Connectedness and Compactness

Lecture 18 - 05/03/2021 Connectedness

Def. Let X be a topological space. A separation of X is a pair U, V of disjoint non-empty open subsets of X whose union is X. X is said to be connected if it does not have a separation.

Prop. (3.1) A space X is connected iff the only subsets of X that are both open and closed ("clopen") are $\emptyset$ and X.

If U, V is a separation, U is clopen.

If $A \neq \emptyset$ is clopen, $A, X \setminus A$ is a separation. $\square$

Prop (3.2) If X and Y are homeomorphic, X is connected iff Y is connected.

Lemma (3.3) If Y is a subspace of X, a separation of Y is pair of disjoint non-empty sets A, B whose union is Y, neither of which contains a limit point of the other The space Y is connected iff there is no separation of Y.

This is easily shown since the sets involved in a separation are clopen (in Y). $\bar{A} \cap Y = A$

$$A \cap B = \emptyset \rightarrow \bar{A} \cap Y \cap B = \emptyset$$
$$\rightarrow \bar{A} \cap B = \emptyset.$$

The other direction is similarly straightforward. $\square$

For example, any topological space with the indiscrete topology is connected.

→ Show that $\mathbb{Q}$ is not connected.

Lemma (3.4). If the sets C, D form a separation of X and Y is a connected subspace of Y, $Y \subseteq C$ or $Y \subseteq D$.

If not, we can write $Y = (Y \cap C) \cup (Y \cap D)$

open in Y

Lemma (3.5). A union of connected spaces is connected if their intersection is non-empty.

Proof. Let (A_α) be a family of connected subspaces of X and $p \in \bigcap_\alpha A_\alpha$.
We claim that $Y = \bigcup_\alpha A_\alpha$ is connected.
Suppose C, D is a separation of Y and wlog that $p \in C$.
Since A_α is connected and $p \in C$, $A_\alpha \subseteq C$.
Therefore, $Y = \bigcup_\alpha A_\alpha \subseteq C$, contradicting the non-emptiness of D. $\square$

Theo. (3.6) Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$, B is also connected.

(We can add any of the limit points without destroying connectedness)

Proof. Suppose C, D is a separation of B. Assume wlog that $A \subseteq C$. Then $B \subseteq \overline{A} \subseteq \overline{C}$. But $\overline{C} \cap D = \emptyset$, yielding a contradiction and proving the claim. $\square$

Theo. (3.7) The image of a connected space under a continuous map is connected.

Proof. Let $f: X \to Y$ be continuous where X is connected and f is surjective. Suppose C, D is a separation of Y. Since f is continuous, $f^{-1}(C)$ and $f^{-1}(D)$ are also open and they form a separation of X, resulting in a contradiction. $\square$

Theo. (3.8) A finite Cartesian product of connected spaces is connected.
(under either the box or product topo., they are equal)

Proof It suffices to show that if X and Y are connected $X \times Y$ is connected. (Use the fact that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic to $(X_1 \times \cdots \times X_{n-1}) \times X_n$)
Fix $x \times y \in X \times Y$. $X \times \{y\}$ is connected (it is homeomorphic to X) and so is $\{x\} \times Y$. The result follows on using Theo 3.5. $\square$

Show that $\mathbb{R}^\omega$ under the box topology is disconnected.

Hint: Let $A = \{(a_n) : (a_n) \text{ is bounded}\}$ and
$B = \{(b_n) : (b_n) \text{ is unbounded}\}$.

Show that $\mathbb{R}^\omega$ under the product topology is connected.

Hint: Show that $\mathbb{R}^\infty$, the set of sequences eventually 0, is connected and that $\mathbb{R}^\omega = \overline{\mathbb{R}^\infty}$.

$\mathbb{R}^\infty = \bigcup_{n \in \mathbb{N}} \tilde{\mathbb{R}}^n$ ↳ 0 after first n co-ordinates.

Theo. (3.9) An arbitrary product of connected spaces is connected in the product topology.
(the proof is nearly identical to that for $\mathbb{R}^\omega$ above)

Def. A simply ordered set L having more than one element is called a linear continuum if

- L has the least upper bound property.
- if $x < y$ in L, there exists z in L such that $x < z < y$.

Clearly, $\mathbb{R}$ is a linear continuum.

Theo. (3.10) If L is a linear continuum, then L, intervals in L, and rays in L, are connected.

Theo. (3.11) [Intermediate Value Theorem]
Let $f: X \to Y$ be continuous, where X is a connected space and Y is an ordered set under the ordered topology. If $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, there exists $c \in X$ such that $f(c) = r$.

Proof. Suppose otherwise. Then $f(X) \cap (-\infty, r)$ and $f(X) \cap (r, \infty)$ form a separation of $f(X)$. However, the image under f of X is connected, resulting in a contradiction. $\square$

Lecture 19 - 10/03/21 Path Connectedness

Def. Given points x, y of the space X, a path from x to y is a continuous function $f: [a,b] \to X$ such that $f(a) = x$ and $f(b) = y$ (for some closed interval $[a,b] \subseteq \mathbb{R}$).
A space X is path-connected if there is a path between any two points in X.

Theo. (3.12) Any path-connected space is connected.

Proof. Suppose otherwise. Let X be path-connected and $f: [a,b] \to X$ be a path in X. Let $X = A \cup B$ be a separation of X.
Since $[a,b]$ is connected and f is continuous, $f([a,b]) \subseteq A$ or $f([a,b]) \subseteq B$, contradicting path-connectedness (across A, B). $\square$

The converse is not true.
Consider

$$S = \{ x \times \sin\left(\tfrac{1}{x}\right) : 0 < x \leq 1 \},$$

known as the topologist's sine curve.
Then $\overline{S} = S \cup (\{0\} \times [-1,1])$. We claim that $\overline{S}$ is not path-connected.
Let continuous $f: [a,c] \to \overline{S}$ beginning at the origin and ending at some point in S. The set

$$\{ t \in [a,c] : f(t) \in \{0\} \times [-1,1] \}$$

is closed (due to continuity), so it has a largest element b.
Then the restriction $f: [b,c] \to \overline{S}$ is a path such that $f(b) \in \{0\} \times [-1,1]$ and $f((a,c]) \subseteq S$.

Wlog, let $[b,c]$ be $[0,1]$ and $f(t) = (x(t), y(t))$. Then
$x(0) = 0$ and for $t>0$, $x(t)>0$ and $y(t) = \sin(1/x(t))$.

For each n, choose a $0<u< x(1/n)$ such that $y(u) = (-1)^n$. Using the IVT (Theo 3.11), there is a $0<t_n<1/n$ such that $x(t_n)=u$. However, then, $t_n \to 0$ but $y(t_n) = (-1)^n$ does not converge, contradicting the continuity of y and proving the claim.

Show that S (and thus $\bar{S}$) is connected, disproving the converse of Theo 3.12

Def. Given X, define an equivalence relation as $x \sim y$ if there exists a connected subspace of X containing both x and y. The resulting equivalence classes are called the **components** or **connected components** of X.

(Check that it is an equivalence relation)

Theo. (3.13) The components of X are connected disjoint subspaces of X whose union is X such that any non-empty connected subspace of X intersects exactly one of them.

Proof left as exercise.

To show that a component C is connected, fix $x \in C$ and for each $y \in C$, let $C_y \subseteq C$ be a connected subspace containing C_y.
by the second part

Then $x \in \bigcap C_y \neq \emptyset$, so $\bigcup C_y = C$ is connected. $\square$

Similar to connected components, we can define the **path components** of X.

($x \sim y$ if there is a path from x to y)

(transitivity can be shown using the pasting lemma)

Theo. (3.14) The path components of X are path-connected disjoint subspaces of X whose union is X such that any non-empty path-connected subspace of X intersects exactly one of them.

Corollary. (3.15) Any connected component of X is closed.

(Use the fact that the closure of a connected space is closed)

It follows that if there are finitely many components, each component is also open.

It need not be true that path-connected components are closed, however. Consider the topologist's sine curve $\bar{S}$. Then S is open in $\bar{S}$ (and not closed) and $\bar{S} \setminus S$ is closed (and not open).

Def A space X is said to be locally connected at $x \in X$ if for every neighbourhood U of x, there is a connected neighbourhood $V \subseteq U$ of x. X is locally connected if it is locally connected at any point of X.
We similarly define local path-connectedness.

Theo. (3.16) A space X is locally connected iff for any open $U \subseteq X$, each component of U is open in X.

Proof. Let X be locally connected, U be open in X, and C be a component of U. Let $x \in C$. There is then a neighbourhood $V \subseteq U$ of x that is connected. It follows that $V \subseteq C$ and therefore, C is open.
On the other hand suppose that the components of open sets in X are open. Let $x \in X$ and U a neighbourhood of x. We can take the component of U containing x, completing the proof $\square$

Theo. (3.17) A space X is locally path-connected iff for any open $U \subseteq X$, each path component of U is open.

The proof is nearly identical to the previous one.

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Theo. (3.18) If X is a topological space, each path component of X lies in a component of X. Moreover, if X is locally path-connected, the components and path components are the same.

Proof. The first part is direct since any path component is connected.
Let C be a component, $x \in C$, and $P \ni x$ be a path component.
Let X be locally path-connected. Suppose $P \subsetneq C$. Let Q be the union of all path components other than P that intersect C.
Then $C = P \cup Q$.
Because X is locally path connected, each path component of X is open in X. In particular, P and Q are open. This contradicts the connectedness of C, proving the claim.
↳ they form a separation of C

Def. A collection $\mathcal{A}$ of subsets of X is said to be a covering if $X = \bigcup_{A \in \mathcal{A}} A$. An open covering is a covering where every subset is open.
X is said to be compact if any open cover contains a finite subcover.

If Y is a subspace of X and $\mathcal{A}$ is a collection of subsets of X, $\mathcal{A}$ is said to cover Y if $Y \subseteq \bigcup_{A \in \mathcal{A}} A$.

Theo. Any closed subspace of a compact space is compact.

Hint Consider the open cover $\mathcal{A} \cup \{X \setminus Y\}$

Theo. Every compact subspace of a Hausdorff space is closed.

Proof. Let X be Hausdorff and Y a compact subspace. Let $x_0 \in X \setminus Y$.
For each $y \in Y$, choose neighbourhoods U_y of x_0 and V_y of y such that $U_y \cap V_y = \emptyset$
Since Y is compact, there exist $y_1, \cdots, y_n$ such that

$$Y \subseteq \bigcup_{1 \leq i \leq n} V_{y_i} = V.$$

But $Y \cap U \subseteq V \cap U = \emptyset$, where

$$U = \bigcap_{1 \leq i \leq n} U_{y_i} \text{ is a neighbourhood of } x_0.$$

Therefore, Y is closed.