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# MA 861 : COMBINATORICS I

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**Amit Rajaraman**

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## §0. Notation and Prerequisites

Given  $n \in \mathbb{N}$ ,  $[n]$  denotes the set  $\{1, \dots, n\}$  and  $[n]_0$  denotes the set  $[n] \cup \{0\}$ .

## §1. Introduction

**Exercise 1.1.** Recall that the number of  $k$ -subsets of  $[n]$  is  $\binom{n}{k}$ . Given a  $k$ -subset  $S = \{x_1, \dots, x_k\}$  of  $[n]$ , we write  $S_{<} = \{x_1, \dots, x_k\}_{<}$  to denote that  $x_1 < x_2 < \dots < x_k$ . Determine the number of  $k$ -subsets  $\{x_1, \dots, x_k\}_{<}$  of  $[n]$  such that  $x_i \equiv i \pmod{2}$ .

For example, for  $n = 6$  and  $k = 3$ , we have the subsets  $\{1, 4, 5\}, \{1, 2, 3\}, \{1, 2, 5\}, \{3, 4, 5\}$ .

Broadly, there are three types of “answers”: a formula, a recurrence, and a generating function. A great example of the second and third is the following.

$p(n)$ , the number of number partitions of  $n$ , is given by the generating function

$$\sum_{n \geq 0} p(n)x^n = \prod_{i \geq 1} \frac{1}{1 - x^i}.$$

Using this, a recursion may be obtained as well. We do *not* plug in values for  $x$  in the above. We merely look at the coefficient of  $x^n$  in it. We want the coefficient to be a finite sum for all  $n$ . If it is an infinite sum, convergence issues may arise.

### 1.1. Counting in $\mathfrak{S}_n$

Recall that  $\mathfrak{S}_n$  is generated by transpositions. A transposition  $(i, j)$  is a permutation  $\sigma$  defined by

$$\sigma(k) = \begin{cases} j, & k = i, \\ i, & k = j, \\ k, & \text{otherwise.} \end{cases}$$

In fact,  $\mathfrak{S}_n$  is generated by the set of just “adjacent transpositions”  $S_i = (i, i + 1)$  for  $1 \leq i < n$ . We have

$$\begin{aligned} S_i^2 &= \text{Id} \\ S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1} \\ S_i S_j &= S_j S_i \text{ if } |i - j| > 2. \end{aligned}$$

**Definition 1.1.** Given a permutation  $\pi \in \mathfrak{S}_n$ , define the *length*  $\ell(\pi)$  of  $\pi$  to be the smallest  $k$  such that there exist adjacent transpositions  $\sigma_1, \dots, \sigma_k$  such that  $\pi = \sigma_1 \cdots \sigma_k$ .

**Proposition 1.1.** Consider the *inversion number*  $\text{inv}(\pi)$  of a permutation, defined by

$$\text{inv}(\pi) = |\{1 \leq i \leq j \leq n : \pi_i > \pi_j\}|.$$

Then,  $\ell(\pi) = \text{inv}(\pi)$ .

**Definition 1.2.** The *sign* of a permutation  $\pi$  is defined by  $\text{sign}(\pi) = (-1)^{\text{inv}(\pi)}$ . Equivalently,

$$\text{sign}(\pi) = \frac{\prod_{1 \leq i < j \leq n} (x_{\pi_i} - x_{\pi_j})}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}.$$

It is straightforward to see that for all  $\pi \in \mathfrak{S}_n$ ,  $0 \leq \text{inv}(\pi) \leq \binom{n}{2}$ .

**Proposition 1.2.** Consider  $\text{inv}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\text{inv}(\pi)}$ . Then,

$$\text{inv}_n(q) = \prod_{1 \leq m \leq n} [m]_q,$$

where

$$[m]_q = \begin{cases} 1 + q + \cdots + q^{m-1}, & m \geq 1, \\ 0, & m = 0. \end{cases}$$

This quantity  $[m]_q$  is called the  $q$ -analogue of  $m$ , and similarly, the  $q$ -analogue of  $n!$  is  $\prod_{i=1}^n [i]_q$  (this is slightly vague). Note in particular that  $n! = \text{inv}_n(1)$ .

*Proof.* We prove this by induction. It is easily verified for  $n = 2$ .

Take  $\sigma \in \mathfrak{S}_{n-1}$ . There are  $n$  “gaps” where  $n$  can be “placed” in  $\sigma$  to get a permutation in  $\mathfrak{S}_n$ . If we place it in the  $i$ th position from the end (for  $0 \leq i \leq n-1$ ), the inversion number of the newly obtained permutation is  $i$  more than the inversion number of  $\sigma$ .

As a result,

$$\text{inv}_n(q) = \text{inv}_{n-1}(q) + q \text{inv}_{n-1}(q) + q^2 \text{inv}_{n-1}(q) + \cdots + q^{n-1} \text{inv}_{n-1}(q) = [n]_q \text{inv}_{n-1}(q),$$

where the  $q^i \text{inv}_{n-1}(q)$  term corresponds to the case where  $n$  is placed in the  $i$ th position from the end. The required follows by the inductive hypothesis. ■

**Definition 1.3** (Descent). For  $\pi \in \mathfrak{S}_n$ , define the *descents*  $\text{DES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$ ,  $\text{des}(\pi) = |\text{DES}(\pi)|$ , and  $\text{maj}(\pi) = \sum_{i \in \text{DES}(\pi)} i$ .

There are central limit theorems for many of these parameters, which we shall not study.

A permutation  $\pi$  has  $\text{des}(\pi) + 1$  many “increasing runs”.

For example, for the permutation  $\pi = (1 \mapsto 5, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 3) \in \mathfrak{S}_6$ ,  $\text{DES}(\pi) = \{1, 4, 5\}$ ,  $\text{des}(\pi) = 3$ , and  $\text{maj}(\pi) = 10$ .

**Proposition 1.3.** The distribution of  $\text{maj}(\pi)$  over  $\mathfrak{S}_n$  is the same as that of  $\text{inv}(\pi)$ . Equivalently,

$$\text{maj}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\text{maj}(\pi)} = \prod_{m=1}^n [m]_q = \text{inv}_n(q).$$

This result took nearly 50 years to prove!

*Proof.* The strategy is similar to that of Proposition 1.2. Let  $\pi \in \mathfrak{S}_{n-1}$ . As before, there are  $n$  positions to insert  $n$ .

- Label the positions of descents of  $\sigma$  and the last position from right to left as  $0, 1, \dots, \text{des}(\pi)$ .
- Label the remaining positions from left to right as  $\text{des}(\pi) + 1, \dots, n-1$ .

We claim that inserting  $n$  at a position increases  $\text{maj}$  by the labelled amount.

If inserted anywhere, all the descent positions starting from there increase by 1. This explains why the increase is equal to the labelled quantity for positions that are descents, since no new descents are introduced. In the case where we insert it in a position of non-descent, we further introduce a new descent at the position of insertion of  $n$ , which explains why the increase is equal to the labelled quantity for positions that are not descents.

The remainder of the proof is identical to that of Proposition 1.2, since the increases are in bijection with  $[n-1]_0$ . ■

## §2. Problem Sheets

### 2.1. Problem Sheet 1

**Problem 2.1.1.** Let  $S(n, k)$  and  $s(n, k)$  be Stirling numbers of the second and first kind respectively. Show that for all  $n, k$ , we have  $s(n, k) \geq S(n, k)$ .

#### Solution

Let  $X_{S(n,k)}$  be the set of partitions of  $[n]$  into exactly  $k$  parts and  $X_{s(n,k)}$  the number of permutations of  $[n]$  with exactly  $k$  cycles. Recall that by definition,  $|X_{S(n,k)}| = S(n, k)$  and  $|X_{s(n,k)}| = s(n, k)$ . It suffices to demonstrate an injection  $f$  from  $X_{S(n,k)}$  to  $X_{s(n,k)}$ . We do so as follows. Let  $\{\{x_{1,1}, \dots, x_{1,n_1}\}, \dots, \{x_{k,1}, \dots, x_{k,n_k}\}\}$  be a partition of  $[n]$  into exactly  $k$  parts, where  $x_{i,j_1} < x_{i,j_2}$  for  $j_1 < j_2$ . Then, we have a corresponding permutation of  $[n]$  with exactly  $k$  cycles given by  $(x_{1,1}, \dots, x_{1,n_1}) \cdots (x_{k,1}, \dots, x_{k,n_k})$ . This map is clearly an injection, so we are done.

**Problem 2.1.2.** Show that

$$S(n, k) = \sum_{r=1}^k (-1)^{k-r} \frac{r^n}{r!(k-r)!}.$$

#### Solution

$S(n, k)$  is merely  $1/k!$  times the number of surjective functions from  $[n]$  to  $[k]$  (because the ordering of the partitions does not matter). The set of functions that are *not* surjective is

$$\bigcup_{i \in [n]} \{f \in [k]^{[n]} : i \notin \text{Im}(f)\}.$$

The size of the above is quite easily determined by the inclusion-exclusion principle to get

$$k^n - k!S(n, k) = \sum_{r=1}^{k-1} (-1)^{r+1} \cdot \underbrace{\binom{k}{r}}_{\text{choosing } r \text{ elements in } [k] \text{ to "avoid" }} \cdot \underbrace{(k-r)^n}_{\text{counting functions that avoid the chosen}},$$

and the desideratum immediately follows.

**Problem 2.1.3.** Let  $A_n(y) = \sum_k S(n, k)y^k$ . Show that  $A_n(y) = (y + yD)^n 1$  where  $D = \frac{d}{dy}$  is the derivative operator.

#### Solution

**Problem 2.1.4.** Let  $D_n$  be the number of derangements in  $\mathfrak{S}_n$  and let  $D(x) = \sum_{n \geq 0} D_n x^n / n!$  be its egf. Determine  $D(x)$ .

#### Solution

A permutation  $\pi \in \mathfrak{S}_n$  is a derangement iff it has no cycles of length 1. Define  $f : \mathbb{N} \rightarrow \mathbb{N}_0$  by

$$f(k) = \begin{cases} 0, & k = 1, \\ 1, & \text{otherwise.} \end{cases}$$

By the earlier observation,  $\pi \in \mathfrak{S}_n$  is a derangement iff  $f(|C_1|) \cdots f(|C_k|) = 1$  where  $C_1, \dots, C_k$  are the cycles of

$\pi$ . Using Corollary 5.1.9 in [SF99], we get that

$$D(x) = \exp \left( \sum_{n \geq 2} \frac{x^n}{n} \right) = \exp(-x - \log(1-x)) = \frac{e^{-x}}{1-x}.$$

**Problem 2.1.5.** Let  $s(n, 2)$  denote the number of  $\pi \in \mathfrak{S}_n$  with 2 cycles in its cyclic decomposition and let  $H_n$  denote the  $n$ th harmonic number. Show that  $s(n+1, 2) = H_n \times n!$ .

### Solution

It is easily checkable that the number of cyclic permutations of  $[k]$  is  $(k-1)!$ . We have

$$\begin{aligned} s(n+1, 2) &= \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \binom{n+1}{k} (k-1)! (n+1-k-1)! \\ &= \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(n+1)!}{k(n+1-k)} \\ &= \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} n! \left( \frac{1}{k} + \frac{1}{n+1-k} \right) \\ &= n! \left( \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{1}{k} + \sum_{k=\lceil (n+1)/2 \rceil}^n \frac{1}{k} \right) \\ &= \begin{cases} n! H_n, & n \text{ is even,} \\ n! (H_n + \frac{2}{n+1}), & n \text{ is odd.} \end{cases} \end{aligned}$$

**Problem 2.1.6.** For a fixed positive integer  $k$ , consider the egf  $f_k(x) = \sum_{n \geq 0} s(n, k) x^n / n!$ . Show that

$$f_k(x) = \frac{1}{k!} \ln \left( \frac{1}{1-x} \right)^k.$$

### Solution

Define  $g : \mathbb{N} \rightarrow \mathbb{N}_0$  by

$$g(r) = \begin{cases} 1, & r = k, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$s(n, k) = \sum_{\pi \in \mathfrak{S}_n} g(r),$$

where  $C_1, \dots, C_r$  are the cycles in  $\pi$ . We now use Corollary 5.1.8 in [SF99] with  $f$  being the function that takes the constant 1. We have  $E_g(x) = \frac{1}{(k-1)!} x^k$ , so we get that

$$f_k(x) = \frac{1}{(k-1)!} \left( \sum_{n \geq 1} \frac{x^n}{n} \right)^k = \frac{1}{(k-1)!} \ln \left( \frac{1}{1-x} \right)^k.$$

**Problem 2.1.7.** Find  $\sum_{k=0}^n (-1)^k s(n, k)$ .

### Solution

**Problem 2.1.8.** Show that  $S(n+1, k+1) = \sum_{m=0}^n \binom{n}{m} S(m, k)$ .

**Solution**

**Problem 2.1.9.** Find the number of ways in which a product of  $n$  distinct primes can be factored.

**Solution**

**Problem 2.1.10.** Show that for  $n \geq 1$ , the  $S_{n,k}$  as  $k$  varies has either a unique maximum value or has at most two equal values.

**Solution**

## References

[SF99] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.