Graph Theory

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§1. Introduction

Definition 1.1.

Theorem 1.1. A graph is bipartite iff it does not have an odd cycle.

Theorem 1.2. A graph is a forest iff for every pair $\{x, y\}$ of distinct vertices, it contains at most one $x - y$ path.

Theorem 1.3. The following are equivalent for a graph G:

(i) G is a tree.

(ii) G is a minimal connected graph. (G is connected and for any $xy \in E(G)$, $G - xy$ is disconnected, every edge is a bridge)

(iii) G is a maximum acyclic graph. (G is acyclic and for any $xy \in \binom{V(G)}{2} \setminus E(G)$, $G + xy$ has a cycle)

Corollary 1.4. Every connected graph contains a spanning tree.

Corollary 1.5. A tree of order n has size $n - 1$; a forest of order n with k components has size $n - k$.

Corollary 1.6. A tree of order 2 contains at least 2 vertices of degree 1.

Theorem 1.7. Given a graph G and a function $f : E(G) \to \mathbb{R}$, an economical spanning tree is a spanning tree H of G such that $\sum_{e \in H} f(e)$ is minimal. The following algorithms produce an economical spanning tree:

(i) Given G and $f : E(G) \to \mathbb{R}$, choose an edge α such that $f(\alpha)$ is minimal. Each subsequent edge is chosen from the cheapest remaining edges of G ensuring that we never form any cycles.

(ii) At each step, delete a costliest edge that does not destroy the connectedness of the graph.

(iii) Pick a vertex x_1 of G. Having found vertices $x_1, \ldots, x_k$ an an edge $x_i x_j$, $i < j$, for each vertex j with $j \leq k$, select a cheapest edge of the form $x_i x$, say $x_i x_{k+1}$, where $1 \leq i \leq k$ and $x_{k+1} \notin \{x_1, \ldots, x_k\}$. The process terminates after we have selected $n - 1$ edges.

(iv) This only works if all the edge costs are distinct. First, for each vertex, select the cheapest edge. After this, repeatedly select a cheapest edge between two distinct connected components until the graph becomes connected.

The second algorithm is destructive in the sense that it removes edges from the original graph whereas the others are constructive, they create a tree by adding edges from the original graph.

Theorem 1.8. For $n \geq 3$, the complete graph K^n is decomposable into edge-disjoint Hamiltonian cycles iff n is odd. For $n \geq 2$, K^n is decomposable into edge-disjoint Hamiltonian paths iff n is even.

Theorem 1.9. A non-trivial connected graph has an Euler circuit if and only if each vertex has even degree. A connected graph has an Euler trail from vertex x to $y \neq x$ iff x and y are the only vertices of odd degree.

Definition 1.2. A graph G is said to be *randomly Eulerian* from a vertex v if every trail of G with initial vertex v can be extended to an Eulerian $v - v$ circuit of G.

This means that it is impossible to "get stuck" when trying to make an Eulerian circuit from v.

1.1. Planar Graphs

Definition 1.3. A graph is a *planar graph* if it can be drawn in the plane in such a way that no two edges intersect.

Theorem 1.10 (Euler's Polyhedron Theorem)**.** If a connected planar graph G has n vertices, m edges, and f faces, then $n - m + f = 2$.

Definition 1.4. The *girth* of a graph is defined to be the number of edges in the shortest cycle.

Theorem 1.11. Denote by f_i the number of faces with exactly i edges on their boundaries. Then if G has no bridge,

$$\sum_i i f_i = 2m.$$

This is because every edge is in the boundary of two faces.

Theorem 1.12. A planar graph of order $n \geq 3$ has at most $3n - 6$ edges. Furthermore, a planar graph of girth at least g, $3 \leq g$, has size at most

$$\max \left\{ \frac{g}{g-2}(n-2), n-1 \right\}.$$

1.2. Exercises

Exercise 1.1.

(i) Show that every graph contains two vertices of equal degree.

(ii) Determine all graphs with one pair of vertices of equal degree.

> **Solution 1.1**
>
> (i) If every vertex has a distinct degree, then the degree sequence of the graph is $0 \leq 1 \leq \cdots \leq n-1$. However, the fact that the vertex of degree $n-1$ is connected to every other vertex, the vertex of degree 0 in particular, yields a contradiction.
>
> (ii) Note that we can infer from the solution to part (i) that if there is only a single pair of vertices with the same degree, then there must be a vertex of degree 0 or $n-1$ (but not both).
> Also, note that there cannot be two vertices of equal degree 0 or $n-1$. This is because the former would imply that $\Delta(G)$ is at most $n-3$ and the latter would imply that $\delta(G)$ is at least 2 and we know that there exist vertices of degree 1 and $n-2$. This argument would fail when $n = 1$ or 2, but in these cases we only have three permissible graphs anyway, namely E^1, E^2, and K^2.

Exercise 1.2. Prove that the complement of a disconnected graph is connected.

> **Solution 1.2**
>
> Let x, y be two vertices. We must show that there is a walk from x to y. Indeed, if x and y are in different connected components of G, then xy is an edge in the complement. Otherwise, we can choose some z that is in a different connected component and then we have the walk x, z, y (such a z exists since there are at least 2 connected components).

Exercise 1.3. Show that in a graph G there exists a set of cycles such that each edge of G belongs to exactly one of these cycles iff every vertex has even degree.

Solution 1.3

It is easy to show the forward implication.
For the backward implication, We can assume that the graph is non-trivial and connected since otherwise, we can just solve each of the connected components separately. We can then consider an Euler circuit (which exists since every vertex has even degree). "Splitting" this cycle into different cycles whenever a vertex is repeated completes the proof.

Exercise 1.4. Show that in an infinite graph G with countably many edges, there exists a set of cycles such that each edge of G belongs to exactly one of these cycles iff for every $X \subseteq V(G)$, the set of edges joining X to $V(G) \setminus X$ is even or infinite.

Solution 1.4

Exercise 1.5. Show that $d_1 \leq d_2 \leq \cdots \leq d_n$ is the degree sequence of a tree iff $d_1 \geq 1$ and $\sum_{i=1}^{n} d_i = 2n - 2$.

Solution 1.5

If it is the degree sequence of a tree G, then since the graph is connected, we have $d_1 \geq 1$ and since $e(G) = n-1$, the sum of the degrees is $2e(G) = 2n - 2$.
For the converse, we must show that a tree with the given degree sequence exists. For Note that $\delta(G) \leq \frac{2n-2}{n}$, and thus $d_1 = 1$. Also note that $\Delta(G) \geq 2$ for $n > 2$. We can then inductively prove that a tree with the given sequence exists. For $n = 2$, the graph is just K^2. If such a graph exists for at most $n - 1$ nodes for some $n > 2$, then for a degree sequence of the form $1 \leq d_2 \leq \cdots \leq d_n$, we can construct a graph with degree sequence $d_2, \ldots, d_n - 1$ (we omit the $\leq$ because $d_n - 1$ may be less than d_{n-1}). We then add a new vertex that has a single edge to the vertex with degree $d_n - 1$. This is well-defined since $d_n - 1$ is positive ($d_n \geq 2$ for $n > 2$).

Exercise 1.6. Show that every integer sequence $d_1 \leq d_2 \leq ... \leq d_n$ with $d_1 \geq 1$ and $\sum_i d_i = 2n - 2k$, $k \geq 1$,is the degree sequence of a forest with k components.

Solution 1.6

Exercise 1.7. Characterize the degree sequence of forests!

Solution 1.7

Exercise 1.8. Prove that a regular bipartite graph of degree at least 2 does not have a bridge.

Solution 1.8

Suppose that such a graph G has a bridge xy and degree $k \geq 2$. Consider the graph $G - xy$ and let the connected component of this graph that contains x (and does not contain y) be H. Obviously, H must be bipartite as well, let it have bipartitions X and Y. Then, we have that $\sum_{v \in X} d(v) = k|X| - 1$ and $\sum_{v \in Y} d(v) = k|Y|$. However, these two must be equal and we thus obtain a contradiction because k cannot divide the former quantity (but it divides the latter).

Exercise 1.9. Let G be a graph of order n. Prove the equivalence of the following.

(i) G is a tree.

(ii) G is connected and has $n - 1$ edges.

(iii) G is acyclic and has $n - 1$ edges.

(iv) $G = K^n$ for $n = 1, 2$, and if $n \geq 3$, then $G \neq K^n$ and the addition of an edge to G produces exactly one new cycle.

Solution 1.9

(i) $\iff$ (iv).

By theorem 1.3, G is a tree if and only if it is a maximum acyclic graph. This is equivalent to (iv).

(i) $\implies$ (ii) and (i) $\implies$ (iii).

This is obvious. If G is a tree, it is connected, acyclic, and has $n-1$ edges by corollary 1.5.

(iii) $\implies$ (ii) and (iii) $\implies$ (i).

Since G is acyclic, it is a forest. It then has $n-k$ edges where k is the number of connected components. However, this implies that it has only 1 connected component, that is, it is connected.
Also, connectedness and acyclicity implies (i).

Exercise 1.10. Let $\mathscr{A} = \{A_1, \ldots, A_n\}$ be a family of $n(\geq 1)$ distinct subset of a set X with n elements. Define a graph G with vertex set $\mathscr{A}$ in which A_iA_j is an edge iff there exists $x \in X$ such that $A_i \triangle A_j = \{x\}$. Label the edge A_iA_j with x. For $H \subseteq G$, let $\mathrm{Lab}(H)$ be the set of labels used for edges in H. Prove that there is a forest $F \subseteq G$ such that $\mathrm{Lab}(F) = \mathrm{Lab}(G)$.

Solution 1.10

We shall show that if $B_1B_2\cdots B_l$ is a cycle, then two edges in the cycle have the same label. Repeatedly removing one of these edges from the graph until there are no cycles remaining gives a suitable H. Assume without loss of generality that $|B_2| = |B_1| + 1$ and $\{x\} = B_2 \setminus B_1$. Now, note that

$$B_2 \setminus \left(B_1 \triangle B_l \cup \bigcup_{i=2}^{l-1} B_i \triangle Bi + 1 \right) \subseteq B_1.$$

However, as $x \notin B_1$ and $x \in B_2$, this gives that x is in the remaining part of the above expression, that is, there is another edge labelled x.
Note that this not only proves the result, but also proves that every cycle in G is of even length and labels within the cycle appear in pairs.

Exercise 1.11. (10 continued) Deduce from the result in the previous exercise that there is an element $x \in X$ such that the sets $A_1 \setminus \{x\}, A_2 \setminus \{x\}, \ldots, A_n \setminus \{x\}$ are all distinct. Show that this need not hold for any n if $|\mathscr{A}| = n+1$.

Solution 1.11

To see this, note that there is an edge between sets A and B if and only if there is some x such that $A \setminus \{x\} = B \setminus \{x\}$. However, there are n elements in X and at most $n-1$ edges. The result follows.
To show that this need not be true if $|\mathscr{A}| = n+1$, take the counter-example with $X = \{1, 2, \ldots, n\}$ and

$$\mathscr{A} = \{\varnothing, \{1\}, \{1,2\}, \ldots, \{1,2,\ldots,n\}\}$$

Exercise 1.12. A *tournament* is a complete oriented graph, that is, a directed graph where for any two distinct vertices x and y, either there is an edge from x to y or there is an edge from y to x, but not both. Prove that every tournament contains a (directed) Hamiltonian path.

Solution 1.12

Exercise 1.13. Let G be a connected graph of order $1 \leq k \leq n$. Show that G contains a connected subgraph of order k.

Solution 1.13

This is easily shown via induction. There is clearly a connected subgraph of order 1. If there is a connected subgraph H of order $k-1$ for some $2 \leq k \leq n$, then there must be an edge from H to some vertex in $G \setminus H$ (if there was not, connectivity would be contradicted). Add this vertex (and all its edges incident on vertices in H) to get a connected subgraph of order k.

Exercise 1.14. Prove that the radius and diameter of a graph satisfy the inequalities

$$\operatorname{rad} G \leq \operatorname{diam} G \leq 2 \operatorname{rad} G,$$

and both inequalities are the best possible.

Solution 1.14

We have

$$\operatorname{rad} G = \min_x \max_y d(x,y) \leq \max_x \max_y d(x,y) = \operatorname{diam} G.$$

On the other hand, let there be vertices x, y such that $d(x,y) = \operatorname{diam} G$ and vertex z such that $\max_w d(z,w) = \operatorname{rad} G$. Then,

$$2 \operatorname{rad} G \geq d(z,x) + d(z,y) \geq d(x,y) = \operatorname{diam} G.$$

Equality is attained in the former case for the graph K^n and in the latter case for the path graph of order n P^n.

Exercise 1.15. Given $d \geq 1$, determine

$$\max_{\operatorname{diam} G = d} \min\{\operatorname{diam} T : T \text{ is a spanning tree of } G\}$$

Solution 1.15

§2. Flow, Connectivity, and Matchings

2.1. Flow

Definition 2.1 (Flow)**.** Let G be a directed graph with vertex set V and edge set E. A *flow* f on G from a vertex s (the *source*) to a vertex t (the *sink*) is a non-negative function defined on E that satisfies Kirchhoff's current law: the total current flowing into each intermediate vertex (a vertex that is not the source or sink) must be equal to the total current leaving the intermediate vertex.

We usually denote $f(\overrightarrow{xy})$ by $f(x,y)$ for a flow f.
For any vertex x, define

$$\Gamma^+(x) = \{y \in V : \overrightarrow{xy} \in E\} \text{ and } \Gamma^-(x) = \{y \in V : \overrightarrow{yx} \in E\}$$

Kirchhoff's current law just says that a flow f from s to t satisfies

$$\sum_{y \in \Gamma^+(x)} f(x,y) = \sum_{y \in \Gamma^-(x)} f(y,x) \text{ for any } x \in V \setminus \{s,t\}.$$

The former quantity represents the flow that exits the vertex and the latter is the flow that enters it.
Now, since the sum of flows (with direction) across all edges is 0, we also have that

$$\sum_{y \in \Gamma^+(s)} f(s,y) - \sum_{y \in \Gamma^-(s)} f(y,s) = \sum_{y \in \Gamma^-(t)} f(y,t) - \sum_{y \in \Gamma^+(s)} f(t,y)$$

This makes sense as well, since the source is the only place where flow can "originate" and the sink is the only place it can "terminate". This common value is known as the *value* of f or the *amount of flow* from s to t and is denoted $v(f)$.

Obviously, there are several real problems that arise when we think of such a problem - maybe we have to transport as much water as we can from a faucet to a sink via a system of pipes. We mainly aim to *maximize the flow* under some constraints.

2.2. Edge Capacities – Max-Flow and Min-Cut

One constraint that immediately comes to mind is that each edge has a *capacity* that restricts the current through the edge (the pipe can only carry so much water). That is, there is an associated non-negative number $c(x,y)$ associated with each edge $\overrightarrow{xy}$ such that any flow must satisfy $f(x,y) \leq c(x,y)$ for all edges.

We now introduce some additional notation to ease the process.

Definition 2.2. Given two subsets X and Y of G, we write

$$E(X,Y) = \{\overrightarrow{xy} \in E : x \in X, y \in Y\}.$$

Whenever $g : E \to \mathbb{R}$ is a function, we write

$$g(X,Y) = \sum_{\overrightarrow{xy} \in E(X,Y)} g(x,y).$$

Definition 2.3 ((Edge-) Cut)**.** If S is a subset of V such that $s \in S$ but $t \notin S$, then $E(S,\overline{S})$ is called a *cut* separating s from t (where $\overline{S} = V \setminus S$).

The capacity of a cut $E(S, \overline{S})$ is $c(S, \overline{S})$.
Note that if we delete (the edges of) a cut from a graph, then $v(f) = 0$ on the remaining graph is 0. Indeed, there will be no path from s to t in this case.
Also, if there is some set of edges after whose deletion $v(f) = 0$, then this set of edges must contain a cut as well.
It is also obvious that the value of any flow is at most the capacity of any cut. In particular, the maximum flow value is at most the minimum capacity of a cut.

We are justified in saying "maximum" and "minimum" here since the flow is $\leq \sum_{\overrightarrow{xy} \in E} c(x, y) < \infty$ so the set of flow values has a supremum, and there are only finitely many cuts (so they have a minimum). We encourage the reader to show that the supremum of the set of flow values is indeed attained (consider a sequence of flows such that their value converges to the supremum v, then the flow of each edge must converge as well).

Theorem 2.1 (Ford and Fulkerson's Max-Flow Min-Cut Theorem)**.** Given a (directed) graph G and a flow f from s to t, the maximum flow value from s to t is equal to the minimum of the capacities of cuts separating s from t.

Proof. Let v be the maximum flow value and f be the corresponding flow. Since we already know that the capacity of any cut is at least v, we only need to prove that a cut with such a capacity exists. Define a subset $S \subseteq V$ as follows. To begin, $s \in S$. If $x \in S$ and

$$c(x, y) > f(x, y) \text{ or } f(y, x) > 0$$

then let $y \in S$. Note that if neither of the above occurs, then $f(x, y) = c(x, y)$ and $f(y, x) = 0$ for every $x \in S$. We claim that $E(S, \overline{S})$ has capacity v.

First of all, we must check that $t \notin S$. If $t \in S$, we can find vertices $s = x_0, x_1, \ldots, x_l = t$ such that

$$\varepsilon_i = \max\{c(x_i, x_{i+1}) - f(x_i, x_{i+1}), f(x_{i+1}, x_i)\} > 0$$

for every $0 \leq i < l$. Put $\varepsilon = \min_i \varepsilon_i$. Then we can construct a flow f^* by increasing the flow in $\overrightarrow{x_i x_{i+1}}$ by ε_i if $\varepsilon_i > f(x_{i+1}, x_i)$ and decreasing the flow in $\overrightarrow{x_{i+1} x_i}$ by ε_i otherwise. But if this is the case, $v(f^*) = v(f) + \varepsilon$, which contradicts the maximality of f.

Next, we must check that the capacity of this cut is indeed equal to v. We have

$$\begin{aligned}
v(f) &= \sum_{x \in S, y \in \overline{S}} f(x, y) - f(y, x) \\
&= \sum_{x \in S, y \in \overline{S}} f(x, y) && \text{(by the construction, } f(y, x) = 0 \text{ for each such } x, y) \\
&= \sum_{x \in S, y \in \overline{S}} c(x, y) = c(S, \overline{S}) && \text{(by the construction, } f(x, y) = c(x, y) \text{ for each such } x, y)
\end{aligned}$$

Therefore, $v(f) = c(S, \overline{S})$ as desired. ■

In the above construction, we essentially just choose those edges in the cut that are really constraining the flow from becoming any larger. All edges in the cut are such that $c(x, y) = f(x, y)$, that is, they are at maximum capacity. We add the $f(y, x) = 0$ condition to ensure that there is no "backflow" occurring. In case we are unable to obtain such a cut, then we are able to increase the flow value along a certain trail from the source to the sink.
Note that this proof does not really depend on the individual capacities being finite, it merely requires that the maximum flow is finite.

This proof also leads to a surprisingly efficient *algorithm* to find the maximal flow value if the capacity function takes only integer values.

Begin with a flow that is 0 everywhere - $f_0(x, y) = 0$ for all edges $\overrightarrow{xy}$. We construct an increasing (in value) sequence of flows that terminates at the maximal flow. Given f_i, we must find the set S belonging to f_i (where S is that mentioned in the above proof). If $t \notin S$, then f_i is a maximal flow so we terminate. Otherwise, we can "augment" it to a flow f_{i+1} by increasing the flow along a (single) path from s to t.
Since each $v(f_i)$ is an integer, we have $v(f_{i+1}) \geq v(f_i) + 1$ and this sequence must terminate in at most v steps, where v is the maximum flow value.

It is possible to find the set S for a given flow in $\mathcal{O}(|E|)$ time, so we can find a maximal flow in $\mathcal{O}(v|E|)$ time, where v is the maximum flow value.

Corollary 2.2 (Integrality Theorem)**.** If the capacity function is integral, then there is an maximal flow that is integral.

It is easy to see that this is the case from the above algorithm since the flow is integral at every step.

Let us now consider a seemingly more complicated situation of flow - one where there can be *multiple* sources and sinks. The definition of a cut varies slightly in this case, S must contain *every* source and *no* sink.
Perhaps surprisingly, this situation is no more complicated than the one we have considered so far! We can just add a single new source with an edge of infinite capacity to every source and a sink with an edge of infinite capacity from every sink. We can then extend the problem with multiple sources and sinks to one with just a single such source and sink and easily solving it as we already have.

Theorem 2.3. The maximum flow value from a set of sources to a set of sinks is equal to the minimum of the capacity of cuts separating the sources from the sinks.

Let us not consider another different problem wherein we are given a function $c : V \setminus \{s, t\} \to \mathbb{R}^+$ such that any flow from s to t must satisfy

$$\sum_{y \in \Gamma^+(x)} f(x, y) = \sum_{y \in \Gamma^-(x)} f(y, x) \leq c(x) \text{ for every } x \in V \setminus \{s, t\}$$

That is, we are given *vertex* capacities so there is a limit to the amount of water that can flow through each vertex.

However, this can easily be reduced to the edge capacity problem as well. Just replace each vertex $x \in V \setminus \{s, t\}$ with two vertices x^- and x^+ such that all incoming edges to x enter x^-, all outgoing edges exit x^+, and there is an edge from x^- to x^+ of capacity $c(x)$.
This problem is easily seen to be equivalent to the vertex capacity problem, so we can now solve it using the Max-Flow Min-Cut Theorem.

Definition 2.4 ((Vertex-) Cut)**.** A *cut* is a subset of $V \setminus \{s, t\}$ such that no positive-valued flow from s to t can be defined on $G \setminus S$.

Formulating the Max-Flow Min-Cut Theorem for vertex capacities in terms of this, we get the following theorem

Theorem 2.4. Let G be a (directed) graph with capacity bounds on the vertices other than the source s and sink t. Then the minimum capacity of a vertex-cut is the maximum of the flow value from s to t.

We can then combine theorem 2.1, theorem 2.3, and theorem 2.4 to get

Theorem 2.5 (Generalized Max-Flow Min-Cut Theorem)**.**

2.3. Menger's Theorem

Let us now attempt to generalize the notion of connectedness of a graph slightly more.

Definition 2.5. Let G be a graph. If G is connected and $G \setminus W$ is disconnected, where W is either a set of vertices or edges, then W is said to *separate* G. If in $G \setminus W$, two vertices s and t are in different components, then W is said to *separate s from t*.
A graph is defined to be *k-connected* ($k \geq 2$) if it has at least $k+1$ vertices and no set of $k-1$ vertices separates it.
A graph is defined to be *k-edge-connected* ($k \geq 2$) if it has at least 2 vertices and no set of $k-1$ edges separates it.
The *connectivity of a graph G* is $\kappa(G) = \max_k\{G \text{ is } k\text{-connected}\}$.
The *edge-connectivity of a graph G* is $\lambda(G) = \max_k\{G \text{ is } k\text{-edge-connected}\}$.

If a graph G is disconnected, we say that $\kappa(G) = \lambda(G) = 0$.
A few points to note are:

- A graph with $k+1$ vertices is k-connected iff it is isomorphic to K^{k+1}.
- The following are equivalent for any graph G with at least 2 vertices:
  - G is connected,
  - G is 1-connected, and
  - G is 1-edge-connected.
- A graph is 2-connected iff it is connected, has at least 3 vertices, and has no cutvertex (Recall that a cutvertex is a vertex whose deletion increases the number of components).
- A graph is 2-edge-connected iff it is connected, has at least 2 vertices, and contains no bridge.
- The connectivity of a graph is the size of a smallest set of vertices that needs to be deleted to disconnect the graph. Edge connectivity is analogous.

We have the following obvious inequalities for any non-trivial graph G, any vertex $x \in V$, and any edge $xy \in E$:

$$\kappa(G) - 1 \leq \kappa(G - x) \text{ and } \lambda(G) - 1 \leq \lambda(G - xy) \leq \lambda(G)$$

Theorem 2.6 (Whitney's Theorem)**.** For any non-trivial graph G,

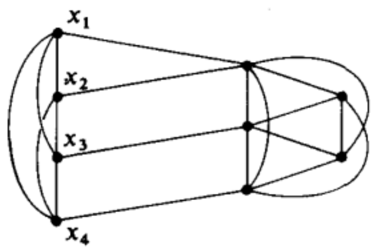
$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Proof. The second part of this inequality is clear since deleting all the edges incident on a certain vertex (a vertex of degree $\delta(G)$ in particular) disconnects the graph.

For the first part, let $\lambda(G) = k$ and $S = \{x_1y_1, \ldots, x_ky_k\}$ be a set of edges whose deletion disconnects G. If there is some vertex v such that v is not incident on any of the edges of S and C is the component of $G - S$ that contains v, then the vertices of C incident on edges of S (of which there are at most k) separates v from $G \setminus C$, so $\kappa(G) \leq k$.
Alternatively, if every vertex is incident on some edge of S, let v be some vertex. Every neighbour w of v such that $vw \notin S$ must be incident on a distinct edge of S (if there were some neighbours w_1, w_2 of v such that $w_1w_2 \in S$, then $S \setminus \{w_1w_2\}$ must be disconnected as well, which contradicts the minimality of k. Therefore, the degree of v is at most k. However, we also have $\delta(G) \geq k$, so the degree of every vertex (v in particular) is *exactly* k. Then, deleting the k vertices adjacent to v separates v from the rest of the graph so $\kappa(G) \leq k$.
The only case where this is not valid is when G has $k+1$ vertices[1], but in this case, G must be the complete graph on $k+1$ vertices(!), where we trivially have $\kappa(G) = \lambda(G) = |G| - 1$. ■

[1] so when we delete the neighbours of v, we delete the rest of the graph

A common (incorrect) "proof" of the above theorem just says that one can delete an arbitrary end-point of each edge to disconnect the graph, but this is not true! When we delete some vertices of the graph, we only care about connectivity among the remaining vertices! If it were indeed true, then if $\{x_1y_1, \ldots, x_ky_k\}$ was a minimal set of edges that separates the graph, then $G \backslash \{x_1, \ldots, x_k\}$ would always be disconnected. However, the following counter-example disproves the claim.



We have already seen that it is quite useful to partition a graph into maximal connected subgraphs – connected components (it is often be the case that proving something for a connected graph proves it for any general graph). Let us extend this to a similar partition into 2-connected graphs.

Definition 2.6. A subgraph B of a graph G is a *block of* G if either B is a bridge (and its endvertices) or is a maximal 2-connected subgraph of G.

Remark. If G_1 and G_2 are k-connected subgraphs ($k \geq 1$) of a graph G with at least k common vertices, then $G_1 \cup G_2$ is k-connected as well. Indeed, if we delete any $k-1$ vertices from $G_1 \cup G_2$, then since what remains of G_1 and G_2 are still connected and they intersect, the entirety of the remaining graph must be connected.

Note that by the above remark, any two blocks have at most one vertex in common. Note that this also implies that distinct blocks have no edges in common.
If $x, y \in B$, then $G - E(B)$ contains no x-y path (the maximality of B is contradicted otherwise).
Therefore, a vertex is a cutvertex iff it belongs to two blocks of G (Why?).

Definition 2.7. Suppose that G is a non-trivial connected graph. Define the *block-cutvertex graph* $\mathsf{bc}(G)$ as the graph with vertices as the blocks and cutvertices of G and an edge from each cutvertex to the (two) blocks containing it.

If G is 2-connected or K^2 (a single edge), then $\mathsf{bc}(G)$ is a single vertex.
We now present an important result on vertex-disjoint s-t paths (paths from s to t with no common vertex).

Theorem 2.7 (Menger's Theorem)**.**

(i) Let s and t be distinct non-adjacent vertices of a graph G. Then the minimal number of vertices separating s from t is equal to the maximum number of independent s-t paths.

(ii) Let s and t be distinct vertices of G. Then the minimal number of edges separating s from t is equal to the maximum number of edge-disjoint s-t paths.

Proof.

(i) Create a directed graph by replacing each edge xy of G with $\overrightarrow{xy}$ and $\overrightarrow{yx}$ and give every vertex other than s and t capacity 1. By the Integrality Theorem, there is a maximal flow (from s to t) with flow either 0 or 1 in each edge. Therefore, the maximum flow value is equal to the number of vertex-disjoint paths from s to t. Now, Theorem 2.4 implies that this maximum flow value is also equal to minimum capacity of a vertex-cut separating s from t, which is just the minimal number of vertices separating s from t (since every vertex has capacity 1).

(ii) Similar to the first part, create a directed graph by replacing each edge xy of G with two edges $\overrightarrow{xy}$ and $\overrightarrow{yx}$ of capacity 1 each. Using the Integrality Theorem, there is an integral maximal flow. Again, this is just equal to the edge-disjoint paths from s to t. Using Theorem 2.1 similar to (i) gives the result.

■

Corollary 2.8. A graph is k-connected ($k \geq 2$) iff it has at least two vertices and between any two vertices there are at least k vertex-disjoint paths.
A graph is k-edge-connected ($k \geq 2$) iff it has at least two vertices and between any two vertices, there are at least k edge-disjoint paths.

Note that similar to how we generalized the max-flow min-cut theorem to an arbitrary number of sources/sinks, we can similarly generalize Menger's Theorem as

Theorem 2.9 (Generalized Menger's Theorem)**.** Let S and T be arbitrary subsets of vertices of G. Then, the maximal number of vertex-disjoint (including end-vertices[2]) S-T paths is equal to

$$\min\{|W| : W \subseteq V(G), G - W \text{ has no } S\text{-}T \text{ path}\}.$$

This is easily proved by adding two new vertices s and t, joining s to every vertex in S, and t to every vertex in T, and applying Menger's Theorem in the new graph.

Exercise 2.1. Let T be a tree. Show that the graph whose vertices are proper 3-colorings of T and whose edges are pairs of colorings which differ at only a single vertex is connected.

[2]the reason for this is obvious from the construction since we add two *new* vertices