Problem Sheet 4

1. Prove the q-Pascal recurrences

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q \text{ and}$$
$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q.$$

2. If $y = f(x)$ and $Dy = p(y)$ for some polynomial p, then show that $D^n y$ is a polynomial of y. If $D^n y = p_n(y)$ for each n, then prove that

$$p_n(y) = \begin{cases} y & n = 0 \\ p_{n-1}(y) \cdot p(y) & n \geq 1. \end{cases}$$

3. Setting $y = \tan x$ and $z = \sec x$, show that

   (a) $D^n y$ is a homogeneous polynomial in y, z of degree $(n+1)$, and

   (b) $D^n y$ has only terms with even exponents of z.

4. Recall that $W_{n,k}$ is the coefficient of $z^{2k+2}y^{n-2k-1}$ in $D^n y$, where $y = \tan x$ and $z = \sec x$. Prove or disprove that

$$\sum_k W_{n,k} = n!.$$

5. Show that

$$W_{n,k} = (2k+2)W_{n-1,k} + (n-2k)W_{n-1,k-1}.$$

6. Show that

   (a) $m_k(C_n) = n/(n-k)\binom{n-k}{k}$.

   (b) $m_k(K_n) = \binom{n}{2k} \cdot (2k)!/(2^k k!)$.

   (c) $m_k(K_{n,n}) = \binom{n}{k}k!$.

7. Given a graph G and edge $e = \{u, v\}$ in G, express Match_G in terms of Match_{G-e} and $\mathrm{Match}_{G-\{u,v\}}$.

8. Show that for a fixed n, $\binom{n}{k}$ as k varies is log-concave.

9. Prove that if $A(x) = \sum_{i=0}^{n} a_i x^i$ is a polynomial with all real roots, then the sequence of coefficients of A is log-concave.