Maps Between Topological Spaces

Def (Continuity) Let X and Y be topological spaces. A function $f: X \to Y$ is said to be **continuous at $b \in X$** if for any open $V \subseteq Y$ with $f(b) \in V$, there exists open $U \ni b$ (in X) such that $f(U) \subseteq V$.
f is **continuous** if for any open V in Y, $f^{-1}(V)$ is open in X.

Note that f is continuous iff it is continuous at all $b \in X$.
(How? Use the fact that an arbitrary union of open sets is open)

Recall that this is equivalent to the usual definition of continuity for metric spaces (taking the metric topology here).

Since the topologies matter as well, note that even the identity map from $\mathbb{R}_c$ to $\mathbb{R}$ is not continuous.

If the topology of Y is given by a basis $\mathcal{B}$ and we want to determine continuity, it suffices to check the pre-images of basis elements of Y. Indeed, use the fact that an arbitrary union of open sets is open.

Further, it suffices to just check subbasis elements!
Indeed, the set of finite intersections of subbasis elements form a basis. (and a finite intersection of open sets is open)

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Theo. (2.1) Let X and Y be topological spaces and $f: X \to Y$. Then the following are equivalent.

i) f is continuous.
ii) For every $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.
iii) For every closed $B \subseteq Y$, $f^{-1}(B)$ is closed in X.
iv) For every $x \in X$ and neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subseteq V$.

Proof $i \Rightarrow ii$

Suppose f is continuous and $A \subseteq X$. Let $x \in \bar{A}$.
Let V be a neighbourhood of $f(x)$. We shall show that $V \cap f(A) \neq \emptyset$, implying that $f(x) \in \overline{f(A)}$.
Since $x \in \bar{A}$ and $x \in f^{-1}(V)$, which is open in X,

$$f^{-1}(V) \cap A \neq \emptyset.$$

Let $y \in f^{-1}(V) \cap A$. Then $f(y) \in \underset{(\text{why?})}{V \cap f(A)}$, proving the claim.

$ii \Rightarrow iii$

Let B be closed in Y and $A = f^{-1}(B)$. Let $x \in \bar{A}$. Then

$$f(x) \in f(\bar{A}) \subseteq \overline{f(A)} \subseteq B$$

Therefore, $x \in f^{-1}(B)$ and the claim is proved.

$iii \Rightarrow i$

Observe that iii is just the definition of continuity but with 'closed' instead of 'open'.
Let $B \subseteq Y$ be open. Then $Y \setminus B$ is closed and $f^{-1}(Y \setminus B)$ is closed. That is, $X \setminus f^{-1}(Y \setminus B)$ is open, and this set is just $f^{-1}(B)$. $\square$

We briefly mentioned $i \Leftrightarrow iv$ earlier. The details are left as an exercise.

Def. Let X and Y be topological spaces and $f: X \to Y$ be a bijection. f is said to be a **homeomorphism** if both f and f^{-1} are continuous.

Homeomorphism

Equivalently, f is a homeomorphism if for any $U \subseteq X$, $f(U)$ is open (in Y) iff U is open (in X).
That is, it is a continuous open bijection.

$$\begin{array}{cc} \downarrow & \downarrow \\ V \text{ open} & U \text{ open} \\ \Downarrow & \Downarrow \\ f^{-1}(V) \text{ open} & f(U) \text{ open} \end{array}$$

A homeomorphism also gives a bijective map between the open sets of X and Y.

So if X has some property that is expressed in terms of the topology on X, Y must have the same property as well.
Such a property is called a topological property of X.
(for example, the space being Hausdorff)

If there is a homeomorphism between two spaces, they are said to be homeomorphic.
(This implicitly uses the fact that if there is a homeomorphism: $X \to Y$, there is a homeomorphism: $Y \to X$ — the inverse of the first)

Homeomorphisms are the topological counterpart of isomorphisms in algebra.

Def. Embedding — Let $f: X \to Y$ be a continuous injective map. Let $Z = f(X) \subseteq Y$ and consider it as a subspace of Y. The function $f': X \to Z$ attained by restricting the codomain is bijective. If f' is a homeomorphism, then f is said to be a topological embedding or just embedding of X in Y.

Note that the "homeomorphic" relation is an equivalence relation. (Why?)

Let X, Y, Z be topological spaces.

1. Any constant map $f: X \to Y$ is continuous.
2. If A is a subspace of X, the inclusion map $f: A \hookrightarrow X$ is continuous.
3. If $f: X \to Y$ and $g: Y \to Z$ are continuous, $g \circ f: X \to Z$ is continuous.
4. If $f: X \to Y$ is continuous and A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.
5. Similarly, we can restrict / expand the range.

to a subspace $Z \supseteq f(X)$ — to a space Z with subspace Y.

Lemma (2.2). Let $f: X \to Y$ and $X = \bigcup_{\alpha \in A} U_\alpha$ for some (U_α). Then f is continuous iff $f|_{U_\alpha}$ is continuous for each α.

Proof. The forward direction is obvious.
For the backward direction, let V be open in Y. Observe that

$$f^{-1}(V) \cap U_\alpha = f|_{U_\alpha}^{-1}(V).$$

$f|_{U_\alpha}^{-1}(V)$ is open in U_α, and thus X (Why?). This implies that

$$f^{-1}(V) = \bigcup_{\alpha \in A} (f^{-1}(V) \cap U_\alpha),$$

which yields the result since an arbitrary union of open sets is open. $\square$

Theo. (2.3) [Pasting Lemma] Let $X = A \cup B$ for closed A, B in X. Let $f: A \to Y$ and $g: B \to Y$ be continuous*. If $f(x) = g(x)$ for all $x \in A \cap B$, then the map $h: X \to Y$ defined by

$$h(x) = \begin{cases} f(x), & x \in A, \\ g(x), & x \in B \end{cases}$$

(* with respect to the subspace topologies)

is continuous

Proof. Let C be closed in Y. Note that

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

Since $f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B, which are in turn closed in X, they are also closed in X. This gives the result because a finite union of closed sets is closed. $\square$

Note that the result holds even if A and B are open.