Maps Between Topological Spaces

Def Let X and Y be topological spaces. A function $f: X \to Y$ is said to be
Continuity **continuous at $b \in X$** if for any open $V \subseteq Y$ with $f(b) \in V$, there exists open $U \ni b$ (in X) such that $f(U) \subseteq V$.
f is **continuous** if for any open V in Y, $f^{-1}(V)$ is open in X.

Note that f is continuous iff it is continuous at all $b \in X$.
(How? Use the fact that an arbitrary union of open sets is open)

Recall that this is equivalent to the usual definition of continuity for metric spaces (taking the metric topology here).

Since the topologies matter as well, note that even the identity map from $\mathbb{R}_c$ to $\mathbb{R}$ is not continuous.

If the topology of Y is given by a basis $\mathcal{B}$ and we want to determine continuity, it suffices to check the pre-images of basis elements of Y. Indeed, use the fact that an arbitrary union of open sets is open.

Further, it suffices to just check subbasis elements!
Indeed, the set of finite intersections of subbasis elements form a basis. (and a finite intersection of open sets is open)

Lecture 10 - 06/02/21 More about Continuous Maps

Theo. (2.1) Let X and Y be topological spaces and $f: X \to Y$. Then the following are equivalent.

i) f is continuous.
ii) For every $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$.
iii) For every closed $B \subseteq Y$, $f^{-1}(B)$ is closed in X.
iv) For every $x \in X$ and neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subseteq V$.

Proof $i \Rightarrow ii$

Suppose f is continuous and $A \subseteq X$. Let $x \in \bar{A}$.
Let V be a neighbourhood of $f(x)$. We shall show that $V \cap f(A) \neq \emptyset$, implying that $f(x) \in \overline{f(A)}$.
Since $x \in \bar{A}$ and $x \in f^{-1}(V)$, which is open in X,

$$f^{-1}(V) \cap A \neq \emptyset.$$

Let $y \in f^{-1}(V) \cap A$. Then $f(y) \in \underset{(\text{why?})}{V \cap f(A)}$, proving the claim.

$ii \Rightarrow iii$

Let B be closed in Y and $A = f^{-1}(B)$. Let $x \in \bar{A}$. Then

$$f(x) \in f(\bar{A}) \subseteq \overline{f(A)} \subseteq B$$

Therefore, $x \in f^{-1}(B)$ and the claim is proved.

$iii \Rightarrow i$

Observe that iii is just the definition of continuity but with 'closed' instead of 'open'.
Let $B \subseteq Y$ be open. Then $Y \setminus B$ is closed and $f^{-1}(Y \setminus B)$ is closed. That is, $X \setminus f^{-1}(Y \setminus B)$ is open, and this set is just $f^{-1}(B)$. $\square$

We briefly mentioned $i \Leftrightarrow iv$ earlier. The details are left as an exercise.

Def. Let X and Y be topological spaces and $f: X \to Y$ be a bijection. f is said to be a homeomorphism if both f and f^{-1} are continuous.

Homeomorphism

Equivalently, f is a homeomorphism if for any $U \subseteq X$, $f(U)$ is open (in Y) iff U is open (in X).
That is, it is a continuous open bijection.

V open $\Downarrow$ $f^{-1}(V)$ open

U open $\Downarrow$ $f(U)$ open

A homeomorphism also gives a bijective map between the open sets of X and Y.

So if X has some property that is expressed in terms of the topology on X, Y must have the same property as well
Such a property is called a topological property of X.
(for example, the space being Hausdorff)

If there is a homeomorphism between two spaces, they are said to be homeomorphic.
(This implicitly uses the fact that if there is a homeomorphism: $X \to Y$, there is a homeomorphism: $Y \to X$ — the inverse of the first)

Homeomorphisms are the topological counterpart of isomorphisms in algebra.

Def. Let $f: X \to Y$ be a continuous injective map. Let $Z = f(X) \subseteq Y$ and consider it as a subspace of Y. The function $f': X \to Z$ attained by restricting the codomain is bijective. If f' is a homeomorphism, then f is said to be a topological embedding or just embedding of X in Y.

Embedding

Note that the "homeomorphic" relation is an equivalence relation. (Why?)

Let X, Y, Z be topological spaces.

1. Any constant map $f: X \to Y$ is continuous.
2. If A is a subspace of X, the inclusion map $f: A \hookrightarrow X$ is continuous.
3. If $f: X \to Y$ and $g: Y \to Z$ are continuous, $g \circ f: X \to Z$ is continuous.
4. If $f: X \to Y$ is continuous and A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.
5. Similarly, we can restrict / expand the range.

to a subspace $Z \supseteq f(X)$ to a space Z with subspace Y.

Lemma. (2.2) Let $f: X \to Y$ and $X = \bigcup_{\alpha \in A} U_\alpha$ for some (U_α). Then f is continuous iff $f|_{U_\alpha}$ is continuous for each α.

Proof. The forward direction is obvious.

For the backward direction, let V be open in Y. Observe that

$$f^{-1}(V) \cap U_\alpha = f|_{U_\alpha}^{-1}(V).$$

$f|_{U_\alpha}^{-1}(V)$ is open in U_α, and thus X (Why?). This implies that

$$f^{-1}(V) = \bigcup_{\alpha \in A} (f^{-1}(V) \cap U_\alpha),$$

which yields the result since an arbitrary union of open sets is open. $\square$

Theo. (2.3) [Pasting Lemma] Let $X = A \cup B$ for closed A, B in X. Let $f: A \to Y$ and $g: B \to Y$ be continuous*. If $f(x) = g(x)$ for all $x \in A \cap B$, then the map $h: X \to Y$ defined by

(* with respect to the subspace topologies)

$$h(x) = \begin{cases} f(x), & x \in A, \\ g(x), & x \in B \end{cases}$$

is continuous

Proof. Let C be closed in Y. Note that

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

Since $f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B, which are in turn closed in X, they are also closed in X. This gives the result because a finite union of closed sets is closed. $\square$

Note that the result holds even if A and B are open.

Lecture 11 - 10/02/21 More about Product Topologies

Theo. (2.4) Let $f: A \to X \times Y$ be given by $f(a) = (f_1(a), f_2(a))$. f is continuous iff the functions $f_1: A \to X$ and $f_2: A \to Y$ are continuous.

Coordinate Function — In this context, f_1 and f_2 are called the coordinate functions of f.

This can easily be proved by considering the basis elements. We omit the proof and shall instead show a more general result later.

Thm. (2.5) Let $A \subseteq X$, $f: A \to Y$ be continuous, and let Y be Hausdorff. Then if f can be extended to a continuous $g: \bar{A} \to Y$, this g is uniquely determined by f.

Proof. Let $g_1, g_2 : \bar{A} \to Y$ be continuous and $g_1(a) = g_2(a) = f(a)$ for all $a \in A$. Let $x \in \bar{A}$ such that $g_1(x) \neq g_2(x)$.
Since Y is Hausdorff, let open $U_1, U_2 \subseteq Y$ such that $g_1(x) \in U_1$, $g_2(x) \in U_2$, and $U_1 \cap U_2 = \emptyset$. We then have

$$A \cap \underbrace{g_1^{-1}(U_1) \cap g_2^{-1}(U_2)}_{\text{open in } \bar{A} \text{ and non-empty.}} \neq \emptyset.$$

Let $z \in A \cap g_1^{-1}(U_1) \cap g_2^{-1}(U_2)$. Then

$$f(z) = g_1(z) \in U_1 \quad \text{and}$$
$$f(z) = g_2(z) \in U_2$$

$\Rightarrow U_1 \cap U_2 \neq \emptyset$, proving the claim. $\square$

Let us revisit the product topology.
How do we generalize the idea to more (than 2) topological spaces?

Suppose $(X_i)_{i=1}^n$ are topological spaces. Consider the topologies on $X_1 \times X_2 \times \cdots \times X_n$ with

1. basis
$$\mathcal{B} = \{ U_1 \times U_2 \times \cdots \times U_n : U_i \text{ is open in } X_i \text{ for each } i \}$$
2. subbasis
$$\mathcal{S} = \bigcup_{i=1}^{n} \{ \pi_i^{-1}(U) : U \text{ open in } X_i \}$$

We even extend the above to a countably infinite number of sets. (we define this better later)
When are the two topologies the same?

It turns out that they are the same for finite products, but not for an infinite product.

↳ In this case, (1) is called the box topology and (2) is called the product topology.

It is easily seen that the described sets are a basis and subbasis.

A general basis element of the product topology is a finite intersection of subbasis elements.

$$\bigcap_{r=1}^{k} \pi_{i_r}^{-1}(U_{i_r}) \quad \text{where } U_{i_r} \text{ is open in } X_{i_r}.$$

(Restriction on a finite number of coordinates)

It is easily seen that the box topology and product topology are equal for a finite number of topological spaces because

$$U_1 \times U_2 \times \cdots \times U_n = \bigcap_{i=1}^{n} \pi_i^{-1}(U_i)$$

↳ finite intersection of subbasis elements

and

$$\bigcap_{r=1}^{k} \pi_{i_r}^{-1}(U_{i_r}) = X_1 \times X_2 \times \cdots \times U_{i_1} \times \cdots \times U_{i_2} \times \cdots \times U_{i_k} \times \cdots \times X_n \in \mathcal{B}$$

Let us define the Cartesian product more concretely in the infinite case.

Let $(X_i)_{i \in \mathbb{N}}$ be sets and $X = \bigcup_{i \in \mathbb{N}} X_i$. Then

Cartesian Product

$$\prod_{i=1}^{\infty} X_i = \{ f : \mathbb{N} \to X : f(j) \in X_j \text{ for each } j \in \mathbb{N} \} \subseteq X^{\mathbb{N}}$$

$$= \{ (x_1, x_2, \dots, x_n, \dots) : x_i \in X_i \text{ for each } i \}$$

$X^{\mathbb{N}}$: $\{f : \mathbb{N} \to X\}$

We can easily extend this definition to any indexing set I as

$$\prod_{i \in I} X_i = \{ f : I \to X : f(i) \in X_i \text{ for each } i \in I \}$$

Let $(X_i)_{i\in I}$ be a set of topological spaces with indexing set I. The box topology on $\prod_{i\in I} X_i$ is that with basis

Box Topology

$$\mathcal{B} = \left\{ \prod_{i\in I} U_i \ : U_i \text{ is open in } X_i \text{ for each } i\in I \right\}$$

and the product topology on $\prod_{i\in I} X_i$ is that with subbasis

Product Topology

$$\mathcal{S} = \left\{ \pi_i^{-1}(U_i) \ : U_i \text{ is open in } X_i \text{ and } i\in I \right\}.$$

For finite I, the box and product topologies are equal.
For infinite I, the box topology is <u>strictly</u> finer than the product topology unless all but finitely many of the topologies are the indiscrete topology on the respective set.