Munkres Solutions

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Contents

§2. Topological Spaces and Continuous Functions

2.13. Basis for a Topology

Exercise 2.13.1. Let X be a topological space and $A \subseteq X$. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subseteq A$. Show that A is open in X.

Solution

For each $x \in A$, denote by U_x an open subset of A that contains A. Then $A = \bigcup_{x \in A} U_x$. However, an arbitrary union of open sets is open and thus, so is A.

Exercise 2.13.5. Show that if $\mathcal{A}$ is a basis for a topology on X, the topology generated by $\mathcal{A}$ equals the intersection of all topologies that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Solution

Let $\mathcal{T}$ be the topology generated by $\mathcal{A}$ and $\mathcal{T}'$ be a topology that contains $\mathcal{A}$. Let $U \in \mathcal{T}$. Then $U = \bigcup_{i \in I} B_i$ for some $(B_i)_{i \in I}$ in $\mathcal{A}$. However, each B_i is also in $\mathcal{T}'$. Since an arbitrary union of open sets is open, $U \in \mathcal{T}'$ as well. Therefore, $\mathcal{T} \subseteq \mathcal{T}'$, proving the result. The solution for the case where $\mathcal{A}$ is a subbasis is very similar and so omitted.

2.16. The Subspace Topology

Exercise 2.16.1. Show that if Y is a subspace of X and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Solution

The topology A inherits as a subspace of X is

$$\begin{aligned} \mathcal{T} &= \{U \cap A : U \text{ open in } X\} \\ &= \{(U \cap Y) \cap A : U \text{ open in } X\} \\ &= \{V \cap A : V \text{ open in } Y\}, \end{aligned}$$

which is the topology it inherits as a subspace of Y.

Exercise 2.16.2. If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X.

Solution

It is easily seen that $\mathcal{T}'_Y$ is finer than $\mathcal{T}_Y$. We further see that it need not be strictly finer by considering the example $X = \{a, b, c\}$, $Y = \{a, b\}$, $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$, and $\mathcal{T}'$ as the discrete topology on X.

Exercise 2.16.3. Consider $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following is open in Y? Which are open in $\mathbb{R}$?

$$\begin{aligned} A &= \left\{x : \frac{1}{2} < x < 1\right\} \\ B &= \left\{x : \frac{1}{2} < x \leq 1\right\} \\ C &= \left\{x : \frac{1}{2} \leq x < 1\right\} \\ D &= \left\{x : \frac{1}{2} \leq x \leq 1\right\} \\ E &= \left\{x : 0 < x < 1 \text{ and } 1/x \notin \mathbb{Z}^+\right\} \end{aligned}$$

Solution

A and B are open in Y and only A is open in $\mathbb{R}$. This is reasonably straightforward to prove.
C is not open in Y (and so not $\mathbb{R}$ either) because there is no basis element U of the order topology such that $1/2 \in U \subseteq C$. A similar argument holds for D as well.
E is open in both $\mathbb{R}$ and Y because it can be written as a union of basis elements

$$E = \bigcap_{n \in \mathbb{Z}^+} \left(\frac{1}{n+1}, \frac{1}{n} \right).$$

Exercise 2.16.4. A map $f : X \to Y$ is said to be an open map if for every open U of X, $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open.

Solution

We shall only show that π_1 is open, the other case is nearly identical. Let

$$U = \bigcup_{i \in I} U_i \times V_i$$

be open in $X \times Y$ for some indexing set I, where each U_i and V_i are open in X and Y respectively. Then,

$$\pi_1(U) = \pi_1 \left(\bigcup_{i \in I} U_i \times V_i = \bigcup_{i \in I} \pi_1(U_i \times V_i) = \bigcup_{i \in I} U_i \right)$$

is open in X.

2.17. Closed Sets and Limit Points

Exercise 2.17.1. Let $\mathcal{C}$ be a collection of subsets of set X. Suppose that $\varnothing$ and X are in $\mathcal{C}$ and that finite unions and arbitrary intersections of elements of $\mathcal{C}$ are in C. Show that the collection $\mathcal{T} = \{X \setminus C : C \in \mathcal{C}\}$ is a topology on X.

Solution

Let $(U_i)_{i \in I}$ be in $\mathcal{T}$ with $U_i = X \setminus C_i$ for each i. Then

$$\bigcup_{i \in I} U_i = X \setminus \bigcap_{i \in I} C_i = X \setminus C \in \mathcal{T}$$

for some $C \in \mathcal{C}$. Closure under finite intersections is shown similarly. We trivially have $\varnothing, X \in \mathcal{T}$ because $X, \varnothing \in \mathcal{C}$.

Exercise 2.17.2. Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Solution

Let U be open in X such that $Y \setminus A = U \cap Y$. Then, we can write A as $X \setminus ((X \setminus Y) \cup U)$. Since $X \setminus Y$ and U are open in X, A is closed in X.

Exercise 2.17.3. Show that if A is closed in X and B is closed in Y, $A \times B$ is closed in $X \times Y$.

Solution

Observe that

$$(X \times Y) \setminus (A \times B) = ((X \setminus A) \times (Y \setminus B)) \cup ((X \setminus A) \times Y) \cup (A \times (Y \setminus B)).$$

Since each of the sets on the right are open in $X \times Y$, $A \times B$ is closed.

Exercise 2.17.4. Show that if U is open in X and A is closed in X, $U \setminus A$ is open in X and $A \setminus U$ is closed in Y.

Solution

This is easily seen on writing $U \setminus A = U \cap (X \setminus A)$ and $A \setminus U = A \cap (X \setminus U)$.

Exercise 2.17.13. Show that if X is Hausdorff iff the *diagonal* $\Delta = \{x \times x : x \in X\}$ is closed in $X \times X$.

Solution

Let us first show the forward direction. Let $(x, y) \in (X \times X) \setminus \Delta$. Then since X is Hausdorff, there are some open $U_1 \ni x$, $U_2 \ni y$ such that $U_1 \cap U_2 = \varnothing$. Since the open set $U_1 \times U_2 \ni (x, y)$ does not intersect Δ, Δ is closed (no other point is in its closure).
For the backward direction, this means that for any $(x, y) \in X \times X$ with $x \neq y$, there exists an open set $U \ni (x, y)$ such that $U \cap \Delta = \varnothing$. We may further assume that U is a basis element of the product topology and can be written as $U_1 \times U_2$. However, $(U_1 \times U_2) \cap \Delta = \varnothing$ iff $U_1 \cap U_2 = \varnothing$, so the required follows.

Exercise 2.17.14. In the finite complement topology on $\mathbb{R}$, what point or points does the sequence $x_n = 1/n$ converge to?

Solution

We claim that x_n converges to any point in $\mathbb{R}$. Let $x \in \mathbb{R}$ and $U \ni x$ be open. U contains all but finitely many x_i since we are in the finite complement topology and therefore, x_n converges to x.

Exercise 2.17.19. If $A \subseteq X$, define the boundary of A by

$$\operatorname{Bd} A = \overline{A} \cap \overline{X \setminus A}.$$

(a) Show that A° and $\operatorname{Bd} A$ are disjoint, and $\overline{A} = A^\circ \cup \operatorname{Bd} A$.

(b) Show that $\operatorname{Bd} A = \varnothing$ iff A is both open and closed.

(c) Show that U is open iff $\operatorname{Bd} U = \overline{U} \setminus U$.

(d) If U is open, is it true that $U = \overline{U}^\circ$? Justify your answer.

Solution

(a) Let $x \in A \setminus A^\circ$. Then for any open $U \ni x$, $U \not\subseteq A$ (otherwise, $A^\circ \cup U \supsetneq A^\circ$ is open and contained in A). That is, $U \cap (X \setminus A) \neq \varnothing$. However, this implies that $x \in \overline{X \setminus A}$, that is, $A \setminus A^\circ \subseteq \overline{X \setminus A}$. Therefore,

$$\begin{aligned} \overline{A} \setminus A^\circ = (\overline{A} \setminus A) \cup (A \setminus A^\circ) &\subseteq \overline{X \setminus A} \\ \overline{A} &\subseteq A^\circ \cup \overline{X \setminus A} \\ &= \overline{A} \cap (A^\circ \cup \overline{X \setminus A}) \\ &= A^\circ \cup (\overline{A} \cup \overline{X \setminus A}) = A^\circ \cup \operatorname{Bd} A. \end{aligned}$$

(b) If A is not closed, $\overline{A} \supsetneq A$ intersects $X \setminus A \subseteq \overline{X \setminus A}$, contradicting $\operatorname{Bd} A = \varnothing$. Similarly, $X \setminus A$ is closed as well, so A is both open and closed.
The other direction is similarly straightforward.

(c) If U is open, $X \setminus U$ is closed so $\operatorname{Bd} U = \overline{U} \cap (X \setminus U) = \overline{U} \setminus U$.
On the other hand, if $\overline{U} \cap (X \setminus U) = \overline{U} \cap \overline{X \setminus U}$, $X \setminus U$ must be closed. Indeed, otherwise, $\overline{X \setminus U} \setminus (X \setminus U) \subseteq$

$U \subseteq \overline{U}$, contradicting the equality.

(d) No, this is not the case. Consider the open set $U = (1,2) \cup (2,3) \subseteq \mathbb{R}$. Then $\overline{U}^{\circ} = (1,3)$.

2.18. Continuous Functions

Exercise 2.18.1. Suppose that $f : X \to Y$ is continuous. If x is a limit point of $A \subseteq X$, is $f(x)$ necessarily a limit point of $f(A)$?

Solution

No. Let $f : \mathbb{R} \to \mathbb{R}$ be the zero function, $A = \{1/n : n \in \mathbb{N}\}$, and $x = 0$.

Exercise 2.18.8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x : f(x) \leq g(x)\}$ is closed in X.

(b) Let $h : X \to Y$ be the function
$$h(x) = \min\{f(x), g(x)\}.$$
Show that h is continuous.

Solution

(a) It suffices to show that $V = \{x : f(x) > g(x)\}$ is open in X. Let $x \in V$. Since Y is Hausdorff, there are open sets U_1, U_2 such that $f(x) \in U_1$, $g(x) \in U_2$, and $U_1 \cap U_2 = \varnothing$.
We can consider a basis element $B_1 \subseteq U_1$ that contains $f(x)$ in lieu of U_1. That is, we may suppose wlog that U_1 and U_2 are disjoint basis elements. Further, we may assume that U_1 is of the form (a, ∞) (if it is of the form (a, c) instead, we can replace it with (a, ∞)). Due to the disjointness assumption, this means that we can consider U_1 to be of the form (a, ∞) and U_2 of the form $(-\infty, b)$ such that for any $c \in U_1, d \in U_2$, $c > d$
Now, let $U = f^{-1}(U_1) \cap g^{-1}(U_2) \ni x$. f and g are continuous so U is open. Further, for any $y \in U$, $f(y) > g(y)$ (by the above assumption), that is, $U \subseteq V$.
It follows that V is open (for any $x \in V$, there is an open $V \subseteq U$ such that $x \in V$).

(b) Let $U_1 = \{x : f(x) \geq g(x)\}$ and $U_2 = \{x : f(x) \leq g(x)\}$. By (a), U_1 and U_2 are both closed. Since g is continuous on U_1, f is continuous on U_2, and $f = g$ on $U_1 \cap U_2$, we can use the pasting lemma to conclude that h is continuous ($h(x) = g(x)$ on U_1 and $f(x)$ on U_2).

2.19. The Product Topology

Exercise 2.19.4. Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic to $X_1 \times \cdots \times X_n$.

Solution

Let the two topological spaces above be denoted by X and Y respectively. Consider the map $f : X \to Y$ with
$$f(x) = (\pi_1(\pi_1(x)), \ldots, \pi_{n-1}(\pi_1(x)), \pi_2(x)).$$
We claim that f is a homomorphism.
If U_i is open in X_i for each i, then $f^{-1}(U_1 \times \cdots \times U_n) = (U_1 \times \cdots \times U_{n-1}) \times U_n$ is open in X ($U_1 \times \cdots \times U_{n-1}$ is open in $X_1 \times \cdots \times X_{n-1}$ and U_n is open in X_n). Therefore, f is continuous.
On the other hand, if U is open in $X_1 \times \cdots \times X_{n-1}$ and U_n is open in X_n, then $f(U \times U_n) = U_1 \times \cdots \times U_n$, where each U_i is open in X_i by Exercise 2.16.4.

Exercise 2.19.6. Let $x_1, x_2, \ldots$ be a sequence of points in the product space $\prod X_\alpha$. Show that this sequence converges to the point x iff the sequence $\pi_\alpha(x_1), \pi_\alpha(x_2), \ldots$ converges to $\pi_\alpha(x)$ for each α. Is this true if we use the box topology instead of the product topology?

> **Solution**
>
> We first show the backward direction. Suppose $(\pi_\alpha(x_i))$ converges to $\pi_\alpha(x)$. Let U be an open set containing x and $B \subseteq U$ be a basis element of the product topology containing x. Let $B = \prod_\alpha U_\alpha$ where $U_\alpha \neq X_\alpha$ for the finite set $\{\alpha_1, \ldots, \alpha_n\}$. Since $(\pi_{\alpha_j}(x_i))$ converges to $\pi_{\alpha_j}(x)$ for each j, each U_{α_j} contains all but finitely many $\pi_\alpha(x_i)$. As a result, B contains all but finitely many x_i and therefore, (x_i) converges to x.
>
> On the other hand, let (x_i) converge to x. Let U_α be an open set in X_α containing $\pi_\alpha(x)$. We wish to show that it contains all but finitely many $\pi_\alpha(x_i)$. Let $U' = \prod U'_\beta$ be a basis element containing x and $V = \prod V_\beta$, where $V_\beta = U'_\beta$ for $\beta \neq \alpha$ and $V_\alpha = U_\alpha \cap U'_\alpha$. Since V is an open set containing x, it contains all but finitely many x_i. In particular, V_α contains all but finitely many $\pi_\alpha(x_i)$. The required follows.
> Observe that the forward proof works even if we use the box topology instead, but the backward direction breaks. To see that the result need not hold for the box, let the product space be $\mathbb{R}^\omega$, $\pi_n(x_i) = n/i$ and $\pi_n(x) = 0$ for each i, n.

Exercise 2.19.7. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually 0 ($x_i \neq 0$ for finitely many i). What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies?

> **Solution**
>
> We claim that the closure under the product topology is $\mathbb{R}^\omega$. Let $x \in \mathbb{R}^\omega$ and $U = \prod U_n$ be a basis element containing x. We wish to determine when $U \cap \mathbb{R}^\infty \neq \varnothing$. Consider $y \in \mathbb{R}^\infty$ such that $y_n = x_n$ if $U_n \neq \mathbb{R}$ and 0 otherwise. Then $y \in U \cap \mathbb{R}^\infty$, thus proving the result.
> For the box topology, we claim that $\mathbb{R}^\infty$ is closed. Indeed, for any $x \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$, consider the open set $U = \prod U_n$, where $U_n = (x_n/2, 3x_n/2)$ if $x_n \neq 0$ and $\mathbb{R}$ otherwise. Then $U \cap \mathbb{R}^\infty$ is empty, completing the proof.

2.20. The Metric Topology

Exercise 2.20.3. Let X be a metric space with metric d.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

(b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, the topology of X' is finer than the topology of X.

This means that if X has a metric d, the metric topology induced by it is the coarsest topology with respect to which d is continuous.

> **Solution**
>
> (a) Let $x = (x_1, x_2) \in X \times X$ and for some $\varepsilon > 0$, $U = B(f(x), \varepsilon)$ be a basis element of $\mathbb{R}$ containing $f(x)$. Consider the open sets $U_1 = B_d(x_1, \varepsilon/4)$ and $U_2 = B_d(x_2, \varepsilon/4)$. If $y_1 \in U_1$ and $y_2 \in U_2$, then
> $$d(y_1, y_2) \leq d(x_1, x_2) + d(x_1, y_1) + d(x_2, y_2) \leq f(x) + \varepsilon/2.$$
> Therefore, $d(y_1, y_2) \in U$. As a result, $U_1 \times U_2$ is an open set such that $x \in f(U_1 \times U_2) \subseteq U$ and d is continuous.
>
> (b) Consider the continuous function $g : X' \to X' \times X'$ given by $y \mapsto (x, y)$ (we proved continuity in Exercise 2.16.4). Since the composition of continuous functions is continuous, so is $d \circ g : X' \to \mathbb{R}$. Note that for any $\varepsilon > 0$, $(d \circ g)^{-1}((0, \varepsilon)) = B_d(x, \varepsilon)$. It follows that $B_d(x, \varepsilon)$ is open in X' and therefore, the topology on X' is finer than that on X.

Exercise 2.20.5. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all infinite sequences that are eventually 0. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology?

Solution

We claim that the closure of $\mathbb{R}^\infty$ is the set of all sequences that converge to 0. Let $x \in \mathbb{R}^\omega$.

- Case 1. The sequence x does not converge to 0. There is then some $\varepsilon > 0$ such that for infinitely many n, $|x_n| > \varepsilon$. We may assume that $\varepsilon < 1$. Consider the open set $U = B_{\overline{\rho}}(x, \varepsilon)$. We claim that $U \cap \mathbb{R}^\infty = \varnothing$. Indeed, for any $y \in \mathbb{R}^\infty$, $|x_n - y| > \varepsilon$ for infinitely many n so $y \notin B_{\overline{d}}(x, \varepsilon)$, thus proving that $x \notin \overline{\mathbb{R}^\infty}$.

- Case 2. The sequence x converges to x. Let $B_{\overline{\rho}}(x, \varepsilon)$ be an arbitrary basis element containing x. Then there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n| < \varepsilon/2$. Consider the element $y \in \mathbb{R}^\infty$ such that $y_n = x_n$ for $n \leq N$ and $y_n = 0$ otherwise. Then $y \in B_{\overline{\rho}}(x, \varepsilon)$, thus proving that $x \in \overline{\mathbb{R}^\infty}$.

Exercise 2.20.6. Let $\overline{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $x = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and $0 < \varepsilon < 1$, let

$$U(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon) \times \cdots .$$

(a) Show that $U(x, \varepsilon)$ is not equal to $B_{\overline{\rho}}(x, \varepsilon)$.

(b) Show that $U(x, \varepsilon)$ is not even open in the uniform topology.

(c) Show that

$$B_{\overline{\rho}}(x, \varepsilon) = \bigcup_{\delta < \varepsilon} U(x, \delta).$$

Solution

(a) Consider

$$x' = (x_1 + \varepsilon/2, x_2 + 2\varepsilon/3, \ldots, x_n + n\varepsilon/(n+1), \ldots).$$

Obviously, $x' \in U(x, \varepsilon)$. However, $\overline{\rho}(x, x') = \sup\{n\varepsilon/(n+1) : n \in \mathbb{N}\} = \varepsilon$, so $x' \notin B_{\overline{\rho}}(x, \varepsilon)$.

(b) Let x' be as in the previous part. Suppose that $U(x, \varepsilon)$ is open in the uniform topology. Then, there is a $\delta > 0$ such that $B_{\overline{\rho}}(x', \delta) \subseteq U(x, \varepsilon)$. In particular, $(x'_n - \delta, x'_n + \delta) \subseteq (x - \varepsilon, x + \varepsilon)$ for any n. This yields a contradiction since it would imply that $\delta < \varepsilon/(n+1)$ for any n and therefore, $U(x, \varepsilon)$ is not open in the uniform topology.

(c) Let $y \in U(x, \delta)$ for some $\delta < \varepsilon$. Then $\overline{\rho}(x, y) \leq \delta < \varepsilon$, so $y \in B_{\overline{\rho}}(x, \varepsilon)$. One direction of the inclusion follows. For the other, let $y \in B_{\overline{\rho}}(x, \varepsilon)$ and $\delta = \overline{\rho}(x, y)$. Since $\delta < \varepsilon$, we can choose a δ' such that $\delta < \delta' < \varepsilon$. Then, $y \in U(x, \delta')$, thus proving the result.

2.21. The Metric Topology (continued)

Exercise 2.21.2. Let X and Y be metric spaces with metrics d_X and d_Y respectively. Let $f : X \to Y$ such that for any $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Show that f is an imbedding. It is called the *isometric imbedding* of X in Y.

Solution

Let $x \in X$ and $V = B_{d_Y}(f(x), \varepsilon)$ be a basis element in Y containing $f(x)$. Then $U = B_{d_X}(x, \varepsilon)$ is an open set in X containing x such that $f(U) \subseteq V$. Therefore, f is continuous. Showing that f is open is similarly straight-forward.

Exercise 2.21.3. Let X_n be a metric space with metric d_n for each $n \in \mathbb{Z}^+$. Let $\overline{d}_i = \min\{d_i, 1\}$. Show that

$$D(x, y) = \sup\{\overline{d}_i(x_i, y_i)/i\}$$

is a metric for the product space $\prod X_i$.

Solution

Let $U = \prod U_i$ be a basis element of the product topology, where $U_i \neq X_i$ for $i = \alpha_1, \ldots, \alpha_n$. Let $x \in U$. For each j, let $B_{\overline{d}_{\alpha_j}}(x_{\alpha_j}, \varepsilon_j) \subseteq U_{\alpha_j}$. Let $\varepsilon = \min\{\varepsilon_j/2\alpha_j\}$. Then $x \in B_D(x, \varepsilon) \subseteq U$, so the metric topology is finer than the product topology.
For the other direction, let $\varepsilon > 0$ and $U = B_D(x, \varepsilon)$. We want to show that there is an element of the product topology containing x that is contained in U. Let $V = \prod V_i$, where $V_i = B_{\overline{d}_i}(x_i, i\varepsilon/2)$ for each i. Note that V is a basis element of the product topology since $V_i = X_i$ for $i > 2/\varepsilon$. Since $x \in V \subseteq U$, the product topology is finer than the metric topology, thus proving the result.

2.22. The Quotient Topology

Exercise 2.22.2.

(a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map on Y, then p is a quotient map.

(b) If $A \subseteq X$, a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Show that a retraction is a quotient map.

Solution

(a) It is obvious that p is surjective since for any $y \in Y$, $p(f(y)) = y$. Let $U \subseteq Y$. We want to show that if $p^{-1}(U)$ is open, then so is U. Indeed, we then have $U = (p \circ f)^{-1}(U) = f^{-1}(p^{-1}(U))$ is open since $p^{-1}(U)$ is open and f is continuous.

(b) It is clear that r is surjective. Let $U \subseteq A$. We want to show that if $r^{-1}(U)$ is open, then U is open in A. Indeed, note that $U = r^{-1}(U) \cap A$, so is open in A by definition.

Exercise 2.22.3. Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points $x \times y$ for which $x \geq 0$ or $y = 0$. Let $q : A \to \mathbb{R}$ be obtained by restricting π_1. Show that q is a quotient map that is neither open nor closed.

Solution

Consider the function $r : A \to \mathbb{R} \times \{0\}$ that maps $x \times y \in A$ to $x \times 0 \in \mathbb{R}^2$ is a retraction, and therefore a quotient map. Also note that $r = g \circ q$, where g is the trivial (identity-like) homemorphism from $\mathbb{R}$ to $\mathbb{R} \times \{0\}$, proving the result.
Consider the open set $[1, 2) \times (1, 2)$ in A. Then $q(A) = [1, 2)$ is not open. On the other hand, consider the closed set $\{x \times y \in \mathbb{R}^2 : xy = 1 \text{ and } x > 0\}$ in A, having image $(0, \infty]$, which is open (Projection maps need not be closed in general and this counter-example works even for π_1).

§3. Connectedness and Compactness

3.23. Connected Spaces

Exercise 3.23.1. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T}' \supseteq \mathcal{T}$, what does connectedness of X in one topology mean about connectedness in the other?

Solution

If $\mathcal{T}'$ is connected, then so is $\mathcal{T}$. Indeed, if $\mathcal{T}$ is disconnected and has separation $X = U \cup V$, then $U \cup V$ serves as a separation of X under $\mathcal{T}'$ as well. The converse need not be true, as can be seen on considering the discrete and indiscrete topology on X.

Exercise 3.23.2. Let $\{A_n\}$ be a sequence of connected subspaces of X such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

Solution

Let $A = \bigcup A_n$. Suppose instead that $A = U \cup V$ is a separation. Since A_1 is connected, we may assume without loss of generality that $A_1 \subseteq U$. We shall now show by induction that for any n, $A_n \subseteq U$. Indeed, if $A_n \subseteq U$ for some $n \geq 1$, then since A_{n+1} is connected, $A_{n+1} \subseteq U$ or $A_{n+1} \subseteq V$. However, $\varnothing \neq A_n \cap A_{n+1} \subseteq U$, so $A_{n+1} \subseteq U$ as well. This contradicts the non-emptiness of V, completing the proof.

Exercise 3.23.3. Let $\{A_\alpha\}$ be a collection of connected subspaces of X and A a connected subspace of X. Show that if $A_\alpha \cap A \neq \varnothing$ for all α, then $A \cup \bigcup A_\alpha$ is connected.

Solution

Denote the space by Y. Suppose instead that $Y = U \cup V$ is a separation. Since A is connected, we may assume wlog that $A \subseteq U$. For any α, $A_\alpha \subseteq U$ or $A_\alpha \subseteq V$. However, $A \cap A_\alpha \neq \varnothing$ so $A_\alpha \subseteq U$. This contradicts the non-emptiness of V, proving the result.

Exercise 3.23.4. Show that if X is an infinite set, it is connected in the finite complement topology.

Solution

Suppose otherwise and let $X = U \cup V$ be a separation. Then $X \setminus U$ and $X \setminus V$ are finite, so $(X \setminus U) \cup (X \setminus V)$ is finite. Then, its complement $U \cap V$ is infinite (in particular, non-empty), contradicting the disjointedness of U and V.

Exercise 3.23.8. Determine whether or not $\mathbb{R}^\omega$ is connected in the uniform topology.

Solution

Let S be the set of all bounded sequences. Then it is not too difficult to show that both S and $\mathbb{R}^\omega \setminus S$ are open, so $\mathbb{R}^\omega$ is disconnected.

Exercise 3.23.9. Let $A \subsetneq X$ and $B \subsetneq Y$. If X and Y are connected, show that $(X \times Y) \setminus (A \times B)$ is connected.

Solution

Denote the space of interest by S. Suppose instead that $S = U \cup V$ is a separation. Let $x \times y, z \times w \in S$. Suppose that $x \times y \in U$. Consider the connected subspaces $A = \{x\} \times Y \subseteq S$ and $B = X \times \{w\} \subseteq S$. Since their intersection is non-empty (it contains $x \times w$), their union is connected as well. Further, because $x \times y \in U$, $A \cup B \subseteq U$. In particular, $w \times z \in U$. Since our choice of $w \times z$ was arbitrary, this contradicts the non-emptiness of V, proving the result.

Exercise 3.23.11. Let $p : X \to Y$ be a quotient map. Show that if each $p^{-1}(\{y\})$ is connected and Y is connected, then X is connected.

Solution

Suppose otherwise and let $X = U \cup V$ be a separation. For any y, either $p^{-1}(\{y\}) \subseteq U$ or $p^{-1}(\{y\}) \subseteq Y$ (due to connectedness). That is, U and V are saturated. But then, $Y = p(X) = p(U) \cup p(V)$ is a separation of Y, yielding a contradiction and proving the result.

3.24. Connected Subspaces of the Real Line

Exercise 3.24.2. Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show that there exist a point x of S_1 such that $f(x) = f(-x)$.

Solution

Consider the continuous map $g : S^1 \to \mathbb{R}$ given by $x \mapsto f(x) - f(-x)$. Choose an arbitrary $x \in S_1$. If $g(x) = 0$, then we are done. Otherwise, let $y = -x$ be the diametrically opposite point to x. Then $g(y) = -g(x)$. Since S^1 is connected and 0 lies between $g(x)$ and $g(y)$, it follows that there exists some $x_0 \in S^1$ such that $g(x_0) = 0$.

Exercise 3.24.3. Let $f : X \to X$ be continuous. Show that if $X = [0, 1]$, there is a point x such that $f(x) = x$. What happens if $X = [0, 1)$ or $(0, 1]$?

Solution

Consider the continuous map $g : X \to \mathbb{R}$ given by $x \mapsto f(x) - x$. Observe that $g(0) \geq 0$ or $g(1) \leq 0$. If equality holds at either place, we are done. Otherwise, 0 lies between them. Since X is connected, the claim follows.
The claim need not hold if $X = [0, 1)$ or $(0, 1]$. Indeed, consider the functions $x \mapsto (x + 1)/2$ and $x \mapsto x/2$.

3.25. Components and Local Connectedness

Exercise 3.25.1. What are the components and path components of $\mathbb{R}_\ell$? What are the continuous maps $\mathbb{R} \to \mathbb{R}_\ell$?

Solution

It is obvious that any singleton in $\mathbb{R}_\ell$ is connected and path-connected. We claim that these are the only non-empty connected (and path-connected) subspaces. Let $A \subseteq \mathbb{R}_\ell$ have at least two elements. Let $x, y \in A$ with $x < y$. Then $(-\infty, y) \cap A$ and $[y, \infty) \cap A$ forms a separation of A, so it is not connected, and thus not a component. Therefore, the components of $\mathbb{R}_\ell$ are the singletons.
Recall that any continuous maps connected subspaces to connected subspaces. In particular, any continuous map $\mathbb{R} \to \mathbb{R}_\ell$ maps $\mathbb{R}$ to a connected subspace of $\mathbb{R}_\ell$. However, this must be a singleton and therefore, the continuous maps $\mathbb{R} \to \mathbb{R}_\ell$ are the singletons.

Exercise 3.25.8. Let $p : X \to Y$ be a quotient map. Show that if X is locally connected, so is Y.

Solution

Let $y \in Y$ and U be a neighbourhood of Y. Let C be a component of U.
It suffices to show that $p^{-1}(C)$ is open in X. Consider the collection of components of $p^{-1}(U)$ that intersect $p^{-1}(C)$. Observe that each of these components is open (by the local connectedness of X). Therefore, their union is open as well. Let D be one of these components.
It suffices to show that $D \subseteq p^{-1}(C)$. Let $d \in D$. Then $p(d) \in p(D) \cap C$. However, C is a component, so $p(D) \subseteq C$. It follows that $D \subseteq p^{-1}(p(D)) \subseteq p^{-1}(C)$, completing the proof.

3.26. Compact Spaces

Exercise 3.26.1.

(a) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X. Suppose that $\mathcal{T}' \supseteq \mathcal{T}$. What does compactness of X under one of these topologies imply about compactness under the other?

(b) Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, either $\mathcal{T}$ and $\mathcal{T}'$ are equal or they are not comparable.

Solution

(a) If X is compact under $\mathcal{T}'$, then it is compact under $\mathcal{T}$. Indeed, any open cover under $\mathcal{T}$ is an open cover under $\mathcal{T}'$, and compactness implies the existence of a finite subcover.
The converse need not hold, as can be seen with the example of $X = \mathbb{R}$, $\mathcal{T}'$ as the discrete topology, and $\mathcal{T}$ as the indiscrete topology.

(b) Suppose $\mathcal{T} \subseteq \mathcal{T}'$. Let $U \in \mathcal{T}'$. Then $X \setminus U$ is closed, and thus compact under $\mathcal{T}'$. By a method similar to part (a), $X \setminus U$ is also compact under $\mathcal{T}$, and thus closed in $\mathcal{T}$. This implies that U is open in $\mathcal{T}$, and therefore $\mathcal{T}' \subseteq \mathcal{T}$, proving the claim.

Exercise 3.26.2.

(a) Show that in the finite complement topology on $\mathbb{R}$, every subspace is compact.

(b) If $\mathbb{R}$ has the topology consisting of all sets A such that $\mathbb{R} \setminus A$ is either countable or all of $\mathbb{R}$, is $[0, 1]$ a compact space?

Solution

(a) Let $A \subseteq \mathbb{R}$ and $\mathcal{A}$ be an open cover of A. Let $U \in \mathcal{A}$. Then $(\mathbb{R} \setminus U) \cap A$ is finite, suppose it is equal to $\{x_1, \ldots, x_n\}$. For each $1 \leq i \leq n$, let $U_i \in \mathcal{A}$ such that $x_i \in U_i$. Then $\{U, U_1, \ldots, U_n\}$ forms a finite subcover of A.

(b) No, consider the open cover

$$\mathcal{A} = \{\mathbb{R} \setminus (\mathbb{Q} \cap [0, 1]) \cup \{x\} : x \in \mathbb{Q} \cap [0, 1]\}.$$

It is easy to show that $\mathcal{A}$ has no finite subcover.

Exercise 3.26.3. Show that a finite union of compact subspaces of X is compact.

Solution

Let $U_1, \ldots, U_n$ be compact subspaces of X, $U = \bigcup_{1 \leq i \leq n} U_i$, and $\mathcal{A}$ be an open cover of U. For each $1 \leq i \leq n$, let $\mathcal{A}_i$ be a finite subcover of $\mathcal{A}$ of U_i (such a subcover exists because U_i is compact and $\mathcal{A}$ is an open cover of U_i). Then $\bigcup_{1 \leq i \leq n} \mathcal{A}_i$ forms a finite subcover of U, proving the required.

Exercise 3.26.5. Let A and B be disjoint compact subspaces of the Hausdorff subspace X. Show that there exist disjoint open subsets U and V containing A and B respectively.

Solution

For each $a \in A$, let U_a and V_a be disjoint open subsets containing $\{a\}$ and B respectively. Then $\{U_a : a \in A\}$ forms an open cover of A, so has a finite subcover, say $\{U_{a_1}, \ldots, U_{a_n}\}$. Then $\bigcup_{1 \leq i \leq n} U_{a_i}$ and $\bigcap_{1 \leq i \leq n} V_{a_i}$ are disjoint open subsets containing A and B respecitvely, completing the proof.

Exercise 3.26.6. Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map.

Solution

Let A be a closed subspace of X. Then A is compact, and since f is continuous, so is $f(A)$. Recall that compact subspaces of Hausdorff spaces are closed. Therefore, $f(A)$ is closed, so f is closed.