MA 861 : Combinatorics I

Amit Rajaraman

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Contents

§0. Notation and Prerequisites

Given $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots, n\}$ and $[n]_0$ denotes the set $[n] \cup \{0\}$.
$S(n,k)$, a Stirling number of the second kind, is the number of partitions of $[n]$ into exactly k parts. $s(n,k)$, a Stirling number of the first kind, is the number of permutations of $[n]$ with exactly k cycles.

§1. Introduction

Exercise 1.1. Recall that the number of k-subsets of $[n]$ is $\binom{n}{k}$. Given a k-subset $S = \{x_1, \dots, x_k\}$ of $[n]$, we write $S_< = \{x_1, \dots, x_k\}_<$ to denote that $x_1 < x_2 < \cdots < x_k$. Determine the number of k-subsets $\{x_1, \dots, x_k\}_<$ of $[n]$ such that $x_i \cong i \mod 2$.

For example, for $n = 6$ and $k = 3$, we have the subsets $\{1, 4, 5\}, \{1, 2, 3\}, \{1, 2, 5\}, \{3, 4, 5\}$.
Broadly, there are three types of "answers": a formula, a recurrence, and a generating function. A great example of the second and third is the following.
$p(n)$, the number of number partitions of n, is given by the generating function

$$\sum_{n \geq 0} p(n) x^n = \prod_{i \geq 1} \frac{1}{1 - x^i}.$$

Using this, a recursion may be obtained as well. We do *not* plug in values for x in the above. We merely look at the coefficient of x^n in it. We want the coefficient to be a finite sum for all n. If it is an infinite sum, convergence issues may arise.

1.1. Counting in $\mathfrak{S}_n$

Recall that $\mathfrak{S}_n$ is generated by transpositions. A transposition (i, j) is a permutation σ defined by

$$\sigma(k) = \begin{cases} j, & k = i, \\ i, & k = j, \\ k, & \text{otherwise.} \end{cases}$$

In fact, $\mathfrak{S}_n$ is generated by the set of just "adjacent transpositions" $S_i = (i, i+1)$ for $1 \leq i < n$. We have

$$\begin{aligned} S_i^2 &= \mathrm{Id} \\ S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1} \\ S_i S_j &= S_j S_i \text{ if } |i - j| > 2. \end{aligned}$$

Definition 1.1. Given a permutation $\pi \in \mathfrak{S}_n$, define the *length* $\ell(\pi)$ of π to be the smallest k such that there exist adjacent transpositions $\sigma_1, \cdots, \sigma_k$ such that $\pi = \sigma_1 \cdots \sigma_k$.

Proposition 1.1. Consider the *inversion number* $\mathrm{inv}(\pi)$ of a permutation, defined by

$$\mathrm{inv}(\pi) = \left|\{1 \leq i \leq j \leq n : \pi_i > \pi_j\}\right|.$$

Then, $\ell(\pi) = \mathrm{inv}(\pi)$.

Definition 1.2. The *sign* of a permutation π is defined by $\mathrm{sign}(\pi) = (-1)^{\mathrm{inv}(\pi)}$. Equivalently,

$$\mathrm{sign}(\pi) = \frac{\prod_{1 \leq i < j \leq n} (x_{pi_i} - x_{\pi_j})}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}.$$

It is straightforward to see that for all $\pi \in \mathfrak{S}_n$, $0 \leq \operatorname{inv}(\pi) \leq \binom{n}{2}$.

Proposition 1.2. Consider $\operatorname{inv}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{inv}(\pi)}$. Then,

$$\operatorname{inv}_n(q) = \prod_{1 \leq m \leq n} [n]_q,$$

where

$$[m]_q = \begin{cases} 1 + q + \cdots + q^{m-1}, & m \geq 1, \\ 0, & m = 0. \end{cases}$$

This quantity $[m]_q$ is called the q-analogue of m, and similarly, the q-analogue of $n!$ is $\prod_{i=1}^{n} [m]_q$ (this is slightly vague). Note in particular that $n! = \operatorname{inv}_n(1)$.

Proof. We prove this by induction. It is easily verified for $n = 2$.
Take $\sigma \in \mathfrak{S}_{n-1}$. There are n "gaps" where n can be "placed" in σ to get a permutation in $\mathfrak{S}_n$. If we place it in the ith position from the end (for $0 \leq i \leq n-1$), the inversion number of the newly obtained permutation is i more than the inversion number of σ.
As a result,

$$\operatorname{inv}_n(q) = \operatorname{inv}_{n-1}(q) + q \operatorname{inv}_{n-1}(q) + q^2 \operatorname{inv}_{n-1}(q) + \cdots + q^{n-1} \operatorname{inv}_{n-1}(q) = [n]_q \operatorname{inv}_{n-1}(q),$$

where the $q^i \operatorname{inv}_{n-1}(q)$ term corresponds to the case where n is placed in the ith position from the end. The required follows by the inductive hypothesis. ■

Definition 1.3 (Descent)**.** For $\pi \in \mathfrak{S}_n$, define the *descents* $\operatorname{DES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$, $\operatorname{des}(\pi) = |\operatorname{DES}(\pi)|$, and $\operatorname{maj}(\pi) = \sum_{i \in \operatorname{DES}(\pi)} i$.

There are central limit theorems for many of these parameters, which we shall not study.
A permutation π has $\operatorname{des}(\pi) + 1$ many "increasing runs".
For example, for the permutation $\pi = (1 \mapsto 5, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 6, 5 \mapsto 4, 6 \mapsto 3) \in \mathfrak{S}_6$, $\operatorname{DES}(\pi) = \{1, 4, 5\}$, $\operatorname{des}(\pi) = 3$, and $\operatorname{maj}(\pi) = 10$.

Proposition 1.3. The distribution of $\operatorname{maj}(\pi)$ over $\mathfrak{S}_n$ is the same as that of $\operatorname{inv}(\pi)$. Equivalently,

$$\operatorname{maj}_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{maj}(\pi)} = \prod_{m=1}^{n} [m]_q = \operatorname{inv}_n(q).$$

This result took nearly 50 years to prove!

Proof. The strategy is similar to that of Proposition 1.2. Let $\pi \in \mathfrak{S}_{n-1}$. As before, there are n positions to insert n.

- Label the positions of descents of σ and the last position from right ot left as $0, 1, \ldots, \operatorname{des}(\pi)$.
- Label the remaining positions from left to right as $\operatorname{des}(\pi) + 1, \ldots, n-1$.

We claim that inserting n at a position increases maj by the labelled amount.
If inserted anywhere, all the descent positions starting from there increase by 1. This explains why the increase is equal to the labelled quantity for positions that are descents, since no new descents are introduced. In the case where we insert it in a position of non-descent, we further introduce a new descent at the position of insertion of n, which explains why the increase is equal tot he labelled quantity for positions that are not descents.
The remainder of the proof is identical to that of Proposition 1.2, since the increases are in bijection with $[n-1]_0$. ■

Definition 1.4. A parameter $f : \mathfrak{S}_n \to \mathbb{R}$ of permutations such that

$$\sum_{\pi \in \mathfrak{S}_n} q^{f(\pi)} = \prod_{1 \le m \le n} [m]_q$$

is said to be *Mahonian*.

As we saw in **??** 1.2**??** 1.3, both inv and maj are Mahonian.

1.2. Counting spanning trees

Problem. Count the number of spanning trees in an arbitrary (finite) graph G.

This was solved by Kirchhoff using the Matrix Tree Theorem.

Theorem 1.4 (Matrix Tree Theorem)**.** Consider the *Laplacian* $L = D - A$ of a graph G, where A is its adjacency matrix and D is a diagonal matrix with the diagonal entries being the degrees of the vertices. The determinant of any $(n-1) \times (n-1)$ submatrix of L obtained by omitting any arbirary row and column is equal to the number of spanning trees of G.

In particular, when $G = K_n$, we end up getting the following.

Theorem 1.5 (Cayley's Theorem)**.** The number of spanningtrees in K_n is n^{n-2}.

One proof by Prüfer gives an explicit bijection between spanning trees and sequences $(v_1, \ldots, v_{n-2})$ of vertices in G. Another proof is of course using the matrix tree theorem, which reduces it to a simple determinant calculation. Joyal gave another bijection between elements of the form (T, u, v) where T is a spanning tree and u, v are vertices in G, and functions from $[n] \to [n]$.

The proof we give uses exponential generating functions. Recall the following result, which we give without proof. Interested readers may consult Corollary 5.1.6 of [SF99] for further details.

Theorem 1.6 (Exponential Formula)**.** Let $\{f_n\}$ be a sequence with exponential generating function

$$F(x) = \sum_{n \ge 1} f_n \frac{x^n}{n!}.$$

Define the sequence h_n by

$$h_n = \sum_{\substack{\pi \in \mathrm{SetPartn}([n]) \\ \pi = \{S_1, \ldots, S_k\}}} f_{|S_1|} f_{|S_2|} \cdots f_{|S_k|}$$

and $h_0 = 1$, and let

$$H(x) = \sum_{n \ge 0} h_n \frac{x^n}{n!}.$$

Then,

$$H(x) = \exp(F(x))$$

Note that the summation of F is for $n \geq 1$, because we may assume that $f_0 = 0$ since f_0 does not appear in the expression of any h_n.

Definition 1.5 (Compositional inverse)**.** Generating functions F and G are said to be *compositional inverses* (of each other) if $F(G(x)) = G(F(x)) = x$.

Let

$$F(x) = \sum_{n \geq 0} f_n x^n \text{ and } G(x) = \sum_{n \geq 0} g_n x^n$$

be compositional inverses of each other. It is reasonably straightforward to show that $f_0 = g_0 = 0$ and $f_1, g_1 \neq 0$. The first condition implies that the coefficient of any x^n in $F \circ G$ (or $G \circ F$) is finite.

Theorem 1.7 (Lagrange Inversion Theorem)**.** Let

$$F(x) = \sum_{n \geq 0} f_n x^n \text{ and } G(x) = \sum_{n \geq 0} g_n x^n$$

be compositional inverses of each other. Then, ng_n is the coefficient of $1/x$ in $(1/F(x))^n$.

Equivalently, ng_n is the coefficient of x^{n-1} in $(x/F(x))^n$.

Proof. We have

$$x = G(F(x)) = \sum_{i \geq 0} g_i F(x)^i.$$

Differentiating,

$$1 = \sum_{i \geq 0} g_i i F(x)^{i-1} F'(x).$$

As a result,

$$\left(\frac{1}{F(x)}\right)^n = \sum_{i \geq 0} g_i i F(x)^{i-1-n} F'(x).$$

Whenever $i \neq n$, the coefficient of $1/x$ in $F(x)^{i-1-n}F'(x) = \left(F(x)^{i-n}/(i-n)\right)'$ is zero. Indeed, recall that the coefficient of $1/x$ in the derivative of any power series with possibly negative exponents is zero.
As a result, the coefficient of $1/x$ in $(1/F(x))^n$ is equal to the coefficient of $1/x$ in $g_n n F'(x)/F(x)$. We have

$$\frac{F'(x)}{F(x)} = \frac{f_1 + 2f_2 x + \cdots}{f_1 x + f_2 x^2 + \cdots}.$$

The constant term in this is $f_1/f_1 = 1$, and the desideratum follows. ∎

At long last, let us return to Cayley's Theorem.

Proof of Cayley's Theorem. Instead of looking at the number T_n of spanning trees, we shall look at RT_n, the number of *rooted* spanning trees. Clearly, $\mathrm{RT}_n = n\mathrm{T}_n$.
Define RF_n to be the number of rooted forests on $[n]$ and let

$$\mathrm{RF}(x) = \sum_{n \geq 0} \mathrm{RF}_n \frac{x^n}{n!}$$

$$\mathrm{RT}(x) = \sum_{n \geq 0} \mathrm{RT}_n \frac{x^n}{n!}.$$

Using Theorem 1.6, it is not too difficult to see that

$$\mathrm{RF}(x) = \exp(\mathrm{RT}(x)). \tag{1.1}$$

Claim (Polya). $\mathrm{RT}_{n+1} = (n+1)\mathrm{RF}_n$.
Indeed, any rooted tree on K_{n+1} may be obtained from a rooted forest F on K_n by adding a new vertex v, adding the edge between each root in F and v to the spanning tree, removing the "root status" from all vertices except v. v can be labelled in $n+1$ ways, so we are done.

As a result,

$$\mathrm{RF}(x) = \sum_{n\geq 0} \frac{\mathrm{RT}_{n+1}}{n+1} \cdot \frac{x^n}{n!} = \frac{1}{x}\mathrm{RT}(x). \tag{1.2}$$

Combining Equations (1.1) and (1.2),

$$\mathrm{RT}(x) = x\exp(\mathrm{RT}(x)).$$

That is, RT is the compositional inverse of $x \mapsto xe^{-x}$. Now, we use the Lagrange Inversion Theorem to get that $n\mathrm{RT}_n/n!$ is equal to the coefficient of x^{n-1} in $(x/xe^{-x})^n = e^{nx}$, which is $n^{n-1}/(n-1)!$. Therefore, $\mathrm{T}_n = \mathrm{RT}_n/n = n^{n-2}$ and we are done. ■

1.3. Chebyshev polynomials

We would like a polynomial $T_n(x)$ such that $T_n(\cos\theta) = \cos(n\theta)$. Why does such a polynomial even exist? Recall that

$$(\cos\theta + \iota\sin\theta)^n = \cos n\theta + \iota\sin n\theta.$$

Since the real part of the left only has even powers of $\sin$, we can convert it to a polynomial of $\cos\theta$s alone.
For example,

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1. \end{aligned}$$

Proposition 1.8.

$$T_n(x) = \begin{cases} 1, & n = 0, \\ x, & n = 1, \\ 2xT_{n-1}(x) - T_{n-2}(x), & n \geq 2. \end{cases}$$

We give the proof of the above in the solution to Problem 2.2.1.

Definition 1.6 (Chebyshev polynomials)**.** The nth *Chebyshev polynomial of the first kind* T_n is defined as above.
The nth *Chebyshev polynomial of the second kind* U_n is defined by

$$U_n(x) = \begin{cases} 1, & n = 0, \\ 2x, & n = 1, \\ 2xU_{n-1}(x) - U_{n-2}(x), & n \geq 2. \end{cases}$$

Consider the number of tilings of a $1 \times n$ board B_n using squares (1×1 pieces) and dimers (1×2 pieces). It is not too difficult to show that this corresponds to the Fibonacci numbers.
Now, instead consider a *weighted* version of this problem, where we give squares a weight of $2x$ and dimers a weight of -1. The weight $\mathrm{wt}(T)$ of a given tiling T is equal to the product of the weights of the pieces used. Then, the Chebyshev polynomial U_n is just the sum of the weights of all tilings of B_n!

$$U_n(x) = \sum_{\text{tilings } T \text{ of } B_n} \mathrm{wt}(T).$$

§2. Problem Sheets

2.1. Problem Sheet 1

Problem 2.1.1. Let $S(n,k)$ and $s(n,k)$ be Stirling numbers of the second and first kind respectively. Show that for all n, k, we have $s(n,k) \geq S(n,k)$.

Solution

Let $X_{S(n,k)}$ be the set of partitions of $[n]$ into exactly k parts and $X_{s(n,k)}$ the number of permutations of $[n]$ with exactly k cycles. Recall that by definition, $|X_{S(n,k)}| = S(n,k)$ and $|X_{s(n,k)}| = s(n,k)$. It suffices to demonstrate an injection f from $X_{S(n,k)}$ to $X_{s(n,k)}$. We do so as follows. Let $\{\{x_{1,1}, \ldots, x_{1,n_1}\}, \ldots, \{x_{k,1}, \ldots, x_{1,n_k}\}\}$ be a partition of $[n]$ into exactly k parts, where $x_{i,j_1} < x_{i,j_2}$ for $j_1 < j_2$. Then, we have a corresponding permutation of $[n]$ with exactly k cycles given by $(x_{1,1}, \ldots, x_{1,n_1}) \cdots (x_{k,1}, \ldots, x_{k,n_k})$. This map is clearly an injection, so we are done.

Problem 2.1.2. Show that

$$S(n,k) = \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{r!(k-r)!}.$$

Solution

$S(n,k)$ is merely $1/k!$ times the number of surjective functions from $[n]$ to $[k]$ (because the ordering of the partitions does not matter). The set of functions that are *not* surjective is

$$\bigcup_{i \in [n]} \{f \in [k]^{[n]} : i \notin \mathrm{Im}(f)\}.$$

The size of the above is quite easily determined by the inclusion-exclusion principle to get

$$k^n - k!S(n,k) = \sum_{r=1}^{k-1} (-1)^{r+1} \cdot \underbrace{\binom{k}{r}}_{\text{choosing } r \text{ elements in } [k] \text{ to "avoid"}} \cdot \underbrace{(k-r)^n}_{\text{counting functions that avoid the chosen}},$$

and the desideratum immediately follows.

Problem 2.1.3. Let $A_n(y) = \sum_k S(n,k) y^k$. Show that $A_n(y) = (y + yD)^n 1$ where $D = \frac{\mathrm{d}}{\mathrm{d}y}$ is the derivative operator.

Solution

First, recall that $S(n+1, k+1) = S(n,k) + (k+1) \cdot S(n, k+1)$ – the $S(n,k)$ corresponds of those partitions where $n+1$ is in a part of its own, and the $(k+1)S(n,k+1)$ corresponds to those partitions where this is not the case, so we can consider any partition of $[n]$ into $k+1$ parts, then decide which of the $k+1$ parts to place $n+1$ in.

We have

$$\begin{aligned}(y+yD)A_n(y) &= (y+yD)\sum_{k=1}^{n} S(n,k)y^k \\ &= \sum_{k=1}^{n} S(n,k)(y^{k+1}+ky^k) \\ &= S(n,1)y + \sum_{k=2}^{n} y^k(S(n,k-1)+(k-1)S(n,k)) + S(n,n)y^{n+1} \\ &= S(n+1,1)y + \sum_{k=2}^{n} S(n+1,k)y^k + S(n+1,n+1)y^{n+1} = A_{n+1}(y).\end{aligned}$$

The required follows inductively.

Problem 2.1.4. Let D_n be the number of derangements in $\mathfrak{S}_n$ and let $D(x) = \sum_{n\geq 0} D_n x^n/n!$ be its egf. Determine $D(x)$.

Solution

A permutation $\pi \in \mathfrak{S}_n$ is a derangement iff it has no cycles of length 1. Define $f : \mathbb{N} \to \mathbb{N}_0$ by

$$f(k) = \begin{cases} 0, & k=1, \\ 1, & \text{otherwise.}\end{cases}$$

By the earlier observation, $\pi \in \mathfrak{S}_n$ is a derangement iff $f(|C_1|)\cdots f(|C_k|) = 1$ where $C_1,\ldots,C_k$ are the cycles of π. Using Corollary 5.1.9 in [SF99], we get that

$$D(x) = \exp\left(\sum_{n\geq 2}\frac{x^n}{n}\right) = \exp\big(-x-\log(1-x)\big) = \frac{e^{-x}}{1-x}.$$

Problem 2.1.5. Let $s(n,2)$ denote the number of $\pi \in \mathfrak{S}_n$ with 2 cycles in its cyclic decomposition and let H_n denote the nth harmonic number. Show that $s(n+1,2) = H_n \times n!$.

Solution

It is easily checkable that the number of cyclic permutations of $[k]$ is $(k-1)!$. We have

$$\begin{aligned} s(n+1,2) &= \sum_{k=1}^{\lfloor (n+1)/2\rfloor} \binom{n+1}{k}(k-1)!(n+1-k-1)! \\ &= \sum_{k=1}^{\lfloor (n+1)/2\rfloor} \frac{(n+1)!}{k(n+1-k)} \\ &= \sum_{k=1}^{\lfloor (n+1)/2\rfloor} n!\left(\frac{1}{k}+\frac{1}{n+1-k}\right) \\ &= n!\left(\sum_{k=1}^{\lfloor (n+1)/2\rfloor}\frac{1}{k} + \sum_{k=\lceil (n+1)/2\rceil}^{n}\frac{1}{k}\right) \\ &= \begin{cases} n!H_n, & n \text{ is even,} \\ n!(H_n + \frac{2}{n+1}), & n \text{ is odd.}\end{cases}\end{aligned}$$

Problem 2.1.6. For a fixed positive integer k, consider the egf $f_k(x) = \sum_{n\geq 0} s(n,k)x^n/n!$. Show that

$$f_k(x) = \frac{1}{k!}\ln\left(\frac{1}{1-x}\right)^k.$$

Solution

Define $g : \mathbb{N} \to \mathbb{N}_0$ by

$$g(r) = \begin{cases} 1, & r = k, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$s(n,k) = \sum_{\pi \in \mathfrak{S}_n} g(r),$$

where $C_1, \ldots, C_r$ are the cycles in π. We now use Corollary 5.1.8 in [SF99] with f being the function that takes the constant 1. We have $E_g(x) = \frac{1}{(k-1)!}x^k$, so we get that

$$f_k(x) = \frac{1}{(k-1)!}\left(\sum_{n\geq 1}\frac{x^n}{n}\right)^k = \frac{1}{(k-1)!}\ln\left(\frac{1}{1-x}\right)^k.$$

Problem 2.1.7. Find $\sum_{k=0}^{n}(-1)^k s(n,k)$.

Solution

Recall that a permutation in $\mathfrak{S}_n$ with k cycles has sign $(-1)^{n-k}$. We have

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k s(n,k) &= \sum_{k=0}^{n} \sum_{\substack{\pi\in\mathfrak{S}_n \\ \pi \text{ has exactly } k \text{ cycles}}} (-1)^k \\ &= (-1)^n \sum_{\pi\in\mathfrak{S}_n} \operatorname{sign}(\pi).\end{aligned}$$

For $n = 1$, this is -1. Otherwise, note that $\pi \mapsto (1,2)\pi$ is a bijection between odd and even permutations. As a result, the above sum is equal to 0.

Problem 2.1.8. Show that $S(n+1,k+1) = \sum_{m=0}^{n}\binom{n}{m}S(m,k)$.

Solution

We shall count the partitions of $[n+1]$ into exactly $k+1$ parts based on the part $n+1$ is in. We have

$$S(n+1,k+1) = \sum_{S\subseteq[n]} S(|[n]\setminus S|, k) = \sum_{m=0}^{n}\binom{n}{m}S(n-m,k) = \sum_{m=0}^{n}\binom{n}{m}S(m,k).$$

Problem 2.1.9. Show that for $n \geq 1$, the $S_{n,k}$ as k varies has either a unique maximum value or has at most two equal values.

Solution

2.2. Problem Sheet 2

Problem 2.2.1. Show that

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

when $n \geq 2$ and $T_0(x) = 1, T_1(x) = x$.

Solution

Let $\cos\theta = x$. We have

$$\begin{aligned}T_n(x) = \cos n\theta &= \cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta \\ &= xT_{n-1}(x) - (\sin(n-2)\theta\cos\theta + \cos(n-2)\theta\sin\theta)\sin\theta \\ &= xT_{n-1}(x) - T_{n-2}(x)(1-x^2) - x(\sin\theta\sin(n-2)\theta) \\ &= xT_{n-1}(x) + x^2T_{n-2}(x) - T_{n-2}(x) - x(\cos\theta\cos(n-2)\theta - \cos(n-1)\theta) \\ &= 2xT_{n-1}(x) - T_{n-2}(x).\end{aligned}$$

Problem 2.2.2. Show that

(a) $T_n(1) = 1$ and

(b) $T_n(-1) = (-1)^n$.

Solution

This immediately follows since $T_n(\cos\theta) = \cos n\theta$, so $T_n(\cos 0) = \cos(n\cdot 0) = 1$ and $T_n(\cos\pi) = \cos(n\pi) = (-1)^n$. They can also be easily proved inductively.

Problem 2.2.3. Show that

(a) $U_n(1) = n+1$.

(b) $U_n(-1) = (-1)^n(n+1)$

Solution

We prove this inductively. Both statements are trivially true for $n = 0, 1$. For $n \geq 2$, inductively, we have

$$U_n(1) = 2U_{n-1}(1) - U_{n-2}(1) = 2n - (n-1) = n+1$$

and

$$U_n(-1) = -2U_{n-1}(1) - U_{n-2}(1) = (-1)^n \cdot 2n + (-1)^{n-1}(n-1) = (-1)^n(n+1).$$

Problem 2.2.4. Show that

$$\frac{1}{\iota^n}U_n(\iota/2) = f_{n+1}.$$

Solution

Again, we prove this inductively. We have $U_0(\iota/2) = 1 = f_1$ and $U_1(\iota/2) = \iota = \iota f_2$. For $n \geq 2$, we inductively have

$$U_n(\iota/2) = \iota U_{n-1}(\iota/2) - U_{n-2}(\iota/2) = \iota^n f_n - \iota^{n-2}f_{n-1} = \iota^n(f_n + f_{n-1}) = \iota^n f_{n+1}.$$

Problem 2.2.5. Show that if $m, n \geq 1$,

$$T_{m+n}(x) = T_m(x)U_n(x) - T_{m-1}(x)U_{n-1}(x).$$

Solution

This may be checked manually for $m+n=2,3$. We perform induction on $m+n$. We have that

$$\begin{aligned}
T_{m+n}(x) &= 2xT_{m+n-1}(x) - T_{m+n-2}(x) \\
&= 2x(T_{m-1}(x)U_n(x) - T_{m-2}(x)U_{n-1}(x)) - (T_{m-1}(x)U_{n-1}(x) - T_{m-2}(x)U_{n-2}(x)) \\
&= 2xT_{m-1}(x)U_n(x) - T_{m-1}(x)U_{n-1}(x) - T_{m-2}(x)(2xU_{n-1}(x) - U_{n-2}(x)) \\
&= 2xT_{m-1}(x)U_n(x) - T_{m-1}(x)U_{n-1}(x) - T_{m-2}(x)U_n(x) \\
&= U_n(x)(2xT_{m-1}(x) - T_{m-2}(x)) - T_{m-1}(x)U_{n-1}(x) \\
&= T_m(x)U_n(x) - T_{m-1}(x)U_{n-1}(x).
\end{aligned}$$

Problem 2.2.6. Similar to the tiling combinatorial model for U_n, get a combinatorial model for T_n.

Solution

One can do this in a manner identical to that of U_n, except that a square piece has weight x if it is at the leftmost $(1,1)$ position.

References

[SF99] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.