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# MA 412: COMPLEX ANALYSIS

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## §1. Introduction

### 1.1. Some basic definitions

Consider the equation  $X^2 + 1 = 0$ . Clearly, this equation has no roots over  $\mathbb{R}$ . Consider the set

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} = \mathbb{R}^2,$$

and define addition and subtraction over  $\mathbb{C}$  as

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b) \cdot (c, d) &= (ac - bd, ad + bc).\end{aligned}$$

It is easy to show that  $(\mathbb{C}, +, \cdot)$  is a field with additive identity  $(0, 0)$  and multiplicative identity  $(1, 0)$ . Further observe that  $\mathbb{R}$  is a subfield of  $\mathbb{C}$  – consider the field homomorphism  $\mathbb{R} \rightarrow \mathbb{C}$  defined by  $a \mapsto (a, 0)$ .

Now, we denote  $\iota = (0, 1)$ , and write  $(a, b)$  as  $a + b\iota$ .

Observe that the equation  $X^2 + 1 = 0$  *does* have roots over  $\mathbb{C}$  since it can be written as  $(X + \iota)(X - \iota)$ . For the sake of completeness, we also note that the multiplicative identity of  $a + b\iota$  is

$$\frac{1}{a + b\iota} = \frac{a - b\iota}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\iota.$$

When writing  $z = a + b\iota$  where  $a, b \in \mathbb{R}$ , we write  $a = \Re z$  (the real part of  $z$ ) and  $b = \Im z$  (the imaginary part of  $z$ ). We also define the absolute value  $|z| = (a^2 + b^2)^{1/2}$  of  $z$ , and the *conjugate*  $\bar{z} = a - b\iota$  of  $z$ . We clearly have

$$\begin{aligned}z\bar{z} &= |z|^2 \\ \Re z &= \frac{z + \bar{z}}{2} \\ \Im z &= \frac{z - \bar{z}}{2\iota}.\end{aligned}$$

It is easy to check that

$$\overline{z + w} = \bar{z} + \bar{w} \text{ and } \overline{z \cdot w} = \bar{z} \cdot \bar{w}.$$

We also have

$$\begin{aligned}\left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \\ |\bar{z}| &= |z|.\end{aligned}$$

**Exercise 1.1.** Check that the set

$$M = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R}$$

with matrix addition and multiplication is a field isomorphic to  $\mathbb{C}$ .

To close out the tedious part of things, we have

$$\begin{aligned}|z + w|^2 &= |z|^2 + |w|^2 + 2\Re(z\bar{w}) \\ |z + w| &\leq |z| + |w|\end{aligned}\tag{1.1}$$

Equation (1.1) is referred to as the *triangle inequality*.

## 1.2. Polar representations and roots

Consider  $z = x + iy \in \mathbb{C}$ . We may then define

$$x = r \cos \theta \quad y = r \sin \theta,$$

where  $|z| = r$  and the angle  $\theta$  is called the *argument* of  $z$  as is denoted  $\theta = \arg z$ . We typically restrict  $\theta$  to  $(-\pi, \pi]$ . We denote  $\text{cis } \theta = \cos \theta + i \sin \theta$ . Therefore, we have

$$z = |z| \text{cis}(\arg z).$$

Observe that rather conveniently,

$$\text{cis } \theta_1 \cdot \text{cis } \theta_2 = \text{cis}(\theta_1 + \theta_2).$$

Therefore, inductively,

$$z_1 z_2 \cdots z_n = \left( \prod_i |z_i| \right) \text{cis} \left( \sum_i \arg z_i \right).$$

In particular,

$$z^n = r^n \text{cis}(n\theta)$$

for any  $n > 0$ . If  $z \neq 0$  (equivalently,  $r \neq 0$ ), the above holds for all  $n \in \mathbb{Z}$ .

In the case where  $r = 1$ , we have

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta) \tag{1.2}$$

Equation (1.2) is referred to as *de Moivre's Formula*.

Let us consider the equation  $z^n = a$ . This equation has  $n$  roots of the form

$$z = |a|^{1/n} \text{cis} \left( \frac{2k\pi + \arg a}{n} \right)$$

for  $k = 0, 1, \dots, n-1$ .

A *line* in the complex plane is a set of the form

$$L = \{z = a + tb : t \in \mathbb{R}\},$$

for some fixed  $a, b \in \mathbb{C}$ , where  $b$  is a *directional* vector whose absolute value may be assumed to be 1. Since  $b \neq 0$ , we equivalently have

$$L = \left\{ z : \Im \left( \frac{z-a}{b} \right) = 0 \right\}.$$

We can also define the half-planes

$$H_a = \left\{ z : \Im \left( \frac{z-a}{b} \right) > 0 \right\}$$

$$K_a = \left\{ z : \Im \left( \frac{z-a}{b} \right) < 0 \right\}.$$

Note that  $H_a = a + H_0$ , where the addition is Minkowski addition:

$$H_a = \{a + z : z \in H_0\}.$$

### 1.3. The extended plane

Define  $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$  and let  $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$  be the unit sphere in  $\mathbb{R}^3$ . We shall show a bijection from  $\mathbb{C}_\infty$  to  $S$ .

Let  $N = (0, 0, 1)$  be the ‘north pole’ of  $S$ , and orient  $\mathbb{C}$  (as  $\mathbb{R}^2$ ) in the horizontal plane in a manner such that  $\mathbb{C}$  cuts  $S$  along the equator. For  $z = x + iy \in \mathbb{C}$ , let us define the corresponding point  $Z = (x_1, x_2, x_3) \in S$ . We shall draw a line connecting  $z$  to  $N$ , and let  $Z$  be the point of intersection (other than  $N$ ) of this line with  $S$ . Finally, we shall map  $\infty$  to  $N$ .

Let us define this more explicitly. The line through  $N$  and  $z$  is

$$L = \{tN + (1-t)z : t \in \mathbb{R}\}.$$

Then, letting  $z = (x, y, 0)$ , we have

$$t^2 + (1-t)^2|z|^2 = 1.$$

So,

$$|z|^2 = \frac{1-t^2}{(1-t)^2} = \frac{1+t}{1-t}$$

and

$$t = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Therefore, we map  $z$  to

$$Z = \left( \frac{2\Re z}{|z|^2 + 1}, \frac{2\Im z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \in S.$$

Based on this, we can define a distance metric between points in  $\mathbb{C}_\infty$ . For  $z, z' \in \mathbb{C}_\infty$  mapping to  $Z, Z' \in S$ , we let  $d(z, z')$  be the Euclidean distance between  $Z, Z'$  in  $\mathbb{R}^3$ . More explicitly,

$$\begin{aligned} d(z, z')^2 &= (x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2 \\ &= 2 - 2(x_1x'_1 + x_2x'_2 + x_3x'_3) \\ &= \frac{2|z - z'|}{(|z|^2 + 1)(|z'|^2 + 1)^{1/2}} \end{aligned}$$

when  $z, z' \in \mathbb{C}$  and if  $z' = \infty$  (so  $Z' = (0, 0, 1)$ ), we have

$$d(z, z') = \frac{4}{|z|^2 + 1}$$

This correspondence between points of  $S$  and  $\mathbb{C}_\infty$  is called the *stereographic projection*.

**Exercise 1.2.** If  $P$  is a plane in  $\mathbb{R}^3$  and  $\Lambda = P \cap S$  is a circle on  $S$ , show that the projection of  $\Lambda$  on  $\mathbb{C}$  under the stereographic projection is a circle as well (possibly a circle of infinite radius, namely a line).

### 1.4. Power series

In this section, we begin discussing convergence of series in  $\mathbb{C}$  and related properties.

**Definition 1.1.** If  $a_n \in \mathbb{C}$  for every  $n \geq 0$ , the series  $\sum_{n=0}^{\infty} a_n$  is said to *converge* to  $z$  iff for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$\left| \sum_{n=0}^m a_n - z \right| < \epsilon$$

for all  $m \geq N$ .

The series  $\sum_{n=0}^{\infty} a_n$  is said to converge *absolutely* if  $\sum_{n=0}^{\infty} |a_n|$  converges.

**Theorem 1.1.**  $\mathbb{C}$  is complete. That is, every Cauchy sequence in  $\mathbb{C}$  is convergent.

*Proof.* Suppose  $\{x_n + iy_n\}$  is a Cauchy sequence in  $\mathbb{C}$ , where  $x_n, y_n \in \mathbb{R}$  for each  $n$ . We then have the existence of  $N \in \mathbb{N}$  such that for all  $m, k > N$ ,  $|(x_m - x_k) + i(y_m - y_k)| < \epsilon$ . Consequently,  $|x_m - x_k| < \epsilon$  and  $|y_m - y_k| < \epsilon$ . However, since  $\mathbb{R}$  is complete, this implies that  $(x_n)$  and  $(y_n)$  are convergent, completing the proof. ■

**Theorem 1.2.** If  $\sum a_n$  converges absolutely,  $\sum a_n$  converges.

*Proof.* Let  $\epsilon > 0$ ,  $z_n = \sum_{i=0}^n a_i$ , and  $S_n = \sum_{i=0}^n |a_i|$ . Because  $\mathbb{C}$  is complete, it suffices to show that  $(z_n)$  is Cauchy. Since  $\sum |a_n|$  is convergent, there exists  $N \in \mathbb{N}$  such that  $|S_m - S_k| < \epsilon$  for all  $m, k > N$ . Supposing  $m > k$ , we have

$$S_m - S_k = \sum_{i=k+1}^m |a_i|.$$

So,

$$\begin{aligned} |z_m - z_k| &= \left| \sum_{i=k+1}^m a_i \right| \\ &\leq \sum_{i=k+1}^m |a_i| < \epsilon, \end{aligned}$$

completing the proof. ■

**Exercise 1.3.** Show that  $\sum_{n=0}^{\infty} z^n$  converges iff  $|z| < 1$ .

**Theorem 1.3.** For a given power series  $\sum_{n=0}^{\infty} a_n(z-a)^n$ , define the number  $R$  ( $0 \leq R \leq \infty$ ) by

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}.$$

Then,

- (a) If  $|z - a| < R$ , the series converges absolutely.
- (b) If  $|z - a| > R$ , the terms of the series become unbounded and the series diverges.
- (b) If  $0 < r < R$ , the series converges uniformly on the set  $\{z : |z - a| \leq r\}$ .

This  $R$  is referred to as the *radius of convergence* of the power series.

*Proof.*

- (a) We assume without loss of generality that  $a = 0$ . If  $|z| < R$ , there exists  $r$  with  $|z| < r < R$ . By the definition of  $R$ , for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$\frac{1}{R} - \epsilon < \sup_{k \geq n} |a_k|^{1/k} < \frac{1}{R} + \epsilon$$

for all  $n > N$ . If we take  $\epsilon = 1/r - 1/R$ , it follows that  $|a_n|^{1/n} < 1/r$  for all  $n > N$ . That is, for all  $n > N$ ,  $|a_n| < 1/r^n$  and so

$$|a_n z^n| < \left( \frac{|z|}{r} \right)^n.$$

Therefore,  $\sum_{n=N}^{\infty} a_n z^n$  is dominated by  $\sum_{n=N}^{\infty} (|z|/r)^n$ . Now however, we can just use the result of Exercise 1.3 to conclude absolute convergence since  $|z|/r < 1$ .

(b) Let  $|z| > R$  and choose  $r$  with  $|z| > r > R$ . For  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$\frac{1}{R} - \epsilon < \sup_{k \geq n} |a_k|^{1/k} \text{ for all } n > N.$$

Choosing  $\epsilon = 1/R - 1/r$ ,

$$|a_n|^{1/n} > 1/r$$

for infinitely many  $n \in \mathbb{N}$ . It follows that  $|a_n z^n| > (|z|/r)^n$  for infinitely many  $n \in \mathbb{N}$ . Since  $|z|/r > 1$ , these terms become unbounded and therefore the series diverges.

(c) Now, suppose  $r < R$  and choose  $\rho$  such that  $r < \rho < R$ . Similar to the argument in (a), we get that

$$|a_n| < \frac{1}{\rho^n} \text{ for all } n \geq N.$$

If  $|z| \leq r$ ,  $|a_n z^n| \leq (r/\rho)^n$  and  $r/\rho < 1$ . The Weierstrass  $M$ -test then gives that the power series converges uniformly on  $\{z : |z| \leq r\}$ . ■

It should be noted that we cannot conclude anything when  $|z - a| = R$ .

**Theorem 1.4.** If  $\sum a_n(z - a)^n$  is a power series with radius of convergence  $R$ , then if it exists,

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = R.$$

*Proof.* Again, assume that  $a = 0$  and let  $\alpha = \lim |a_n/a_{n+1}|$ , which we assume exists. Suppose that  $|z| < \alpha$  and take  $r \in \mathbb{R}$  such that  $|z| < r < \alpha$ . For all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for  $n \geq N$ ,

$$\alpha - \epsilon < \left| \frac{a_n}{a_{n+1}} \right| < \alpha + \epsilon.$$

Taking  $\epsilon = \alpha - r$ ,  $|a_n/a_{n+1}| > r$  for all  $n \geq N$ . Let  $B = |a_N| r^N$ . Then,

$$a_{N+1} r^{N+1} = |a_{N+1}| r \cdot r^N < |a_N| r^N = B.$$

Similarly, we get that  $|a_n| r^n < B$  for all  $n \geq N$ . Therefore,

$$|a_n z^n| < B \left( \frac{|z|}{r} \right)^n$$

for all  $n \geq N$ . Thus, the sequence converges absolutely since  $|z| < r$ . Since  $r < \alpha$  was arbitrary, this implies that  $\alpha \leq R$ .

On the other hand, if  $|z| > \alpha$ , take  $r \in \mathbb{R}$  such that  $|z| > r > \alpha$ . Taking  $\epsilon = r - \alpha$ , we get  $N \in \mathbb{N}$  such that

$$\left| \frac{a_n}{a_{n+1}} \right| < r$$

for all  $n \geq N$ . Letting  $B = |a_N| r^N$  again, we once more obtain that  $|a_n| r^n > B$  for all  $n \geq N$ . This gives that

$$|a_n z^n| > B \left( \frac{|z|}{r} \right)^n$$

for all  $n \geq N$ , and since  $|z| > r$ , the sequence diverges (we may assume that  $B \neq 0$  by making  $N$  larger if required to ensure that  $a_N \neq 0$  – if this is not possible, the problem is trivial since it means that  $(a_n)$  is eventually 0). Since the choice of  $r$  was arbitrary, this implies that  $R \leq \alpha$ , completing the proof. ■

Now, consider the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The radius of convergence of this series is  $\infty$ . So, it converges for any complex number  $z$ , and convergence is uniform on every compact subset of  $\mathbb{C}$ .

The above defines a function  $\exp : \mathbb{C} \rightarrow \mathbb{C}$ .

We also denote  $e^z = \exp(z)$ .

**Definition 1.2** (Differentiability). If  $G$  is an open set in  $\mathbb{C}$  and  $f : G \rightarrow \mathbb{C}$ , then  $f$  is said to be *differentiable* at a point  $a \in G$  if the limit

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

exists. If it exists, the value of this limit is denoted  $f'(a)$  and is called the *derivative* of  $f$  at  $a$ .

If  $f$  is differentiable at each point of  $G$ , we say that  $f$  is differentiable on  $G$ . Note that if  $f$  is differentiable on  $G$ , then  $f' : G \rightarrow \mathbb{C}$  is a function. If  $f'$  is continuous,  $f$  is said to be *continuously differentiable*.

**Theorem 1.5.** If  $f : G \rightarrow \mathbb{C}$  is differentiable at a point  $a \in G$ ,  $f$  is continuous at  $a$ .

*Proof.* The proof of this is direct:

$$\begin{aligned} \lim_{z \rightarrow a} |f(z) - f(a)| &= \left( \lim_{z \rightarrow a} \frac{|f(z) - f(a)|}{|z - a|} \right) \cdot \lim_{z \rightarrow a} |z - a| \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

■

**Definition 1.3.** A function  $f : G \rightarrow \mathbb{C}$  is said to be *analytic* if  $f$  is continuously differentiable on  $G$ .

Let  $f, g$  be analytic on  $G$  and  $\Omega$  respectively, and suppose that  $f(G) \subseteq \Omega$ . Then,  $g \circ f$  is analytic on  $G$  and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$$

for all  $z \in G$ . This is called the *chain rule*.

We shall show later that if  $f$  is differentiable then its derivative is continuous, and so  $f$  is analytic.

**Theorem 1.6.** Let  $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$  have radius of convergence  $R > 0$ . Then

(a) For each  $k \geq 1$ , the series

$$\sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (z-a)^{n-k}$$

has radius of convergence  $R$ .

(b) The function  $f$  is infinitely differentiable on  $B(a, R)$  (the open ball of radius  $R$  centered at  $a$ ), and further,  $f^{(k)}(z)$  is given by the series in (a) for all  $k \geq 1$  and  $|z-a| < R$ .

(c) For  $n \geq 0$ ,  $a_n = \frac{1}{n!} f^{(n)}(a)$ .

*Proof.* Assume that  $a = 0$ .

(a) Note that it suffices to prove the result for  $k = 1$  (Why?). To show this, it is enough to show that

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} = \limsup_{n \rightarrow \infty} |na_n|^{1/(n-1)}$$

First, it is not difficult to show that  $\lim_{n \rightarrow \infty} n^{1/(n-1)} = 1$ . It may be shown that for any sequences  $(c_n), (d_n)$  in  $\mathbb{R}$  where  $c_n \geq 0$ , if  $\lim c_n = c$  and  $\limsup d_n = d$ , then  $\limsup c_n d_n = cd$ . Therefore, we are done if we show that  $\limsup_{n \rightarrow \infty} |a_n|^{1/n} = \limsup_{n \rightarrow \infty} |a_n|^{1/(n-1)}$ .

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + z \sum_{n=0}^{\infty} a_{n+1} z^n.$$

Let  $R'$  be the radius of convergence of  $\sum_{n=0}^{\infty} a_{n+1} z^n$ . We want to show that  $R' = R$ .

If  $|z| < R'$ , then

$$\sum |a_n z^n| \leq |a_0| + |z| \sum_{n=0}^{\infty} |a_{n+1} z^n| < \infty,$$

so  $R' \leq R$ . On the other hand, if  $|z| < R$  and  $z \neq 0$ ,

$$\sum |a_{n+1} z^n| < \frac{1}{|z|} \left( \sum |a_n z^n| + |a_0| \right) < \infty,$$

so  $R \leq R'$  and we are done.

(b) Once again, it suffices to prove the result for  $k = 0$ . For  $|z| < R$  and  $g(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$ ,

$$s_n(z) = \sum_{k=0}^n a_k z^k \text{ and } R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k,$$

fix a point  $w \in B(0, R)$  and  $r$  such that  $|w| < r < R$ . We wish to show that  $f'(w)$  exists and is equal to  $g(w)$ . Let  $\delta > 0$  be arbitrary with  $\overline{B(w, \delta)} \subseteq B(0, r)$ . Letting  $z \in B(w, \delta)$ , we have

$$\frac{f(z) - f(w)}{z - w} - g(w) = \frac{s_n(z) - s_n(w)}{z - w} - s'_n(w) + s'_n(w) - g(w) + \frac{R_n(z) - R_n(w)}{z - w}.$$

We have

$$|z^k - w^k| = |z - w| |z^{k-1} + z^{k-2}w + \cdots + w^{k-1}| \leq |z - w| k r^{k-1}.$$

Therefore,

$$\left| \frac{R_n(z) - R_n(w)}{z - w} \right| = \left| \sum_{k=n+1}^{\infty} a_k \frac{z^k - w^k}{z - w} \right| \leq \sum_{k=n+1}^{\infty} |a_k| k r^{k-1}.$$

Since  $r < R$ ,  $\sum_{k=1}^{\infty} |a_k| k r^{k-1}$  converges and so for any  $\epsilon > 0$ , there exists  $N_1 \in \mathbb{N}$  such that for  $n \geq N_1$ ,

$$\left| \frac{R_n(z) - R_n(w)}{z - w} \right| < \epsilon/3.$$

Since  $\lim s'_n(w) = g(w)$ , there exists  $N_2 \in \mathbb{N}$  such that

$$|s'_n(w) - g(w)| < \epsilon/3$$

for  $n \geq N_2$ . Choose  $n \geq \max(N_1, N_2)$ . Then, there exists  $\delta > 0$  such that whenever  $0 < |z - w| < \delta$ ,

$$\left| \frac{s_n(z) - s_n(w)}{z - w} - s'_n(w) \right| < \epsilon/3.$$

Putting all these together, we get the desideratum.



(c) This is straightforward using the explicit expression for  $f^{(k)}(a)$ . ■

If the series  $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$  has radius of convergence  $R > 0$ , then  $f$  is analytic on  $B(a, R)$ . Therefore,  $\exp$  is analytic on  $\mathbb{C}$ .

Further, letting  $g = \exp$ ,

$$g'(z) = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = g(z).$$

Define the functions  $\cos$  and  $\sin$  using power series as

$$\begin{aligned}\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^k \frac{z^{2k}}{(2k)!} + \cdots \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^k \frac{z^{2k+1}}{(2k+1)!} + \cdots\end{aligned}$$

Note that

$$\cos z = \frac{e^{\iota z} + e^{-\iota z}}{2} \quad \text{and} \quad \sin z = \frac{e^{\iota z} - e^{-\iota z}}{2\iota}.$$

Therefore,

$$e^{\iota z} = \cos z + \iota \sin z.$$

In particular, if  $z = \theta \in \mathbb{R}$ ,

$$e^{\iota \theta} = \cos \theta + \iota \sin \theta.$$

It is direct to show next that  $\cos^2 z + \sin^2 z = 1$  for  $z \in \mathbb{C}$ .

**Definition 1.4.** A function  $f$  is said to be *periodic* with period  $c$  if  $f(z) = f(z + c)$  for all  $z \in \mathbb{C}$ .

$e^z$  is periodic with period  $2\pi\iota$ .

Similar to  $\cos$  and  $\sin$ , one can define the function  $\log$  as

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

$\log z$  is defined only when  $|z-1| < 1$ . Further note that we cannot define  $\log$  as the inverse of  $\exp$  (as we do over the reals) since  $\exp$  is not injective here.

We would like to define  $\log$  such that  $w = \exp z$  when  $z = \log w$ . Since  $\exp$  is non-zero, also suppose that  $w \neq 0$ . If  $z = x + \iota y$ , then  $|w| = e^x$  and  $\arg w = y + 2\pi k\iota$  for some  $k \in \mathbb{Z}$ . Therefore, the solution set for  $e^z = w$  is

$$\{\log |w| + \iota(\arg w + 2\pi k) : k \in \mathbb{Z}\}.$$

**Definition 1.5.** If  $G$  is an open connected set in  $\mathbb{C}$  and  $f : G \rightarrow \mathbb{C}$  is a continuous function such that  $z = \exp(f(z))$  for all  $z \in G$ , then  $f$  is a *branch of the logarithm*.

**Lemma 1.7.** If  $G \subseteq \mathbb{C}$  is open and connected and  $f$  is a branch of the logarithm on  $G$ , then the totality of the branches of  $\log z$  are the functions  $f(z) + 2\pi k\iota$  for  $k \in \mathbb{Z}$ .

*Proof.* If  $g(z) = f(z) + 2\pi k\iota$  for some  $k \in \mathbb{Z}$ , then  $\exp(g(z)) = \exp(f(z)) = z$ , so  $g$  is also a branch of the logarithm. On the other hand, suppose that  $g$  is a branch of the logarithm. For  $z \in G$ ,  $\exp(f(z)) = \exp(g(z)) = z$ , so  $g(z) = f(z) + 2\pi k\iota$ . However, note that this  $k$  depends on  $z$ . We must show that the same  $k$  works for all  $z$ . Indeed,  $h(z) = (g(z) - f(z))/2\pi\iota$  is continuous on  $G$  and  $h(G) \subseteq \mathbb{Z}$ , so the required follows. ■

Now, let  $G = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ . Clearly,  $G$  is connected and each  $z \in G$  can be uniquely denoted by  $|z|e^{\iota\theta}$ , where  $-\pi < \theta < \pi$ . For  $\theta$  in this range, define

$$f(re^{\iota\theta}) = \log r + \iota\theta.$$

This is a branch of the logarithm on  $G$ , and is commonly referred to as the *principal branch*.

**Theorem 1.8.** Let  $G, \Omega$  be open subsets of  $\mathbb{C}$ . Suppose that  $f : G \rightarrow \mathbb{C}$  and  $g : \Omega \rightarrow \mathbb{C}$  are continuous such that  $g(f(z)) = z$  for all  $z \in G$ . If  $G$  is differentiable and  $g'(z) \neq 0$ ,  $f$  is differentiable and

$$f'(z) = \frac{1}{g'(f(z))}.$$

If  $g$  is analytic, so is  $f$ .

*Proof.* Fix  $a \in G$  and let  $h \in \mathbb{C} \setminus \{0\}$  with  $a + h \in G$ . Since  $g(f(a)) = a \neq a + h = g(f(a + h))$ ,  $f(a) \neq f(a + h)$ . Also,

$$1 = \frac{g(f(a + h)) - g(f(a))}{h} = \frac{g(f(a + h)) - g(f(a))}{f(a + h) - f(a)} \cdot \frac{f(a + h) - f(a)}{h}.$$

Take the limit of either side as  $h \rightarrow 0$ . The first fraction is equal to  $g'(f(a))$  since  $\lim_{h \rightarrow 0} (f(a + h) - f(a)) = 0$ , and therefore  $\lim_{h \rightarrow 0} (f(a + h) - f(a))/h = f'(a)$  exists, and  $1 = g'(f(a)) \cdot f'(a)$ . The required follows.

If  $g$  is analytic, then  $g'$  is continuous so  $f$  is analytic. ■

**Corollary 1.9.** Any branch of the logarithm function is analytic and has derivative  $z \mapsto 1/z$ .

Given a branch of the logarithm  $f$  on an open connected set  $G$  and fixed  $b \in \mathbb{C}$ , define  $g(z) = \exp(bf(z))$ . If  $b \in \mathbb{Z}$ ,  $g(z) = z^b$ . In general, this defines a branch of  $z^b$  ( $b \in \mathbb{C}$ ) for any open connected set on which there is a branch of  $\log z$ .

If we write  $z^b$  as a function, it is implicitly understood that the  $f$  in  $\exp(bf(z))$  is the principal branch of the logarithm. Since  $\log$  is analytic, so is  $z \mapsto z^b$ .

## 1.5. Cauchy-Riemann Equations

Let  $f : G \rightarrow \mathbb{C}$  be analytic and let

$$u(x, y) = \Re(f(x + iy)), v(x, y) = \Im(f(x + iy))$$

for  $x + iy \in G$ . Let us evaluate the limit

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z + h) - f(z)}{h}.$$

in two different ways.

First, if we let  $h \rightarrow 0$  through real values, we get

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y).$$

Along the imaginary axis, we get

$$f'(z) = -i \frac{\partial u}{\partial y}(x, y) + \frac{\partial v}{\partial y}(x, y).$$

Therefore,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Supposing that  $u$  and  $v$  have continuous second derivative (we shall later show that they are infinitely differentiable), we have that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Therefore, since the second derivatives are continuous,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.3}$$

A function  $u$  with continuous second partial derivatives satisfying Equation (1.3) is said to be *harmonic*. Similarly,  $v$  is also harmonic.

**Theorem 1.10.** Let  $u, v$  be real-valued functions defined on an open connected set (a *region*)  $G$  and suppose that they have continuous second partial derivatives. Then,  $f : G \rightarrow \mathbb{C}$  defined by  $f(z) = u(z) + \iota v(z)$  is analytic iff  $u$  and  $v$  satisfy the Cauchy-Riemann equations.

*Proof.* We have already shown the forward direction.

For the other direction, let  $z = x + \iota y \in G$  and  $B(z, r) \subseteq G$ . Let  $h = s + \iota t \in B(0, r)$ . Our goal is to show that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\left| \frac{f(z+h) - f(z) - f'(z)h}{h} \right| < \epsilon$$

for all  $h \in B(0, \delta)$  for some  $f'(z) \in \mathbb{C}$ . Note that

$$u(x+s, y+t) - u(x, y) = (u(x+s, y+t) - u(x, y+t)) + (u(x, y+t) - u(x, y)).$$

Now, for fixed  $t \in (-r, r)$ ,  $s \mapsto u(x+s, y+t)$  is a differentiable function on  $(-r, r)$ . We apply the mean value theorem to conclude that there exist  $s_1, t_1 \in (-r, r)$  for each  $s + \iota t \in B(0, r)$  such that  $|s_1| < |s|$ ,  $|t_1| < |t|$ , and

$$\begin{aligned} u(x+s, y+t) - u(x, y+t) &= u_x(x+s_1, y+t)s \\ u(x, y+t) - u(x, y) &= u_y(x, y+t_1)t. \end{aligned}$$

Now, let

$$\varphi(s, t) = (u(x+s, y+t) - u(x, y)) - (u_x(x, y)s + u_y(x, y)t).$$

We get that

$$\varphi(s, t) = (su_x(x+s_1, y+t) - su_x(x, y)) + (tu_y(x, y+t_1) - tu_y(x, y)).$$

So,

$$\frac{\varphi(s, t)}{s + \iota t} = \frac{s}{s + \iota t} (u_x(x+s_1, y+t) - u_x(x, y)) + \frac{t}{s + \iota t} (u_y(x, y+t_1) - u_y(x, y))$$

and on taking the limit of both sides as  $s + \iota t \rightarrow 0$ , we can use the fact that  $|s| \leq |s + \iota t|$ ,  $|t| \leq |s + \iota t|$ ,  $|s_1| < |s|$ ,  $|t_1| < |t|$ , and the continuity of  $u_x, u_y$ , to conclude that

$$\lim_{s+\iota t \rightarrow 0} \frac{\varphi(s, t)}{s + \iota t} = 0.$$

Therefore,

$$u(x+s, y+t) - u(x, y) = u_x(x, y)s + u_y(x, y)t + \varphi(s, t).$$

We get a similar equation for  $v$  as well, with a function  $\psi$  (in place of  $\varphi$ ). Combining the two,

$$\begin{aligned} \frac{f(z+s+\iota t) - f(z)}{s + \iota t} &= \frac{u(x+s, y+t) - u(x, y)}{s + \iota t} + \iota \frac{v(x+s, y+t) - v(x, y)}{s + \iota t} \\ &= \frac{su_x(x, y) + tu_y(x, y) + \varphi(s, t) + \iota (sv_x(x, y) + tv_y(x, y) + \psi(s, t))}{s + \iota t} \\ &= \frac{u_x(x, y)(s + \iota t) + \iota v_x(x, y)(s + \iota t) + \varphi(s, t) + \iota \psi(s, t)}{s + \iota t}, \end{aligned}$$

where we used Cauchy-Riemann equations in the final step and thus,

$$\lim_{s+\iota t \rightarrow 0} \frac{f(z+s+\iota t) - f(z)}{s + \iota t} = u_x(x, y) + \iota v_x(x, y),$$

completing the proof. Since  $u_x$  and  $v_x$  are continuous,  $f'$  is continuous and  $f$  is analytic. ■

A next question is: given some  $u$  such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

when does there exist harmonic  $v$  such that  $u + iv$  is analytic? Such a  $v$  is referred to as a *harmonic conjugate* of  $u$ . It turns out that the answer is not always. Indeed,  $u(x, y) = \log((x^2 + y^2)^{1/2})$  on  $\mathbb{C} \setminus \{0\}$ , despite being harmonic, does not have a harmonic conjugate.

**Theorem 1.11.** Let  $G$  be either the entirety of  $\mathbb{C}$  or some open disk. If  $u : G \rightarrow \mathbb{R}$  is a harmonic function, then  $u$  has a harmonic conjugate.

*Proof.* Let  $G = B(0, R)$  for some  $0 < R \leq \infty$  and let  $u : G \rightarrow \mathbb{R}$  be analytic. Define

$$v(x, y) = \int_0^y u_x(x, t) dt + \varphi(x)$$

so that  $u_x = v_y$ . We shall determine  $\varphi$  such that  $v_x = -u_y$ . Differentiating with respect to  $x$ , we get

$$\begin{aligned} v_x(x, y) &= \int_0^y u_{xx}(x, t) dt + \varphi'(x) \\ &= - \int_0^y u_{yy}(x, t) dt + \varphi'(x) \\ &= -u_y(x, y) + u_y(x, 0) + \varphi'(x). \end{aligned}$$

Therefore,  $\varphi'(x) = -u_y(x, 0)$ , and the function

$$v(x, y) = \int_0^y u_x(x, t) dt - \int_0^x u_y(s, 0) ds$$

is a harmonic conjugate of  $u$ . ■

The above proof requires that the entire segments  $[(0, 0), (x, 0)]$   $[(x, 0), (x, y)]$  are contained in  $G$ , which is true when we are on a disk.

## 1.6. Transformations

Consider the two hyperbolas defined by

$$\begin{aligned} x^2 - y^2 &= c \\ 2xy &= d, \end{aligned}$$

where  $c, d \neq 0$ .

This gives

$$y = \pm \sqrt{\frac{-c \pm \sqrt{d^2 + c^2}}{2}}.$$

Consider the functions

$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy. \end{aligned}$$

The two hyperbolas above are mapped by this  $f = u + iv$  to the straight lines  $u = c$  and  $v = d$ .

**Definition 1.6.** A *path* in a region  $G \subseteq \mathbb{C}$  is a continuous function  $\gamma : [a, b] \rightarrow G$  for some interval  $[a, b]$  in  $\mathbb{R}$ . If  $\gamma'(t)$  exists for each  $t \in [a, b]$  and  $\gamma' : [a, b] \rightarrow \mathbb{C}$  is continuous, then  $\gamma$  is said to be *smooth*.  $\gamma$  is said to be *piecewise smooth* if there is a partition  $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$  of  $[a, b]$  such that  $\gamma$  is smooth on each subinterval  $[t_{i-1}, t_i]$  for  $1 \leq i \leq n$ .

For a path  $\gamma : [a, b] \rightarrow \mathbb{C}$ ,  $\gamma([a, b])$  is sometimes referred to as the *trace* of  $\gamma$  and denoted  $\{\gamma\}$ .

By the existence of  $\gamma'$ , we mean that the two-sided limit

$$\lim_{h \rightarrow 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

exists for  $t \in (a, b)$  and the right and left sided limits exist for  $t = a, b$  respectively. This is equivalent to saying that  $\Re \gamma$  and  $\Im \gamma$  have derivatives.

Suppose  $\gamma : [a, b] \rightarrow G$  is a smooth path and for some  $t_0 \in (a, b)$ ,  $\gamma'(t_0) \neq 0$ . Then,  $\gamma$  has a *tangent line* at the point  $z_0 = \gamma(t_0)$ . This line goes through the point  $z_0$  in the direction of the vector  $\gamma'(t_0)$ , that is, the slope of the line is  $\tan(\arg \gamma'(t_0))$ .

If  $\gamma_1$  and  $\gamma_2$  are two smooth paths with  $\gamma_1(t_1) = \gamma_2(t_2) = z_0$  and  $\gamma'_1(t_1), \gamma'_2(t_2) \neq 0$ , then define the *angle* between the paths  $\gamma_1, \gamma_2$  at  $z_0$  to be  $\arg(\gamma'_2(t_2)) - \arg(\gamma'_1(t_1))$ .

Suppose  $\gamma$  is a smooth path in  $G$  and  $f : G \rightarrow \mathbb{C}$  is analytic. Then,  $\sigma = f \circ \gamma$  is also a smooth path and  $\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t)$ . Further, if  $z_0$  is a fixed point of  $f$  with  $\gamma(t_0) = z_0$ ,

$$\arg(\sigma'(t_0)) - \arg(\gamma'(t_0)) = \arg(f'(z_0)).$$

Let  $\gamma_1, \gamma_2$  be smooth paths with  $\gamma_1(t_1) = \gamma_2(t_2) = z_0$  with non-zero derivatives at  $t_1, t_2$  respectively, and let  $\sigma_1 = f \circ \gamma_1, \sigma_2 = f \circ \gamma_2$ . Further suppose that the two paths  $\gamma_1, \gamma_2$  are not tangent to each other at  $z_0$ . Then,

$$\arg(\gamma'_2(t_2)) - \arg(\gamma'_1(t_1)) = \arg(\sigma'_2(t_2)) - \arg(\sigma'_1(t_1)).$$

This says that the angle between two paths are preserved after applying an analytic function to both. A function  $f$  satisfying this is said to have the *angle-preserving property*.

**Definition 1.7.** A function  $f : G \rightarrow \mathbb{C}$  which has the angle-preserving property and also has

$$\lim_{z \rightarrow a} \left| \frac{f(z) - f(a)}{z - a} \right|$$

existing for all  $a \in G$  is called a *conformal map*.

It turns out that a function  $f$  is a conformal map if and only if it is analytic and  $f'(z) \neq 0$  for all  $z$  (How?).

**Definition 1.8.** A mapping of the form

$$S(z) = \frac{az + b}{cz + d}$$

is called a *linear fractional transformation*. If we further have that  $ad - bc \neq 0$ , then  $S(z)$  is called a *Möbius transformation*.

We have

$$S'(z) = \frac{ad - bc}{(cz + d)^2}.$$

If  $w = S(z)$ , it is relatively simple to show that

$$z = S^{-1}(w) = \frac{dw - b}{-cw + a}.$$

Therefore, the inverse of a Möbius transformation is a Möbius transformation. The composition of two Möbius transformations is a Möbius transformation as well.

Also observe that the coefficients  $a, b, c, d$  for a given Möbius transformation are not unique since we can multiply them by a constant. We may also extend  $S$  to  $\mathbb{C}_\infty$  with  $S(\infty) = a/c$  and  $S(-d/c) = \infty$ .

$S(z) = z + a$  is called a *translation*,  $S(z) = az$  with  $a \neq 0$  is called a *dilation*,  $S(z) = e^{i\theta}z$  is called a *rotation*, and  $S(z) = 1/z$  is called the *inversion*. We shall see later that any Möbius transformation is a composition of these five types of transformations.

What are the fixed points of a Möbius transformation  $S$ ?  $S(z) = z$  gives

$$cz^2 + (a - d)z + b = 0.$$

Therefore, a Möbius transformation has at most two fixed points unless  $S(z) = z$  for all  $z \in \mathbb{C}_\infty$ .

Let  $a, b, c \in \mathbb{C}_\infty$  be distinct with  $S(a) = \alpha$ ,  $S(b) = \beta$ ,  $S(c) = \gamma$ . Let  $T$  be another Möbius transformation with  $T(a) = \alpha$ ,  $T(b) = \beta$ ,  $T(c) = \gamma$ . Then  $T^{-1} \circ S$  has three (distinct) fixed points, and therefore  $S = T$ .

Therefore, any Möbius transformation is uniquely determined by its value at any three distinct points.

Let  $z_2, z_3, z_4 \in \mathbb{C}_\infty$  be distinct. Define  $S : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  by

$$S(z) = \begin{cases} \frac{(z - z_3)/(z - z_4)}{(z_2 - z_3)/(z_2 - z_4)}, & z_2, z_3, z_4 \in \mathbb{C}, \\ \frac{z_2 - z_4}{z - z_4}, & z_3 = \infty, \\ \frac{z - z_3}{z_2 - z_3}, & z_4 = \infty. \end{cases}$$

In any case,  $S(z_2) = 1$ ,  $S(z_3) = 0$ ,  $S(z_4) = \infty$ , and  $S$  is the only transformation having this property.

**Definition 1.9.** If  $z_1 \in \mathbb{C}_\infty$ , then  $(z_1, z_2, z_3, z_4)$  is referred to as the *cross-ratio* of  $z_1, z_2, z_3, z_4$  and is the image of  $z_1$  under the Möbius transformation described above, which is the unique Möbius transformation taking  $z_2$  to 1,  $z_3$  to 0, and  $z_4$  to  $\infty$ .

For example,  $(z_2, z_2, z_3, z_4) = 1$  and  $(z, 1, 0, \infty) = z$ .

If  $M$  is any Möbius transformation with  $M(w_2) = 1$ ,  $M(w_3) = 0$ ,  $M(w_4) = \infty$ , then  $M(z) = (z, w_2, w_3, w_4)$  for all  $z \in \mathbb{C}_\infty$ .

**Theorem 1.12.** If  $z_2, z_3, z_4$  are distinct points and  $T$  is any Möbius transformation, then

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4).$$

*Proof.* Let  $S(z) = (z, z_2, z_3, z_4)$ . If  $M = ST^{-1}$ , then

$$M(T(z_2)) = 1, \quad M(T(z_3)) = 0, \quad M(T(z_4)) = \infty.$$

Therefore,  $M = (z, Tz_2, Tz_3, Tz_4)$ . That is,

$$ST^{-1}z = (z, Tz_2, Tz_3, Tz_4)$$

for all  $z \in \mathbb{C}_\infty$ . Setting  $z = Tz_1$  yields the required. ■

**Lemma 1.13.** If  $\{z_2, z_3, z_4\}, \{w_2, w_3, w_4\} \subseteq \mathbb{C}_\infty$ , then there exists a unique Möbius transformation  $S$  with  $Sz_i = w_i$  for each  $i$ .

We omit the proof of the above.

**Lemma 1.14.** Let  $\{z_1, z_2, z_3, z_4\} \subseteq \mathbb{C}_\infty$ . Then,  $(z_1, z_2, z_3, z_4)$  is real iff the four points lie on a circle.

*Proof.* Define  $S : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  by  $Sz = (z, z_2, z_3, z_4)$ . We are done if we show that  $S^{-1}(\mathbb{R}_\infty)$  is a circle (since a circle is uniquely determined by three distinct points on it).

Let  $S(z) = (az + b)/(cz + d)$ .

First, let us show that  $S^{-1}(\mathbb{R}_\infty) \subseteq \Gamma$  for a circle  $\Gamma$  in  $\mathbb{C}_\infty$ . Let  $w \in S^{-1}(\mathbb{R}_\infty)$ . Then,  $Sw = \overline{Sw}$  so

$$\frac{aw + b}{cw + d} = \frac{\overline{aw + b}}{\overline{cw + d}}.$$

This gives that

$$(a\bar{c} - \bar{a}c)|w|^2 + (a\bar{d} - \bar{a}d)w + (b\bar{c} - \bar{b}c)\bar{w} + (b\bar{d} - \bar{b}d) = 0. \quad (*)$$

If  $a\bar{c}$  is real, we get that

$$\Im \left( (a\bar{d} - \bar{a}d)w + b\bar{d} \right) = 0,$$

which is a circle through  $\infty$  (a line).

If on the other hand  $a\bar{c}$  is not real, then  $(*)$  becomes

$$2\iota \underbrace{\Im(a\bar{c})}_{\alpha \neq 0} |w|^2 + (a\bar{d} - \bar{a}d)w + (b\bar{c} - \bar{b}c)\bar{w} + (b\bar{d} - \bar{b}d) = 0.$$

Dividing by  $2\iota\alpha$ ,

$$|w|^2 + \frac{(a\bar{d} - \bar{a}d)w}{2\iota\alpha} + \frac{(b\bar{c} - \bar{b}c)\bar{w}}{2\iota\alpha} + \frac{(b\bar{d} - \bar{b}d)}{2\iota\alpha} = 0.$$

Since  $\alpha$  is real,

$$\frac{\overline{(b\bar{c} - \bar{b}c)\bar{w}}}{2\iota\alpha} = \frac{(a\bar{d} - \bar{a}d)w}{2\iota\alpha}$$

and

$$\frac{(b\bar{d} - \bar{b}d)}{2\iota\alpha}$$

is real. This gives

$$|w|^2 + \bar{\gamma}w + \gamma\bar{w} - \delta = 0$$

for some  $\gamma \in \mathbb{C}, \delta \in \mathbb{R}$ . This is equivalent to  $|w + \gamma| = (|\gamma|^2 + \delta)^{1/2}$ , which is the equation of a circle<sup>1</sup>.

Letting  $T = S^{-1}$  and  $\Gamma$  be the circle obtained in the previous part of the proof, we must now show that  $T(\mathbb{R}_\infty) = \Gamma$ . Since  $\mathbb{R}_\infty$  is connected and compact and  $T$  is a homeomorphism,  $T(\mathbb{R}_\infty)$  is a closed arc, say  $\Gamma_1$ , of  $\Gamma$ . If  $\Gamma_1 \neq \Gamma$ , let  $z_1, z_2$  be the endpoints of this arc. If  $T(\infty) = z_3$  which is distinct from  $z_1, z_2$ , then  $\mathbb{R}_\infty \setminus \{\infty\}$  is connected but  $\Gamma_1 \setminus \{z_1\}$  is disconnected, which is a contradiction. So, suppose  $T(\infty) = z_1$ . Then,  $\mathbb{R}_\infty \setminus \{\infty, T^{-1}(z_2)\}$  is disconnected but  $\Gamma_1 \setminus \{z_1, z_2\}$  is connected, yielding a contradiction once more and completing the proof. ■

Next, we give a more general version of the above.

**Theorem 1.15.** A Möbius transformation takes circles to circles.

Note that Lemma 1.14 follows from this since  $\mathbb{R}_\infty$  is a circle (of infinite radius) in  $\mathbb{C}_\infty$ .

<sup>1</sup>it may be checked that  $|\gamma|^2 + \delta$  is a positive real by substituting their values.

*Proof.* Let  $\Gamma$  be a circle in  $\mathbb{C}_\infty$  and  $S$  a Möbius transformation. Let  $z_2, z_3, z_4$  be three distinct points on  $\Gamma$ , and set  $w_j = Sz_j$  for each  $j$ . We claim that  $S(\Gamma)$  is the circle  $\Gamma'$  determined by  $w_2, w_3, w_4$ . Indeed,

$$(z, z_2, z_3, z_4) = (Sz, w_2, w_3, w_4)$$

for any  $z$ , and if  $z \in \Gamma$ , the LHS is real by Lemma 1.14, and using the same theorem on the RHS completes the proof.  $\blacksquare$

**Definition 1.10.** Let  $\Gamma$  be a circle through  $z_2, z_3, z_4$ . The points  $z, z^* \in \mathbb{C}_\infty$  are said to be *symmetric* with respect to  $\Gamma$  if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

*Remark.* The above definition only depends on  $\Gamma$ , not the choice of  $z_2, z_3, z_4$ .

Observe that  $z$  is symmetric with respect to itself with respect to  $\Gamma$  if and only if  $z \in \Gamma$ . Indeed, it implies that  $(z, z_2, z_3, z_4)$  is real, which by Lemma 1.14 implies that  $z \in \Gamma$ .

What does it mean for  $z, z^*$  to be symmetric?

If  $\Gamma$  is a straight line,  $z, z^*$  are symmetric with respect to  $\Gamma$  iff their perpendicular bisector is equal to  $\Gamma$ . That is, the line joining  $z, z^*$  is perpendicular to  $\Gamma$  and they are the same distance from  $\Gamma$  (but on opposite sides). Indeed, choosing  $z_4 = \infty$ , we get that

$$\frac{z^* - z_3}{z_2 - z_3} = \frac{\bar{z} - \bar{z}_3}{\bar{z}_2 - \bar{z}_3},$$

so

$$|z - z_3| = |z^* - z_3|$$

for all  $z_3 \in \Gamma$ .

Now, suppose that  $\Gamma = \{z : |z - a| = R\}$  for some  $0 < R < \infty$ . We extensively use Theorem 1.12 and the five types of Möbius translations in the following sequence of equations. Then,

$$\begin{aligned} (z^*, z_2, z_3, z_4) &= \overline{(z, z_2, z_3, z_4)} \\ &= \overline{(z - a, z_2 - a, z_3 - a, z_4 - a)} \\ &= \left( \bar{z} - \bar{a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}, \frac{R^2}{z_4 - a} \right) \\ &= \left( \frac{R^2}{\bar{z} - \bar{a}}, z_2 - a, z_3 - a, z_4 - a \right) \\ &= \left( \frac{R^2}{\bar{z} - \bar{a}} + a, z_2, z_3, z_4 \right). \end{aligned}$$

Therefore,  $z^* = a + \frac{R^2}{\bar{z} - \bar{a}}$ , that is,

$$(z^* - a)(\bar{z} - \bar{a}) = R^2.$$

Since

$$\frac{z^* - a}{z - a} = \frac{R^2}{|z - a|^2} > 0$$

is real, it follows that  $z^*$  is on the ray  $\{a + t(z - a) : 0 < t < \infty\}$ . We also have that

$$|z^* - a||z - a| = R^2,$$

so one can easily obtain  $z^*$  from  $z$  or vice-versa.



**Lemma 1.16** (Symmetry Principle). If a Möbius transformation takes a circle  $\Gamma_1$  to the circle  $\Gamma_2$ , then any pair of points symmetric with respect to  $\Gamma_1$  is mapped to a pair of points symmetric with respect to  $\Gamma_2$ .

*Proof.* The proof of this is near-direct.

$$\begin{aligned} (Tz, Tz_2, Tz_3, Tz_4) &= (z^*, z_2, z_3, z_4) \\ &= \overline{(z, z_2, z_3, z_4)} \\ &= \overline{(Tz, Tz_2, Tz_3, Tz_4)}. \end{aligned}$$

■

**Definition 1.11.** If  $\Gamma$  is a circle, then an *orientation* for  $\Gamma$  is an ordered triple  $(z_1, z_2, z_3)$  of points in  $\Gamma$ .

An orientation is used to represent a “direction” of the circle, where we “go” from  $z_1$  to  $z_2$  to  $z_3$ .

Let  $\Gamma = \mathbb{R}$  and  $z_1, z_2, z_3 \in \mathbb{R}$ . Also put  $Tz = (z, z_1, z_2, z_3)$ . Since  $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ ,  $a, b, c, d$  can be chosen to be reals. Then,

$$\begin{aligned} Tz &= \frac{az + b}{cz + d} \\ &= \frac{az + b}{|cz + d|^2} (c\bar{z} + d) \\ &= \frac{1}{|cz + d|^2} (ac|z|^2 + bd + bc\bar{z} + adz). \end{aligned}$$

So,

$$\Im(z, z_1, z_2, z_3) = \frac{ad - bc}{|cz + d|^2} \Im z$$

and thus,  $\{z : \Im(z, z_1, z_2, z_3) > 0\}$  is either the upper or lower half-plane depending on whether  $ad - bc$  is negative or positive. Note that  $ad - bc$  is the determinant of  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

Let  $\Gamma$  be an arbitrary circle and suppose that  $z_1, z_2, z_3 \in \Gamma$ . Then, for any Möbius transformation  $S$ ,

$$\begin{aligned} \{z : \Im(z, z_1, z_2, z_3) > 0\} &= \{z : \Im(Sz, Sz_1, Sz_2, Sz_3) > 0\} \\ &= S^{-1}\{z : \Im(z, Sz_1, Sz_2, Sz_3) > 0\}. \end{aligned}$$

So, if  $S$  is chosen to map  $\Gamma$  to  $\mathbb{R}_\infty$ , then the above set is equal to  $S^{-1}$  of either the upper or lower halfspace.

**Definition 1.12.** If  $z_1, z_2, z_3$  is an orientation of  $\Gamma$ , we denote the *right side* and *left side* of  $\Gamma$  (with respect to  $(z_1, z_2, z_3)$ ) to be

$$\{z : \Im(z, z_1, z_2, z_3) > 0\} \text{ and } \{z : \Im(z, z_1, z_2, z_3) < 0\}$$

respectively.

**Theorem 1.17** (Orientation Principle). Let  $\Gamma_1, \Gamma_2$  be circles in  $\mathbb{C}_\infty$  such that  $T\Gamma_1 = \Gamma_2$  for some Möbius transformation  $T$ . Let  $(z_1, z_2, z_3)$  be an orientation of  $\Gamma_1$ . Then,  $T$  takes the right side (resp. left side) of  $\Gamma_1$  with respect to the orientation  $(z_1, z_2, z_3)$  to the right side (resp. left side) of  $\Gamma_2$  with respect to the orientation  $(Tz_1, Tz_2, Tz_3)$ .

The proof of the above is left as an exercise to the reader.

Since  $(z, 1, 0, \infty) = z$  by definition, the right side of  $\mathbb{R}_\infty$  with respect to the orientation  $(1, 0, \infty)$  is the upper half-plane.

**Exercise 1.4.** Find an analytic function  $f : G \rightarrow \mathbb{C}$  where  $G = \{z : \Re z > 0\}$ , such that  $f(G) = \{z : |z| < 1\}$ .

Similar to the above exercise, one may show that

$$g(z) = \frac{e^z - 1}{e^z + 1}$$

maps the infinite strip  $\{z : |\Im z| < \pi/2\}$  to the open unit disk  $D$ .

## §2. Integration

### 2.1. Basic definitions

#### 2.1.1. Integrals of real functions

First, let us recall the definition of the Riemann integral<sup>2</sup> of functions on  $\mathbb{R}$ .

**Definition 2.1** (Riemann Integral). Let  $[a, b]$  be a given interval. A *partition*  $\mathcal{P}$  of  $[a, b]$  is a finite set of points  $x_0, x_1, \dots, x_n$  where

$$a = x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n = b.$$

We also write  $\Delta x_i = x_i - x_{i-1}$  for  $i = 1, 2, \dots, n$ .

For a bounded real function  $f$  on  $[a, b]$  and each partition  $\mathcal{P}$  of  $[a, b]$ , we set

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x), \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x).$$

Further, set

$$U(\mathcal{P}, f) = \sum_{i=1}^n M_i \Delta x_i, \quad L(\mathcal{P}, f) = \sum_{i=1}^n m_i \Delta x_i$$

as the upper and lower Riemann sum respectively, and finally,

$$\overline{\int_a^b} f \, dx = \inf_{\mathcal{P}} U(\mathcal{P}, f), \quad \underline{\int_a^b} f \, dx = \sup_{\mathcal{P}} L(\mathcal{P}, f)$$

as the upper and lower Riemann integrals of  $f$ .

Next, we define the slightly more general Riemann-Stieltjes integral. Note that this is the same as the usual Riemann integral when  $\alpha$  is the identity function.

**Definition 2.2** (Riemann-Stieltjes Integral). Let  $\alpha : [a, b] \rightarrow \mathbb{R}$  be a monotonically increasing function on  $[a, b]$ . Corresponding to each partition  $\mathcal{P}$  of  $[a, b]$ , write  $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ . Clearly,  $\Delta \alpha_i \geq 0$  for each  $i$ . For any real function  $f$  which is bounded on  $[a, b]$ , we put

$$U(\mathcal{P}, f, \alpha) = \sum_{i=1}^n M_i \Delta \alpha_i, \quad L(\mathcal{P}, f, \alpha) = \sum_{i=1}^n m_i \Delta \alpha_i,$$

where  $M_i, m_i$  are defined as in the definition of the Riemann integral. We then define

$$\overline{\int_a^b} f \, d\alpha = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha), \quad \underline{\int_a^b} f \, d\alpha = \sup_{\mathcal{P}} L(\mathcal{P}, f, \alpha).$$

If these two are equal, we say that  $f$  is *Riemann-Stieltjes integrable* with respect to  $\alpha$  on  $[a, b]$  and denote the common value as  $\int_a^b f \, d\alpha$ .

---

<sup>2</sup>technically the Darboux integral?

We also remark that

$$\int_a^b f \, d\alpha = \lim_{\max \Delta\alpha_k \rightarrow 0} \sum_{k=1}^n f(\tau_k) \Delta\alpha_k,$$

where  $x_{k-1} \leq \tau_k \leq x_k$  for each  $k$ .

More generally, we define the *mesh* of  $\mathcal{P}$  with respect to  $\alpha$  as

$$\|\mathcal{P}\| = \max\{\Delta\alpha_i : 1 \leq i \leq n\}.$$

So for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any partition  $\mathcal{P}$  of  $[a, b]$  with  $\|\mathcal{P}\| < \delta$ , then

$$\left| \sum_{k=1}^n f(\tau_k) \Delta\alpha_k - \int_a^b f \, d\alpha \right| < \epsilon$$

for any choice of points  $x_{k-1} \leq \tau_k \leq x_k$ .

### 2.1.2. Riemann-Stieltjes integrals of complex-valued functions

**Definition 2.3.** A function  $\gamma : [a, b] \rightarrow \mathbb{C}$  for  $[a, b] \subseteq \mathbb{R}$  is said to be of *bounded variation* if there exists  $M > 0$  such that for any partition  $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b\}$  of  $[a, b]$ ,

$$v(\gamma; \mathcal{P}) = \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \leq M.$$

The *total variation*  $V(\gamma)$  of  $\gamma$  is defined by

$$V(\gamma) = \sup\{v(\gamma; \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}.$$

Clearly,  $V(\gamma) \leq M < \infty$ .

**Lemma 2.1.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be of bounded variation. Then,

1. If  $\mathcal{P}, \mathcal{Q}$  are partitions of  $[a, b]$  with  $\mathcal{P} \subseteq \mathcal{Q}$ , then  $v(\gamma; \mathcal{P}) \leq v(\gamma; \mathcal{Q})$ .
2. If  $\sigma : [a, b] \rightarrow \mathbb{C}$  is also of bounded variation and  $\alpha, \beta \in \mathbb{C}$ , then  $\alpha\gamma + \beta\sigma$  is of bounded variation and

$$V(\alpha\gamma + \beta\sigma) \leq |\alpha|V(\gamma) + |\beta|V(\sigma).$$

We omit the proof of the above, which is direct on using the triangle inequality on the definition of  $v(\gamma; \mathcal{P})$ .

**Lemma 2.2.** If  $\gamma : [a, b] \rightarrow \mathbb{C}$  is piecewise smooth,  $\gamma$  is of bounded variation and

$$V(\gamma) = \int_a^b |\gamma'(t)| \, dt.$$

*Proof.* It suffices to show the required in the case where  $\gamma$  is smooth, since in general we can consider the refinement of any partition that splits along the pieces along which  $\gamma$  is smooth.

The right hand side is well-defined since  $\gamma'$  is continuous. Let  $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b\}$ . By definition,

$$\begin{aligned} v(\gamma, \mathcal{P}) &= \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) \, dt \right| \\ &\leq \sum_{k=1}^m \int_{t_{k-1}}^{t_k} |\gamma'(t)| \, dt = \int_a^b |\gamma'(t)| \, dt. \end{aligned}$$

Therefore,  $V(\gamma) \leq \int_a^b |\gamma'(t)| dt$ , so  $\gamma$  is of bounded variation.

Since  $\gamma'$  is continuous, it is uniformly continuous. So, if  $\epsilon > 0$ , we may choose  $\delta_1 > 0$  such that

$$|s - t| < \delta_1 \implies |\gamma'(s) - \gamma'(t)| < \epsilon.$$

Also, let  $\delta_2 > 0$  such that if  $\|P\| < \delta_2$ , then

$$\left| \int_a^b |\gamma'(t)| dt - \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \epsilon,$$

where  $\tau_k$  is any point in  $[t_{k-1}, t_k]$ . Therefore,

$$\begin{aligned} \int_a^b |\gamma'(t)| dt &\leq \epsilon + \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \\ &= \epsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k) dt \right| \\ &\leq \epsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} (\gamma'(\tau_k) - \gamma'(t)) dt \right| + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) dt \right|. \end{aligned}$$

If  $\|P\| < \delta = \min(\delta_1, \delta_2)$ , then  $|\gamma'(\tau_k) - \gamma'(t)| < \epsilon$  for all  $t \in [t_{k-1}, t_k]$  and

$$\begin{aligned} \left| \int_a^b |\gamma'(t)| dt \right| &\leq \epsilon + \epsilon(b - a) + \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \epsilon(1 + b - a) + V(\gamma; P) \leq \epsilon(1 + b - a) + V(\gamma), \end{aligned}$$

so we are done since  $1 + b - a > 0$  is finite and  $\epsilon$  can be made arbitrarily small. ■

**Theorem 2.3.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be of bounded variation and suppose that  $f : [a, b] \rightarrow \mathbb{C}$  is continuous. Then, there exists a (unique) complex number  $\mathcal{I}$  such that for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that when  $\mathcal{P} = \{t_0 < t_1 < \dots < t_m\}$  is a partition of  $[a, b]$  with  $\|P\| = \max_{1 \leq k \leq m} (t_k - t_{k-1}) < \delta$ ,

$$\left| \mathcal{I} - \sum_{k=1}^m f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \right| < \epsilon$$

for any choice of points  $\tau_k$  with  $t_{k-1} \leq \tau_k \leq t_k$ .

This  $\mathcal{I}$  is called the integral of  $f$  with respect to  $\gamma$  over  $[a, b]$  and is denoted by

$$\mathcal{I} = \int_a^b f d\gamma = \int_a^b f(t) d\gamma(t).$$

*Proof.* First of all, note that it suffices to consider the case where  $\gamma$  is real-valued, since we can write  $\gamma = \gamma_1 + i\gamma_2$ , where  $\gamma_1, \gamma_2$  are real-valued, to get two integrals  $\mathcal{I}_1, \mathcal{I}_2$  (for  $\gamma_1, \gamma_2$  respectively), and finally use the triangle inequality to get  $\mathcal{I} = \mathcal{I}_1 + i\mathcal{I}_2$ .

Since  $f$  is continuous, it is uniformly continuous. We can (inductively) find positive numbers  $\delta_1 > \delta_2 > \dots$  such that if  $|s - t| < \delta_m$ ,  $|f(s) - f(t)| < 1/m$ . For each  $M \geq 1$ , let  $\mathcal{P}_m$  be the collection of all partitions  $P$  of  $[a, b]$  with  $\|P\| \leq \delta_m$ , so  $\mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \dots \supseteq \mathcal{P}_m \supseteq \dots$ . Finally, define  $F_m$  to be the closure of the set

$$\left\{ \sum_{k=1}^n f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) : P \in \mathcal{P}_m \text{ and } t_{k-1} \leq \tau_k \leq t_k \right\}.$$

Because  $\mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \cdots$ , it follows trivially that

$$F_1 \supseteq F_2 \supseteq \cdots.$$

We claim that

$$\text{diam } F_m \leq \frac{2}{m} V(\gamma). \quad (2.1)$$

If we do this, then Cantor's Theorem (since  $\mathbb{C}$  is complete) implies that there is precisely one complex number  $\mathcal{I}$  such that  $\mathcal{I} \in F_m$  for all  $m \geq 1$ . Then, for any  $\epsilon > 0$ , we may let  $m > (2/\epsilon)V(\gamma)$  so  $\epsilon > (2/m)V(\gamma) \geq \text{diam } F_m$ . Since  $\mathcal{I} \in F_m$ ,  $F_m \subseteq B(\mathcal{I}, \epsilon)$ . Therefore,  $\delta = \delta_m$  gets the job done.

So, we must show that

$$\text{diam} \left\{ f(\tau_k) (\gamma(t_k) - \gamma(t_{k-1})) : P \in \mathcal{P}_m \text{ and } t_{k-1} \leq \tau_k \leq t_k \right\} \leq \frac{2}{m} V(\gamma).$$

To do this, if  $P = \{t_0 < \cdots < t_n\}$  is a partition, denote by  $S(P)$  a sum of the form  $\sum f(\tau_k) (\gamma(t_k) - \gamma(t_{k-1}))$  where  $t_{k-1} \leq \tau_k \leq t_k$  for each  $k$ . Fixing  $m \geq 1$ , let  $P \in \mathcal{P}_m$ . If  $P \subseteq Q$  (so  $Q \in \mathcal{P}_m$  as well), then

$$|S(P) - S(Q)| < \frac{1}{m} V(\gamma).$$

We only show this in the case where  $Q$  is obtained from  $P$  by adding a single extra partition point (the general case follows similarly). Let  $Q = \{t_0 < t_1 < \cdots < t_{p-1} < t^* < t_p < \cdots < t_n\}$ . If  $t_{p-1} \leq \sigma \leq t^*$  and  $t^* \leq \sigma' \leq t_p$ . Then,

$$S(Q) = \sum_{k \neq p} f(\sigma_k) (\gamma(t_k) - \gamma(t_{k-1})) + f(\sigma) (\gamma(t^*) - \gamma(t_{p-1})) + f(\sigma') (\gamma(t_p) - \gamma(t^*)).$$

Then, using the definition of  $\delta_m$ ,

$$\begin{aligned} |S(P) - S(Q)| &= \left| \sum_{k \neq p} (f(\tau_k) - f(\sigma_k)) (\gamma(t_k) - \gamma(t_{k-1})) \right. \\ &\quad \left. + f(\tau_p) (\gamma(t_p) - \gamma(t_{p-1})) - f(\sigma) (\gamma(t^*) - \gamma(t_{p-1})) - f(\sigma') (\gamma(t_p) - \gamma(t^*)) \right| \\ &\leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k-1})| + \left| (f(\tau_p) - f(\sigma)) (\gamma(t^*) - \gamma(t_{p-1})) + (f(\tau_p) - f(\sigma')) (\gamma(t_p) - \gamma(t^*)) \right| \\ &\leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k-1})| + \frac{1}{m} |\gamma(t^*) - \gamma(t_{p-1})| + \frac{1}{m} |\gamma(t_p) - \gamma(t^*)| \\ &\leq \frac{1}{m} V(\gamma). \end{aligned}$$

Next, let  $P, R$  be any two partitions in  $\mathcal{P}_m$ , and  $Q = P \cup R$  a partition that contains  $P$  and  $R$ . Using the first part,

$$|S(P) - S(Q)| \leq |S(P) - S(Q)| + |S(Q) - S(R)| \leq \frac{2}{m} V(\gamma).$$

It follows that the diameter of the set of interest is at most  $(2/m)V(\gamma)$ , completing the proof. ■

**Theorem 2.4.** Let  $f, g$  be continuous functions on  $[a, b]$  and let  $\gamma, \sigma$  be functions of bounded variation on  $[a, b]$ . Then for any scalars  $\alpha, \beta$ ,

$$\begin{aligned} \int_a^b (\alpha f + \beta g) d\gamma &= \alpha \int_a^b f d\gamma + \beta \int_a^b g d\gamma \\ \int_a^b f d(\alpha \gamma + \beta \sigma) &= \alpha \int_a^b f d\gamma + \beta \int_a^b f d\sigma. \end{aligned}$$

**Proposition 2.5.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be of bounded variation and let  $f : [a, b] \rightarrow \mathbb{C}$  be continuous. If  $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ , then

$$\int_a^b f \, d\gamma = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f \, d\gamma.$$

We omit the proofs of the above.

**Theorem 2.6.** If  $\gamma$  is piecewise smooth and  $f : [a, b] \rightarrow \mathbb{C}$  is continuous, then  $\int_a^b f \, d\gamma = \int_a^b f(t)\gamma'(t) \, dt$ .

*Proof.* It suffices to consider the case where  $\gamma$  is smooth by Proposition 2.5. Also, by looking at the real and imaginary parts of  $\gamma$  separately, it suffices to consider the case where  $\gamma$  is real-valued on  $[a, b]$ . Let  $\epsilon > 0$  and choose  $\delta > 0$  such that if  $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$  has  $\|P\| < \delta$ , then

$$\left| \int_a^b f \, d\gamma - \sum_{k=1}^n f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \right| < \epsilon/2$$

and

$$\left| \int_a^b f(t)\gamma'(t) \, dt - \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \epsilon/2$$

for any  $t_{k-1} \leq \tau_k \leq t_k$  for each  $k$ .

Applying the mean value theorem on  $\gamma$  (this requires that  $\gamma$  be real-valued), one gets that there exists  $\tau_k \in [t_{k-1}, t_k]$  for each  $k$  such that

$$\gamma'(\tau_k) = \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}}.$$

Using these  $\tau_k$  specifically,

$$\left| \int_a^b f \, d\gamma - \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \epsilon/2,$$

so

$$\left| \int_a^b f \, d\gamma - \int_a^b f(t)\gamma'(t) \, dt \right| < \epsilon,$$

completing the proof. ■

## 2.2. Integrals On Curves

**Definition 2.4.**  $\gamma : [a, b] \rightarrow \mathbb{C}$  is called a *rectifiable path* if it is continuous and of bounded variation. Note that if  $\gamma$  is piecewise smooth, then it is rectifiable and its length is

$$\int_a^b |\gamma'(t)| \, dt = V(\gamma).$$

**Definition 2.5.** If  $\gamma : [a, b] \rightarrow \mathbb{C}$  is a rectifiable path and  $f$  is a function continuous on  $\{\gamma\}$ , then the (line) integral of  $f$  along  $\gamma$  is

$$\int_a^b f(\gamma(t)) d\gamma(t).$$

This line integral is also denoted as

$$\int_{\gamma} f = \int_{\gamma} f(z) dz.$$

For example, if  $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$  as  $\gamma(t) = e^{it}$ ,

$$\int_{\gamma} \frac{1}{z} dz = \int_0^{2\pi} e^{-it} (ie^{it}) dt = 2\pi i.$$

and

$$\int_{\gamma} z^m dz = \int_0^{2\pi} e^{imt} (ie^{it}) dt = i \int_0^{2\pi} \cos((m+1)t) dt - \int_0^{2\pi} \sin((m+1)t) dt = 0.$$

**Theorem 2.7.** If  $\gamma : [a, b] \rightarrow \mathbb{C}$  is a rectifiable path and  $\varphi : [c, d] \rightarrow [a, b]$  is a continuous non-decreasing function with  $\varphi(c) = a, \varphi(d) = b$ , then for any function  $f$  continuous on  $\gamma$ ,

$$\int_{\gamma} f = \int_{\gamma \circ \varphi} f.$$

*Remark.* The above uses the fact that  $\gamma \circ \varphi$  is also rectifiable (Why is this true?).

*Proof.* Let  $\epsilon > 0$  and choose  $\delta_1 > 0$  such that for a partition  $\{s_0 < s_1 < \dots < s_n\}$  of  $[c, d]$  with  $(s_k - s_{k-1}) < \delta_1$  and any  $s_{k-1} \leq \sigma_k \leq s_k$ ,

$$\left| \int_{\gamma \circ \varphi} f - \sum_{k=1}^n f((\gamma \circ \varphi)(s_k)) - f((\gamma \circ \varphi)(s_{k-1})) \right| < \epsilon/2.$$

Similarly, choose  $\delta_2 > 0$  such that for a partition  $\{t_0 < t_1 < \dots < t_m\}$  of  $[a, b]$  with  $(t_k - t_{k-1}) < \delta_2$  and  $t_{k-1} \leq \tau_k \leq t_k$ ,

$$\left| \int_{\gamma} f - \sum_{k=1}^m f(\gamma(t_k)) - f(\gamma(t_{k-1})) \right| < \epsilon/2.$$

Since  $\varphi$  is uniformly continuous on  $[c, d]$ , there exists  $\delta > 0$  less than  $\delta_1$  such that  $|\varphi(s) - \varphi(t)| < \delta_2$  whenever  $|s - t| < \delta$ . So, if  $\{s_0 < s_1 < \dots < s_n\}$  is a partition of  $[c, d]$  with  $(s_k - s_{k-1}) < \delta < \delta_1$  and  $t_k = \varphi(s_k)$ , then  $\{t_0 < t_1 < \dots < t_n\}$  is a partition of  $[a, b]$  with  $(t_k - t_{k-1}) < \delta_2$ . If  $s_{k-1} \leq \sigma_k \leq s_k$  and  $\tau_k = \varphi(\sigma_k)$ , then we can use the two earlier inequalities to conclude that

$$\left| \int_{\gamma} f - \int_{\gamma \circ \varphi} f \right| < \epsilon,$$

completing the proof. ■

**Definition 2.6.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a rectifiable path, and for  $a \leq t \leq b$ , set  $|\gamma|(t) = V(\gamma; [a, t])$ . That is,

$$|\gamma|(t) = \sup \left\{ \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| : \{t_0 < t_1 < \dots < t_n\} \text{ is a partition of } [a, t] \right\}.$$

Clearly,  $|\gamma|$  is increasing on  $[a, b]$  and of bounded variation. In fact,  $V(|\gamma|; [a, b]) = |\gamma|(b) - |\gamma|(a)$ . If  $f$  is continuous on  $[a, b]$ , define

$$\int f |dz| = \int_a^b f(\gamma(t)) d|\gamma|(t).$$



**Theorem 2.8.** Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a rectifiable curve and suppose that  $f$  is a function continuous on  $\{\gamma\}$ . Then,

$$\int_{\gamma} f = - \int_{-\gamma} f \quad (2.2)$$

where  $(-\gamma)(t) = \gamma(a + b - t)$ ,

$$\left| \int_{\gamma} f \right| \leq \int_{\gamma} |f| |dz| \leq V(\gamma) \sup\{|f(z)| : z \in \{\gamma\}\}, \quad (2.3)$$

and for  $c \in \mathbb{C}$ ,

$$\int_{\gamma} f(z) dz = \int_{\gamma+c} f(z - c) dz. \quad (2.4)$$

*Proof.* Equations (2.2) and (2.4) follow near-directly from the definition, so we prove only Equation (2.3). Let  $\epsilon > 0$ . Then, there exists  $\delta > 0$  such that if  $P = \{t_0 < t_1 < \dots < t_n\}$  is a partition of  $[a, b]$  with  $\|P\| < \delta$ , then

$$\left| \int_{\gamma} f(z) dz - \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) \right| \leq \left| \int_{\gamma} f(z) dz - \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) \right| < \epsilon/2$$

for any  $t_{k-1} \leq \tau_k \leq t_k$ . That is,

$$\begin{aligned} \left| \int_{\gamma} f(z) dz \right| &< \left| \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) \right| + \epsilon/2 \\ &\leq \sum_{k=1}^n |f(\gamma(\tau_k))| |\gamma(t_k) - \gamma(t_{k-1})| + \epsilon/2. \end{aligned}$$

We may also assume that for this same  $\delta$ ,

$$\sum_{k=1}^n |f(\gamma(t_k))| (|\gamma|(t_k) - |\gamma|(t_{k-1})) < \int_{\gamma} |f(z)| |dz| + \epsilon/2.$$

Recall that  $|\gamma|(t)$  is an increasing function. So,

$$|\gamma|(t_k) - |\gamma|(t_{k-1}) \geq |\gamma(t_k) - \gamma(t_{k-1})|$$

Therefore,

$$\begin{aligned} \left| \int_{\gamma} f(z) dz \right| &< \sum_{k=1}^n |f(\gamma(\tau_k))| (|\gamma|(t_k) - |\gamma|(t_{k-1})) + \epsilon/2 \\ &< \int_{\gamma} |f(z)| |dz| + \epsilon. \end{aligned}$$

It follows that

$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz|.$$

To conclude the proof, note that

$$\int_{\gamma} |dz| = |\gamma|(b) - |\gamma|(a) = |\gamma|(b) = V(\gamma),$$

so

$$\int_{\gamma} |f(z)| |dz| \leq V(\gamma) \sup_{z \in \{\gamma\}} |f(z)|.$$

■

**Lemma 2.9.** If  $G$  is an open set in  $\mathbb{C}$ ,  $\gamma : [a, b] \rightarrow G$  is a rectifiable path, and  $f : G \rightarrow \mathbb{C}$  is continuous, then for every  $\epsilon > 0$  there exists a polygonal path  $\Gamma$  in  $G$  such that  $\Gamma(a) = \gamma(a)$ ,  $\Gamma(b) = \gamma(b)$ , and

$$\left| \int_{\gamma} f - \int_{\Gamma} f \right| < \epsilon$$

*Proof.* We prove the result in the case where  $G$  is an open disk. In the general case where  $G$  need not be a disk, since  $\{\gamma\}$  is compact, there exists a number  $r$  with  $0 < r < d(\{\gamma\}, \partial G)$ . Choose  $\delta > 0$  such that  $|\gamma(s) - \gamma(t)| < r$  when  $|s - t| < \delta$ . The idea is that we shall take several smaller disks and stitch together the polygonal paths on each of these sections.

If  $P = \{t_0 < t_1 < \dots < t_n\}$  is a partition of  $[a, b]$  with  $\|P\| < \delta$ , then  $|\gamma(t_k) - \gamma(t_{k-1})| < r$  for  $t_{k-1} \leq t \leq t_k$ . That is, if  $\gamma_k : [t_{k-1}, t_k] \rightarrow G$  is defined by  $\gamma_k(t) = \gamma(t)$ , then  $\{\gamma_k\} \subseteq B(\gamma(t_{k-1}), r)$  for  $1 \leq k \leq n$ . Getting a polygonal path  $\Gamma_k$  for each  $k$  such that

$$\left| \int_{\gamma_k} f - \int_{\Gamma_k} f \right| < \epsilon/n,$$

defining  $\Gamma(t) = \Gamma_k(t)$  on  $[t_{k-1}, t_k]$  does the job.

Now, let us prove the result in the disk case.

Because  $\{\gamma\}$  is a compact set,  $d = d(\{\gamma\}, \partial G) > 0$ . It follows that if  $G = B(c, r)$ , then  $\{\gamma\} \subseteq B(c, \rho)$  where  $\rho = r - d/2$ .

Now, observe that  $f$  is uniformly continuous on  $\overline{B}(c, \rho) \subseteq G$ . Thus, we may assume without loss of generality that  $f$  is uniformly continuous on  $G$ . Now, choose  $\delta > 0$  such that if  $|z - w| < \delta$ , then  $|f(z) - f(w)| < \epsilon$ . If  $\gamma : [a, b] \rightarrow G$ , then  $\gamma$  is uniformly continuous so there is a partition  $P = \{t_0 < t_1 < \dots < t_n\}$  of  $[a, b]$  such that if  $t_{k-1} \leq s, t \leq t_k$ ,  $|\gamma(s) - \gamma(t)| < \delta$ , and such that for  $t_{k-1} \leq \tau_k \leq t_k$ ,

$$\left| \int_{\gamma} f - \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) \right| < \epsilon.$$

Now, define  $\Gamma : [a, b] \rightarrow G$  by

$$\Gamma(t) = \frac{(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k)}{t_k - t_{k-1}}$$

if  $t_{k-1} \leq t \leq t_k$ . This is the polygonal path we shall consider. Indeed,

$$\Gamma(t) - \gamma(\tau_k) = \frac{t_k - t}{t_k - t_{k-1}}(\gamma(t_{k-1}) - \gamma(\tau_k)) + \frac{t - t_{k-1}}{t_k - t_{k-1}}(\gamma(t_k) - \gamma(\tau_k)),$$

so

$$\begin{aligned} |\Gamma(t) - \gamma(\tau_k)| &\leq \left| \frac{t_k - t}{t_k - t_{k-1}} \right| |\gamma(t_{k-1}) - \gamma(\tau_k)| + \left| \frac{t - t_{k-1}}{t_k - t_{k-1}} \right| |\gamma(t_k) - \gamma(\tau_k)| \\ &\leq |\gamma(t_{k-1}) - \gamma(\tau_k)| + |\gamma(t_k) - \gamma(\tau_k)| < 2\delta. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\Gamma} f &= \int_a^b f(\Gamma(t))\Gamma'(t) dt \\ &= \sum_{k=1}^n \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t)) dt \end{aligned}$$

and

$$\begin{aligned}
\left| \int_{\gamma} f - \int_{\Gamma} f \right| &= \left| \int_{\gamma} f - \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) \right| + \left| \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) - \int_{\Gamma} f \right| \\
&\leq \epsilon + \left| \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) - \int_{\Gamma} f \right| \\
&\leq \epsilon + \sum_{k=1}^n \frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\Gamma(t)) - f(\gamma(\tau_k))| dt \\
&\leq \epsilon + \epsilon \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| \\
&\leq \epsilon(1 + V(\gamma)),
\end{aligned}$$

which can be made arbitrarily small, thus completing the proof.  $\blacksquare$

The following can be thought of as an analogue of the Fundamental Theorem of Calculus for complex functions.

**Theorem 2.10.** Let  $G$  be open in  $\mathbb{C}$  and  $\gamma$  be a rectifiable path in  $G$  with initial and end points  $\alpha, \beta$  respectively. If  $f : G \rightarrow \mathbb{C}$  is a continuous function with a primitive  $F : G \rightarrow \mathbb{C}$  ( $F$  is differentiable and  $F' = f$ ), then

$$\int_{\gamma} f = F(\beta) - F(\alpha).$$

*Proof.* When  $\gamma : [a, b] \rightarrow \mathbb{C}$  is piecewise smooth,

$$\begin{aligned}
\int_{\gamma} f &= \int_a^b f(\gamma(t))\gamma'(t) dt \\
&= \int_a^b F'(\gamma(t))\gamma'(t) dt \\
&= \int_a^b (F \circ \gamma)'(t) dt \\
&= (F \circ \gamma)(b) - (F \circ \gamma)(a) && \text{(by the Fundamental Theorem of Calculus)} \\
&= F(\beta) - F(\alpha).
\end{aligned}$$

In general, we may use Lemma 2.9. For  $\epsilon > 0$ , let  $\Gamma$  be a polygonal path of the described form. Since  $\Gamma$  is piecewise smooth,  $\int_{\Gamma} f = F(\beta) - F(\alpha)$ , so

$$\left| \int_{\gamma} f - (F(\beta) - F(\alpha)) \right| < \epsilon.$$

Since  $\epsilon$  was chosen arbitrarily, the desideratum follows.  $\blacksquare$

The fundamental theorem of calculus says that each continuous function has a primitive. However, this is not true for functions of complex variables. For example, letting  $f(z) = |z|^2$ , if  $F$  is a primitive of  $f$ , then  $F$  is analytic. So, if  $F = U + \iota V$ ,  $x^2 + y^2 = F'(x + \iota y)$ . Consequently,

$$\begin{aligned}
\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} = x^2 + y^2 \\
\frac{\partial U}{\partial y} &= \frac{\partial V}{\partial x} = 0.
\end{aligned}$$

However,  $\frac{\partial U}{\partial y} = 0$  implies that  $U(x, y) = u(x)$  for some function  $u$ , which implies that  $u'(x) = x^2 + y^2$ , a contradiction.

### 2.3. Power series representation of analytic functions

Recall the following result which we had used in the proof of Theorem 1.11.

**Theorem 2.11.** Let  $\varphi : [a, b] \times [c, d] \rightarrow \mathbb{C}$  be a continuous function and defined  $g : [c, d] \rightarrow \mathbb{C}$  by

$$g(t) = \int_a^b \varphi(s, t) \, ds.$$

Then  $g$  is continuous. Moreover, if  $\frac{\partial \varphi}{\partial t}$  exists and is a continuous function on  $[a, b] \times [c, d]$ , then  $g$  is continuously differentiable and

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t) \, ds.$$

This is referred to as the Leibniz rule.

For example, this may be used to prove that if  $|z| < 1$ ,

$$\int_0^{2\pi} \frac{e^{\iota s}}{e^{\iota s} - z} \, ds = 2\pi.$$

To do so, let  $\varphi(s, t) = e^{\iota s} / (e^{\iota s} - tz)$  for  $0 \leq t \leq 1$  and  $0 \leq s \leq 2\pi$ . Observe that  $\varphi$  is continuously differentiable since  $|z| < 1$ . Thus,

$$g(t) = \int_0^{2\pi} \varphi(s, t) \, ds$$

is continuously differentiable. Since  $\varphi(s, 0) = 1$ ,  $g(0) = 2\pi$ . Now,

$$\begin{aligned} g'(t) &= \int_0^{2\pi} \frac{\partial \varphi}{\partial t}(s, t) \, ds \\ &= \int_0^{2\pi} \frac{ze^{\iota s}}{(e^{\iota s} - tz)^2} \, ds. \end{aligned}$$

For fixed  $t$ ,  $\Phi(s) = ze^{\iota s} / (e^{\iota s} - tz)$  satisfies

$$\Phi'(s) = -\frac{\iota z}{(e^{\iota s} - tz)^2} \cdot \iota e^{\iota s} = \frac{ze^{\iota s}}{(e^{\iota s} - tz)^2}.$$

Therefore,  $g'(t) = \Phi(2\pi) - \Phi(0) = 0$ , so  $g$  is a constant and  $g(t) = g(0) = 2\pi$  for any  $t$ , 1 in particular.

**Theorem 2.12.** Let  $f : G \rightarrow \mathbb{C}$  be analytic and suppose that  $\overline{B(a, r)} \subseteq G$  for some  $r > 0$ . If  $\gamma(t) = a + re^{\iota t}$  for  $0 \leq t \leq 2\pi$ , then

$$f(z) = \frac{1}{2\pi \iota} \int_{\gamma} \frac{f(w)}{w - z} \, dw$$

for  $|z - a| < r$ .

*Proof.* Defining  $G_1 = \{(z - a)/r : z \in G\}$  and  $g(z) = f(a + rz)$ , it suffices to consider the case where  $a = 0$  and  $r = 1$ .

Fix  $z$  with  $|z| < 1$ . It must be shown that

$$f(z) = \int_{2\pi} \int_{\gamma} \frac{f(w)}{w - z} \, dw = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{\iota s})e^{\iota s}}{e^{\iota s} - z} \, ds.$$

That is, we want to show that

$$\begin{aligned} 0 &= \int_0^{2\pi} \frac{f(e^{\iota s})e^{\iota s}}{e^{\iota s} - z} ds - 2\pi f(z) \\ &= \int_0^{2\pi} \left( \frac{f(e^{\iota s})e^{\iota s}}{e^{\iota s} - z} - f(z) \right) ds. \end{aligned}$$

For this, let

$$\varphi(s, t) = \frac{f(z + t(e^{\iota s} - z))e^{\iota s}}{e^{\iota s} - z} - f(z)$$

for  $0 \leq t \leq 1$  and  $0 \leq s \leq 2\pi$ , and

$$g(t) = \int_0^{2\pi} \varphi(s, t) ds.$$

We wish to show that  $g(1) = 0$ . Observe that

$$g(0) = \int_0^{2\pi} \frac{f(z)e^{\iota s}}{e^{\iota s} - z} - f(z) ds = f(z) \int_0^{2\pi} \frac{e^{\iota s}}{e^{\iota s} - z} ds - 2\pi f(z) = 0.$$

Also,

$$\begin{aligned} g'(t) &= \int_0^{2\pi} \frac{\partial \varphi}{\partial t}(s, t) ds \\ &= \int_0^{2\pi} \frac{e^{\iota s}}{e^{\iota s} - z} f'(z + t(e^{\iota s} - z))(e^{\iota s} - z) ds \\ &= \int_0^{2\pi} e^{\iota s} f'(z + t(e^{\iota s} - z)) ds \\ &= \frac{1}{t} f(z + t(e^{\iota s} - z)) \Big|_{s=0}^{s=2\pi} \\ &= 0, \end{aligned}$$

completing the proof. ■

If  $|z - a| < r$  and  $w$  is such that  $|w - a| = r$ , then

$$\frac{1}{w - z} = \frac{1}{w - a} \cdot \frac{1}{1 - \frac{z-a}{w-a}} = \frac{1}{w - a} \sum_{i=0}^{\infty} \left( \frac{z-a}{w-a} \right)^i.$$

since  $|z - a| < |w - a|$ .

Now, multiplying by  $f(w)/2\pi\iota$  and integrating around the circle  $\gamma$  defined by  $|w - a| = r$ , we get that

$$f(z) = \int_{\gamma} \frac{f(w)}{2\pi\iota} \sum_{i=0}^{\infty} \frac{(z-a)^i}{(w-a)^{i+1}} dw.$$

But how do we simplify the right hand side? We do not know (*yet*) that the integral and summation may be switched. So, let us get to showing this.

**Lemma 2.13.** Let  $\gamma$  be a rectifiable curve in  $\mathbb{C}$  and suppose that  $F_n$  and  $F$  are continuous functions on  $\{\gamma\}$ . If  $(F_n)$  uniformly converges to  $F$  on  $\{\gamma\}$ , then

$$\int_{\gamma} = \lim_{n \rightarrow \infty} \int_{\gamma} F_n.$$

*Proof.* Let  $\epsilon > 0$  and let  $N \in \mathbb{N}$  such that

$$|F_n(w) - F(w)| < \frac{\epsilon}{V(\gamma)}$$

for  $n \geq N$ . This implies that

$$\left| \int_{\gamma} F_n - \int_{\gamma} F \right| \leq V(\gamma) \sup_w |F_n(w) - F(w)| \leq \epsilon$$

for  $n \geq N$ , completing the proof. ■

**Theorem 2.14.** Let  $f$  be analytic on  $B(a, R)$ . Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$$

for all  $|z - a| < R$ , where  $a_n = f^{(n)}(a)/n!$  and this series has radius of convergence at least  $R$ .

*Proof.* Let  $0 < r < R$  such that  $\overline{B(a, r)} \subseteq B(a, R)$ . Let  $\gamma(t) = a + re^{it}$  ( $0 \leq t \leq 2\pi$ ). Since  $|z - a| < r$ , if  $M = \max\{|f(w)| : |w - a| = r\}$ ,

$$\frac{|f(w)||z - a|^n}{|w - a|^{n+1}} \leq \frac{M}{r} \left( \frac{|z - a|}{r} \right)^n.$$

Since  $|z - a| < r$ ,

$$\sum_{n=0}^{\infty} f(w) \frac{(z - a)^n}{(w - a)^{n+1}}$$

converges uniformly for  $w$  on  $\{\gamma\}$ . By the discussion before the previous lemma together with the lemma itself,

$$f(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} \right) (z - a)^n. \quad (*)$$

Since

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}}.$$

is independent of  $z$ ,  $(*)$  converges for  $|z - a| < R$ . However, we now know from Theorem 1.6(c) that  $a_n = f^{(n)}(a)/n!$ , completing the proof. ■

**Corollary 2.15.** If  $f$  is analytic,

$$f^{(n)}(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} dw$$

where  $\gamma = a + re^{it}$  and  $r < R$ , the radius of convergence of the series.

**Corollary 2.16.** If  $f : G \rightarrow \mathbb{C}$  is analytic, then  $f$  is infinitely differentiable.

Indeed, this follows directly from the fact that

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} dw$$

where  $\gamma(t) = a + re^{it}$  for  $0 \leq t \leq 2\pi$ .

**Corollary 2.17** (Cauchy's Estimate). Let  $f$  be analytic on  $B(a, R)$  and suppose  $|f(z)| \leq M$  for all  $z \in B(a, R)$ . Then

$$|f^{(n)}(a)| \leq \frac{n!M}{R^n}.$$

Indeed, the above applies with  $r < R$  so we get that

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi} \int_{\gamma} \frac{|f(w)|}{|w-a|^{n+1}} |dw| \leq \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n!M}{r^n}.$$

Since  $r < R$  is arbitrary, we may let  $r \rightarrow R^-$ .

**Proposition 2.18.** Let  $f$  be analytic on the disk  $B(a, R)$  and suppose that  $\gamma$  is a closed rectifiable curve in  $B(a, R)$ . Then  $\int_{\gamma} f = 0$ .

*Proof.* Due to Theorem 2.10, it suffices to show that  $f$  has a primitive. We know that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for  $|z-a| < R$ , where  $a_n = f^{(n)}(a)/n!$ . Consider the function

$$F(z) = (z-a) \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-a)^n.$$

Since  $\lim_{n \rightarrow \infty} (n+1)^{1/n} = 1$ , this power series has the same radius of convergence as  $\sum a_n (z-a)^n$ . Therefore,  $F$  is defined on  $B(a, R)$ . Moreover,  $F'(z) = f(z)$  for  $|z-a| < R$  by Theorem 1.6(b), completing the proof. ■

**Definition 2.7.** An *entire* function is a function which is defined and analytic on the whole complex plane  $\mathbb{C}$ .

**Proposition 2.19.** If  $f$  is entire, then it has a power series expansion with infinite radius of convergence.

Therefore, entire functions may be considered as polynomials of “infinite degree”. Polynomials of finite non-zero degree are typically unbounded. These two insights lead to the following result.

**Theorem 2.20** (Liouville's Theorem). If  $f$  is a bounded entire function, then  $f$  is constant.

*Proof.* Suppose that  $|f(z)| \leq M$  for all  $z \in \mathbb{C}$ . We shall show that  $f'(z) = 0$  for all  $z \in \mathbb{C}$ . By **Cauchy's Estimate**, since  $f$  is analytic on any disk  $B(z, R)$ ,  $|f'(z)| \leq M/R$ . However,  $R$  is arbitrary so  $f'(z) = 0$  for any  $z \in \mathbb{C}$ . ■

**Theorem 2.21** (Fundamental Theorem of Algebra). If  $p$  is a non-constant polynomial with coefficients in  $\mathbb{C}$ , then there exists  $a \in \mathbb{C}$  with  $p(a) = 0$ .

*Proof.* Suppose  $p(z) \neq 0$  for all  $z \in \mathbb{C}$ . Consider the entire function  $f(z) = 1/p(z)$ . This function is then bounded as  $p(z)$  goes to  $\infty$  as  $z$  goes to infinity. By **Liouville's Theorem**,  $f$  (and thus  $p$ ) is constant, which is a contradiction. ■

Due to the above,  $\mathbb{C}$  is an algebraically closed field.

**Corollary 2.22.** If  $p(z)$  is a polynomial and its roots are  $(p_j)$  with multiplicity  $k_j$  (for  $1 \leq j \leq m$ ), then  $p(z) = C(z-a_1)^{k_1}(z-a_2)^{k_2} \cdots (z-a_m)^{k_m}$  for some constant  $C$ , where  $\sum k_j$  is the degree of  $p$ .

It is not too difficult to show that if  $p(z)$  is a non-constant polynomial, then  $p$  is a surjective analytic function on  $\mathbb{C}$ . However, we know that the map  $z \mapsto e^z$  is an entire function but there is no  $b \in \mathbb{C}$  such that  $e^b = 0$ . So, power series (“polynomials of infinite degree”) cannot be thought of in the same way as ordinary polynomials (of finite degree). However, we shall see later that given a non-constant entire function  $f$ , there exists at most one  $a \in \mathbb{C}$  that is not in the image of  $f$ . This is referred to as Little Picard’s Theorem.

**Theorem 2.23.** Let  $G$  be a connected open set and  $f : G \rightarrow \mathbb{C}$  be analytic. Then, the following are equivalent statements.

- (a)  $f$  is identically zero.
- (b) There exists  $a \in \mathbb{C}$  such that for all  $n \geq 0$ ,  $f^{(n)}(a) = 0$ .
- (c)  $\{z \in G : f(z) = 0\}$  has a limit point in  $G$ .

*Proof.* Clearly, (a) implies (b) and (c).

Next, let us show that (c) implies (b). Let  $a \in G$  be a limit point of the zero set of  $f$ . Let  $R > 0$  such that  $B(a, R) \subseteq G$ . Since  $a$  is a limit point of  $z$  and  $f$  is continuous,  $f(a) = 0$ . Let  $n \geq 1$  such that  $f^{(k)}(a) = 0$  for  $k < n$  and  $f^{(n)}(a) \neq 0$ . Expanding  $f$  as a power series about  $a$  gives that

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k$$

for  $|z-a| < R$  and  $a_n \neq 0$ . Let

$$g(z) = \sum_{k=n}^{\infty} a_k (z-a)^{k-n}.$$

Since  $g$  is continuous in  $B(a, R)$  and  $g(a) \neq 0$ , let  $r < R$  such that  $g(z) \neq 0$  when  $|z-a| < r$ . Since  $a$  is a limit point of  $z$ , there exists  $b$  with  $f(b) = 0$  and  $0 < |a-b| < r$ . This gives  $0 = (b-a)^n g(b)$ , so  $g(b) = 0$ , a contradiction. Therefore, no such  $n$  can be found and (b) is true.

Finally, let us show that (b) implies (a). Let

$$A = \{z \in G : f^{(n)}(z) = 0 \text{ for all } n \geq 0\}.$$

By the definition of (b),  $A \neq \text{empty set}$ . We shall show that  $A$  is both open and closed in  $G$ , and by the connectedness of  $G$  it follows that  $A$  is the entirety of  $G$ . Showing that  $A$  is closed is direct – if  $z \in \overline{A}$  and  $(z_k)$  a sequence such that  $z_k \rightarrow z$ , then since each  $f^{(k)}$  is continuous,  $f^{(n)}(z) = \lim f^{(n)}(z_k) = 0$  for all  $n \geq 0$ , and so  $z \in A$ . On the other hand, if  $a \in A$ , we can write  $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n = 0$  on  $B(a, R)$  (for some  $R > 0$ ), so  $B(a, R) \subseteq A$  and  $A$  is open, completing the proof. ■

**Corollary 2.24.** If  $f, g$  are analytic on a region  $G$ , then  $f \equiv g$  iff  $\{z \in G : f(z) = g(z)\}$  has a limit point in  $G$ .

**Corollary 2.25.** If  $f$  is non-trivial and analytic on an open connected set  $G$ , then each zero of  $f$  has finite multiplicity. More explicitly, for each  $a \in G$  with  $f(a) = 0$ , there is an integer  $n \geq 1$  and an analytic function  $g : G \rightarrow \mathbb{C}$  such that  $g(a) \neq 0$  and  $f(z) = (z-a)^n g(z)$  for all  $z \in G$ .

*Proof.* It is clear that there exists a largest  $n \geq 1$  such that  $f^{(k)}(a) = 0$  for all  $k \leq n-1$ . ■

**Corollary 2.26.** If  $f : G \rightarrow \mathbb{C}$  is non-trivial and analytic, and  $a \in G$  with  $f(a) = 0$ , then there exists  $R > 0$  such that  $B(a, R) \subseteq G$ , and  $f(z) \neq 0$  for all  $0 < |z-a| < R$ .

The above follows from the fact that the zeros of  $f$  are isolated.



**Theorem 2.27** (Maximum Modulus Theorem). If  $G$  is a region and  $f : G \rightarrow \mathbb{C}$  is an analytic function such that there is a point  $a \in G$  with  $|f(a)| \geq |f(z)|$  for all  $z \in G$ , then  $f$  is constant.

That is, if  $|f|$  attains its maximum,  $f$  is constant.

*Proof.* Let  $\overline{B(a, r)} \subseteq G$  and  $\gamma(t) = a + re^{it}$  for  $0 \leq t \leq 2\pi$ . Then,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} dw \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) dt. \end{aligned}$$

Therefore,

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})| dt \leq |f(a)|.$$

Therefore,

$$0 = \int_0^{2\pi} (|f(a)| - |f(a + re^{it})|) dt.$$

Since the integrand is continuous and non-negative,  $|f(a)| = |f(a + re^{it})|$  for all  $t \in [0, 2\pi]$ . If  $f(a) = 0$ , we are clearly done. Otherwise, since  $r$  was arbitrary,  $f$  maps any disk  $B(a, R)$  to the circle  $|z| = |f(a)|$ . It may then be shown using the Cauchy-Riemann equations that  $f$  is constant on  $B(a, R)$  and is equal to  $f(a)$  for all  $|z-a| < R$ . Therefore,  $f(z) = f(a)$  for all  $z \in G$  since the zeros of  $f - f(a)$  are not isolated. ■

Recall that

$$\int_{\gamma} \frac{1}{z-a} dz = 2\pi i n$$

if  $\gamma(t) = a + e^{int}$  for  $t \in [0, 2\pi]$ . However, this property is not peculiar to the path  $\gamma$ , as shown by the following result.

**Theorem 2.28.** If  $\gamma : [0, 1] \rightarrow \mathbb{C}$  is a closed rectifiable curve and  $a \notin \{\gamma\}$ , then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-a} dz$$

is an integer.

*Proof.* Using Lemma 2.9, we may assume that  $\gamma$  is piecewise smooth (Why?).

Let us assume that  $\gamma$  is smooth. Define  $g : [0, 1] \rightarrow \mathbb{C}$  by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a} ds.$$

Then,  $g(0) = 0$  and  $g(1) = \int_{\gamma} 1/(z-a) dz$ . We also have that

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - a}$$

for  $0 \leq t \leq 1$ . This gives that

$$\frac{d}{dt} \left( e^{-g(t)} (\gamma(t) - a) \right) = e^{-g(t)} \gamma'(t) - g'(t) e^{-g(t)} (\gamma(t) - a) = 0.$$

Therefore,

$$e^{-g(0)}(\gamma(0) - a) = e^{-g(1)}(\gamma(1) - a).$$

Because  $\gamma(0) = \gamma(1)$  (the curve is closed) and  $g(0) = 0$ ,  $g(1) = 2\pi i n$  for some integer  $n$ . In the case where  $\gamma$  is piecewise-smooth, we can define  $g$  by integrating over each of the smooth intervals and the result follows near-identically. ■

**Definition 2.8.** If  $\gamma$  is a closed rectifiable curve in  $\mathbb{C}$  then for  $a \notin \{\gamma\}$ ,

$$n(\gamma; a) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - a} dz$$

is called the *index* of  $\gamma$  with respect to the point  $a$ . It is also sometimes referred to as the *winding number* of  $\gamma$  around  $a$ .

Recall the definition of  $(-\gamma)$  from (2.2), also denoted  $\gamma^{-1}$ . If  $\gamma$  and  $\sigma$  are curves on  $[0, 1]$  with  $\gamma(1) = \sigma(0)$ ,  $\gamma + \sigma$  is the curve

$$(\gamma + \sigma)(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq 1/2, \\ \sigma(2t - 1), & 1/2 \leq t \leq 1. \end{cases}$$

**Proposition 2.29.** If  $\sigma, \gamma$  are closed rectifiable curves with the same initial (and final) points, then

$$n(\gamma; a) = -n(-\gamma; a) \tag{2.5}$$

for all  $a \notin \{\gamma\}$  and

$$n(\gamma + \sigma; a) = n(\gamma; a) + n(\sigma; a) \tag{2.6}$$

for all  $a \notin \{\sigma\} \cup \{\gamma\}$ .

We omit the proof of the above.

The reason for  $n(\cdot; \cdot)$  being called the winding number is clear from what happens in the case of a circle. For  $a + e^{2\pi i n t}$ , then  $n(\gamma; a) = n$  is the number of times this curve “winds” or “wraps” around  $a$ . In fact, if  $|b - a| < 1$ ,  $n(\gamma; b) = n$  and if  $|b - a| > 1$ ,  $n(\gamma; b) = 0$ .

Recall that the components of a set are its maximal connected subsets.

**Theorem 2.30.** Let  $\gamma$  be a closed rectifiable curve in  $\mathbb{C}$ . Then  $n(\gamma; a)$  is constant for  $a$  belonging to a component of  $G = \mathbb{C} \setminus \{\gamma\}$ . Also,  $n(\gamma; a) = 0$  for  $a$  belonging to the unbounded component of  $G$ .

*Remark.* Since  $\{\gamma\}$  is compact, the connected set  $\{z : |z| > R\} \subseteq G$  for sufficiently large  $R$ , so  $\gamma$  has precisely one unbounded component.

*Proof.* Define  $f : G \rightarrow \mathbb{C}$  by  $f(a) = n(\gamma; a)$ . If we manage to show that  $f$  is continuous on  $G$ , we are done since the image of this map is a subset of the integers and each component is connected by definition, so  $f$  is constant on each component.

Recall that components of  $G$  are open. Fix  $a \in G$  and let  $r = d(a, \{\gamma\}) > 0$ . If  $|a - b| < \delta \leq r/2$  (we shall fix  $\delta$  more precisely later), then

$$\begin{aligned} |f(a) - f(b)| &= \frac{1}{2\pi} \left| \int_{\gamma} \left( \frac{1}{z - a} - \frac{1}{z - b} \right) dz \right| \\ &\leq \frac{|a - b|}{2\pi} \int_{\gamma} \frac{1}{|z - a||z - b|} |dz|. \end{aligned}$$

By definition,  $|z - a| \geq r$  for any  $a \in \{\gamma\}$  and  $|z - b| \geq |z - a| - |a - b| \geq r/2$ . So,

$$\begin{aligned} |f(a) - f(b)| &\leq \frac{|a - b|}{2\pi} \int_{\gamma} \frac{2}{r^2} |dz| \\ &\leq \frac{\delta}{\pi r^2} V(\gamma). \end{aligned}$$

For a given  $\epsilon > 0$ , setting  $\delta = \min\{r/2, \epsilon\pi r^2/V(\gamma)\}$  does the job, completing the first part of the proof.

It remains to show that  $\lim_{a \rightarrow \infty} f(a) = 0$  (Why does this imply the required?). Let  $U$  be the unbounded component of  $G$ . For a given  $R > 0$ , let  $a \in U$  such that  $d(a; \gamma) > R$ . Then,

$$|f(a)| = \frac{1}{2\pi} \int_{\gamma} \left| \frac{1}{z - a} \right| |dz| \leq \frac{1}{2\pi R} \int_{\gamma} |dz| = \frac{V(\gamma)}{2\pi R}.$$

$R$  can be made arbitrarily large (as  $a \rightarrow \infty$ ), so we are done. ■