Connectedness and Compactness

Lecture 18 - 05/03/2021 Connectedness

Def. Let X be a topological space. A separation of X is a pair U, V of disjoint non-empty open subsets of X whose union is X. X is said to be connected if it does not have a separation.

Prop. (3.1) A space X is connected iff the only subsets of X that are both open and closed ("clopen") are $\emptyset$ and X.

If U, V is a separation, U is clopen.

If $A \neq \emptyset$ is clopen, $A, X \setminus A$ is a separation. $\square$

Prop (3.2) If X and Y are homeomorphic, X is connected iff Y is connected.

Lemma (3.3) If Y is a subspace of X, a separation of Y is pair of disjoint non-empty sets A, B whose union is Y, neither of which contains a limit point of the other. The space Y is connected iff there is no separation of Y.

This is easily shown since the sets involved in a separation are clopen (in Y). $\bar{A} \cap Y = A$

$$A \cap B = \emptyset \rightarrow \bar{A} \cap Y \cap B = \emptyset$$
$$\rightarrow \bar{A} \cap B = \emptyset.$$

The other direction is similarly straightforward. $\square$

For example, any topological space with the indiscrete topology is connected.

→ Show that $\mathbb{Q}$ is not connected.

Lemma (3.4). If the sets C, D form a separation of X and Y is a connected subspace of Y, $Y \subseteq C$ or $Y \subseteq D$.

If not, we can write $Y = \underbrace{(Y \cap C) \cup (Y \cap D)}_{\text{open in } Y}$

Lemma (3.5). A union of connected spaces is connected if their intersection is non-empty.

Proof. Let (A_α) be a family of connected subspaces of X and $p \in \bigcap_\alpha A_\alpha$.
We claim that $Y = \bigcup_\alpha A_\alpha$ is connected.
Suppose C, D is a separation of Y and wlog that $p \in C$.
Since A_α is connected and $p \in C$, $A_\alpha \subseteq C$.
Therefore, $Y = \bigcup_\alpha A_\alpha \subseteq C$, contradicting the non-emptiness of D. $\square$

Theo. (3.6) Let A be a connected subspace of X. If $A \subseteq B \subseteq \bar{A}$, B is also connected.

(We can add any of the limit points without destroying connectedness)

Proof. Suppose C, D is a separation of B. Assume wlog that $A \subseteq C$. Then $B \subseteq \bar{A} \subseteq \bar{C}$. But $\bar{C} \cap D = \emptyset$, yielding a contradiction and proving the claim. $\square$

Theo. (3.7) The image of a connected space under a continuous map is connected.

Proof. Let $f : X \to Y$ be continuous where X is connected and f is surjective. Suppose C, D is a separation of Y. Since f is continuous, $f^{-1}(C)$ and $f^{-1}(D)$ are also open and they form a separation of X, resulting in a contradiction. $\square$

Theo. (3.8) A finite Cartesian product of connected spaces is connected.
(under either the box or product topo., they are equal)

Proof It suffices to show that if X and Y are connected $X \times Y$ is connected. (Use the fact that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic to $(X_1 \times \cdots \times X_{n-1}) \times X_n$
Fix $x \times y \in X \times Y$. $X \times \{y\}$ is connected (it is homeomorphic to X) and so is $\{x\} \times Y$. The result follows on using Theo 3.5. $\square$

Show that $\mathbb{R}^\omega$ under the box topology is disconnected.

Hint: Let $A = \{(a_n) : (a_n)$ is bounded$\}$ and
$B = \{(b_n) : (b_n)$ is unbounded$\}$.

Show that $\mathbb{R}^\omega$ under the product topology is connected.

Hint: Show that $\mathbb{R}^\infty$, the set of sequences eventually 0, is connected and that $\mathbb{R}^\omega = \overline{\mathbb{R}^\infty}$.

$\mathbb{R}^\infty = \bigcup_{n \in \mathbb{N}} \tilde{\mathbb{R}}^n$ $\rightarrow$ 0 after first n co-ordinates.

Theo. (3.9) An arbitrary product of connected spaces is connected in the product topology.
(the proof is nearly identical to that for $\mathbb{R}^\omega$ above)

Def. A simply ordered set L having more than one element is called a linear continuum if

- L has the least upper bound property.
- if $x < y$ in L, there exists z in L such that $x < z < y$.

Clearly, $\mathbb{R}$ is a linear continuum.

Theo. (3.10) If L is a linear continuum, then L, intervals in L, and rays in L, are connected.

Theo. [Intermediate Value Theorem]

(3.11) Let $f: X \to Y$ be continuous, where X is a connected space and Y is an ordered set under the ordered topology. If $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, there exists $c \in X$ such that $f(c) = r$.

Proof. Suppose otherwise. Then $f(X) \cap (-\infty, r)$ and $f(X) \cap (r, \infty)$ form a separation of $f(X)$. However, the image under f of X is connected, resulting in a contradiction. $\square$

Lecture 19 - 10/03/21 Path Connectedness

Def. Given points x, y of the space X, a path from x to y is a continuous function $f: [a,b] \to X$ such that $f(a) = x$ and $f(b) = y$ (for some closed interval $[a,b] \subseteq \mathbb{R}$).

A space X is path-connected if there is a path between any two points in X.

Theo. Any path-connected space is connected.
(3.12)

Proof. Suppose otherwise. Let X be path-connected and $f: [a,b] \to X$ be a path in X. Let $X = A \cup B$ be a separation of X.
Since $[a,b]$ is connected and f is continuous, $f([a,b]) \subseteq A$ or $f([a,b]) \subseteq B$, contradicting path-connectedness (across A, B). $\square$

The converse is not true.
Consider

$$S = \{ x \times \sin\left(\tfrac{1}{x}\right) : 0 < x \leq 1 \},$$

known as the topologist's sine curve.
Then $\bar{S} = S \cup (\{0\} \times [-1,1])$. We claim that $\bar{S}$ is not path-connected.
Let continuous $f: [a,c] \to \bar{S}$ beginning at the origin and ending at some point in S. The set

$$\{ t \in [a,c] : f(t) \in \{0\} \times [-1,1] \}$$

is closed (due to continuity), so it has a largest element b.
Then the restriction $f: [b,c] \to \bar{S}$ is a path such that $f(b) \in \{0\} \times [-1,1]$ and $f((a,c]) \subseteq S$.

Wlog, let $[b,c]$ be $[0,1]$ and $f(t) = (x(t), y(t))$. Then $x(0) = 0$ and for $t>0$, $x(t)>0$ and $y(t) = \sin(1/x(t))$.

For each n, choose a $0<u<x(1/n)$ such that $y(u) = (-1)^n$. Using the IVT (Theo 3.11), there is a $0<t_n<1/n$ such that $x(t_n)=u$. However, then, $t_n \to 0$ but $y(t_n) = (-1)^n$ does not converge, contradicting the continuity of y and proving the claim.

Show that S (and thus $\bar{S}$) is connected, disproving the converse of Theo 3.12

Def. Given X, define an equivalence relation as $x \sim y$ if there exists a connected subspace of X containing both x and y. The resulting equivalence classes are called the components or connected components of X.

Component

(Check that it is an equivalence relation)

Theo. (3.13) The components of X are connected disjoint subspaces of X whose union is X such that any non-empty connected subspace of X intersects exactly one of them.

Proof left as exercise.

To show that a component C is connected, fix $x \in C$ and for each $y \in C$, let $C_y \subseteq C$ be a connected subspace containing C_y.

by the second part

Then $x \in \bigcap C_y \neq \emptyset$, so $\bigcup C_y = C$ is connected. $\square$

Similar to connected components, we can define the path components of X.

Path Component

($x \sim y$ if there is a path from x to y)

(transitivity can be shown using the pasting lemma)

Theo. (3.14) The path components of X are path-connected disjoint subspaces of X whose union is X such that any non-empty path-connected subspace of X intersects exactly one of them.

Corollary (3.15). Any connected component of X is closed.

(Use the fact that the closure of a connected space is closed)

It follows that if there are finitely many components, each component is also open.

It need not be true that path-connected components are closed, however. Consider the topologist's sine curve $\bar{S}$. Then S is open in $\bar{S}$ (and not closed) and $\bar{S} \setminus S$ is closed (and not open).

Def A space X is said to be locally connected at $x \in X$ if for every neighbourhood U of x, there is a connected neighbourhood $V \subseteq U$ of X. X is locally connected if it is locally connected at any point of X.

Locally Connected

We similarly define local path-connectedness.

Theo. (3.16) A space X is locally connected iff for any open $U \subseteq X$, each component of U is open in X.

Proof. Let X be locally connected, U be open in X, and C be a component of U Let $x \in C$. There is then a neighbourhood $V \subseteq U$ of x that is connected. It follows that $V \subseteq C$ and therefore, C is open.

On the other hand suppose that the components of open sets in X are open. Let $x \in X$ and U a neighbourhood of x. We can take the component of U containing x, completing the proof $\square$

Theo. (3.17) A space X is locally path-connected iff for any open $U \subseteq X$, each path component of U is open.

The proof is nearly identical to the previous one.

__Lecture 20__ - 10/03/21 Introduction to Compactness

Theo. (3.18) If X is a topological space, each path component of X lies in a component of X. Moreover, if X is locally path-connected, the components and path components are the same.

Proof. The first part is direct since any path component is connected.
Let C be a component, $x \in C$, and $P \ni x$ be a path component.
Let X be locally path-connected. Suppose $P \subsetneq C$. Let Q be the union of all path components other than P that intersect C.
Then $C = P \cup Q$.
Because X is locally path connected, each path component of X is open in X. In particular, P and Q are open. This contradicts the connectedness of C, proving the claim.
↳ they form a separation of C

Def. A collection $\mathcal{A}$ of subsets of X is said to be a covering if $X = \bigcup_{A \in \mathcal{A}} A$. An open covering is a covering where every subset is open.
Cover

Compact

X is said to be compact if any open cover contains a finite subcover.

If Y is a subspace of X and $\mathcal{A}$ is a collection of subsets of X, $\mathcal{A}$ is said to cover Y if $Y \subseteq \bigcup_{A \in \mathcal{A}} A$.

Theo. (3.19) Let Y be a subspace of X. Then Y is compact iff every covering of Y by sets open in X contains a finite subcollection covering Y.

Theo. (3.20) Any closed subspace of a compact space is compact.

Hint: Consider the open cover $\mathcal{A} \cup \{X \setminus Y\}$

Theo. (3.21) Every compact subspace of a Hausdorff space is closed.

Proof. Let X be Hausdorff and Y a compact subspace. Let $x_0 \in X \setminus Y$.
For each $y \in Y$, choose neighbourhoods U_y of x_0 and V_y of y such that $U_y \cap V_y = \emptyset$
Since Y is compact, there exist $y_1, \dots, y_n$ such that

$$Y \subseteq \bigcup_{1 \leq i \leq n} V_{y_i} = V.$$

But $Y \cap U \subseteq V \cap U = \emptyset$, where

$$U = \bigcap_{1 \leq i \leq n} U_{y_i} \text{ is a neighbourhood of } x_0.$$

Therefore, Y is closed. $\square$

The above need not be true for non-Hausdorff spaces.
(Consider $\mathbb{R}$ under finite complement topology)

Lemma. (3.22) If Y is a compact subspace of the Hausdorff space X, and $x_0 \in X \setminus Y$, there are disjoint open sets U and V of X such that $x_0 \in U$ and $Y \subseteq V$.

Lecture 21 - 12/03/21 More on Compact Spaces

Theo. (3.23) The image of a compact space under a continuous map is compact.

Theo (3.24) Let $f: X \to Y$ be bijective and continuous. If X is compact and Y is Hausdorff, f is a homeomorphism.

(Show that f is a closed map)

Lemma (3.25) [Tube Lemma] Let X and Y be spaces with Y compact. Suppose $x_0 \in X$ and $N \supseteq \{x_0\} \times Y$ is an open subset of $X \times Y$. Then, there is a neighbourhood W of x_0 in X such that $W \times Y \subseteq N$.

$\underbrace{W \times Y}_{\text{tube about } x_0 \times Y}$

Tube Lemma

Proof. Let us cover $x_0 \times Y$ with the basis elements $U \times V$ (for the topology of $X \times Y$) lying in N.

Since $\{x_0\} \times Y$ is compact, there is a finite subcover

$$U_1 \times V_1, \ \ldots, \ U_n \times V_n$$

We may assume that $x_0 \in U_i$ for each i (Why?). Let

$$W = \bigcap_{1 \leq i \leq n} U_i .$$

Then W is open in X and contains x_0.

It is easily shown that $W \times Y$ is covered by the $(U_i \times V_i)$. Since each $U_i \times V_i \subseteq N$, $W \times Y \subseteq N$. $\square$

Theo. (3.26) The product of finitely many compact spaces is compact.

(It is in fact true for arbitrary products, which we shall see later in Tychonoff's Theorem)

Proof It suffices to show the result for two spaces. Let X, Y be compact and $\mathcal{A}$ an open subcover of $X \times Y$

For each $x_0 \in X$, $\{x_0\} \times Y$ can be covered by finitely many $A_1, \ldots, A_m \in \mathcal{A}$.

Then $\{x_0\} \times Y \subseteq A_1 \cup \cdots \cup A_m = N$, so N contains a tube $W \times Y$ containing $\{x_0\} \times Y$. $W \times Y$ is covered by the A_i $(1 \leq i \leq m)$.

That is, for each $x \in X$, there is a neighbourhood W_x of x such that $W_x \times Y$ can be covered by finitely many elements.

The collection of all W_x forms an open cover of X, so it has a finite subcover $W_1, W_2, \ldots, W_k$.

Then, $X \times Y \subseteq (W_1 \times Y) \cup (W_2 \times Y) \cup \cdots \cup (W_k \times Y)$.

Each is covered by finitely many elements from $\mathcal{A}$, so $X \times Y$ can as well. $\square$

Def. A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property if for any finite $\{C_1, C_2, \dots, C_k\} \subseteq \mathcal{C}$,

$$C_1 \cap C_2 \cap \dots \cap C_k \neq \emptyset.$$

Finite Intersection Property

Theo. (3.27) Let X be a topological space X is compact iff for every collection $\mathcal{C}$ of closed subsets of X with the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ is non-empty.

Proof. Forward direction

Suppose otherwise. Then $\mathcal{A} = \{X \setminus C : C \in \mathcal{C}\}$ is an open cover so has a finite subcover $A_1, \dots, A_m$. But then, $(X \setminus A_1) \cap \dots \cap (X \setminus A_m) = \emptyset$, contradicting the finite intersection property and proving the result.

Backward direction.

Let $\mathcal{A}$ be an open cover and $\mathcal{C} = \{X \setminus A : A \in \mathcal{A}\}$. Suppose $\mathcal{A}$ does not have a finite subcover. Then $\mathcal{C}$ has the finite intersection property so $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$. This contradicts $\mathcal{A}$ being a cover, proving the result. $\square$

Let us look at what the compact subspaces of the real line.

Theo. (3.28) Let X be a simply ordered set having the least upper bound property. In the order topology, each closed and bounded interval of X is compact.

Proof. Let $a < b$ and $\mathcal{A}$ be an open cover of $[a,b]$ in the subspace topology (which is the same as the order topology since $[a,b]$ is compact).

Claim 1. If $x \in [a,b)$, there exists $y > x$ in $[a,b]$ such that $[x,y]$ can be covered by atmost two elements of $\mathcal{A}$.

- → If x has an immediate successor, let y be this element. Then $[x,y] = \{x,y\}$, so the claim is obvious.
- → Otherwise, choose $A \in \mathcal{A}$ containing x. Since A is open, it contains an interval of the form $[x,c)$. We can then let y be any element of (x,c).

Now, let C be the set of all points $y>a$ such that $[a,y]$ can be covered by finitely many elements of $\mathcal{A}$. (We want to show that $b\in C$.)
By the claim, $C\neq\emptyset$. Let c be the least upper bound of C.
Claim 2. $c\in C$.
Choose $A\in\mathcal{A}$ containing c. A contains an interval of the form $(d,c]$.
If c does not have an immediate predecessor (even otherwise, it is similarly shown)
Let $z\in C$ such that $z\in(d,c)$ (Why does such a z exist?).
Then $[a,z)$ can be covered by finitely many elements $A_1,\dots,A_k$ of $\mathcal{A}$, so it follows that $\{A_1,\dots,A_k\}\cup\{A\}$ is a finite subset of $\mathcal{A}$ covering $[a,c]$, proving the result.

Finally, let us prove that $b=c$. Suppose otherwise Then there exists $y\in(c,b]$ such that $[c,y]$ can be covered by atmost two elements. This implies that $[a,y]$ can be covered by finitely many elements, contradicting the fact that c is an upper bound of C, proving the result. $\square$

Corollary (3.29). Any closed and bounded interval in $\mathbb{R}$ is compact.
↳ w.r.t. Euclidean metric

Lecture 21 - 17/03/21 Local Compactness

Theo. (3.30) A subspace A of $\mathbb{R}^n$ iff it is closed and bounded in the Euclidean metric or square metric ρ.

Proof It suffices to consider the metric ρ (Why?).
Suppose $A\subseteq\mathbb{R}^n$ is compact.
→ Since $\mathbb{R}^n$ is Hausdorff, A is closed.
→ Consider the collection of open sets $\{B_\rho(\mathbf{0},m): m\in\mathbb{N}\}$, which forms a covering of A. Since it has a finite subcover, it follows that $A\subseteq B_\rho(\mathbf{0},n)$ for some $n\in\mathbb{N}$ and A is bounded
Suppose $A\subseteq\mathbb{R}^n$ is closed and bounded. Suppose $\rho(x,y)<M$ for $x,y\in A$.
Fix some $x_0\in A$. Then A is a subset of $\prod_{i=1}^n [(x_0)_i-M,(x_0)_i+M]$, which is compact (finite product of compact sets) Since A is closed in this subspace, it is compact. $\square$

Def. If X is a space, $x \in X$ is said to be an isolated point of X if $\{x\}$ is open in X.

Isolated Point

Theo. (3-31) Let X be a non-empty compact Hausdorff space. If X has no isolated points, it is uncountable

Proof. → We show that given any non-empty open $U \subseteq X$ and $x \in X$, there is a non-empty open $V \subseteq U$ such that $x \notin \overline{V}$.

Choose a $y \in U$ different from x (Why does such a y exist?). Let W_1 and W_2 be disjoint neighbourhoods of x and y. $V = W_2 \cap U$ is the required set — $x \notin \overline{V}$ because $x \in W_1$ and $W_1 \cap V = \emptyset$.

→ We show that given any $f: \mathbb{N} \to X$, f is not surjective.

Let $x_n = f(n)$. Let $V_1 \subseteq X$ be an open set such that $x \in \overline{V_1}$ (choosing $U = X$). In general, given non-empty open V_{n-1}, let $V_n \subseteq V_{n-1}$ be a non-empty open set such that $x_n \notin \overline{V_n}$. Then

$$\overline{V_1} \supseteq \overline{V_2} \supseteq \cdots \supseteq \overline{V_n} \supseteq \cdots.$$

Let $V = \bigcap_{n=1}^{\infty} V_n$. Observe that the above collection of subsets has the finite intersection property. Therefore V is non-empty and closed. Let $x \in V$. Then $\underset{\in \overline{V_n}}{x} \neq \underset{\notin \overline{V_n}}{x_n}$ for any n, completing the proof. $\blacksquare$

Def. A space X is said to be locally compact at x if there is some compact subspace C of X that contains a neighbourhood of x.

Locally Compact

If X is locally compact at each $x \in X$, X is said to be locally compact.

Observe that any compact space is locally compact (taking $C = X$).

A slightly more non-trivial example is that $\mathbb{R}$ is locally compact.

If X is Hausdorff and $\mathcal{B}$ forms a basis, then X is locally compact iff for all $x \in X$, there is $B \in \mathcal{B}$ such that $x \in B$ and $\overline{B}$ is compact.

→ Show that $\mathbb{Q}$ is not locally compact.

Theo. (3.32) Let X be a space. X is locally compact and Hausdorff iff there is a space Y such that

One-Point Compactification

1. X is a subspace of Y.
2. $Y \setminus X$ is a singleton.
3. Y is compact and Hausdorff.

Further, if Y and Y' are two spaces satisfying the above, they are homeomorphic with a homeomorphism whose restriction to X is the identity map.

Proof → We first check uniquness up to homeomorphism. Let $h: Y \to Y'$ such that $h(x) = x$ for $x \in X$ and the single point $p \in Y$ is mapped to the single point $p' \in Y'$. h is clearly bijective.

Let U be open in Y.

- If $p \notin U$, $h(U) = U$ is clearly open. (X is open in Y or Y' since $\{p\}$ and $\{p'\}$ are closed)
- Suppose $p \in U$. Since $C = Y \setminus U$ is closed in Y, it is compact in Y. Further, it is a compact subspace of X. Since X is a subspace of Y', C is a compact subspace of Y'. Because Y' is Hausdorff, C is closed in Y' so $h(U) = Y' \setminus C$ is open in Y'.

Therefore, h is open. Similarly, h is also continuous so it is a homeomorphism.

→ Suppose X is locally compact and Hausdorff. Take some point $\infty \notin X$. Let $Y = X \cup \{\infty\}$. Give Y a topology as:

i) all sets U open in X are open in Y.

ii) all sets of the form $Y \setminus C$ are open in Y, where C is a compact subspace of X.

Why is this a topology?

- $\emptyset$, Y are open by i, ii respectively.
- $U_1 \cap U_2$ is open by i

  $(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2)$ is open by ii

  $U_1 \cap (Y \setminus C_1) = U_1 \cap (X \setminus C_1)$ is open by i

- Similarly, an arbitrary union of open sets is open
  $\bigcup_\alpha U_\alpha$ is open by i
  $\bigcup_\alpha (Y\setminus C_\alpha) = Y\setminus(\bigcap_\alpha C_\alpha) = Y\setminus C$ is open by ii
  $\bigcup_\alpha U_\alpha \cup \bigcup_\beta (Y\setminus C_\beta) = U \cup (Y\setminus C) = Y\setminus \underbrace{(C\setminus U)}_{\text{closed}}$ is open
  $C\cap(X\setminus U)$

Next, let us show that X is a subspace of Y. Let U be open in Y.
- If U is of type i, $U\cap X$ is open in X.
- If $U = Y\setminus C$ is of type ii, $(Y\setminus C)\cap X = X\setminus C$ is open in X because C is closed in X.

(it is obvious that for any open V in X, there is an open $\underset{\supseteq V}{U}\subseteq Y$ s.t $V = U\cap X$)

Let $\mathcal{A}$ be an open covering of Y. $\mathcal{A}$ must contain some open set $(Y\setminus C)$ of type ii (∞ is not in any set of type i).
Consider all the elements of $\mathcal{A}$ other than $Y\setminus C$ and intersect them with X. This forms an open cover of C.
Because C is compact, finitely many of them cover C. The corresponding elements of $\mathcal{A}$ together with $Y\setminus C$ form a finite subcover of Y.
Therefore, Y is compact.

Let $x,y\in Y$ with $x\neq y$. If $x,y\in X$, then there are clearly disjoint open sets containing them. Otherwise, suppose $x\in X$ and $y=\infty$. Let C be a compact subspace of X containing a neighbourhood U of x.
Then U and $Y\setminus C$ are disjoint neighbourhoods of x and y.

—

→ Let us now prove the converse. We claim that X is locally compact and Hausdorff. Denote by ∞ the element of $Y\setminus X$.
X is Hausdorff by Theo. 1.20 (c).
Let $x\in X$. Let U, V be disjoint neighbourhoods of x and ∞. Then $(Y\setminus V)$ is closed in Y, and thus compact. The required follows since $U\subseteq (Y\setminus V)$.
as a subspace of X
since $Y\setminus V\subseteq X$.

$\square$

Lecture 22 - 19/03/21 Countability Axioms

Def. If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y, then Y is said to be a compactification of X. If $Y \setminus X$ is a singleton, Y is called a one-point compactification.

Compactification

In Theo 3.32, if X is not compact, Y is its one-point compactification.

→ Show that the one-point compactification of $\mathbb{R}^2$ (or $\mathbb{C}$) is homeomorphic to S^2.
$\mathbb{C} \cup \{\infty\}$ is called the Riemann sphere or extended complex plane.
If N is the north pole of S^2, $S^2 \setminus \{N\} \simeq \mathbb{C}$.
(by the stereographic projection)

Theo. (3.33) Let X be a Hausdorff space. Then X is locally compact iff given $x \in X$ and a neighbourhood U of x, there exists a neighbourhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$

Proof The backward direction is obvious since $x \in V \subseteq \overline{V}$
To prove the converse, let U open and $x \in U$. Let Y be the one-point compactification of X and $C = Y \setminus U$.
Then C is compact in Y. Therefore, there exist open sets $V \ni x$ and $W \supseteq C$ with $\overline{V} \cap W = \emptyset$.
But $\overline{V}$ is compact in Y (and thus X), and $\overline{V} \cap C = \emptyset \Rightarrow \overline{V} \subseteq U$ as required. ◻

Def. A space X is said to have a countable basis at $x \in X$ if there is a countable collection $\mathcal{B}$ of neighbourhoods of x such that any neighbourhood of x contains an element of $\mathcal{B}$.
A space that has a countable basis at each of its points is said to be first countable.

First countable

For example, any metrizable space is first countable — consider
$$\{B_d(x, 1/n) : n \in \mathbb{N}\}$$

Theo. (3.34) Let X be a topological space.

a) Let $A \subseteq X$. If there is a sequence of points in A that converges to $x \in X$, then $x \in \overline{A}$. The converse holds if X is first countable.

b) Let $f: X \to Y$ be continuous. If (x_n) be a sequence of points in X that converges to x, then $f(x_n) \to f(x)$. The converse holds if X is first countable.

Proof Recall that we have proved the above for metrizable X by considering $B_d(x, 1/n)$.

Here, just consider

$$B_n = U_1 \cap \cdots \cap U_n \text{ instead,}$$

where $\{U_n\}$ forms a countable basis at x.

The proof is relatively straightforward — let $x_n \in B_n \cap A$ for each n.

For (b), show that $f(\overline{A}) \subseteq \overline{f(A)}$. $\blacksquare$

Def A topological space X is said to be second countable if it has a countable basis (for the topology).

Second countable

Prop. (3.35) Any second countable space is first countable.

The proof is direct.

For example, $\mathbb{R}$ is second countable — consider $\{(a,b) : a, b \in \mathbb{Q}\}$.

Similarly, $\mathbb{R}^n$ is second countable as well.

$\mathbb{R}^\omega$ under the product topology is second countable as well.

$$\left\{ \prod_{n \in \mathbb{N}} U_n : U_n = (a,b) \text{ for } a,b \in \mathbb{Q} \text{ for finitely many } n \text{ and } \mathbb{R} \text{ otherwise} \right\}$$

↳ Why is this set countable?

→ Show that $\mathbb{R}^\omega$ under the uniform topology is not second countable.

Hint: Show that the subspace containing sequences of 0s and 1s has the discrete topology so does not have a countable basis. Use this to prove the required.

Theo. (3.36) A subspace of a first (resp. second) countable space is first (second) countable and a countable product of first (resp second) countable spaces is first (second) countable.

Proof. We prove the above for the second countable case. If X has a countable basis $\mathcal{B}$, then $\{B \cap A : B \in \mathcal{B}\}$ is a countable basis for $A \subseteq X$ as a subspace.
If X_i has countable basis $\mathcal{B}_i$, then

$$\{\Pi U_i : U_i \in \mathcal{B}_i \text{ for finitely many } i \text{ and } U_i = X_i \text{ otherwise}\}$$

forms a countable basis of $\Pi_i X_i$. $\square$

Recall that a subset A of a space X is dense in X iff $\overline{A} = X$.

Theo. (3.37) Let X be a second countable space.
(a) Every open covering of X contains a countable subcover.
(b) X has a countable dense subset.

Proof. (a) Let $\mathcal{A}$ be an open covering of X. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis of X. For each n, if possible, choose an $A_n \supseteq B_n$. Let $\mathcal{A}'$ be the collection of these A_n. It is clearly countable. Further, it covers X. Given $x \in X$, let $A \in \mathcal{A}$ such that $x \in A$ and $B_n \supseteq A$ be a basis element. Then $x \in A_n$, proving the claim.

(b) For each B_n, let $x_n \in B_n$. Let $D = \{x_n : n \in \mathbb{N}\}$. Then D is a countable dense subset since any basis element of X (and so any open set) intersects D. $\square$

Def. A space for which every open cover contains a countable subcover is called a Lindelöf space.

Lindelöf space

A space having a countable dense subset is said to be separable.

Separable space

Lecture 23 - 24/03/21 Separation Axioms

Obviously, any compact set is Lindelöf.

→ Show that $\mathbb{R}_\ell$ satisfies all countability axioms except the second.

The product of two Lindelöf spaces need not be Lindelöf.

→ Show that $\mathbb{R}_\ell$ is Lindelöf but $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not.

Hint: Consider $\mathbb{R}_\ell^2 \setminus \{(x,-x) . x \in \mathbb{R}_\ell\}$.

A subspace of a Lindelöf space need not be Lindelöf either.

→ Consider the ordered square $I_o^2 = [0,1]^2$ (under the dictionary order). Show that I_o^2 is compact (and Lindelöf) but the subspace $A = I_o \times (0,1)$ is not Lindelöf.

Def. Suppose that one point sets are closed in X. Then X is said to be regular if for each $x \in X$ and closed B disjoint from x, there exist disjoint open sets containing x and B.

Regular

The space is said to be normal if for disjoint closed A, B, there exist disjoint open sets containing A and B.

Normal

Observe that any normal space is regular and any regular space is Hausdorff.

Lemma (3.38). Let one-point sets be closed in X.

a) X is regular iff given $x \in X$ and a nbd. U of x, there exists a neighbourhood V of x such that $\bar{V} \subseteq U$.

b) X is normal iff given a closed $A \subseteq X$ and open $U \supseteq A$, there is an open set $V \supseteq A$ with $\bar{V} \subseteq U$.

Proof a) (Forward) Let $B = X \setminus U$ be closed. There exist disjoint open V and W containing x and B. Then $\bar{V} \cap B = \emptyset$. Therefore, $\bar{V} \subseteq U$.

(Backward) Let $x \in X$ and B disjoint from $\{x\}$ be closed. Let $U = X \setminus B$. Let V be a nbd. of x such that $\overline{V} \subseteq U$. Then V and $X \setminus \overline{V}$ are disjoint open sets containing x and B respectively. Therefore, X is regular.

The argument for (b) is nearly identical, taking A instead of $\{x\}$. $\blacksquare$

Theo. (3.39) A subspace of a regular space is regular. A product of regular spaces is regular.

Proof. Let X be regular and $Y \subseteq X$. One-point sets are closed in Y. Let $x \in Y$ and B a closed subset of Y disjoint from $\{x\}$. Then $\overline{B} \cap Y = B$. Therefore, $x \notin \overline{B}$. Using regularity, let U, V be disjoint open sets containing x and $\overline{B}$. Then $(U \cap Y)$ and $(V \cap Y)$ are disjoint open sets of Y containing x and B.

Let (X_α) be a family of regular spaces and $X = \prod_\alpha X_\alpha$.
X is Hausdorff, so singletons are closed in X. Let $x \in X$ and U be a neighbourhood of x. Let $\prod_\alpha U_\alpha$ be a basis element containing x and contained in U. For each α, let V_α be a nbd. of x_α such that $\overline{V_\alpha} \subseteq U_\alpha$. If $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$. Then $V = \prod_\alpha V_\alpha$ is a neighbourhood of x in X. Since $\overline{V} = \prod_\alpha \overline{V_\alpha}$, $x \in \overline{V} \subseteq U$, so X is regular. $\blacksquare$
(by Theo 2.9)

→ Show that $\mathbb{R}_K$ is Hausdorff but not regular

Hint: Consider 0 and K.

→ Show that $\mathbb{R}_\ell$ is normal.

Theo. (3.40) Any second countable regular space is normal

Proof Let X be regular with countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. Each $x \in A$ has a nbd. U disjoint from B. Choose a nbd. V of x such that $\overline{V} \subseteq U$. Then, choose an element of $\mathcal{B}$ containing x and contained in V.

This gives a countable covering (U_n) of A by open sets whose closures do not intersect B. Similarly, choose a countable covering (V_n) of B.
$\bigcup U_n$ and $\bigcup V_n$ are open sets containing A and B, but need not be disjoint.
For each n, let

$$U_n' = U_n \setminus \bigcup_{i=1}^{n} \overline{V_i} \quad \text{and} \quad V_n' = V_n \setminus \bigcup_{i=1}^{n} \overline{U_i}.$$

Each U_i' and V_i' is open. Also, (U_n') covers A because for any $x \in A$, $x \in U_n$ for some n but $x \notin \overline{V_i}$ for $1 \leq i \leq n$.

The open sets

$$U' = \bigcup U_n' \quad \text{and} \quad V' = \bigcup V_n'$$

are disjoint. It is easy to show that U' and V' are disjoint. $\square$

(3.41) **Theo.** Any metrizable space is normal.

Proof. Let X be metrizable with metric d. Let A, B be disjoint closed subsets of X. For each $a \in A$, choose ε_a such that $B_d(a, \varepsilon_a) \cap B = \emptyset$.

Choose ε_b similarly. Then, let

$$U = \bigcup_{a \in A} B_d(a, \varepsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B_d(b, \varepsilon_b/2).$$

It is easy to show that these are disjoint (they are clearly open and contain A, B), completing the proof. $\square$

Theo. Any compact Hausdorff space is normal.
(3.42)

Left as exercise.

Lecture 24 - 26/03/21 Urysohn Lemma and Completely Regular Spaces

Theo. [Urysohn Lemma]

(3.43) Let A, B be disjoint closed subsets of X. If X is normal, then for a closed interval $[a,b]$ in the real line, there exists a continuous map $f: X \to [a,b]$ such that $f(x) = a$ for every $a \in A$ and $f(x) = b$ for every $b \in B$.

Proof Clearly, it suffices to take $[a,b] = [0,1]$.

Let $P = \mathbb{Q} \cap [0,1]$. For each $p \in P$, we define open U_p such that if $p < q$, $\overline{U_p} \subseteq U_q$.

Arrange P as an infinite sequence (p_n) and for convenience, let $p_1 = 0$ and $p_2 = 1$. Let $U_1 = X \setminus B$. Because A is closed and $A \subseteq U_1$, we may choose (by normality) an open U_0 such that $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$.

In general, let $P_n = \{p_k : 1 \leq k \leq n\}$ and suppose open U_p is defined for $p \in P_n$ $(n \geq 2)$ such that $p < q \Rightarrow \overline{U_p} \subseteq U_q$.

Let $r = p_{n+1}$. Since P_{n+1} is finite, it has a simple ordering $<$ (derived from the usual ordering).
Let p_i and p_j be the immediate predecessor and successor respectively in P_{n+1} (Why do these exist?)
Now, choose U_r as an open set such that $\overline{U_{p_i}} \subseteq U_r \subseteq \overline{U_r} \subseteq U_{p_j}$ — such a U_r exists by using normality on the sets $\overline{U_{p_i}}$ and $X \setminus U_{p_j}$.
$\rightarrow$ This defines U_p for $p \in P$ such that $p<q \Rightarrow \overline{U_p} \subseteq U_q$.
Extend this to define U_p for all $p \in \mathbb{Q}$ as $U_p = \emptyset$ if $p<0$ and $U_p = X$ if $p>1$

Given $x \in X$, let $\mathbb{Q}(x) = \{p \in \mathbb{Q} : x \in U_p\}$.
Observe that $\mathbb{Q}(x)$ is bounded below (by, say, -1)
$\rightarrow$ Let $f(x) = \inf \mathbb{Q}(x) = \inf\{p \in \mathbb{Q} : x \in U_p\}$.
We claim that f is the desired function.
Note that $f(x) \in [0,1]$ for any $x \in X$ (Why?). (*)
For any $x \in A \subseteq U_0$, $f(x) = 0$.
For any $x \in B$, $x \notin p$ for any $p \leq 1$ ($U_1 = X \setminus B$). By (*), $f(x) = 1$.
It remains to show that f is continuous.
Observe that

- if $x \in \overline{U_r}$, $f(x) \leq r$
- if $x \notin U_r$, $f(x) \geq r$

This follows from the denseness of rationals.

Let $x_0 \in X$ and (c,d) be in $\mathbb{R}$ containing $f(x_0)$.
Choose rationals p,q such that $c<p<f(x_0)<q<d$.
Let $U = U_q \setminus \overline{U_p}$.
Then,

$$\left.\begin{array}{l} f(x_0)<q \Rightarrow x_0 \in U_q \\ f(x_0)>p \Rightarrow x_0 \notin \overline{U_p} \end{array}\right\} \Rightarrow x_0 \in U$$

- Let $x \in U$. Then $f(x) \in U_q \subseteq \overline{U_q} \Rightarrow f(x) \leq q < d$

$$f(x) \notin \overline{U_p} \supseteq U_p \Rightarrow f(x) \geq p > c$$

$\Rightarrow f(x_0) \in \underset{\text{open}}{U} \subseteq (c,d)$, so f is continuous by Theo 2.1 (iv). $\square$

Observe that the converse holds too — we may take $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$

Def A space X is *completely regular* if one-point sets are closed in X and for each $x_0 \in X$ and closed $A \not\ni x_0$, there is a continuous function $f: X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

By the Urysohn Lemma, normality implies complete regularity

The axioms are labelled as

- T_1: for any x, y, there are open U, V such that $x \in U \not\ni y$ and $y \in V \not\ni x$.
- T_2: Hausdorff
- T_3: Regular
- $T_{3\frac{1}{2}}$: Completely regular
- T_4: Normal

Theo. A subspace of a completely regular space is completely regular.
(3.44) A product of completely regular spaces is completely regular.

Lecture 25 - 31/03/21 The Urysohn Metrization Theorem and Tychonoff's Theorem

Theo. [Urysohn Metrization Theorem]

A regular second countable space X is metrizable

Proof. We shall embed X in a metrizable space Y

Let $Y = \mathbb{R}^\omega$ under the product topology. We have already seen that Y is metrizable under the metric

$$D(x,y) = \sup_i \left\{ \frac{\min\{|x_i - y_i|, 1\}}{i} \right\}.$$

(The proof can also be carried out by taking Y as $\mathbb{R}^\omega$ under the uniform topology)

We will in fact embed X in $[0,1]^\omega$.

→ Claim 1 There exists a countable collection of continuous functions $f_n : X \to [0,1]$ such that for any $x_0 \in X$ and nbd. U of x_0, there is some n such that $f_n(x_0) \neq 0$ and $f(x) = 0$ for $x \in X \setminus U$.

- Let (B_n) be a countable basis for x. For each pair n,m with $\overline{B_n} \subseteq B_m$, use Theo. 3.43 to get a continuous function $g_{n,m} : X \to [0,1]$ such that $g_{n,m}(B_n) = \{1\}$ and $g_{n,m}(X \setminus B_m) = \{0\}$.

  Then given any $x_0 \in X$ and neighbourhood U of x_0, we can choose a basis element B_m with $x_0 \in B_m \subseteq U$. By regularity and Lemma 3.38, we can let B_n be a basis element with $x_0 \in B_n \subseteq \overline{B_n} \subseteq B_m$.

  $(g_{n,m})$ then satisfies our requirements.

→ For this (f_n), define $F : X \to Y$ by

$$F(x) = (f_1(x), f_2(x), f_3(x), \dots)$$

- Because Y has the product topology and each f_n is continuous, F is continuous.
- For $x \neq y$, there is some index n such that $f_n(x) \neq 0$ and $f_n(y) = 0$. So, $F(x) \neq F(y)$ and F is injective.
- We must show that F is a homeomorphism of X to $f(x)$. We have already shown that it is a continuous bijection. Let U be open in X. Let $z_0 \in F(U)$ and $x_0 \in X$ with $F(x_0) = z_0$. Let N be such that $f_N(x_0) \neq 0$ and $f_N(X \setminus U) = \{0\}$.

  Let $V = \pi_N^{-1}((0,\infty)) \subseteq \mathbb{R}^\omega$. Let $W = V \cap F(X)$ be open in $F(X)$. Now, $\pi_N(z_0) = f_N(x_0) > 0$, so $z_0 \in W$. Further, $W \cap F(X \setminus U) = \emptyset$, so $W \subseteq F(U)$. Therefore, $F(U)$ is open and F is a homeomorphism. $\square$

Theo. [Embedding Theorem]

Let X be a space in which one-point sets are closed. Let $(f_\alpha)_{\alpha\in J}$ be a family of continuous function $X \to \mathbb{R}$ such that for any $x_0 \in X$ and nbd. U of x_0, there is $\alpha \in J$ such that $f_\alpha(x_0) > 0$ and $f_\alpha(X \setminus U) = \{0\}$. Then $F : X \to \mathbb{R}^J$ defined by $(F(x))_\alpha = f_\alpha(x)$ is an embedding of X in $\mathbb{R}^J$.

A family of continuous functions that satisfies the hypothesis of the above theorem is said to separate points from closed sets.
For a space in which one-point sets are closed, this is seen to be equivalent to X being completely regular.

Corollary. A space X is completely regular iff it is homeomorphic to $[0,1]^J$ for some J.

Theo. [Tietze Extension Theorem]

Let X be normal and A be closed in X.

(a) Any continuous map $A \to [a,b] \subseteq \mathbb{R}$ may be extended to a continuous map $X \to [a,b]$.

(b) Any continuous map $A \to \mathbb{R}$ may be extended to a continuous $X \to \mathbb{R}$.

The Tietze Extension Theorem can be used to prove the Urysohn Lemma.
(but its proof uses the Urysohn Lemma)

Theo. [Tychonoff's Theorem]
An arbitrary product of compact spaces is compact.

Let $(X_\alpha)_{\alpha \in J}$ be compact and $X = \prod_\alpha X_\alpha$.
We first prove a couple of lemmas.

Claim 1. Let X be a set and $\mathcal{A}$ a collection of subsets having the finite intersection property. Then there is a $\mathcal{D}$ such that $\mathcal{A} \subseteq \mathcal{D} \subseteq 2^X$, $\mathcal{D}$ has the finite intersection property, and no $\mathcal{F}$ with $\mathcal{D} \subsetneq \mathcal{F} \subseteq 2^X$ has the finite intersection property.

Proof. We use Zorn's Lemma to prove this.

↳ Given a strictly partially ordered set A in which every simply ordered subset has an upper bound, A has a maximal element.

The strict poset we consider is a set of collections of subsets of X. Let

$$\mathbb{C} = \{ \mathcal{B} \subseteq 2^X : \mathcal{A} \subseteq \mathcal{B} \text{ and } \mathcal{B} \text{ has the finite intersection property} \}$$

with the strict partial order strict inclusion $\subsetneq$.
We want to show that $\mathbb{C}$ has a maximal element $\mathcal{D}$.
Let $\mathbb{B} \subseteq \mathbb{C}$ be a simply ordered subset. It suffices to show that

$$\mathcal{C} = \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B} \in \mathbb{C}$$

and is an upper bound of $\mathbb{B}$ (which is obvious)
It is clear that $\mathcal{A} \subseteq \mathcal{C}$. Let $C_1, C_2, \cdots, C_n \in \mathcal{C}$. For each i, choose $\mathcal{B}_i \in \mathbb{B}$ such that $C_i \in \mathcal{B}_i$.
$\{\mathcal{B}_i : 1 \le i \le n\}$ is simply ordered by proper inclusion and is finite, so has a maximal element $\mathcal{B}_k$. Then $C_1, C_2, \cdots, C_n \in \mathcal{B}_k$. Since $\mathcal{B}_k$ has the finite intersection property, $\bigcap_{1 \le i \le n} C_i \neq \emptyset$, so $\mathcal{C}$ has the finite intersection property.
Using Zorn's Lemma completes the proof. $\blacksquare$

Lecture 26 - 31/03/21 Completing the proof of Tychonoff's Theorem

Claim 2. Let X be a set and $\mathcal{D} \subseteq 2^X$ be as defined in the previous claim.

a) If B is a finite intersection of elements of $\mathcal{D}$, $B \in \mathcal{D}$.

b)

Proof a) Let B equal the intersection of finitely many elements in $\mathcal{D}$ and $\mathcal{E} = \mathcal{D} \cup \{B\}$. We show that $\mathcal{E}$ has the finite intersection property, so $\mathcal{E} = \mathcal{D}$.

Take finitely many elements of $\mathcal{E}$.

→ If none of them is B, their intersection is clearly nonempty.

→ If B is one of them, we can expand B as a finite intersection to get that the overall intersection is non-empty.

b) Left as exercise (idea similar to a)

We now come to the main proof of Tychonoff's Theorem. Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property.

We show that $\bigcap_{A \in \mathcal{A}} \overline{A} \neq \emptyset$

By Claim 1, choose $\mathcal{D} \supseteq \mathcal{A}$ as defined.

It suffices to show that $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$.

Consider for each $\alpha \in J$

$$\mathcal{D}_\alpha = \{\pi_\alpha(D) \cdot D \in \mathcal{D}\} \subseteq 2^{X_\alpha}$$

Because $\mathcal{D}$ has the finite intersection property, so does $\mathcal{D}_\alpha$.

By compactness, we may choose for each α, $x_\alpha \in X_\alpha$ such that

$$x_\alpha \in \bigcap_{D_\alpha \in \mathcal{D}_\alpha} \overline{D_\alpha}$$

Let $x = (x_\alpha)_{\alpha \in J} \in X$. If we show that $x \in \overline{D}$ for any $D \in \mathcal{D}$, we are done.

Let $D \in \mathcal{D}$ and U_β be a nbd of $x_\beta \in X_\beta$. Since $x_\beta \in \overline{\pi_\beta(D)}$, we can choose $y \in D$ such that $\pi_\beta(y) \in U_\beta \cap \pi_\beta(D)$.

Then, $y \in \pi_\beta^{-1}(U_\beta) \cap D$.

From (b) of Claim 2, every subbasis element containing x belongs to $\mathcal{D}$.

By (a) of Claim 2, every basis element containing x belongs to $\mathcal{D}$ and intersects every element of $\mathcal{D}$. Therefore, $x \in \overline{D}$ for all $D \in \mathcal{D}$.