CS 6001: Game Theory and Algorithmic Mechanism Design

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Contents

§1. Introduction

In typical linear programming, we have an objective that a *single* individual is trying to maximize (subject to some constraints). In game theory, we typically study systems where there are *multiple* individuals with objectives that they are trying to maximize, but each individual can only set some of the variables (the set of variables is shared). An optimal solution for one individual might not be the optimal solution for another. As a result, it is now better to look at *equilibria* instead of pure black-and-white optimality.
These equilibria are what we shall study.
The course is broadly split in two parts:

- *Game theory*, where we study games, interactions between agents who want to maximize their utilities. This is a predictive approach. We wish to find the most probably outcomes or responses of the agents/players.
- *Mechanism design*, where we try to design a game with desirable outcomes. This is a prescriptive approach. Given a reasonable outcome, we wish to build a game that yields this as a probable outcome.

The above is essentially analysis versus synthesis.

A well-known example of game theory is the *Prisoner's Dilemma*, which we slightly rephrase in the following.

Example 1. Two kingdoms can each decide to dedicate all their resources to either agriculture or warfare. If both kingdoms choose warfare, they each earn one unit of joy. If one chooses agriculture and the other chooses warfare, the former earns nothing while the latter earns six units of joy. Finally, if both choose agriculture, they each earn five units of joy.
This can be represented by the following matrix.

A, B	Agriculture	War
Agriculture	$5, 5$	$0, 6$
War	$6, 0$	$1, 1$

The above matrix is called a *game matrix*. Each of the two numbers in the cells are referred to as the *utilities* of the respective kingdoms.
Suppose that kingdom A decides to invest in agriculture. In this case, note that irrespective of what the other kingdom decides to choose, A can increase its payoff by switching to investing in warfare. Consequently, it makes more sense to invest in warfare (since the game is symmetric, this is true for both players). Even though both kingdoms heading to war is not the most profitable outcome, it appears to be the most likely outcome.

A *game* is a formal representation of the strategic interaction between multiple agents called *players*. The choices available to the players are called *actions*. A mapping from the state of the game to the set of actions is referred to as a *strategy*.
Depending on the context, games can be represented in different ways, normal form, extensive form, repeated form, stochastic form, etcetera.

Game theory is the formal study of the strategic interactions between players who are rational and intelligent.
A player is rational if they pick actions to maximize their payoff.
A player is intelligent if they know the rules of the game perfectly and picks actions assuming that the remaining players are also rational and intelligent. This also assumes that the player has sufficient computational ability to find the "optimal" action.

Let us now look at an example of mechanism design.

Example 2. Suppose we want to split a cake in two parts (for two children, say) in an "envy-free" fashion. That is, neither child would prefer the other piece of cake. We do not see the children's preferences, so we do not even know what a fair division might involve. This well-known problem has a well-known solution – make one child cut the cake and the other choose the piece. Why does this work? The first child splits it in a way that is exactly half from their perspective, and they are indifferent to the two pieces. The second child on the other hand gets a larger piece in their perspective.

While game theory allows us to study existing games, mechanism design helps us design games to attain specific goals.

1.1. Chess

The reader is no doubt familiar with the rules of chess. It has two players, *White* and *Black*, with 16 pieces each. Each piece has some legal moves (the players' actions are these moves). The game starts with White and players take alternating turns. White wins if they capture Black's king, and Black wins they capture White's king. There are a couple of conditions under which a draw can occur, which we do not detail.
There are numerous natural questions that arise. First and foremost, does White (or Black) have a winning strategy? A winning strategy is a plan of moves such that it wins irrespective of the moves performed by Black. Alternatively, is it possible to guarantee a draw? It is possible for none of these exist.

What is a strategy? Denote a board position by x_k. A *game situation* is a finite sequence $(x_0, \dots, x_k)$, such that x_0 is the opening board position and $x_k \to x_{k+1}$ for even (resp. odd) k is created by a single action of White (resp. Black).
This set of game situations can be naturally represented by a tree, referred to as the *game tree*. The nodes are labelled with board positions (x_i), and arrows between nodes are labelled with actions. A *strategy* is a mapping from game situation to action, which describes what action to take at every vertex of this game tree. This is something of a contingency plan for every possible situation.
A strategy pair (s_W, s_B) which describes strategies for both players determines an outcome, also called a *play* of the game. This describes a path through the game tree.
All leaves in the game tree correspond to either White's victory, Black's victory, or a draw.

A *winning strategy* for White is a strategy s_W^* such that for any strategy s_B, (s_W^*, s_B) ends in a win for White. Similarly, we can define a *strategy guaranteeing at least a draw for White*, denoted s'_W.
It is not immediately obvious if such strategies exist.

Theorem 1.1 (von Neumann, 1928)**.** In chess, exactly one of the following statements is true.

(1) White has a winning strategy.

(2) Black has a winning strategy.

(3) Each player has a strategy guaranteeing a draw.

If any such strategy was known, the game would become boring.

Proof sketch. Each vertex of the game tree is a game situation. Denote by $\Gamma(x)$ the subtree rooted at x (including x itself) and by n_x the number of vertices in $\Gamma(x)$. $n_x = 1$ implies that x is a terminal vertex.
Using induction on n_x, we prove that one of the three statements must hold at any game situation. The theorem is clearly true for $n_x = 1$. Suppose x is a vertex with $n_x > 1$. By the inductive hypothesis, for all $y \in \Gamma(y) \setminus \{x\}$, the statement holds $(n_y < n_x)$. Let $C(x)$ be the vertices reachable from x in one step, and assume wlog that it is White's turn.

(a) If there exists $y_0 \in C(x)$ such that (1) is true, then (1) is true at x as well.

(b) If (2) is true for all $y \in C(x)$, then Black will win irrespective of White's move.

(c) Otherwise, because (a) does not hold, White does not have a winning strategy for any $y \in C(x)$. As a result, for every $y \in C(x)$, either Black has a winning strategy or both have a draw-guaranteeing strategy.
Because (b) does not hold, there is some $y \in C(x)$ where Black does not have a winning strategy. By the preceding argument, both players must have a draw-guaranteeing strategy at this node. ■

Remark. The above proof is technically incorrect since the game tree for chess is infinite, so $\Gamma(x)$ need not be finite. This issue can be fixed through some clever manipulation of the "cycles" in the game tree to make the graph finite, but we do not go into more detail on this.

1.2. Normal form games

Normal form is a representation technique for games. The set $N = \{1, \ldots, n\}$ is the set of players. S_i is the set of strategies for player i. A particular strategy is denoted $s_i \in S_i$. The set of *strategy profiles* is $S = \bigtimes_{i \in N} S_i$, with specific elements $s = (s_1, \ldots, s_n) \in S$. A strategy profile without i is $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. $u_i : S \to \mathbb{R}$ is the utility function of player i.
The *normal form game* (NFG) representation of a game is the tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$.
If S_i is finite for all $i \in N$, the game is said to be a *finite* game.

As mentioned earlier, a player is rational if they pick actions that maximize their utility. A player is intelligent if they know the rules of the game perfectly, and pick actions assuming that all other players are rational and intelligent.

A fact is said to be *common knowledge* if

1. all players know the fact,
2. the fact that "all players know the fact" is also common knowledge.

Example 3 (Propagation of common knowledge)**.** There is an isolated island (with a hundred people, say) where all inhabitants have eye color either blue or black. There is no reflecting surface on the island (people cannot figure out their own eye color) and nobody can communicate with each other.
One day, a truth-telling god comes to the island and declares that all blue-eyed people are bad for the island and must leave as soon as possible. He also says that there is at least one blue-eyed person on the island. The inhabitants, being deeply devout, do listen to him and leave at the end of the day if they discover that their eyes are blue. In this setting, the fact that there is at least one blue-eyed person on the island is common knowledge.

If there was only one blue-eyed person, he would see that all other people have black eyes. Because the god said that there is a blue-eyed person, he infers that he must be the only blue-eyed person and leaves at the end of the first day.
If there were two, then on the second day everyone would notice that all people remain on the island, so they would infer that there are at least two blue-eyed people on the island. If one of the inhabitants sees that exactly one of the other four people is blue-eyed, then he, along with the other blue-eyed person, leaves on the second day.
This goes on, and it is seen that if there are exactly n blue-eyed people, then all of these n people leave at the end of the nth day.

Now, we discuss the concept of *domination* in NFGs. Consider the following game matrix:

$1, 2$	L	M	R
U	$1, 0$	$1, 3$	$3, 2$
D	$-1, 6$	$0, 5$	$3, 3$

Observe that player 2 has no reason to ever play R. Indeed, no matter what player 1 chooses, they can increase their payoff by switching to M instead. In such a scenario, we say that R is dominated by M.

Definition 1.1. A strategy $s_i' \in S_i$ of player i is said to be *strictly dominated* if there exists another strategy $s_i \in S_i$ such that for every strategy profile $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}).$$

A strategy $s_i' \in S_i$ of player i is said to be *weakly dominated* if there exists another strategy $s_i \in S_i$ such that for every strategy profile $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

and in addition, there exists some $\tilde{s}_{-i} \in S_{-i}$ such that

$$u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i}).$$

So, in the earlier example R is strictly dominated and D is weakly dominated.
There is no reason for a rational player to ever play a (weakly or strictly) dominated strategy.

Definition 1.2. A strategy $s_i \in S_i$ is *strictly (weakly) dominant* if it strictly (weakly) dominates all $s_i' \in S_i \setminus \{s_i\}$.

Recall the agriculture/defense example we looked at towards the beginning of this section. In this, warfare strictly dominates agriculture, which is precisely what we said there, albeit in more explicit terms.

Let us give another example of this.

Example 4. Suppose there are two players having "values" v_1, v_2 respectively. Each player can choose a number in $[0, M]$, where $M \gg v_1, v_2$. The player who quotes the larger number wins the object (with ties being broken in favour of player 1, say), and pays the losing player's chosen number. The utility of the winning player is their value v_i minus their payment (the amount bid by the other player), and the utility of the losing player is 0.
As a NFG representation, $N = \{1, 2\}$, $S_1 = S_2 = [0, M]$, and

$$u_1(s_1, s_2) = \begin{cases} v_1 - s_2, & s_1 \geq s_2, \\ 0, & \text{otherwise,} \end{cases} \text{ and } u_1(s_1, s_2) = \begin{cases} v_2 - s_1, & s_1 < s_2, \\ 0, & \text{otherwise,} \end{cases}$$

It turns out that the strategy s_i where player i chooses v_i is a weakly dominant strategy! Let us check this for player 1.
Suppose that $s_1' > v_1$ is in S_1 and $s_2 \in S_2$. We would like to show that $u_1(s_1, s_2) \geq u_1(s_1', s_2)$. If $s_2 > s_1'$, both payoffs are zero. If $s_2 \leq s_1 < s_1'$, then both payoffs are equal to the same value. The remaining case is when $s_1 \leq s_2 \leq s_1'$. The payoff for s_1 is zero, but the payoff for s_1' is non-positive since we are paying more than we value the item.

§2. Equilibria in NFGs

In this section, we look at various types of "equilibria" – strategies which make sense to be played by rational players.

2.1. Some types of equilibria

Definition 2.1. A strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ is a *strictly (weakly) dominant strategy equilibrium* if each s_i^* is a strictly (weakly) dominant strategy for each i.

We abbreviate the above as SDSE or WDSE.
No rational player would play dominated strategies, so we can eliminate dominated strategies one-by-one. A point of note here is that after eliminating a strategy, we get a reduced game with fewer strategies, and this game may have dominated strategies that were not there earlier.
For strictly dominated strategies, the order of elimination does not matter. For weakly dominated strategies however, some reasonably outcomes may be eliminated (since this effectively says how we break ties).

Example 5 (Order of eliminating weakly dominated strategies matters)**.** Consider the following.

	L	C	R
T	1,2	2,3	0,3
M	2,2	2,1	3,2
B	2,1	0,0	1,0

Right off the bat, it is seen that T, B, and C are weakly dominated strategies. Suppose we start by imposing that player 1 does not play T. If we do this, then R becomes weakly dominated as well, so it makes sense to eliminate it. Similarly, we can go on to eliminate B and C. Finally, the payoff of (M, L) is $2, 2$.
On the other hand, if we eliminate strategies in the order of B, L, C, T, then the final remaining strategies are (M, R), which gives a payoff of $3, 2$, which is not the same as the previous $2, 2$!

Dominant strategies (and dominant strategy equilibria) need not exist in games in general, as can be seen in the following game.

	L	R
L	$1, 1$	$0, 0$
R	$0, 0$	$1, 1$

As a result, dominance is not enough to give a reasonable outcome, so we must give a more refined notion.

Definition 2.2 (Nash Equilibrium)**.** A strategy profile (s_i^*, s_{-i}^*) is a *pure strategy Nash equilibrium* (PSNE) if for all $s_i \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

That is, fixing the remaining players' strategies, no player can increase their payoff by moving to another strategy. Unilateral deviation is not beneficial.
In the above example, (L, L) and (R, R) are both PSNEs.
PSNEs need not exist either!

Example 6 (PSNEs need not exist)**.** The following game has no PSNE.

	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$1, -1$

Definition 2.3. A *best response* of player i against a strategy profile s_{-i} is a strategy that gives the maximum utility. That is,

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \text{ for all } s'_i \in S_i\}.$$

So, a PSNE is a strategy profile (s^*_i, s^*_{-i}) such that $s^*_i \in B_i(s^*_{-i})$ for all $i \in N$.
A PSNE gives some sort of stability. Once there, no rational player has a reason to change their strategy.

One of our biggest assumptions thus far is that all players are rational and intelligent. There are, however, other types of rationality.
One is risk-aversion, where each player makes pessimistic estimates of others. This worst case optimal choice is called a max-min strategy.

Definition 2.4. A strategy s_i^{maxmin} is a *max-min strategy* for player i if

$$s_i^{\text{maxmin}} \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

The *max-min value* is defined by

$$\underline{v}_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

We have that for any $t_{-i} \in S_{-i}$,

$$u_i(s_i^{\text{maxmin}}, t_{-i}) = \underline{v}_i.$$

Theorem 2.1. Any dominant strategy is a maxmin strategy.

Proof. Let s^*_i be a dominant strategy for player i. We have that for any $s_{-i} \in S_{-i}$ and $s'_i \in S_i \setminus \{s^*_i\}$,

$$u_i(s^*_i, s_{-i}) \geq u_i(s'_i, s_{-i}).$$

In particular, when we set s_{-i} as any $s_{-i}^{\text{min}}(s^*_i) \in \arg\min_{s'_{-i} \in S_{-i}} u_i(s^*_i, s'_{-i})$, we get

$$\min_{s_{-i} \in S_{-i}} u_i(s^*_i, s_{-i}) = u_i(s^*_i, s_{-i}^{\text{min}}(s^*_i)) \geq u_i(s'_i, s_{-i}^{\text{min}}(s'_i)) \geq \min_{s_{-i} \in S_{-i}} u_i(s'_i, s_{-i})$$

and as a result,

$$s^*_i \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

■

Theorem 2.2. Every PSNE $s^* = (s^*_1, \ldots, s^*_n)$ of an NFG satisfies

$$u_i(s^*) \geq \underline{v}_i$$

for all $i \in N$.

Proof. We have

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \quad \text{(by definition of PSNE)}$$
$$\geq \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) = \underline{v}_i. \quad \blacksquare$$

What happens to stability and security when games are eliminated? Recall that depending on the order in which dominant strategies are eliminated, the final value can change.

Example 7. The game matrix involved in this example is very similar to that in Example 5.

	L	C	R
T	$1,2$	$2,3$	$0,3$
M	$2,2$	$2,1$	$3,2$
B	$2,0$	$0,0$	$1,0$

The initial maxmin values for the two players are 2 for player 1 and 0 for player 2, but after we eliminate B, the values go to 2 and 2.
It is not a coincidence that the maxmin value of 2 for player 1 is unchanged.

Theorem 2.3. Consider an NFG $G = \langle N, (S_i)_{i\in N}, (u_i)_{i \in N}\rangle$. Let $\hat{s}_j \in S_j$ be a dominated strategy, and $\hat{G}$ be the residual game after removing $\hat{s}_j$. The maxmin value of player j in $\hat{G}$ is equal to that in G.

The idea is that the eliminated strategy cannot be the unique maxmin strategy in G since it is dominated.

Proof. We are done if we show that there is a maxmin strategy in G in $S_j \setminus \{s_{-j}\}$ (Why?).
Let $\hat{s}_j$ be dominated by $t_j \in S_j \setminus \{\hat{s}_j\}$. Then, for all $s_{-j} \in S_{-j}$,

$$u_j(t_j, s_{-j}) \geq u_j(\hat{s}_j, s_{-j}).$$

We have

$$\min_{s_{-j} \in S_{-j}} u_j(t_j, s_{-j}) \geq \min_{s_{-j} \in S_{-j}} u_j(\hat{s}_j, s_{-j}),$$

so

$$\max_{s_j \in S_j \setminus \{\hat{s}_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) \geq \min_{s_{-j} \in S_{-j}} u_j(\hat{s}_j, s_{-j}),$$

completing the proof – there is another strategy whose value is at least that of $\hat{s}_j$. $\blacksquare$

To summarize,

- eliminating strictly dominated strategies has no effect on the PSNEs,
- eliminating weakly dominated strategies may make the set of PSNEs smaller, but does not add new PSNEs, and
- the maxmin value is unaffected by eliminating (strictly or weakly) dominated strategies.

2.2. Matrix games

Definition 2.5. A *matrix game* or *two player zero-sum game* is a normal form game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and $u_1 + u_2 \equiv 0$.

Example 8. Examples of matrix games are the following.

	L	C	R
T	$3, -3$	$-5, 5$	$-2, 2$
M	$1, -1$	$4, -4$	$1, -1$
B	$6, -6$	$-3, 3$	$-5, 5$

	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

A point of note here that we shall soon examine in more detail is that the first game has PSNEs while the second does not.

A matrix game can be represented by a single *utility matrix*, by considering the utilities of only one of the players, say player 1. Player 2's utilities are then just the negative of the matrix.
Given a utility matrix u, $u_1 \equiv u$ and $u_2 \equiv -u$.

What are the PSNEs, if any, of a matrix game?

Definition 2.6. A *saddle point* of a matrix A is an index A_{ij} that is the largest in the ith row and the smallest in the ith column.

Theorem 2.4. In a matrix game with utility matrix u, (s_1^*, s_2^*) is a PSNE if and only if it is a saddlepoint.

Proof. Indeed, (s_1^*, s_2^*) is a saddle point iff

$$u(s_1^*, s_2^*) \geq u(s_1, s_2*) \text{ for all } s_1 \in S_2 \text{ and}$$
$$u(s_1^*, s_2^*) \leq u(s_1^*, s_2) \text{ for all } s_2 \in S_2,$$

which is equivalent to being a PSNE since $u_1 \equiv u$ and $u_2 \equiv -u$. ■

Definition 2.7. Given a two player game with utility matrix u, define the *maxmin* value $\underline{v}$ by

$$\underline{v} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$$

and the *minmax* value $\overline{v}$ by

$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2).$$

The above can be slightly rephrased to say

$$\underline{v} = -\min_{s_1 \in S_1} \max_{s_2 \in S_2} u_2(s_1, s_2)$$
$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_2} u_1(s_1, s_2).$$

Lemma 2.5. For matrix games, $\overline{v} \geq \underline{v}$.

Proof. Let $\overline{v}$ and $\underline{v}$ be attained by (s_1, s_2) and (t_1, t_2) respectively. Then,

$$\overline{v} = u(s_1, s_2) \overset{(1)}{\geq} u(t_1, s_2) \overset{(2)}{\geq} u(t_1, t_2) = \underline{v},$$

where (1) is because (s_1, s_2) is a minmax strategy and (2) is because (t_1, t_2) is a maxmin strategy. ∎

Going back to Example 8, check that in the first game, $\overline{v} = 1 = \underline{v}$, while in the second game, $\overline{v} = 1$ and $\underline{v} = -1$.

Theorem 2.6. A matrix game has a PSNE iff $\overline{v} = \underline{v}$.

Proof. We wish to show that a utility matrix u has a saddle point iff its maxmin and minmax values are equal.

Suppose that (s_1^*, s_2^*) is a saddle point. Then,

$$u(s_1^*, s_2^*) \geq \max_{s_1 \in S_1} u(s_1, s_2^*) \geq \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2) = \overline{v}.$$

An identical argument for player 2 (keeping in mind that $u_2 \equiv -u$!) yields that $\underline{u} \geq u(s_1^*, s_2^*)$.
Combining the two,

$$\underline{v} \geq u(s_1^*, s_2^*) \geq \overline{v}.$$

Recalling Lemma 2.5, we must have that $\overline{v} = \underline{v}$.

Now, suppose that $\overline{v} = \underline{v}$. There is a strategy (s_1^*, s_2^*) that is both a maxmin and minmax strategy (Why?).
For any $s_2 \in S_2$,

$$u(s_1^*, s_2) \geq \min_{t_2 \in S_2} u(s_1^*, t_2) = \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) = \underline{v}.$$

Similarly, for any $s_1 \in S_1$,

$$u(s_1, s_2^*) \leq \max_{t_1 \in S_1} u(t_1, s_2^*) = \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2) = \overline{v}.$$

The two equations above imply that (s_1^*, s_2^*) is a saddle point, so we are done. ∎

2.3. Mixed strategies

So far, one issue is that PSNEs may not exist. We are also limiting ourselves to "pure" strategies, in the sense that there is a certain strategy that we definitely play.

Given a finite set A, define the set of probability distributions on A

$$\Delta A = \{p \in [0, 1]^A \sum_{a \in A} p_a = 1\}.$$

A *mixed strategy* of player i is some $\sigma_i \in \Delta(S_i)$.
Because we are looking at non-cooperative games, mixed strategies of distinct players are independent.
The utility of the ith player for a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_1 \in S_1} \cdots \sum_{s_n \in S_n} \left(\prod_{i \in [n]} \sigma_i(s_i) \right) u_i(s_1, \ldots, s_n).$$

That is, the utility of a mixed strategy is the expectation of the utility.

Example 9. Consider the following game.

	U	D
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

Suppose the mixed strategy σ_1 chooses L and R with probabilities $2/3$ and $1/3$ respectively, and σ_2 chooses U and D with probabilities $4/5$ and $1/5$ respectively. Then,

$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1) + \frac{2}{3}\frac{1}{5} \cdot 1 + \frac{1}{3}\frac{4}{5} \cdot 1 + \frac{1}{3}\frac{1}{5} \cdot (-1) = -1/5.$$

By linearity of expectation, we have

$$u_i(\lambda\sigma_i + (1-\lambda)\sigma_i', \sigma_{-i}) = \lambda u_i(\sigma_i, \sigma_{-i}) + (1-\lambda) u_i(\sigma_i', \sigma_{-i}). \tag{2.1}$$

This is referred to as mixing strategies.

Definition 2.8 (Mixed Strategy Nash Equilibrium)**.** A *mixed strategy nash equilibrium* (MSNE) is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ such that for all $i \in N$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*)$$

for any $\sigma_i' \in \Delta(S_i)$.

A PSNE is just a special case of a MSNE where all the mixed strategies are degenerate distributions.

Theorem 2.7. A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE iff

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*)$$

for all $s_i \in S_i$.

The forward implication is direct on setting the σ_i' appropriately, and the backward implication by an extension of Equation (2.1) to more than two strategies (How?).

Therefore, any PSNE is a MSNE. In any MSNE, there must be some amount of "balance" – in Example 9, the probability distributions that assign $1/2$ to each strategy together constitute a MSNE (check this!).
How do we make this notion of balance more formal?

Definition 2.9. Given a mixed strategy σ_i, the *support* of σ_i is

$$\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}.$$

We now characterize MSNEs.

Theorem 2.8. A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is a MSNE iff

(a) $u_i(s_i, \sigma_{-i}^*)$ is equal for all $s_i \in \delta(\sigma_i^*)$, and

(b) $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$, $s_i' \notin \delta(\sigma_i^*)$.

We encourage the reader to use the above theorem to determine MSNEs in earlier examples.

Proof. We have

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) && \text{(similar to Theorem 2.7)} \\ &= \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \\ &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*). \end{aligned}$$

It is possible for the maximum and weighted average to be equal iff all the utilities are the same, so (a) is proved. Now, suppose that there is some $s_i \in \delta(\sigma_i^*)$, $s_i' \notin \delta(\sigma_i^*)$ such that

$$u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*).$$

Considering the mixed strategy that is identical to σ_i^* except that the weight that was on s_i is shifted to s_i'. This gives a strictly larger utility, contradicting the fact that we have an MSNE and proving (b).

Now, let us prove the backward direction. Let $u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ (using (a)). Using (b), we have $m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$. Then,

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \\ &= m_i(\sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*), \end{aligned}$$

so we are done by Theorem 2.7. ■

We now try to convert the above theorem to an algorithm. Suppose we have an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$. The number of possible supports of $S_1 \times \cdots \times S_n$ is $K = (2^{|S_1|-1})(2^{|S_2|-1}) \cdots 2^{|S_n|-1}$.
For every possible support profile $X_1 \times \cdots \times X_n$, solve the following feasibility program.

$$\begin{aligned} w_i &= \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}) \text{ for all } s_i \in X_i, i \in N \text{ and} \\ w_i &\geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}) \text{ for all } s_i \in (S_i \setminus X_i), i \in N, \end{aligned}$$

where for all $j \in N$, $\sigma_j(s_j) \geq 0$ for all $s_j \in S_j$ and $\sum_{s_j \in S_j} \sigma_j(s_j) = 1$.
This program is not linear unless there are only two players.
For games in general, there is no polynomial time algorithm known for this. In fact, the problem of finding an MSNE

is *PPAD-complete*[1]. The interested reader may see [DGP09] for more details.

The previous algorithm may be applied to a smaller set of strategies by removing dominated strategies. Now, we can even talk about domination by a mixed strategy. As we saw earlier, removing a weak dominated strategy can upset the equilibrium.

Theorem 2.9. If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta(S_i)$, any MSNE of the game chooses s_i with probability zero.

We can remove strictly dominated strategies without consequences.

Theorem 2.10 (Nash, [Nas51])**.** Any finite game has a mixed Nash equilibrium.

Above, "finite" means that the number of players and strategies are finite.

2.4. Correlated equilibria

So far, we have worked in the setting where each agent independently picks their own strategy. Now, we look at an alternative approach with a mediating agent or device. This is merely another version of rationality. It may lead to results where the utility of all players is improved, and is computationally tractable.

Example 10. Consider the following game, modelling the choices of cars at a crossroads.

	Wait	Go
Wait	$0, 0$	$1, 2$
Go	$2, 1$	$-10, -10$

It is clear that both Wait, Go and Go, Wait are PSNEs. It can also be seen however that a MSNE assigns some nonzero probability to the event where both players choose Go. In practice, a traffic light guides the players. This trusted third party is called the *mediator*. It randomizes over the strategy *profiles* (and not just strategies like earlier) and suggests the corresponding strategies to the players. If the startegies are enforceable, then it is an equilibrium.

Definition 2.10. A *correlated strategy* is a mapping $\pi \in \Delta S$.

In the setting of the previous example, an example of a sensible correlated strategy is that which chooses Wait, Go or Go, Wait with equal probability $1/2$.

Definition 2.11. A *correlated equilibrium* is a correlated strategy π such that

$$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i', s_{-i})$$

for all $s_i, s_i' \in S_i$ for all $i \in N$.

[1] PPAD stands for "Polynomial Parity Argument on Directed graphs".

That is, no player benefits from (unilaterally) changing their strategy. This is largely similar to the definition of an MSNE, but the distribution is over strategy profiles and not strategies.

One massive advantage of correlated equilibria is that they can be computed efficiently! We merely need to find a solution to the set of constraints

$$\begin{aligned}\sum_{s_{-i}\in S_{-i}} \pi(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} \pi(s_i, s_{-i})u_i(s'_i, s_{-i}) &\text{ for all } s_i, s'_i \in S_i \text{ and } i \in N \\ \pi(s) \geq 0 &\text{ for all } s \in S \\ \sum_{s\in S} \pi(S) = 1.&\end{aligned}$$

If $|S_i| = m$ for all i, this gives $O(nm^2)$ inequalities in the first part and $O(m^n)$ inequalities for the second.
All together, they represent a feasibility linear program.

In MSNEs, the total number of support profiles was $O(2^{mn})$. Here, we just have $O(m^n)$ inequalities, which is exponentially smaller.

Theorem 2.11. Given a MSNE σ^*, consider the strategy profile distribution π^* defined by

$$\pi^*(s_1, \ldots, s_n) = \prod_{i=1}^{n} \sigma_i^*(s_i).$$

Then, π^* is a correlated equilibrium.

We omit the proof of the above as it is straightforward.
Summarizing much of our discussion thus far, we have that the set of SDSEs is contained in the set of WDSEs, which is contained in the set of PSNEs, which is contained in the set of MSNEs, which is contained in the set of CEs.

§3. Long-form games

Thus far, we have only looked at "single-step" games in NFGs. Not all games can be represented by this however, with an obvious example that we have discussed being chess.

3.1. Perfect information extensive form games

Definition 3.1 (Perfect Information Extensive Form Game)**.** A perfect information extensive form game (PIEFG/EFG) is a 7-tuple $\langle N, A, \mathcal{H}, Z, \mathcal{X}, P, (u_i)_{i\in N}\rangle$ where

- N is the set of players,
- A is the set of all possible actions (of all players)
- $\mathcal{H} \subseteq \bigcup_{k=0}^{\infty} A^k$ is the set of all sequences of actions (*histories*) satisfying
  - $\varnothing \in \mathcal{H}$ and
  - if $h = (a^{(0)}, a^{(1)}, \dots, a^{(\tau)}) \in \mathcal{H}$, any subsequence $h' = (a^{(0)}, a^{(1)}, \dots, a^{(t)})$ of h (for $t \leq \tau$) starting at the root is in $\mathcal{H}$,
- $Z \subseteq \mathcal{H}$ is the set of all *terminal histories*, where a history $h = (a^{(0)}, a^{(1)}, \dots, a^{(\tau-1)})$ is terminal if there exists no $a^{(\tau)} \in A$ with $(a^{(0)}, a^{(1)}, \dots, a^{(\tau)}) \in \mathcal{H}$,
- $\mathcal{X} : \mathcal{H} \setminus Z \to 2^A$, called the *action set selection function*, gives the set of all valid actions given a non-terminal history,
- $P : \mathcal{H} \setminus Z \to N$ is the *player function* which gives the player who plays at a given non-terminal history, and
- $u_i : Z \to \mathbb{R}$ is the utility of player i.

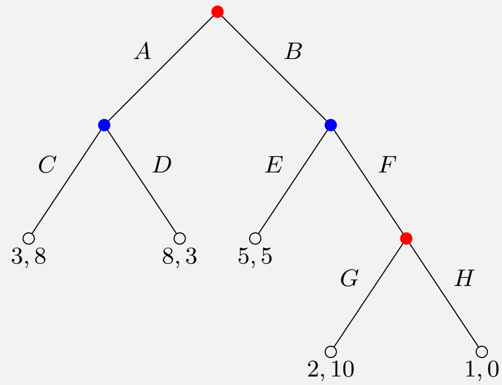
A history is essentially a path from the root in the game tree (recall our discussion of this from the first section).
A natural next question is: what is a strategy in a PIEFG?

Definition 3.2 (Strategy)**.** A *strategy* of a player in an EFG is a tuple of actions at every history where the player plays. That is, a strategy of a player i is an element of

$$S_i = \bigtimes_{h\in\mathcal{H}:P(h)=i} \mathcal{X}(h).$$

It turns out that we can transform EFGs to NFGs! Indeed, the above definition explicitly describes the strategy set of each player, and associated to any tuple of strategies for all the players, we can determine the payoff.
This conversion has a massive explosion in size however, and equilibria in the converted NFG do not necessarily make sense in the context of the original EFG.

Example 11 (PSNEs in PIEFGs)**.** Consider the following game with two players, red and blue.



At the node BF, red has no reason to play H since choosing G results in a higher payoff. However, it is still seen that (AH, CF) (at each node, the player chooses whichever of these actions is available) is a PSNE! This is a consequence of the fact that this strategy profile never even reaches the node where red chooses G or H, so it never comes into play.

Henceforth, we talk about PSNEs of EFGs as PSNEs of their NFG conversion.

Definition 3.3. A *subgame* of a game is the restriction of the game to the descendants of a history.

The idea behind an equilibrium should be *subgame perfection*, where each player chooses the best possible action at each subgame where they play.

Definition 3.4 (Subgame Perfect Nash Equilibrium)**.** A *subgame perfect nash equilibrium* (SPNE) of an EFG G is a strategy profile $s \in S$ such that for any subgame G' of G, the restriction of s to G' is a PSNE of G'.

Similar to PSNEs, SPNEs are guaranteed to exist in finite PIEFGs. Observe that any SPNE is a PSNE. As we shall now see, the algorithm to find an SPNE is quite simple.

Algorithm 1: Backward induction to determine SPNEs

Input: An EFG G
Output: The utility and action to be taken by a given player at a certain history

backInd(*history* h)
  if $h \in Z$ **then**
    return $u(h), \varnothing$
  $\mathsf{bestUtil}_{P(h)} \leftarrow -\infty$
  foreach $a \in \mathcal{X}(h)$ **do**
    $\mathsf{utilAtChild}_{P(h)} \leftarrow \mathtt{backInd}((h, a))$
    if $\mathsf{utilAtChild}_{P(h)} > \mathsf{bestUtil}_{P(h)}$ **then**
      $\mathsf{bestUtil}_{P(h)} \leftarrow \mathsf{utilAtChild}_{P(h)}$
      $\mathsf{bestAction}_{P(h)} \leftarrow a$
  return $\mathsf{bestUtil}_{P(h)}, \mathsf{bestAction}_{P(h)}$

The idea of subgame perfection is intrinsically tied to the above algorithm.
The issue however is that we are essentially parsing the entire tree, so the algorithm is computationally very expensive. Further, some criticize the idea of SPNEs for assuming that the cognitive limit of the players is infinite (which is not realistic).
It is easy to find an SPNE in simple games such as tic-tac-toe.

Example 12 (Centipede game)**.** Players 1 and 2 alternate, and each can play a move from $\{\mathsf{take}, \mathsf{push}\}$, with a maximum of N rounds, say. The game terminates when take is played for the first time or the limit of N rounds if take is never played.
The game also fixes some quantities m_0, m_1 with $m_0 > m_1$. Suppose that the game ends on round $t \in \{0, \ldots, N-1\}$ with player p making the final move; let p' be the other player. Then, the payoffs for the two players are as follows:

- if p played take, then p, p' have payoffs of $2^t m_0$ and $2^t m_1$ respectively.
- if p played push, then p, p' have payoffs of $2^{t+1} m_1$ and $2^{t+1} m_0$ respectively.

That is, if a player plays push, they increase the size of the pot to be won, and if they play take, the game ends, with them getting a larger amount of money.
Most players except grandmasters play for a few rounds, with some of the reasons claimed for this being altruism, the difference in computational capacity of individuals and incentive difference.

There are some other criticisms of SPNEs as well, such as that it discusses what to do if the game reaches a certain history, but the equilibrium in earlier stages might show that we cannot actually reach this history.

3.2. Imperfect information extensive form games

EFGs are perfect information systems, where every player has perfect information about all developments in the game until that round. This is not useful in many practical scenarios, such as card games. EFGs also do not allow the representation of games with simultaneous moves. For example, the neighbouring kingdom dilemma we discussed in Example 1 depends significantly on the fact that both kingdoms move simultaneously.
This can be modeled as a sequential game, where the second player does not known for certain which history the game is in. This is called an *information set*. Such games where players deal with information sets instead of particular histories are called imperfect information extensive form games. When the information sets are singletons, we get back PIEFGs.

Definition 3.5. An *imperfect information extensive form games* (IIEFG) is an 8-tuple $\langle N, A, \mathcal{H}, Z, \mathcal{X}, P, (u_i)_{i\in N}, (I_i)_{i\in N}\rangle$ where

- $\langle N, A, \mathcal{H}, Z, \mathcal{X}, P, (u_i)_{i\in N}\rangle$ is a PIEFG,
- for every $i \in N$, $I_i = (I_i^1, I_i^2, \ldots, I_i^{k(i)})$ is a partition of $\{h \in \mathcal{H} \setminus Z : P(h) = i\}$ with the property that $\mathcal{X}(h) = \mathcal{X}(h')$ and $P(h) = P(h') = i$ whenever there exists some j such that $h, h' \in I_i^j$. The I_i^js are called the *information sets* of player i, and I_i the collection of information sets of i.

At a given information set, the player and all their available actions ar ethe same. The player is not certain which history in the information set the game is at.
Since the actions now depend on the information sets and not the histories, we denote $\mathcal{X}(I_i^j) := \mathcal{X}(h)$ for any $h \in I_i^j$.
Strategies are also defined over information sets. The strategy set of player $i \in N$ is defined by

$$S_i = \bigtimes_{\tilde{I} \in I_i} \mathcal{X}(\tilde{I}).$$

With IIEFGs, NFGs can be represented using EFGs, though this is not very succinct. The representation used is typically chosen on the basis of the game we are working with.
In NFGs, we had mixed strategies where we randomized over pure strategies. In EFGs, randomization can be done in several ways:

- randomize over the strategies defined at the very beginning of the game, and
- randomize over the actions at a given information set – we call such a strategy a behavioral strategy. This takes advantage of all the information a player has at a certain point of time.

Definition 3.6. A *behavioral strategy* of a player in an IIEFG is a function that maps each of their information sets to a probability distribution over the set of actions at that information set

A couple of questions come to mind seeing this: can a player attain a higher payoff using one type of strategy? Is there any sort of equivalence between mixed and behavioral strategies.
Given a node x, mixed/behavioral strategy σ_i, and a mixed/behavioral strategy vector ξ_{-i}, denote by $\rho(x; \sigma_i, \chi_{-i})$ the probability of going to the node x in a game.

Definition 3.7 (Equivalence of mixed and behavioral strategies)**.** A mixed strategy σ_i and a behavioral strategy b_i of a player i in an IIEFG are *equivalent* if for every mixed/behavioral strategy vector ξ_{-i} of the other players and every vertex x in the game tree,

$$\rho(x; \sigma_i, \xi_{-i}) = \rho(x; b_i, \xi_{-i}).$$

Now, by definition, equivalence implies that the above equality holds at leaf nodes.
In fact, it suffices to check that it holds at leaf nodes! Given an arbitrary non-leaf nodes, the probability of reaching that node is equal to the sum of the probabilities of reaching the leaf nodes in the subtree, so equality at the leaves implies equality at non-leaves.
In fact, we get the following more generally.

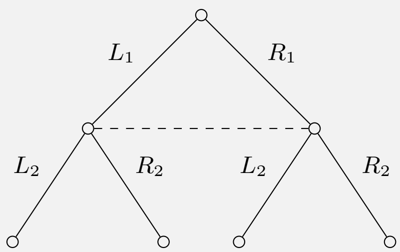
Theorem 3.1 (Utility Equivalence)**.** If σ_i, b_i are equivalent, then for every mixed/behavioral strategy vector of the other players ξ_{-i} and every $j \in N$,

$$u_j(\sigma_i, \xi_{-i}) = u_j(b_i, \xi_{-i}).$$

Corollary 3.2. Let σ and b be equivalent. Then, for all $i \in N$, $u_i(\sigma) = u_i(b)$.

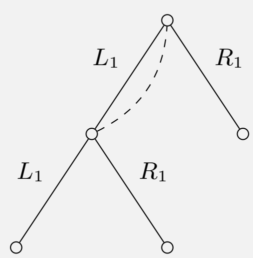
Behavioral strategies are more natural in IIEFGs. Players plan at a given stage (i.e. information set), and don't have a master plan from the very beginning. They are also far simpler computationally, since we only need to deal with fewer variables. For example, if a player has 4 information sets with 2 actions each, a mixed strategy would require $2^4 - 1$ variables whereas a behavioral strategy requires only 4.
It turns out that behavioral strategies and mixed strategies have incomparable power, as the following two examples demonstrate.

Example 13 (There exist mixed strategies without equivalent behavioral strategies)**.** Consider the following game tree with a single player.



Observe that there does not exist any behavioral strategy b for the player such that $\rho(b; L_1R_2) = \rho(b; R_1L_2) = 0$. Indeed, this would require $b_1(L_2) = b_1(R_2) = 0$.
On the other hand, there does exist a mixed strategy σ that sets $\sigma(L_1R_2) = \sigma(R_1L_2) = 0$. This is easily seen on realizing that $S_1 = \{L_1, L_2\} \times \{R_1, R_2\}$, so we have complete freedom in assigning probabilities to all four leaf ndoes.

Example 14 (There exist behavioral strategies without equivalent mixed strategies)**.** Consider the following game tree with a single player.



Observe that there exists no mixed strategy that goes to L_1R_1 with nonzero probability. Indeed, here, $S_1 = \{L_1, R_1\}$, so the path L_1R_1 is not even an option.
On the other hand, the behavioral strategy that picks L_1 or R_1 with equal probability $1/2$ does go to L_1R_1 with probability $1/4$.

In the first example above, the player remembers that they made a move but does not remember which move they made. In the second, the player does not remember that they made a move.

So, when exactly does a behavioral strategy not have an equivalent mixed strategy?

Lemma 3.3. If there exists a path from the root to some vertex x that passes through the same information set twice, and if the action leading to x is not the same at each of these vertices, then the player of the information set has a behavioral strategy that has no equivalent strategy.

Theorem 3.4. Consider an IIEFG such that every vertex has at least two actions. Every behavioral strategy has an equivalent mixed strategy iff each information set of a player intersects every path emanating from the root at most once.

When does a mixed strategy not have an equivalent mixed strategy?
We need to figure out some way of formalizing the forgetfulness in Example 13.

Let $X = (x_0, x_1, \ldots, x_K)$ and $\hat{X} = (x_0, \hat{x}_1, \ldots, \hat{x}_L)$ be two paths in the game tree and I_i^j an information set of player i that intersects the two paths are precisely one vertex, say x_k and $\hat{x}_\ell$ respectively where $k < K$, $\ell < L$. Then, the two paths choose the same action at I_i^j if the two actions at these vertices are identical, that is,

$$a_i(x_k \to x_{k+1}) = a_i(\hat{x}_\ell \to \hat{x}_{\ell+1}).$$

In general, we also consider $a_i(x \to y)$ (where y is a descendant but not necessarily a child of x), defined by the action $a_i(x \to x')$, where x' is the child of x on the path from x to y.
Player i is said to have perfect recall if

1. any path from the root to a leaf intersects an information set of player i at most once, and
2. any two paths that end in the same information set of player i pass through the same information sets of i in the same order, and in every such information set, the two paths choose the same action.

Let us state this more formally.

Definition 3.8 (Perfect recall)**.** Player i is said to have *perfect recall* if for every I_i^j and pair of vertices $x, \hat{x} \in I_i^j$, if the decision vertices of i are $x_i^1, x_i^2, \ldots, x_i^L = x$ and $\hat{x}_i^1, \hat{x}_i^2, \ldots, \hat{x}_i^{\hat{L}}$ respectively for the two paths from the root to x and $\hat{x}$, then

1. $L = \hat{L}$,
2. for any $1 \leq \ell \leq L$, $x_i^\ell, \hat{x}_i^\ell \in I_i^k$ for some k, and
3. for any $1 \leq \ell \leq L$, $a_i(x_i^\ell \to x_i^{\ell+1}) = a_i(\hat{x}_i^\ell \to \hat{x}_i^{\ell+1})$.

A game is said to be of perfect recall if every player has perfect recall.

Observe that perfect recall subsumes the condition in Theorem 3.4.
Let us look at some consequences of perfect recall.
Given a node x, let $S_i^*(x)$ be the set of pure strategies of player i at which he chooses the actions leading to x. This is the intersections of members of S_i with the paths from the root to x.

Theorem 3.5. If i is a player with perfect recall and $x, \hat{x}$ are two vertices in the same information set of i, then $S_i^*(x) = S_i^*(\hat{x})$.

The above conclusion follows because the same information sets are seen and we take the same actions at these sets.

Theorem 3.6 (Kuhn)**.** In an IIEFG, if i is a player with perfect recall, then for any mixed strategy of i, there exists an equivalent behavioral strategy.

The proof is constructive in nature, with the construction using perfect recall.

What are some notions of equilibria in IIEFGs? We could extend the subgame perfection in PIEFGs, but because the histories are uncertain, we are forced to extend to mixed strategies. Due to the information sets, a best response cannot be defined without the "belief" of each player.

Definition 3.9. *Belief* is a conditional probability distribution over the histories in an information set, where the conditioning is done over reaching the information set.

Henceforth, we only discuss IIEFGs with perfect recall.

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