Connectedness and Compactness

Lecture 18 - 05/03/2021

Def. Let X be a topological space. A separation of X is a pair U, V of disjoint non-empty open subsets of X whose union is X. X is said to be connected if it does not have a separation.

Prop. (3.1) A space X is connected iff the only subsets of X that are both open and closed ("clopen") are $\emptyset$ and X.

If U, V is a separation, U is clopen.

If $A \neq \emptyset$ is clopen, $A, X \setminus A$ is a separation.

Prop (3.2) If X and Y are homeomorphic, X is connected iff Y is connected.

Lemma (3.3) If Y is a subspace of X, a separation of Y is pair of disjoint non-empty sets A, B whose union is Y, neither of which contains a limit point of the other. The space Y is connected iff there is no separation of Y.

This is easily shown since the sets involved in a separation are clopen (in Y). $\bar{A} \cap Y = A$

$$A \cap B = \emptyset \Rightarrow \bar{A} \cap Y \cap B = \emptyset$$
$$\Rightarrow \bar{A} \cap B = \emptyset.$$

The other direction is similarly straightforward.

For example, any topological space with the indiscrete topology is connected.

$\rightarrow$ Show that $\mathbb{Q}$ is not connected.

Lemma. (3.4) If the sets C, D form a separation of X and Y is a connected subspace of Y, $Y \subseteq C$ or $Y \subseteq D$.

If not, we can write $Y = \underbrace{(Y \cap C) \cup (Y \cap D)}_{\text{open in } Y}$

Lemma. (3.5) A union of connected spaces is connected if their intersection is non-empty.

Proof. Let (A_α) be a family of connected subspaces of X and $p \in \bigcap_\alpha A_\alpha$.
We claim that $Y = \bigcup_\alpha A_\alpha$ is connected.
Suppose C, D is a separation of Y and wlog that $p \in C$.
Since A_α is connected and $p \in C$, $A_\alpha \subseteq C$.
Therefore, $Y = \bigcup_\alpha A_\alpha \subseteq C$, contradicting the non-emptiness of D. $\square$

Theo. (3.6) Let A be a connected subspace of X If $A \subseteq B \subseteq \overline{A}$, B is also connected.

(We can add any of the limit points without destroying connectedness)

Proof Suppose C, D is a separation of B. Assume wlog that $A \subseteq C$. Then $B \subseteq \overline{A} \subseteq \overline{C}$. But $\overline{C} \cap D = \emptyset$, yielding a contradiction and proving the claim. $\square$

Theo. (3.7) The image of a connected space under a continuous map is connected.

Proof Let $f: X \to Y$ be continuous where X is connected and f is surjective. Suppose C, D is a separation of Y. Since f is continuous, $f^{-1}(C)$ and $f^{-1}(D)$ are also open and they form a separation of X, resulting in a contradiction. $\square$

Theo. (3.8) A finite Cartesian product of connected spaces is connected. (under either the box or product topo., they are equal)

Proof It suffices to show that if X and Y are connected $X \times Y$ is connected. (Use the fact that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic to $(X_1 \times \cdots \times X_{n-1}) \times X_n$)
Fix $x \times y \in X \times Y$. $X \times \{y\}$ is connected (it is homeomorphic to X) and so is $\{x\} \times Y$. The result follows on using Theo 3.5. $\blacksquare$

Show that $\mathbb{R}^{\omega}$ under the box topology is disconnected.

Hint: Let $A = \{(a_n) : (a_n) \text{ is bounded}\}$ and
$B = \{(b_n) : (b_n) \text{ is unbounded}\}$.

Show that $\mathbb{R}^{\omega}$ under the product topology is connected.

Hint: Show that $\mathbb{R}^{\infty}$, the set of sequences eventually 0, is connected and that $\mathbb{R}^{\omega} = \overline{\mathbb{R}^{\infty}}$.

$$\mathbb{R}^{\infty} = \bigcup_{n \in \mathbb{N}} \widetilde{\mathbb{R}}^n$$
$\widetilde{\mathbb{R}}^n$ ↳ 0 after first n co-ordinates.

Theo. (3.9) An arbitrary product of connected spaces is connected in the product topology.
(the proof is nearly identical to that for $\mathbb{R}^{\omega}$ above)

Def. A simply ordered set L having more than one element is called a linear continuum if

- L has the least upper bound property.
- if $x < y$ in L, there exists z in L such that $x < z < y$.

Clearly, $\mathbb{R}$ is a linear continuum.

Theo. (3.10) If L is a linear continuum, then L, intervals in L, and rays in L, are connected.

Theo. [Intermediate Value Theorem]

Let $f: X \to Y$ be continuous, where X is a connected space and Y is an ordered set under the ordered topology. If $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, there exists $c \in X$ such that $f(c) = r$.

Proof. Suppose otherwise. Then $f(X) \cap (-\infty, r)$ and $f(X) \cap (r, \infty)$ form a separation of $f(X)$. However, the image under f of X is connected, resulting in a contradiction.