Munkres Solutions

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§2. Topological Spaces and Continuous Functions

2.13. Basis for a Topology

Exercise 2.13.1. Let X be a topological space and $A \subseteq X$. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subseteq A$. Show that A is open in X.

Solution

For each $x \in A$, denote by U_x an open subset of A that contains A. Then $A = \bigcup_{x \in A} U_x$. However, an arbitrary union of open sets is open and thus, so is A.

Exercise 2.13.5. Show that if $\mathcal{A}$ is a basis for a topology on X, the topology generated by $\mathcal{A}$ equals the intersection of all topologies that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Solution

Let $\mathcal{T}$ be the topology generated by $\mathcal{A}$ and $\mathcal{T}'$ be a topology that contains $\mathcal{A}$. Let $U \in \mathcal{T}$. Then $U = \bigcup_{i \in I} B_i$ for some $(B_i)_{i \in I}$ in $\mathcal{A}$. However, each B_i is also in $\mathcal{T}'$. Since an arbitrary union of open sets is open, $U \in \mathcal{T}'$ as well. Therefore, $\mathcal{T} \subseteq \mathcal{T}'$, proving the result. The solution for the case where $\mathcal{A}$ is a subbasis is very similar and so omitted.

Exercise 2.13.6. Show that the collection

$$\mathcal{B} = \{(a, b) : a < b, a \text{ and } b \text{ are rational}\}.$$

2.16. The Subspace Topology

Exercise 2.16.1. Show that if Y is a subspace of X and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Solution

The topology A inherits as a subspace of X is

$$\begin{aligned} \mathcal{T} &= \{U \cap A : U \text{ open in } X\} \\ &= \{(U \cap Y) \cap A : U \text{ open in } X\} \\ &= \{V \cap A : V \text{ open in } Y\}, \end{aligned}$$

which is the topology it inherits as a subspace of Y.

Exercise 2.16.2. If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X.

Solution

It is easily seen that $\mathcal{T}'_Y$ is finer than $\mathcal{T}_Y$. We further see that it need not be strictly finer by considering the example $X = \{a, b, c\}$, $Y = \{a, b\}$, $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$, and $\mathcal{T}'$ as the discrete topology on X.