MA 412: Complex Analysis

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§1. Introduction

1.1. Some basic definitions

Consider the equation $X^2 + 1 = 0$. Clearly, this equation has no roots over $\mathbb{R}$. Consider the set

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} = \mathbb{R}^2,$$

and define addition and subtraction over $\mathbb{C}$ as

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b) \cdot (c, d) &= (ac - bd, ad + bc).\end{aligned}$$

It is easy to show that $(\mathbb{C}, +, \cdot)$ is a field with additive identity $(0, 0)$ and multiplicative identity $(1, 0)$. Further observe that $\mathbb{R}$ is a subfield of $\mathbb{C}$ – consider the field homomorphism $\mathbb{R} \to \mathbb{C}$ defined by $a \mapsto (a, 0)$.
Now, we denote $\iota = (0, 1)$, and write (a, b) as $a + b\iota$.

Observe that the equation $X^2 + 1 = 0$ *does* have roots over $\mathbb{C}$ since it can be written as $(X + \iota)(X - \iota)$. For the sake of completeness, we also note that the multiplicative identity of $a + \iota b$ is

$$\frac{1}{a + \iota b} = \frac{a - \iota b}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\iota.$$

When writing $z = a + b\iota$ where $a, b \in \mathbb{R}$, we write $a = \Re z$ (the real part of z) and $b = \Im z$ (the imaginary part of z). We also define the absolute value $|z| = (a^2 + b^2)^{1/2}$ of z, and the *conjugate* $\overline{z} = a - \iota b$ of z. We clearly have

$$\begin{aligned}z\overline{z} &= |z|^2\\ \Re z &= \frac{z + \overline{z}}{2}\\ \Im z &= \frac{z - \overline{z}}{2\iota}.\end{aligned}$$

It is easy to check that

$$\overline{z + w} = \overline{z} + \overline{w} \text{ and } \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

We also have

$$\begin{aligned}\left|\frac{z}{w}\right| &= \frac{|z|}{|w|}\\ |\overline{z}| &= |z|.\end{aligned}$$

Exercise 1.1. Check that the set

$$M = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R}$$

with matrix addition and multiplication is a field isomorphic to $\mathbb{C}$.

To close out the tedious part of things, we have

$$\begin{aligned}|z + w|^2 &= |z|^2 + |w|^2 + 2\Re(z\overline{w})\\ |z + w| &\le |z| + |w| \qquad (1.1)\end{aligned}$$

Equation (1.1) is referred to as the *triangle inequality*.

1.2. Polar representations and roots

Consider $z = x + \iota y \in \mathbb{C}$. We may then define

$$x = r\cos\theta \qquad y = r\sin\theta,$$

where $|z| = r$ and the angle θ is called the *argument* of z as is denoted $\theta = \arg z$. We typically restrict θ to $(-\pi, \pi]$. We denote $\operatorname{cis}\theta = \cos\theta + \iota\sin\theta$. Therefore, we have

$$z = |z|\operatorname{cis}(\arg z).$$

Observe that rather conveniently,

$$\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2).$$

Therefore, inductively,

$$z_1 z_2 \cdots z_n = \left(\prod_i |z_i|\right) \operatorname{cis}\left(\sum_i \arg z_i\right).$$

In particular,

$$z^n = r^n \operatorname{cis}(n\theta)$$

for any $n > 0$. If $z \neq 0$ (equivalently, $r \neq 0$), the above holds for all $n \in \mathbb{Z}$.
In the case where $r = 1$, we have

$$(\cos\theta + \iota\sin\theta)^n = \cos(n\theta) + \iota\sin(n\theta) \tag{1.2}$$

Equation (1.2) is referred to as *de Moivre's Formula.*

Let us consider the equation $z^n = a$. This equation has n roots of the form

$$z = |a|^{1/n}\operatorname{cis}\left(\frac{2k\pi + \arg a}{n}\right)$$

for $k = 0, 1, \ldots, n-1$.
A *line* in the complex plane is a set of the form

$$L = \{z = a + tb : t \in \mathbb{R}\},$$

for some fixed $a, b \in \mathbb{C}$, where b is a *directional* vector whose absolute value may be assumed to be 1. Since $b \neq 0$, we equivalently have

$$L = \left\{z : \Im\left(\frac{z-a}{b}\right) = 0\right\}.$$

We can also define the half-planes

$$H_a = \left\{z : \Im\left(\frac{z-a}{b}\right) > 0\right\}$$
$$K_a = \left\{z : \Im\left(\frac{z-a}{b}\right) < 0\right\}.$$

Note that $H_a = a + H_0$, where the addition is Minkowski addition:

$$H_a = \{a + z : z \in H_0\}.$$

1.3. The extended plane

Define $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and let $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. We shall show a bijection from $\mathbb{C}_\infty$ to S.
Let $N = (0, 0, 1)$ be the 'north pole' of S, and orient $\mathbb{C}$ (as $\mathbb{R}^2$) in the horizontal plane in a manner such that $\mathbb{C}$ cuts S along the equator. For $z = x + \iota y \in \mathbb{C}$, let us define the corresponding point $Z = (x_1, x_2, x_3) \in S$. We shall draw a line connecting z to N, and let Z be the point of intersection (other than N) of this line with S. Finally, we shall map ∞ to N.
Let us define this more explicitly. The line through N and z is

$$L = \{tN + (1 - t)z : t \in \mathbb{R}\}.$$

Then, letting $z = (x, y, 0)$, we have

$$t^2 + (1 - t)^2|z|^2 = 1.$$

So,

$$|z|^2 = \frac{1 - t^2}{(1 - t)^2} = \frac{1 + t}{1 - t}$$

and

$$t = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Therefore, we map z to

$$Z = \left(\frac{2\Re z}{|z|^2 + 1}, \frac{2\Im z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}.\right) \in S.$$

Based on this, we can define a distance metric between points in $\mathbb{C}_\infty$. For $z, z' \in \mathbb{C}_\infty$ mapping to $Z, Z' \in S$, we let $d(z, z')$ be the Euclidean distance between Z, Z' in $\mathbb{R}^3$. More explicitly,

$$\begin{aligned} d(z, z')^2 &= (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 \\ &= 2 - 2(x_1x_1' + x_2x_2' + x_3x_3') \\ &= \frac{2|z - z'|}{\left((|z|^2 + 1)(|z'|^2 + 1)\right)^{1/2}} \end{aligned}$$

when $z, z' \in \mathbb{C}$ and if $z' = \infty$ (so $Z' = (0, 0, 1)$), we have

$$d(z, z') = \frac{4}{|z|^2 + 1}$$

This correspondence between points of S and $\mathbb{C}_\infty$ is called the *stereographic projection*.

Exercise 1.2. If P is a plane in $\mathbb{R}^3$ and $\Lambda = P \cap S$ is a circle on S, show that the projection of Λ on $\mathbb{C}$ under the stereographic projection is a circle as well (possibly a circle of infinite radius, namely a line).

1.4. Power series

In this section, we begin discussing convergence of series in $\mathbb{C}$ and related properties.

Definition 1.1. If $a_n \in \mathbb{C}$ for every $n \geq 0$, the series $\sum_{n=0}^\infty a_n$ is said to *converge* to z iff for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\sum_{n=0}^m a_n - z\right| < \epsilon$$

for all $m \geq N$.
The series $\sum_{n=0}^\infty a_n$ is said to converge *absolutely* if $\sum_{n=0}^\infty |a_n|$ converges.

Theorem 1.1. $\mathbb{C}$ is complete. That is, every Cauchy sequence in $\mathbb{C}$ is convergent.

Proof. Suppose $\{x_n + \iota y_n\}$ is a Cauchy sequence in $\mathbb{C}$, where $x_n, y_n \in \mathbb{R}$ for each n. We then have the existence of $N \in \mathbb{N}$ such that for all $m, k > N$, $|(x_m - x_k) + \iota(y_m - y_k)| < \epsilon$. Consequently, $|x_m - x_k| < \epsilon$ and $|y_m - y_k| < \epsilon$. However, since $\mathbb{R}$ is complete, this implies that (x_n) and (y_n) are convergent, completing the proof. ■

Theorem 1.2. If $\sum a_n$ converges absolutely, $\sum a_n$ converges.

Proof. Let $\epsilon > 0$, $z_n = \sum_{i=0}^n a_i$, and $S_n = \sum_{i=0}^n |a_i|$. Because $\mathbb{C}$ is complete, it suffices to show that (z_n) is Cauchy. Since $\sum |a_n|$ is convergent, there exists $N \in \mathbb{N}$ such that $|S_m - S_k| < \epsilon$ for all $m, k > N$. Supposing $m > k$, we have

$$S_m - S_k = \sum_{i=k+1}^{m} |a_i|.$$

So,

$$\begin{aligned} |z_m - z_k| &= \left| \sum_{i=k+1}^{m} a_i \right| \\ &\leq \sum_{i=k+1}^{m} |a_i| < \epsilon, \end{aligned}$$

completing the proof. ■

Exercise 1.3. Show that $\sum_{n=0}^\infty z^n$ converges iff $|z| < 1$.

Theorem 1.3. For a given power series $\sum_{n=0}^\infty a_n (z-a)^n$, define the number R $(0 \leq R \leq \infty)$ by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Then,

(a) If $|z - a| < R$, the series converges absolutely.

(b) If $|z - a| > R$, the terms of the series become unbounded and the series diverges.

(b) If $0 < r < R$, the series converges uniformly on the set $\{z : |z - a| \leq r\}$.

This R is referred to as the *radius of convergence* of the power series.

Proof.

(a) We assume without loss of generality that $a = 0$. If $|z| < R$, there exists r with $|z| < r < R$. By the definition of R, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{R} - \epsilon < \sup_{k \geq n} |a_k|^{1/k} < \frac{1}{R} + \epsilon$$

for all $n > N$. If we take $\epsilon = 1/r - 1/R$, it follows that $|a_n|^{1/n} < 1/r$ for all $n > N$. That is, for all $n > N$, $|a_n| < 1/r^n$ and so

$$|a_n z^n| < \left(\frac{|z|}{r}\right)^n.$$

Therefore, $\sum_{n=N}^\infty a_n z^n$ is dominated by $\sum_{n=N}^\infty (|z|/r)^n$. Now however, we can just use the result of Exercise 1.3 to conclude absolute convergence since $|z|/r < 1$.

(b) Let $|z| > R$ and choose r with $|z| > r > R$. For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{R} - \epsilon < \sup_{k \geq n} |a_k|^{1/k} \text{ for all } n > N.$$

Choosing $\epsilon = 1/R - 1/r$,

$$|a_n|^{1/n} > 1/r$$

for infinitely many $n \in \mathbb{N}$. It follows that $|a_n z^n| > (|z|/r)^n$ for infinitely many $n \in \mathbb{N}$. Since $|z|/r > 1$, these terms become unbounded and therefore the series diverges.

(c) Now, suppose $r < R$ and choose ρ such that $r < \rho < R$. Similar to the argument in (a), we get that

$$|a_n| < \frac{1}{\rho^n} \text{ for all } n \geq N.$$

If $|z| \leq r$, $|a_n z^n| \leq (r/\rho)^n$ and $r/\rho < 1$. The Weierstrass M-test then gives that the power series converges uniformly on $\{z : |z| \leq r\}$. ∎

It should be noted that we cannot conclude anything when $|z - a| = R$.

Theorem 1.4. If $\sum a_n(z-a)^n$ is a power series with radius of convergence R, then if it exists,

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R.$$

Proof. Again, assume that $a = 0$ and let $\alpha = \lim |a_n/a_{n+1}|$, which we assume exists. Suppose that $|z| < \alpha$ and take $r \in \mathbb{R}$ such that $|z| < r < \alpha$. For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\alpha - \epsilon < \left| \frac{a_n}{a_{n+1}} \right| < \alpha + \epsilon.$$

Taking $\epsilon = \alpha - r$, $|a_n/a_{n+1}| > r$ for all $n \geq N$. Let $B = |a_N| r^N$. Then,

$$a_{N+1} r^{N+1} = |a_{N+1}| r \cdot r^N < |a_N| r^N = B.$$

Similarly, we get that $|a_n| r^n < B$ for all $n \geq N$. Therefore,

$$|a_n z^n| < B \left(\frac{|z|}{r} \right)^n$$

for all $n \geq N$. Thus, the sequence converges absolutely since $|z| < r$.
Since $r < \alpha$ was arbitrary, this implies that $\alpha \leq R$.

On the other hand, if $|z| > \alpha$, take $r \in \mathbb{R}$ such that $|z| > r > \alpha$. Taking $\epsilon = r - \alpha$, we get $N \in \mathbb{N}$ such that

$$\left| \frac{a_n}{a_{n+1}} \right| < r$$

for all $n \geq N$. Letting $B = |a_N| r^N$ again, we once more obtain that $|a_n| r^n > B$ for all $n \geq N$. This gives that

$$|a_n z^n| > B \left(\frac{|z|}{r} \right)^n$$

for all $n \geq N$, and since $|z| > r$, the sequence diverges (we may assume that $B \neq 0$ by making N larger if required to ensure that $a_N \neq 0$ – if this is not possible, the problem is trivial since it means that (a_n) is eventually 0). Since the choice of r was arbitrary, this implies that $R \leq \alpha$, completing the proof. ∎

Now, consider the series
$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$
The radius of convergence of this series is ∞. So, it converges for any complex number z, and convergence is uniform on every compact subset of $\mathbb{C}$.
The above defines a function $\exp : \mathbb{C} \to \mathbb{C}$.
We also denote $e^z = \exp(z)$.

Definition 1.2 (Differentiability)**.** If G is an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$, then f is said to be *differentiable* at a point $a \in G$ if the limit
$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
exists. If it exists, the value of this limit is denoted $f'(a)$ and is called the *derivative* of f at a.

If f is differentiable at each point of G, we say that f is differentiable on G. Note that if f is differentiable on G, then $f' : G \to \mathbb{C}$ is a function. If f' is continuous, f is said to be *continuously differentiable*.

Theorem 1.5. If $f : G \to \mathbb{C}$ is differentiable at a point $a \in G$, f is continuous at a.

Proof. The proof of this is direct:
$$\begin{aligned}\lim_{z\to a} |f(z) - f(a)| &= \left(\lim_{z\to a} \frac{|f(z) - f(a)|}{|z-a|}\right) \cdot \lim_{z\to a} |z-a| \\ &= f'(a) \cdot 0 = 0.\end{aligned}$$
■

Definition 1.3. A function $f : G \to \mathbb{C}$ is said to be *analytic* if f is continuously differentiable on G.

Let f, g be analytic on G and Ω respectively, and suppose that $f(G) \subseteq \Omega$. Then, $g \circ f$ is analytic on G and
$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$$
for all $z \in G$. This is called the *chain rule*.

We shall show later in Theorem 3.15 that if f is differentiable then its derivative is continuous, and so f is analytic.

Theorem 1.6. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$. Then

(a) For each $k \geq 1$, the series
$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k}$$
has radius of convergence R.

(b) The function f is infinitely differentiable on $B(a, R)$ (the open ball of radius R centered at a), and further, $f^{(k)}(z)$ is given by the series in (a) for all $k \geq 1$ and $|z - a| < R$.

(c) For $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

Proof. Assume that $a = 0$.

(a) Note that it suffices to prove the result for $k = 1$ (Why?). To show this, it is enough to show that

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |na_n|^{1/(n-1)}$$

First, it is not difficult to show that $\lim_{n\to\infty} n^{1/(n-1)} = 1$. It may be shown that for any sequences $(c_n), (d_n)$ in $\mathbb{R}$ where $c_n \geq 0$, if $\lim c_n = c$ and $\limsup d_n = d$, then $\limsup c_n d_n = cd$. Therefore, we are done if we show that $\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/(n-1)}$.

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + z\sum_{n=0}^{\infty} a_{n+1} z^n.$$

Let R' be the radius of convergence of $\sum_{n=0}^{\infty} a_{n+1} z^n$. We want to show that $R' = R$.
If $|z| < R'$, then

$$\sum |a_n z^n| \leq |a_0| + |z| \sum_{n=0}^{\infty} |a_{n+1} z^n| < \infty,$$

so $R' \leq R$. On the other hand, if $|z| < R$ and $z \neq 0$,

$$\sum |a_{n+1} z^n| < \frac{1}{|z|}\left(\sum |a_n z^n| + |a_0|\right) < \infty,$$

so $R \leq R'$ and we are done.

(b) Once again, it suffices to prove the result for $k = 0$. For $|z| < R$ and $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$,

$$s_n(z) = \sum_{k=0}^{n} a_k z^k \text{ and } R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k,$$

fix a point $w \in B(0, R)$ and r such that $|w| < r < R$. We wish to show that $f'(w)$ exists and is equal to $g(w)$. Let $\delta > 0$ be arbitrary with $\overline{B(w, \delta)} \subseteq B(0, r)$. Letting $z \in B(w, \delta)$, we have

$$\frac{f(z) - f(w)}{z - w} - g(w) = \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) + s_n'(w) - g(w) + \frac{R_n(z) - R_n(w)}{z - w}.$$

We have

$$|z^k - w^k| = |z - w||z^{k-1} + z^{k-2}w + \cdots + w^{k-1}| \leq |z - w| k r^{k-1}.$$

Therefore,

$$\left|\frac{R_n(z) - R_n(w)}{z - w}\right| = \left|\sum_{k=n+1}^{\infty} a_k \frac{z^k - w^k}{z - w}\right| \leq \sum_{k=n+1}^{\infty} |a_k| k r^{k-1}.$$

Since $r < R$, $\sum_{k=1}^{\infty} |a_k| k r^{k-1}$ converges and so for any $\epsilon > 0$, there exists $N_1 \in N$ such that for $n \geq N_1$,

$$\left|\frac{R_n(z) - R_n(w)}{z - w}\right| < \epsilon/3.$$

Since $\lim s_n'(w) = g(w)$, there exists $N_2 \in \mathbb{N}$ such that

$$|s_n'(w) - g(w)| < \epsilon/3$$

for $n \geq N_2$. Choose $n \geq \max(N_1, N_2)$. Then, there exists $\delta > 0$ such that whenever $0 < |z - w| < \delta$,

$$\left|\frac{s_n(z) - s_n(w)}{z - w} - s_n'(w)\right| < \epsilon/3.$$

Putting all these together, we get the desideratum.

(c) This is straightforward using the explicit expression for $f^{(k)}(a)$. ■

If the series $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence $R > 0$, then f is analytic on $B(a, R)$. Therefore, exp is analytic on $\mathbb{C}$.
Further, letting $g = \exp$,

$$g'(z) = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = g(z).$$

Define the functions cos and sin using power series as

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + (-1)^k \frac{z^{2k}}{(2k)!} + \cdots$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^k \frac{z^{2k+1}}{(2k+1)!} + \cdots$$

Note that

$$\cos z = \frac{e^{\iota z} + e^{-\iota z}}{2} \text{ and } \sin z = \frac{e^{\iota z} - e^{-\iota z}}{2\iota}.$$

Therefore,

$$e^{\iota z} = \cos z + \iota \sin z.$$

In particular, if $z = \theta \in \mathbb{R}$,

$$e^{\iota\theta} = \cos\theta + \iota\sin\theta.$$

It is direct to show next that $\cos^2 z + \sin^2 z = 1$ for $z \in \mathbb{C}$.

Definition 1.4. A function f is said to be *periodic* with period c if $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

e^z is periodic with period $2\pi\iota$.

Similar to cos and sin, one can define the function log as

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots.$$

$\log z$ is defined only when $|z-1| < 1$. Further note that we cannot define log as the inverse of exp (as we do over the reals) since exp is not injective here.
We would like to define log such that $w = \exp z$ when $z = \log w$. Since exp is non-zero, also suppose that $w \neq 0$. If $z = x + \iota y$, then $|w| = e^x$ and $\arg w = y + 2\pi k\iota$ for some $k \in \mathbb{Z}$. Therefore, the solution set for $e^z = w$ is

$$\{\log|w| + \iota(\arg w + 2\pi k) : k \in \mathbb{Z}\}.$$

Definition 1.5. If G is an open connected set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ is a continuous function such that $z = \exp(f(z))$ for all $z \in G$, then f is a *branch of the logarithm.*

Lemma 1.7. If $G \subseteq \mathbb{C}$ is open and connected and f is a branch of the logarithm on G, then the totality of the branches of $\log z$ are the functions $f(z) + 2\pi k\iota$ for $k \in \mathbb{Z}$.

Proof. If $g(z) = f(z) + 2\pi k\iota$ for some $k \in \mathbb{Z}$, then $\exp(g(z)) = \exp(f(z)) = z$, so g is also a branch of the logarithm. On the other hand, suppose that g is a branch of the logarithm. For $z \in G$, $\exp(f(z)) = \exp(g(z)) = z$, so $g(z) = f(z) + 2\pi k\iota$. However, note that this k depends on z. We must show that the same k works for all z. Indeed, $h(z) = (g(z) - f(z))/2\pi\iota$ is continuous on G and $h(G) \subseteq \mathbb{Z}$, so the required follows. ■

Now, let $G = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Clearly, G is connected and each $z \in G$ can be uniquely denoted by $|z|e^{\iota\theta}$, where $-\pi < \theta < \pi$. For θ in this range, define

$$f(re^{\iota\theta}) = \log r + \iota\theta.$$

This is a branch of the logarithm on G, and is commonly referred to as the *principal branch.*

Theorem 1.8. Let G, Ω be open subsets of $\mathbb{C}$. Suppose that $f : G \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are continuous such that $g(f(z)) = z$ for all $z \in G$. If G is differentiable and $g'(z) \neq 0$, f is differentiable and

$$f'(z) = \frac{1}{g'(f(z))}.$$

If g is analytic, so is f.

Proof. Fix $a \in G$ and let $h \in \mathbb{C} \setminus \{0\}$ with $a + h \in G$. Since $g(f(a)) = a \neq a + h = g(f(a+h))$, $f(a) \neq f(a+h)$. Also,

$$1 = \frac{g(f(a+h)) - g(f(a))}{h} = \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \cdot \frac{f(a+h) - f(a)}{h}.$$

Take the limit of either side as $h \to 0$. The first fraction is equal to $g'(f(a))$ since $\lim_{h \to 0}(f(a+h) - f(a)) = 0$, and therefore $\lim_{h\to 0}(f(a+h) - f(a))/h = f'(a)$ exists, and $1 = g'(f(a)) \cdot f'(a)$. The required follows.
If g is analytic, then g' is continuous so f is analytic. ∎

Corollary 1.9. Any branch of the logarithm function is analytic and has derivative $z \mapsto 1/z$.

Given a branch of the logarithm f on an open connected set G and fixed $b \in \mathbb{C}$, define $g(z) = \exp(bf(z))$. If $b \in \mathbb{Z}$, $g(z) = z^b$. In general, this defines a branch of z^b ($b \in \mathbb{C}$) for any open connected set on which there is a branch of $\log z$.
If we write z^b as a function, it is implicitly understood that the f in $\exp(bf(z))$ is the principal branch of the logarithm. Since log is analytic, so is $z \mapsto z^b$.

1.5. Cauchy-Riemann Equations

Let $f : G \to \mathbb{C}$ be analytic and let

$$u(x,y) = \Re(f(x + \iota y)), v(x,y) = \Im(f(x + \iota y))$$

for $x + \iota y \in G$. Let us evaluate the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

in two different ways.
First, if we let $h \to 0$ through real values, we get

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + \iota \frac{\partial v}{\partial x}(x,y).$$

Along the imaginary axis, we get

$$f'(z) = -\iota \frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y).$$

Therefore,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Supposing that u and v have continuous second derivative (we shall later show that they are infinitely differentiable), we have that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Therefore, since the second derivatives are continuous,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.3}$$

A function u with continuous second partial derivatives satisfying Equation (1.3) is said to be *harmonic*. Similarly, v is also harmonic.

Theorem 1.10. Let u, v be real-valued functions defined on an open connected set (a *region*) G and suppose that they have continuous second partial derivatives. Then, $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + \iota v(z)$ is analytic iff u and v satisfy the Cauchy-Riemann equations.

Proof. We have already shown the forward direction.
For the other direction, let $z = x + \iota y \in G$ and $B(z, r) \subseteq G$. Let $h = s + \iota t \in B(0, r)$. Our goal is to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(z+h) - f(z) - f'(z)h}{h} \right| < \epsilon$$

for all $h \in B(0, \delta)$ for some $f'(z) \in \mathbb{C}$. Note that

$$u(x+s, y+t) - u(x, y) = \big(u(x+s, y+t) - u(x, y+t)\big) + \big(u(x, y+t) - u(x, y)\big).$$

Now, for fixed $t \in (-r, r)$, $s \mapsto u(x+s, y+t)$ is a differentiable function on $(-r, r)$. We apply the mean value theorem to conclude that there exist $s_1, t_1 \in (-r, r)$ for each $s + \iota t \in B(0, r)$ such that $|s_1| < |s|$, $|t_1| < |t|$, and

$$\begin{aligned} u(x+s, y+t) - u(x, y+t) &= u_x(x+s_1, y+t)s \\ u(x, y+t) - u(x, y) &= u_y(x, y+t_1)t. \end{aligned}$$

Now, let

$$\varphi(s, t) = \big(u(x+s, y+t) - u(x, y)\big) - \big(u_x(x, y)s + u_y(x, y)t\big).$$

We get that

$$\varphi(s, t) = \big(su_x(x+s_1, y+t) - su_x(x, y)\big) + \big(tu_y(x, y+t_1) - tu_y(x, y)\big).$$

So,

$$\frac{\varphi(s, t)}{s + \iota t} = \frac{s}{s + \iota t}\big(u_x(x+s_1, y+t) - u_x(x, y)\big) + \frac{t}{s + \iota t}\big(u_y(x, y+t_1) - u_y(x, y)\big)$$

and on taking the limit of both sides as $s + \iota t \to 0$, we can use the fact that $|s| \le |s + \iota t|$, $|t| \le |s + \iota t|$, $|s_1| < |s|$, $|t_1| < t$, and the continuity of u_x, u_y, to conclude that

$$\lim_{s+\iota t \to 0} \frac{\varphi(s, t)}{s + \iota t} = 0.$$

Therefore,

$$u(x+s, y+t) - u(x, y) = u_x(x, y)s + u_y(x, y)t + \varphi(s, t).$$

We get a similar equation for v as well, with a function ψ (in place of φ). Combining the two,

$$\begin{aligned} \frac{f(z+s+\iota t) - f(z)}{s + \iota t} &= \frac{u(x+s, y+t) - u(x, y)}{s + \iota t} + \iota \frac{v(x+s, y+t) - v(x, y)}{s + \iota t} \\ &= \frac{su_x(x, y) + tu_y(x, y) + \varphi(s, t) + \iota\left(sv_x(x, y) + tv_y(x, y) + \psi(s, t)\right)}{s + \iota t} \\ &= \frac{u_x(x, y)(s + \iota t) + \iota v_x(x, y)(s + \iota t) + \varphi(s, t) + \iota\psi(s, t)}{s + \iota t}, \end{aligned}$$

where we used Cauchy-Riemann equations in the final step and thus,

$$\lim_{s+\iota t \to 0} \frac{f(z+s+\iota t) - f(z)}{s + \iota t} = u_x(x, y) + \iota v_x(x, y),$$

completing the proof. Since u_x and v_x are continuous, f' is continuous and f is analytic. ∎

A next question is: given some u such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

when does there exist harmonic v such that $u + \iota v$ is analytic? Such a v is referred to as a *harmonic conjugate* of u. It turns out that the answer is not always. Indeed, $u(x,y) = \log((x^2+y^2)^{1/2})$ on $\mathbb{C} \setminus \{0\}$, despite being harmonic, does not have a harmonic conjugate.

Theorem 1.11. Let G be either the entirety of $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is a harmonic function, then u has a harmonic conjugate.

Proof. Let $G = B(0,R)$ for some $0 < R \leq \infty$ and let $u : G \to \mathbb{R}$ be analytic. Define

$$v(x,y) = \int_0^y u_x(x,t)\,\mathrm{d}t + \varphi(x)$$

so that $u_x = v_y$. We shall determine φ such that $v_x = -u_y$. Differentiating with respect to x, we get

$$\begin{aligned} v_x(x,y) &= \int_0^y u_{xx}(x,t)\,\mathrm{d}t + \varphi'(x) \\ &= -\int_0^y u_{yy}(x,t)\,\mathrm{d}t + \varphi'(x) \\ &= -u_y(x,y) + u_y(x,0) + \varphi'(x). \end{aligned}$$

Therefore, $\varphi'(x) = -u_y(x,0)$, and the function

$$v(x,y) = \int_0^y u_x(x,t)\,\mathrm{d}t - \int_0^x u_y(s,0)\,\mathrm{d}s$$

is a harmonic conjugate of u. ∎

The above proof requires that the entire segments $[(0,0),(x,0)]$ $[(x,0),(x,y)]$ are contained in G, which is true when we are on a disk.

1.6. Transformations

Consider the two hyperbolas defined by

$$\begin{aligned} x^2 - y^2 &= c \\ 2xy &= d, \end{aligned}$$

where $c, d \neq 0$.
This gives

$$y = \pm\sqrt{\frac{-c \pm \sqrt{d^2+c^2}}{2}}.$$

Consider the functions

$$\begin{aligned} u(x,y) &= x^2 - y^2 \\ v(x,y) &= 2xy. \end{aligned}$$

The two hyperbolas above are mapped by this $f = u + \iota v$ to the straight lines $u = c$ and $v = d$.

Definition 1.6. A *path* in a region $G \subseteq \mathbb{C}$ is a continuous function $\gamma : [a, b] \to G$ for some interval $[a, b]$ in $\mathbb{R}$. If $\gamma'(t)$ exissts for each $t \in [a, b]$ and $\gamma' : [a, b] \to \mathbb{C}$ is continuous, then γ is said to be *smooth.* γ is said to be *piecewise smooth* if there is a partition $a = t_0 < t_1 < \cdots t_{n-1} < t_n = b$ of $[a, b]$ such that γ is smooth on each subinterval $[t_{i-1}, t_i]$ for $1 \leq i \leq n$.
For a path $\gamma : [a, b] \to \mathbb{C}$, $\gamma([a, b])$ is sometimes referred to as the *trace* of γ and denoted $\{\gamma\}$.

By the existence of γ', we mean that the two-sided limit

$$\lim_{h \to 0} \frac{\gamma(t + h) - \gamma(t)}{h}$$

exists for $t \in (a, b)$ and the right and left sided limits exist for $t = a, b$ respectively. This is equivalent to saying that $\Re\gamma$ and $\Im\gamma$ have derivatives.
Suppose $\gamma : [a, b] \to G$ is a smooth path and for some $t_0 \in (a, b)$, $\gamma'(t_0) \neq 0$. Then, γ has a *tangent line* at the point $z_0 = \gamma(t_0)$. This line goes through the point z_0 in the direction of the vector $\gamma'(t_0)$, that is, the slope of the line is $\tan(\arg \gamma'(t_0))$.

If γ_1 and γ_2 are two smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1), \gamma_2'(t_2) \neq 0$, then define the *angle* between the paths γ_1, γ_2 at z_0 to be $\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1))$.

Suppose γ is a smooth path in G and $f : G \to \mathbb{C}$ is analytic. Then, $\sigma = f \circ \gamma$ is also a smooth path and $\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t)$. Further, if z_0 is a fixed point of f with $\gamma(t_0) = z_0$,

$$\arg(\sigma'(t_0)) - \arg(\gamma'(t_0)) = \arg(f'(z_0)).$$

Let γ_1, γ_2 be smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ with non-zero derivatives at t_1, t_2 respectively, and let $\sigma_1 = f \circ \gamma_1, \sigma_2 = f \circ \gamma_2$. Further suppose that the two paths γ_1, γ_2 are not tangent to each other at z_0. Then,

$$\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1)) = \arg(\sigma_2'(t_2)) - \arg(\sigma_1'(t_1)).$$

This says that the angle between two paths are preserved after applying an analytic function to both. A function f satisfying this is said to have the *angle-preserving property.*

Definition 1.7. A function $f : G \to \mathbb{C}$ which has the angle-preserving property and also has

$$\lim_{z \to a} \left| \frac{f(z) - f(a)}{z - a} \right|$$

existing for all $a \in G$ is called a *conformal map.*

It turns out that a function f is a conformal map if and only if it is analytic and $f'(z) \neq 0$ for all z (How?).

Definition 1.8. A mapping of the form

$$S(z) = \frac{az + b}{cz + d}$$

is called a *linear fractional transformation.* If we further have that $ad - bc \neq 0$, then $S(z)$ is called a *Möbius transformation.*

We have

$$S'(z) = \frac{ad - bc}{(cz + d)^2}.$$

If $w = S(z)$, it is relatively simple to show that

$$z = S^{-1}(w) = \frac{dw - b}{-cw + a}.$$

Therefore, the inverse of a Möbius transformation is a Möbius transformation. The composition of two Möbius transformations is a Möbius transformation as well.
Also observe that the coefficients a, b, c, d for a given Möbius transformation are not unique since we can multiply them by a constant. We may also extend S to $\mathbb{C}_\infty$ with $S(\infty) = a/c$ and $S(-d/c) = \infty$.

$S(z) = z + a$ is called a *translation*, $S(z) = az$ with $a \neq 0$ is called a *dilation*, $S(z) = e^{\iota\theta} z$ is called a *rotation*, and $S(z) = 1/z$ is called the *inversion*. It is not too difficult to see that any Möbius transformation is a composition of these five types of transformations.
What are the fixed points of a Möbius transformation S? $S(z) = z$ gives

$$cz^2 + (a - d)z + b = 0.$$

Therefore, a Möbius transformation has at most two fixed points unless $S(z) = z$ for all $z \in \mathbb{C}_\infty$.

Let $a, b, c \in \mathbb{C}_\infty$ be distinct with $S(a) = \alpha$, $S(b) = \beta$, $S(c) = \gamma$. Let T be another Möbius transformation with $T(a) = \alpha$, $T(b) = \beta$, $T(c) = \gamma$. Then $T^{-1} \circ S$ has three (distinct) fixed points, and therefore $S = T$.
Therefore, any Möbius transformation is uniquely determined by its value at any three distinct points.

Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Define $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$S(z) = \begin{cases} \frac{(z-z_3)/(z-z_4)}{(z_2-z_3)/(z_2-z_4)}, & z_2, z_3, z_4 \in \mathbb{C}, \\ \frac{z_2-z_4}{z-z_4}, & z_3 = \infty, \\ \frac{z-z_3}{z_2-z_3}, & z_4 = \infty. \end{cases}$$

In any case, $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$, and S is the only transformation having this property.

Definition 1.9. If $z_1 \in \mathbb{C}_\infty$, then (z_1, z_2, z_3, z_4) is referred to as the *cross-ratio* of z_1, z_2, z_3, z_4 and is the image of z_1 under the Möbius transformation described above, which is the unique Möbius transformation taking z_2 to 1, z_3 to 0, and z_4 to ∞.

For example, $(z_2, z_2, z_3, z_4) = 1$ and $(z, 1, 0, \infty) = z$.
If M is any Möbius transformation with $M(w_2) = 1$, $M(w_3) = 0$, $M(w_4) = \infty$, then $M(z) = (z, w_2, w_3, w_4)$ for all $z \in \mathbb{C}_\infty$.

Theorem 1.12. If z_2, z_3, z_4 are distinct points and T is any Möbius transformation, then

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4).$$

Proof. Let $S(z) = (z, z_2, z_3, z_4)$. If $M = ST^{-1}$, then

$$M(T(z_2)) = 1, \quad M(T(z_3)) = 0, \quad M(T(z_4)) = \infty.$$

Therefore, $M = (z, Tz_2, Tz_3, Tz_4)$. That is,

$$ST^{-1}z = (z, Tz_2, Tz_3, Tz_4)$$

for all $z \in \mathbb{C}_\infty$. Setting $z = Tz_1$ yields the required. ∎

Lemma 1.13. If $\{z_2, z_3, z_4\}, \{w_2, w_3, w_4\} \subseteq \mathbb{C}_\infty$, then there exists a unique Möbius transformation S with $Sz_i = w_i$ for each i.

We omit the proof of the above.

Lemma 1.14. Let $\{z_1, z_2, z_3, z_4\} \subseteq \mathbb{C}_\infty$. Then, (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle.

Proof. Define $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $Sz = (z, z_2, z_3, z_4)$. We are done if we show that $S^{-1}(\mathbb{R}_\infty)$ is a circle (since a circle is uniquely determined by three distinct points on it).
Let $S(z) = (az + b)/(cz + d)$.

First, let us show that $S^{-1}(\mathbb{R}_\infty) \subseteq \Gamma$ for a circle Γ in $\mathbb{C}_\infty$. Let $w \in S^{-1}(\mathbb{R}_\infty)$. Then, $Sw = \overline{Sw}$ so

$$\frac{aw + b}{cw + d} = \frac{\overline{a}\overline{w} + \overline{b}}{\overline{c}\overline{w} + \overline{d}}.$$

This gives that

$$(a\overline{c} - \overline{a}c)|w|^2 + (a\overline{d} - \overline{b}c)w + (b\overline{c} - d\overline{a})\overline{w} + (b\overline{d} - \overline{b}d) = 0. \tag{$*$}$$

If $a\overline{c}$ is real, we get that

$$\Im\left((a\overline{d} - \overline{b}c)w + b\overline{d}\right) = 0,$$

which is a circle through ∞ (a line).
If on the other hand $a\overline{c}$ is not real, then ($*$) becomes

$$2\iota\underbrace{\Im(a\overline{c})}_{\alpha\neq 0}|w|^2 + (a\overline{d} - b\overline{c})w + (b\overline{c} - \overline{a}d)\overline{w} + (b\overline{d} - \overline{b}d) = 0.$$

Dividing by $2\iota\alpha$,

$$|w|^2 + \frac{(a\overline{d} - b\overline{c})w}{2\iota\alpha} + \frac{(b\overline{c} - \overline{a}d)\overline{w}}{2\iota\alpha} + \frac{(b\overline{d} - \overline{b}d)}{2\iota\alpha} = 0.$$

Since α is real,

$$\overline{\frac{(b\overline{c} - \overline{a}d)\overline{w}}{2\iota\alpha}} = \frac{(a\overline{d} - b\overline{c})w}{2\iota\alpha}$$

and

$$\frac{(b\overline{d} - \overline{b}d)}{2\iota\alpha}$$

is real. This gives

$$|w|^2 + \overline{\gamma}w + \gamma\overline{w} - \delta = 0$$

for some $\gamma \in \mathbb{C}, \delta \in \mathbb{R}$. This is equivalent to $|w + \gamma| = (|\gamma|^2 + \delta)^{1/2}$, which is the equation of a circle[1].

Letting $T = S^{-1}$ and Γ be the circle obtained in the previous part of the proof, we must now show that $T(\mathbb{R}_\infty) = \Gamma$. Since $\mathbb{R}_\infty$ is connected and compact and T is a homeomorphism, $T(\mathbb{R}_\infty)$ is a closed arc, say Γ_1, of Γ. If $\Gamma_1 \neq \Gamma$, let z_1, z_2 be the endpoints of this arc. If $T(\infty) = z_3$ which is distinct from z_1, z_2, then $\mathbb{R}_\infty \setminus \{\infty\}$ is connected but $\Gamma_1 \setminus \{z_1\}$ is disconnected, which is a contradiction. So, suppose $T(\infty) = z_1$. Then, $\mathbb{R}_\infty \setminus \{\infty, T^{-1}(z_2)\}$ is disconnected but $\Gamma_1 \setminus \{z_1, z_2\}$ is connected, yielding a contradiction once more and completing the proof. ■

Next, we give a more general version of the above.

Theorem 1.15. A Möbius transformation takes circles to circles.

Note that Lemma 1.14 follows from this since $\mathbb{R}_\infty$ is a circle (of infinite radius) in $\mathbb{C}_\infty$.

[1] it may be checked that $|\gamma|^2 + \delta$ is a positive real by substituting their values.

Proof. Let Γ be a circle in $\mathbb{C}_\infty$ and S a Möbius transformation. Let z_2, z_3, z_4 be three distinct points on Γ, and set $w_j = Sz_j$ for each j. We claim that $S(\Gamma)$ is the circle Γ' determined by w_2, w_3, w_4. Indeed,

$$(z, z_2, z_3, z_4) = (Sz, w_2, w_3, w_4)$$

for any z, and if $z \in \Gamma$, the LHS is real by Lemma 1.14, and using the same theorem on the RHS completes the proof. ■

Definition 1.10. Let Γ be a circle through z_2, z_3, z_4. The points $z, z^* \in \mathbb{C}_\infty$ are said to be *symmetric* with respect to Γ if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

Remark. The above definition only depends on Γ, not the choice of z_2, z_3, z_4.

Observe that z is symmetric with respect to itself with respect to Γ if and only if $z \in \Gamma$. Indeed, it implies that (z, z_2, z_3, z_4) is real, which by Lemma 1.14 implies that $z \in \Gamma$.

What does it mean for z, z^* to be symmetric?
If Γ is a straight line, z, z^* are symmetric with respect to Γ iff their perpendicular bisector is equal to Γ. That is, the line joining z, z^* is perpendicular to Γ and they are the same distance from Γ (but on opposite sides). Indeed, choosing $z_4 = \infty$, we get that

$$\frac{z^* - z_3}{z_2 - z_3} = \frac{\overline{z} - \overline{z_3}}{\overline{z_2} - \overline{z_3}},$$

so

$$|z - z_3| = |z^* - z_3|$$

for all $z_3 \in \Gamma$.
Now, suppose that $\Gamma = \{z : |z - a| = R\}$ for some $0 < R < \infty$. We extensively use Theorem 1.12 and the five types of Möbius translations in the following sequence of equations. Then,

$$\begin{aligned}
(z^*, z_2, z_3, z_4) &= \overline{(z, z_2, z_3, z_4)} \\
&= \overline{(z - a, z_2 - a, z_3 - a, z_4 - a)} \\
&= \left(\overline{z} - \overline{a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}, \frac{R^2}{z_4 - a}\right) \\
&= \left(\frac{R^2}{\overline{z} - \overline{a}}, z_2 - a, z_3 - a, z_4 - a\right) \\
&= \left(\frac{R^2}{\overline{z} - \overline{a}} + a, z_2, z_3, z_4\right).
\end{aligned}$$

Therefore, $z^* = a + \frac{R^2}{\overline{z} - \overline{a}}$, that is,

$$(z^* - a)(\overline{z} - \overline{a}) = R^2.$$

Since

$$\frac{z^* - a}{z - a} = \frac{R^2}{|z - a|^2} > 0$$

is real, it follows that z^* is on the ray $\{a + t(z - a) : 0 < t < \infty\}$. We also have that

$$|z^* - a||z - a| = R^2,$$

so one can easily obtain z^* from z or vice-versa.

Lemma 1.16 (Symmetry Principle)**.** If a Möbius transformation takes a circle Γ_1 to the circle Γ_2, then any pair of points symmetric with respect to Γ_1 is mapped to a pair of points symmetric with respect to Γ_2.

Proof. The proof of this is near-direct.

$$\begin{aligned}(Tz, Tz_2, Tz_3, Tz_4) &= (z^*, z_2, z_3, z_4) \\ &= \overline{(z, z_2, z_3, z_4)} \\ &= \overline{(Tz, Tz_2, Tz_3, Tz_4)}.\end{aligned}$$

■

Definition 1.11. If Γ is a circle, then an *orientation* for Γ is an ordered triple (z_1, z_2, z_3) of points in Γ.

An orientation is used to represent a "direction" of the circle, where we "go" from z_1 to z_2 to z_3.

Let $\Gamma = \mathbb{R}$ and $z_1, z_2, z_3 \in \mathbb{R}$. Also put $Tz = (z, z_1, z_2, z_3)$. Since $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$, a, b, c, d can be chosen to be reals. Then,

$$\begin{aligned}Tz &= \frac{az+b}{cz+d} \\ &= \frac{az+b}{|cz+d|^2}(c\overline{z}+d) \\ &= \frac{1}{|cz+d|^2}\left(ac|z|^2 + bd + bc\overline{z} + adz\right).\end{aligned}$$

So,

$$\Im(z, z_1, z_2, z_3) = \frac{ad-bc}{|cz+d|^2}\Im z$$

and thus, $\{z : \Im(z, z_1, z_2, z_3) > 0\}$ is either the upper or lower half-plane depending on whether $ad - bc$ is negative or positive. Note that $ad - bc$ is the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let Γ be an arbitrary circle and suppose that $z_1, z_2, z_3 \in \Gamma$. Then, for any Möbius transformation S,

$$\begin{aligned}\{z : \Im(z, z_1, z_2, z_3) > 0\} &= \{z : \Im(Sz, Sz_1, Sz_2, Sz_3) > 0\} \\ &= S^{-1}\{z : \Im(z, Sz_1, Sz_2, Sz_3) > 0\}.\end{aligned}$$

So, if S is chosen to map Γ to $\mathbb{R}_\infty$, then the above set is equal to S^{-1} of either the upper or lower halfspace.

Definition 1.12. If z_1, z_2, z_3 is an orientation of Γ, we denote the *right side* and *left side* of Γ (with respect to (z_1, z_2, z_3)) to be

$$\{z : \Im(z, z_1, z_2, z_3) > 0\} \text{ and } \{z : \Im(z, z_1, z_2, z_3) < 0\}$$

respectively.

Theorem 1.17 (Orientation Principle)**.** Let Γ_1, Γ_2 be circles in $\mathbb{C}_\infty$ such that $T\Gamma_1 = \Gamma_2$ for some Möbius transformation T. Let (z_1, z_2, z_3) be an orientation of Γ_1. Then, T takes the right side (resp. left side) of Γ_1 with respect to the orientation (z_1, z_2, z_3) to the right side (resp. left side) of Γ_2 with respect to the orientation (Tz_1, Tz_2, Tz_3).

The proof of the above is left as an exercise to the reader.

Since $(z, 1, 0, \infty) = z$ by definition, the right side of $\mathbb{R}_\infty$ with respect to the orientation $(1, 0, \infty)$ is the upper half-plane.

Exercise 1.4. Find an analytic function $f : G \to \mathbb{C}$ where $G = \{z : \Re z > 0\}$, such that $f(G) = \{z : |z| < 1\}$.

Similar to the above exercise, one may show that

$$g(z) = \frac{e^z - 1}{e^z + 1}$$

maps the infinite strip $\{z : |\Im z| < \pi/2\}$ to the open unit disk D.

§2. Integration

2.1. Basic definitions

2.1.1. Integrals of real functions

First, let us recall the definition of the Riemann integral[2] of functions on $\mathbb{R}$.

Definition 2.1 (Riemann Integral)**.** Let $[a, b]$ be a given interval. A *partition* $\mathcal{P}$ of $[a, b]$ is a finite set of points $x_0, x_1, \ldots, x_n$ where

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b.$$

We also write $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, \ldots, n$.
For a bounded real function f on $[a, b]$ and each partition $\mathcal{P}$ of $[a, b]$, we set

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x), \qquad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x).$$

Further, set

$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i, \qquad L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

as the upper and lower Riemann sum respectively, and finally,

$$\overline{\int_a^b} f \,\mathrm{d}x = \inf_{\mathcal{P}} U(\mathcal{P}, f), \qquad \underline{\int_a^b} f \,\mathrm{d}x = \sup_{\mathcal{P}} L(\mathcal{P}, f)$$

as the upper and lower Riemann integrals of f.

Next, we define the slightly more general Riemann-Stieltjes integral. Note that this is the same as the usual Riemann integral when α is the identity function.

Definition 2.2 (Riemann-Stieltjes Integral)**.** Let $\alpha : [a, b] \to \mathbb{R}$ be a monotonically increasing function on $[a, b]$. Corresponding to each partition $\mathcal{P}$ of $[a, b]$, write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Clearly, $\Delta\alpha_i \geq 0$ for each i.
For any real function f which is bounded on $[a, b]$, we put

$$U(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i, \qquad L(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i,$$

where M_i, m_i are defined as in the definition of the Riemann integral. We then define

$$\overline{\int_a^b} f \,\mathrm{d}\alpha = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha), \qquad \underline{\int_a^b} f \,\mathrm{d}\alpha = \sup_{\mathcal{P}} L(\mathcal{P}, f, \alpha).$$

If these two are equal, we say that f is *Riemann-Stieltjes integrable* with respect to α on $[a, b]$ and denote the common value as $\int_a^b f \,\mathrm{d}\alpha$.

[2] technically the Darboux integral?

We also remark that

$$\int_a^b f\,\mathrm{d}\alpha = \lim_{\max \Delta\alpha_k \to 0} \sum_{k=1}^n f(\tau_k)\Delta\alpha_k,$$

where $x_{k-1} \leq \tau_k \leq x_k$ for each k.
More generally, we define the *mesh* of $\mathcal{P}$ with respect to α as

$$\|\mathcal{P}\| = \max\{\Delta\alpha_i : 1 \leq i \leq n\}.$$

So for all $\epsilon > 0$, there exists $\delta > 0$ such that for any partition $\mathcal{P}$ of $[a,b]$ with $\|P\| < \delta$, then

$$\left| \sum_{k=1}^n f(\tau_k)\Delta\alpha_k - \int_a^b f\,\mathrm{d}\alpha \right| < \epsilon$$

for any choice of points $x_{k-1} \leq \tau_k \leq x_k$.

2.1.2. Riemann-Stieltjes integrals of complex-valued functions

Definition 2.3. A function $\gamma : [a,b] \to \mathbb{C}$ for $[a,b] \subseteq \mathbb{R}$ is said to be of *bounded variation* if there exists $M > 0$ such that for any partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b\}$ of $[a,b]$,

$$v(\gamma; \mathcal{P}) = \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \leq M.$$

The *total variation* $V(\gamma)$ of γ is defined by

$$V(\gamma) = \sup\{v(\gamma; \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a,b]\}.$$

Clearly, $V(\gamma) \leq M < \infty$.

Lemma 2.1. Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation. Then,

1. If $\mathcal{P}, \mathcal{Q}$ are partitions of $[a,b]$ with $\mathcal{P} \subseteq \mathcal{Q}$, then $v(\gamma;\mathcal{P}) \leq v(\gamma;\mathcal{Q})$.
2. If $\sigma : [a,b] \to \mathbb{C}$ is also of bounded variation and $\alpha, \beta \in \mathbb{C}$, then $\alpha\gamma + \beta\sigma$ is of bounded variation and
$$V(\alpha\gamma + \beta\sigma) \leq |\alpha| V(\gamma) + |\beta| V(\sigma).$$

We omit the proof of the above, which is direct on using the triangle inequality on the definition of $v(\gamma;\mathcal{P})$.

Lemma 2.2. If $\gamma : [a,b] \to \mathbb{C}$ is piecewise smooth, γ is of bounded variation and

$$V(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t.$$

Proof. It suffices to show the required in the case where γ is smooth, since in general we can consider the refinement of any partition that splits along the pieces along which γ is smooth.
The right hand side is well-defined since γ' is continuous. Let $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b\}$. By definition,

$$\begin{aligned} v(\gamma, \mathcal{P}) &= \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t)\,\mathrm{d}t \right| \\ &\leq \sum_{k=1}^m \int_{t_{k-1}}^{t_k} |\gamma'(t)|\,\mathrm{d}t = \int_a^b |\gamma'(t)|\,\mathrm{d}t. \end{aligned}$$

Therefore, $V(\gamma) \leq \int_a^b |\gamma'(t)| \, \mathrm{d}t$, so γ is of bounded variation.
Since γ' is continuous, it is uniformly continuous. So, if $\epsilon > 0$, we may choose $\delta_1 > 0$ such that

$$|s - t| < \delta_1 \implies |\gamma'(s) - \gamma'(t)| < \epsilon.$$

Also, let $\delta_2 > 0$ such that if $\|P\| < \delta_2$, then

$$\left| \int_a^b |\gamma'(t)| \, \mathrm{d}t - \sum_{k=1}^m |\gamma'(\tau_k)|(t_k - t_{k-1}) \right| < \epsilon,$$

where τ_k is any point in $[t_{k-1}, t_k]$. Therefore,

$$\begin{aligned} \int_a^b |\gamma'(t)| \, \mathrm{d}t &\leq \epsilon + \sum_{k=1}^m |\gamma'(t_k)|(t_k - t_{k-1}) \\ &= \epsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(\tau_k) \, \mathrm{d}t \right| \\ &\leq \epsilon + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} (\gamma'(\tau_k) - \gamma'(t)) \, \mathrm{d}t \right| + \sum_{k=1}^m \left| \int_{t_{k-1}}^{t_k} \gamma'(t) \, \mathrm{d}t \right|. \end{aligned}$$

If $\|P\| < \delta = \min(\delta_1, \delta_2)$, then $|\gamma'(\tau_k) - \gamma'(t)| < \epsilon$ for all $t \in [t_{k-1}, t_k]$ and

$$\begin{aligned} \int_a^b |\gamma'(t) \, \mathrm{d}t| &\leq \epsilon + \epsilon(b - a) + \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \\ &= \epsilon(1 + b - a) + V(\gamma; P) \leq \epsilon(1 + b - a) + V(\gamma), \end{aligned}$$

so we are done since $1 + b - a > 0$ is finite and ϵ can be made arbitrarily small. ∎

Theorem 2.3. Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation and suppose that $f : [a, b] \to \mathbb{C}$ is continuous. Then, there exists a (unique) complex number $\mathcal{I}$ such that for every $\epsilon > 0$, there exists $\delta > 0$ such that when $\mathcal{P} = \{t_0 < t_1 < \cdots < t_m\}$ is a partition of $[a, b]$ with $\|P\| = \max_{1 \leq k \leq m}(t_k - t_{k-1}) < \delta$,

$$\left| \mathcal{I} - \sum_{k=1}^m f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \right| < \epsilon$$

for any choice of points τ_k with $t_{k-1} \leq \tau_k \leq t_k$.
This $\mathcal{I}$ is called the integral of f with respect to γ over $[a, b]$ and is denoted by

$$\mathcal{I} = \int_a^b f \, \mathrm{d}\gamma = \int_a^b f(t) \, \mathrm{d}\gamma(t).$$

Proof. First of all, note that it suffices to consider the case where γ is real-valued, since we can write $\gamma = \gamma_1 + \iota\gamma_2$, where γ_1, γ_2 are real-valued, to get two integrals $\mathcal{I}_1, \mathcal{I}_2$ (for γ_1, γ_2 respectively), and finally use the triangle inequality to get $\mathcal{I} = \mathcal{I}_1 + \iota\mathcal{I}_2$.
Since f is continuous, it is uniformly continuous. We can (inductively) find positive numbers $\delta_1 > \delta_2 > \cdots$ such that if $|s - t| < \delta_m$, $|f(s) - f(t)| < 1/m$. For each $M \geq 1$, let $\mathcal{P}_m$ be the collection of all partitions P of $[a, b]$ with $\|P\| \leq \delta_m$, so $\mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \cdots \supseteq \mathcal{P}_m \supseteq \cdots$. Finally, define F_m to be the closure of the set

$$\left\{ \sum_{k=1}^n f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) : P \in \mathcal{P}_m \text{ and } t_{k-1} \leq \tau_k \leq t_k \right\}.$$

Because $\mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \cdots$, it follows trivially that

$$F_1 \supseteq F_2 \supseteq \cdots .$$

We claim that

$$\operatorname{diam} F_m \leq \frac{2}{m} V(\gamma). \tag{2.1}$$

If we do this, then Cantor's Theorem (since $\mathbb{C}$ is complete) implies that there is precisely one complex number $\mathcal{I}$ such that $\mathcal{I} \in F_m$ for all $m \geq 1$. Then, for any $\epsilon > 0$, we may let $m > (2/\epsilon)V(\gamma)$ so $\epsilon > (2/m)V(\gamma) \geq \operatorname{diam} F_m$. Since $\mathcal{I} \in F_m$, $F_m \subseteq B(\mathcal{I}, \epsilon)$. Therefore, $\delta = \delta_m$ gets the job done.
So, we must show that

$$\operatorname{diam}\left\{ f(\tau_k)\left(\gamma(t_k) - \gamma(t_{k-1})\right) : P \in \mathcal{P}_m \text{ and } t_{k-1} \leq \tau_k \leq t_k \right\} \leq \frac{2}{m} V(\gamma).$$

To do this, if $P = \{t_0 < \cdots < t_n\}$ is a partition, denote by $S(P)$ a sum of the form $\sum f(\tau_k)\left(\gamma(t_k) - \gamma(t_{k-1})\right)$ where $t_{k-1} \leq \tau_k \leq t_k$ for each k. Fixing $m \geq 1$, let $P \in \mathcal{P}_m$. If $P \subseteq Q$ (so $Q \in \mathcal{P}_m$ as well), then

$$|S(P) - S(Q)| < \frac{1}{m} V(\gamma).$$

We only show this in the case where Q is obtained from P by adding a single extra partition point (the general case follows similarly). Let $Q = \{t_0 < t_1 < \cdots < t_{p-1} < t^* < t_p < \cdots t_n\}$. If $t_{p-1} \leq \sigma \leq t^*$ and $t^* \leq \sigma' \leq t_p$. Then,

$$S(Q) = \sum_{k \neq p} f(\sigma_k)(\gamma(t_k) - \gamma(t_{k-1})) + f(\sigma)\left(\gamma(t^*) - \gamma(t_{p-1})\right) + f(\sigma')\left(\gamma(t_p) - \gamma(t^*)\right).$$

Then, using the definition of δ_m,

$$\begin{aligned}
\left|S(P) - S(Q)\right| &= \left| \sum_{k \neq p} \left(f(\tau_k) - f(\sigma_k)\right)\left(\gamma(t_k) - \gamma(t_{k-1})\right) \right. \\
&\qquad \left. + f(\tau_p)(\gamma(t_p) - \gamma(t_{p-1})) - f(\sigma)(\gamma(t^*) - \gamma(t_{p-1})) - f(\sigma')(\gamma(t_p) - \gamma(t^*)) \right| \\
&\leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k-1}| + \left| \left(f(\tau_p) - f(\sigma)\right)\left(\gamma(t^*) - \gamma(t_{p-1})\right) + \left(f(\tau_p) - f(\sigma')\right)\left(\gamma(t_p) - \gamma(t^*)\right) \right| \\
&\leq \frac{1}{m} \sum_{k \neq p} \left|\gamma(t_k) - \gamma(t_{k-1})\right| + \frac{1}{m}\left|\gamma(t^*) - \gamma(t_{p-1})\right| + \frac{1}{m}\left|\gamma(t_p) - \gamma(t^*)\right| \\
&\leq \frac{1}{m} V(\gamma).
\end{aligned}$$

Next, let P, R be any two partitions in $\mathcal{P}_m$, and $Q = P \cup R$ a partition that contains P and R. Using the first part,

$$|S(P) - S(Q)| \leq |S(P) - S(Q)| + |S(Q) - S(R)| \leq \frac{2}{m} V(\gamma).$$

It follows that the diameter of the set of interest is at most $(2/m)V(\gamma)$, completing the proof. ■

Theorem 2.4. Let f, g be continuous functions on $[a, b]$ and let γ, σ be functions of bounded variation on $[a, b]$. Then for any scalars α, β,

$$\begin{aligned}
\int_a^b (\alpha f + \beta g)\, \mathrm{d}\gamma &= \alpha \int_a^b f\, \mathrm{d}\gamma + \beta \int_a^b g\, \mathrm{d}\gamma \\
\int_a^b f\, \mathrm{d}(\alpha\gamma + \beta\sigma) &= \alpha \int_a^b f\, \mathrm{d}\gamma + \beta \int_a^b f\, \mathrm{d}\sigma.
\end{aligned}$$

Proposition 2.5. Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and let $f : [a,b] \to \mathbb{C}$ be continuous. If $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, then

$$\int_a^b f \, \mathrm{d}\gamma = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f \, \mathrm{d}\gamma.$$

We omit the proofs of the above.

Theorem 2.6. If γ is piecewise smooth and $f : [a,b] \to \mathbb{C}$ is continuous, then $\int_a^b f \, \mathrm{d}\gamma = \int_a^b f(t)\gamma'(t) \, \mathrm{d}t$.

Proof. It suffices to consider the case where γ is smooth by Proposition 2.5. Also, by looking at the real and imaginary parts of γ separately, it suffices to consider the case where γ is real-valued on $[a,b]$. Let $\epsilon > 0$ and choose $\delta > 0$ such that if $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ has $\|P\| < \delta$, then

$$\left| \int_a^b f \, \mathrm{d}\gamma - \sum_{k=1}^n f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \right| < \epsilon/2$$

and

$$\left| \int_a^b f(t)\gamma'(t) \, \mathrm{d}t - \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \epsilon/2$$

for any $t_{k-1} \leq \tau_k \leq t_k$ for each k.
Applying the mean value theorem on γ (this requires that γ be real-valued), one gets that there exists $\tau_k \in [t_{k-1}, t_k]$ for each k such that

$$\gamma'(\tau_k) = \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}}.$$

Using these τ_k specifically,

$$\left| \int_a^b f \, \mathrm{d}\gamma - \sum_{k=1}^n f(\tau_k)\gamma'(\tau_k)(t_k - t_{k-1}) \right| < \epsilon/2,$$

so

$$\left| \int_a^b f \, \mathrm{d}\gamma - \int_a^b f(t)\gamma'(t) \, \mathrm{d}t \right| < \epsilon,$$

completing the proof. ∎

2.2. Integrals On Curves

Definition 2.4. $\gamma : [a,b] \to \mathbb{C}$ is called a *rectifiable path* if it is continuous and of bounded variation. Note that if γ is piecewise smooth, then it is rectifiable and its length is

$$\int_a^b |\gamma'(t)| \, \mathrm{d}t = V(\gamma).$$

Definition 2.5. If $\gamma : [a,b] \to \mathbb{C}$ is a rectifiable path and f is a function continuous on $\{\gamma\}$, then the *(line) integral* of f along γ is
$$\int_a^b f(\gamma(t))\,\mathrm{d}\gamma(t).$$
This line integral is also denoted as
$$\int_\gamma f = \int_\gamma f(z)\,\mathrm{d}z.$$

For example, if $\gamma : [0, 2\pi] \to \mathbb{C}$ as $\gamma(t) = e^{\iota t}$,
$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^{2\pi} e^{-\iota t}(\iota e^{\iota t})\,\mathrm{d}t = 2\pi\iota.$$
and
$$\int_\gamma z^m\,\mathrm{d}z = \int_0^{2\pi} e^{\iota m t}(\iota e^{\iota t})\,\mathrm{d}t = \iota\int_0^{2\pi}\cos((m+1)t)\,\mathrm{d}t - \int_0^{2\pi}\sin((m+1)t)\,\mathrm{d}t = 0.$$

Theorem 2.7. If $\gamma : [a,b] \to \mathbb{C}$ is a rectifiable path and $\varphi : [c,d] \to [a,b]$ is a continuous non-decreasing function with $\varphi(c) = a, \varphi(d) = b$, then for any function f continuous on γ,
$$\int_\gamma f = \int_{\gamma\circ\varphi} f.$$

Remark. The above uses the fact that $\gamma \circ \varphi$ is also rectifiable (Why is this true?).

Proof. Let $\epsilon > 0$ and choose $\delta_1 > 0$ such that for a partition $\{s_0 < s_1 < \cdots < s_n\}$ of $[c,d]$ with $(s_k - s_{k-1}) < \delta_1$ and any $s_{k-1} \le \sigma_k \le s_k$,
$$\left|\int_{\gamma\circ\varphi} f - \sum_{k=1}^n f((\gamma\circ\varphi)(s_k)) - f((\gamma\circ\varphi)(s_{k-1}))\right| < \epsilon/2.$$
Similarly, choose $\delta_2 > 0$ such that for a partition $\{t_0 < t_1 < \cdots < t_m\}$ of $[a,b]$ with $(t_k - t_{k-1}) < \delta_2$ and $t_{k-1} \le \tau_k \le t_k$,
$$\left|\int_\gamma f - \sum_{k=1}^m f(\gamma(t_k)) - f(\gamma(t_{k-1}))\right| < \epsilon/2.$$
Since φ is uniformly continuous on $[c,d]$, there exists $\delta > 0$ less than δ_1 such that $|\varphi(s) - \varphi(t)| < \delta_2$ whenever $|s - t| < \delta$. So, if $\{s_0 < s_1 < \cdots < s_n\}$ is a partition of $[c,d]$ with $(s_k - s_{k-1}) < \delta < \delta_1$ and $t_k = \varphi(s_k)$, then $\{t_0 < t_1 < \cdots < t_n\}$ is a partition of $[a,b]$ with $(t_k - t_{k-1}) < \delta_2$. If $s_{k-1} \le \sigma_k \le s_k$ and $\tau_k = \varphi(\sigma_k)$, then we can use the two earlier inequalities to conclude that
$$\left|\int_\gamma f - \int_{\gamma\circ\varphi} f\right| < \epsilon,$$
completing the proof. ∎

Definition 2.6. Let $\gamma : [a,b] \to \mathbb{C}$ be a rectifiable path, and for $a \le t \le b$, set $|\gamma|(t) = V(\gamma; [a,t])$. That is,
$$|\gamma|(t) = \sup\left\{\sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| : \{t_0 < t_1 < \cdots < t_n\} \text{ is a partition of } [a,t]\right\}.$$
Clearly, $|\gamma|$ is increasing on $[a,b]$ and of bounded variation. In fact, $V(|\gamma|; [a,b]) = |\gamma|(b) - |\gamma|(a)$. If f is continuous on $[a,b]$, define
$$\int f|\mathrm{d}z| = \int_a^b f(\gamma(t))\,\mathrm{d}|\gamma|(t).$$

Theorem 2.8. Let $\gamma : [a, b] \to \mathbb{C}$ be a rectifiable curve and suppose that f is a function continuous on $\{\gamma\}$. Then,

$$\int_\gamma f = -\int_{-\gamma} f \tag{2.2}$$

where $(-\gamma)(t) = \gamma(a + b - t)$,

$$\left|\int_\gamma f\right| \leq \int_\gamma |f||\mathrm{d}z| \leq V(\gamma) \sup\{|f(z)| : z \in \{\gamma\}\}, \tag{2.3}$$

and for $c \in \mathbb{C}$,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma+c} f(z - c)\,\mathrm{d}z. \tag{2.4}$$

Proof. Equations (2.2) and (2.4) follow near-directly from the definition, so we prove only Equation (2.3). Let $\epsilon > 0$. Then, there exists $\delta > 0$ such that if $P = \{t_0 < t_1 < \cdots t_n\}$ is a partition of $[a, b]$ with $\|P\| < \delta$, then

$$\left|\left|\int_\gamma f(z)\,\mathrm{d}z\right| - \left|\sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1}))\right|\right| \leq \left|\int_\gamma f(z)\,\mathrm{d}z - \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1}))\right| < \epsilon/2$$

for any $t_{k-1} \leq \tau_k \leq t_k$. That is,

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &< \left|\sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1}))\right| + \epsilon/2 \\ &\leq \sum_{k=1}^n \big|f(\gamma(\tau_k))\big|\,\big|\gamma(t_k) - \gamma(t_{k-1})\big| + \epsilon/2.\end{aligned}$$

We may also assume that for this same δ,

$$\sum_{k=1}^n |f(\gamma(t_k))|(|\gamma|(t_k) - |\gamma|(t_{k-1})) < \int_\gamma |f(z)||\mathrm{d}z| + \epsilon/2.$$

Recall that $|\gamma|(t)$ is an increasing function. So,

$$|\gamma|(t_k) - |\gamma|(t_{k-1}) \geq |\gamma(t_k) - \gamma(t_{k-1})|$$

Therefore,

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &< \sum_{k=1}^n |f(\gamma(\tau_k))|\big(|\gamma|(t_k) - |\gamma|(t_{k-1})\big) + \epsilon/2 \\ &< \int_\gamma |f(z)||\mathrm{d}z| + \epsilon.\end{aligned}$$

It follows that

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \int_\gamma |f(z)||\mathrm{d}z|.$$

To conclude the proof, note that

$$\int_\gamma |\mathrm{d}z| = |\gamma|(b) - |\gamma|(a) = |\gamma|(b) = V(\gamma),$$

so

$$\int_\gamma |f(z)||\mathrm{d}z| \leq V(\gamma) \sup_{z\in\{\gamma\}} |f(z)|.$$

∎

Lemma 2.9. If G is an open set in $\mathbb{C}$, $\gamma : [a,b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous, then for every $\epsilon > 0$ there exists a polygonal path Γ in G such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$, and

$$\left| \int_\gamma f - \int_\Gamma f \right| < \epsilon$$

Proof. We prove the result in the case where G is an open disk. In the general case where G need not be a disk, since $\{\gamma\}$ is compact, there exists a number r with $0 < r < d(\{\gamma\}, \partial G)$. Choose $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ when $|s - t| < \delta$. The idea is that we shall take several smaller disks and stitch together the polygonal paths on each of these sections.
If $P = \{t_0 < t_1 < \cdots < t_n\}$ is a partition of $[a,b]$ with $\|P\| < \delta$, then $|\gamma(t_k) - \gamma(t_{k-1})| < r$ for $t_{k-1} \le t \le t_k$. That is, if $\gamma_k : [t_{k-1}, t_k] \to G$ is defined by $\gamma_k(t) = \gamma(t)$, then $\{\gamma_k\} \subseteq B(\gamma(t_{k-1}), r)$ for $1 \le k \le n$. Getting a polygonal path Γ_k for each k such that

$$\left| \int_{\gamma_k} f - \int_{\Gamma_k} f \right| < \epsilon/n,$$

defining $\Gamma(t) = \Gamma_k(t)$ on $[t_{k-1}, t_k]$ does the job.

Now, let us prove the result in the disk case.
Because $\{\gamma\}$ is a compact set, $d = d(\{\gamma\}, \partial G) > 0$. It follows that if $G = B(c,r)$, then $\{\gamma\} \subseteq B(c,\rho)$ where $\rho = r - d/2$.
Now, observe that f is uniformly continuous on $\overline{B}(c,\rho) \subseteq G$. Thus, we may assume without loss of generality that f is uniformly continuous on G. Now, choose $\delta > 0$ such that if $|z - w| < \delta$, then $|f(z) - f(w)| < \epsilon$. If $\gamma : [a,b] \to G$, then γ is uniformly continuous so there is a partition $P = \{t_0 < t_1 < \cdots < t_n\}$ of $[a,b]$ such that if $t_{k-1} \le s, t \le t_k$, $|\gamma(s) - \gamma(t)| < \delta$, and such that for $t_{k-1} \le \tau_k \le t_k$,

$$\left| \int_\gamma f - \sum_{k=1}^{n} f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) \right| < \epsilon.$$

Now, define $\Gamma : [a,b] \to G$ by

$$\Gamma(t) = \frac{(t_k - t)\gamma(t_{k-1}) + (t - t_{k-1})\gamma(t_k)}{t_k - t_{k-1}}$$

if $t_{k-1} \le t \le t_k$. This is the polygonal path we shall consider. Indeed,

$$\Gamma(t) - \gamma(\tau_k) = \frac{t_k - t}{t_k - t_{k-1}}(\gamma(t_{k-1}) - \gamma(\tau_k)) + \frac{t - t_{k-1}}{t_k - t_{k-1}}(\gamma(t_k) - \gamma(\tau_k)),$$

so

$$\begin{aligned} |\Gamma(t) - \gamma(\tau_k)| &\le \left| \frac{t_k - t}{t_k - t_{k-1}} \right| \left| \gamma(t_{k-1}) - \gamma(\tau_k) \right| + \left| \frac{t - t_{k-1}}{t_k - t_{k-1}} \right| \left| \gamma(t_k) - \gamma(\tau_k) \right| \\ &\le \left| \gamma(t_{k-1}) - \gamma(\tau_k) \right| + \left| \gamma(t_k) - \gamma(\tau_k) \right| < 2\delta. \end{aligned}$$

Thus,

$$\begin{aligned} \int_\Gamma f &= \int_a^b f(\Gamma(t))\Gamma'(t)\,\mathrm{d}t \\ &= \sum_{k-1}^{n} \frac{\gamma(t_k) - \gamma(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\Gamma(t))\,\mathrm{d}t \end{aligned}$$

and

$$\begin{aligned}\left|\int_\gamma f - \int_\Gamma f\right| &= \left|\int_\gamma f - \sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1}))\right| + \left|\sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) - \int_\Gamma f\right| \\ &\leq \epsilon + \left|\sum_{k=1}^n f(\gamma(\tau_k))(\gamma(t_k) - \gamma(t_{k-1})) - \int_\Gamma f\right| \\ &\leq \epsilon + \sum_{k=1}^n \frac{|\gamma(t_k) - \gamma(t_{k-1})|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\Gamma(t)) - f(\gamma(\tau_k))| \,\mathrm{d}t \\ &\leq \epsilon + \epsilon \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})| \\ &\leq \epsilon(1 + V(\gamma)),\end{aligned}$$

which can be made arbitrarily small, thus completing the proof. ■

The following can be thought of as an analogue of the Fundamental Theorem of Calculus for complex functions.

Theorem 2.10. Let G be open in $\mathbb{C}$ and γ be a rectifiable path in G with initial and end points α, β respectively. If $f : G \to \mathbb{C}$ is a continuous function with a primitive $F : G \to \mathbb{C}$ (F is differentiable and $F' = f$), then

$$\int_\gamma f = F(\beta) - F(\alpha).$$

Proof. When $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth,

$$\begin{aligned}\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t) \,\mathrm{d}t \\ &= \int_a^b F'(\gamma(t))\gamma'(t) \,\mathrm{d}t \\ &= \int_a^b (F \circ \gamma)'(t) \,\mathrm{d}t \\ &= (F \circ \gamma)(b) - (F \circ \gamma)(a) && \text{(by the Fundamental Theorem of Calculus)} \\ &= F(\beta) - F(\alpha).\end{aligned}$$

In general, we may use Lemma 2.9. For $\epsilon > 0$, let Γ be a polygonal path of the described form. Since Γ is piecewise smooth, $\int_\Gamma f = F(\beta) - F(\alpha)$, so

$$\left|\int_\gamma f - (F(\beta) - F(\alpha))\right| < \epsilon.$$

Since ϵ was chosen arbitrarily, the desideratum follows. ■

The fundamental theorem of calculus says that each continuous function has a primitive. However, this is not true for functions of complex variables. For example, letting $f(z) = |z|^2$, if F is a primitive of f, then F is analytic. So, if $F = U + \iota V$, $x^2 + y^2 = F'(x + \iota y)$. Consequently,

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} = x^2 + y^2 \\ \frac{\partial U}{\partial y} &= \frac{\partial V}{\partial x} = 0.\end{aligned}$$

However, $\frac{\partial U}{\partial y} = 0$ implies that $U(x, y) = u(x)$ for some function u, which implies that $u'(x) = x^2 + y^2$, a contradiction.

2.3. Power series representation of analytic functions

Recall the following result which we had used in the proof of Theorem 1.11.

Theorem 2.11. Let $\varphi : [a,b] \times [c,d] \to \mathbb{C}$ be a continuous function and defined $g : [c,d] \to \mathbb{C}$ by

$$g(t) = \int_a^b \varphi(s,t) \, \mathrm{d}s.$$

Then g is continuous. Moreover, if $\frac{\partial \varphi}{\partial t}$ exists and is a continuous function on $[a,b] \times [c,d]$, then g is continuously differentiable and

$$g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s,t) \, \mathrm{d}s.$$

This is referred to as the Leibniz rule.

For example, this may be used to prove that if $|z| < 1$,

$$\int_0^{2\pi} \frac{e^{\iota s}}{e^{\iota s} - z} = 2\pi.$$

To do so, let $\varphi(s,t) = e^{\iota s}/(e^{\iota s} - tz)$ for $0 \le t \le 1$ and $0 \le s \le 2\pi$. Observe that φ is continuously differentiable since $|z| < 1$. Thus,

$$g(t) = \int_0^{2\pi} \varphi(s,t) \, \mathrm{d}s$$

is continuously differentiable. Since $\varphi(s,0) = 1$, $g(0) = 2\pi$. Now,

$$\begin{aligned} g'(t) &= \int_0^{2\pi} \frac{\partial \varphi}{\partial t}(s,t) \, \mathrm{d}s \\ &= \int_0^{2\pi} \frac{z e^{\iota s}}{(e^{\iota s} - tz)^2} \, \mathrm{d}s. \end{aligned}$$

For fixed t, $\Phi(s) = z\iota/(e^{\iota s} - tz)$ satisfies

$$\Phi'(s) = -\frac{\iota z}{(e^{\iota s} - tz)^2} \cdot \iota e^{\iota s} = \frac{z e^{\iota s}}{(e^{\iota s} - tz)^2}.$$

Therefore, $g'(t) = \Phi(2\pi) - \Phi(0) = 0$, so g is a constant and $g(t) = g(0) = 2\pi$ for any t, 1 in particular.

Theorem 2.12. Let $f : G \to \mathbb{C}$ be analytic and suppose that $\overline{B(a,r)} \subseteq G$ for some $r > 0$. If $\gamma(t) = a + re^{\iota t}$ for $0 \le t \le 2\pi$, then

$$f(z) = \frac{1}{2\pi\iota} \int_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w$$

for $|z - a| < r$.

Proof. Defining $G_1 = \{(z-a)/r : z \in G\}$ and $g(z) = f(a + rz)$, it suffices to consider the case where $a = 0$ and $r = 1$.
Fix z with $|z| < 1$. It must be shown that

$$f(z) = \int_{2\pi\iota} \int_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{\iota s}) e^{\iota s}}{e^{\iota s} - z} \, \mathrm{d}s.$$

That is, we want to show that

$$\begin{aligned}0 &= \int_0^{2\pi} \frac{f(e^{\iota s})e^{\iota s}}{e^{\iota s}-z}\,\mathrm{d}s - 2\pi f(z) \\ &= \int_0^{2\pi}\left(\frac{f(e^{\iota s})e^{\iota s}}{e^{\iota s}-z} - f(z)\right)\mathrm{d}s.\end{aligned}$$

For this, let

$$\varphi(s,t) = \frac{f(z+t(e^{\iota s}-z))e^{\iota s}}{e^{\iota s}-z} - f(z)$$

for $0 \leq t \leq 1$ and $0 \leq s \leq 2\pi$, and

$$g(t) = \int_0^{2\pi} \varphi(s,t)\,\mathrm{d}s.$$

We wish to show that $g(1) = 0$. Observe that

$$g(0) = \int_0^{2\pi} \frac{f(z)e^{\iota s}}{e^{\iota s}-z} - f(z)\,\mathrm{d}s = f(z)\int_0^{2\pi}\frac{e^{\iota s}}{e^{\iota s}-z}\,\mathrm{d}s - 2\pi f(z) = 0.$$

Also,

$$\begin{aligned}g'(t) &= \int_0^{2\pi} \frac{\partial\varphi}{\partial t}(s,t)\,\mathrm{d}s \\ &= \int_0^{2\pi} \frac{e^{\iota s}}{e^{\iota s}-z} f'(z+t(e^{\iota s}-z))(e^{\iota s}-z)\,\mathrm{d}s \\ &= \int_0^{2\pi} e^{\iota s} f'(z+t(e^{\iota s - z}))\,\mathrm{d}s \\ &= \frac{1}{t} f(z+t(e^{\iota s}-z))\Big|_{s=0}^{s=2\pi} \\ &= 0,\end{aligned}$$

completing the proof. ■

If $|z-a| < r$ and w is such that $|w-a| = r$, then

$$\frac{1}{w-z} = \frac{1}{w-a}\cdot\frac{1}{1-\frac{z-a}{w-a}} = \frac{1}{w-a}\sum_{i=0}^{\infty}\left(\frac{z-a}{w-a}\right)^i.$$

since $|z-a| < |w-a|$.
Now, multiplying by $f(w)/2\pi\iota$ and integrating around the circle γ defined by $|w-a| = r$, we get that

$$f(z) = \int_\gamma \frac{f(w)}{2\pi\iota}\sum_{i=0}^{\infty}\frac{(z-a)^i}{(w-a)^{i+1}}\,\mathrm{d}w.$$

But how do we simplify the right hand side? We do not know (*yet*) that the integral and summation may be switched. So, let us get to showing this.

Lemma 2.13. Let γ be a rectifiable curve in $\mathbb{C}$ and suppose that F_n and F are continuous functions on $\{\gamma\}$. If (F_n) uniformly converges to F on $\{\gamma\}$, then

$$\int_\gamma F = \lim_{n\to\infty}\int_\gamma F_n.$$

Proof. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ such that

$$|F_n(w) - F(w)| < \frac{\epsilon}{V(\gamma)}$$

for $n \geq N$. This implies that

$$\left| \int_\gamma F_n - \int_\gamma F \right| \leq V(\gamma) \sup_w |F_n(w) - F(w)| \leq \epsilon$$

for $n \geq N$, completing the proof. ■

Theorem 2.14. Let f be analytic on $B(a, R)$. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for all $|z - a| < R$, where $a_n = f^{(n)}(a)/n!$ and this series has radius of convergence at least R.

Proof. Let $0 < r < R$ such that $\overline{B(a,r)} \subseteq B(a,R)$. Let $\gamma(t) = a + re^{\iota t}$ $(0 \leq t \leq 2\pi)$. Since $|z-a| < r$, if $M = \max\{|f(w)| : |w - a| = r\}$,

$$\frac{|f(w)||z-a|^n}{|w-a|^{n+1}} \leq \frac{M}{r}\left(\frac{|z-a|}{r}\right)^n.$$

Since $|z - a| < r$,

$$\sum_{n=0}^{\infty} f(w) \frac{(z-a)^n}{(w-a)^{n+1}}$$

converges uniformly for w on $\{\gamma\}$. By the discussion before the previous lemma together with the lemma itself,

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi\iota} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \right) (z-a)^n. \tag{$*$}$$

Since

$$a_n = \frac{1}{2\pi\iota} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, \mathrm{d}w.$$

is independent of z, $(*)$ converges for $|z - a| < R$. However, we now know from Theorem 1.6(c) that $a_n = f^{(n)}(a)/n!$, completing the proof. ■

Corollary 2.15. If f is analytic on $B(a, R)$,

$$f^{(n)}(a) = \frac{n!}{2\pi\iota} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, \mathrm{d}w$$

where $\gamma = a + re^{\iota t}$ and $r < R$.

Corollary 2.16. If $f : G \to \mathbb{C}$ is analytic, then f is infinitely differentiable.

Indeed, this follows directly from the fact that

$$f^{(n)}(a) = \frac{n!}{2\pi\iota} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} \, \mathrm{d}w$$

where $\gamma(t) = a + re^{\iota t}$ for $0 \leq t \leq 2\pi$.
That is, continuous differentiability implies infinite differentiability.

Corollary 2.17 (Cauchy's Estimate)**.** Let f be analytic on $B(a, R)$ and suppose $|f(z)| \leq M$ for all $z \in B(a, R)$. Then

$$|f^{(n)}(a)| \leq \frac{n!M}{R^n}.$$

Indeed, the above applies with $r < R$ so we get that

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi} \int_\gamma \frac{|f(w)|}{|w-a|^{n+1}} |dw| \leq \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n!M}{r^n}.$$

Since $r < R$ is arbitrary, we may let $r \to R^-$.

Proposition 2.18. Let f be analytic on the disk $B(a, R)$ and suppose that γ is a closed rectifiable curve in $B(a, R)$. Then $\int_\gamma f = 0$.

Proof. Due to Theorem 2.10, it suffices to show that f has a primitive. We know that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for $|z - a| < R$, where $a_n = f^{(n)}(a)/n!$. Consider the function

$$F(z) = (z-a) \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-a)^n.$$

Since $\lim_{n\to\infty}(n+1)^{1/n} = 1$, this power series has the same radius of convergence as $\sum a_n(z-a)^n$. Therefore, F is defined on $B(a, R)$. Moreover, $F'(z) = f(z)$ for $|z - a| < R$ by Theorem 1.6(b), completing the proof. ■

Definition 2.7. An *entire* function is a function which is defined and analytic on the whole complex plane $\mathbb{C}$.

Proposition 2.19. If f is entire, then it has a power series expansion with infinite radius of convergence.

Therefore, entire functions may be considered as polynomials of "infinite degree". Polynomials of finite non-zero degree are typically unbounded. These two insights lead to the following result.

Theorem 2.20 (Liouville's Theorem)**.** If f is a bounded entire function, then f is constant.

Proof. Suppose that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We shall show that $f'(z) = 0$ for all $z \in \mathbb{C}$. By Cauchy's Estimate, since f is analytic on any disk $B(z, R)$, $|f'(z)| \leq M/R$. However, R is arbitrary so $f'(z) = 0$ for any $z \in \mathbb{C}$. ■

Theorem 2.21 (Fundamental Theorem of Algebra)**.** If p is a non-constant polynomial with coefficients in $\mathbb{C}$, then there exists $a \in \mathbb{C}$ with $p(a) = 0$.

Proof. Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Consider the entire function $f(z) = 1/p(z)$. This function is then bounded as $p(z)$ goes to ∞ as z goes to infinity. By Liouville's Theorem, f (and thus p) is constant, which is a contradiction. ■

Due to the above, $\mathbb{C}$ is an algebraically closed field.

Corollary 2.22. If $p(z)$ is a polynomial and its roots are (p_j) with multiplicity k_j (for $1 \leq j \leq m$), then $p(z) = C(z-a_1)^{k_1}(z-a_2)^{k_2} \cdots (z-a_m)^{k_m}$ for some constant C, where $\sum k_j$ is the degree of p.

Given the fundamental theorem of algebra, it is easy to show that any non-constant polynomial p on $\mathbb{C}$ is surjective and analytic. On the other hand, we know that the map $z \mapsto e^z$ is an entire function but there is no $b \in C$ such that $e^b = 0$. So, power series ("polynomials of infinite degree") cannot be thought of in the same way as ordinary polynomials (of finite degree). In fact, given a non-constant entire function f, there exists at most one $a \in \mathbb{C}$ that is not in the image of f. This is referred to as Little Picard's Theorem.

Theorem 2.23. Let G be a connected open set and $f : G \to \mathbb{C}$ be analytic. Then, the following are equivalent statements.

(a) f is identically zero.

(b) There exists $a \in \mathbb{C}$ such that for all $n \geq 0$, $f^{(n)}(a) = 0$.

(c) $\{z \in G : f(z) = 0\}$ has a limit point in G.

Proof. Clearly, (a) implies (b) and (c).
Next, let us show that (c) implies (b). Let $a \in G$ be a limit point of the zero set of f. Let $R > 0$ such that $B(a, R) \subseteq G$. Since a is a limit point of z and f is continuous, $f(a) = 0$. Let $n \geq 1$ such that $f^{(k)}(a) = 0$ for $k < n$ and $f^{(n)}(a) \neq 0$. Expanding f as a power series about a gives that

$$f(z) = \sum_{k=n}^{\infty} a_k (z - a)^k$$

for $|z - a| < R$ and $a_n \neq 0$. Let

$$g(z) = \sum_{k=n}^{\infty} a_k (z - a)^{k-n}.$$

Since g is continuous in $B(a, R)$ and $g(a) \neq 0$, let $r < R$ such that $g(z) \neq 0$ when $|z - a| < r$. Since a is a limit point of z, there exists b with $f(b) = 0$ and $0 < |a - b| < r$. This gives $0 = (b - a)^n g(b)$, so $g(b) = 0$, a contradiction. Therefore, no such n can be found and (b) is true.
Finally, let us show that (b) implies (a). Let

$$A = \{z \in G : f^{(n)}(z) = 0 \text{ for all } n \geq 0\}.$$

By the definition of (b), $A \neq \varnothing$. We shall show that A is both open and closed in G, and by the connectedness of G it follows that A is the entirety of G. Showing that A is closed is direct – if $z \in \overline{A}$ and (z_k) a sequence such that $z_k \to z$, then since each $f^{(k)}$ is continuous, $f^{(n)}(z) = \lim f^{(n)}(z_k) = 0$ for all $n \geq 0$, and so $z \in A$. On the other hand, if $a \in A$, we can write $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z - a)^n = 0$ on $B(a, R)$ (for some $R > 0$), so $B(a, R) \subseteq A$ and A is open, completing the proof. ■

Corollary 2.24. If f, g are analytic on a region G, then $f \equiv g$ iff $\{z \in G : f(z) = g(z)\}$ has a limit point in G.

Corollary 2.25. If f is non-trivial and analytic on an open connected set G, then each zero of f has finite multiplicity. More explicitly, for each $a \in G$ with $f(a) = 0$, there is an integer $n \geq 1$ and an analytic function $g : G \to \mathbb{C}$ such that $g(a) \neq 0$ and $f(z) = (z - a)^n g(z)$ for all $z \in G$.

Corollary 2.26. If $f : G \to \mathbb{C}$ is non-trivial and analytic, and $a \in G$ with $f(a) = 0$, then there exists $R > 0$ such that $B(a, R) \subseteq G$, and $f(z) \neq 0$ for all $0 < |z - a| < R$.

The above follows from the fact that the zeros of f are isolated.

Theorem 2.27 (Maximum Modulus Theorem)**.** If G is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point $a \in G$ with $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

That is, if $|f|$ attains its maximum, f is constant.

Proof. Let $\overline{B(a,r)} \subseteq G$ and $\gamma(t) = a + re^{\iota t}$ for $0 \le t \le 2\pi$. Then,

$$\begin{aligned} f(a) &= \frac{1}{2\pi\iota}\int_\gamma \frac{f(w)}{w-a}\,\mathrm{d}w \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{\iota t})\,\mathrm{d}t. \end{aligned}$$

Therefore,

$$|f(a)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{\iota t})|\,\mathrm{d}t \le |f(a)|.$$

Therefore,

$$0 = \int_0^{2\pi} \left(|f(a)| - |f(a+re^{\iota t})|\right)\mathrm{d}t.$$

Since the integrand is continuous and non-negative, $|f(a)| = |f(a+re^{\iota t})|$ for all $t \in [0, 2\pi]$. If $f(a) = 0$, we are clearly done. Otherwise, since r was arbitrary, f maps any disk $B(a,R)$ to the circle $|z| = |f(a)|$. It may then be shown using the Cauchy-Riemann equations that f is constant on $B(a,R)$ and is equal to $f(a)$ for all $|z-a| < R$. Therefore, $f(z) = f(a)$ for all $z \in G$ since the zeros of $f - f(a)$ are not isolated. ■

§3. Integrals along closed curves

3.1. Winding Number

Recall that

$$\int_\gamma \frac{1}{z-a}\,\mathrm{d}z = 2\pi\iota n$$

if $\gamma(t) = a + e^{\iota nt}$ for $t \in [0, 2\pi]$. However, this property is not peculiar to the path γ, as shown by the following result.

Theorem 3.1. If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$, then

$$\frac{1}{2\pi\iota}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z$$

is an integer.

Proof. Using Lemma 2.9, we may assume that γ is piecewise smooth (Why?).
Let us assume that γ is smooth. Define $g : [0,1] \to \mathbb{C}$ by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a}\,\mathrm{d}s.$$

Then, $g(0) = 0$ and $g(1) = \int_\gamma 1/(z-a)\,\mathrm{d}z$. We also have that

$$g'(t) = \frac{\gamma'(t)}{\gamma(t)-a}$$

for $0 \le t \le 1$. This gives that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-g(t)}(\gamma(t)-a)\right) = e^{-g(t)}\gamma'(t) - g'(t)e^{-g(t)}(\gamma(t)-a) = 0.$$

Therefore,

$$e^{-g(0)}(\gamma(0)-a) = e^{-g(1)}(\gamma(1)-a).$$

Because $\gamma(0) = \gamma(1)$ (the curve is closed) and $g(0) = 0$, $g(1) = 2\pi\iota n$ for some integer n. In the case where γ is piecewise-smooth, we can define g by integrating over each of the smooth intervals and the result follows near-identically. ∎

Definition 3.1. If γ is a closed rectifiable curve in $\mathbb{C}$ then for $a \notin \{\gamma\}$,

$$n(\gamma; a) = \frac{1}{2\pi\iota}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z$$

is called the *index* of γ with respect to the point a. It is also sometimes referred to as the *winding number* of γ around a.

Recall the definition of $(-\gamma)$ from (2.2), also denoted γ^{-1}. If γ and σ are curves on $[0,1]$ with $\gamma(1) = \sigma(0)$, $\gamma + \sigma$ is the curve

$$(\gamma+\sigma)(t) = \begin{cases} \gamma(2t), & 0 \le t \le 1/2, \\ \sigma(2t-1), & 1/2 \le t \le 1. \end{cases}$$

Proposition 3.2. If σ, γ are closed rectifiable curves with the same initial (and final) points, then

$$n(\gamma; a) = -n(-\gamma; a) \tag{3.1}$$

for all $a \notin \{\gamma\}$ and

$$n(\gamma + \sigma; a) = n(\gamma; a) + n(\sigma; a) \tag{3.2}$$

for all $a \notin \{\sigma\} \cup \{\gamma\}$.

We omit the proof of the above.

The reason for $n(\cdot;\cdot)$ being called the winding number is clear from what happens in the case of a circle. For $a+e^{2\pi\iota nt}$, then $n(\gamma; a) = n$ is the number of times this curve "winds" or "wraps" around a. In fact, if $|b-a| < 1$, $n(\gamma; b) = n$ and if $|b-a| > 1$, $n(\gamma; b) = 0$.

Recall that the components of a set are its maximal connected subsets.

Theorem 3.3. Let γ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma; a)$ is constant for a belonging to a component of $G = \mathbb{C} \setminus \{\gamma\}$. Also, $n(\gamma; a) = 0$ for a belonging to the unbounded component of G.

Remark. Since $\{\gamma\}$ is compact, the connected set $\{z : |z| > R\} \subseteq G$ for sufficiently large R, so γ has precisely one unbounded component.

Proof. Define $f : G \to \mathbb{C}$ by $f(a) = n(\gamma; a)$. If we manage to show that f is continuous on G, we are done since the image of this map is a subset of the integers and each component is connected by definition, so f is constant on each component.
Recall that components of G are open. Fix $a \in G$ and let $r = d(a, \{\gamma\}) > 0$. If $|a - b| < \delta \leq r/2$ (we shall fix δ more precisely later), then

$$\begin{aligned} |f(a) - f(b)| &= \frac{1}{2\pi}\left|\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right) \mathrm{d}z\right| \\ &\leq \frac{|a-b|}{2\pi} \int_\gamma \frac{1}{|z-a||z-b|} |\mathrm{d}z|. \end{aligned}$$

By definition, $|z - a| \geq r$ for any $a \in \{\gamma\}$ and $|z - b| \geq |z - a| - |a - b| \geq r/2$. So,

$$\begin{aligned} |f(a) - f(b)| &\leq \frac{|a-b|}{2\pi} \int_\gamma \frac{2}{r^2} |\mathrm{d}z| \\ &\leq \frac{\delta}{\pi r^2} V(\gamma). \end{aligned}$$

For a given $\epsilon > 0$, setting $\delta = \min\{r/2, \epsilon\pi r^2/V(\gamma)\}$ does the job, completing the first part of the proof.

It remains to show that $\lim_{a\to\infty} f(a) = 0$ (Why does this imply the required?). Let U be the unbounded component of G. For a given $R > 0$, let $a \in U$ such that $d(a; \gamma) > R$. Then,

$$|f(a)| = \frac{1}{2\pi} \int_\gamma \left|\frac{1}{z-a}\right| |\mathrm{d}z| \leq \frac{1}{2\pi R} \int_\gamma |\mathrm{d}z| = \frac{V(\gamma)}{2\pi R}.$$

R can be made arbitrarily large (as $a \to \infty$), so we are done. ∎

Now, one would expect to see that for a "nice" f defined on a nice region G, for closed rectifiable paths γ, $\int_\gamma f$ is zero. Indeed, this is evidenced by how we saw that $n(\gamma; a)$ is zero on the unbounded component of $\mathbb{C} \setminus \{\gamma\}$. Even before that, we had seen that $\int_\gamma f = 0$ if $f : G \to \mathbb{C}$ is analytic, γ is a closed rectifiable curve, and $G = B(a, R)$.

It turns out that the last of the above statements is true for a more general class of regions, not just disks. It is not true on any region however, since the winding number of a path about a point can be nonzero. It turns out that this winding number situation is the only real problematic case, and we shall see in Cauchy's Theorem that this "general class of regions" is the set of regions without any "hole".
On the other hand, one may ask the question: for a fixed domain G and f analytic on G, for what γ inside G is $\int_\gamma f = 0$?

Lemma 3.4. Let γ be a rectifiable curve and suppose $\varphi : \{\gamma\} \to \mathbb{C}$ is continuous. Then, for each $m \geq 1$, defining

$$F_m(z) = \int_\gamma \frac{\varphi(w)}{(w-z)^m},$$

F_m is analytic on $\mathbb{C} \setminus \{\gamma\}$ and $F_m'(z) = mF_{m+1}(z)$.

Note that this matches the power series expansion for a general function we had got earlier, where a_n, which is related to the nth derivative of f at that point, was evaluated as an integral of the above form.

Proof. Let us first show that F_m is continuous for each m. Let $a \in \mathbb{C} \setminus \{\gamma\}$. We have

$$\begin{aligned} F_m(z) - F_m(a) &= \int_\gamma \varphi(w) \left(\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} \right) \mathrm{d}w \\ &= \int_\gamma \left(\frac{1}{w-z} - \frac{1}{w-a} \right) \sum_{k=1}^m \frac{1}{(w-z)^{m-k}(w-a)^{k-1}} \mathrm{d}w \\ &= \int_\gamma (z-a) \sum_{k=1}^m \frac{1}{(w-z)^{m-k+1}(w-a)^k} \mathrm{d}w \end{aligned} \tag{3.3}$$

So,

$$|F_m(z) - F_m(a)| \leq \int_\gamma |\varphi(w)||z-a| \sum_{k=1}^m \frac{1}{|w-z|^{m+1-k}|w-a|^k} |\mathrm{d}w|$$

Since φ is continous on $\{\gamma\}$ and $\{\gamma\}$ is compact, there exists $M > 0$ such that $|\varphi(w)| \leq M$ for all $w \in \{\gamma\}$. Because $a \notin \{\gamma\}$, $r = d(a, \{\gamma\}) > 0$. Let $\delta \leq r/2$. Then, for $z \in \mathbb{C} \setminus \{\gamma\}$ with $|z-a| < \delta$, we have that $|w-z| \geq r$ and $|w-a| \geq |w-z| - |z-a| \geq r/2$. So,

$$\begin{aligned} |F_m(z) - F_m(a)| &\leq \int_\gamma |\varphi(w)||z-a| \sum_{k=1}^m \frac{1}{|w-z|^{m+1-k}|w-a|^k} |\mathrm{d}w| \\ &= M\delta \int_\gamma \sum_{k=1}^m \frac{1}{|w-z|^{m+1-k}|w-a|^k} |\mathrm{d}w| \\ &\leq M\delta \int_\gamma \sum_{k=1}^m \frac{1}{(r/2)^{m+1}} |\mathrm{d}w| \\ &= \delta \cdot Mm \left(\frac{2}{r} \right)^{m+1} V(\gamma). \end{aligned}$$

Taking δ appropriately small, we are done with the first part of the proof.
Now, let us show the differentiability of F_m. Rewriting (3.3),

$$\frac{F_m(z) - F_m(a)}{z-a} = \sum_{k=1}^m \int_\gamma \frac{\varphi(w)(w-a)^{-k}}{(w-z)^{m+1-k}} \mathrm{d}w.$$

The limit of this as $z \to a$ is clearly well-defined, so F_m is differentiable. Because $a \notin \gamma$ by definition, each of the m integrands above is a continuous function of w.
Therefore,

$$\begin{aligned}\lim_{z\to a}\frac{F_m(z)-F_m(a)}{z-a} &= \sum_{k=1}^{m}\int_\gamma \frac{\varphi(w)(w-a)^{-k}}{(w-a)^{m+1-k}}\,\mathrm{d}w \\ &= \sum_{k=1}^{m}\int_\gamma \frac{\varphi(w)}{(w-a)^{m+1}}\,\mathrm{d}w = mF_{m+1}(a).\end{aligned}$$

■

Definition 3.2. If γ is a closed rectifiable curve and G is a region, we say that γ is *homologous* to 0 on G and write $\gamma \approx 0$ in G if $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$.

Theorem 3.5 (Cauchy's Integral Formula, version 1)**.** Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ be analytic. If $\gamma \approx 0$ in G, then for $a \in G \setminus \{\gamma\}$,

$$\frac{1}{2\pi\iota}\int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z = n(\gamma; a)f(a).$$

In particular, if $n(\gamma; a) = 0$, the integral on the left is zero.

Proof. Define $\varphi : G \times G \to \mathbb{C}$ as

$$\varphi(z,w) = \begin{cases}\frac{f(z)-f(w)}{z-w}, & z \neq w\\ f'(z), & z = w.\end{cases}$$

Observe that if we show that $\int_\gamma \varphi(z,w)\,\mathrm{d}z = 0$, then

$$f(z)\int_\gamma \frac{1}{w-z}\,\mathrm{d}w = \int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}z,$$

which implies the required since the left-hand side is just $2\pi\iota n(\gamma; z)f(z)$.
It is not too difficult to show that φ is continuous $G \times G$ (this uses the continuity of f'!).
Fix some $w \in G$. We shall first show that ψ_w that maps $z \mapsto \varphi(z,w)$ is analytic on G. First, let us check at $a \neq w$. We have

$$\begin{aligned}\lim_{h\to 0}\frac{\varphi(a+h,w)-\varphi(a,w)}{h} &= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(a+h)-f(w)}{a+h-w} - \frac{f(a)-f(w)}{a-w}\right)\\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{(a-w)(f(a+h)-f(w)) - (a-w)f(a) - hf(a) + (a+h-w)f(w)}{(a+h+w)(a-w)}\right)\\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{(a-w)(f(a+h)-f(a)) - h(f(a)-f(w))}{(a+h-w)(a-w)}\right)\\ \psi_w'(a) &= \frac{f'(a)}{a-w} - \frac{f(a)-f(w)}{(a-w)^2}.\end{aligned}$$

Since f is analytic, ψ_w is analytic on $G \setminus \{w\}$.
For $a = w$ on the other hand,

$$\begin{aligned}\lim_{h\to 0}\frac{\varphi(w+h,w)-\varphi(w,w)}{h} &= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(w+h)-f(w)}{h} - f'(w)\right)\\ &= \lim_{h\to 0}\frac{f(w+h)-f(w)-hf'(w)}{h^2} \qquad (3.4)\\ \psi_w'(w) &= \frac{1}{2}f''(w),\end{aligned}$$

where the final step is direct on using the fact that f has a power series expansion on some $B(w, r)$ for small r. Checking that ψ_w is analytic at G is not too difficult on using the power series expansion of f about w (we in fact get a limit similar to (3.4)).
So, we now have that ψ_w is analytic. Define

$$H = \{w \in \mathbb{C} : n(\gamma; w) = 0\}.$$

By Theorem 3.3, H is open. Moreover, $G \cup H = \mathbb{C}$. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \int_\gamma \psi_w(z)\,\mathrm{d}w, & z \in G \\ \int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w, & z \in H. \end{cases}$$

We shall show that g is bounded and entire, and thus constant. If we then show that $\lim_{z\to 0} g(z) = 0$ (this involves only the second part of the definition of g), we have $g(z) = 0$ on G as well, which is exactly what we want.
Boundedness of the first part is straightforward as G may be assumed to be bounded. For the second part,

$$\int_\gamma \frac{|f(w)|}{|w-z|}|\mathrm{d}w| \leq M \int_\gamma \frac{1}{|w-z|}|\mathrm{d}w|,$$

where M is the supremum of f. However, the integral is clearly bounded, and the integrand (so the integral) may even be made infinitely small as $z \to \infty$. If we show now that g is entire, then g is zero everywhere on $\mathbb{C}$ and we are home. ■

Theorem 3.6 (Cauchy's Integral Formula, version 2)**.** Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ be analytic. If $\gamma_1, \ldots, \gamma_m$ are closed rectifiable curves in G such that $\sum_k n(\gamma_k; w) = 0$ for $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \bigcup_k \{\gamma_k\}$,

$$\sum_{k=1}^m \frac{1}{2\pi\iota} \int_{\gamma_i} \frac{f(z)}{z-a}\,\mathrm{d}z = f(a) \sum_{k=1}^m n(\gamma_k; a).$$

The idea of the proof is very similar to that of Cauchy's Integral Formula, version 1, with the only difference being that we define

$$H = \{z \in \mathbb{C} : \sum_k n(\gamma_k; z) = 0\}$$

and

$$g(z) = \begin{cases} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w)}{w-z}\,\mathrm{d}w, & z \in H, \\ \sum_{k=1}^m \varphi(z, w)\,\mathrm{d}w, & z \in G. \end{cases}$$

Corollary 3.7 (Cauchy's Theorem)**.** Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ be analytic. If $(\gamma_k)_{k=1}^m$ are closed rectifiable curves in G such that $\sum_k n(\gamma_k; w) = 0$ for $w \in \mathbb{C} \setminus G$, then

$$\sum_{k=1}^m \int_{\gamma_k} f = 0.$$

The above follows directly from Cauchy's Integral Formula, version 2 on setting $g(z) = f(z)(z-a)$ for some $a \in G \setminus \bigcup_k \{\gamma_k\}$. Indeed, such an a exists since $\bigcup_k \{\gamma_k\}$ is a finite union of compact sets so is closed and bounded, but G is open (if it is closed, it must be $\mathbb{C}$, which is not bounded).
In fact, we may even prove Theorem 3.6 from Corollary 3.7 by using it on the analytic function

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a, \\ f'(a), & z = a. \end{cases}$$

Going back to Lemma 3.4, we have that

$$F(z) = n(\gamma; z) = \frac{1}{2\pi\iota}\int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w$$

for $z \in G \setminus \{\gamma\}$. The result there says that

$$F^{(m)}(a) = m!\frac{1}{2\pi\iota}\int_\gamma \frac{f(w)}{(w-a)^{m+1}}\,\mathrm{d}w.$$

Further,

$$F^{(m)}(a) = n(\gamma; a)f^{(m)}(a)$$

since $n(\gamma, \cdot)$ is constant on components.

Theorem 3.8. Let G be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ be analytic. If $(\gamma_k))_{k=1}^m$ are closed rectifiable curves in G such that $\sum_k n(\gamma_k; w) = 0$ for $w \in \mathbb{C} \setminus G$, then for $a \in G \setminus \bigcup\{\gamma_k\}$ and $r \geq 1$,

$$f^{(r)}(a)\sum_{k=1}^m n(\gamma_k; a) = r!\sum_{k=1}^m \frac{1}{2\pi\iota}\int_{\gamma_k} \frac{f(w)}{(w-a)^{r+1}}\,\mathrm{d}w.$$

Theorem 3.9 (Morera's Theorem)**.** Let G be a region and $f : G \to \mathbb{C}$ be continuous such that for any triangular path T in G, $\int_T f = 0$. Then f is analytic.

Above, a triangular path is a closed polygonal curve that consists of three "edges". That is, it looks like a triangle.

Proof. It is enough to show that f is analytic on each open disk contained inside G, so assume wlog that G is an open disk $B(a, R)$. We are done if we find a primitive F of f. Indeed, this would mean that F, and thus f, is analytic. For $z \in G$, define

$$F(z) = \int_{[a,z]} f,$$

where $[a, z]$ is the segment joining a and z. More concretely, $[a, z]$ is the curve given by

$$\gamma(t) = a + t(z-a)$$

for $t \in [0, 1]$.
Fix some $z_0 \in G$. We shall show that $F'(z_0) = f(z_0)$ For any $z \in G$,

$$F(z) = \int_{[a,z_0]} f + \int_{[z_0,z]} f = F(z_0) + \int_{[z_0,z]} f.$$

Then,

$$\begin{aligned}\frac{F(z)-F(z_0)}{z-z_0} &= \frac{1}{z-z_0}\int_{[z_0,z]} f \\ &= f(z_0) + \frac{1}{z-z_0}\int_{z_0,z} f(w) - f(z_0)\,\mathrm{d}w.\end{aligned}$$

Now, fixing $\epsilon > 0$, use the continuity of f to get $\delta > 0$ such that if $|z_0 - w| < \delta$, then $|f(z_0) - f(w)| < \epsilon$. Then, when $|z_0 - z| < \delta$,

$$\begin{aligned}\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| &\leq \frac{1}{|z-z_0|}\int_{[z_0,z]} |f(w)-f(z_0)||\mathrm{d}w| \\ &\leq \frac{1}{|z-z_0|}\int_{[z_0,z]} \epsilon|\mathrm{d}w| \\ &= \epsilon,\end{aligned}$$

completing the proof. ■

Recall Corollary 2.25. Similarly, if $a_1, \ldots, a_k$ are the zeroes of f (repeated according to multiplicity), then

$$f(z) = (z - a_1)(z - a_2) \cdots (z - a_k) g(z),$$

where $g(a_i) \neq 0$ for all i. Therefore,

$$\frac{f'(z)}{f(z)} = \frac{1}{z - a_1} + \frac{1}{z - a_2} + \cdots + \frac{1}{z - a_k} + \frac{g'(z)}{g(z)}.$$

Proposition 3.10. Let G be a region and f an analytic function on G with finitely many zeroes $a_1, \ldots, a_k$ (repeated according to multiplicity). If γ is a closed rectifiable curve in G that does not pass through any a_j and $\gamma \approx 0$ on G, then

$$\frac{1}{2\pi\iota} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = \sum_j n(\gamma; a_j).$$

We do not prove the above since it follows directly from the prior discussion and Corollary 3.7.

Corollary 3.11. Let G be a region and f an analytic function on G with finitely many points $a_1, \ldots, a_k$ with $f(a_i) = \alpha$. If γ is a closed rectifiable curve in G that does not pass through any a_j and $\gamma \approx 0$ on G, then

$$\frac{1}{2\pi\iota} \int_\gamma \frac{f'(z)}{f(z) - \alpha} \, \mathrm{d}z = \sum_j n(\gamma; a_j).$$

The above follows directly on applying the previous proposition to $f - \alpha$.

Recall that if $\gamma : [0, 1] \to G$ is rectifiable and $f : G \to \mathbb{C}$ is analytic, then $\sigma = (f \circ \gamma)$ is rectifiable.
Suppose we further have that γ is closed and smooth and $\gamma \approx 0$ in G. Let $\alpha \in \mathbb{C} \setminus \{\sigma\}$. Then,

$$n(\sigma; \alpha) = \frac{1}{2\pi\iota} \int_\sigma \frac{1}{w - \alpha} \, \mathrm{d}w = \frac{1}{2\pi\iota} \int_0^1 \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t)) - \alpha} \, \mathrm{d}t = \frac{1}{2\pi\iota} \int_\gamma \frac{f'(z)}{f(z) - \alpha},$$

which we just evaluated above. As might be expected, this is in fact true for any closed rectifiable γ.

What about the scenario of Corollary 3.11 where f has infinitely many zeroes in G? By Theorem 2.23, any limit point of the zero set $Z(f)$ of f is in ∂G. Since G is open, $\partial G \cap G = \varnothing$.
We shall show that

$$Z(f) \cap \underbrace{\{z \in \mathbb{C} : n(\gamma; z) \neq 0\}}_{H} \subseteq G$$

is finite, so Corollary 3.11 is still true, since all but finitely many of the terms are zero.
Observe that H is closed and bounded in $\mathbb{C}$, and is thus compact. Further note that since the zeroes of f are isolated, $Z(f) \cap H$ is a discrete closed subset of H. But any discrete compact set is finite(!), so we are done.
Therefore, even in the infinite zero case of Corollary 3.11, we have

$$\frac{1}{2\pi\iota} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z = \sum n(\gamma; a),$$

where the sum is over all points in $Z(f)$, taken with multiplicity.

Proposition 3.12. Let $f : G \to \mathbb{C}$ be analytic and non-constant, $\alpha \in f(G) \setminus \{\sigma\}$, and $\gamma \approx 0$ in G is such that $n(\gamma; a) = 1$ for all $a \in f^{-1}(\alpha)$ (this also assumes that $\{\gamma\}$ does not contain any such a). Then, $f(G)$ contains the component of $\mathbb{C} \setminus \{\sigma\}$ containing α, where $\sigma = f \circ \gamma$.

Proof. Let β belong to the mentioned component. We must show show the existence of a $z \in G$ with $f(z) = \beta$.
By Theorem 3.3, $n(\sigma; \alpha) = n(\sigma; \beta)$. The first quantity is equal to $m = \sum_k n(\gamma; z_k(\alpha))$, and the second is equal to $\frac{1}{2\pi\iota} \int_\gamma \frac{f'(z)}{f(z) - \beta} \, \mathrm{d}z$, where $z_k(\alpha)$ are the finitely many points inside G for which $n(\gamma; z_k(\alpha)) \neq 0$ and $f(z_k(\alpha)) = \alpha$. Note that $m \neq 0$ by the $n(\gamma; a) = 1$ condition. If $\beta \notin f(G)$, then the function $z \mapsto f'(z)/(f(z) - \beta)$ is analytic on G. Since $\gamma \approx 0$ on G, this must then be zero by Cauchy's Theorem so we have arrived at a contradiction, proving the claim. ∎

Further note that above, we get that the number m of points $z_k(\alpha)$ (taken with multiplicity) is equal to the number of point $z_k(\beta)$.

Lemma 3.13. Suppose f is analytic and non-constant on $B(a, R)$. If $f - \alpha$ has a zero at a of order m, then there exist $\epsilon, \delta > 0$ such that for $0 < |\zeta - \alpha| < \delta$, the equation $f - \zeta$ has exactly m simple roots in $B(a, \epsilon)$.

This also implies that $f(B(a, \epsilon)) \supseteq B(\alpha, \delta)$.

Theorem 3.14 (Open Mapping Theorem)**.** Let G be a region and let f be a non-constant analytic function on G. Then for any open $U \subseteq G$, $f(U)$ is open in $\mathbb{C}$.

Proof. Fix $\alpha \in f(U)$. We shall demonstrate the existence of $\delta > 0$ such that $B(\alpha, \delta) \subseteq U$. Since $\alpha \in f(U)$, let $a \in U$ such that $f(a) = \alpha$. There also exists $R > 0$ such that $B(a, R) \subseteq U$. The result directly follows on using the remark after the previous lemma. ∎

Theorem 3.15 (Goursat's Theorem)**.** Let $G \subseteq \mathbb{C}$ be open and $f : G \to \mathbb{C}$ be differentiable. Then, f is analytic.

Proof. It suffices to consider the case where G is an open disk. By Morera's Theorem, it suffices to show that $\int_T f = 0$ for any triangular curve T in G. Fix some such $T = [a, b, c, a]$, and let $\triangle$ be the closed set formed by its convex hull. Joining the midpoints of each side of T, we get four triangles $(\triangle_i)_{i=1}^4$ as follows. Let $T_i = \partial\triangle_i$ be paths having the following 'directions'. Then,

$$\int_T f = \sum_{i=1}^{4} \int_{T_i} f.$$

Let $T^{(1)} \in \{T_i\}_{i=1}^4$ such that

$$\left|\int_{T^{(1)}} f\right| = \max_i \left|\int_{T_i} f\right|.$$

Observe that $\ell(T_i) = (1/2)\ell(T)$ and $\operatorname{diam}(\triangle_i) = (1/2)\operatorname{diam}(\triangle)$. Now,

$$\left|\int_T f\right| \leq 4\left|\int_{T^{(1)}} f\right|.$$

We may perform the same process on $T^{(1)}$ to get $T^{(2)}$, and in general on $T^{(i)}$ to get $T^{(i+1)}$. This sequence of triangles is such that if $\triangle^{(n)} = \operatorname{Conv}(T^{(n)})$, then

$$\triangle^{(1)} \supseteq \triangle^{(2)} \supseteq \cdots \triangle^{(n)} \supseteq \cdots,$$

$$\left|\int_{T^{(n)}} f\right| \leq 4\left|\int_{T^{(n+1)}} f\right|,$$

$$\ell(T^{(n+1)}) = \frac{1}{2}\ell(T^{(n)}), \text{ and } \operatorname{diam}(\triangle^{(n+1)}) = \frac{1}{2}\operatorname{diam}(\triangle^{(n)}).$$

This yields that

$$\left|\int_T f\right| \leq 4^n \left|\int_{T^{(n)}} f\right|,$$

$$\ell(T^{(n)}) = \frac{1}{2^n}\ell(T),$$

$$\operatorname{diam}(\triangle^{(n)}) = \frac{1}{2^n}\operatorname{diam}(\triangle).$$

Using Cantor's Theorem, since $\mathbb{C}$ is complete, $\bigcap_n \triangle^{(n)}$ is a singleton, say $\{z_0\}$. Fix $\epsilon > 0$. Because f is differentiable at z_0, let $\delta > 0$ such that $B(z_0, \delta) \subseteq G$ and

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$$

whenever $|z - z_0| < \delta$. Now, choose n such that $\operatorname{diam}(\triangle^{(n)}) < \delta$. Since $z_0 \in \triangle^{(n)}$, $\triangle^{(n)} \subseteq B(z_0, \delta)$. We now have that

$$\begin{aligned}
\left| \int_T f \right| &\leq 4^n \left| \int_{T^{(n)}} f \right| \\
&= 4^n \left| \int_T f(z) - f(z_0) - f'(z_0)(z - z_0)\, \mathrm{d}z \right| \\
&\leq 4^n \int_T \left| f(z) - f(z_0) - f'(z_0)(z - z_0) \right| |\mathrm{d}z| \\
&\leq 4^n \int_T \epsilon |z - z_0| |\mathrm{d}z| \\
&\leq 4^n \int_T \epsilon \operatorname{diam}(\triangle^{(n)}) |\mathrm{d}z| \\
&= 4^n \epsilon \operatorname{diam}(\triangle^{(n)}) \ell(T^{(n)}) \\
&= \epsilon \operatorname{diam}(\triangle) \ell(T).
\end{aligned}$$

Since ϵ can be made arbitrarily small, we are done. ■

3.2. Homotopy

Definition 3.3. Let $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ be two closed rectifiable curves. Then, a *homotopy* between γ_1, γ_2 is a continuous function $\Gamma : [0,1] \times [0,1] \to \mathbb{C}$ such that

1. $\Gamma(s, 0) = \gamma_1(s)$,
2. $\Gamma(s, 1) = \gamma_2(s)$, and
3. $\Gamma(0, t) = \Gamma(1, t)$ for any s, t.

If there exists a homotopy between two curves, they are said to be *homotopic* and we write $\gamma_1 \sim \gamma_2$.

When a curve is homotopic to a constant curve, we write $\gamma \sim 0$.

In a convex set, any two paths are homotopic. as is seen by the homotopy

$$\Gamma(s, t) = t\gamma(s) + (1 - t)a$$

between any curve and a constant curve. Similarly, any two paths are homotopic in a star-shaped set, as can be seen by using the above homotopy with a as the point that is in sight of everything.

Theorem 3.16 (Homotopic version of Cauchy's theorem)**.** If γ_0, γ_1 are closed rectifiable curves in a region G and $\gamma_0 \sim \gamma_1$, then

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

for every analytic f on G.

Proof. We only prove the result in the case where there exists a homotopy Γ between γ_0, γ_1 with continuous second partial derivatives. Then, throughout the unit square $I^2 = [0,1]^2$,

$$\frac{\partial^2 \Gamma}{\partial s\ \partial t} = \frac{\partial^2 \Gamma}{\partial t\ \partial s}.$$

Define

$$g(t) = \int_0^1 f(\Gamma(s,t)) \frac{\partial \Gamma}{\partial s}(s,t)\,\mathrm{d}s.$$

We have that

$$g(0) = \int_0^1 f(\gamma_0(s)) \cdot \frac{\partial \gamma_0}{\partial s}(s)\,\mathrm{d}s = \int_{\gamma_0} f$$

and similarly, $g(1) = \int_{\gamma_1} f$. If we show that g is constant, we are done. To show this,

$$g'(t) = \int_0^1 \left(f'(\Gamma(s,t)) \frac{\partial \Gamma}{\partial t}(s,t) \frac{\partial \Gamma}{\partial s}(s,t) + f(\Gamma(s,t)) \tfrac{\partial^2 \Gamma}{\partial t\ \partial s}(s,t) \right) \mathrm{d}s.$$

Now,

$$\frac{\partial}{\partial s}\left((f \circ \Gamma)(s,t) \frac{\partial \Gamma}{\partial t}(s,t) \right) = f'(\Gamma(s,t)) \frac{\partial \Gamma}{\partial t}(s,t) \frac{\partial \Gamma}{\partial s}(s,t) + f(\Gamma(s,t)) \frac{\partial^2 \Gamma}{\partial t\ \partial s}(s,t).$$

Therefore,

$$g'(t) = (f \circ \Gamma)(1,t) \frac{\partial \Gamma}{\partial t}(1,t) - (f \circ \Gamma)(0,t) \frac{\partial \Gamma}{\partial t}(0,t).$$

Since $\Gamma(0,t) = \Gamma(1,t)$ for all t, this is zero. Therefore, g is constant on $[0,1]$ and $\int_{\gamma_0} f = \int_{\gamma_1} f$. ■

Corollary 3.17. If $\gamma \sim 0$, then $\gamma \approx 0$.

Above, 0 refers to any constant curve that maps every $t \in [0,1]$ to some fixed $a \in \mathbb{C}$. Similar to how we define $\gamma \approx 0$, we may define $\gamma_1 \approx \gamma_2$ on G in general, asserting that the winding numbers at the relevant points with respect to the two curves are equal.

Proof. Letting γ_0 be a constant curve, for any $w \notin G$,

$$\begin{aligned} n(\gamma; w) &= \frac{1}{2\pi\iota} \int_\gamma \frac{1}{z-w}\,\mathrm{d}z \\ &= \frac{1}{2\pi\iota} \int_{\gamma_0} \frac{1}{z-w}\,\mathrm{d}z \\ &= \frac{1}{2\pi\iota} \int_{\gamma_0} \frac{1}{a-w}\,\mathrm{d}z = 0. \end{aligned}$$

■

The converse of the above is not true in general.

Definition 3.4 (Simply connected domain)**.** An open set G is said to be *simply connected* if G is connected and every closed curve γ on G is homotopic to a constant curve.

Theorem 3.18. G is simply connected if and only if $\mathbb{C}_\infty \setminus G$ is connected.

Corollary 3.19. If G is simply connected, $\int_\gamma f = 0$ for every closed rectifiable curve γ in G and analytic function f on G.

Recall that if a function has a primitive, then its integral along any closed rectifiable curve is zero.
Further, the proof of Morera's Theorem showed that the converse is true on open disks.

Proposition 3.20. If G is simply connected and $f : G \to \mathbb{C}$ is analytic in G, then f has a primitive in G.

Proof. Fix $a \in G$. Define

$$F(z) = \int_{\gamma_z} f(z)\,\mathrm{d}z,$$

where γ_z is any rectifiable path from a to z. Simple connectedness implies that the value of the above is the same for any choice of γ_z. Indeed, if γ_1, γ_2 are two such choices, then the path $(\gamma_1 * \gamma_2^{-1}) \sim 0$, so the integral along it is 0. This integral is just equal to $\int_{\gamma_1} f - \int_{\gamma_2} f$. For $z_0 \in G$, let $R > 0$ such that $B(z_0, R) \subseteq G$. For any $\epsilon > 0$, we must demonstrate a $\delta > 0$ such that whenever $|z - z_0| < \delta$,

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| < \epsilon.$$

It suffices to show this for $z \in B(z_0, R)$. To do this, we can let γ_1 be a path from a to z_0 and $\gamma_2 = [z_0, z]$. Then,

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} &= \frac{1}{z - z_0} \left(\int_{\gamma_1 * \gamma_2} f - \int_{\gamma_1} f \right) \\ &= \frac{1}{z - z_0} \int_{[z_0, z]} f. \end{aligned}$$

The rest of the proof follows exactly as that of Morera's Theorem, and we can show that $F' = f$. ■

Corollary 3.21. If G is simply connected and $f : G \to \mathbb{C}$ is analytic such that $f(z) \neq 0$ for all $z \in G$, there exists analytic $g : G \to \mathbb{C}$ such that $e^{g(z)} = f(z)$ for all $z \in G$.

In particular, there exists a branch of the log on any simply connected domain that does not contain 0.

Proof. Then, there exists an analytic function $h : G \to \mathbb{C}$ such that $h(z) = f'(z)/f(z)$. Now,

$$\frac{\mathrm{d}}{\mathrm{d}z}(e^{-h(z)} f(z)) = e^{-h(z)} f'(z) - f(z) h'(z) e^{-h(z)} = 0.$$

Therefore, $e^{-h(z)} f(z)$ is some non-zero constant α, so $f(z) = \alpha e^{h(z)}$. We easily get some β such that $e^\beta = \alpha$, so the analytic function $\beta + h(z)$ does the job. ■

§4. Singularities

4.1. Poles and singularities

A function f is said to have a singularity at

Definition 4.1. A function f is said to have an *isolated singularity* at a if there exists $R > 0$ such that f is defined and analytic on $B(a, R) \setminus \{a\}$ but not on $B(a, R)$.
Further, f is said to have a *removable singularity* at a if there is an analytic function $g : B(a, R) \to \mathbb{C}$ such that $g = f$ on $B(a, R) \setminus \{a\}$.

The punctured disk $B(a, R) \setminus \{a\}$ is typically denoted $B(a, R)^*$.

Theorem 4.1. If f has an isolated singularity at a, then a is a removable singularity of f iff $\lim_{z\to a}(z - a)f(z) = 0$.

Proof. If f has a removable singularity at $z = a$, there exists $R > 0$ and analytic $g : B(a, R) \to \mathbb{C}$ such that $g = f$ on $B(a, R)^*$. This implies that

$$\lim_{z\to a}(z - a)f(z) = \lim_{z\to a}(z - a)g(z) = 0.$$

For the converse, define

$$g(z) = \begin{cases} (z - a)f(z), & z \neq a \\ 0, & z = a. \end{cases}$$

Clearly, g is continuous. We are done if we show that g is analytic on $B(a, R)$. Indeed, we can then write $g(z) = h(z)$ for some analytic h $B(a, R)$ that is equal to f on $B(a, R)^*$.
To prove analyticity, we shall use Morera's Theorem. For a T in $B(a, R)$, if a is not inside T, then $T \sim 0$ in $B(a, R)^*$ so $\int_T g = 0$ (g is analytic on the punctured disk).
For the remaining case, it suffices to consider the scenario where a is one of the vertices of $T = [a, b, c, a]$. Indeed, we may in general "split" the triangle into three subtriangles, each of which has a as a vertex. Let $x \in [a, b]$ and $y \in [a, c]$. Observe that the integral of g along $T = [a, b, c, a]$ is equal to that along $T_1 = [a, x, y, a]$. However, $\left|\int_{T_1} g\right|$ is at most $MV(T_1) = M(|a - x| + |a - y| + |x - y|)$, where M is the supremum of $|g|$ over some $\overline{B(a, \delta)} \subseteq B(a, R)$ that contains T, and this can be made arbitrarily small by bringing x, y close to a, completing the proof. ∎

Interestingly, the above says that if f has an isolated singularity at a and $\lim_{z\to a}(z - a)f(z) = 0$, then $\lim_{z\to a} f(z)$ exists!

Definition 4.2 (Pole)**.** If $z = a$ is an isolated singularity of f, a is said to be a *pole* of f if $\lim_{z\to a} f(z) = \infty$. That is, for any $M > 0$, there exists $\delta > 0$ such that $|f(z)| \geq M$ whenever $0 < |z - a| < \delta$.

An isolated singularity that is neither a removable singularity nor a pole is referred to as an *essential singularity.*

Proposition 4.2. If G is a region with $a \in G$ and f is analytic on $G \setminus \{a\}$ with a pole at a, then there is $m \in \mathbb{Z}^+$ and analytic $g : G \to \mathbb{C}$ such that

$$f(z) = \frac{g(z)}{(z - a)^m}.$$

This is equivalent to asserting that there exists $m \in \mathbb{Z}^+$ such that $f(z)(z - a)^m$ has a removable singularity at a.

Proof. We have that

$$\lim_{z\to a} \frac{1}{f(z)} = 0.$$

Define $h(z) = 1/f(z)$ on some $B(a, R)^*$ where f is nowhere 0. It is not too difficult to show using Theorem 4.1 that h has a removable singularity at $z = a$. Thus, there exists analytic $h_1 : B(a, R) \to \mathbb{C}$ such that $h_1 = 1/f$ on $B(a, R^*)$. Further, let $m \geq 1$ such that $h_1(z) = (z-a)^m h_2(z)$, where $h_2(a) \neq 0$ and h_2 is analytic on $B(a, R)$.
Then, $(z-a)^m f(z) = 1/h_2(z)$ on $B(a, R')$ for some R'. Let $g = 1/h_2$. This g may be extended to an analytic function from $G \to \mathbb{C}$ as

$$g(z) = \begin{cases} (z-a)^m f(z), & z \neq a, \\ 1/h_2(a), & z = a. \end{cases}$$

■

Definition 4.3. If f has a pole at a and m is the smallest positive integer such that $f(z)/(z-a)^m$ has a removable singularity at a, then f is said to have a pole of *order* m at a.

It is seen that if we take m as the order of the pole in the previous proposition, then $g(a) \neq 0$.
If $G = B(a, R)$, then the obtained g (for the order) is analytic, and hence,

$$g(z) = a_{-m} + a_{-(m-1)}(z-a) + \cdots + a_{-1}(z-a)^{m-1} + (z-a)^m \sum_{k=0}^{\infty} a_k (z-a)^k.$$

Consequently,

$$f(z) = \sum_{k=-m}^{\infty} a_k (z-a)^k.$$

Note that the function obtained by taking only the non-negative values of k in the above summation is analytic on $B(a, R)$. Also, since m is the order of the pole, $a_{-m} = g(a) \neq 0$.

Definition 4.4. If f has a pole of order m at a and

$$f(z) = \sum_{k=-m}^{\infty} a_k (z-a)^k,$$

then

$$\sum_{k=-m}^{-1} a_k (z-a)^k$$

is referred to as the *singular part* of f at a.

4.2. Laurent Series

Let us now look at how to deal with doubly infinite summations in general.

Definition 4.5. If $\{z_n : n \in \mathbb{Z}\}$ is a doubly infinite sequence of complex numbers, then

$$\sum_{n=-\infty}^{\infty} z_n$$

is *absolutely convergent* if both $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$ are absolutely convergent.
If u_n is a function on a set S for $n \in \mathbb{Z}$ and $\sum_{n=-\infty}^{\infty} u_n(s)$ is absolutely convergent for each $s \in S$, then the sum $\sum_{n=-\infty}^{\infty} u_n$ is *uniformly convergent* on S if both $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=1}^{\infty} u_{-n}$ converge uniformly on S.

Given $0 \leq R_1 < R_2 \leq \infty$ and $a \in \mathbb{C}$, define the *annulus*

$$\operatorname{ann}(a, R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z-a| < R_2\}.$$

In particular, $\operatorname{ann}(a, 0, R) = B(a, R)^*$.

Theorem 4.3. Let f be analytic on $\operatorname{ann}(a, R_1, R_2)$. Then,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where the convergence is absolute and uniform over $\overline{\operatorname{ann}(a, r_1, r_2)}$ for any $R_1 < r_1 < r_2 < R_2$.
Further, the coefficients a_n are given by the formula

$$a_n = \frac{1}{2\pi\iota} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \,\mathrm{d}z,$$

where γ is the circle $a + re^{2\pi\iota t}$ for some $R_1 < r < R_2$. Moreover, this integral is independent of r and the choice of coefficients is unique.

This summation is referred to as the function's *Laurent series expansion.*

Proof. It is clear that the integrals for paths of distinct r are the same since two such paths are path-homotopic.
Define $f_2 : B(a, R_2) \to \mathbb{C}$ be

$$f_2(z) = \frac{1}{2\pi\iota} \int_\gamma \frac{f(w)}{w-z} \,\mathrm{d}w,$$

where $\gamma(t) = a + re^{2\pi\iota t}$ for some $R_1 < r < R_2$ with $r > |z-a|$. Notice that f_2 is well-defined and analytic on $B(a, R_2)$ by Lemma 3.4. Similarly, consider the function $f_1 : \operatorname{ann}(a, R_1, \infty) \to \mathbb{C}$ defined by

$$f_1(z) = -\frac{1}{2\pi\iota} \int_\gamma \frac{f(w)}{w-z} \,\mathrm{d}w$$

where $\gamma(t) = a + re^{2\pi\iota t}$ for some $R_1 < r < R_2$ with $r < |z-a|$. Once more, f_1 is well-defined and analytic on $\operatorname{ann}(a, R_1, \infty)$.
Let $z \in \operatorname{ann}(a, R_1, R_2)$ and let $r_1, r_2 \in (R_1, R_2)$ with $r_1 < |z-a| < r_2$. Let $\gamma_i(t) = a + r_i e^{2\pi\iota t}$ for $i = 1, 2$, and λ be a straight line curve joining $a + r_1$ to $a + r_2$, and assume this does not pass through z. Finally, let $\Gamma = \gamma_1 + \lambda - \gamma_2 - \lambda$.
Then, for any $b \notin \operatorname{ann}(a, R_1, R_2)$,

$$\begin{aligned} n(\Gamma; b) &= \frac{1}{2\pi\iota} \int_\Gamma \frac{1}{w-b} \,\mathrm{d}w \\ &= \frac{1}{2\pi\iota} \left(\int_{\gamma_1} \frac{1}{w-b} \,\mathrm{d}w - \int_{\gamma_2} \frac{1}{w-b} \,\mathrm{d}w \right) \\ &= n(\gamma_1; b) - n(\gamma_2; b) = 0. \end{aligned}$$

That is, $\Gamma \approx 0$ in $\operatorname{ann}(a, R_1, R_2)$. By Cauchy's integral formula,

$$\begin{aligned} \frac{1}{2\pi\iota} \int_\Gamma \frac{f(w)}{w-z} \,\mathrm{d}w &= n(\Gamma; z) f(z) \\ &= \big(n(\gamma_1; z) - n(\gamma_2; z)\big) f(z) \\ &= -f(z). \end{aligned}$$

Therefore,

$$\begin{aligned} f(z) &= -\frac{1}{2\pi\iota} \int_{\gamma_1} \frac{f(w)}{w-z} \,\mathrm{d}w + \frac{1}{2\pi\iota} \int_{\gamma_2} \frac{f(w)}{w-z} \,\mathrm{d}w \\ &= f_1(z) + f_2(z). \end{aligned}$$

We shall now expand f_1 and f_2 as power series. Since f_2 is analytic on $B(a, R_2)$, it is equal to a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$, where

$$a_n = \frac{f_2^{(n)}(a)}{n!} = \frac{1}{2\pi\iota}\int_{\gamma_2} \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z$$

by Lemma 3.4.
f_1 on the other hand is problematic because the region of analyticity is not an open disk. To resolve this, let

$$g(z) = \begin{cases} f_1(a+\frac{1}{z}), & 0 < |z| < 1/R_1, \\ 0, & z = 0. \end{cases}$$

It may be shown that $\lim_{z\to 0} g(z) = 0$, and g is thus analytic on $B(a, 1/R_1)$ by a method similar to what we did in the proof of Theorem 4.1. Therefore, g has a power series, so

$$g(z) = \sum_{n=1}^{\infty} a_{-n} z^n,$$

and thus, for $z \in \operatorname{ann}(a, R_1, \infty)$,

$$f_1(z) = \sum_{n=1}^{\infty} a_{-n}\frac{1}{(z-a)^n},$$

and the desideratum follows because $f = f_1 + f_2$.

It remains to show that the coefficients are unique. Suppose that we can write

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a)^n$$

Let $\gamma(t) = a + re^{2\pi\iota t}$ for some $R_1 < r < R_2$. We wish to show that

$$b_n = \frac{1}{2\pi\iota}\int_{\gamma} \frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w.$$

for $z \in \operatorname{ann}(a, R_1, R_2)$, such that the convergence is absolute and uniform on $\overline{\operatorname{ann}(a, r_1, r_2)}$ for $R_1 < r_1 < r_2 < R_2$. Due to absolute convergence,

$$f(w) = \lim_{m\to\infty} \sum_{k=-m}^{m} b_k (w-a)^k = \lim_{m\to\infty} S_m(w)$$

Due to uniform convergence,

$$\begin{aligned}
\frac{1}{2\pi\iota}\int_{\gamma} \frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w &= \lim_{m\to\infty} \frac{1}{2\pi\iota}\int_{\gamma} \frac{S_m(w)}{(w-a)^{n+1}}\,\mathrm{d}w \\
&= \lim_{m\to\infty} \frac{1}{2\pi\iota}\int_{\gamma} \frac{1}{(w-a)^{n+1}} \sum_{k=-m}^{m} b_k (w-a)^k\,\mathrm{d}w \\
&= \lim_{m\to\infty} \sum_{k=-m}^{m} b_k \frac{1}{2\pi\iota}\int_{\gamma} (w-a)^{k-n-1}\,\mathrm{d}w \\
&= b_n n(\gamma; a) = b_n,
\end{aligned}$$

where the second-to-last term follows from the fact that every other summand has a primitive so integrates to 0, thus completing the proof. ■

Corollary 4.4. Let a be an isolated singularity of f and let $f(z) = \sum_{-\infty}^{\infty} a_n(z-a)^n$ be its Laurent series expansion in $\operatorname{ann}(a, 0, R)$. Then,

(a) a is a removable singularity iff $a_n = 0$ for $n \leq -1$.

(b) a is a pole of order m iff $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$.

(c) a is an essential singularity iff $a_n \neq 0$ for infinitely many negative integers n.

We omit the proof of the above.

Theorem 4.5 (Casoratti-Weierstrass Theorem)**.** If f has an essential singularity at a, then for any $\delta > 0$, $\overline{f(\operatorname{ann}(a, 0, \delta))} = \mathbb{C}$.

Proof. Fix some $R > 0$. We wish to show that $f(\operatorname{ann}(a, 0, R))$ is dense in $\mathbb{C}$. Suppose instead that there exist $c \in \mathbb{C}$ and $\epsilon > 0$ such that $f(\operatorname{ann}(a, 0, R)) \cap B(c, \epsilon) = \varnothing$. That is,

$$|f(z) - c| \geq \epsilon$$

for all $z \in \operatorname{ann}(a, 0, R)$. Consequently,

$$\lim_{z \to a} \frac{|f(z) - c|}{|z - a|} = \infty.$$

This means that the function $g : z \mapsto (f(z) - c)/(z - a)$ has a pole at a of order $m \geq 1$ (say). Therefore, $(z - a)^{m-1}(f(z) - c)$ has a removable singularity at $z = a$, so

$$\lim_{z \to a} (z - a)^m (f(z) - c) = 0.$$

Because $m \geq 1$, this means that $\lim_{z \to a} (z - a)^m f(z) = 0$. This means that f has a pole at a, which contradicts the fact that it has an essential singularity. ■

We now define what it means to have a singularity at ∞.

Definition 4.6. Let $R > 0$ and $G = \{z : |z| > R\}$, $f : G \to \mathbb{C}$ is said to have an isolated singularity at ∞ if $z \mapsto f(1/z)$ has an isolated singularity at 0.

We have similar definitions for a removable singularity, pole (of order m), or essential singularity at ∞.
It may be shown that an entire function has a removable singularity at ∞ iff it is constant. An entire function $f : \mathbb{C} \to \mathbb{C}$ has a pole ∞ of order m iff it is a polynomial of degree m.

4.3. Residues

Definition 4.7. Let f have an isolated singularity at a and let $f(z) = \sum_{-\infty}^{\infty} a_n (z - a)^n$ be its Laurent series expansion about a. Then, the *residue* of f at a is the coefficient a_{-1}. We denote this by $\operatorname{Res}(f; a)$.

That is, if f is analytic on $\operatorname{ann}(a, 0, R_1)$,

$$\operatorname{Res}(f; a) = \frac{1}{2\pi\iota} \int_\gamma f(z) \, \mathrm{d}z,$$

where $\gamma = a + re^{2\pi\iota t}$ for any $0 < r < R_1$.
A natural question to ask is: what is the value of the above integral if γ contains more than one singularity?

Theorem 4.6. Let f be analytic on a region $G \setminus \{b_1, \ldots, b_m\}$, where each $b_k \in G$ is an isolated singularity of f. If γ is a closed rectifiable curve that does not pass through any of the points a_k and $\gamma \approx 0$ in G, then

$$\frac{1}{2\pi\iota}\int_\gamma f = \sum_{k=1}^{m} n(\gamma; a_k)\operatorname{Res}(f; a_k).$$

Just like zeros in the discussion after Corollary 3.11, singularities are isolated here. As a result, the above works out even if there are infinitely many isolated singularities, because only finitely many of the $n(\gamma; a_k)$ are non-zero.

Proof. ■

Improper integrals are of two types: either infinite integrals or when the integrand is discontinuous.
Theorem 4.6 is incredibly useful at times. For example, let us show that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}.$$

Define $f(z) = z^2/(1+z^4)$. Clearly, f is analytic on $\mathbb{C} \setminus Z(1+z^4)$. Let $a_k = e^{(2k+1)\pi\iota/4}$ for each $k = 1,2,3,4$ be the elements of $Z(1+z^4)$. Now,

$$\operatorname{Res}(f; a_1) = \lim_{z \to a_1} f(z) = \frac{a_1^2}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)}.$$

We can compute each of the residues to get

$$\operatorname{Res}(f; a_1) = \frac{1}{4}e^{-\iota\pi/4},$$
$$\operatorname{Res}(f; a_2) = \frac{1}{4}e^{-3\iota\pi/4}.$$

Let $R > 1$ and consider the curves $\gamma_1(t) = Re^{\pi\iota t}$ and $\gamma_2(t) = (2t-1)R$. Now,

$$\begin{aligned}\frac{1}{2\pi\iota}\int_{\gamma_1 * \gamma_2} f &= n(\gamma; a_1)\operatorname{Res}(f; a_1) + n(\gamma; a_2)\operatorname{Res}(f; a_2) \\ &= \frac{1}{4}\left(e^{-\iota\pi/4} + e^{-3\iota\pi/4}\right) = -\frac{\iota}{2\sqrt{2}}.\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}} - \lim_{R\to\infty}\int_{\gamma_2} f.$$

It remains to compute the final quantity.

$$\begin{aligned}\int_{\gamma_1} f &= \int_0^\pi \frac{(Re^{\iota t})^2}{1+(Re^{\iota t})^4} R\iota e^{\iota t}\,\mathrm{d}t \\ &= \iota R^3 \int_0^\pi \frac{e^{3\iota t}}{1+R^4 e^{4\iota t}}.\end{aligned}$$

We have

$$\left|\iota R^3 \int_0^\pi \frac{e^{3\iota t}}{1+R^4e^{4\iota t}}\right| \le R^3 \int_0^\pi \frac{1}{R^4-1}|\mathrm{d}t| = \frac{R^3}{R^4-1}\pi.$$

Therefore, $\lim_{R\to\infty}\int_{\gamma_2} f = 0$, and the value of the required integral is just $]pi/\sqrt{2}$!

Recall that if f is analytic and has a zero of multiplicty m at a, then $f(z) = (z-a)^m g(z)$, where g is analytic and $g(a) \neq 0$. Consequently,

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}$$

on $B(a, R)^*$ for some $R > 0$.
On the other hand, if f has a pole at a of order m, then $f(z) = (z-a)^{-m} g(z)$ where g is analytic and $g(a) \neq 0$. So,

$$\frac{f'(z)}{f(z)} = \frac{-m}{z-a} + \frac{g'(z)}{g(z)}$$

on $B(a, R)^*$ for some $R > 0$.

Definition 4.8. If G is open and f an analytic function on G except for poles, then f is said to be *meromorphic* on G.

Theorem 4.7 (Argument Principle)**.** Let f be meromorphic on G with poles at $p_1, \ldots, p_m$ and zeros at $z_1, \ldots, z_n$ (counted multiple times according to multiplicity). If γ is a closed rectifiable curve inside G with $\gamma \approx 0$ in G that does not pass through any p_i or z_i, then

$$\frac{1}{2\pi\iota} \int_\gamma \frac{f'(z)}{f(z)} = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{k=1}^{m} n(\gamma; p_k).$$

The proof is straightforward using the observation before the previous definition.

Recall that if f is one-one and analytic on some open set G and $f(G) = \Omega$, then $f^{-1} : \Omega \to G$ is analytic.

Proposition 4.8. Let f be analytic on an open set containing $\overline{B(a, R)}$, and suppose f is one-one on $B(a, R)$. If $\Omega = f(B(a, R))$ and γ is the circle $z = a + Re^{2\pi\iota t}$, then for each $w \in \Omega$,

$$f^{-1}(w) = \frac{1}{2\pi\iota} \int_\gamma \frac{z f'(z)}{f(z) - w} \, \mathrm{d}z.$$

Proof. For $w \in \Omega$, $f(z) - w$ has a zero in $B(a, R)$ of multiplicity 1 (due to Lemma 3.13). It follows that

$$\frac{f'(z)}{f(z) - w} = \frac{1}{z - f^{-1}(w)} + \frac{g'(z)}{g(z)}$$

where $g(z) \neq 0$ for all $z \in B(a, R)$.
So,

$$\frac{1}{2\pi\iota} \int_\gamma \frac{z f'(z)}{f(z) - w} = \underbrace{\frac{1}{2\pi\iota} \int_\gamma \frac{z}{z - f^{-1}(w)}}_{f^{-1}(w)} + \underbrace{\frac{1}{2\pi\iota} \int_\gamma \frac{z g'(z)}{g(z)}}_{0},$$

by Theorem 3.5, completing the proof. ∎

Theorem 4.9 (Rouche's Theorem)**.** Suppose f and g are meromorphic in an open set containing $\overline{B(a, R)}$ with no zeros or poles on the circle $\gamma : t \mapsto a + Re^{2\pi\iota t}$. If Z_f, Z_g (resp. P_f, P_g) are the number of zeros (resp. poles) of f and g respectively inside γ counted according to multiplicity, and $|f(z) + g(z)| < |f(z)| + |g(z)|$ on γ, then $Z_f - P_f = Z_g - P_g$.

Proof. It follows that on γ, $f(z)/g(z)$ is never a positive real. That is, it takes values inside $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ on an open set U containing $\{\gamma\}$. Let log be a branch of the log ion $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Then, $\frac{(f/g)'}{(f/g)}$ has primitive $\log(f/g)$ on U. As a result,

$$\begin{aligned} 0 &= \frac{1}{2\pi\iota} \int_\gamma \frac{(f/g)'}{(f/g)} \\ &= \int_\gamma \frac{f'g - fg'}{g^2} \cdot \frac{g}{f} \\ &= \int_\gamma \frac{f'}{f} - \int_\gamma \frac{g'}{g}. \end{aligned}$$

The result follows from the Argument Principle. ■