Maps Between Topological Spaces

Def Let X and Y be topological spaces. A function $f: X \to Y$ is said to be
Continuity **continuous at $b \in X$** if for any open $V \subseteq Y$ with $f(b) \in V$, there exists open $U \ni b$ (in X) such that $f(U) \subseteq V$.
f is **continuous** if for any open V in Y, $f^{-1}(V)$ is open in X.

Note that f is continuous iff it is continuous at all $b \in X$.
(How? Use the fact that an arbitrary union of open sets is open)

Recall that this is equivalent to the usual definition of continuity for metric spaces (taking the metric topology here).

Since the topologies matter as well, note that even the identity map from $\mathbb{R}_c$ to $\mathbb{R}$ is not continuous.

If the topology of Y is given by a basis $\mathcal{B}$ and we want to determine continuity, it suffices to check the pre-images of basis elements of Y. Indeed, use the fact that an arbitrary union of open sets is open.

Further, it suffices to just check subbasis elements!
Indeed, the set of finite intersections of subbasis elements form a basis. (and a finite intersection of open sets is open)

Lecture 10 - 06/02/21 More about Continuous Maps

Theo. (2.1) Let X and Y be topological spaces and $f: X \to Y$. Then the following are equivalent.

i) f is continuous.
ii) For every $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.
iii) For every closed $B \subseteq Y$, $f^{-1}(B)$ is closed in X.
iv) For every $x \in X$ and neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subseteq V$.

Proof

i $\Rightarrow$ ii

Suppose f is continuous and $A \subseteq X$. Let $x \in \overline{A}$.
Let V be a neighbourhood of $f(x)$. We shall show that $V \cap f(A) \neq \emptyset$, implying that $f(x) \in \overline{f(A)}$.
Since $x \in \overline{A}$ and $x \in f^{-1}(V)$, which is open in X,

$$f^{-1}(V) \cap A \neq \emptyset.$$

Let $y \in f^{-1}(V) \cap A$. Then $f(y) \in \underset{(\text{why?})}{V \cap f(A)}$, proving the claim.

ii $\Rightarrow$ iii

Let B be closed in Y and $A = f^{-1}(B)$. Let $x \in \overline{A}$. Then

$$f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq B$$

Therefore, $x \in f^{-1}(B)$ and the claim is proved.

iii $\Rightarrow$ i

Observe that iii is just the definition of continuity but with 'closed' instead of 'open'.
Let $B \subseteq Y$ be open. Then $Y \setminus B$ is closed and $f^{-1}(Y \setminus B)$ is closed. That is, $X \setminus f^{-1}(Y \setminus B)$ is open, and this set is just $f^{-1}(B)$. $\square$

We briefly mentioned i $\Leftrightarrow$ iv earlier. The details are left as an exercise.

Def. Let X and Y be topological spaces and $f: X \to Y$ be a bijection. f is said to be a homeomorphism if both f and f^{-1} are continuous.

Homeomorphism

Equivalently, f is a homeomorphism if for any $U \subseteq X$, $f(U)$ is open (in Y) iff U is open (in X).
That is, it is a continuous open bijection.

continuous: V open $\Downarrow$ $f^{-1}(V)$ open

open: U open $\Downarrow$ $f(U)$ open

A homeomorphism also gives a bijective map between the open sets of X and Y.

So if X has some property that is expressed in terms of the topology on X, Y must have the same property as well.
Such a property is called a topological property of X.
(for example, the space being Hausdorff)

If there is a homeomorphism between two spaces, they are said to be homeomorphic.
(This implicitly uses the fact that if there is a homeomorphism: $X \to Y$, there is a homeomorphism: $Y \to X$ — the inverse of the first)

Homeomorphisms are the topological counterpart of isomorphisms in algebra.

Def. Let $f: X \to Y$ be a continuous injective map. Let $Z = f(X) \subseteq Y$ and consider it as a subspace of Y. The function $f': X \to Z$ attained by restricting the codomain is bijective. If f' is a homeomorphism, then f is said to be a topological embedding or just embedding of X in Y.

Embedding

Note that the "homeomorphic" relation is an equivalence relation. (Why?)

Let X, Y, Z be topological spaces.

1. Any constant map $f: X \to Y$ is continuous.
2. If A is a subspace of X, the inclusion map $f: A \hookrightarrow X$ is continuous.
3. If $f: X \to Y$ and $g: Y \to Z$ are continuous, $g \circ f: X \to Z$ is continuous.
4. If $f: X \to Y$ is continuous and A is a subspace of X, then the restricted function $f|_A: A \to Y$ is continuous.
5. Similarly, we can restrict / expand the range.

to a subspace $Z \supseteq f(X)$ to a space Z with subspace Y.

Lemma. (2.2) Let $f: X \to Y$ and $X = \bigcup_{\alpha \in A} U_\alpha$ for some open (U_α). Then f is continuous iff $f|_{U_\alpha}$ is continuous for each α.

Proof. The forward direction is obvious.
For the backward direction, let V be open in Y. Observe that

$$f^{-1}(V) \cap U_\alpha = f|_{U_\alpha}^{-1}(V).$$

$f|_{U_\alpha}^{-1}(V)$ is open in U_α, and thus X (Why?). This implies that

$$f^{-1}(V) = \bigcup_{\alpha \in A} \left(f^{-1}(V) \cap U_\alpha\right),$$

which yields the result since an arbitrary union of open sets is open. $\square$

Theo. (2.3) [Pasting Lemma] Let $X = A \cup B$ for closed A, B in X. Let $f: A \to Y$ and $g: B \to Y$ be continuous*. If $f(x) = g(x)$ for all $x \in A \cap B$, then the map $h: X \to Y$ defined by

(* with respect to the subspace topologies)

$$h(x) = \begin{cases} f(x), & x \in A, \\ g(x), & x \in B \end{cases}$$

is continuous

Proof. Let C be closed in Y. Note that

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

Since $f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B, which are in turn closed in X, they are also closed in X. This gives the result because a finite union of closed sets is closed. $\square$

Note that the result holds even if A and B are open.

Lecture 11 - 10/02/21 More about Product Topologies

Theo. (2.4) Let $f: A \to X \times Y$ be given by $f(a) = (f_1(a), f_2(a))$. f is continuous iff the functions $f_1: A \to X$ and $f_2: A \to Y$ are continuous.

Coordinate Function — In this context, f_1 and f_2 are called the coordinate functions of f.

This can easily be proved by considering the basis elements. We omit the proof and shall instead show a more general result later. (Theorem 2.10)

Theo. (2.5) Let $A \subseteq X$, $f: A \to Y$ be continuous, and let Y be Hausdorff. Then if f can be extended to a continuous $g: \bar{A} \to Y$, this g is uniquely determined by f.

Proof. Let $g_1, g_2 : \bar{A} \to Y$ be continuous and $g_1(a) = g_2(a) = f(a)$ for all $a \in A$. Let $x \in \bar{A}$ such that $g_1(x) \neq g_2(x)$.
Since Y is Hausdorff, let open $U_1, U_2 \subseteq Y$ such that $g_1(x) \in U_1$, $g_2(x) \in U_2$, and $U_1 \cap U_2 = \emptyset$. We then have

$$A \cap \underbrace{g_1^{-1}(U_1) \cap g_2^{-1}(U_2)}_{\text{open in } \bar{A} \text{ and non-empty.}} \neq \emptyset.$$

Let $z \in A \cap g_1^{-1}(U_1) \cap g_2^{-1}(U_2)$. Then

$$f(z) = g_1(z) \in U_1 \quad \text{and}$$
$$f(z) = g_2(z) \in U_2$$

$\Rightarrow U_1 \cap U_2 \neq \emptyset$, proving the claim. $\blacksquare$

Let us revisit the product topology.
How do we generalize the idea to more (than 2) topological spaces?

Suppose $(X_i)_{i=1}^n$ are topological spaces. Consider the topologies on $X_1 \times X_2 \times \cdots \times X_n$ with

1. basis
$$\mathcal{B} = \{ U_1 \times U_2 \times \cdots \times U_n : U_i \text{ is open in } X_i \text{ for each } i \}$$
2. subbasis
$$\mathcal{S} = \bigcup_{i=1}^{n} \{ \pi_i^{-1}(U) : U \text{ open in } X_i \}$$

We even extend the above to a countably infinite number of sets. (we define this better later)
When are the two topologies the same?

It turns out that they are the same for finite products, but not for an infinite product.

↳ In this case, (1) is called the box topology and (2) is called the product topology.

It is easily seen that the described sets are a basis and subbasis.

A general basis element of the product topology is a finite intersection of subbasis elements.

$$\bigcap_{r=1}^{k} \pi_{i_r}^{-1}(U_{i_r}) \quad \text{where } U_{i_r} \text{ is open in } X_{i_r}.$$

(Restriction on a finite number of coordinates)

It is easily seen that the box topology and product topology are equal for a finite number of topological spaces because

$$U_1 \times U_2 \times \cdots \times U_n = \bigcap_{i=1}^{n} \pi_i^{-1}(U_i)$$

↳ finite intersection of subbasis elements

and $$\bigcap_{r=1}^{k} \pi_{i_r}^{-1}(U_{i_r}) = X_1 \times X_2 \times \cdots \times U_{i_1} \times \cdots \times U_{i_2} \times \cdots \times U_{i_k} \times \cdots \times X_n \in \mathcal{B}$$

Let us define the Cartesian product more concretely in the infinite case.

Let $(X_i)_{i\in\mathbb{N}}$ be sets and $X = \bigcup_{i\in\mathbb{N}} X_i$. Then

Cartesian Product

$$\prod_{i=1}^{\infty} X_i = \{ f : \mathbb{N} \to X : f(j) \in X_j \text{ for each } j \in \mathbb{N} \} \subseteq X^{\mathbb{N}}$$

$$= \{ (x_1, x_2, \dots, x_n, \dots) : x_i \in X_i \text{ for each } i \}$$

$\{f : \mathbb{N} \to X\}$

is the Cartesian product of the (X_i).

Def. We can easily extend this definition to any indexing set I as

$$\prod_{i\in I} X_i = \{ f : I \to X : f(i) \in X_i \text{ for each } i \in I \}$$

We often denote f as $(x_i)_{i\in I}$. x_i is the ith coordinate of f.

Def. Let $(X_i)_{i \in I}$ be a set of topological spaces with indexing set I. The box topology on $\prod_{i \in I} X_i$ is that with basis

Box Topology
$$\mathcal{B} = \left\{ \prod_{i \in I} U_i \;:\; U_i \text{ is open in } X_i \text{ for each } i \in I \right\}$$

and the product topology on $\prod_{i \in I} X_i$ is that with subbasis

Product Topology
$$\mathcal{S} = \left\{ \pi_i^{-1}(U_i) \;:\; U_i \text{ is open in } X_i \text{ and } i \in I \right\}.$$

→ For finite I, the box and product topologies are equal.

→ For infinite I, the box topology is strictly finer than the product topology unless all but finitely many of the topologies are the indiscrete topology on the respective set (in which case they are equal)

Lecture 12 - 10/02/21

For indexing set I, an element of $X^I = \prod_{i \in I} X$ is known as an I-tuple of elements of X.

In the product topology, $\prod_{i \in I} X_i$ is called a product space.

Note that if U_i, V_i open in X_i,
$$\pi_i^{-1}(U_i) \cap \pi_i^{-1}(V_i) = \pi_i^{-1}(\underbrace{U_i \cap V_i}_{\text{open in } X_i}) \in \mathcal{S}.$$

→ A typical element in a basis of the product topology is
$$B = \pi_{i_1}^{-1}(U_{i_1}) \cap \pi_{i_2}^{-1}(U_{i_2}) \cap \cdots \cap \pi_{i_n}^{-1}(U_{i_n})$$
where the (i_r) are distinct and U_{i_r} is open in X_{i_r} for each r.

$$\Rightarrow B = \prod_{i \in I} U_i, \quad \text{where } U_j = X_j \text{ if } j \neq i_r \text{ for some } r.$$

That is, the product topology has as basis $\prod_{i \in I} X_i$, where U_i is open in X_i for each i and all but finitely many U_i are equal to X_i.

Theo. (2.6) Let X_i $(i \in I)$ be given by a basis $\mathcal{B}_i$. Then

$$\mathcal{B}_1 = \left\{ \prod_{i\in I} B_i : B_i \in \mathcal{B}_i \text{ for each } i \right\}$$

is a basis of the box topology and

$$\mathcal{B}_2 = \left\{ \prod_{i\in I} B_i : B_i \in \mathcal{B}_i \text{ for finitely many } i \text{ and } = X_i \text{ otherwise} \right\}$$

is a basis of the product topology.

Proof. One direction of the containment easily follows.

1. Let U_i be open in X_i for each i and $x \in \prod_{i\in I} U_i$. For each i, let $B_i \in \mathcal{B}_i$ such that $x_i \in B_i \subseteq U_i$. Then

$$x \in \prod_{i\in I} B_i \subseteq \prod_{i\in I} U_i . \qquad \square$$

The second part is left as an exercise.

Theo. (2.7) Let A_i be a subspace of X_i for each $i \in I$. $\prod A_i$ is a subspace of $\prod X_i$ if both sets are given the box topology or both are given the product topology.

Theo. (2.8) For each $i \in I$, let X_i be Hausdorff. Then $\prod X_i$ is Hausdorff under both the box and product topologies.

Theo. (2.9) Let $A_i \subseteq X_i$ for each $i \in I$. If $\prod X_i$ is given either the product or box topologies,

$$\prod \overline{A_i} = \overline{\prod A_i} .$$

Proof. Let $x \in \prod \overline{A_i}$. Let $U = \prod U_i$ be a basis element of either the box or product topology containing x. Then for each i,

$$x_i \in \overline{A_i} \quad \text{and} \quad \emptyset \neq U_i \cap A_i \ni y_i$$

Then $(y_i) = y \in U \cap \left(\prod A_i\right) \neq \emptyset$.

Since U is arbitrary, $x \in \overline{\prod A_i}$.

Conversely, let $x \in \overline{\prod A_i}$. We shall show that for each i, $x_i \in \overline{A_i}$. Let $V_i \ni x_i$ be open in X_i. Then $\pi_i^{-1}(V_i)$ is open and contains $y = (y_i) \in \prod A_i$. Therefore, $y_i \in V_i \cap A_i$ and $x_i \in \overline{A_i}$. $\square$

Theo. (2.10) Let $f : A \to \prod X_i$ be given by $f(a) = (f_i(a))_{i \in I}$, where $f_i : A \to X_i$. Let $\prod X_i$ have the product topology. Then f is continuous iff each f_i is continuous.

Proof. For each i, $f_i = \pi_i \circ f$. If f is continuous, then since each π_i is continuous, so is each f_i.
On the other hand, if each f_i is continuous and U_i is open in X_i,

$$f^{-1}(\underbrace{\pi_i^{-1}(U_i)}_{\text{open}}) = \underbrace{f_i^{-1}(U_i)}_{\text{open}}.$$

The result follows because as we have seen, it suffices to check that the preimage of any subbasis element is open.

To see why the result does not hold for the box topology, consider $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ where for each i, $f_i(t) = t$. Then f is not continuous.
Consider $\left((-1,1) \times (-1/2, 1/2) \times \cdots \times (-1/n, 1/n) \times \cdots\right)$ open in $\mathbb{R}^{\mathbb{N}}$ but it does not have open preimage — 0 is in the set but for any $\delta > 0$, $(-\delta, \delta)$ is not.

Now, let us revisit the metric topology.

A set U is open in the metric topology iff for any $y \in U$, there is $\delta > 0$ s.t. $B_d(y, \delta) \subseteq U$.
(in the topology definition, we get a ball that needn't be centered at y. They are still equivalent though.)

Def. Let X be a topological space. X is said to be metrizable if there is some metric on X that induces the topology of X.

It should be noted that given a metrizable space, the corresponding metric is NOT unique. They can even be very different; for example, a set is bounded in one metric may be unbounded in another.

Indeed,

Let d be a metric on X. Define $\bar{d}: X \times X \to \mathbb{R}$ by

$$\bar{d}(x,y) = \min\{1, d(x,y)\}$$

Then $\bar{d}$ is a metric that induces the same topology as d.

In this case, $\bar{d}$ is called the standard bounded metric corresponding to d.

Lecture 14 - 17/02/21 (Quiz) Nothing new, just a recap of the previous lecture

Lecture 15 - 19/02/21

Theo (2.11) Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^\omega$, define

$$D(x,y) = \sup\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}.$$

Then D is a metric that induces the product topology on $\mathbb{R}$.

Proof The triangle inequality is satisfied because

$$\frac{\bar{d}(x_i,z_i)}{i} \leq \frac{\bar{d}(x_i,y_i)}{i} + \frac{\bar{d}(y_i,z_i)}{i} \leq D(x,y) + D(y,z)$$

then taking the sup on the left. The first two conditions are easily proved so D is a metric.

Let U be open in the metric topology and $x \in U$. Consider $B_D(x,\varepsilon) \subseteq U$ for $\varepsilon > 0$ and choose N large enough that $\frac{1}{N} < \varepsilon$. Finally let V be the basis element (of the product topology)

$$V = (x_1-\varepsilon, x_1+\varepsilon) \times \dots \times (x_N-\varepsilon, x_N+\varepsilon) \times \mathbb{R} \times \mathbb{R} \times \dots$$

We claim that $V \subseteq B_D(x,\varepsilon)$. Given $y \in \mathbb{R}^\omega$,

$$\frac{\bar{d}(x_i,y_i)}{i} < \frac{1}{N} \quad \text{for } i > N.$$

Therefore,

$$D(x,y) \leq \max\left\{\frac{\bar{d}(x_1,y_1)}{1}, \dots, \frac{\bar{d}(x_N,y_N)}{N}, \frac{1}{N}\right\}$$

If $y \in V$, the above is less than ε so $V \subseteq B_D(x,\varepsilon)$.

On the other hand, let U be a basis element of the product topology. $U = \prod U_i$, where $U_i \neq \mathbb{R}$ is open for $i = \alpha_1, \alpha_2, \dots, \alpha_n$ and $U_i = \mathbb{R}$ otherwise. Let $x \in U$.

Choose an interval $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subseteq U_i$ for $i = \alpha_1, \dots, \alpha_n$. Further take each $\varepsilon_i \leq 1$. Let

$$\varepsilon = \min\{\varepsilon_i/i : i = \alpha_1, \dots, \alpha_n\}.$$

We claim that $x \in B_D(x,\varepsilon) \subseteq U$. Indeed, for $y \in B_D(x,\varepsilon)$, $|x_i - y_i| < \varepsilon \leq \varepsilon_i$ for $i = \alpha_1, \alpha_2, \dots, \alpha_n$ so $y \in \prod U_i$. $\square$

for $i = \alpha_1, \alpha_2, \dots, \alpha_n$ so $y \in \prod U_i$. $\blacksquare$

Lemma. (2.12) Let X be a topological space and $A \subseteq X$. If there is a sequence of points of A converging to $x \in X$, then $x \in \bar{A}$. The converse holds if $\bar{A}$ is metrizable.

Sequence Lemma

Proof. Let $x_n \to x$ where $x_n \in A$. Then any neighbourhood U of x contains some $x_n \in A$, so $x \in \bar{A}$.

The converse is obvious since we can take X as a metric space. (Let $x_n \in A \cap B(x, 1/n)$ for each n) $\blacksquare$

For example, try showing that $\mathbb{R}^\omega$ under the box topology is not metrizable and does not satisfy the sequence lemma

Theo. (2.13) Let $f: X \to Y$. If f is continuous, then for every $x_n \to x$ in X, $f(x_n) \to f(x)$. The converse holds if X is metrizable

Left as exercise.

Recall the definition of uniform convergence

Theo. (2.14) Let $f_n: X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If $f_n \to f$ uniformly, then f is continuous.

The proof is nearly identical to that in real analysis. (Using an $\varepsilon/3$ trick)

$\mathbb{R}^J$ for any uncountable set J is not metrizable under the product topology: Let A be the collection of elements of $\mathbb{R}^J$ with finitely many elements as 0 and the remaining elements 1 and $x = \mathbf{0}$. Show that $x \in \bar{A}$ but no sequence in A converges to x.