

# Topology

Lecture 1 - 06/01/21 Introduction and examples of topologies

Def. A topology on a set  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  such that

Topology i)  $\emptyset \in \mathcal{T}$  and  $X \in \mathcal{T}$ .

ii) If  $U_i \in \mathcal{T}$  for all  $i \in I$ , where  $I$  is some indexing set, then  $\bigcup_{i \in I} U_i \in \mathcal{T}$ .

Equivalently, iii) If  $U_j \in \mathcal{T}$  for all  $j \in J$ , where  $J$  is some finite indexing set, then for  $U_1, U_2 \in \mathcal{T}$ ,  $U_1 \cap U_2 \in \mathcal{T}$ .  $\leftarrow \bigcap_{j \in J} U_j \in \mathcal{T}$ .

Unless mentioned otherwise, assume  $X \neq \emptyset$ .

Recall the definition of a metric space and an open set. (from Real Analysis)

Since the set of open sets is closed under arbitrary unions and finite intersections, observe that the set of open subsets of a metric space  $(X, d)$  is a topology. That is,

$$\mathcal{T} = \{ U \subseteq X : U \text{ is open in } (X, d) \}$$

is a topology. ( $\emptyset$  and  $X$  are trivially open)

Topologies essentially extend the idea of open sets. How?

Def. A topological space  $(X, \mathcal{T})$  is a set  $X$  along with a topology  $\mathcal{T}$  on  $X$ .

Topological Space Open Set For a topological space, we call the elements of  $\mathcal{T}$  open.

$(X, \{\emptyset, X\})$  is a trivial topological space on a set  $X$ .

We now introduce the analogues of interior points, closed sets, etc. Since we don't have "balls" in topological spaces, we have to define everything in an alternate way that remains consistent.

For a metric space  $(X, d)$ , the topology  
 $T = \{U \subseteq X : U \text{ is open}\}$

is called the **metric topology** induced by the metric  $d$ .

For a set  $X$ , the topology  $P(X)$  is called the **discrete topology** on  $X$ .

Observe that this is the metric topology induced by the discrete metric. (for  $x, y \in X$ ,  $d(x, y) = 0$  if  $x = y$  and 1 otherwise)

For a set  $X$ , the topology  $\{\emptyset, X\}$  is called the **indiscrete topology** on  $X$ .

Let  $X$  be a set and

$$T_f = \{\emptyset\} \cup \{U \subseteq X : X \setminus U \text{ is finite}\}.$$

$T_f$  is a topology on  $X$  and is called the **finite complement topology** or the **co-finite topology**.

- Clearly,  $\emptyset$  and  $X$  are in  $T_f$ .
- For  $(U_i)_{i \in I}$  in  $T_f$ ,

$$\left(\bigcup_{i \in I} U_i\right)^c = \bigcap_{i \in I} U_i^c \text{ is finite (since each } U_i^c \text{ is finite)}$$

- For  $(U_i)_{i=1}^n$  in  $T_f$ ,

$$\left(\bigcap_{i=1}^n U_i\right)^c = \bigcup_{i=1}^n U_i^c \text{ is finite (a finite union of finite sets)}$$

We have seen that any metric defines a topology. Is the converse true?

No!

Topologies that are induced by a metric are said to be **metrizable**.

→ Consider the indiscrete topology  $\{\emptyset, X\}$ . (for  $|X| > 1$ )

Use the fact that distinct points are separable by neighbourhoods.

If  $X$  is a finite set, the finite complement topology is the discrete topology.

Co-countable Topology Similar to the co-finite topology  $\tau_f$ , we can define  $\tau_c$ , the co-countable topology.

$$\left( \{\emptyset\} \cup \{U \subseteq X : X \setminus U \text{ is countable}\} \right)$$

Lecture 2 - 08/01/21 Bases of topologies

Def. Suppose  $\tau$  and  $\tau'$  are two topologies on a set  $X$ . If  $\tau' \supseteq \tau$ , we say that  $\tau'$  is finer than  $\tau$  and  $\tau$  is coarser than  $\tau'$ . We can also define strictly finer and strictly coarser if there is a strict containment.

$\tau$  and  $\tau'$  are said to be comparable if  $\tau \subseteq \tau'$  or  $\tau' \subseteq \tau$ .

(This is similar to the refinement of partitions in the Darboux integral)

Def. Basis If  $X$  is a set, a basis (for a topology) on  $X$  is a collection  $\mathcal{B}$  of subsets of  $X$  (called basis elements) such that

- $\forall x \in X, \exists B \in \mathcal{B}$  such that  $x \in B$  (that is,  $\bigcup_{B \in \mathcal{B}} B = X$ )
- if  $x \in B_1 \cap B_2$  for  $B_1, B_2 \in \mathcal{B}$ , then there exists  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .

If  $\mathcal{B}$  is a basis, the topology  $\tau$  generated by  $\mathcal{B}$  is defined as

Generated Topology  $\tau = \left\{ U \subseteq X : U = \bigcup_{\substack{B \in \mathcal{B} \\ B \subseteq U}} B \right\}$   $\left( \tau = \left\{ U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U \right\} \right)$

Alternatively, (why?)

$$\tau = \left\{ U \subseteq X : U = \bigcup_{i \in I} B_i \text{ for some } (B_i)_{i \in I} \text{ in } \mathcal{B} \right\}$$

$\mathcal{B}$  is then said to be a basis of  $\tau$ .

We take by convention that  $\bigcup_{s \in \emptyset} s = \emptyset$ .

Observe that

- we trivially have  $\emptyset \in \tau$
- the first condition implies that  $x \in U$ .
- closure under (finite) intersections follows from the second condition. (Why?)
- closure under arbitrary unions follows from the way we define the topology.

Also note that  $B \subseteq \tau$ .

Note that bases here are extremely different from bases in linear algebra. A better analogue would be a spanning set.

How do we find a "smallest" basis though?

(an analogue of linear independence, perhaps?)  
(for example,  $\{(a,b) : a,b \in \mathbb{Q}\}$  generates the Euclidean metric topology of  $\mathbb{R}^2$ )

Lecture 3 - 13/01/21 More about Bases and Topologies on  $\mathbb{R}$

Also, how do we find a (non-trivial) basis for a topology?

Lemma. Let  $(X, \tau)$  be a topological space. Suppose that  $\mathcal{C}$  is a collection of open subsets of  $X$  such that for each open set  $U$  of  $X$  and each  $x \in U$ , there is  $C \in \mathcal{C}$  such that  $x \in C \subseteq U$ . Then  $\mathcal{C}$  is a basis of  $\tau$ .

Proof.

- Given  $x \in X$ , there is, by hypothesis,  $C \in \mathcal{C}$  such that  $x \in C \subseteq X$
- Next, let  $x \in C_1 \cap C_2$  for  $C_1, C_2 \in \mathcal{C}$ . Since  $C_1$  and  $C_2$  are open, so is  $C_1 \cap C_2$ . Therefore,  $\exists C_3 \in \mathcal{C}$  s.t.  $x \in C_3 \subseteq C_1 \cap C_2$ .  
 $\Rightarrow \mathcal{C}$  is a basis on  $X$ .

Let  $\tau'$  be the topology generated by  $\mathcal{C}$ .

- Let  $U \in \tau$ . Then  $\forall x \in U, \exists C \in \mathcal{C}$  s.t.  $x \in C \subseteq U$ .  
 $\Rightarrow \tau \subseteq \tau'$ . (by the definition of a generated topology)

- Let  $U \in \tau'$ . Then  $\forall x \in U$ ,  $\exists C_x \in \mathcal{C}$  s.t.  $x \in C_x \subseteq U$ .  
 $\Rightarrow U = \bigcup_{x \in U} C_x$ .

However, each  $C_x \in \mathcal{C} \subseteq \tau$ .

$\Rightarrow U \in \tau$ . Therefore,  $\tau' \subseteq \tau$

so  $\tau = \tau'$ . □

Lemma.  
(1.2) Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for the topologies  $\tau$  and  $\tau'$  on  $X$ .  
The following are equivalent.

- (i)  $\tau'$  is finer than  $\tau$ .
- (ii) for each  $x \in X$  and  $B \in \mathcal{B}$  with  $x \in B$ , there is a  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .

Proof. (ii)  $\Rightarrow$  (i)

Let  $U \in \tau$  and  $x \in U$ . Let  $B \in \mathcal{B}$  s.t.  $x \in B \subseteq U$  (since  $\mathcal{B}$  generates  $\tau$ ).

Let  $B' \in \mathcal{B}'$  s.t.  $x \in B' \subseteq B \subseteq U$ .

$\Rightarrow U \in \tau'$  by definition and therefore,  $\tau \subseteq \tau'$ .

(i)  $\Rightarrow$  (ii)

Let  $x \in X$  and  $B \in \mathcal{B}$  with  $x \in B$ . By definition,  $B \in \tau$ .

$\Rightarrow B \in \tau'$ . Since  $\tau'$  is generated by  $\mathcal{B}'$ , there exists  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ . □

Example. The collection  $\mathcal{B}$  of <sup>open</sup> circular regions in  $\mathbb{R}^2$  generates the same topology as the collection  $\mathcal{B}'$  of all <sup>open</sup> rectangular regions in  $\mathbb{R}^2$ .  
(Show that each is finer than the other using the above lemma)

Def. If  $\mathcal{B}$  is the collection of all open intervals in the real line, then the topology generated by  $\mathcal{B}$  is called the standard topology on  $\mathbb{R}$ . Unless mentioned otherwise,  $\mathbb{R}$  is taken to have this topology.

(This is the topology induced by the Euclidean metric)

If  $B'$  is the collection of all half-open intervals of the form  $[a, b) : a, b \in \mathbb{R}$  where  $a < b$ , the topology generated by  $B'$  is called the **lower limit topology** on  $\mathbb{R}$ .

Lower Limit Topology When  $\mathbb{R}$  is given by this topology, it is denoted  $\mathbb{R}_l$ .

K-Topology Let  $K = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ . Let  $B''$  be the collection of all open intervals  $(a, b)$  along with sets of the form  $(a, b) \setminus K$ . The topology generated by  $B''$  is called the **K-topology** on  $\mathbb{R}$ . When  $\mathbb{R}$  is given by this topology, it is denoted  $\mathbb{R}_K$ .

(Do check that the above collections are bases)

Lemma: The topologies of  $\mathbb{R}_l$  and  $\mathbb{R}_K$  are strictly finer than the standard topology on  $\mathbb{R}$ , but are not comparable with each other.

Proof- Let the topologies of  $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_K$  be  $\tau, \tau', \tau''$  with bases  $B, B', B''$ . Given  $(a, b) \in B$  and  $x \in (a, b)$ ,  $x \in [x, b] \subseteq (a, b)$

$$\hookrightarrow \in B' \Rightarrow \tau \subseteq \tau'.$$

Also, given  $[x, d) \in B'$ , there are no  $(a, b)$  such that  $x \in (a, b) \subseteq [x, d)$

$$\Rightarrow \tau \subsetneq \tau'$$

$\tau \subseteq \tau''$  is easily shown since  $B \subseteq B''$ .

To show that it is strictly finer, consider  $B'' = (-1, 1) \setminus K$  and  $0 \in B$ . There is no  $(a, b)$  such that  $0 \in (a, b) \subseteq B''$ .

$$\Rightarrow \tau \subsetneq \tau''.$$

To show that  $\tau'$  and  $\tau''$  are not comparable, consider  $2 \in [2, 3) \in B'$  and  $0 \in [-1, 1] \setminus K \in B''$ . The details are left to the reader.

$$\Rightarrow \text{Neither } \tau' \subseteq \tau'' \text{ nor } \tau'' \subseteq \tau'.$$



Def. A subbasis  $S$  for a topology  $\tau$  on  $X$  is a collection of subsets of  $X$  whose union is  $X$ .

Sub-basis

$$(S \subseteq P(X) \text{ such that } \bigcup_{U \in S} U = X.)$$

The topology generated by  $S$  is the collection  $\tau$  of all unions of finite intersections of elements of  $S$ .

Let  $B = \left\{ \bigcap_{i=1}^n S_i : (S_i)_i^n \in S \right\}$ .

The topology generated by  $S$  is just that generated by the basis  $B$ .

↳ Why? This is easily checked using the definition (of a basis)

Lecture 4 - 15/01/21

## Order Topologies

Def. We say that a relation  $C$  is a complete order on a set  $X$  if

Order

1. For  $x, y \in X$ ,  $xCy$  or  $yCx$ .
2. For no  $x \in X$ ,  $xCx$ .
3. If  $xCy$  and  $yCz$ , then  $xCz$ .

Same Order Type

Def. Let  $A$  and  $B$  be two sets with orders  $<_A$  and  $<_B$ . We say that  $A$  and  $B$  have the same order type if there is a bijection  $f: A \rightarrow B$  such that for any  $a_1, a_2 \in A$ ,

$$a_1 <_A a_2 \iff f(a_1) <_B f(a_2)$$

Dictionary Order

Def. The dictionary order relation  $<$  on  $A \times B$  is defined by

$$a_1 \times b_1 < a_2 \times b_2 \iff a_1 <_A a_2 \text{ or } (a_1 = a_2 \text{ and } b_1 <_B b_2)$$

Two finite ordered sets of the same cardinality always have same order type. (why?).

Given an order  $<$ , we write  $a \leq b$  if  $a < b$  or  $a = b$ ,  
 $a > b$  if  $b < a$ , and  
 $a \geq b$  if  $a > b$  or  $a = b$ .

Let  $X$  be an ordered set with order  $\leq$ . Given  $a, b \in X$ , we define the four intervals

$$\begin{aligned}(a, b) &= \{x \in X : a < x < b\} \\ [a, b) &= \{x \in X : a \leq x < b\} \quad (a, b] : \{x \in X : a < x \leq b\} \\ [a, b] &= \{x \in X : a \leq x \leq b\}\end{aligned}$$

with at least two elements

Lemma. Let  $X$  be an ordered set. Let  $B$  be the collection of all sets of the types

(1.4) 1.  $(a, b)$ ,  $a, b \in X$

2.  $[a_0, b)$ , where  $a_0$  is the smallest element (if any) of  $X$ .

3.  $(a, b_0]$ , where  $b_0$  is the largest element (if any) of  $X$ .

$B$  is a basis for a topology on  $X$ , which is known as the **order topology** on  $X$ .

Order Topology

First of all, every  $x \in X$  is contained in one of the above intervals.

$\rightarrow$  if  $x \neq b_0$ ,  $x \in [a_0, b_0] \in B$

$\rightarrow$  if  $x = b_0$ ,  $x \in (a_0, b_0] \in B$

Second, note that the intersection of two sets in  $B$  is either empty or in  $B$ .

The result follows. □

Observe that the standard topology on  $\mathbb{R}$  is the order topology under the usual order.

Another example: the dictionary relation on  $\mathbb{R} \times \mathbb{R}$  has

$$B = \{(a \times b, c \times d) : \underbrace{a < c}_{(a,c) \times \mathbb{R}} \text{ or } \underbrace{a=c \text{ and } b < d}_{a \times (b,d)}\}$$

Let  $B' = \{a \times (b, d) : b < d\} \subseteq B$ .

Show that  $B'$  is a basis and that the topology generated by  $B'$  is the same as that generated by  $B$ .

Hint. Use the theorem given last class to show the non-trivial containment

If  $X$  is an ordered set and  $a \in X$ , we define the rays determined by  $a$

$$\begin{aligned} (a, \infty) &= \{x \in X : x > a\} \\ (-\infty, a) &= \{x \in X : x < a\} \\ [a, \infty) &= \{x \in X : x \geq a\} \\ (-\infty, a] &= \{x \in X : x \leq a\} \end{aligned} \quad \left. \begin{array}{l} \text{Open rays} \\ \text{Closed rays} \end{array} \right\}$$

Note that any open ray is open in the order topology.

Def. Let  $X$  and  $Y$  be topological spaces. The product topology on  $X \times Y$  is the topology with basis

$$\mathcal{B} = \{U \times V : U \text{ open in } X \text{ and } V \text{ open in } Y\}$$

Product Topology  
Why is  $\mathcal{B}$  a basis?

$$\rightarrow X \times Y \in \mathcal{B}$$

$$\rightarrow \text{For } U_1 \times V_1, U_2 \times V_2 \in \mathcal{B},$$

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$$

Also,  $\mathcal{B}$  need not be a topology

Consider  $\mathbb{R} \times \mathbb{R}$ :

$$((1, 3) \times (1, 3)) \cup ((2, 4) \times (2, 4)) \notin \mathcal{B}.$$

Theo. Let  $\mathcal{B}$  and  $\mathcal{C}$  bases for the topologies on  $X$  and  $Y$  respectively. Then  
(1.5)  $P = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$

is a basis for the product topology on  $X \times Y$ .

Proof Let  $U = U_1 \times U_2$  be open in  $X \times Y$  and  $(x_1, x_2) \in U$ . By definition, there are  $B \in \mathcal{B}$  and  $C \in \mathcal{C}$  such that  $x_1 \in B \subseteq U_1$  and  $x_2 \in C \subseteq U_2$ . Then  $(x_1, x_2) \in B \times C \subseteq U_1 \times U_2$  so  $P$  is a basis of the product topology.  
(by Lemma 1.1) □

Def. Let  $\pi_1: X \times Y \rightarrow X$  be defined by  $(x, y) \mapsto x$  and  $\pi_2: X \times Y \rightarrow Y$  by projection  $(x, y) \mapsto y$ . The maps  $\pi_1$  and  $\pi_2$  are called the **projections** of  $X \times Y$  onto its first and second factors respectively.

Note that for  $U \subseteq X$  and  $V \subseteq Y$ ,  $\pi_1^{-1}(U) = U \times Y$  and  $\pi_2^{-1}(V) = X \times V$ .

→ If  $U$  is open in  $X$ ,  $\pi_1^{-1}(U)$  is open in  $X \times Y$ .

If  $V$  is open in  $Y$ ,  $\pi_2^{-1}(V)$  is open in  $X \times Y$

→ This becomes relevant when we define continuous maps.

Theo. The collection

(1.6)  $S = \{ \pi_1^{-1}(U) : U \text{ open in } X \} \cup \{ \pi_2^{-1}(V) : V \text{ open in } Y \}$   
is a subbasis for the product topology on  $X \times Y$ .

Proof.  $S$  is obviously a subbasis as  $(X \times Y) \in S$ .

For any open  $U \subseteq X, V \subseteq Y$ ,

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$$

The required easily follows. □

Def. Let  $(X, \tau)$  be a topological space. If  $Y \subseteq X$ ,

$$\tau_Y = \{ Y \cap U : U \in \tau \}$$

Subspace Topology is a topology on  $Y$ , called the **subspace topology** on  $Y$  and  $Y$  is then called a **subspace** of  $X$ . (Proving that it is a topology is trivial)

Note that if  $Y$  is open,  $\tau_Y = \{ U \cap Y : U \in \tau \}$ .

Lemma If  $B$  is a basis for the topology of  $X$ , the collection

$$B_Y = \{ B \cap Y : B \in B \}$$

is a basis for the subspace topology on  $Y$ .

(Proof left as exercise)

If  $Y$  is a subspace of  $X$ , we say that a set  $U$  is open in  $Y$  if it belongs to the topology of  $Y$ .

A set open in  $Y$  need not be open in  $X$ .  
(but it is if  $Y$  is open)

Theorem  
(1.8) If  $A$  is a subspace of  $X$  and  $B$  is a subspace of  $Y$ , then the product topology on  $A \times B$  is equal to the topology it inherits as a subspace of  $X \times Y$ .

Indeed, the basis of the former is

$$\{ (U \cap A) \times (V \cap B) : U \text{ open in } X, V \text{ open in } Y \}$$

and that of the latter is

$$\{ (U \times V) \cap (A \times B) : U \text{ open in } X, V \text{ open in } Y \}$$

which are both equal.