Function Spaces

Lecture 27 - 07/04/21 Cauchy Sequences and Complete Spaces

Recall the definition of a Cauchy sequence in a metric space.
A metric space (X,d) is said to be complete if every Cauchy seq. in X converges.

→ Any convergent sequence is Cauchy.

→ Let A be a closed subspace of a complete metric space (X,d). Then A is complete in the restricted metric.

→ X is complete under the metric d iff it is complete under the standard bounded metric $\bar{d} = \min\{d, 1\}$.

Try to prove the above basic facts.

(5.1) **Lemma.** X is complete iff every Cauchy sequence in X has a convergent subsequence.

Proof. Let (x_n) be Cauchy and the subseq. $x_{n_k} \to x$.
For $\varepsilon > 0$, let N such that $d(x_n, x_m) < \varepsilon/2$ for all $n, m > N$.
For sufficiently large $k > N$, let $d(x_{n_k}, x) < \varepsilon/2$.
Then $d(x_n, x) < \varepsilon$ for all $n > N$, proving the claim.
The other direction is direct.

Theo. (5.2) $\mathbb{R}^k$ is complete in the Euclidean metric d and the square metric ρ.

Proof Since $\rho < d < \sqrt{k}\,\rho$, $\mathbb{R}^k$ is complete wrt one metric iff it is complete wrt the other.
We shall show that $(\mathbb{R}^k, \rho)$ is complete. Let (x_n) be Cauchy. Then $\{x_n\}$ is a bounded subset of $\mathbb{R}^k$ (Why?).
Let $x_n < M$ for all $n \in \mathbb{N}$. Then $x_n \in [-M, M]^k$ for any k. Since this subspace is compact (it is closed and bounded), any sequence in it has a convergent subsequence, (x_n) in particular. Using Lemma 5.1 completes the proof.

Lemma (5.3). Let $X = \prod_{\alpha \in J} X_\alpha$ and (x_n) a sequence of points in X. Then $x_n \to x$ iff $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for all $\alpha \in J$.

Proof The forward direction is immediate since π_α is a continuous map.
Suppose $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for all $\alpha \in J$
Let $U = \prod_\alpha U_\alpha$ be a basis element of X containing x. For each α with $U_\alpha \neq X_\alpha$, let N_α such that $\pi_\alpha(x_n) \in U_\alpha$ for $n \geq N_\alpha$. Letting N be the largest of these N_α, then for all $n \geq N$, $x_n \in U$. Therefore, $x_n \to x$

Theo. (5.4) There is a metric for the product space $\mathbb{R}^\omega$ with respect to which it is complete.

Proof. Let $D(x,y) = \sup_i \left\{ \min\{d(x,y),1\}/_i \right\}$. D induces the product topology.

We claim that $\mathbb{R}^\omega$ under D is complete. Let (x_n) be Cauchy in $\mathbb{R}^\omega$. For fixed i, $(\pi_i(x_n))$ is Cauchy because $\bar{d}(\pi_i(x), \pi_i(y)) \leq i D(x,y)$. Therefore, $(\pi_i(x_n))$ converges to some a_i. The result follows on using Lemma 5.3.

Completeness is <u>not</u> a topological property. For example, $(-1,1)$ is not complete and $\mathbb{R}$ is, but the two are homeomorphic.

Def Let (Y,d) be a metric space. If $x = (x_\alpha)_{\alpha\in J}$ and $y = (y_\alpha)_{\alpha \in J}$ are points in Y^J, then

$$\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\}$$

defines a metric on Y^J. It is called the uniform metric on Y^J corresponding to the metric d on Y^J.

Recall that $Y^J = \{f : J \to Y\}$. Then

$$\bar{\rho}(f,g) = \sup\{\bar{d}(f(\alpha), g(\alpha)) : \alpha \in J\}.$$

Theo. (5.5) With the above defined notation, if (Y,d) is complete, then so is $(Y^J, \bar{\rho})$.

Proof. Since (Y,d) is complete, so is $(Y,\bar{d})$.
Let (f_n) be Cauchy in Y^J For $\alpha \in J$,

$$\bar{d}(f_n(\alpha), f_m(\alpha)) \leq \bar{\rho}(f_n, f_m).$$

Therefore, $(f_n(\alpha))$ is Cauchy in $(Y,\bar{d})$, and thus convergent (Why?).
Let $f_n(\alpha) \to f(\alpha)$ for each α.
Let $\varepsilon > 0$. Cauchyness implies that for sufficiently large N,

$$\bar{d}(f_n(\alpha), f_m(\alpha)) < \varepsilon/2 \quad \text{for } n,m \geq N \text{ and } \alpha \in J.$$

Then making m arbitrarily large and using convergence of $(f_m(\alpha))$,

$$\bar{d}(f_n(\alpha), f(\alpha)) \leq \varepsilon/2 \quad \text{for } n \geq N \text{ and } \alpha \in J.$$

Therefore,

$$\bar{\rho}(f_n, f) \leq \varepsilon/2 < \varepsilon \quad \text{for } n \geq N,$$

proving the claim.

Henceforth, denote

$$Y^X = \{f : Y \to X\}$$
$$\mathcal{C}(X,Y) = \{f : Y \to X : f \text{ is continuous}\}$$
$$\mathcal{B}(X,Y) = \{f : Y \to X : f \text{ is bounded}\}$$

Theo. (5.6) Let X be a topological space and (Y,d) be a metric space Then $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$ are closed in Y^X under the uniform metric.
In particular, if Y is complete, so are $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$.

Proof. Let $f_n \to f$ in Y^X. We claim that f_n converges uniformly.
For each $\varepsilon > 0$, choose N such that

$$\bar{\rho}(f, f_n) < \varepsilon \quad \text{for all } n > N.$$

Then for all $x \in X$,

$$\bar{d}(f(x), f_n(x)) \leq \bar{\rho}(f, f_n) < \varepsilon,$$

so (f_n) converges uniformly to f.

Now, we show that $\mathcal{C}(X,Y)$ is closed in Y^X relative to $\bar{\rho}$.
Let $(f_n) \in \mathcal{C}(X,Y)$ and $f_n \to f$. By the uniform limit theorem, $f \in \mathcal{C}(X,Y)$

Showing that $\mathcal{B}(X,Y)$ is closed is straightforward (using the triangle inequality).