Function Spaces

Lecture 27 - 07/04/21 Cauchy Sequences and Complete Spaces

Recall the definition of a Cauchy sequence in a metric space.
A metric space (X,d) is said to be **complete** if every Cauchy seq. in X converges.

→ Any convergent sequence is Cauchy.
→ Let A be a closed subspace of a complete metric space (X,d). Then A is complete in the restricted metric.
→ X is complete under the metric d iff it is complete under the standard bounded metric $\bar{d} = \min\{d, 1\}$.

Try to prove the above basic facts.

(5.1)
Lemma. X is complete iff every Cauchy sequence in X has a convergent subsequence.

Proof. Let (x_n) be Cauchy and the subseq. $x_{n_k} \to x$.
For $\varepsilon > 0$, let N such that $d(x_n, x_m) < \varepsilon/2$ for all $n,m > N$.
For sufficiently large $k > N$, let $d(x_{n_k}, x) < \varepsilon/2$.
Then $d(x_n, x) < \varepsilon$ for all $n > N$, proving the claim.
The other direction is direct.

Theo. $\mathbb{R}^k$ is complete in the Euclidean metric d and the square metric ρ.
(5.2)

Proof Since $\rho < d < \sqrt{k}\,\rho$, $\mathbb{R}^k$ is complete wrt one metric iff it is complete wrt the other.
We shall show that $(\mathbb{R}^k, \rho)$ is complete. Let (x_n) be Cauchy. Then $\{x_n\}$ is a bounded subset of $\mathbb{R}^k$ (Why?).
Let $x_n < M$ for all $n \in \mathbb{N}$. Then $x_n \in [-M, M]^k$ for any k. Since this subspace is compact (it is closed and bounded), any sequence in it has a convergent subsequence, (x_n) in particular. Using Lemma 5.1 completes the proof.

Lemma (5.3). Let $X = \prod_{\alpha \in J} X_\alpha$ and (x_n) a sequence of points in X. Then $x_n \to x$ iff $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for all $\alpha \in J$.

Proof. The forward direction is immediate since π_α is a continuous map.
Suppose $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for all $\alpha \in J$
Let $U = \prod_\alpha U_\alpha$ be a basis element of X containing x. For each α with $U_\alpha \neq X_\alpha$, let N_α such that $\pi_\alpha(x_n) \in U_\alpha$ for $n \geq N_\alpha$. Letting N be the largest of these N_α, then for all $n \geq N$, $x_n \in U$. Therefore, $x_n \to x$

Theo. (5.4) There is a metric for the product space $\mathbb{R}^\omega$ with respect to which it is complete.

Proof. Let $D(x,y) = \sup_i \left\{ \min\{d(x,y),1\}/i \right\}$. D induces the product topology.

We claim that $\mathbb{R}^\omega$ under D is complete. Let (x_n) be Cauchy in $\mathbb{R}^\omega$. For fixed i, $(\pi_i(x_n))$ is Cauchy because $\bar{d}(\pi_i(x), \pi_i(y)) \leq i D(x,y)$. Therefore, $(\pi_i(x_n))$ converges to some a_i. The result follows on using Lemma 5.3.

Completeness is <u>not</u> a topological property. For example, $(-1,1)$ is not complete and $\mathbb{R}$ is, but the two are homeomorphic.

Def. Let (Y,d) be a metric space. If $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ are points in Y^J, then

$$\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\}$$

defines a metric on Y^J. It is called the uniform metric on Y^J corresponding to the metric d on Y^J.

Recall that $Y^J = \{f : J \to Y\}$. Then

$$\bar{\rho}(f,g) = \sup\{\bar{d}(f(\alpha), g(\alpha)) : \alpha \in J\}.$$

Theo. (5.5) With the above defined notation, if (Y, d) is complete, then so is $(Y^J, \bar{\rho})$.

Proof. Since (Y,d) is complete, so is $(Y, \bar{d})$.

Let (f_n) be Cauchy in Y^J For $\alpha \in J$,

$$\bar{d}\,(f_n(\alpha), f_m(\alpha)) \leq \bar{\rho}\,(f_n, f_m).$$

Therefore, $(f_n(\alpha))$ is Cauchy in $(Y, \bar{d})$, and thus convergent (Why?).

Let $f_n(\alpha) \to f(\alpha)$ for each α.

Let $\varepsilon > 0$. Cauchyness implies that for sufficiently large N,

$$\bar{d}\,(f_n(\alpha), f_m(\alpha)) < \varepsilon/2 \quad \text{for } n,m \geqslant N \text{ and } \alpha \in J.$$

Then making m arbitrarily large and using convergence of $(f_m(\alpha))$,

$$\bar{d}\,(f_n(\alpha), f(\alpha)) \leq \varepsilon/2 \quad \text{for } n \geqslant N \text{ and } \alpha \in J.$$

Therefore,

$$\bar{\rho}\,(f_n, f) \leq \varepsilon/2 < \varepsilon \quad \text{for } n \geqslant N,$$

proving the claim.

Henceforth, denote

$$Y^X = \{f : Y \to X\}$$
$$\mathcal{C}(X, Y) = \{f : Y \to X : f \text{ is continuous}\}$$
$$\mathcal{B}(X, Y) = \{f : Y \to X : f \text{ is bounded}\}$$

Theo. (5.6) Let X be a topological space and (Y, d) be a metric space Then $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ are closed in Y^X under the uniform metric. In particular, if Y is complete, so are $\mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$.
(in the uniform metric)

Proof. Let $f_n \to f$ in Y^X. We claim that f_n converges uniformly.

For each $\varepsilon > 0$, choose N such that

$$\bar{\rho}\,(f, f_n) < \varepsilon \quad \text{for all } n > N.$$

Then for all $x \in X$,

$$\bar{d}\,(f(x), f_n(x)) \leq \bar{\rho}\,(f, f_n) < \varepsilon,$$

so (f_n) converges uniformly to f.

Now, we show that $\mathcal{C}(X,Y)$ is closed in Y^X relative to $\bar{\rho}$.
Let $(f_n) \in \mathcal{C}(X,Y)$ and $f_n \to f$. By the uniform limit theorem, $f \in \mathcal{C}(X,Y)$

Showing that $\mathcal{B}(X,Y)$ is closed is straightforward (using the triangle inequality and uniform convergence)

Lecture 28 - 09/04/21 More on Function Spaces

Def If (Y,d) is a metric space and X is a topological space,
$$\rho(f,g) = \sup_{x\in X} \{d(f(x),g(x))\}$$
defines a metric on $\mathcal{B}(X,Y)$, known as the sup metric.

(Why is it a metric?)

1. For $f,g \in \mathcal{B}(X,Y)$,
$$\underbrace{\bar{\rho}(f,g)}_{\text{uniform metric}} = \min\{\underbrace{\rho(f,g)}_{\text{sup metric}}, 1\}.$$

2. If X is compact, every continuous function $f: X \to Y$ is bounded.
($f(X)$ is compact, and thus bounded)

Theo. (5.7) Let (X,d) be a metric space. There is an isometric embedding of X into a complete metric space.
(distance-preserving)

Proof Consider $\mathcal{B}(X,\mathbb{R})$ and let $x_0 \in X$. For $a \in X$, define $\varphi_a : X \to \mathbb{R}$ by
$$\varphi_a(x) = d(x,a) - d(x,x_0)$$
φ_a is bounded for any $a \in X$ because $|\varphi_a(x)| \leq d(x_0,a)$.
Define $\varphi \; X \to \mathcal{B}(X,\mathbb{R})$ by $\varphi(a) = \varphi_a$.
Because $\mathbb{R}$ is complete, so is $\mathcal{B}(X,\mathbb{R})$ under the uniform metric, and thus the sup metric.
We show that φ is an isometric embedding of X in $\mathcal{B}(X,\mathbb{R})$.

For $a, b \in X$,

$$\begin{aligned}\rho(\varphi_a, \varphi_b) &= \sup_{x\in X} \{|\varphi_a(x) - \varphi_b(x)|\} \\ &= \sup_{x\in X} \{|d(x,a) - d(x,b)|\} \\ &= d(a,b).\end{aligned}$$

so φ is isometric

φ is injective, continuous, and open (on its image), completing the proof.

Def. Let X be a metric space. If $h: X \to Y$ is an isometric embedding of X into a complete metric space Y, $\overline{h(X)}$ is a complete metric space (as a subspace of Y), known as the completion of X.

The completion is unique up to isometry.

Def. A metric space (X,d) is said to be totally bounded if for any $\varepsilon > 0$, there is a finite covering of X by ε-balls.

1. Any totally bounded space is bounded.
   (The converse does not hold – consider $\mathbb{R}$ under $\bar{d}$)
2. A compact space is totally bounded

(5.8)
Theo. A metric space (X,d) is compact iff it is complete and totally bounded.

Proof. We have already seen that any compact space is complete and totally bounded.

Let X be complete and totally bounded. It suffices to show that X is sequentially compact.

(for a metric space, compact $\Leftrightarrow$ sequentially compact $\Leftrightarrow$ limit pt. compact)

Let (x_n) be a sequence in X. We shall construct a Cauchy subsequence of (x_n).

Cover X by finitely many balls of radius 1. Suppose B_1 is one of these balls that contains infinitely many x_n. Let $J_1 \subseteq \mathbb{N}$ be the set of indices contained in B_1.

Given a set J_k of naturals, let J_{k+1} be an infinite subset of J_k such that there is a $(1/k+1)$-ball B_{k+1} that contains x_i for all $i \in J_{k+1}$. (Such a J_{k+1} exists by total boundedness)
Choose $n_1 \in J_1$. Given n_k, choose $n_{k+1} \in J_{k+1}$ such that $n_{k+1} > n_k$ (because J_{k+1} is infinite). For any $i, j \geq k$, $n_i, n_j \in J_k$. That is, x_{n_i} and x_{n_j} are contained in a $(1/k)$-ball.
It follows that (x_{n_k}) is Cauchy, and thus convergent, completing the proof.

Def. Let (Y,d) be a metric space and $\mathcal{F} \subseteq \mathcal{C}(X,Y)$ For $x_0 \in X$, $\mathcal{F}$ is said to be equicontinuous at x_0 if given $\varepsilon > 0$, there is a neighbourhood U of x_0 such that for all $x \in X$ and $f \in \mathcal{F}$,

$$d(f(x), f(x_0)) < \varepsilon.$$

If $\mathcal{F}$ is equicontinuous at x_0 for each $x_0 \in X$, it is said to be equicontinuous.

If (Y,d) is a metric space, $\mathcal{F} \subseteq \mathcal{C}(X,Y)$ is said to be pointwise bounded if for each $a \in X$, $\mathcal{F}_a = \{f(a) : f \in \mathcal{F}\}$ is bounded.

Theo. (5.9) [Ascoli's Theorem, Classical Version]
Let X be a compact space, $(\mathbb{R}^n, d)$ denote Euclidean space in the square metric or Euclidean metric, and give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. Then $\mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R}^n)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and pointwise bounded

Lemma. (5.10) Let X be any space and (Y,d) be a metric space. If $\mathcal{F} \subseteq \mathcal{C}(X,Y)$ is totally bounded under the uniform metric, $\mathcal{F}$ is equicontinuous.

Proof. Let $\mathcal{F}$ be totally bounded. Let $0 < \varepsilon < 1$ and $x_0 \in X$.
Let $\delta = \varepsilon/3$. Cover $\mathcal{F}$ by finitely many δ-balls $B(f_1, \delta), \ldots, B(f_n, \delta)$ in $\mathcal{C}(X,Y)$.
Choose a nbd. U of x_0 such that

$$d(f_i(x), f_i(x_0)) < \delta \text{ for all } x \in U \text{ and } 1 \leq i \leq n.$$

Let $f \in \mathcal{F}$. For some i, $f \in B(f_i, \delta)$.

Therefore,

$$\bar{d}(f(x), f(x_0)) < \bar{d}(f(x), f_i(x)) + \bar{d}(f_i(x), f_i(x_0)) + \bar{d}(f_i(x_0), f(x_0))$$
$$= 3\delta = \varepsilon,$$

completing the proof.

Lecture 29 - 14/04/21 Completing the proof of Ascoli's Theorem

Lemma (5.11) Let X be a space, (Y, d) a metric space, and suppose both are compact. If $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is equicontinuous, then $\mathcal{F}$ is totally bounded under the uniform and sup metrics corresponding to d.

Proof Total boundedness under ρ is equivalent to total boundedness under $\bar{\rho}$.

Let $\mathcal{F}$ be equicontinuous. Given $\varepsilon > 0$, we shall cover $\mathcal{F}$ by ε-balls in ρ.

Let $\delta = \varepsilon/4$. For each $a \in X$, choose a nbd $U_a \ni a$ such that $d(f(x), f(a)) < \delta$ for all $x \in U_a$, $f \in \mathcal{F}$.

Cover X by finitely many of the U_a, say $U_{a_1}, \dots, U_{a_k}$, using compactness.

Cover Y by finitely many open sets $V_1, \dots, V_m$ of diameter $< \delta$ using compactness.

Let J be the set of all functions $\alpha : \{1, \dots, k\} \to \{1, \dots, m\}$.

Given $\alpha \in J$, if there exists a $f \in \mathcal{F}$ s.t. $f(a_i) \in V_{\alpha(i)}$ for each $i = 1, \dots, k$, choose one such f and label it f_α.

Since J is finite, so is the set of f_α. Let J' be the set of $\alpha \in J$ for which f_α is defined.

We claim that $\{B_\rho(f_\alpha, \varepsilon) : \alpha \in J'\}$ covers $\mathcal{F}$.

Let $f \in \mathcal{F}$. For each $i = 1, \dots, k$, choose $\alpha(i)$ s.t. $f(a_i) \in V_{\alpha(i)}$. Then $\alpha \in J'$. We shall show that $f \in B_\rho(f_\alpha, \varepsilon)$.

Let $x \in X$ and i such that $x \in U_i$ Then

$$d(f(x), f(a_i)) < \delta \qquad (x \in U_i)$$
$$d(f(a_i), f_\alpha(a_i)) < \delta \qquad (f(a_i), f_\alpha(a_i) \in V_{\alpha(i)})$$
$$d(f_\alpha(a_i), f_\alpha(x)) < \delta \qquad (x \in U_i)$$

$\Rightarrow d(f(x), f_\alpha(x)) < 3\delta \Rightarrow \rho(f, f_\alpha) \leq 3\delta < \varepsilon.$

completing the proof.

Let us now get to the proof of Ascoli's Theorem.
Since X is compact, the sup metric is defined on $\mathcal{C}(X, \mathbb{R}^n)$ and gives the uniform topology. Let $\mathcal{G} = \overline{\mathcal{F}}$ in $\mathcal{C}(X, \mathbb{R}^n)$

$\to$ Suppose $\mathcal{G}$ is compact.

Then $\mathcal{G}$ is totally bounded under ρ and $\bar{\rho}$, so it is equicontinuous under d by Lemma 5.10.

Compactness also implies that $\mathcal{G}$ is bounded under ρ, so pointwise bounded under d.

$\Rightarrow$ $\mathcal{G}$ is equicontinuous and pointwise bounded.

$\Rightarrow$ $\mathcal{F}$ is equicontinuous and pointwise bounded.

$\to$ Suppose $\mathcal{F}$ is equicontinuous and pointwise bounded.

- Given $x_0 \in X$, choose a nbd. U of x_0 such that

$$d(f(x), f(x_0)) < \varepsilon/3 \quad \text{for all } x \in U, f \in \mathcal{F}.$$

Given $g \in \mathcal{G}$, choose $f \in \mathcal{F}$ s.t. $\rho(f, g) < \varepsilon/3$. (because $\mathcal{G} = \overline{\mathcal{F}}$)
Then $d(g(x), g(x_0)) < \varepsilon$ for all $x \in U$.
Thus, $\mathcal{G}$ is equicontinuous. (since g, x_0 were arbitrary)

- Given a, choose M such that $\operatorname{diam} \mathcal{F}_a \leq M$.
Given $g, g' \in \mathcal{G}$, choose $f, f' \in \mathcal{F}$ such that

$$\rho(f, g) < 1 \quad \text{and} \quad \rho(f', g') < 1 \qquad (\text{because } \mathcal{G} = \overline{\mathcal{F}})$$

Then $d(g(a), g'(a)) \leq M+2 \Rightarrow \operatorname{diam} \mathcal{G}_a \leq M+2$.
Thus, $\mathcal{G}$ is pointwise bounded. (since a is arbitrary)

- We show that there is a compact subspace Y of $\mathbb{R}^n$ that contains the union of the sets $g(X) : g \in \mathcal{G}$. (so we can use Lemma 5.11)
For each $a \in X$, choose a nbd. U_a of a s.t. $d(g(x), g(a)) < 1$ for $x \in U_a$ and $g \in \mathcal{G}$. Since X is compact, we can cover it by finitely many $a_1, \dots, a_k$. Because the $\mathcal{G}_{a_i}$ are bounded, their union is also bounded. Suppose it lies in the ball of radius M in $\mathbb{R}^n$. Then for all $g \in \mathcal{G}$, $g(X)$ is contained in the ball of radius $N+1$ centered at the origin. Let Y be the closure of this ball. Y is closed and bounded, so is compact.

- Because $\mathcal{G}$ is a closed subspace of the complete space $(\mathcal{C}(X,\mathbb{R}^n), \rho)$, it is complete.
- Equicontinuity of $\mathcal{G}$, together with compactness of X, Y, implies that $\mathcal{G}$ is totally bounded by Lemma 5.11.

Therefore, $\mathcal{G}$ is compact by Theo. 5.8

Corollary (5.12) Let X be compact and d denote either the square metric or Euclidean metric on $\mathbb{R}^n$. Give $\mathcal{C}(X,\mathbb{R}^n)$ the uniform topology. Then $\mathcal{F} \subseteq \mathcal{C}(X,\mathbb{R}^n)$ if and only if it is closed, bounded under the sup metric ρ, and equicontinuous under d.

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