# MA 861 : Combinatorics I

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## Contents

# §0. Notation and Prerequisites

Given $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \dots, n\}$ and $[n]_0$ denotes the set $[n] \cup \{0\}$.
$S(n,k)$, a Stirling number of the second kind, is the number of partitions of $[n]$ into exactly $k$ parts. $s(n,k)$, a Stirling number of the first kind, is the number of permutations of $[n]$ with exactly $k$ cycles.
Given a graph $G$ and edge $e \in G$, $G - e$ is the graph obtained by deleting $e$ (it has the same vertex set), and $G \setminus e$ is the graph obtained by "contracting" $e$, that is, merging the two vertices of $e$ and having a vertex adjacent to the new vertex if they are adjacent to either of the earlier vertices.

# §1. Introduction

**Exercise 1.1.** Recall that the number of $k$-subsets of $[n]$ is $\binom{n}{k}$. Given a $k$-subset $S = \{x_1, \dots, x_k\}$ of $[n]$, we write $S_< = \{x_1, \dots, x_k\}_<$ to denote that $x_1 < x_2 < \dots < x_k$. Determine the number of $k$-subsets $\{x_1, \dots, x_k\}_<$ of $[n]$ such that $x_i \cong i \mod 2$.

For example, for $n = 6$ and $k = 3$, we have the subsets $\{1,4,5\}, \{1,2,3\}, \{1,2,5\}, \{3,4,5\}$.
Broadly, there are three types of "answers": a formula, a recurrence, and a generating function. A great example of the second and third is the following.
$p(n)$, the number of number partitions of $n$, is given by the generating function

$$\sum_{n \geq 0} p(n)x^n = \prod_{i \geq 1} \frac{1}{1 - x^i}.$$

Using this, a recursion may be obtained as well. We do *not* plug in values for $x$ in the above. We merely look at the coefficient of $x^n$ in it. We want the coefficient to be a finite sum for all $n$. If it is an infinite sum, convergence issues may arise.

## 1.1. Counting in $\mathfrak{S}_n$

Recall that $\mathfrak{S}_n$ is generated by transpositions. A transposition $(i, j)$ is a permutation $\sigma$ defined by

$$\sigma(k) = \begin{cases} j, & k = i, \\ i, & k = j, \\ k, & \text{otherwise.} \end{cases}$$

In fact, $\mathfrak{S}_n$ is generated by the set of just "adjacent transpositions" $S_i = (i, i+1)$ for $1 \leq i < n$. We have

$$\begin{aligned} S_i^2 &= \mathrm{Id} \\ S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1} \\ S_i S_j &= S_j S_i \text{ if } |i - j| > 2. \end{aligned}$$

**Definition 1.1.** Given a permutation $\pi \in \mathfrak{S}_n$, define the *length* $\ell(\pi)$ of $\pi$ to be the smallest $k$ such that there exist adjacent transpositions $\sigma_1, \cdots, \sigma_k$ such that $\pi = \sigma_1 \cdots \sigma_k$.

**Proposition 1.2.** Consider the *inversion number* $\mathrm{inv}(\pi)$ of a permutation, defined by

$$\mathrm{inv}(\pi) = \left|\{1 \leq i \leq j \leq n : \pi_i > \pi_j\}\right|.$$

Then, $\ell(\pi) = \mathrm{inv}(\pi)$.

**Definition 1.3.** The *sign* of a permutation $\pi$ is defined by $\mathrm{sign}(\pi) = (-1)^{\mathrm{inv}(\pi)}$. Equivalently,

$$\mathrm{sign}(\pi) = \frac{\prod_{1 \leq i < j \leq n}(x_{\pi_i} - x_{\pi_j})}{\prod_{1 \leq i < j \leq n}(x_i - x_j)}.$$

It is straightforward to see that for all $\pi \in \mathfrak{S}_n$, $0 \leq \operatorname{inv}(\pi) \leq \binom{n}{2}$.

**Proposition 1.4.** Consider $\operatorname{inv}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{inv}(\pi)}$. Then,

$$\operatorname{inv}_n(q) = \prod_{1 \leq m \leq n} [n]_q,$$

where

$$[m]_q = \begin{cases} 1 + q + \cdots + q^{m-1}, & m \geq 1, \\ 0, & m = 0. \end{cases}$$

This quantity $[m]_q$ is called the $q$-analogue of $m$, and similarly, the $q$-analogue of $n!$ is $\prod_{i=1}^n [m]_q$ (this is slightly vague). Note in particular that $n! = \operatorname{inv}_n(1)$.

*Proof.* We prove this by induction. It is easily verified for $n = 2$.
Take $\sigma \in \mathfrak{S}_{n-1}$. There are $n$ "gaps" where $n$ can be "placed" in $\sigma$ to get a permutation in $\mathfrak{S}_n$. If we place it in the $i$th position from the end (for $0 \leq i \leq n-1$), the inversion number of the newly obtained permutation is $i$ more than the inversion number of $\sigma$.
As a result,

$$\operatorname{inv}_n(q) = \operatorname{inv}_{n-1}(q) + q \operatorname{inv}_{n-1}(q) + q^2 \operatorname{inv}_{n-1}(q) + \cdots + q^{n-1} \operatorname{inv}_{n-1}(q) = [n]_q \operatorname{inv}_{n-1}(q),$$

where the $q^i \operatorname{inv}_{n-1}(q)$ term corresponds to the case where $n$ is placed in the $i$th position from the end. The required follows by the inductive hypothesis. ■

**Definition 1.5** (Descent)**.** For $\pi \in \mathfrak{S}_n$, define the *descents* $\operatorname{DES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$, $\operatorname{des}(\pi) = |\operatorname{DES}(\pi)|$, and $\operatorname{maj}(\pi) = \sum_{i \in \operatorname{DES}(\pi)} i$.

Some books define the number of descents as $\operatorname{des}(\pi) + 1$ instead.
There are central limit theorems for many of these parameters, which we shall not study.
A permutation $\pi$ has $\operatorname{des}(\pi) + 1$ many "increasing runs".
For example, for the permutation $\pi = (1 \mapsto 5, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 3) \in \mathfrak{S}_6$, $\operatorname{DES}(\pi) = \{1, 4, 5\}$, $\operatorname{des}(\pi) = 3$, and $\operatorname{maj}(\pi) = 10$.

**Proposition 1.6.** The distribution of $\operatorname{maj}(\pi)$ over $\mathfrak{S}_n$ is the same as that of $\operatorname{inv}(\pi)$. Equivalently,

$$\operatorname{maj}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{maj}(\pi)} = \prod_{m=1}^n [m]_q = \operatorname{inv}_n(q).$$

This result took nearly 50 years to prove!

*Proof.* The strategy is similar to that of Proposition 1.4. Let $\pi \in \mathfrak{S}_{n-1}$. As before, there are $n$ positions to insert $n$.

- Label the positions of descents of $\sigma$ and the last position from right ot left as $0, 1, \ldots, \operatorname{des}(\pi)$.
- Label the remaining positions from left to right as $\operatorname{des}(\pi) + 1, \ldots, n-1$.

We claim that inserting $n$ at a position increases $\operatorname{maj}$ by the labelled amount.
If inserted anywhere, all the descent positions starting from there increase by $1$. This explains why the increase is equal to the labelled quantity for positions that are descents, since no new descents are introduced. In the case where we insert it in a position of non-descent, we further introduce a new descent at the position of insertion of $n$, which explains why the increase is equal tot he labelled quantity for positions that are not descents.
The remainder of the proof is identical to that of Proposition 1.4, since the increases are in bijection with $[n-1]_0$. ■

**Definition 1.7.** A parameter $f : \mathfrak{S}_n \to \mathbb{R}$ of permutations such that

$$\sum_{\pi \in \mathfrak{S}_n} q^{f(\pi)} = [n]_q! = \prod_{1 \le m \le n} [m]_q$$

is said to be *Mahonian*.

As we saw in **??** 1.4**??** 1.6, both $\mathrm{inv}$ and $\mathrm{maj}$ are Mahonian.

## 1.2. Counting spanning trees

**Problem.** Count the number of spanning trees in an arbitrary (finite) graph $G$.

This was solved by Kirchhoff using the Matrix Tree Theorem.

**Theorem 1.8** (Matrix Tree Theorem)**.** Consider the *Laplacian* $L = D - A$ of a graph $G$, where $A$ is its adjacency matrix and $D$ is a diagonal matrix with the diagonal entries being the degrees of the vertices. The determinant of any $(n-1) \times (n-1)$ submatrix of $L$ obtained by omitting any arbitrary row and column is equal to the number of spanning trees of $G$.

In particular, when $G = K_n$, we end up getting the following.

**Theorem 1.9** (Cayley's Theorem)**.** The number of spanning trees in $K_n$ is $n^{n-2}$.

One proof by Prüfer gives an explicit bijection between spanning trees and sequences $(v_1, \ldots, v_{n-2})$ of vertices in $G$. Another proof is of course using the matrix tree theorem, which reduces it to a simple determinant calculation. Joyal gave another bijection between elements of the form $(T, u, v)$ where $T$ is a spanning tree and $u, v$ are vertices in $G$, and functions from $[n] \to [n]$.

The proof we give uses exponential generating functions. Recall the following result, which we give without proof. Interested readers may consult Corollary 5.1.6 of [SF99] for further details.

**Theorem 1.10** (Exponential Formula)**.** Let $\{f_n\}, \{g_n\}$ be a sequence with exponential generating functions

$$F(x) = \sum_{n \ge 1} f_n \frac{x^n}{n!} \text{ and } G(n) = \sum_{n \ge 0} g_n \frac{x^n}{n!}.$$

Define the sequence $h_n$ by

$$h_n = \sum_{\substack{\pi \in \mathrm{SetPartn}([n]) \\ \pi = \{S_1, \ldots, S_k\}}} f_{|S_1|} f_{|S_2|} \cdots f_{|S_k|} g_k$$

and $h_0 = 1$, and let

$$H(x) = \sum_{n \ge 0} h_n \frac{x^n}{n!}.$$

Then,

$$H(x) = G(F(x)).$$

**Theorem 1.11** (Permutation Formula II)**.** Let $\{f_n\}$ be a sequence and define the sequence $h_n$ by

$$h_n = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \pi \text{ has cycles } \{C_1,\ldots,C_k\}}} f_{|C_1|} f_{|C_2|} \cdots f_{|C_k|}$$

and $h_0 = 1$, and let

$$H(x) = \sum_{n \geq 0} h_n \frac{x^n}{n!}.$$

Then,

$$H(x) = \exp\left( \sum_{n \geq 1} f_n \frac{x^n}{n} \right).$$

Note that the summation of $F$ is for $n \geq 1$, because we may assume that $f_0 = 0$ since $f_0$ does not appear in the expression of any $h_n$.

**Definition 1.12** (Compositional inverse)**.** Generating functions $F$ and $G$ are said to be *compositional inverses* (of each other) if $F(G(x)) = G(F(x)) = x$.

Let

$$F(x) = \sum_{n \geq 0} f_n x^n \text{ and } G(x) = \sum_{n \geq 0} g_n x^n$$

be compositional inverses of each other. It is reasonably straightforward to show that $f_0 = g_0 = 0$ and $f_1, g_1 \neq 0$. The first condition implies that the coefficient of any $x^n$ in $F \circ G$ (or $G \circ F$) is finite.

**Theorem 1.13** (Lagrange Inversion Theorem)**.** Let

$$F(x) = \sum_{n \geq 0} f_n x^n \text{ and } G(x) = \sum_{n \geq 0} g_n x^n$$

be compositional inverses of each other. Then, $ng_n$ is the coefficient of $1/x$ in $(1/F(x))^n$.

Equivalently, $ng_n$ is the coefficient of $x^{n-1}$ in $(x/F(x))^n$.

*Proof.* We have

$$x = G(F(x)) = \sum_{i \geq 0} g_i F(x)^i.$$

Differentiating,

$$1 = \sum_{i \geq 0} g_i i F(x)^{i-1} F'(x).$$

As a result,

$$\left( \frac{1}{F(x)} \right)^n = \sum_{i \geq 0} g_i i F(x)^{i-1-n} F'(x).$$

Whenever $i \neq n$, the coefficient of $1/x$ in $F(x)^{i-1-n}F'(x) = \left(F(x)^{i-n}/(i-n)\right)'$ is zero. Indeed, recall that the coefficient of $1/x$ in the derivative of any power series with possibly negative exponents is zero.
As a result, the coefficient of $1/x$ in $(1/F(x))^n$ is equal to the coefficient of $1/x$ in $g_n nF'(x)/F(x)$. We have

$$\frac{F'(x)}{F(x)} = \frac{f_1 + 2f_2x + \cdots}{f_1x + f_2x^2 + \cdots}.$$

The coefficient of $1/x$ in this is $f_1/f_1 = 1$, and the desideratum follows. ■

At long last, let us return to Cayley's Theorem.

*Proof of Cayley's Theorem.* Instead of looking at the number $\mathrm{T}_n$ of spanning trees, we shall look at $\mathrm{RT}_n$, the number of *rooted* spanning trees. Clearly, $\mathrm{RT}_n = n\mathrm{T}_n$.
Define $\mathrm{RF}_n$ to be the number of rooted forests on $[n]$ and let

$$\mathrm{RF}(x) = \sum_{n\geq 0} \mathrm{RF}_n \frac{x^n}{n!}$$
$$\mathrm{RT}(x) = \sum_{n\geq 0} \mathrm{RT}_n \frac{x^n}{n!}.$$

Using Theorem 1.10, it is not too difficult to see that

$$\mathrm{RF}(x) = \exp(\mathrm{RT}(x)). \tag{1.1}$$

**Claim** (Polya). $\mathrm{RT}_{n+1} = (n+1)\mathrm{RF}_n$.
Indeed, any rooted tree on $K_{n+1}$ may be obtained from a rooted forest $F$ on $K_n$ by adding a new vertex $v$, adding the edge between each root in $F$ and $v$ to the spanning tree, removing the "root status" from all vertices except $v$. $v$ can be labelled in $n+1$ ways, so we are done.

As a result,

$$\mathrm{RF}(x) = \sum_{n\geq 0} \frac{\mathrm{RT}_{n+1}}{n+1} \cdot \frac{x^n}{n!} = \frac{1}{x}\mathrm{RT}(x). \tag{1.2}$$

Combining Equations (1.1) and (1.2),

$$\mathrm{RT}(x) = x\exp(\mathrm{RT}(x)).$$

That is, RT is the compositional inverse of $x \mapsto xe^{-x}$. Now, we use the Lagrange Inversion Theorem to get that $n\mathrm{RT}_n/n!$ is equal to the coefficient of $x^{n-1}$ in $(x/xe^{-x})^n = e^{nx}$, which is $n^{n-1}/(n-1)!$. Therefore, $\mathrm{T}_n = \mathrm{RT}_n/n = n^{n-2}$ and we are done. ■

## 1.3. Chebyshev polynomials

We would like a polynomial $T_n(x)$ such that $T_n(\cos\theta) = \cos(n\theta)$. Why does such a polynomial even exist? Recall that

$$(\cos\theta + \iota\sin\theta)^n = \cos n\theta + \iota\sin n\theta.$$

Since the real part of the left only has even powers of sin, we can convert it to a polynomial of $\cos\theta$s alone.
For example,

$$T_0(x) = 1,$$
$$T_1(x) = x,$$
$$T_2(x) = 2x^2 - 1.$$

**Proposition 1.14.** $T_0(x) = 1$, $T_1(x) = x$, and for $n \geq 2$,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

*Proof.* Let $\cos\theta = x$. We have

$$\begin{aligned} T_n(x) = \cos n\theta &= \cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta \\ &= xT_{n-1}(x) - (\sin(n-2)\theta\cos\theta + \cos(n-2)\theta\sin\theta)\sin\theta \\ &= xT_{n-1}(x) - T_{n-2}(x)(1-x^2) - x(\sin\theta\sin(n-2)\theta) \\ &= xT_{n-1}(x) + x^2T_{n-2}(x) - T_{n-2}(x) - x(\cos\theta\cos(n-2)\theta - \cos(n-1)\theta) \\ &= 2xT_{n-1}(x) - T_{n-2}(x). \end{aligned}$$

■

**Definition 1.15** (Chebyshev polynomials)**.** The $n$th *Chebyshev polynomial of the first kind* $T_n$ is defined as above. The $n$th *Chebyshev polynomial of the second kind* $U_n$ is defined by

$$U_n(x) = \begin{cases} 1, & n = 0, \\ 2x, & n = 1, \\ 2xU_{n-1}(x) - U_{n-2}(x), & n \geq 2. \end{cases}$$

Consider the number of tilings of a $1 \times n$ board $B_n$ using squares ($1 \times 1$ pieces) and dimers ($1 \times 2$ pieces). It is not too difficult to show that this corresponds to the Fibonacci numbers.
Now, instead consider a *weighted* version of this problem, where we give squares a weight of $2x$ and dimers a weight of $-1$. The weight $\mathrm{wt}(T)$ of a given tiling $T$ is equal to the product of the weights of the pieces used. Then, the Chebyshev polynomial $U_n$ is just the sum of the weights of all tilings of $B_n$!

$$U_n(x) = \sum_{\text{tilings } T \text{ of } B_n} \mathrm{wt}(T).$$

Similar to this, we can get a combinatorial model for $T_n$ as well, with the only difference being that a square piece has weight $x$ if it is at the lefmost $(1,1)$ position.
Given a tiling $T$, let $S(T)$ and $D(T)$ be the number of squares and dimers in the tiling respectively. In general, define

$$F_n(s,t) = \sum_{\text{tilings } T \text{ of } B_n} s^{S(T)} t^{D(T)}. \tag{1.3}$$

Then,

$$\begin{aligned} F_0(s,t) &= 1, \\ F_1(s,t) &= s, \\ F_n(s,t) &= sF_{n-1}(s,t) + tF_{n-2}(s,t). \end{aligned}$$

## 1.4. More on $q$-analogues

Recall the definition of $[n]_q! = \prod_{i=1}^n [i]_q$. Inspired by this, define

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

This is clearly a rational function of $q$. It turns out that this is a polynomial in $q$! For example,

$$\binom{5}{2}_q = \frac{[5]_q[4]_q}{[2]_q[1]_q} = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

Recall that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

**Proposition 1.16** ($q$-Pascal's recurrences)**.** It holds that

$$\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q.$$

*Proof.* We show only the first recurrence. The second follows similarly.

$$\begin{aligned} q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q &= q^k\frac{[n-1]_q!}{[k]_q![n-k-1]_q!} + \frac{[n-1]_q!}{[k-1]_q![n-k]_q!} \\ &= q^k\binom{n}{k}_q\frac{[n-k]_q}{[n]_q} + \binom{n}{k}_q\frac{[k]_q}{[n]_q} \\ &= \binom{n}{k}_q\left(\frac{q^k[n-k]_q + [k]_q}{[n]_q}\right) = \binom{n}{k}_q. \end{aligned}$$

■

**Corollary 1.17.** $\binom{n}{k}_q$ is a polynomial in $q$ with non-negative coefficients.

It turns out that the coefficients of the polynomial are unimodal and symmetric! We do not prove this, the reader can see _ for more details.
A natural question to ask then is: what do the coefficients of this polynomial count?
Let $\binom{n}{k}_q = f_{n,k}(q) = \sum_{r\geq 0} a_{n,k}^{(r)} q^r$. Can we have

$$\binom{n}{k}_q = \sum_{T\in\binom{[n]}{k}} q^{\text{parameter}(T)}?$$

$a_{n,k}^{(r)}$ then just counts the number of $T$ with the given parameter value.
Recall that $\binom{n}{k}$ is the number of paths from $(0,0)$ to $(n-k,k)$ if only upwards and rightwards movements on the integer lattice $\mathbb{Z}^2$ are allowed. Let $P$ be such a path.
Consider the portion of the box above $P$. This can be viewed as the Ferrer diagram of some number partition $\lambda(P)$. $\lambda(P)$ has at most $k$ parts, and no part is of size more than $n-k$. In fact, *all* such partitions correspond to some path! What number is $\lambda(P)$ a number partition of? Denote this number as $|\lambda(P)|$. Let $\mathcal{S}_{n,k}$ be the set of all paths of the mentioned form.

**Theorem 1.18.**

$$\sum_{P\in\mathcal{S}_{n,k}} q^{|\lambda(P)|} = \binom{n}{k}_q.$$

Perhaps surprisingly, the proof of the above is near-straightforward using the $q$-Pascal recurrence – merely consider two cases depending on whether the first step of the path is right or upwards.

## 1.5. Derivative polynomials

We begin this section by recalling the following rather interesting result.
Define the *Bell polynomial* $B_{n,k}$ by

$$B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) = \sum \frac{n!}{j_1!(1!)^{j_1}j_2!(2!)^{j_2}\cdots j_{n-k+1}!((n-k+1)!)^{n-k+1}}\cdot x_1^{j_1}x_2^{j_2}\cdots x_{n-k+1}^{j_{n-k+1}},$$

where the summation is taken over all indices $j_1, \ldots, j_{n-k+1}$ of non-negative integers such that

$$k = j_1 + j_2 + \cdots + j_{n-k+1} \text{ and}$$
$$n = j_1 + 2j_2 + 3j_3 + \cdots + (n-k+1)j_{n-k+1}.$$

This has a natural correspondence to the Stirling numbers of the second kind, with $j_i$ representing the number of partitions of size $i$. In particular, the sum of coefficients of $B_{n,k}$ is $S_{n,k}$.

**Proposition 1.19** (Faà di Bruno's Formula,[dB55]).

$$D^n f(g(x)) = \sum f^{(k)}(g(x)) \cdot B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)).$$

To illustrate this better, let us look at the first few derivatives explicitly. Dropping the $(x)$ on the right to make the notation more succinct, we have

$$\begin{aligned} Df(g(x)) &= f'(g)g' \\ D^2 f(g(x)) &= f''(g)(g')^2 + f'(g)g'' \\ D^3 f(g(x)) &= f'''(g)(g')^3 + 3f''(g)g'g'' + f''(g)g'''. \end{aligned}$$

Consider the partitions of $\{1,2,3\}$, given by $1|2|3$, $12|3$, $13|2$, $23|1$, and $123$. The number of partitions of $[n]$ with $n_i$ parts of size $i$ for each $i$ neatly corresponds to the coefficient of $\prod_i (g^{(i)})^{n_i}$!
Let $y = f(x)$. If $Dy = p(f(x))$ for some polynomial $p$, then $D^n y$ is a polynomial of $f$ as well.
Suppose that $D^n y = p_n(y)$ for some sequence of polynomials $(p_n)$. It is straightforward to see that

$$\begin{aligned} p_0(y) &= y \\ p_n(y) &= \begin{cases} y & n = 0 \\ p_{n-1}(y) \cdot p_1(y) & n \geq 1. \end{cases} \end{aligned}$$

For the remainder of this section, set $y = \tan x$ and $z = \sec x$. Then, $Dy = 1 + y^2 = z^2$ and $Dz = yz$. It is not difficult to see that

$$\begin{aligned} D^2 y &= 2yz^2 \\ D^3 y &= 4y^2 z^2 + 2z^4 \\ D^4 y &= 8y^3 z^2 + 16yz^4 \end{aligned}$$

**Exercise 1.2.** With $y, z$ defined as above, show that

1. $D^n y$ is a homogeneous polynomial in $y, z$ of degree $(n+1)$.
2. $D^n y$ has only terms with even exponents of $z$.

**Corollary 1.20.** We can write $D^n y = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} W_{n,k} z^{2k+2} y^{n-2k-1}$.

Again, we ask the question: is there some parameter on $\pi \in \mathfrak{S}_n$ such that

$$W_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} W_{n,k} x^k = \sum_{\pi \in \mathfrak{S}_n} x^{\text{parameter}(\pi)}?$$

**Definition 1.21** (Peak)**.** Given a permutation $\pi \in \mathfrak{S}_n$, we say that $i \in [n] \setminus \{1, n\}$ is a *peak* of $\pi$ if $\pi_i > \pi_{i-1}$ and $\pi_i > \pi_{i+1}$. Denote the set of peaks of $\pi$ by $\mathrm{Peak}(\pi)$, $\mathrm{pk}(\pi) = |\mathrm{Peak}(\pi)|$ the number of peaks.

**Lemma 1.22.** For $n, k \geq 1$,

$$W_{n,k} = (2k+2)W_{n-1,k} + (n-2k)W_{n-1,k-1}.$$

*Proof.* We have

$$\begin{aligned}
D^n y &= D \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} W_{n-1,k} z^{2k+2} y^{n-2k-2} \\
&= \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (2k+2)W_{n-1,k} z^{2k+1} \cdot zy \cdot y^{n-2k-2} + (n-2k-2)W_{n-1,k} z^{2k+2} y^{n-2k-3} \cdot z^2 \\
&= \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (2k+2)W_{n-1,k} z^{2k+2} y^{n-2k-1} + (n-2k-2) z^{2k+4} y^{n-2k-3}.
\end{aligned}$$

The required follows. ∎

**Theorem 1.23.**

$$W_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{pk}(\pi)}.$$

*Proof.* Let $Y_n$ be the polynomial on the right, and let $Y_{n,k}$ be its coefficients. It is easily checked that $Y_{n,k}$ and $W_{n,k}$ are equal for $n = 0$ or $k = 0$. To prove the statement, we shall merely show that $Y_{n,k}$ satisfies the recurrence of Lemma 1.22 too.
Similar to what we did in earlier proofs such as those of **??** 1.4**??** 1.6, let $\sigma$ be a permutation in $\mathfrak{S}_{n-1}$.
We shall use it to get a permutation $\pi \in \mathfrak{S}_n$ by "inserting" $n$ at one of the $n$ possible positions. If we insert it at the position of a non-peak of $\sigma$, the number of peaks increases by one. If we insert it before or after the position of a peak, the number of peaks stays the same. Since peaks cannot occur immediately after each other, we can insert it at precisely $2k+2$ positions while ensuring that the number of peaks does not increase (the extra 2 is for the extreme positions), and so at $n-2k-2$ positions which increases the number of peaks by one. Therefore,

$$Y_{n,k} = (2k+2)Y_{n-1,k} + (n-2(k-1)-2)Y_{n-1,k-1} = (2k+2)Y_{n-1,k} + (n-2k)Y_{n-1,k-1}.$$ ∎

**Corollary 1.24.** It is true that

$$\sum_k W_{n,k} = n!.$$

## 1.6. Matching theory

**Definition 1.25** (Matching)**.** Given a graph $G = (V, E)$, a *matching* in $G$ is a collection $M \subseteq E$ of edges such that for any distinct $e_1, e_2 \in M$, $e_1 \cap e_2 = \varnothing$.
The number of $k$-sized matchings is denoted $m_k(G)$. Define the *matching polynomial*

$$\mathrm{Match}_G(x) = \sum_{k \geq 0} (-1)^k m_k(G) x^{n-2k}$$

Some books call the above the "defect" matching polynomial, taking the actual matching polynomial as $p(x) = \sum_k m_k(G)x^k$. Note that $\mathrm{Match}_G(x) = x^n p(-1/x^2)$.
Clearly, $\mathrm{Match}_G(x) = x^n \cdot p(-1/x^2)$.

There is a very rich literature regarding matching theory. One work that set off a frenzy of results in related areas was [Edm65], which gave a polynomial-time algorithm to get a maximum weight matching in any graph. It does so by looking at the polytope in $\mathbb{R}^E$ that is the convex hull of the indicator functions of all matchings. It is worth noting that while there is a polynomial time algorithm to find a maximum weight matching, the problem of determining the number of maximum matchings in a graph is #P-complete. Consequently, no polynomial time algorithm is known to determine $m_k(G)$ given a graph $G$.

Before moving on, we give some simple lemmas about the matching polynomial.

**Lemma 1.26.**

(a) If $G$ and $H$ are vertex-disjoint graphs,

$$\mathrm{Match}_{G \cup H}(x) = \mathrm{Match}_G(x)\,\mathrm{Match}_H(x).$$

(b) Given a graph $G$ and vertex $v \in G$,

$$\mathrm{Match}_G(x) = x\,\mathrm{Match}_{G-\{v\}}(x) - \sum_{u : u \leftrightarrow v} \mathrm{Match}_{G-\{u,v\}}(x).$$

(c) Given a graph $G$ and edge $e = \{u, v\} \in G$,

$$\mathrm{Match}_G(x) = \mathrm{Match}_{G-e}(x) - \mathrm{Match}_{G-\{u,v\}}(x).$$

*Proof.* We omit the proof of (a) as it is straightforward.

(b) Let $M$ be a matching of size $k$ on $G$. If $M$ does not have an edge incident on $v$, it is a matching of size $k$ on $G - \{v\}$. Otherwise, there is some edge $e = \{u, v\} \ni M$, and $M \setminus \{e\}$ is a matching on $G - \{u, v\}$. As a result,

$$m_k(G) = m_k(G - \{v\}) + \sum_{u : u \leftrightarrow v} m_{k-1}(G - \{u, v\}).$$

Multiplying with $(-1)^k x^{n-2k}$ and summing over $k$,

$$\begin{aligned} \mathrm{Match}_G(x) &= \sum_k (-1)^k x \cdot x^{(n-1)-2k} m_k(G - \{v\}) - \sum_{u : u \leftrightarrow v} (-1)^{k-1} x^{(n-2)-2(k-1)} m_k(G - \{u, v\}) \\ &= x\,\mathrm{Match}_{G-\{v\}} - \sum_{u : u \leftrightarrow v} \mathrm{Match}_{G-\{u,v\}}(x). \end{aligned}$$

(c) Similar to (b), let $M$ be a matching of size $k$ on $G$. If $M$ does not have $e$, it is a matching of size $k$ on $G-e$. Otherwise, $M \setminus \{e\}$ is a matching on $G - \{u, v\}$. So,

$$m_k(G) = m_k(G-e) + m_{k-1}(G - \{u,v\}).$$

Multiplying with $(-1)^k x^{n-2k}$ and summing over $k$,

$$\begin{aligned}\mathrm{Match}_G(x) &= \sum_k (-1)^k x^{n-2k} m_k(G-e) - (-1)^{k-1} x^{(n-2)-2(k-1)} m_{k-1}(G - \{u,v\}) \\ &= \mathrm{Match}_{G-e}(x) - \mathrm{Match}_{G-\{u,v\}}(x).\end{aligned}$$ ■

**Proposition 1.27.**

1. $m_k(P_n) = \binom{n-k}{k}$.
2. $m_k(C_n) = \frac{n}{n-k}\binom{n-k}{k}$.
3. $m_k(K_n) = \binom{n}{2k} \cdot \frac{(2k)!}{2^k k!}$.
4. $m_k(K_{n,n}) = \binom{n}{k}^2 k!$.

*Proof.*

1. Collapse every edge in a matching to its left endpoint, and "mark" the collapsed vertices. This results in a path with $n-k$ vertices with $k$ marked vertices. This process of marking the vertices using the matching is reversible, and $m_k(G) = \binom{n-k}{k}$.
2. Fix some edge $e$. $e$ is absent in exactly $(n-k)/n$ of the $k$-matchings of $C_n$. In this case, the remaining matching forms a matching on $C_n - e$, which is isomorphic to $P_n$. Therefore, $(n-k)/n m_k(C_n) = m_k(P_n) = \binom{n-k}{k}$.
3. A $k$-matching of $K_n$ is obtained by choosing $2k$ vertices (done in $\binom{n}{2k}$) ways, putting the $2k$ vertices in $k$ indistinguishable "boxes" by putting 2 in each (this can be done in $(2k)!/k!2^k$ ways).
4. A $k$-matching is obtained by choosing $k$ vertices from each side of the bipartite graph (done in $\binom{n}{k}^2$ ways), then assigning each vertex on the left side a vertex on the right that it is joined to in the matching (done in $k!$ ways). ■

**Theorem 1.28.** Given a graph $G$, all roots of $\mathrm{Match}_G(x)$ are real.

The version of the proof of the above we give is due to Godsil [GG81].

*Proof.* Using Lemma 1.26(a), we may assume that $G$ is connected.
We first prove the result for the case where $G$ is a tree $T$. To prove this, we shall prove that $\mathrm{Match}_T(x)$ is the characteristic polynomial $\det(xI - A)$ of the adjacency matrix $A$ of $T$(!); the result then follows since $A$ is a real symmetric matrix and thus has real eigenvalues.
Let $xI - A = (b_{ij})$. We have

$$\mathrm{Charpoly}(A) = \sum_{\pi \in \mathfrak{S}_n} \mathrm{sign}(\pi) \prod_{i=1}^n b_{i\pi(i)}.$$

First, we claim that if $\pi \in \mathfrak{S}_n$ has a cycle of length greater than 2, then the term corresponding to $\pi$ on the right will be zero. In other words, the term is zero if $\pi$ is not an involution. Indeed, this follows immediately since $G$ has no

cycles (of length $\geq 3$). As a result, if $(i_1, i_2, \ldots, i_t)$ were a cycle in $\pi$, then there must be some $j$ such that $\{i_j i_{j+1}\}$ is not an edge in $G$ and $b_{i_j\sigma(i_j)} = 0$.
Suppose that some $\pi \in \mathfrak{S}_n$ has $k$ 2-cycles and $(n-2k)$ fixed points, and also has the term on the right being nonzero. We have $\operatorname{sign}(\pi) = (-1)^{(n-(k+n-2k))} = (-1)^k$. Suppose that the $k$ 2-cycles are $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$. We have $b_{i_r j_r} = b_{j_r i_r} = (-1)$ so $b_{i_r j_r} b_{j_r i_r} = 1$, and also that no $i_r$ (or $j_r$) is equal to any other $i_s$ (or $j_s$). That is, the edges constituted by $\{i_r, j_r\}$ form a matching of size $k$! Therefore,

$$\operatorname{Charpoly}(A) = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \pi \text{ an involution}}} \operatorname{sign}(\pi) \prod_{i=1}^{n} b_{i\pi(i)} = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \pi \text{ an involution}}} (-1)^k x^{n-2k} = \sum_{\text{matchings } M} (-1)^{|M|} x^{n-2|M|} = \operatorname{Match}_T(x).$$

For a general graph, we come up with a tree $T_a(G)$ that depends on a "starting vertex" $a \in G$. We then show that the matching polynomial of our graph divides the matching polynomial of the tree.
To define $T_a$, we need paths starting at $a$ in $G$ without repeated vertices. These paths are the vertices of $T_a(G)$. There is an edge between two paths if one is an extension of another by a single vertex – for example, the paths $abdc$ and $abdce$ would have an edge between them.
It is straightforward to see that $G$ is isomorphic to $T_a(G)$ for any vertex $a \in G$ if $G$ is a tree. Indeed, there is precisely one path from $a$ to any vertex $b$ in the tree. The heart of the argument is the fact that for any $a \in G$,

$$\frac{\operatorname{Match}_{G-a}(x)}{\operatorname{Match}_G(x)} = \frac{\operatorname{Match}_{T_a(G)-a}(x)}{\operatorname{Match}_{T_a(g)}(x)}. \tag{1.4}$$

To prove this, we use induction on the number of vertices $n$. The base case is $n = 2$, which forces $G$ to be a tree. Now,

$$\begin{aligned}
\frac{\operatorname{Match}_G(x)}{\operatorname{Match}_{G-a}(x)} &= \frac{x \operatorname{Match}_{G-a}(x) - \sum_{b \leftrightarrow a} \operatorname{Match}_{G-a-b}(x)}{\operatorname{Match}_{G-a}(x)} \\
&= x - \sum_{b \leftrightarrow a} \frac{\operatorname{Match}_{G-a-b}(x)}{\operatorname{Match}_{G-a}(x)}.
\end{aligned}$$

By the inductive hypothesis,

$$\frac{\operatorname{Match}_{G-a-b}(x)}{\operatorname{Match}_{G-a}(x)} = \frac{\operatorname{Match}_{T_b(G-a)-b}(x)}{\operatorname{Match}_{T_b(G-a)}(x)}. \tag{1.5}$$

Now, very carefully observe that

$$\begin{aligned}
T_b(G-a) - b &= \bigcup_{\substack{c \leftrightarrow b \\ c \neq a}} T_c(G-a) \\
T_a(G) - a &= \bigcup_{c \leftrightarrow a} T_c(G-a) \\
T_a(G) - a - ab &= \left( \bigcup_{\substack{c \leftrightarrow a \\ a \neq b}} T_c(G-a) \right) \cup \left( \bigcup_{\substack{c \leftrightarrow b \\ c \neq a}} T_c(G-a-b) \right) = \left( \bigcup_{\substack{c \leftrightarrow a \\ a \neq b}} T_c(G-a) \right) \cup \big( T_b(G-a) - b \big)
\end{aligned}$$

We can then use Lemma 1.26(a) to get that

$$\begin{aligned}
\operatorname{Match}_{T_a(G)-a}(x) &= \prod_{c \leftrightarrow a} \operatorname{Match}_{T_c(G-a)}(x). \\
\operatorname{Match}_{T_a(G)-a-ab}(x) &= \prod_{\substack{c \leftrightarrow a \\ c \neq b}} \operatorname{Match}_{T_c(G-a)}(x) \times \operatorname{Match}_{T_b(G-a)-b}(x).
\end{aligned}$$

Dividing the two,

$$\frac{\operatorname{Match}_{T_a(G)-a-ab}(x)}{\operatorname{Match}_{T_a(G)-a}(x)} = \frac{\operatorname{Match}_{T_b(G-a)-b}(x)}{\operatorname{Match}_{T_b(G-a)}}. \tag{1.6}$$

Combining Equations (1.5) and (1.6),

$$\frac{\mathrm{Match}_{G-a-b}(x)}{\mathrm{Match}_{G-a}(x)} = \frac{\mathrm{Match}_{T_a(G)-a}(x)}{\mathrm{Match}_{T_a(G)-a-ab}(x)}$$

and

$$\begin{aligned}\frac{\mathrm{Match}_G(x)}{\mathrm{Match}_{G-a}(x)} &= x - \sum_{b \leftrightarrow a} \frac{\mathrm{Match}_{T_a(G)-a-ab}(x)}{\mathrm{Match}_{T_a(G)-a}}(x) \\ &= \frac{x\,\mathrm{Match}_{T_a(G)-a}(x) - \sum_{b\leftrightarrow a} \mathrm{Match}_{T_a(G)-a-ab}(x)}{\mathrm{Match}_{T_a(G)-a}(x)} \\ &= \frac{\mathrm{Match}_{T_a(G)}(x)}{\mathrm{Match}_{T_a(G)-a}(x)}. && \text{(by Lemma 1.26(b))}\end{aligned}$$

Now, to complete the proof, we shall show that $\mathrm{Match}_{T_a(G)}(x)$ divides $\mathrm{Match}_G(x)$. We do so by induction. To do this, by Equation (1.4), it suffices to show that $\mathrm{Match}_{G-a}(x)$ divides $\mathrm{Match}_{T_a(G)-a}(x)$. Recalling that $T_a(G) - a = \bigcup_{b\leftrightarrow a} T_b(G-a)$, the inductive hypothesis implies that $\mathrm{Match}_{G-a}(x)$ divides each $\mathrm{Match}_{T_b(G-a)}(x)$, so divides their product $\mathrm{Match}_{T_a(G)-a}(x)$ as well, completing the proof. ■

This result has some consequences.

**Definition 1.29** (Log-concave)**.** A sequence $(a_n)_{n\geq 0}$ is said to be *log-concave* if $a_n^2 \geq a_{n-1}a_{n+1}$ for all $n$.

**Example 1.** For a fixed $n$, $\binom{n}{k}$ as $k$ varies is log-concave. Indeed,

$$\binom{n}{k}^2 = \frac{(n!)^2}{(k!)!((n-k)!)^2} \geq \frac{(n!)^2}{(k-1)!(k+1)!\cdot(n-k-1)!(n-k+1)!} = \binom{n}{k-1}\binom{n}{k+1}.$$

The Stirling numbers of the first and second kind are also log-concave.

**Proposition 1.30.** If $A(x) = \sum_{i=0}^n a_i x^i$ is a polynomial with all real roots, then the sequence of coefficients of $A$ is log-concave.

**Exercise 1.3.** Prove the above.

The above has an even stronger version.

**Proposition 1.31.** If $A(x) = \sum_{i=0}^n a_i x^i$ is a polynomial with all real roots, then $\left(a_i/\binom{n}{i}\right)_{i\geq 0}$ is log-concave. This is referred to as *ultralogconcavity*.

**Corollary 1.32.** For any graph $G$, $(m_k(G))_{k\geq 0}$ is log-concave. That is, for all $k$, $m_k(G)^2 \geq m_{k-1}(G)m_{k+1}(G)$.

## 1.7. Colorings

**Definition 1.33.** Given a graph $G = (V, E)$, a *$k$-coloring* (sometimes called *proper coloring*) is a function $c : V(G) \to [k]$ such that if $u \leftrightarrow v$, $c(u) \neq c(v)$.
The *chromatic number* $\chi(G)$ of a graph is the minimum number of colours required to colour it. We denote by $c_k(G)$ the number of $k$-colorings of $G$.

Determining the chromatic number of a graph is (or rather, determining if there exists a $k$-coloring) is NP-hard. The best known algorithm today outputs a colouring which is a $n/\text{polylog}(n)$-approximation of the minimum coloring.

**Proposition 1.34.** A graph $G$ has chromatic number $1$ iff it is an empty graph. It has chromatic number $\leq 2$ iff it is bipartite.

It is difficult to approximate a minimum colouring of even $3$-colorable graphs!

**Theorem 1.35** (Four-Color Theorem, [AH89])**.** A planar graph is $4$-colorable.

We omit the proof of the above (for reasons obvious to anyone familiar with the result).

**Theorem 1.36.** For any graph $G$, there exists a polynomial $\text{Chrom}_G$, known as the *chromatic polynomial*, such that $\text{Chrom}_G(k) = c_k(G)$.

For example, $\text{Chrom}_{\overline{K_n}}(x) = x^n$. For any tree $T$ on $n$ vertices, $\text{Chrom}_T(x) = t(t-1)^{n-1}$.
Observe that the chromatic polynomial is unique for a given graph since we know its value at an infinite number of points.
The proof of the above follows near-directly from the following result using an inductive argument on the number of edges – both $G \setminus E$ and $G - e$ have fewer edges than $G$.

**Proposition 1.37** (Deletion-contraction recurrence)**.** For any graph $G$, we have for $e \in G$ that

$$\text{Chrom}_G(x) = \text{Chrom}_{G-e}(x) - \text{Chrom}_{G\setminus e}(x)$$

*Proof.* Take any $k$-coloring of $G - e$. If the endpoints of $e$ have the same colour, it corresponds to a $k$-colouring of $G \setminus e$, and if the endpoints have distinct colours, it corresponds to a $k$-coloring of $G$. Therefore,

$$c_k(G) = c_k(G - e) = c_k(G \setminus e)$$

and the result follows. ■

This leads to a method to find the chromatic polynomial of any graph. Since we know the chromatic polynomial of the empty graph, we can repeatedly delete and contract edges until from our graph until we get to an empty graph. A natural question based on what we've done so far is: what do the coefficients of the chromatic polynomial mean?

**Definition 1.38.** Let $G = (V, E)$ be a graph and $<$ a total order on $E$. Given a cycle $C$, the corresponding *broken circuit* $B$ is given by $E(C) \setminus \{e\}$, where $e$ is the smallest edge in $E(C)$ in the total order. A set $A$ of edges is said to be an *NBC set* (no broken circuit) if $B \not\subseteq A$ for any broken circuit $B$. Also let

$$\mathrm{NBC}_k(G) = \{A \subseteq E : |A| = k \text{ and } A \text{ is an NBC set}\}$$

and $\mathrm{nbc}_k(G) = |\mathrm{NBC}_k(G)|$.

**Theorem 1.39.** If $|V| = n$, then for any ordering of $E$,

$$\mathrm{Chrom}_G(t) = \sum_{k=0}^{n} (-1)^k \mathrm{nbc}_k(G) t^{n-k}.$$

*Proof.* Fix some $t$. For each $A \in \mathrm{NBC}_k(A)$, consider the corresponding subgraph with edge set $A$. This describes a forest on $V$ with $n-k$ components. The number of functions $c : V \to [t]$ such that $c$ is constant on components of $A$ is $t^{n-k}$. Call such a coloring an $A$-improper coloring. For each such pair $(A, c)$, assign the sign $\mathrm{sign}(A, c) = (-1)^k$. Denote the set of all such pairs by $S$.
We would like to show that

$$\sum_{(A,c)\in S} (-1)^{\mathrm{sign}(A)} = \mathrm{Chrom}_G(t) = c_t(G).$$

Observe that $(\varnothing, c) \in S$ for any proper coloring $c$, so it suffices to show that the summation of the remaining terms is $0$. Call this set of remaining terms $S'$. We shall come up with an sign-reversing involution $\iota$ on $S'$ to prove that the sum of signs is $0$.
Observe that given any $(A, c) \in S'$, there must exist edges that are monochromatic. Let $e$ be the smallest such edge, and let $\iota(A, c) = (A \triangle \{e\}, c) = (A', c)$. It is evident that $\iota$ is sign-reversing, and it is an involution because $c$ does not change, so the smallest monochromatic edge $e$ does not either. We are done if we manage to show that the expression on the right is indeed in $S$.
If $A' = A \setminus \{e\}$, then it is clearly an NBC set and the coloring is $A'$-improper (since it is $A$-improper). If $A' = A \cup \{e\}$, since $e$ joined two vertices of the same color, $c$ is $A'$-improper. Suppose instead that $A'$ is not an NBC set. Then, it contains a broken circuit $B$ such that $e \in B$. Let $C$ be the cycle causing $B$ to be a broken circuit. Because $c$ is $A'$-improper, all vertices in $C$ have the same colour. However, $e$ is the smallest monochromatic edge in $C$, contradicting the fact that a smaller edge was removed to get $B$ and completing the proof. ■

**Corollary 1.40.** $\mathrm{Chrom}_G$ is a monic polynomial with coefficients of alternating sign.

Another question (that is not so natural) is: is $\mathrm{Chrom}_G(x)$ meaningful for $x \notin \mathbb{N}$?

**Definition 1.41.** Given a graph $G = (V, E)$, an *acyclic orientation* of $G$ is obtained by replacing each edge $uv$ with one of the directed arcs $\overrightarrow{uv}$ or $\overrightarrow{vu}$.

**Theorem 1.42.** $\mathrm{Chrom}_G(-1) = (-1)^n a(G)$, where $n = |V(G)|$ and $a(G)$ is the number of acyclic orientations of $G$.

Also observe that by Theorem 1.39, $a(G)$ is equal to the number of NBC sets in $G$.

*Proof.* Let $e = \{u, v\}$ be an arbitrary edge in the graph. Using the Deletion-contraction recurrence, it suffices to show that

$$a(G) = a(G - e) + a(G \setminus e).$$

Let $A(G)$ be the set of acyclic orientations on $G$, so $a(G) = |A(G)|$.
Define $f : A(G) \to A(G - e)$ as the natural restriction map. We claim that $f$ is surjective.
Suppose instead that some $O' \in A(G - e)$ is not in the image of $f$. Consider the two orientations $O_1, O_2$ which are identical to $O'$ except that the edge $e$ has orientation $u \to v$ or $v \to u$ respectively. Since neither of these is in $A(G)$, there must be a directed path $v \to u$ in $O'$ (because $O_1$ has a cycle) and another directed path $u \to v$ in $O'$ (because $O_2$ has a cycle). Concatenating the two, we get a directed closed walk which must contain a cycle, contradicting the fact that $O' \in A(G - e)$.

Because $f$ is a restriction map, for any $O' \in A(G - e)$, $|f^{-1}(O')| \in \{1, 2\}$. Let $X$ and be the set of all $O'$ with $f^{-1}(O') = 1$ and $Y$ the set of all $O'$ with $f^{-1}(O') = 2$. Also let $x = |X|$ and $y = |Y|$.
We have $a(G - e) = x + y$ and $a(G) = x + 2y$, so we are done if we show that $a(G \setminus e) = y$.

Because we can orient $e$ in either way for any $O' \in A(G - e)$ to get an orientation in $A(G)$, there is no directed path from $u$ to $v$ or $v$ to $u$ in $O'$. In particular, when merging $u$ and $v$ in $O'$ to get an orientation of $A(G \setminus e)$, there is no issue in assigning orientations (there cannot be a vertex $w$ such that $\overrightarrow{uw}$ and $\overrightarrow{wv}$ are edges).
This merging procedure can be reversed as well, so there is a bijection between $Y$ and $A(G \setminus e)$ and we are done. ■

This idea of plugging negative values into polynomials of interest has led to the fascinating subject of combinatorial reciprocity. This includes the idea of "running recurrences backward". For example, we have the Fibonacci numbers defined by $f_0 = 0, f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$. Can we extend this to negative numbers such that $f_{m-1} = f_{m+1} - f_m$, where $m - 1$ is negative. This leads to a signed version of the Fibonacci numbers, with $f_{-n} = (-1)^{n+1} f_n$ for $n > 0$.

We encourage the reader to try Problem 3.3.5 to get that on running the binomial recurrence backwards (for negative $n$), one gets

$$\binom{-(n+1)}{k} = (-1)^k \binom{n+k}{k}.$$

**Exercise 1.4.** Run the recurrence of $S_{n,k}$ backwards (to negative $n, k$).

*Proof.* We have $S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}$. For negative $n$,

$$S_{-(n+1),k} = S_{-n,k+1} - (k+1)S_{-(n+1),k+1}.$$

For $k = -1$, we get $S_{-(n+1),-1} = S_{-n,0}$. In particular, $S_{-1,-1} = 1$. ■

The interested reader may refer to [BS] for more details on reciprocity theorems.

## 1.8. Increasing spanning forests

**Definition 1.43** (Increasing spanning forest)**.** Let $G = (V, E)$ and let the elements of $V$ be totally ordered. Let $F$ be a forest in $G$. Root each connected component $T$ of $F$ at its minimum vertex. $F$ is called an *increasing spanning forest* if each tree is increasing, that is, the "children" of any vertex in the tree with respect to the rooting are larger than

the vertex. Equivalently, any path from a root to a vertex in the tree is increasing.
Let $\mathrm{isf}_m(G)$ be the number of increasing spanning forests on $G$ with $m$ edges, and define

$$\mathrm{ISF}_G(x) = \sum_{m \geq 0} (-1)^m \,\mathrm{isf}_m(G) x^{n-m}.$$

**Lemma 1.44.** Let $G = (V = [n], E)$ with the total order on $V$ being the natural order. For each $k \in V$, let

$$E_k = \{\{i, k\} \in E : i < k\}.$$

$F$ is an increasing spanning forest of $G$ iff it has at most one edge from each $E_k$.

Observe that the $E_k$ are mutually disjoint and their union is all of $E$.

*Proof.* Suppose that there are two edges $\{i, k\}$ and $\{j, k\}$ in some $E_k$ in an isf $E$, where $i < j$. Then, the root of the tree containing $\{i, j, k\}$ is at most $i$. In particular, one of the paths from this root to $i$ or $j$ must contain the subpath $ikj$ or $jki$, which contradicts the fact that it is increasing.

On the other hand, let $F \subseteq E$ such that $|F \cap E_k| \leq 1$ for each $k$.
First, let us show that $F$ is acyclic. Suppose instead that $v_1 v_2 \cdots v_r v_1$ is a cycle, and let $v_j = \max_{1 \leq i \leq r} v_i$. Then, both $v_{j-1} v_j$ and $v_{j+1} v_j$ are edges in $E_{v_j}$, contradicting the fact that $|F \cap E_{v_j}| \leq 1$.
Now, we must show that it is increasing. Suppose instead that $v_1 v_2 \cdots v_r$ is a non-increasing path in $F$, with $v_1 \leq v_i$ for $i > 1$, and that $v_{j-1} < v_j > v_{j+1}$. This again contradicts the fact that $|F \cap E_{v_j}| \leq 1$, completing the proof. ∎

**Theorem 1.45.** With $E_i$ defined as above,

$$\mathrm{ISF}_G(x) = \sum_{i=1}^{n} (x - |E_i|).$$

*Proof.* Let the polynomial on the right be $p$. Then, the coefficient of $x^{n-m}$ in $p$ is

$$\sum_{\{i_1, \ldots, i_m\} \subseteq [n]} |E_{i_1}||E_{i_2}| \cdots |E_{i_m}|.$$

This is precisely equal to $\mathrm{isf}_m(G)$ by Lemma 1.44. ∎

It has been observed that for certain classes of graphs and orderings, this is equal to the chromatic polynomial – the interested reader may look at [Sag20] for more details.

## 1.9. Linear recurrences and rational generating functions

Consider sequences $(a_n)$ such that

$$A(x) = \sum a_n x^n = \frac{p(x)}{q(x)},$$

where $q$ is a polynomial of degree $d$ and $p$ is a polynomial of degree $< d$.

**Definition 1.46.** A sequence $(a_n)_{n\geq 0}$ of reals is said to satisfy a *linear constant term recurrence* (of length $d$) if there exist complex numbers $c_1, \dots, c_d$ with $c_d \neq 0$ such that

$$a_{n+d} + c_1 a_{n+d-1} + \cdots + c_d a_n = 0$$

for all $n \geq 0$.

A simple example of this is the recurrence satisfied by the Fibonacci numbers. Given $f_{n+2} - f_{n+1} - f_n = 0$, the recurrence is

$$F(x) = \frac{p(x)}{1 - x - x^2},$$

where $p$ depends on the initial values. Observe that the coefficients $1, -1, -1$ in the denominator match those in the recurrence!

**Theorem 1.47.** Let $(a_n)_{n\geq 0}$ be a sequence with generating function $A$. The following are equivalent.

(a) $(a_n)_{n\geq 0}$ satisfies a linear constant term recurrence of length $d$ with constants $c_1, \dots, c_d$.

(b) There exist

$$A(x) = \sum a_n x^n = \frac{p(x)}{q(x)},$$

where $q(x) = 1 + c_1 x + \cdots + c_d x^d$ and $\deg(p) < d$.

(c) If the roots of the polynomial $q$ in (b) are $(1/r_i)_{i=1}^k$, with $(1/r_i)$ having multiplicity $d_i$, then there are polynomials $(p_i)_{i=1}^k$ of degree $< d_i$ such that

$$a_n = \sum_{i=1}^k p_i(n) r_i^n$$

for all $n \geq 0$.

*Proof.*

(a)⇒(b) Multiply the linear constant term recurrence by $x^{n+d}$ and sum over all $n \geq 0$. Setting $c_0 = 1$, we get

$$0 = \sum_{t=0}^{d} c_t x^t \left( A(x) - \sum_{i=1}^{d-t-1} a_i x^i \right),$$

so

$$A(x) = \frac{\sum_{t=0}^{d} c_t x^t \left( \sum_{i=1}^{d-t-1} a_i x^i \right)}{\sum_{t=0}^{d} c_t x^t} = \frac{p(x)}{q(x)},$$

where $\deg(p) < d$.

(b)⇒(a) We have $q(x)A(x) = p(x)$. Because $\deg(p) < d$, the coefficient of $x^{n+d}$ on the right hand side is zero. On the left, this coefficient is precisely equal to the left hand side of the desired recurrence.

(b)⇔(c) Let us check that (b) implies (c) first. We have

$$A(x) = \frac{p(x)}{\prod_{i=1}^k (1 - r_i x)^{d_i}}.$$

Checking that (c) holds true amounts to expanding the denominator to the numerator and matching coefficients. The argument is reversible as well, so (c) implies (b). ■

Also observe that running a linear constant term recurrence backwards (extending it to negative $n$) gives another linear constant term recurrence!

**Theorem 1.48.** Let $(a_n)_{n\geq 0}$ be a sequence satisfying a linear constant term recurrence. Obtain $a(-n)$ by running the recurrence backwards, and set

$$A^{\text{rev}}(x) = \sum_{n\geq 1} a(-n)x^n.$$

Then,

$$A^{\text{rev}}(x) = -A\left(\frac{1}{x}\right).$$

*Proof.* ■

There are numerous multivariate extensions of this proved by Stanley.

## 1.10. Exponential Generating Functions

**Definition 1.49.** Given a sequence $(a_n)_{n\geq 0}$, its *exponential generating function* (egf) is defined by

$$A(x) = \sum_{n\geq 0} \frac{a_n x^n}{n!}.$$

**Lemma 1.50.** If $A(x)$ and $B(x)$ are the egfs of $(a_n)$ and $(b_n)$ respectively, then $A(x)B(x)$ is the egf of $(c_n)_{n\geq 0}$ defined by

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

The above is very simple to prove, and we omit the proof.
Let us use the above to determine the egf for $d_n$, the number of derangements in $\mathfrak{S}_n$. Take by convention that $d_0 = 1$.

**Proposition 1.51.** It is true that

$$\sum_{k=0}^{n} \binom{n}{k} d_{n-k} = n!.$$

*Proof.* Given a bijection $\pi \in \mathfrak{S}_n$, let $S \subseteq [n]$ be the set of fixed points $\{i : \pi_i = i\}$ of the permutation. This results in a derangement of $[n] \setminus S$, and this idea results in a bijection between

$$\mathfrak{S}_n \text{ and } \{(S, \sigma) : S \subseteq [n], \sigma \text{ is a derangement of } [n] \setminus S\}.$$

■

**Corollary 1.52.** The egf of $(d_n)_{n\geq 0}$ is equal to

$$D(x) = \frac{1}{e^x(1-x)}.$$

*Proof.* By the previous proposition,

$$n! = \sum_{k=0} \binom{n}{k} \cdot d_{n-k} \cdot 1.$$

The egf of the constant sequence 1 is $e^x$ and the egf of $(n!)$ is $1/(1-x)$. By Lemma 1.50, $D(x)e^x = 1/(1-x)$, completing the proof. ∎

**Proposition 1.53.** If $(a_n)_{n\geq 0}$ has egf $A(x)$, then

$$A'(x) = \sum_{n\geq 0} a_n \frac{x^{n-1}}{(n-1)!}.$$

We now encourage the reader to do Problem 3.3.1.

## 1.11. Another equidistributed pair of parameters

Recall how we had seen in Proposition 1.6 a pair of parameters that have the same distribution. Now, we shall look at another such pair of parameters.
Recall descents in $\mathfrak{S}_n$ from Definition 1.5.

**Definition 1.54** (Eulerian polynomial)**.** Define the *Eulerian polynomial*

$$A_n(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}(\pi)}. \tag{1.7}$$

For example, $A_3(t) = 1+4t+t^2$. Let $A_{n,k}$ be the coefficient of $x^k$ in $A_n$ – this is the number of $\pi \in \mathfrak{S}_n$ with $\mathrm{des}(\pi) = k$. One gets the recurrence

$$A_{n,k} = (k+1)A_{n-1,k} + (n-k)A_{n-1,k-1}$$

for $n \geq 1$ with $A_{0,0} = 1$. This may be proved by the (hopefully) now standard trick of seeing how the number of descents changes on inserting $n+1$ at various positions in $\pi \in \mathfrak{S}_n$.

Also observe that on reversing a permutation, we have $\mathrm{des}(\pi^{\mathsf{rev}}) = (n-1) - \mathrm{des}(\pi)$. This means that the coefficients of $A_n$ are palindromic. Further, descents and "ascents" are equidistributed.

**Definition 1.55** (Excedances)**.** Given $\pi \in \mathfrak{S}_n$, define the set of *excedances*

$$\mathrm{EXC}(\pi) = \{i \in [n] : \pi_i > i\}$$

and the number of excedances $\mathrm{exc}(\pi) = |\,\mathrm{EXC}(\pi)|$.

Excedances are something of a cyclic counterpart of descents.

**Theorem 1.56.** Descents and excedances are equidistributed. Equivalently,

$$A_n(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{exc}(\pi)}.$$

Similar to excedances, consider non-excedances – points where $\pi_i < i$.

**Exercise 1.5.** Show that $\operatorname{exc}(\pi) = \operatorname{nonexc}(\pi^{-1})$.

*Proof.* ■

*Proof of Theorem 1.56.* Let $\pi \in \mathfrak{S}_n$ have cycles $c_1, \ldots, c_k$. For each $i$, let $m_i$ be the maximal element of $c_i$, and assume that $c_i$ is written starting with $m_i$. Also assume that we write the cycles in increasing order of $m_i$. We refer to this as the *canonical* cycle decomposition of $\pi$. Suppose we write the permutation in this form, as

$$((c_1)_1(c_1)_2 \cdots (c_1)_{r_1}) \cdots ((c_k)_1(c_k)_2 \cdots (c_k)_{r_k}).$$

Observe that the parenthesising is redundant! Reading the string $(c_1)_1(c_1)_2 \cdots (c_1)_{r_1} \cdots (c_k)_1(c_k)_2 \cdots (c_k)_{r_k}$ from left to right, we can recover the cycles – a certain position is the beginning of a cycle iff it is greater than all the elements before it. This gives a bijection between canonical cycle representations and $\mathfrak{S}_n$ (which is not the obvious one). Therefore, composing the two maps, let $\Phi : \mathfrak{S}_n \to \mathfrak{S}_n$ be the map that given a permutation, reads it from left to right to get a canonical cycle decomposition, and yields as output the permutation corresponding to this cycle decomposition.
This map gives some sort of conversion from a linear form to a cyclic form. Recall how excedances were something of the cyclic analogue of the linear phenomenon of descents.

We claim that $\operatorname{nonexc}(\Phi(\pi)) = \operatorname{des}(\pi)$ for any $\pi \in \mathfrak{S}_n$.
Let $\pi \in \mathfrak{S}_n$ and $\sigma = \Phi(\pi)$. Let $i \in \operatorname{DES}(\pi)$. That is, $\pi_i > \pi_{i+1}$. Observe that $\pi_i, \pi_{i+1}$ are forced to be in the same cycle in $\sigma$ (due to the "left to right maxima" conversion). That is, $\sigma(\pi_i) = \pi_{i+1}$. Because this is less than $\pi_i$, any descent in $\pi$ map to a non-excedance in $\sigma$.
It remains to show that ascents do not map to non-excedances. If $\pi_i < \pi_{i+1}$, then $\pi_i, \pi_{i+1}$ are either

- in different cycles of $\sigma$, in which case it is either an excedance if the size of the cycle is $> 1$ (because it maps to the first element of the cycle, which is the maximum) or a fixed point otherwise, or
- within a cycle, in which case they contribute to $\operatorname{exc}(\sigma)$ (and so definitely not $\operatorname{nonexc}(\sigma)$),

completing the proof. ■

Consider a *palindromic polynomial*

$$f(t) = f_0 + f_1 t + \cdots + f_d t^d,$$

where $f_r = f_{d-r}$. Let $k$ be the "center of palindromicity" which is roughly $d/2$. For a fixed $d$, observe that the sum of palindromic polynomials is palindromic, and the scalar multiple of a palindromic polynomial is palindromic as well. That is, the set of palindromic polynomials form a vector space.
One obvious basis of this vector space is

$$\{t^r + t^{d-r} : 0 \le r \le k\}.$$

Another (less obvious) basis of this space is

$$\operatorname{span}\left\{t^j(1+t)^{d-2j} : 0 \le j \le k\right\}.$$

This is referred to as the *Gamma basis* – why is it a basis?

**Theorem 1.57** ([DS20])**.** The Eulerian polynomial is $\gamma$-positive.

That is, when the Eulerian polynomial is represented as a linear combination of the elements of the Gamma basis, all coefficients are positive. For example,

$$A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4 = (1+t)^4 + 22t(1+t)^2 + 16t^2.$$

We do not prove the above.

## 1.12. Carlitz' identity

Consider the generating function

$$P_n(t) = \sum_{j \geq 0} j^n t^j$$

**Theorem 1.58** (Carlitz' Identity)**.** It is true that

$$P(t) = \frac{tA_n(t)}{(1-t)^{n+1}}.$$

Equivalently,

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{j \geq 0} (j+1)^n t^j.$$

*Proof.* Define a *barred permutation* to be a permutation of length $n$ with "vertical bars" between elements such that there is a bar between $\pi_i, \pi_{i+1}$ if $i \in \mathrm{DES}(\pi)$. There are allowed to be more than one bar between any elements of the permutation, the only constraint is that there must be at least one bar at positions of descent. Given a barred permutation $\overline{\pi}$, let $\mathrm{bars}(\overline{\pi})$ be the number of bars in it. We shall look at

$$\sum_{\overline{\pi} \text{ is a barred permutation of length } n} t^{\mathrm{bars}(\overline{\pi})}. \tag{1.8}$$

Also denote $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ to be the number of $k$-sized multisets of $[n]$. So,

$$\frac{1}{(1-t)^{n+1}} = \sum_{k \geq 0} \left(\!\binom{n+1}{k}\!\right) t^k.$$

Observe that $t^{\mathrm{des}(\pi)}/(1-t)^{n+1}$ for some permutation $\pi$ is precisely the contribution of $\pi$ to Equation (1.8)! The coefficient of $t^k$ in $1/(1-t)^{n+1}$ is precisely the number of ways to insert $k$ bars in the $n$ positions. Therefore,

$$\sum_{\overline{\pi} \text{ is a barred permutation of length } n} t^{\mathrm{bars}(\overline{\pi})} = \sum_{\pi \in \mathfrak{S}_n} \frac{t^{\mathrm{des}(\pi)}}{(1-t)^{n+1}} = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Let us now count this expression in another way, looking at the coefficient of $t^k$. This amounts to just putting $n$ "distinct balls" in $k+1$ "distinct bins", which is $(k+1)^n$. Therefore,

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (k+1)^n t^k. \qquad \blacksquare$$

*Proof due to class.* Observe that

$$P_{n+1}(t) = \sum_{j \geq 0} j^{n+1} t^j = t \sum_{j \geq 0} j^n (jt^j) = tP_n'(t).$$

It is also seen that

$$P_0(t) = \sum_{j \geq 0} t^j = \frac{1}{1-t}.$$

The two equations above are seen to imply that $P_n(t)$ is a rational function for any $n$. Further, the denominator of any $P_n$ is some exponent of $(1-t)$.

In general, suppose $P_k(t) = p_k(t)/(1-t)^{r_k}$ for a polynomial $p_k$ such that $p_k(1) \neq 0$. Then,

$$\begin{aligned} tP_k'(t) &= \frac{t\left(p_k'(t)(1-t)^{r_k} - r_k(1-t)^{r_k-1}p_k(t)\right)}{(1-t)^{2r_k}} \\ &= \frac{t(1-t)p_k'(t) - r_k t p_k(t)}{(1-t)^{r_k+1}}. \end{aligned}$$

The numerator is nonzero at 1, so $r_{k+1} = r_k + 1$. Since $r_0 = 1$, $r_k$ is just $k+1$. We also have

$$p_{k+1}(t) = t(1-t)p_k'(t) - (k+1)tp_k(t).$$

It is seen that this recurrence is satisfied by $tA_n(t)$ and $p_1(t) = tA_1(t)$, completing the proof. ■

Now, consider the egf

$$S(t,u) = \sum_{n\geq 0} A_n(t)\frac{u^n}{n!}.$$

We have by Theorem 1.58 that

$$\begin{aligned} \frac{S(t,u)}{1-t} &= \sum_{n,m\geq 0}(m+1)^n t^m \frac{\left(u(1-t)\right)^n}{n!} \\ &= \sum_{m\geq 0} t^m \sum_{n\geq 0}\frac{\left(u(m+1)(1-t)\right)}{n!} \\ &= \sum_{m\geq 0} t^m \exp(u(m+1)(1-t)) \\ &= \exp(u(1-t))\sum_{m\geq 0}\left(t\exp(u(1-t))\right)^m \\ &= \frac{e^{u(1-t)}}{1-te^{u(1-t)}} \\ &= \frac{1}{e^{u(t-1)}-t} \\ S(t,u) &= \frac{1-t}{e^{u(t-1)}-t}. \end{aligned}$$

Despite not having anything explicit for the Eulerian polynomial itself, we do get an explicit egf for it!

Recall $\gamma$-positivity of palindromic polynomials.

**Definition 1.59.** If $f(t) = \sum_{k=r}^n a_k t^k$ with $a_r, a_n \neq 0$ then $f$ is said to be *palindromic* if $t^{n+r}f(1/t) = f(t)$.

Typically, we discuss palindromic polynomials for a fixed $r$.
The center of symmetry of the polynomial is roughly $(n+r)/2$, and the set of palindromic polynomials for a fixed $r$ form a vector space of dimension roughly $(n-r)/2$.
Some bases of this space are

$$\begin{aligned} B_1 &= \left\{t^\ell + t^{n+r-\ell} : k = r, \ldots, (n-r)/2\right\} \\ B_2 &= \Gamma_{(n-r)/2} = \left\{t^{r+\ell}(1+t)^{n-r-2\ell} : \ell = r, \ldots, (n-r)/2\right\} \\ B_3 &= G_{(n-r)/2} = \left\{[b+1-\ell]_t - [r+\ell]_t : \ell = 0, \ldots, (n-r)/2\right\}. \end{aligned}$$

**Example 2.** It is true that

$$\begin{aligned} [5]_q &= 1+q+q^2+q^3+q^4 \\ &= (1)(1+4q+6q^2+4q^3+q^4) + (-3)(q+2q^2+q^3) + q^2. \end{aligned}$$

Let us string the coefficients that appear above as a polynomial. For example,

$$\gamma_{[5]_q}(t) = 1 - 3t + t^2 \text{ and } \gamma_{[4]_q}(t) = 1 - 2t. \tag{1.9}$$

Do the $\gamma$-coefficients of $[n]_q$ alternate in sign?
Recall the $1 \times n$ board $B_n$, which we saw how to tile with squares and dimers. Also recall Equation (1.3) and the recurrence thereafter. It is easily verified that

$$\begin{aligned} F_3(s,t) &= s^3 + 2st \\ F_4(s,t) &= s^4 + 3s^2t + t^2. \end{aligned}$$

Observe that these coefficients match (up to sign) the coefficients in Equation (1.9)!

**Lemma 1.60.**

$$F_n(1+q, -q) = [n+1]_q.$$

*Proof.* We prove this inductively. The base cases $n = 1, 2$ are trivially true as $F_1(1+q,-q) = 1 + q = [2]_q$ and $F_2(1+q,-q) = (1+q)^2 + (-q) = [3]_q$. In general, using the inductive hypothesis,

$$\begin{aligned} F_n(1+q,-q) &= (1+q)F_{n-1}(1+q,-q) - qF_{n-2}(1+q,-q) \\ &= (1+q)[n]_q - q[n-1]_q \\ &= \left(1 + 2\sum_{1 \le i \le n-1} q^i + q^n\right) - \sum_{1 \le i \le n-1} q^i = [n+1]_q. \end{aligned}$$ ■

Let

$$[n]_q = F_{n-1}(1+q,-q) = \sum_{k=0}^{n-1} f_{n-1,k}(1+q)^{n-1-2k}(-q)^k.$$

What are these $f_{n-1,k}$?

Given a graph $G$ of size $n$, consider the bivariate polynomial

$$\mathrm{Match}_G(s,t) = \sum_{\substack{\mathcal{M} \text{ matching in } G \\ |\mathcal{M}| = k}} s^{n-2k}t^k = \sum_{k \ge 0} m_k(G)s^{n-2k}t^k.$$

There is an straightforward correspondence between matchings on $P_n$ and tilings $F_n$. For each edge $\{i, i+1\}$ in a matching, consider the tiling on $B_n$ with dimers precisely at positions $\{i, i+1\}$. This is sensible because the edges form a matching.
As a result,

$$F_n(s,t) = \mathrm{Match}_{P_n}(s,t) = \sum_{k \ge 0} \binom{n-k}{k} s^{n-2k}t^k$$

and $f_{n,k} = \binom{n-k}{k}$. Consequently,

$$[n]_q = \sum_{k \ge 0} (-1)^k \binom{n-1-k}{k} \underbrace{(1+q)^{n-1-2k}q^k}_{\text{element of the } \Gamma\text{-basis!}}.$$

The $\gamma$-coefficients of $[n]_q$ do alternate in sign.

# §2. Symmetric functions

## 2.1. Introduction

Let $G = \{x_1, x_2, \dots, \}$ be a countably infinite abelian group.
Consider the group action induced by the set of permutations of $\mathbb{N}$ on the set of monomials $x_{i_1}^{\alpha_{i_1}} \cdots x_{i_k}^{\alpha_{i_k}}$. For example, if $\sigma = (1,2)$ and $m = x_1^2 x_2 x_4^3$, then $\sigma(m) = x_1 x_2^2 x_4^3$. $\sigma$ thus acts on $\mathbb{Q}[x_1, \dots, ]$ by extending this linearly. A function $f$ is said to be symmetric if $\sigma(f) = f$ for all $\sigma$. The collection of homogeneous degree $d$ symmetric functions (with the zero polynomial) forms a vector space over $\mathbb{Q}$.

**Example 3.** The unique symmetric function (up to scaling) of degree 1 is $f = \sum_{i \geq 1} x_i$.
For $d = 2$, we get $\sum_{i<j} x_i x_j$ and $\sum_i x_i^2$ as a basis.

Denote the vector space of $\Lambda_{\mathbb{Q}}^d$. It is not too difficult to show that $\dim(\Lambda_{\mathbb{Q}}^d) = p(d)$, the number of number-partitions of $d$.
The basis of $\Lambda_{\mathbb{Q}}^d$ suggested by the above example is as follows. Let $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$ be a partition of $d$ (denoted $\lambda \vdash d$). Define the monomial symmetric function

$$m_\lambda = \sum_{\text{symmetric}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} = \sum \mathbf{x}^\lambda.$$

The "symmetric" means that we sum over all distinct ways to permute the exponents $\lambda_1, \dots, \lambda_k$. The summation above is slightly strange, because we want to ensure that every monomial in $m_\lambda$ appears with coefficient 1. $\{m_\lambda\}_{\lambda \vdash d}$ is a basis of $\Lambda_{\mathbb{Q}}^d$.

**Question 1.** Given a matrix $M_{r\times k}$, on summing each row of $M$, we get a vector $\operatorname{rowsum}(M) = (r_1, \dots, r_\ell)$ and on summing the columns we get a vector $\operatorname{colsum}(M) = (c_1, \dots, c_k)$. When does there exist a 0-1 matrix $M$ such that $\operatorname{rowsum}(M)$ and $\operatorname{colsum}(M)$ are each equal to given vectors $(r_1, \dots, r_\ell)$ and $(c_1, \dots, c_k)$?

For starters, we clearly require $\sum r_i = \sum c_j =: S$. Assume without loss of generality that $r_1 \geq \cdots \geq r_\ell$ and $c_1 \geq \cdots \geq c_k$ – if a matrix for this exists, we can first reorder the rows to ensure the correct order of row sums, then reorder the columns. That is, we have two number partitions $\lambda = (r_1, \dots, r_\ell)$ and $\mu = (c_1, \dots, c_k)$ of $S$.
We shall return to the problem later.

Now, we define some partial orders on the set of number partitions.
First, we consider an order that allows the comparison of number partitions of different numbers.

**Definition 2.1** (Young's order)**.** Under *Young's order*, given two number partitions $\lambda, \mu$, $\lambda \preccurlyeq \mu$ iff the Ferrer diagram of $\lambda$ is contained in that of $\mu$.

Observe that under this order, two number partitions of the same number are *not* comparable.

Next, we define a total order over number partitions of a fixed number.

**Definition 2.2** (Lexicographic order)**.** Fix some $d$ and two partitions $\lambda = (\lambda_1, \dots, \lambda_k), \mu = (\mu_1, \dots, \mu_\ell)$ of $d$. Under the *lexicographic order*, we have $\lambda < \mu$ iff for some $r \leq k$, $\lambda_r < \mu_r$ and $\lambda_i = \mu_i$ for all $i < r$.

Finally, we define a partial order over number partitions of a fixed number.

**Definition 2.3** (Dominance/Majorisation Order)**.** Fix some $d$ and two partitions $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)}), \mu = (\mu_1, \dots, \mu_{\ell(\mu)})$ of $d$. Padding zeros at the end of one of the partitions, assume that $\ell(\lambda) = \ell(\mu) =: p$. Under the *majorisation order*, we have $\lambda \preccurlyeq \mu$ iff for $1 \leq j \leq p$,

$$\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i.$$

## 2.2. Elementary symmetric functions

We shall now see another basis of $\Lambda_{\mathbb{Q}}^d$. Consider the *elementary symmetric functions*

$$e_n = m_{1^n} = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Recall that $p(d)$ grows exponentially with $d$. Observe that if $f, g$ are symmetric functions, then $fg$ is also symmetric. For example,

$$m_1^2 = \left(\sum_i x_i\right)\left(\sum_i x_i\right) = m_2 + m_{1,1}.$$

Multiplying increases the degree as well, which is the number the partition corresponds to. As another example,

$$m_2 m_1 = m_3 + m_{2,1}.$$

For $\lambda \vdash d$ with $\lambda = \lambda_1, \cdots, \lambda_\ell$, where $\ell \geq 2$ (we have already looked at the case where $\lambda = d$), define

$$e_\lambda = \prod_{i=1}^{\ell} e_{\lambda_i}.$$

For example,

$$\begin{aligned} e_3 = m_{1^3} &= \sum_{i<j<k} x_i x_j x_k \text{ and} \\ e_{2,1} = e_2 e_1 &= \left(\sum_{i<j} x_i x_j\right)\left(\sum_k x_k\right) = 3m_{1,1,1} + m_{2,1}. \end{aligned}$$

We claim that $\{e_\lambda : \lambda \vdash d\}$ is a basis of the set of symmetric functions.
In the case of $d = 4$, we have

$$\begin{aligned} e_4 &= 1m_{1^4} \\ e_{3,1} &= 4m_{1^4} + 1m_{2,1,1} \\ e_{2,2} &= 6m_{1^4} + 2m_{2,1,1} + 1m_{2,2} \\ e_{2,1,1} &= 12m_{1^4} + 5m_{2,1,1} + 2m_{2,2} + 1m_{3,1} \\ e_{1^4} &= 24m_{1^4} + 12m_{2,1,1} + 6m_{2,2} + 4m_{3,1} + 1m_4. \end{aligned}$$

As the coefficients seem to be non-negative and integral, one can ask for a combinatorial interpretation. If

$$e_\lambda = \sum_{\mu \vdash d} M_{\lambda\mu} m_\mu,$$

what is $M_{\lambda\mu}$?

**Theorem 2.4.** Let $\lambda, \mu \vdash d$. Then, $M_{\lambda\mu}$ is the number of $\ell(\lambda) \times \ell(\mu)$ $0,1$-matrices $A$ with $\operatorname{rowsum}(A) = \lambda$ and $\operatorname{colsum}(A) = \mu$.

Recall Question 1.

*Proof.* Let $\ell(\lambda) = r, \ell(\mu) = k$.
Consider the $r \times k$ matrix

$$\begin{bmatrix} x_1 & x_2 & \cdots \\ x_1 & x_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Choose a $0,1$ vector with $\lambda_i$ 1s and paste it in in $M$ at the $i$th row. The coefficient of $x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$ in $\prod_i e_{\lambda_i}$ is given by choosing for each $i \in [r]$ a set of indices in $[k]$ such that ■

Consider the $p(d) \times p(d)$ matrix $M$ with the $\lambda, \mu$th entry equal to the $M_{\lambda\mu}$ that we defined earlier.

**Corollary 2.5.** $M$ is symmetric.

This follows immediately from Theorem 2.4 upon taking the transpose of any matrix.

**Theorem 2.6** (Gale-Ryser)**.** Let $\lambda, \mu \vdash d$. There exists a $0,1$-matrix $A$ with $\operatorname{rowsum}(A) = \lambda, \operatorname{colsum}(A) = \mu$ iff $\lambda \preccurlyeq \mu^*$ under the majorisation order.

Here, $\lambda^*$ is the *conjugate* partition of $\lambda$, which is such that its Ferrer diagram is the transpose of that of $\lambda$. More precisely,

$$(\lambda^*)_i = \left|\{j : \lambda_j \geq i\}\right|.$$

For a proof of the above, one can see this link. Note that the notation used here swaps our definitions of row sum and column sum.

**Exercise 2.1.** Show that $\lambda \preccurlyeq \mu^*$ iff $\mu \preccurlyeq \lambda^*$.

Let us now return to the question of whether the $e_\lambda$ form a basis of $\Lambda^d_{\mathbb{Q}}$. We would like to show that $M$ is invertible.

1. First, suppose that the partitions that index the and the rows lexicographically, starting with the partition $1^n$ and ending with $n$. Totally order the columns as $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(p(d))}$ such that it is compatible with majorisation – if $\mu \succcurlyeq \theta$, then $\mu \geq \theta$ (in our new order), and the reverse conjugate order is also compatible with majorisation. It turns out that the reverse lexicographic order satisfies this.

2. Show that this new matrix is upper triangular. To prove this, show that the lexicographic order is a topological sorting of the majorisation partial order.

3. Argue that all the diagonal elements of this matrix are nonzero (in fact, they are 1). This amounts to showing that the diagonal elements correspond to partition pairs of the form $(\lambda, \lambda^*)$.

**Proposition 2.7.** The lexicographic order is a topological sorting of the majorisation partial order.

*Proof.* This amounts to showing that if $\lambda \succcurlyeq \mu$, then $\lambda \geq_{\text{lex}} \mu$. ■

**Corollary 2.8.** The lexicographic order is compatible with majorisation, and so is its reverse conjugate.

*Proof.* The first part is precisely what we showed in the previous proposition. For the second part, we have that if $\lambda \succcurlyeq \mu$, then $\mu^* \succcurlyeq \lambda^*$, so $\lambda^* \geq_{\text{revlex}} \mu^*$. Indeed, if $r$ is the first index where $\lambda_r, \mu_r$ differ, we have

$$\lambda_r = \sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i = \mu_r. \qquad \blacksquare$$

**Corollary 2.9.** $M$ is invertible and has an integral inverse. Consequently, $\{e_\lambda\}_{\lambda \vdash d}$ is a basis of $\Lambda_{\mathbb{Q}}^d$.

*Proof.* By Proposition 2.7 and theorem 2.6, $M$ is upper triangular. Further, $M_{\lambda\lambda^*} = 1$ for all $\lambda$. Therefore, $M$ is invertible. As it has determinant $1$, it has an integral inverse. ■

**Definition 2.10** (Algebraic independence over $\mathbb{R}$)**.** Let $\mathbb{A}$ be an algebra. $\alpha_1, \alpha_2, \ldots \in \mathbb{A}$ are said to be *algebraically independent* over $\mathbb{R}$ if there exists no polynomial $f$ with coefficents in $\mathbb{R}$ such that $f(y_1, \ldots)\big|_{y_i = \alpha_i} = 0$.

Recall that $\Lambda_{\mathbb{Q}}^d$ is an algebra over $\mathbb{R}$.

**Corollary 2.11.** $\{e_n\}_{n \geq 1}$ are algebraically independent.

*Proof.* Suppose instead that there exists a polynomial $f(y_1, \ldots)$ such that $f(y_1, \ldots)\big|_{y_i = e_i} = 0$.
Because monomials of different degree do not interact, we may assume that $\deg(m_j\big|_{y_i = e_i}) = d$ for all monomials $m_j$ for some $d$. Therefore,

$$f(y_1, \ldots)\big|_{y_i = e_i} = \sum_{\lambda \vdash d} c_\lambda e_\lambda$$

where some $c_\lambda \neq 0$. This immediately contradicts Corollary 2.9, which says that the $\{e_\lambda\}_{\lambda \vdash d}$ are linearly independent. ■

Linear independence implies algebraically independence!

Consider the following, where we look at the products of finitely many terms on the left.

$$\prod_{i,j \geq 1} \left(1 + x_i y_j\right).$$

This gives meaningful terms when we choose finitely many terms of the form $x_i y_j$ and $1$ from the rest. Now, any such choice can be though of as an $r \times s$ $0,1$-matrix $A$, where $A_{ij} = 1$ if we choose $x_i y_j$ and $0$ otherwise. The contribution of this matrix $A$ is just $x^{\text{rowsum}(A)} y^{\text{colsum}(A)}$! Therefore,

$$\begin{aligned}
\prod_{i,j \geq 1} \left(1 + x_i y_j\right) &= \sum_{A \text{ finite } 0,1\text{-matrix}} x^{\text{rowsum}(A)} y^{\text{colsum}(A)} \\
&= \sum_{d \geq 0} \sum_{\lambda, \mu \vdash d} M_{\lambda\mu} m_\lambda(x) m_\mu(y) \\
\prod_{i,j \geq 1} \left(1 + x_i y_j\right) &= \sum_{d \geq 0} \sum_{\lambda \vdash d} m_\lambda(x) e_\lambda(y). \qquad (2.1)
\end{aligned}$$

## 2.3. Homogenous symmetric functions

Define a new class of symmetric functions, known as the *complete homogenous symmetric functions*, by

$$h_n = \sum_{\mu \vdash n} m_\mu$$

and for $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash d$,

$$h_\lambda = \prod_{i=1}^{\ell} h_{\lambda_i}.$$

For example,

$$\begin{aligned}
h_2 &= m_2 + m_{1,1} \\
h_{1,1} &= (m_1)^2 = m_2 + 2m_{1,1} \\
h_3 &= m_3 + m_{2,1} + m_{1,1,1} \\
h_{2,1} &= (m_2 + m_{1,1})m_1 = m_3 + 2m_{2,1} + 3m_{1,1,1} \\
h_{1,1,1} &= m_1^3 = m_3 + 3m_{2,1} + 6m_{1,1,1}.
\end{aligned}$$

As in the $e_\lambda$, is it true that $h_\lambda$ is a non-negative integer combination of the $m_\mu$?

**Lemma 2.12.** Let $\lambda \vdash d$. If

$$h_\lambda = \sum_{\mu \vdash d} N_{\lambda\mu} m_\mu,$$

then $N_{\lambda\mu}$ is the number of $\mathbb{N}$-matrices $A$ with $\mathrm{rowsum}(A) = \lambda, \mathrm{colsum}(A) = \mu$.

*Proof.* As before, we shall establish a bijection between the set of described matrices and the terms in the product that contribute to $m_\mu$ (each way of taking products).
Let $\lambda = \lambda_1, \ldots, \lambda_\ell$ and $\mu = \mu_1, \ldots, \mu_k$. Given a way of taking products, taking $m_{\theta_i} = \sum x^{\theta_i}$ from $h_{\lambda_i}$, consider the $\ell \times k$ matrix that assigns the power of $x_r$ to the $(r, i)$th entry.
Because $h_n = \sum_{\theta \vdash n} m_\theta$, we now have the advantage of choosing any entries in the $r$th row, as long as the sum is $\lambda_r$. ■

**Exercise 2.2.** Prove that $N_{\lambda\mu} \neq 0$ for any $\lambda, \mu \vdash d$. That is, show that there always exists an $\mathbb{N}$-matrix $A$ such that $\mathrm{rowsum}(A) = \lambda, \mathrm{colsum}(A) = \mu$.
Try to write a program that takes $\lambda, \mu$ as input and outputs such a matrix.

It turns out, however, that the problem of determining $N_{\lambda\mu}$ is #P-complete.
The next natural question to ask is: are the $(h_\lambda)$ a basis as well?
Before moving to this, let us quickly look at analogues of Equation (2.1). We jave

$$\begin{aligned}
\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} &= \prod_{i,j \geq 1} \left(1 + (x_i y_j) + (x_i y_j)^2 + \cdots\right) \\
&= \sum_{A \text{ finite } \mathbb{N}\text{-matrix}} x^{\mathrm{rowsum}(A)} y^{\mathrm{colsum}(A)} \\
&= \sum_d \sum_{\lambda, \mu \vdash d} N_{\lambda\mu} m_\lambda(x) m_\mu(y) \\
\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} &= \sum_d \sum_{\lambda \vdash d} m_\lambda(x) h_\lambda(y). \qquad (2.2)
\end{aligned}$$

Recall that $\Lambda_{\mathbb{Q}}$ is generated by $\{e_n\}_{n \geq 1}$.

**Theorem 2.13.** Consider the algebra homomorphism $\omega : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ defined by $\omega(e_n) = h_n$ for all $n$, so $\omega(e_\lambda) = h_\lambda$. $\omega$ is an involution. In particular, $(h_\lambda)_{\lambda \vdash d}$ form a basis of $\Lambda_{\mathbb{Q}}^d$.

*Proof.* Let $E(t) = \sum_{n\geq 0} e_n t^n$ be the ogf of the $e_i$, and similarly $H(t) = \sum_{n \geq 0} h_n t^n$. By Equations (2.1) and (2.2), setting $x = (t, 0, 0, \ldots)$ and noting that $m_\lambda(x) \neq 0$ iff $\lambda$ is $n$, we have that

$$\begin{aligned} E(t) &= \prod_{i\geq 1}(1 + x_i t) \\ H(t) &= \prod_{i \geq 1} \frac{1}{(1 - x_i t)}. \end{aligned}$$

That is, $H(t)E(-t) = 1$. Therefore, for $r > 0$,

$$\sum_{0 \leq k \leq r} (-1)^{r-k} e_k h_{r-k} = 0.$$

Now, let us prove by induction that $\omega(h_i) = e_i$. The base case, $n = 1$, is trivial since $e_1 = h_1$. Using the above equation for $r = n$, assuming that $\omega(h_k) = e_k$ for $k < n$, we get

$$\begin{aligned} 0 &= \omega\left(\sum_{0\leq k\leq n} (-1)^{n-k} e_k h_{n-k}\right) \\ &= \sum_{0 \leq k \leq n} (-1)^{n-k} h_k \omega(h_{n-k}) \\ &= (-1)^n \omega(h_n) + \sum_{1\leq k \leq n} (-1)^{n-k} h_k e_{n-k} \\ &= (-1)^n \omega(h_n) + (-1)^n \sum_{1 \leq k \leq n} (-1)^k h_k e_{n-k} \\ &= (-1)^n \omega(h_n) - (-1)^n e_n + (-1)^n \sum_{0\leq k\leq m} (-1)^k h_k e_{n-k} \\ 0 &= (-1)^n (\omega(h_n) - e_n). \end{aligned}$$

Therefore, $\omega$ is an involution. The second part of the theorem follows directly, because if the $\{h_\lambda\}_{\lambda\vdash d}$ were not linearly independent, then applying $\omega$ yields a violation to the linear independence of $\{e_\lambda\}_{\lambda \vdash d}$. ∎

*Remark.* $\omega : \Lambda_{\mathbb{Q}}^d \to \Lambda_{\mathbb{Q}}^d$ is a linear transformation satisfying $\omega^2 - \mathrm{Id} = 0$. Therefore, the minimal polynomial of $\omega$ is $x^2 - 1 = 0$, and the characteristic polynomial of $\omega$ is $(x-1)^\alpha (x+1)^{p(d)-\alpha}$ for some $\alpha$. We shall figure out the value of $\alpha$ later, after Theorem 2.20.

## 2.4. Power sum symmetric functions

Let us now look at one more class of symmetric functions, known as the *power sum symmetric functions*. For $n > 0$, define

$$p_n = m_n = \sum_{i \geq 1} x_i^n.$$

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash d$, define

$$p_\lambda = \prod_{i=1}^{\ell} p_{\lambda_i}.$$

Let

$$p_\lambda = \sum_{\mu \vdash d} R_{\lambda\mu} m_\mu.$$

For example,

$$\begin{aligned} p_3 &= m_3 \\ p_{2,1} &= m_3 + m_{2,1} \\ p_{1,1,1} &= m_3 + 3m_{2,1} + 6m_{1^3} \end{aligned}$$

and

$$\begin{aligned} p_4 &= m_4 \\ p_{3,1} &= m_4 + m_{3,1} \\ p_{2,2} &= m_4 + 0m_{3,1} + 2m_{2,2} \\ p_{2,1^2} &= m_4 + 2m_{3,1} + 2m_{2,2} + 2m_{2,1^2} \\ p_{1^4} &= m_4 + 4m_{3,1} + 6m_{2,2} + 12m_{2,1^2} + 24m_{1^4}. \end{aligned}$$

It is obvious that $R_{\lambda\mu}$ is a non-negative integer for each $\lambda, \mu$. As before, is there some combinatorial interpretation of these coefficients?
Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\mu = (\mu_1, \dots, \mu_k)$. We want to find the number of ways to get $x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$ on expanding

$$\left( \sum_{i_1 \geq 1} x_{i_1}^{\lambda_1} \right) \left( \sum_{i_2 \geq 1} x_{i_2}^{\lambda_2} \right) \cdots \left( \sum_{i_\ell \geq 1} x_{i_\ell}^{\lambda_\ell} \right).$$

This number of ways is equal to the number of ordered partitions (sometimes called *preferential arrangements*) $(S_1, \dots, S_k)$ of $[\ell]$ such that for each $1 \leq r \leq k$,

$$\sum_{j \in S_r} \lambda_j = \mu_r.$$

**Theorem 2.14.** If $R_{\lambda\mu} > 0$, then $\mu \succcurlyeq \lambda$.

Note that unlike Gale-Ryser, this is not an iff statement.

*Proof.* Let $(S_1, \dots, S_k)$ be an ordered partition of $[\ell]$ such that for each $1 \leq r \leq k$, $\sum_{j \in S_r} \lambda_j = \mu_r$. In particular, $\lambda_j \leq \mu_r$ for any $j \in S_r$.
Let $i \in S_{p_i}$ for each $i \in [\ell]$. Then, for any $1 \leq r \leq \ell$,

$$\sum_{i=1}^r \lambda_i \leq \sum \mu_{p_i} \leq \sum_{i=1}^r \mu_i,$$

so $\mu \succcurlyeq \lambda$. ∎

Therefore, the $p(d) \times p(d)$ matrix $R$ with rows and columns ordered reverse lexicographically is lower-triangular! Further, it is not too difficult to see that for $\lambda \vdash d$, $R_{\lambda\lambda} \neq 0$.

**Exercise 2.3.** Find a formula for $R_{\lambda\lambda}$.

**Corollary 2.15.** $\{p_\lambda\}_{\lambda \vdash d}$ is a basis of $\Lambda^d_{\mathbb{Q}}$.

Recall that two permutations in $\mathfrak{S}_d$ are conjugate iff they have the same cycle type, so a conjugacy class can be determined ("indexed") by a partition of $[d]$. For $\lambda \vdash d$, define $z_\lambda$ by

$$\frac{d!}{z_\lambda} = \text{size of conjugacy class of } \mathfrak{S}_d \text{ indexed by } \lambda.$$

Recall Equations (2.1) and (2.2).

**Proposition 2.16.**

$$\prod_{i,j\geq 1} \frac{1}{1 - x_i y_j} = \sum_{d\geq 0} \sum_{\lambda \vdash d} \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y).$$

*Proof.* We have

$$\begin{aligned}
\log\left(\prod_{i,j\geq 1} \frac{1}{1-x_iy_j}\right) &= \sum_{i,j\geq 1} \log\left(\frac{1}{1-x_iy_j}\right) \\
&= \sum_{i,j\geq 1}\sum_{n\geq 1} \frac{(x_iy_j)^n}{n} \\
&= \sum_{n\geq 1} \frac{1}{n} \sum_{i,j\geq 1} (x_iy_j)^n \\
&= \sum_{n\geq 1} \frac{1}{n} p_n(x)p_n(y).
\end{aligned}$$

Now, we shall use the permutation formula Theorem 1.11. Set $f_n = p_n(x)p_n(y)$. Then, the corresponding $g_n$ in the formula is given by

$$g_n = \sum_{\lambda \vdash n} \left(\frac{n!}{z_\lambda}\right) \cdot p_\lambda(x)p_\lambda(y).$$

Therefore, setting $x = 1$ in the permutation formula, we get

$$\begin{aligned}
\prod_{i,j\geq 1} \frac{1}{1-x_iy_j} &= \exp\left(\sum_{n\geq 1} \frac{1}{n} p_n(x)p_n(y)\right) \\
&= \exp\left(\sum_{n\geq 1} \frac{1}{n} f_n\right) \\
&= \sum_{n\geq 0} \frac{g_n}{n!} = \sum_{n\geq 0}\sum_{\lambda\vdash n} \frac{1}{z_\lambda} p_\lambda(x)p_\lambda(y).
\end{aligned}$$

■

We can also get an explicit formula for $z_\lambda$. Let $c_\lambda = n!/z_\lambda$ be the size of conjugacy class of $\mathfrak{S}_d$ indexed by $\lambda$. Let $\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n}$, where $m_i \geq 0$ is the multiplicity of $i$ in $\lambda$. Clearly, $\sum_i i m_i = n$. Any permutation with cycle type $\lambda$ can be attained by taking a permutation in $\mathfrak{S}_n$, then grouping together the $\sum_{i\leq r} i m_i + j + 1$ to $\sum_{i \leq r} i m_i + j + r$ elements in a cycle of size $r$ for $0 \leq j \leq m_r - 1$. Further, each permutation is overcounted (repeated) precisely $\prod_{i\geq 1} i^{m_i} m_i!$ times, so

$$c_\lambda = \frac{n!}{\prod_{i\geq 1} i^{m_i} m_i!} = \frac{n!}{z_\lambda}.$$

**Corollary 2.17.**

$$h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda.$$

*Proof.* Set $y_1 = t$ and everything else as $0$ in the previous proposition. Then,

$$\sum_{d\geq 1} h_d t^d = \prod_{i\geq 1} \frac{1}{1-tx_i} = \sum_d \left( \sum_{\lambda \vdash d} \frac{1}{z_\lambda} p_\lambda(x) \right) t^d.$$

Comparing the coefficients of $t^d$ completes the proof. ■

**Exercise 2.4.** Show that

$$\prod_{i,j\geq 1} (1 + x_i y_j) = \sum_{d\geq 0} \sum_{\lambda \vdash d} \frac{\epsilon_\lambda}{z_\lambda} p_\lambda(x) p_\lambda(y),$$

where $\epsilon_\lambda = (-1)^{d-\ell(\lambda)} = \operatorname{sign}(\pi)$ for any $\pi \in \mathfrak{S}_n$ with cycle type $\lambda$.

**Corollary 2.18.**

$$e_n = \sum_{\lambda \vdash n} \frac{\epsilon_\lambda}{z_\lambda} p_\lambda.$$

The following may be proved in precisely the same way as Corollary 2.11.

**Porism 2.19.** $\{p_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ are algebraically independent families.

Recall the involution $\omega$ from Theorem 2.13. What is $\omega(p_n)$, or $\omega(p_\lambda)$ more generally?

**Theorem 2.20.** For all $\lambda \vdash d$, $\omega(p_\lambda) = \epsilon_\lambda p_\lambda$. Therefore, $\{p_\lambda\}_{\lambda \vdash d}$ forms an eigenbasis of $\omega$, with $p_\lambda$ having eigenvalue $\epsilon_\lambda$, and $\omega$ is diagonalisable.

Therefore, with respect to the $\{p_\lambda\}_{\lambda \vdash d}$ basis, $\omega$ as a linear transformation is diagonal. This gives us an alternate proof that $\omega^2 = \mathrm{Id}$. Further, it gives that

$$\operatorname{Charpoly}(\omega) = (x-1)^{\alpha(d)} (x+1)^{p(d)-\alpha(d)},$$

where $\alpha(d)$ is the number of partitions $\lambda \vdash d$ such that $d - \ell(\lambda)$ is even.

*Proof.* We think of $\omega$ acting on $m_\lambda(x)$ while symmetric functions in the variables $y_1, y_2, \ldots$ are thought of as scalars. Recalling that $\omega(e_\lambda) = h_\lambda$,

$$\begin{aligned}
\sum_d \sum_{\lambda \vdash d} \frac{1}{z_\lambda} p_\lambda(y) \omega(p_\lambda(x)) &= \omega \left( \prod_{i,j\geq 1} \frac{1}{1 - x_i y_j} \right) \\
&= \omega \left( \sum_{d\geq 0} \sum_{\lambda \vdash d} m_\lambda(y) h_\lambda(x) \right) \\
&= \sum_{d\geq 0} \sum_{\lambda \vdash d} m_\lambda(y) e_\lambda(x) \\
&= \prod_{i,j\geq 1} (1 + x_i y_j) \\
&= \sum_{d\geq 0} \sum_{\lambda \vdash d} \frac{\epsilon_\lambda}{z_\lambda} p_\lambda(x) p_\lambda(y).
\end{aligned}$$

Matching the coefficients of $p_\lambda(y)$, we get $\omega(p_\lambda) = \epsilon_\lambda p_\lambda$. ■

## 2.5. Dual bases

We now endow $\Lambda^d_{\mathbb{Q}}$ with a scalar product (a bilinear form over $\mathbb{Q}$) by

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} = \begin{cases} 1, & \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Given this inner product, one can ask for an orthogonal basis of $\Lambda^d_{\mathbb{Q}}$.

**Definition 2.21** (Dual basis)**.** If $\{u_\lambda\}$ and $\{v_\mu\}$ are a pair of bases for $\Lambda^d_{\mathbb{Q}}$ (for all $d$), we say that they are *dual pair* of bases if $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$.

**Theorem 2.22.** If $\{u_\lambda\}_{\lambda \vdash d}$ and $\{v_\mu\}_{\mu \vdash d}$ are a pair of bases of $\Lambda^d_{\mathbb{Q}}$, then they form a dual pair iff

$$\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_{d \geq 0} \sum_{\lambda \vdash d} u_\lambda(x) v_\lambda(y).$$

*Proof.* Let

$$m_\lambda = \sum_{\varphi \vdash d} A_{\lambda\varphi} u_\varphi \text{ and } h_\mu = \sum_{\theta \vdash d} B_{\mu\theta} v_\theta.$$

Now,

$$\begin{aligned} \delta_{\lambda\mu} &= \langle m_\lambda, h_\mu \rangle \\ &= \left\langle \sum_{\varphi \vdash d} A_{\lambda\varphi} u_\varphi, \sum_{\theta \vdash d} B_{\mu\theta} v_\theta \right\rangle \\ &= \sum_{\varphi, \theta} A_{\lambda\varphi} B_{\mu\theta} \langle u_\varphi, v_\theta \rangle. \end{aligned}$$

Define a $p(d) \times p(d)$ matrix $T$ by $T_{\varphi\theta} = \langle u_\varphi, v_\theta \rangle$, then the above gives that

$$I_{p(d)} = ATB^\top.$$

Suppose that $\{u_\lambda\}, \{v_\mu\}$ form a pair of dual basis, so $T = I_{p(d)}$, and $I = AB^\top = A^\top B$. That is,

$$\delta_{\lambda\mu} = \sum_{\theta \vdash d} A_{\theta\lambda} B_{\theta\mu}.$$

Now,

$$\begin{aligned} \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} &= \sum_{d \geq 0} \sum_{\lambda \vdash d} m_\lambda(x) h_\lambda(y) \\ &= \sum_{d \geq 0} \sum_{\lambda \vdash d} \left( \sum_{\varphi \vdash d} A_{\lambda\varphi} u_\varphi(x) \right) \left( \sum_{\theta \vdash d} A_{\lambda\theta} v_\theta(y) \right) \\ &= \sum_{d \geq 0} \sum_{\varphi, \theta \vdash d} u_\varphi(x) v_\theta(y) \left( \sum_{\lambda \vdash d} A_{\lambda\varphi} B_{\lambda\theta} \right) \\ &= \sum_{d \geq 0} \sum_{\varphi \vdash d} u_\varphi(x) v_\varphi(y), \end{aligned}$$

completing the forward direction of the proof.
The entire argument is reversible, and we omit the details of checking this. ■

**Exercise 2.5.** Complete the above proof.

Recall that

$$\prod_{i,j\geq 1} \frac{1}{1-x_iy_j} = \sum_{d\geq 0}\sum_{\lambda\vdash d} \frac{1}{z_\lambda} p_\lambda(x)p_\lambda(y),$$

so

$$\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda\mu}.$$

**Corollary 2.23.**

1. $\{p_\lambda\}_{\lambda\vdash d}$ is an orthogonal basis of $\Lambda^d_{\mathbb{Q}}$.
2. $\{p_\lambda/\sqrt{z_\lambda}\}_{\lambda\vdash d}$ is a self-dual basis of $\Lambda^d_{\mathbb{R}}$.
3. $\{p_\lambda\}_{\lambda\vdash d}$ and $\{p_\lambda/z_\lambda\}_{\lambda\vdash d}$ form a pair of dual bases of $\Lambda^d_{\mathbb{Q}}$.

The above also allows us to show properties of symmetry, non-negativity etc.

**Corollary 2.24.** Let $f, g \in \Lambda^d_{\mathbb{Q}}$. Then,

1. $\langle f, g\rangle = \langle g, f\rangle$.
2. $\langle f, f\rangle \geq 0$, with equality iff $f = 0$.

*Proof.* Let $f = \sum_{\lambda\vdash d} A_\lambda p_\lambda$ and $g = \sum_{\lambda\vdash d} B_\lambda p_\lambda$.

1. We have
$$\langle f, g\rangle = \sum_{\lambda\vdash d} z_\lambda A_\lambda B_\lambda = \langle g, f\rangle.$$
2. We have
$$\langle f, f\rangle = \sum_{\lambda\vdash d} z_\lambda A_\lambda^2,$$
which is non-negative and equal to $0$ iff all $A_\lambda$ are $0$. ■

**Exercise 2.6.** Start with $\{m_\lambda\}_{\lambda\vdash d}$ and run Gram-Schmidt Orthogonalization. What is the output orthogonal basis?

## 2.6. Schur symmetric functions

Now, let us define another basis (which will turn out to be orthogonal).

Let $\lambda \vdash d$. Consider the Ferrer's diagram of $\lambda$. We shall fill the boxes with positive integers such that

1. each row is weakly increasing from left to right and
2. all columns are strictly increasing.

**Example 4.** For the partition $\lambda = (3, 1)$ of 4,

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

is such a filling.

Such a filling is called a *semi-standard Young Tableaux* (SSYT) of shape $\lambda$. Let $F$ be an SSYT. Let $F$ have $m_i$ occurrences of $i$ in it – this yields a vector $(m_1, m_2, \ldots, )$, called the *content* of $F$ and denoted $\mathrm{content}(F)$. Clearly, $\sum_i m_i = d$. Define

$$S_\lambda = \sum_{\text{SSYTs } F \text{ of shape } \lambda} x^{\mathrm{content}(F)}$$

For example, when $\lambda = (2, 1)$,

$$S_\lambda = m_{2,1} + 2m_{1^3}$$

**Theorem 2.25** (Knuth-Bender)**.** Let $d \geq 0$ and $\lambda \vdash d$. Then, $S_\lambda$ is a symmetric function.

*Proof.* Fix some content vector $(m_1, m_2, \ldots)$. Let $\mathcal{F}$ be the set of SSYTs $F$ of shape $\lambda$ with $\mathrm{content}(F) = (m_1, m_2, \ldots)$. Fix arbitrary $i$, and let $\mathcal{F}_i$ be the set of SSYTs $F$ of shape $\lambda$ with $\mathrm{content}(F) = (m_1, m_2, \ldots, m_{i-1}, m_{i+1}, m_i, m_{i+2}, m_{i+3}, \ldots)$ (the $i$th and $(i+1)$th coordinates are swapped).
We shall demonstrate a bijection $\varphi_i$ between $\mathcal{F}$ and $\mathcal{F}_i$ – this implies the result because $i$ is arbitrary (Why?).
Let $F \in \mathcal{F}$. In a given column, there are four possibilities.

1. both $i$ and $i+1$ occurs – in this case, there must be precisely one of each in the column and they are consecutive.
2. neither $i$ not $i+1$ occur.
3. only $i$ or only $i+1$ occur – in this case, there is precisely one of it in the column.

In the first two cases, we say that the occurences of $i, (i+1)$ are "paired". In the third case, we say that the occurence is free. Suppose that some row of $F$ has $a$ $i$s and $b$ $(i+1)$s. Note that these $(a+b)$ occurrences are consecutive! Showing this requires the fact that $F$ is an SSYT – show that all paired $i$s are before free $i$s, then that all free $(i+1)$s are before paired $(i+1)$s.
We then define $\varphi_i(F)$ to be that filling where this consecutive block of $(a+b)$ elements is changed to have $b$ $i$s then $a$ $(i+1)$s; the remainder of the diagram is left unchanged compared to $F$. Note that this filling is indeed in $\mathcal{F}_i$! ■

The $(S_\lambda)$ are known as the *Schur symmetric functions*.
Note that for any $n$, $S_n = h_n$ and $S_{1^n} = e_n$.
Let us now try to show, as per usual, that the $(S_\lambda)$ form a basis of $\Lambda^d_{\mathbb{Q}}$. We have

$$\begin{aligned} S_3 &= m_{1^3} + m_{2,1} + m_3 \\ S_{2,1} &= 2m_{1^3} + m_{2,1} \\ S_{1^3} &= m_{1^3}. \end{aligned}$$

Let

$$S_\lambda = \sum_{\mu \vdash d} K_{\lambda\mu} m_\mu. \tag{2.3}$$

Here, the $K_{\lambda\mu}$ are known as the *Kostka numbers*. By definition, $K_{\lambda\mu}$ is the number of SSYTs with shape $\lambda$ and content $\mu$. In a spirit much like Gale-Ryser, we shall show that $K$ is upper-triangular when we order both the rows and columns in the reverse-lexicographic order.

**Theorem 2.26.** $K_{\lambda\mu}$ is nonzero if and only if $\lambda \succcurlyeq \mu$ (under the majorisation partial order).

*Proof.* First, suppose that $K_{\lambda\mu} \neq 0$. Note that the $i$th row cannot have any element less than $i+1$. In particular, all the $i$s for $i \leq k$ are contained in the first $k$ rows. This precisely says that $\sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i$, since the first term is the number of cells in the first $k$ rows and the second is the number of $i \leq k$ in the first $k$ rows.
Using a constructive argument, we can show the converse. ■

**Exercise 2.7.** Complete the above proof.

Now, note that the diagonal entries of $K$ are equal to 1 by an argument similar to that of the previous theorem.

**Corollary 2.27.** $(S_\lambda)_{\lambda \vdash d}$ forms a basis of $\Lambda_{\mathbb{Q}}^{d}$.

Unfortunately, $K_{\lambda\mu}$ has no good formula in general. However, there are formulae in certain special cases.
When $\mu = 1^d$, we wish to determine the number of SSYTs with shape $\lambda$ such that each element of $\{1, 2, \ldots, d\}$ occurs precisely once.

**Definition 2.28.** A *Standard Young Tableaux* (SYT) of shape $\lambda \vdash d$ is an SSYT of shape $\lambda$ where each element of $[d]$ occurs precisely once. The number of SYTs of shape $\lambda$ is denoted $f_\lambda = K_{\lambda,1^d}$.

Note that this implies that rows are strictly increasing. The number of SYTs of a given shape has some very neat formulae!

**Theorem 2.29** (Hook length formula)**.** Given a cell $c$ in the Ferrer diagram of $\lambda \vdash n$, the *hook* at cell $c$ consists of $c$, cells to the right of $c$, and cells below $c$. Denote by $h_c$ the size of the hook at $\lambda$. Then,

$$f_\lambda = \frac{n!}{\prod_{c \in \text{cell}(\lambda)} h_c}.$$

We do not prove the above. The original proof was due to Frame, Robinson, Thrall, and Wilf, Nijenhuis later gave an easier proof.
Using the above, note in particular that the number of SYTs of shape $(n, n)$ is

$$f_{n,n} = \frac{1}{n+1} \frac{2n}{n},$$

the $n$th Catalan number $C_n$!

## 2.7. The RSK algorithm

**Theorem 2.30.** Let $\lambda \vdash n$. Recall that $\mathfrak{S}_n$ has $p(n)$ distinct irreducible representations (up to isomorphism), indexed by the conjugacy classes of $\mathfrak{S}_n$. Further, the irreducible representation indexed by $\lambda$ has dimension $f_\lambda$.
Consequently,

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!.$$

Let us give an alternate bijective proof of this. We would like to come up with a bijection

$$\varphi : \{(P, Q) : P, Q \text{ are SYTs of the same shape}\} \to \mathfrak{S}_n.$$

We use the *RSK algorithm*. We start with a finite matrix $M$ with non-negative integer entries. We encode this as a $2 \times r$ matrix, where $r$ is the sum of entries of $M$. To construct this matrix, we iterate through the entries of $M$ in row-major form, and insert the column $\begin{pmatrix} i \\ j \end{pmatrix}$ $M_{ij}$ many times. Let the first row of this matrix be $Q$ and the second be $P$.
By definition, the lengths of both $P$ and $Q$ is the sum of entries of $M$. Because we are iterating through the entries in row-major order, $P$ is weakly increasing. For example, in the case where $M$ is a permutation matrix, $Q$ is just $1, 2, \ldots, n$ and $P$ some permutation of $[n]$.
Now, using this new $2 \times r$ matrix, we shall form an SSYT. The algorithm just repeats the following. Iterating through $[r]$, suppose we are at the $i$th entry.

1. If it is possible to add $P_i$ at the end of the first row, we append it.
2. If this is not possible, we insert it in the first row at the earliest possible position, "kicking out" some other entry. We then try to insert this entry in the part of the filling below the first row, recursively in the exact same manner as just described.
3. Insert $Q_i$ at an appropriate position in order to ensure that the shapes of the tableauxs remain the same.

By the construction, it is easy to see that $P, Q$ have the same shape and $P$ is an SSYT. We must show that $Q$ is an SSYT as well.
This is easy to see in the case where $M$ is a permutation matrix, since the element of $Q$ we add is always strictly greater than previous elements so there is no chance of violating column strictness or row strictness. The issue arises with column strictness in non-permutation matrices since the element we are adding might coincide with other elements in the tableaux.
In general, we prove this with the following lemma.

**Lemma 2.31.** Given an tableaux $R$ and $j$, denote by $R \leftarrow j$ the set of cells modified in $P$, including the newly formed cell, on the insertion of $j$ into $R$ in the above described manner. Letting $R'$ be the tableaux attained after inserting $j$ into $R$, denote by $(R \leftarrow j) \leftarrow k$ the set $R' \leftarrow k$.
If $j \leq k$, every cell of $P \leftarrow j$ is strictly to the left of any cell in $(P \leftarrow j) \leftarrow k$. That is, if $(r, s) \in P \leftarrow j$ and $(r, t) \in (P \leftarrow j) \leftarrow k$, then $s < t$.

**Exercise 2.8.** Prove the above.

We know that every $Q^{(i)}$ is strictly increasing in both rows and columns. What we must show is that if we insert the pairs $\begin{pmatrix} i \\ j \end{pmatrix}$ and $\begin{pmatrix} i \\ k \end{pmatrix}$ into $P$ and $Q$ where $j \leq k$,
In the case where $M$ is a permutation matrix, it is quite easy to see that this transformation is bijective as well! Indeed, the SSYT of $Q$ precisely describes the order in which the cells were formed. Given $Q^{(i)}$, the location of the largest element of $Q^{(i)}$ is the location of the newly formed cell. If this is in the first row, this is indeed the element of $P$ that was inserted as well. Otherwise, we must go to higher rows and "reverse-kick" entries, by determining the largest

element smaller than this reverse-kicking number.
In the permutation matrix, we do the reversal by looking at the position of the largest entry in $Q^{(i)}$. In general, it turns out that we must look at the position of the *rightmost* largest entry in $Q^{(i)}$, and this will be unique, due to the column strictness of SSYTs.

**Theorem 2.32.** The RSK algorithm is a bijection between $\mathbb{N}$-valued matrices $M$ and ordered pairs $(P, Q)$ of SSYTs such that

$$\mathrm{content}(P) = \mathrm{rowsum}(M) \text{ and } \mathrm{content}(Q) = \mathrm{colsum}(M).$$

A better proof of SSYT RSK reversibility case is by using the SYT permutation matrix case, by performing the following transformation. Iterating upwards, change repeating elements from left to right to distinct values. For example, if we have three 1s, change them to $1, 2, 3$ in order from left to right.
The permutation matrix case yields the desired bijection we mentioned earlier, so

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!.$$

**Exercise 2.9.** Show that if running the RSK algorithm on $M$ gives $(P, Q)$, running it on $M^\top$ gives $(Q, P)$.
In particular, show that if running it on $\pi \in \mathfrak{S}_n$ gives $(P, Q)$, running it on $\pi^{-1}$ gives $(Q, P)$. Further, $P = Q$ here iff $\pi^2 = \mathrm{Id}$, so the number of involutions $I_n$ is equal to $\sum_{\lambda \vdash n} f_\lambda$.

**Exercise 2.10.** Show that

$$\sum_{n \geq 0} I_n \cdot \frac{x^n}{n!} = \exp\left(x + \frac{x^2}{2}\right).$$

**Exercise 2.11.** Show that $I_n = I_{n-1} + (n-1)I_{n-2}$ for $n \geq 3$.

Let us look at a couple of interesting corollaries of the above construction.

**Theorem 2.33** (Schenstead)**.** The length of the first row of $P$ is the length of the largest weakly increasing subsequence of $\pi$. The length of the first column of $P$ is the length of the largest weakly decreasing subsequence of $\pi$.

**Corollary 2.34** (Erdős-Szekeres)**.** Given a sequence of $(mn + 1)$ elements, there exists an increasing subsequence of length $(m + 1)$ or a decreasing subsequence of length $(n + 1)$.

**Corollary 2.35** (Cauchy's Identity)**.** It is true that

$$\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_{d \geq 0} \sum_{\lambda \vdash d} s_\lambda(x) s_\lambda(y).$$

*Proof.* The coefficient of $x^\alpha y^\beta$ on the left is equal to $N_{\alpha,\beta}$, the number of non-negative matrices with $\mathrm{rowsum}(M) = \alpha$ and $\mathrm{colsum}(M) = \beta$. The coefficient of $x^\alpha y^\beta$ on the right is equal to the number of SSYT pairs $(P, Q)$, where $P, Q$ have the same shape and $\mathrm{content}(P) = \alpha, \mathrm{content}(Q) = \beta$. Since the RSK algorithm is a bijection, we get that the two are equal and are done. ∎

Theorem 2.22 immediately yields the following.

**Corollary 2.36.** $\{S_\lambda\}_{\lambda\vdash d}$ is an orthonormal basis of $\Lambda^d_{\mathbb{Q}}$.

Recall the Kostka numbers $K_{\lambda,\mu}$ from Equation (2.3). The next corollary immediately follows by looking at the coefficient of $x^\alpha y^\beta$ in Cauchy's identity, as we did in the proof.

**Corollary 2.37.** Let $\alpha, \beta \vdash d$. Then,

$$N_{\alpha\beta} = \sum_{\lambda\vdash d} K_{\lambda\alpha} K_{\lambda\beta}.$$

That is, $N = K^\top K$.

It turns out that the change-of-basis matrix from $\{m_\lambda\}$ to $\{s_\lambda\}$ is the change of basis-matrix from $\{s_\lambda\}$ to $\{h_\lambda\}$!

**Corollary 2.38.** For $\mu \vdash d$,

$$h_\mu = \sum_{\lambda\vdash d} K_{\lambda\mu} s_\lambda.$$

*Proof.* Let $h_\mu = \sum_{\lambda\vdash d} a_{\lambda\mu} s_\lambda$, using the fact that the $s_\lambda$ form a basis. Because the $\{s_\lambda\}_{\lambda\vdash d}$ form an orthonormal basis,

$$a_{\lambda\mu} = \langle h_\mu, s_\lambda\rangle = \langle h_\mu, \sum_{\theta\vdash d} K_{\lambda\theta} m_\theta\rangle = K_{\lambda\mu},$$

where we use that $\langle h_\mu, m_\theta\rangle = \delta_{\mu\theta}$. ∎

**Theorem 2.39.** Given any $0,1$-matrix $M$, we can bijectively get a pair $(P, Q)$ of tableauxs of the same shape, where $P^\top, Q$ are SSYTs, and $\mathrm{content}(P) = \mathrm{colsum}(M)$ and $\mathrm{content}(Q) = \mathrm{rowsum}(M)$.

## 2.8. The dual RSK algorithm

This proof uses the *dual RSK algorithm*. Instead of going from non-negative matrices to SSYT pairs $(P, Q)$ of the same shape, we shall go from $0,1$-matrices $M$ to pairs $(P, Q)$ of the same shape where $P^\top, Q$ are SSYTs.
The algorithm is mostly similar to the RSK algorithm, with one minor modification – in the kicking step, instead of kicking the first element that is (strictly) greater than the current element, we insert the first element that is *at least* the current element. This ensures row strictness of $P$, which is needed for its conjugate to be an SSYT.
By construction, $P$ and $Q$ have the same shape, and $P^\top$ is an SSYT. What about $Q$? Much like the earlier proof, we must look at the rightmost occurrence of the largest element in $Q$.

**Corollary 2.40** (Dual Cauchy Identity)**.** Denoting by $\lambda'$ the conjugate of $\lambda \vdash d$, we have

$$\prod_{i,j\geq 1} (1 + x_i y_j) = \sum_{d\geq 0} \sum_{\lambda\vdash d} s_{\lambda'}(x) s_\lambda(y).$$

**Corollary 2.41.** For $\lambda \vdash d$, $\omega(s_\lambda) = s_{\lambda'}$.

This follows immediately on applying $\omega$ to the Cauchy identity and using the dual Cauchy identity. This presents another proof of $\omega^2 = \mathrm{Id}$.

## 2.9. Symmetric functions in $n$ variables

Let $\Lambda_{\mathbb{Q}}^{d,n}$ be the vector space over $\mathbb{Q}$ of degree $d$ symmetric functions in $n$ variables. Clearly, $\dim(\Lambda_{\mathbb{Q}}^{d,n}) \le \dim(\Lambda_{\mathbb{Q}}^{d,n})$. In fact, the dimension is the number of partitions of $d$ with at most $n$ parts. Note that if we have at least $d$ variables, the two become equal.
The classical definition of the Schur symmetric function is on $n$ variables, a function $s_\lambda(x_1, \dots, x_n)$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$. $\mathfrak{S}_n$ acts on monomials $x^\alpha$ as $\pi(x^\alpha) = \prod_{i=1}^n x_i^{\alpha_{\pi(i)}}$. Define

$$f_\alpha(x) = \sum_{pi \in \mathfrak{S}_n} \mathrm{sign}(\pi)\pi(x^\alpha).$$

Note that $f_\alpha$ is not symmetric, since $\sigma(f_\alpha) = \sum_{\pi \in \mathfrak{S}_n} \mathrm{sign}(\pi)\sigma(\pi(x^\alpha))$. However, this is equal to $\sigma(f_\alpha) = \mathrm{sign}(\sigma) f_\alpha$.

**Definition 2.42** (Skew-symmetric function)**.** A function $f$ such that $\sigma(f) = \mathrm{sign}(\sigma) f$ is called a *skew-symmetric function*.

**Proposition 2.43.** $f_\alpha \neq 0$ iff all the $\alpha_i$ are distinct. Further, $(x_i - x_j) \mid f_\alpha$ for distinct $i, j$.

The first part above is easily seen by considering $\sigma(f_\alpha)$ where $\sigma$ is just a transposition of two equal exponents.

Note that the form of the $f_\alpha$ is quite reminiscent of the determinant of a matrix! In fact $f_\alpha$ is precisely equal to the determinant of the matrix

$$M_\alpha = \begin{pmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \cdots & x_n^{\alpha_n} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \cdots x_n^{\alpha_2} & \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_n} & x_2^{\alpha_n} & \cdots & x_n^{\alpha_n} \end{pmatrix}.$$

That is, $(M_\alpha)_{ij} = x_i^{\alpha_j}$. This presents an easy proof of the above proposition.

Assume henceforth that $\alpha$ has all distinct parts, and $\alpha_1 > \alpha_2 > \cdots > \alpha_n$.
Because the $(x_i - x_j)$ are linear monomials, some ring theory yields that

$$\prod_{i<j} (x_i - x_j) \mid f_\alpha.$$

The expression on the left is precisely the Vandermonde determinant

$$\det \begin{pmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots x_n^{n-2} & \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \end{pmatrix}.$$

Setting $\delta_n = (n-1, n-2, \dots, 0)$ (denoting it as just $\delta$ if $n$ is clear from context), the above is just $f_\delta$. Now, $f_\alpha$ is a polynomial of $\sum \alpha_i$ and $f_{\delta_n}$ is a polynomial of degree $\binom{n}{2}$. As a result, $f_\alpha / f_{\delta_n}$, which is a polynomial, has degree $\left(\sum_i \alpha_i\right) - \binom{n}{2}$. Note that because $f_\alpha$ and $f_\delta$ are skew-symmetric, their quotient is symmetric!
Set $\lambda_i = \alpha_i - (n-i)$ so $\alpha = \lambda + \delta$, and note that $f_\alpha / f_\delta$ has terms like $x^\lambda$. Note that the $\lambda_i$ are weakly decreasing, so they form a partition of some number with at most $n$ parts (some of which may be $0$). We are interested in $f_{\lambda+\delta}/f_\delta$.

**Definition 2.44** (Schur symmetric function)**.** For a partition $\lambda$ with at most $n$ parts, define the *Schur symmetric function* $s_\lambda(x_1, \dots, x_n)$ by

$$s_\lambda = \frac{f_{\lambda+\delta}}{f_\delta}.$$

The above is Cauchy's bialternant definition of the $s_\lambda$. We must now show that this coincides with the earlier Schur symmetric function we have studied, on setting $x_j = 0$ for $j > n$.
Recall that $h_\mu = \sum_{\lambda \vdash d} K_{\lambda\mu} s_\lambda$. Applying $\omega$, we get $e_\mu = \sum_{\lambda \vdash d} K_{\lambda'\mu} s_\lambda$. Set all variables $x_j$ for $j > n$ as $0$. We now wish to show that

$$f_\delta e_\mu = \sum_{\lambda \vdash d} K_{\lambda'\mu} f_{\lambda+\delta}.$$

Let $\mu = (\mu_1, \dots, \mu_r)$. Note that the expression on the left is skew-symmetric. We wish to find the number of ways of choosing indices (where for a fixed $k$, the $i_{k,r}$ are distinct) such that

$$(x_1^{n-1} x_2^{n-2} \cdots x_{n-1})(x_{i_{1,1}} x_{i_{1,2}} \cdots x_{1,\mu_1}) \cdots (x_{i_{r,1}} \cdots x_{i_{r,\mu_r}}) = (x_1^{\lambda_1+n-1} x_2^{\lambda_2+n-2} \cdots x_n^{\lambda_n}).$$

Consider each such way of choosing indices such that at no point in the above $n$ steps of multiplication do two indices become equal. Consider the matrix $M$ as defined in the above example, where $M_{ij} = 1$ if the $i$th step chooses the index $j$ and is $0$ otherwise. Encode this matrix as the tableaux with the $j$th column having those indices where $M_{ij} = 1$, arranged in increasing order of $i$ from top-to-bottom. It is quite easy to see that this tableaux is column-strict. It also clearly has shape $\lambda'$ and content $\mu$. We would like to show that this is an SSYT, that is, that weak row ordering holds as well.

**Example 5.** Suppose we have $\lambda = (5, 3, 2, 1)$ and $\mu = (3, 2, 2, 1, 1, 1, 1)$. That is, we would like to go from $x_1^3 x_2^2 x_3^1$ to $x_1^8 x_2^4 x_3^3 x_4$ in 7 steps, with the $i$th step multiplying by exactly $\mu_i$ indices. One way of doing this is encoded by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the $ij$th element of the matrix being 1 denotes that the index $j$ is present in the indices in the $\mu_i$th step. We encode this as the tableaux

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 7 \\ \hline 2 & 3 & 2 \\ \cline{1-3} 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}.$$

This has shape $\lambda'$ and content $\mu$. This is not an SSYT, but note that this multiplication gives 0 due to skew-symmetry! Indeed, $(x_1^3 x_2^2 x_3^1)(x_1 x_2 x_3)(x_1 x_3) = x_1^5 x_2^3 x_3^3$.
For another example, consider the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

with corresponding tableaux

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 4 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}.$$

Again, this has shape $\lambda'$ and content $\mu$. One can check that this multiplication does indeed give something nonzero, and the above is clearly an SSYT.

We would like to show that we get an SSYT iff no two exponents are equal in the process.

**Lemma 2.45.** We can encode multiplication of terms from $e_{\mu_1}, e_{\mu)2}, \ldots$ to the monomial $x^\delta$ to get $x^{\lambda+\delta}$ as a filling of shape $\lambda'$ and content $\mu$. Further, this filling is an SSYT iff at each stage of the multiplication, all the exponents are distinct.

*Proof sketch.* The argument is via induction. Show that the "ordering" is violated at a step iff weak row ordering is violated. ∎

## 2.10. Determinants and paths

Consider the lattice $\mathbb{Z}^2$, and the standard problem of going from $A = (\beta, \gamma)$ to $B = (\alpha, \delta)$ with steps only E (east) and S (south) of unit length, where $\alpha \geq \beta$ and $\gamma \geq \delta$. We encode any such path $P$ by a monomial $x_P = \prod_{i \in [\delta,\gamma]} x_i^{P_i}$, where $P_i = |\{j \in \mathbb{Z} : P$ takes an E step at (j,i)$\}|$.
Clearly, this transformation is injective. This monomial $x_P$ has total degree $(\alpha - \beta)$ and only involves $x_i$ for $i \in [\delta, \gamma]$. As remarked, the set of all such monomials is in bijection with set of all paths.
In more generality, we can have a *path system* with $n$ pairs of points denoted by $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n), \overline{\beta}, \overline{\gamma}, \overline{\delta}$ satisfying $\alpha_i \geq \beta_i$ and $\gamma_i \geq \delta_i$ for each $i$. Now, let us look at the set of non-intersecting (vertex-disjoint) paths $(P_1, \ldots, P_n)$, where $P_i$ is from $A_i = (\alpha_i, \gamma_i)$ to $B_i = (\beta_i, \delta_i)$.

# §3. Problem Sheets

## 3.1. Problem Sheet 1

**Problem 3.1.1.** Let $S(n,k)$ and $s(n,k)$ be Stirling numbers of the second and first kind respectively. Show that for all $n,k$, we have $s(n,k) \geq S(n,k)$.

**Solution**

Let $X_{S(n,k)}$ be the set of partitions of $[n]$ into exactly $k$ parts and $X_{s(n,k)}$ the number of permutations of $[n]$ with exactly $k$ cycles. Recall that by definition, $|X_{S(n,k)}| = S(n,k)$ and $|X_{s(n,k)}| = s(n,k)$. It suffices to demonstrate an injection $f$ from $X_{S(n,k)}$ to $X_{s(n,k)}$. We do so as follows. Let $\{\{x_{1,1},\ldots,x_{1,n_1}\},\ldots,\{x_{k,1},\ldots,x_{1,n_k}\}\}$ be a partition of $[n]$ into exactly $k$ parts, where $x_{i,j_1} < x_{i,j_2}$ for $j_1 < j_2$. Then, we have a corresponding permutation of $[n]$ with exactly $k$ cycles given by $(x_{1,1},\ldots,x_{1,n_1})\cdots(x_{k,1},\ldots,x_{k,n_k})$. This map is clearly an injection, so we are done.

**Problem 3.1.2.** Show that

$$S(n,k) = \sum_{r=1}^{k}(-1)^{k-r}\frac{r^n}{r!(k-r)!}.$$

**Solution**

$S(n,k)$ is merely $1/k!$ times the number of surjective functions from $[n]$ to $[k]$ (because the ordering of the partitions does not matter). The set of functions that are *not* surjective is

$$\bigcup_{i\in[n]}\{f \in [k]^{[n]} : i \notin \mathrm{Im}(f)\}.$$

The size of the above is quite easily determined by the inclusion-exclusion principle to get

$$k^n - k!S(n,k) = \sum_{r=1}^{k-1}(-1)^{r+1}\cdot \underbrace{\binom{k}{r}}_{\text{choosing } r \text{ elements in } [k] \text{ to ``avoid''}} \cdot \underbrace{(k-r)^n}_{\text{counting functions that avoid the chosen}},$$

and the desideratum immediately follows.

**Problem 3.1.3.** Let $A_n(y) = \sum_k S(n,k)y^k$. Show that $A_n(y) = (y+yD)^n 1$ where $D = \frac{\mathrm{d}}{\mathrm{d}y}$ is the derivative operator.

**Solution**

First, recall that $S(n+1,k+1) = S(n,k) + (k+1)\cdot S(n,k+1)$ – the $S(n,k)$ corresponds of those partitions where $n+1$ is in a part of its own, and the $(k+1)S(n,k+1)$ corresponds to those partitions where this is not the case, so we can consider any partition of $[n]$ into $k+1$ parts, then decide which of the $k+1$ parts to place $n+1$ in.

We have

$$\begin{aligned}(y+yD)A_n(y) &= (y+yD)\sum_{k=1}^{n} S(n,k)y^k \\ &= \sum_{k=1}^{n} S(n,k)(y^{k+1}+ky^k) \\ &= S(n,1)y + \sum_{k=2}^{n} y^k(S(n,k-1)+(k-1)S(n,k)) + S(n,n)y^{n+1} \\ &= S(n+1,1)y + \sum_{k=2}^{n} S(n+1,k)y^k + S(n+1,n+1)y^{n+1} = A_{n+1}(y).\end{aligned}$$

The required follows inductively.

**Problem 3.1.4.** Let $D_n$ be the number of derangements in $\mathfrak{S}_n$ and let $D(x) = \sum_{n\geq 0} D_n x^n/n!$ be its egf. Determine $D(x)$.

**Solution**

A permutation $\pi \in \mathfrak{S}_n$ is a derangement iff it has no cycles of length $1$. Define $f : \mathbb{N} \to \mathbb{N}_0$ by

$$f(k) = \begin{cases} 0, & k = 1, \\ 1, & \text{otherwise.} \end{cases}$$

By the earlier observation, $\pi \in \mathfrak{S}_n$ is a derangement iff $f(|C_1|)\cdots f(|C_k|) = 1$ where $C_1, \ldots, C_k$ are the cycles of $\pi$. Using Corollary 5.1.9 in [SF99], we get that

$$D(x) = \exp\left(\sum_{n\geq 2} \frac{x^n}{n}\right) = \exp\big(-x - \log(1-x)\big) = \frac{e^{-x}}{1-x}.$$

**Problem 3.1.5.** Let $s(n,2)$ denote the number of $\pi \in \mathfrak{S}_n$ with $2$ cycles in its cyclic decomposition and let $H_n$ denote the $n$th harmonic number. Show that $s(n+1,2) = H_n \times n!$.

**Solution**

It is easily checkable that the number of cyclic permutations of $[k]$ is $(k-1)!$. We have

$$\begin{aligned}s(n+1,2) &= \sum_{k=1}^{\lfloor (n+1)/2\rfloor} \binom{n+1}{k}(k-1)!(n+1-k-1)! \\ &= \sum_{k=1}^{\lfloor (n+1)/2\rfloor} \frac{(n+1)!}{k(n+1-k)} \\ &= \sum_{k=1}^{\lfloor (n+1)/2\rfloor} n!\left(\frac{1}{k} + \frac{1}{n+1-k}\right) \\ &= n!\left(\sum_{k=1}^{\lfloor (n+1)/2\rfloor} \frac{1}{k} + \sum_{k=\lceil (n+1)/2\rceil}^{n} \frac{1}{k}\right) \\ &= \begin{cases} n!H_n, & n \text{ is even,} \\ n!(H_n + \frac{2}{n+1}), & n \text{ is odd.} \end{cases}\end{aligned}$$

**Problem 3.1.6.** For a fixed positive integer $k$, consider the egf $f_k(x) = \sum_{n\geq 0} s(n,k)x^n/n!$. Show that

$$f_k(x) = \frac{1}{k!}\ln\left(\frac{1}{1-x}\right)^k.$$

**Solution**

Define $g : \mathbb{N} \to \mathbb{N}_0$ by

$$g(r) = \begin{cases} 1, & r = k, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$s(n,k) = \sum_{\pi\in\mathfrak{S}_n} g(r),$$

where $C_1, \ldots, C_r$ are the cycles in $\pi$. We now use Corollary 5.1.8 in [SF99] with $f$ being the function that takes the constant 1. We have $E_g(x) = \frac{1}{k!}x^k$, so we get that

$$f_k(x) = \frac{1}{k!}\left(\sum_{n\geq 1}\frac{x^n}{n}\right)^k = \frac{1}{k!}\ln\left(\frac{1}{1-x}\right)^k.$$

**Problem 3.1.7.** Find $\sum_{k=0}^n (-1)^k s(n,k)$.

**Solution**

Recall that a permutation in $\mathfrak{S}_n$ with $k$ cycles has sign $(-1)^{n-k}$. We have

$$\begin{aligned}\sum_{k=0}^n (-1)^k s(n,k) &= \sum_{k=0}^n \sum_{\substack{\pi\in\mathfrak{S}_n \\ \pi \text{ has exactly } k \text{ cycles}}} (-1)^k \\ &= (-1)^n \sum_{\pi\in\mathfrak{S}_n} \operatorname{sign}(\pi).\end{aligned}$$

For $n = 1$, this is $-1$. Otherwise, note that $\pi \mapsto (1,2)\pi$ is a bijection between odd and even permutations. As a result, the above sum is equal to $0$.

**Problem 3.1.8.** Show that $S(n+1,k+1) = \sum_{m=0}^n \binom{n}{m} S(m,k)$.

**Solution**

We shall count the partitions of $[n+1]$ into exactly $k+1$ parts based on the part $n+1$ is in. We have

$$S(n+1,k+1) = \sum_{T\subseteq[n]} S(|[n]\setminus T|, k) = \sum_{m=0}^n \binom{n}{m} S(n-m,k) = \sum_{m=0}^n \binom{n}{m} S(m,k).$$

**Problem 3.1.9.** Show that for $n \geq 1$, the $S_{n,k}$ as $k$ varies has either a unique maximum value or has at most two equal values.

**Solution**

## 3.2. Problem Sheet 2

**Problem 3.2.1.** Show that

(a) $T_n(1) = 1$ and

(b) $T_n(-1) = (-1)^n$.

**Solution**

This immediately follows since $T_n(\cos\theta) = \cos n\theta$, so $T_n(\cos 0) = \cos(n\cdot 0) = 1$ and $T_n(\cos\pi) = \cos(n\pi) = (-1)^n$. They can also be easily proved inductively.

**Problem 3.2.2.** Show that

(a) $U_n(1) = n+1$.

(b) $U_n(-1) = (-1)^n(n+1)$

**Solution**

We prove this inductively. Both statements are trivially true for $n = 0, 1$. For $n \geq 2$, inductively, we have

$$U_n(1) = 2U_{n-1}(1) - U_{n-2}(1) = 2n - (n-1) = n+1$$

and

$$U_n(-1) = -2U_{n-1}(1) - U_{n-2}(1) = (-1)^n \cdot 2n + (-1)^{n-1}(n-1) = (-1)^n(n+1).$$

**Problem 3.2.3.** Show that

$$\frac{1}{\iota^n} U_n(\iota/2) = f_{n+1}.$$

**Solution**

Again, we prove this inductively. We have $U_0(\iota/2) = 1 = f_1$ and $U_1(\iota/2) = \iota = \iota f_2$. For $n \geq 2$, we inductively have

$$U_n(\iota/2) = \iota U_{n-1}(\iota/2) - U_{n-2}(\iota/2) = \iota^n f_n - \iota^{n-2} f_{n-1} = \iota^n(f_n + f_{n-1}) = \iota^n f_{n+1}.$$

**Problem 3.2.4.** Show that if $m, n \geq 1$,

$$T_{m+n}(x) = T_m(x)U_n(x) - T_{m-1}(x)U_{n-1}(x).$$

**Solution**

This may be checked manually for $m + n = 2, 3$. We perform induction on $m + n$. We have that

$$\begin{aligned}
T_{m+n}(x) &= 2xT_{m+n-1}(x) - T_{m+n-2}(x) \\
&= 2x(T_{m-1}(x)U_n(x) - T_{m-2}(x)U_{n-1}(x)) - (T_{m-1}(x)U_{n-1}(x) - T_{m-2}(x)U_{n-2}(x)) \\
&= 2xT_{m-1}(x)U_n(x) - T_{m-1}(x)U_{n-1}(x) - T_{m-2}(x)(2xU_{n-1}(x) - U_{n-2}(x)) \\
&= 2xT_{m-1}(x)U_n(x) - T_{m-1}(x)U_{n-1}(x) - T_{m-2}(x)U_n(x) \\
&= U_n(x)(2xT_{m-1}(x) - T_{m-2}(x)) - T_{m-1}(x)U_{n-1}(x) \\
&= T_m(x)U_n(x) - T_{m-1}(x)U_{n-1}(x).
\end{aligned}$$

## 3.3. Problem Sheet 3

**Problem 3.3.1.** Let $(E_n)$ be the Euler numbers – $E_n$ is the number of alternating permutations of $[n]$. Show that its egf

$$E(x) = \sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

**Solution**

Let $n \geq 1$. Suppose that $n+1$ is at the $(k+1)$th position of an alternating permutation in $\mathfrak{S}_{n+1}$ for some odd $k$. The section before this as well as the section after this must be equal to the alternating permutations of some subset (of size $k$ or $n-k$) of $[n]$. Consequently,

$$E_{n+1} = \sum_{\substack{k \in [n]_0 \\ k \text{ odd}}} \binom{n}{k} E_k E_{n-k}.$$

Repeating the same activity but considering the placement of 1 instead (observe that the number of "reverse-alternating" permutations which start with a decrease instead of an increase is equal to the number of alternating permutations) gives that

$$E_{n+1} = \sum_{\substack{k \in [n]_0 \\ k \text{ even}}} \binom{n}{k} E_k E_{n-k}.$$

Therefore,

$$\begin{aligned} 2E_{n+1} &= \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k} \\ 2(n+1)\frac{E_{n+1}}{(n+1)!} &= \sum_{k=0}^{n} \frac{E_k}{k!} \frac{E_{n-k}}{(n-k)!}. \end{aligned}$$

Now, observe that the coefficient of $x^n$ in $E(x)^2$ is $F_n = \sum_{k=0}^{n} \frac{E_k}{k!} \frac{E_{n-k}}{(n-k)!}$. As a result,

$$\begin{aligned} 2(n+1)\frac{E_{n+1}}{(n+1)!} &= F_n \\ 2\sum_{n \geq 1} (n+1)x^n \frac{E_{n+1}}{(n+1)!} &= \sum_{n \geq 1} x^n F_n \\ 2(E'(x) - 1) &= E(x)^2 - 1 \\ 2E'(x) &= E(x)^2 + 1 \end{aligned}$$

Solving this gives $E(x) = \sec x + \tan x$.

**Problem 3.3.2.** A recurrence relation of the following type is called a *three term recurrence* for a sequence $p_n$ of polynomials with $n \geq 0$ (where each $p_n$ is a polynomial of degree $n$) if there exist real numbers $(a_n), (b_n)$ such that

$$p_{n+1}(x) = (x - a_n)p_n(x) - b_n p_{n-1}(x).$$

Show that the matching polynomials of $(P_n)$, $(C_n)$, and $(K_n)$ each satisfy a three term recurrence.

**Solution**

(a) Setting $v$ as one of the "end" vertices in Lemma 1.26(b), we get

$$\mathrm{Match}_{P_n}(x) = x\,\mathrm{Match}_{P_{n-1}}(x) - \mathrm{Match}_{P_{n-2}}(x).$$

(b) Fixing some edge $e$ and using Lemma 1.26(c), then using the recurrence from (a), gives

$$\begin{aligned} \mathrm{Match}_{C_n}(x) &= \mathrm{Match}_{P_n}(x) - \mathrm{Match}_{P_{n-2}}(x) \\ &= x\,\mathrm{Match}_{P_{n-1}}(x) - \mathrm{Match}_{P_{n-2}}(x) - x\,\mathrm{Match}_{P_{n-3}}(x) + \mathrm{Match}_{P_{n-2}}(x) \\ &= x\,\mathrm{Match}_{C_{n-1}}(x) - \mathrm{Match}_{C_{n-2}}(x). \end{aligned}$$

(c) Fixing some vertex $v$ and using Lemma 1.26(b), we get

$$\mathrm{Match}_{K_n}(x) = x\,\mathrm{Match}_{K_{n-1}}(x) - (n-1)\,\mathrm{Match}_{K_{n-2}}(x)$$

**Problem 3.3.3.** For $n \geq 1$, show that $\mathrm{Match}_{C_n}(2x) = 2T_n(x)$.

**Solution**

The above is easily verified for $n = 1, 2$, and we saw in Problem 3.3.2 that $\mathrm{Match}_{C_n}$ satisfies the same recurrence as $2T_n$, which is the same as that of $T_n$ from Proposition 1.14.

**Problem 3.3.4.** Let $T$ be a tree with maximum degree $d$. Show that any root of $\mathrm{Match}_T(x)$ has absolute value at most $2\sqrt{d-1}$.

**Solution**

By the proof of Theorem 1.28, it suffices to show that the eigenvalues of the adjacency matrix $A$ of $T$ are bounded from above by $2\sqrt{d-1}$.
Fix a "root" vertex $r$ in $T$ and denote by $d(\cdot, r)$ the distance from a vertex to $r$. Consider the diagonal matrix $D$ defined by $D_{u,u} = (d-1)^{-d(u,r)/2}$. Consider $B = DAD^{-1}$. Clearly, the spectrum of $A$ and $B$ are equal, so it suffices to show that the eigenvalues of $B$ are at most $2\sqrt{d-1}$. We have

$$B_{uv} = \begin{cases} (d-1)^{(d(u,r)-d(v,r))/2}, & u \leftrightarrow v, \\ 0, & \text{otherwise.} \end{cases}$$

Recall the Perron-Frobenius Theorem, one of whose consequences gives that the largest eigenvalue of $B$ is at most the largest row sum of $B$.
Now, observe that for any vertex $u$ with neighbour $v$, $d(v, r) \in \{d(u, r) - 1, d(u, r) + 1\}$. Further, because $T$ is a tree, there can be at most one neighbour $v$ with $d(v, r) = d(u, r) - 1$ – were there more, we would have a cycle. Therefore,

$$\lambda_{\max} \leq \max_u \sum_v B_{u,v} = \max_u \sum_{v : v \leftrightarrow u} (d-1)^{d(u,r)-d(v,r)} \leq 1 \cdot (d-1)^{1/2} + \frac{d-1}{(d-1)^{1/2}} = 2\sqrt{d-1}.$$

The idea of the above proof is similar to that of the Alon-Boppana bound, and the $2\sqrt{d-1}$ bound occurs in the definition of *Ramanujan graphs*. For more details, the interested reader may read Theorem 2.27 and following discussion in the author's Topics in Graph Theory notes.

**Problem 3.3.5.** Show that when one runs the recurrence relation of the binomial coefficients backwards, we get signed versions of the binomial coefficients.

**Solution**

The usual binomial recurrence is

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

For negative coefficients, we have

$$\binom{-(n+1)}{k} = \binom{-n}{k} - \binom{-(n+1)}{k-1}.$$

We have $\binom{0}{k} = 1$ if $k = 0$ and $0$ otherwise, and also that $\binom{-n}{0} = 1$. For $n \geq 0$, let $f_n(x) = \sum_{k \geq 0} \binom{-n}{k} x^k$. Then,

$$\begin{aligned}\sum_{k \geq 1} \binom{-(n+1)}{k} x^k &= \sum_{k \geq 1} \binom{-n}{k} x^k - x \sum_{k \geq 1} \binom{-(n+1)}{k-1} x^{k-1} \\ f_{n+1}(x) - 1 &= f_n(x) - 1 - x f_{n+1}(x) \\ f_{n+1}(x) &= \frac{1}{1+x} f_n(x) \\ f_n(x) &= \frac{1}{(1+x)^n}.\end{aligned}$$

Finally,

$$\frac{1}{(1+x)^n} = \sum_{k \geq 0} (-1)^k \binom{n+k}{k} x^k,$$

so $\binom{-n}{k} = (-1)^k \binom{n+k}{k}$.

**Problem 3.3.6.** A restricted growth function (RGF) of length $n$ is a sequence $w_1, w_2, \ldots, w_n$ of positive integers such that $w_1 = 1$ and $w_i \leq 1 + \max(w_1, \ldots, w_{i-1})$ when $i \geq 2$. Determine the number of RGFs of length $n$.

**Solution**

Let $r_{n,k}$ be the number of RGFs $w_1, \ldots, w_n$ of length $n$, where $\max w_i = k$. Then,

$$r_{n,k} = r_{n-1,k-1} + k r_{n-1,k}.$$

We also have $r_{1,1} = 1$ and $r_{1,k} = 0$ for $k > 1$. Since the recurrence is identical to that of the Stirling numbers of the second kind and the base values coincide as well, $r_{n,k} = S_{n,k}$. So, the number of RGFs of length $n$ is $\sum_k S_{n,k} = B_n$, the $n$th Bell number.

**Problem 3.3.7.** Consider a staircase board $\mathsf{St}_n$ with $n$ columns where the $i$th column has $i-1$ cells all stacked from the same height. We place "rooks" on this board and recall that rooks can travel on the same row or column. Two rooks can attack each other if they are on the same row or column and a placement of rooks is said to be non-attacking if no two rooks in the placement can attack each other. Determine the number of ways to place $n-k$ non-attacking rooks in $\mathsf{St}_n$.

**Solution**

Denote by $\mathsf{Rook}_{n,k}$ the desired quantity.
If there is no rook in the final column, we get $\mathsf{Rook}_{n-1,k-1}$ placements, and if there is a rook, we have $\mathsf{Rook}_{n-1,k} \cdot (n-1-(n-k-1)) = k\mathsf{Rook}_{n-1,k}$ – placing the $n-k-1$ rooks on the rest of the board, we get $k$ possible positions to place the final rook in the $n$th column. In all, we have

$$\mathsf{Rook}_{n,k} = \mathsf{Rook}_{n-1,k-1} + k\mathsf{Rook}_{n-1,k}.$$

Note that this is the same as the recurrence for Stirling numbers of the second kind! Verifying that the base values are the same, we get $\mathsf{Rook}_{n,k} = S_{n,k}$.

## 3.4. Problem Sheet 5

**Problem 3.4.1.** Let $P$ be a set of polynomials in commuting variables $x_1, \ldots, x_n$ and $\sigma \in \mathfrak{S}_n$ such that $\sigma(f) = f$ for all $f \in P$. Prove or disprove that for $\pi \in \mathfrak{S}_n$, $(\pi\sigma\pi^{-1})(f) = f$ for all $f \in P$.

**Solution**

The statement is false. For example, if $f(x) = x_1 x_2$ and $\sigma = (1,2)$, clearly $f = \sigma(f)$. However, $\sigma$ is in the same conjugacy class as $(\pi\sigma\pi^{-1}) = (1,3)$ (for an appropriate $\pi$), and clearly the desired property does not hold.

# References


[AH89] Kenneth Appel and Wolfgang Haken. Every planar map is four colorable. *Contemporary Mathematics*, 98, 1989.

[BS] Matthias Beck and Raman Sanyal. *Combinatorial Reciprocity Theorems*.

[dB55] Faa di Bruno. Note sur une nouvelle formule de calcul differentiel. *Quart. J. Math*, (1):359–360, 1855.

[DS20] Hiranya Kishore Dey and Sivaramakrishnan Sivasubramanian. *The Electronic Journal of Combinatorics*, 2020.

[Edm65] Jack Edmonds. Maximum matching and a polyhedron with 0,1-vertices. *Journal of Research of the National Bureau of Standards Section B Mathematics and Mathematical Physics*, page 125, 1965.

[GG81] C. D. Godsil and I. Gutman. On the theory of the matching polynomial. *Journal of Graph Theory*, 5(2):137–144, 1981.

[Sag20] Bruce E. Sagan. *Combinatorics: The Art of Counting*. American Mathematical Society, 2020.

[SF99] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.