Hausdorff Spaces

Recall that $\{x\}$ and $\{(x,y)\}$ for $x, y \in \mathbb{R}^2$ are closed in $\mathbb{R}$ and $\mathbb{R}^2$ respectively.

However, singletons need not be closed in general topological spaces.

$$X = \{a,b,c\} \quad \mathcal{T} = \{\emptyset, X\}$$

$$\text{or } \mathcal{T} = \{\emptyset, X, \{a,b\}, \{b,c\}, \{b\}\}$$

Obviously, $\{b\}$ is not closed in X.

(for either topology)

Also, recall that (in $\mathbb{R}$ or $\mathbb{R}^2$), a sequence (x_n) converges, if at all, to a unique point.

How would we extend this notion to general topological spaces? Can we extend this in a meaningful way in general?

Def. Let $(X, \mathcal{T})$ be a topological space and (x_n) a sequence in X. x_n is

Convergence said to *converge* to $x \in X$ if for all open sets $U \ni x$ in X, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

However, sequences need not have unique limits in general!

Let $X = \{a,b,c\}$ and $\mathcal{T} = \{\emptyset, X, \{a,b\}, \{b,c\}, \{b\}\}$.

Consider the constant sequence $x_n = b$.

By our above definition, x_n converges to any of a, b, c.

In what spaces are limits unique? When showing uniqueness of limits in metric spaces, we only really used the fact that there is a separating open set.

Def. Hausdorff Space

A topological space X is called a Hausdorff space if for any $x_1, x_2 \in X$ $(x_1 \neq x_2)$, there are open sets $U_1 \ni x_1$ and $U_2 \ni x_2$ such that $U_1 \cap U_2 = \emptyset$.

Observe that metric spaces are Hausdorff.

Theo. (2.1) Every finite point set in a Hausdorff space X is closed.

Proof. It suffices to show that singletons are closed (closed sets are closed under finite unions). Let $x_0 \in X$. If $x \in X \setminus \{x_0\}$, there are disjoint neighbourhoods U, V of x_0, x. Since V does not intersect $\{x_0\}$, x cannot belong to the closure of $\{x_0\}$. As a result, using Theo 1.15, the closure of $\{x_0\}$ is itself and it is closed. $\square$

Is the converse true? No, consider $\mathbb{R}$ under the cofinite topology.

The condition that finite point sets are closed is called the T_1 axiom.

Theo. (2.2) Let X be a topological space satisfying the T_1 axiom and $A \subseteq X$. Then x is a limit point of A iff every neighbourhood of x contains infinitely many points of A.

Proof. The backward direction is trivial.

For the forward direction, suppose $x \in A'$ and some neighbourhood U of x intersects A at finitely many points. Then

$$U \cap (X \setminus \underbrace{(U \cap (A \setminus \{x\}))}_{\text{finite so closed and complement is open.}})$$

is a neighbourhood of x that does not intersect A, thus giving a contradiction and proving the claim. $\square$

Theo. (2.3) If X is a Hausdorff space, then a sequence of points of X converges, if at all, to a unique point in X.

Proof. Let x_n be a sequence in X that converges to x. Let $y \neq x$. Let U, V be disjoint neighbourhoods of x, y. Since U contains all but finitely many x_n, V contains a finite number of (x_n). Then y is not a limit point of $\{x_n\}$ (as a set) by Theo 2.2, so (x_n) obviously cannot converge to y.