(g(x)) &= f''(g)(g')^2 + f'(g)g'' \\ D^3f(g(x)) &= f'''(g)(g')^3 + 3f''(g)g'g'' + f''(g)g'''. \end{aligned}$$

Consider the partitions of $\{1,2,3\}$, given by $1|2|3$, $12|3$, $13|2$, $23|1$, and 123. The number of partitions of $[n]$ with n_i parts of size i for each i neatly corresponds to the coefficient of $\prod_i (g^{(i)})^{n_i}$!
Let $y = f(x)$. If $Dy = p(f(x))$ for some polynomial p, then $D^n y$ is a polynomial of f as well.
Suppose that $D^n y = p_n(y)$ for some sequence of polynomials (p_n).

Exercise 1.3. Show that

$$\begin{aligned} p_0(y) &= y \\ p_n(y) &= \begin{cases} y & n = 0 \\ p_{n-1}(y) \cdot p_1(y) & n \geq 1. \end{cases} \end{aligned}$$

For the remainder of this section, set $y = \tan x$ and $z = \sec x$. Then, $Dy = 1 + y^2 = z^2$ and $Dz = yz$. It is not difficult to see that

$$\begin{aligned} D^2 y &= 2yz^2 \\ D^3 y &= 4y^2z^2 + 2z^4 \\ D^4 y &= 8y^3z^2 + 16yz^4 \end{aligned}$$

Exercise 1.4. With y, z defined as above, show that

1. $D^n y$ is a homogeneous polynomial in y, z of degree $(n+1)$.
2. $D^n y$ has only terms with even exponents of z.

Corollary 1.13. We can write $D^n y = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} W_{n,k} z^{2k+2} y^{n-2k-1}$.

Exercise 1.5. Prove or disprove that

$$\sum W_{n,k} = n!.$$

Again, we ask the question: is there some parameter on $\pi \in \mathfrak{S}_n$ such that

$$W_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} W_{n,k} x^k = \sum_{\pi \in \mathfrak{S}_n} x^{\text{parameter}(\pi)}?$$

Definition 1.7 (Peak)**.** Given a permutation $\pi \in \mathfrak{S}_n$, we say that $i \in [n] \setminus \{1, n\}$ is a *peak* of π if $\pi_i > \pi_{i-1}$ and $\pi_i > \pi_{i+1}$. Denote the set of peaks of π by $\mathrm{Peak}(\pi)$, $\mathrm{pk}(\pi) = |\mathrm{Peak}(\pi)|$ the number of peaks.

Lemma 1.14. Show that

$$W_{n,k} = (2k+2)W_{n-1,k} + (n-2k)W_{n-1,k-1}.$$

for $n, k \geq 1$.

Proof. We have

$$\begin{aligned}
D^n y &= D \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} W_{n-1,k} z^{2k+2} y^{n-2k-2} \\
&= \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (2k+2) W_{n-1,k} z^{2k+1} \cdot zy \cdot y^{n-2k-2} + (n-2k-2) W_{n-1,k} z^{2k+2} y^{n-2k-3} \cdot z^2 \\
&= \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (2k+2) W_{n-1,k} z^{2k+2} y^{n-2k-1} + (n-2k-2) z^{2k+4} y^{n-2k-3}.
\end{aligned}$$

The required follows. ■

Theorem 1.15.

$$W_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{pk}(\pi)}.$$

Proof. Let Y_n be the polynomial on the right, and let $Y_{n,k}$ be its coefficients. It is easily checked that $Y_{n,k}$ and $W_{n,k}$ are equal for $n = 0$ or $k = 0$. To prove the statement, we shall merely show that $Y_{n,k}$ satisfies the recurrence of Lemma 1.14 too.
Similar to what we did in earlier proofs such as those of **??** 1.2**??** 1.3, let σ be a permutation in $\mathfrak{S}_{n-1}$.
We shall use it to get a permutation $\pi \in \mathfrak{S}_n$ by "inserting" n at one of the n possible positions. If we insert it at the position of a non-peak of σ, the number of peaks increases by one. If we insert it before or after the position of a peak, the number of peaks stays the same. Since peaks cannot occur immediately after each other, we can insert it at precisely $2k+2$ positions while ensuring that the number of peaks does not increase (the extra 2 is for the extreme positions), and so at $n-2k-2$ positions which increases the number of peaks by one. Therefore,

$$Y_{n,k} = (2k+2)Y_{n-1,k} + (n-2(k-1)-2)Y_{n-1,k-1} = (2k+2)Y_{n-1,k} + (n-2k)Y_{n-1,k-1}.$$

■

1.6. Matching theory

Definition 1.8 (Matching)**.** Given a graph $G = (V, E)$, a *matching* in G is a collection $M \subseteq E$ of edges such that for any distinct $e_1, e_2 \in M$, $e_1 \cap e_2 = \varnothing$.
The number of k-sized matchings is denoted $m_k(G)$. Define the *matching polynomial*

$$\mathrm{Match}_G(x) = \sum_{k \geq 0} (-1)^k m_k(G) x^{n-2k}$$

Some books call the above the "defect" matching polynomial, taking the actual matching polynomial as $p(x) = \sum_k m_k(G) x^k$. Note that $\mathrm{Match}_G(x) = x^n p(-1/x^2)$.
Clearly, $N_G(x) = x^n \cdot \mathrm{Match}_G(-1/x^2)$.

There is a very rich literature regarding matching theory. One work that set off a frenzy of results in related areas was [Edm65], which gave a polynomial-time algorithm to get a maximum weight matching in any graph. It does so by looking at the polytope in $\mathbb{R}^E$ that is the convex hull of the indicator functions of all matchings. It is worth noting that while there is a polynomial time algorithm to find a maximum weight matching, the problem of determining the number of maximum matchings in a graph is #P-complete. Consequently, no polynomial time algorithm is known to determine $m_k(G)$ given a graph G.

Before moving on, we give some simple lemmas about the matching polynomial.

Lemma 1.16.

(a) If G and H are vertex-disjoint graphs,

$$\mathrm{Match}_{G\cup H}(x) = \mathrm{Match}_G(x)\,\mathrm{Match}_H(x).$$

(b) Given a graph G and vertex $v \in G$,

$$\mathrm{Match}_G(x) = x\,\mathrm{Match}_{G\setminus\{v\}}(x) - \sum_{u:u\leftrightarrow v}\mathrm{Match}_{G\setminus\{u,v\}}(x).$$

(c) Given a graph G and edge $e = \{u,v\} \in G$,

$$\mathrm{Match}_G(x) = \mathrm{Match}_{G\setminus\{e\}}(x) - \mathrm{Match}_{G\setminus\{u,v\}}(x).$$

Proof. We omit the proof of (a) as it is straightforward.

(b) Let M be a matching of size k on G. If M does not have an edge incident on v, it is a matching of size k on $G\setminus\{v\}$. Otherwise, there is some edge $e = \{u,v\} \ni M$, and $M\setminus\{e\}$ is a matching on $G\setminus\{u,v\}$. As a result,

$$m_k(G) = m_k(G\setminus\{v\}) + \sum_{u:u\leftrightarrow v} m_{k-1}(G\setminus\{u,v\}).$$

Multiplying with $(-1)^k x^{n-2k}$ and summing over k,

$$\begin{aligned}\mathrm{Match}_G(x) &= \sum_k (-1)^k x\cdot x^{(n-1)-2k} m_k(G\setminus\{v\}) - \sum_{u:u\leftrightarrow v}(-1)^{k-1}x^{(n-2)-2(k-1)}m_k(G\setminus\{u,v\})\\ &= x\,\mathrm{Match}_{G\setminus\{v\}} - \sum_{u:u\leftrightarrow v}\mathrm{Match}_{G\setminus\{u,v\}}(x).\end{aligned}$$

(c) Similar to (b), let M be a matching of size k on G. If M does not have e, it is a matching of size k on $G\setminus e$. Otherwise, $M\setminus\{e\}$ is a matching on $G\setminus\{u,v\}$. So,

$$m_k(G) = m_k(G\setminus\{e\}) + m_{k-1}(G\setminus\{u,v\}).$$

Multiplying with $(-1)^k x^{n-2k}$ and summing over k,

$$\begin{aligned}\mathrm{Match}_G(x) &= \sum_k (-1)^k x^{n-2k} m_k(G\setminus\{e\}) - (-1)^{k-1}x^{(n-2)-2(k-1)}m_{k-1}(G\setminus\{u,v\})\\ &= \mathrm{Match}_{G\setminus\{e\}}(x) - \mathrm{Match}_{G\setminus\{u,v\}}(x).\end{aligned}$$

■

Proposition 1.17.

1. $m_k(P_n) = \binom{n-k}{k}$.
2. $m_k(C_n) = \frac{n}{n-k}\binom{n-k}{k}$.

3. $m_k(K_n) = \binom{n}{2k} \cdot \frac{(2k)!}{2^k k!}$.
4. $m_k(K_{n,n}) = \binom{n}{k}^2 k!$.

Proof.

1. Collapse every edge in a matching to its left endpoint, and "mark" the collapsed vertices. This results in a path with $n-k$ vertices with k marked vertices. This process of marking the vertices using the matching is reversible, and $m_k(G) = \binom{n-k}{k}$.
2. Fix some edge e. e is absent in exactly $(n-k)/n$ of the k-matchings of C_n. In this case, the remaining matching forms a matching on $C_n \setminus \{e\}$, which is isomorphic to P_n. Therefore, $(n-k)/n m_k(C_n) = m_k(P_n) = \binom{n-k}{k}$.
3. A k-matching of K_n is obtained by choosing $2k$ vertices (done in $\binom{n}{2k}$) ways, putting the $2k$ vertices in k indistinguishable "boxes" by putting 2 in each (this can be done in $(2k)!/k!2^k$ ways).
4. A k-matching is obtained by choosing k vertices from each side of the bipartite graph (done in $\binom{n}{k}^2$ ways), then assigning each vertex on the left side a vertex on the right that it is joined to in the matching (done in $k!$ ways). ∎

Theorem 1.18. Given a graph G, all roots of $\mathrm{Match}_G(x)$ are real.

The version of the proof of the above we give is due to Godsil [GG81].

Proof. Using Lemma 1.16(a), we may assume that G is connected.
We first prove the result for the case where G is a tree T. To prove this, we shall prove that $\mathrm{Match}_T(x)$ is the characteristic polynomial $\det(xI - A)$ of the adjacency matrix A of T(!); the result then follows since A is a real symmetric matrix and thus has real eigenvalues.
Let $xI - A = (b_{ij})$. We have

$$\mathrm{Charpoly}(A) = \sum_{\pi \in \mathfrak{S}_n} \mathrm{sign}(\pi) \prod_{i=1}^{n} b_{i\pi(i)}.$$

First, we claim that if $\pi \in \mathfrak{S}_n$ has a cycle of length greater than 2, then the term corresponding to π on the right will be zero. In other words, the term is zero if π is not an involution. Indeed, this follows immediately since G has no cycles (of length ≥ 3). As a result, if $(i_1, i_2, \ldots, i_t)$ were a cycle in π, then there must be some j such that $\{i_j i_{j+1}\}$ is not an edge in G and $b_{i_j\sigma(i_j)} = 0$.
Suppose that some $\pi \in \mathfrak{S}_n$ has k 2-cycles and $(n-2k)$ fixed points, and also has the term on the right being nonzero. We have $\mathrm{sign}(\pi) = (-1)^{(n-(k+n-2k))} = (-1)^k$. Suppose that the k 2-cycles are $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$. We have $b_{i_r j_r} = b_{j_r i_r} = (-1)$ so $b_{i_r j_r} b_{j_r i_r} = 1$, and also that no i_r (or j_r) is equal to any other i_s (or j_s). That is, the edges constituted by $\{i_r, j_r\}$ form a matching of size k! Therefore,

$$\mathrm{Charpoly}(A) = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \pi \text{ an involution}}} \mathrm{sign}(\pi) \prod_{i=1}^{n} b_{i\pi(i)} = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \pi \text{ an involution}}} (-1)^k x^{n-2k} = \sum_{\text{matchings } M} (-1)^{|M|} x^{n-2|M|} = \mathrm{Match}_T(x).$$

For a general graph, we come up with an associated tree, and show that the matching polynomial of our graph divides the matching polynomial of the tree. We do not give this proof. ∎

§2. Problem Sheets

2.1. Problem Sheet 1

Problem 2.1.1. Let $S(n,k)$ and $s(n,k)$ be Stirling numbers of the second and first kind respectively. Show that for all n,k, we have $s(n,k) \geq S(n,k)$.

Solution

Let $X_{S(n,k)}$ be the set of partitions of $[n]$ into exactly k parts and $X_{s(n,k)}$ the number of permutations of $[n]$ with exactly k cycles. Recall that by definition, $|X_{S(n,k)}| = S(n,k)$ and $|X_{s(n,k)}| = s(n,k)$. It suffices to demonstrate an injection f from $X_{S(n,k)}$ to $X_{s(n,k)}$. We do so as follows. Let $\{\{x_{1,1},\ldots,x_{1,n_1}\},\ldots,\{x_{k,1},\ldots,x_{1,n_k}\}\}$ be a partition of $[n]$ into exactly k parts, where $x_{i,j_1} < x_{i,j_2}$ for $j_1 < j_2$. Then, we have a corresponding permutation of $[n]$ with exactly k cycles given by $(x_{1,1},\ldots,x_{1,n_1})\cdots(x_{k,1},\ldots,x_{k,n_k})$. This map is clearly an injection, so we are done.

Problem 2.1.2. Show that

$$S(n,k) = \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{r!(k-r)!}.$$

Solution

$S(n,k)$ is merely $1/k!$ times the number of surjective functions from $[n]$ to $[k]$ (because the ordering of the partitions does not matter). The set of functions that are *not* surjective is

$$\bigcup_{i\in[n]} \{f \in [k]^{[n]} : i \notin \operatorname{Im}(f)\}.$$

The size of the above is quite easily determined by the inclusion-exclusion principle to get

$$k^n - k!S(n,k) = \sum_{r=1}^{k-1} (-1)^{r+1} \cdot \underbrace{\binom{k}{r}}_{\text{choosing } r \text{ elements in } [k] \text{ to ``avoid''}} \cdot \underbrace{(k-r)^n}_{\text{counting functions that avoid the chosen}},$$

and the desideratum immediately follows.

Problem 2.1.3. Let $A_n(y) = \sum_k S(n,k)y^k$. Show that $A_n(y) = (y + yD)^n 1$ where $D = \frac{\mathrm{d}}{\mathrm{d}y}$ is the derivative operator.

Solution

First, recall that $S(n+1,k+1) = S(n,k) + (k+1)\cdot S(n,k+1)$ – the $S(n,k)$ corresponds of those partitions where $n+1$ is in a part of its own, and the $(k+1)S(n,k+1)$ corresponds to those partitions where this is not the case, so we can consider any partition of $[n]$ into $k+1$ parts, then decide which of the $k+1$ parts to place $n+1$ in.

We have

$$\begin{aligned}(y+yD)A_n(y) &= (y+yD)\sum_{k=1}^{n} S(n,k)y^k \\ &= \sum_{k=1}^{n} S(n,k)(y^{k+1}+ky^k) \\ &= S(n,1)y + \sum_{k=2}^{n} y^k(S(n,k-1)+(k-1)S(n,k)) + S(n,n)y^{n+1} \\ &= S(n+1,1)y + \sum_{k=2}^{n} S(n+1,k)y^k + S(n+1,n+1)y^{n+1} = A_{n+1}(y).\end{aligned}$$

The required follows inductively.

Problem 2.1.4. Let D_n be the number of derangements in $\mathfrak{S}_n$ and let $D(x) = \sum_{n\geq 0} D_n x^n/n!$ be its egf. Determine $D(x)$.

Solution

A permutation $\pi \in \mathfrak{S}_n$ is a derangement iff it has no cycles of length 1. Define $f : \mathbb{N} \to \mathbb{N}_0$ by

$$f(k) = \begin{cases} 0, & k = 1, \\ 1, & \text{otherwise.} \end{cases}$$

By the earlier observation, $\pi \in \mathfrak{S}_n$ is a derangement iff $f(|C_1|)\cdots f(|C_k|) = 1$ where $C_1, \ldots, C_k$ are the cycles of π. Using Corollary 5.1.9 in [SF99], we get that

$$D(x) = \exp\left(\sum_{n\geq 2} \frac{x^n}{n}\right) = \exp\big(-x - \log(1-x)\big) = \frac{e^{-x}}{1-x}.$$

Problem 2.1.5. Let $s(n,2)$ denote the number of $\pi \in \mathfrak{S}_n$ with 2 cycles in its cyclic decomposition and let H_n denote the nth harmonic number. Show that $s(n+1,2) = H_n \times n!$.

Solution

It is easily checkable that the number of cyclic permutations of $[k]$ is $(k-1)!$. We have

$$\begin{aligned}s(n+1,2) &= \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n+1}{k}(k-1)!(n+1-k-1)! \\ &= \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(n+1)!}{k(n+1-k)} \\ &= \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} n!\left(\frac{1}{k} + \frac{1}{n+1-k}\right) \\ &= n!\left(\sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{1}{k} + \sum_{k=\lceil (n+1)/2 \rceil}^{n} \frac{1}{k}\right) \\ &= \begin{cases} n!H_n, & n \text{ is even,} \\ n!(H_n + \frac{2}{n+1}), & n \text{ is odd.} \end{cases}\end{aligned}$$

Problem 2.1.6. For a fixed positive integer k, consider the egf $f_k(x) = \sum_{n \geq 0} s(n,k)x^n/n!$. Show that

$$f_k(x) = \frac{1}{k!} \ln\left(\frac{1}{1-x}\right)^k.$$

Solution

Define $g : \mathbb{N} \to \mathbb{N}_0$ by

$$g(r) = \begin{cases} 1, & r = k, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$s(n,k) = \sum_{\pi \in \mathfrak{S}_n} g(r),$$

where $C_1, \ldots, C_r$ are the cycles in π. We now use Corollary 5.1.8 in [SF99] with f being the function that takes the constant 1. We have $E_g(x) = \frac{1}{(k-1)!}x^k$, so we get that

$$f_k(x) = \frac{1}{(k-1)!}\left(\sum_{n \geq 1} \frac{x^n}{n}\right)^k = \frac{1}{(k-1)!} \ln\left(\frac{1}{1-x}\right)^k.$$

Problem 2.1.7. Find $\sum_{k=0}^{n} (-1)^k s(n,k)$.

Solution

Recall that a permutation in $\mathfrak{S}_n$ with k cycles has sign $(-1)^{n-k}$. We have

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k s(n,k) &= \sum_{k=0}^{n} \sum_{\substack{\pi \in \mathfrak{S}_n \\ \pi \text{ has exactly } k \text{ cycles}}} (-1)^k \\ &= (-1)^n \sum_{\pi \in \mathfrak{S}_n} \operatorname{sign}(\pi). \end{aligned}$$

For $n = 1$, this is -1. Otherwise, note that $\pi \mapsto (1,2)\pi$ is a bijection between odd and even permutations. As a result, the above sum is equal to 0.

Problem 2.1.8. Show that $S(n+1, k+1) = \sum_{m=0}^{n} \binom{n}{m} S(m,k)$.

Solution

We shall count the partitions of $[n+1]$ into exactly $k+1$ parts based on the part $n+1$ is in. We have

$$S(n+1,k+1) = \sum_{S \subseteq [n]} S(|[n] \setminus S|, k) = \sum_{m=0}^{n} \binom{n}{m} S(n-m,k) = \sum_{m=0}^{n} \binom{n}{m} S(m,k).$$

Problem 2.1.9. Show that for $n \geq 1$, the $S_{n,k}$ as k varies has either a unique maximum value or has at most two equal values.

Solution

2.2. Problem Sheet 2

Problem 2.2.1. Show that

(a) $T_n(1) = 1$ and

(b) $T_n(-1) = (-1)^n$.

Solution

This immediately follows since $T_n(\cos\theta) = \cos n\theta$, so $T_n(\cos 0) = \cos(n\cdot 0) = 1$ and $T_n(\cos\pi) = \cos(n\pi) = (-1)^n$. They can also be easily proved inductively.

Problem 2.2.2. Show that

(a) $U_n(1) = n+1$.

(b) $U_n(-1) = (-1)^n(n+1)$

Solution

We prove this inductively. Both statements are trivially true for $n = 0, 1$. For $n \geq 2$, inductively, we have

$$U_n(1) = 2U_{n-1}(1) - U_{n-2}(1) = 2n - (n-1) = n+1$$

and

$$U_n(-1) = -2U_{n-1}(1) - U_{n-2}(1) = (-1)^n \cdot 2n + (-1)^{n-1}(n-1) = (-1)^n(n+1).$$

Problem 2.2.3. Show that

$$\frac{1}{\iota^n} U_n(\iota/2) = f_{n+1}.$$

Solution

Again, we prove this inductively. We have $U_0(\iota/2) = 1 = f_1$ and $U_1(\iota/2) = \iota = \iota f_2$. For $n \geq 2$, we inductively have

$$U_n(\iota/2) = \iota U_{n-1}(\iota/2) - U_{n-2}(\iota/2) = \iota^n f_n - \iota^{n-2} f_{n-1} = \iota^n(f_n + f_{n-1}) = \iota^n f_{n+1}.$$

Problem 2.2.4. Show that if $m, n \geq 1$,

$$T_{m+n}(x) = T_m(x)U_n(x) - T_{m-1}(x)U_{n-1}(x).$$

Solution

This may be checked manually for $m + n = 2, 3$. We perform induction on $m + n$. We have that

$$\begin{aligned}
T_{m+n}(x) &= 2xT_{m+n-1}(x) - T_{m+n-2}(x) \\
&= 2x(T_{m-1}(x)U_n(x) - T_{m-2}(x)U_{n-1}(x)) - (T_{m-1}(x)U_{n-1}(x) - T_{m-2}(x)U_{n-2}(x)) \\
&= 2xT_{m-1}(x)U_n(x) - T_{m-1}(x)U_{n-1}(x) - T_{m-2}(x)(2xU_{n-1}(x) - U_{n-2}(x)) \\
&= 2xT_{m-1}(x)U_n(x) - T_{m-1}(x)U_{n-1}(x) - T_{m-2}(x)U_n(x) \\
&= U_n(x)(2xT_{m-1}(x) - T_{m-2}(x)) - T_{m-1}(x)U_{n-1}(x) \\
&= T_m(x)U_n(x) - T_{m-1}(x)U_{n-1}(x).
\end{aligned}$$

2.3. Problem Sheet 3

Problem 2.3.1. Let (E_n) be the Euler numbers. Show that its egf

$$E(x) = \sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

Problem 2.3.2. A recurrence relation of the following type is called a *three term recurrence* for a sequence p_n of polynomials with $n \geq 0$ (where each p_n is a polynomial of degree n) if there exist real numbers $(a_n), (b_n)$ such that

$$p_{n+1}(x) = (x - a_n)p_n(x) - b_n p_{n-1}(x).$$

Show that the matching polynomials of (P_n), (C_n), and (K_n) each satisfy a three term recurrence.

Solution

(a) Setting v as one of the "end" vertices in Lemma 1.16(b), we get

$$\mathrm{Match}_{P_n}(x) = x\,\mathrm{Match}_{P_{n-1}}(x) - \mathrm{Match}_{P_{n-2}}(x).$$

(b) Fixing some edge e and using Lemma 1.16(c), then using the recurrence from (a), gives

$$\begin{aligned}\mathrm{Match}_{C_n}(x) &= \mathrm{Match}_{P_n}(x) - \mathrm{Match}_{P_{n-2}}(x) \\ &= x\,\mathrm{Match}_{P_{n-1}}(x) - \mathrm{Match}_{P_{n-2}}(x) - x\,\mathrm{Match}_{P_{n-3}}(x) + \mathrm{Match}_{P_{n-2}}(x) \\ &= x\,\mathrm{Match}_{C_{n-1}}(x) - \mathrm{Match}_{C_{n-2}}(x).\end{aligned}$$

(c) Fixing some vertex v and using Lemma 1.16(b), we get

$$\mathrm{Match}_{K_n}(x) = x\,\mathrm{Match}_{K_{n-1}}(x) - (n-1)\,\mathrm{Match}_{K_{n-2}}(x)$$

Problem 2.3.3. For $n \geq 1$, show that $\mathrm{Match}_{C_n}(2x) = 2T_n(x)$.

Solution

The above is easily verified for $n = 1, 2$, and we saw in Problem 2.3.2 that Match_{C_n} satisfies the same recurrence as $2T_n$, which is the same as that of T_n Proposition 1.8.

Problem 2.3.4. Let T be a tree with maximum degree d. Show that any root of $\mathrm{Match}_T(x)$ has absolute value at most $2\sqrt{d-1}$.

Solution

By the proof of Theorem 1.18, it suffices to show that the eigenvalues of the adjacency matrix A of T are bounded from above by $2\sqrt{d-1}$.
Fix a "root" vertex r in T and denote by $d(\cdot, r)$ the distance from a vertex to r. Consider the diagonal matrix D defined by $D_{u,u} = (d-1)^{-d(u,r)/2}$. Consider $B = DAD^{-1}$. Clearly, the spectrum of A and B are equal, so it suffices to show that the eigenvalues of B are at most $2\sqrt{d-1}$. We have

$$B_{uv} = \begin{cases} (d-1)^{(d(u,r)-d(v,r))/2}, & u \leftrightarrow v, \\ 0, & \text{otherwise.} \end{cases}$$

Recall the Perron-Frobenius Theorem, one of whose consequences gives that the largest eigenvalue of B is at most the largest row sum of B.
Now, observe that for any vertex u with neighbour v, $d(v, r) \in \{d(u, r) - 1, d(u, r) + 1\}$. Further, because T is a tree, there can be at most one neighbour v with $d(v, r) = d(u, r) - 1$ – were there more, we would have a cycle. Therefore,

$$\lambda_{\text{max}} \leq \max_u \sum_v B_{u,v} = \max_u \sum_{v : v \leftrightarrow u} (d-1)^{d(u,r)-d(v,r)} \leq 1 \cdot (d-1)^{1/2} + \frac{d-1}{(d-1)^{1/2}} = 2\sqrt{d-1}.$$

The idea of the above proof is similar to that of the Alon-Boppana bound, and the $2\sqrt{d-1}$ bound occurs in the definition of *Ramanujan graphs*. For more details, the interested reader may read Theorem 2.27 and following discussion in the author's Topics in Graph Theory notes.

Problem 2.3.5. Show that when one runs the recurrence relation of the binomial coefficients backwards, we get signed versions of the binomial coefficients.

Problem 2.3.6. A restricted growth function (RGF) of length n is a sequence $w_1, w_2, \ldots, w_n$ of positive integers such that $w_1 = 1$ and $w_i \leq 1 + \max(w_1, \ldots, w_{i-2})$ when $i \geq 2$. Determine the number of RGFs of length n.

Problem 2.3.7. Consider a staircase board St_n with n columns where the ith column has $i-1$ cells all stacked from the same height. We place "rooks" on this board and recall that rooks can travel on the same row or column. Two rooks can attack each other if they are on the same row or column and a placement of rooks is said to be non-attacking if no two rooks in the placement can attack each other. Determine the number of ways to place $n-k$ non-attacking rooks in St_n.

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