

Connectedness and Compactness

Lecture 18 - 05/03/2021 Connectedness

Def Let X be a topological space. A separation of X is a pair U, V of disjoint non-empty open subsets of X whose union is X . X is said to be connected if it does not have a separation.

Prop (3.1) A space X is connected iff the only subsets of X that are both open and closed ("clopen") are \emptyset and X .

If U, V is a separation, U is clopen.

If $A \neq \emptyset$ is clopen, $A, X \setminus A$ is a separation. \square

Prop (3.2) If X and Y are homeomorphic, X is connected iff Y is connected.

Lemma (3.3) If Y is a subspace of X , a separation of Y is pair of disjoint non-empty sets A, B whose union is Y , neither of which contains a limit point of the other. The space Y is connected iff there is no separation of Y .

This is easily shown since the sets involved in a separation are clopen (in Y). $\bar{A} \cap Y = A$

$$\begin{aligned} A \cap B = \emptyset &\rightarrow \bar{A} \cap Y \cap B = \emptyset \\ &\rightarrow \bar{A} \cap B = \emptyset. \end{aligned}$$

The other direction is similarly straightforward. \square

For example, any topological space with the indiscrete topology is connected.

\rightarrow Show that \mathbb{Q} is not connected.

Lemma. (3.4) If the sets C, D form a separation of X and Y is a connected subspace of X , $Y \subseteq C$ or $Y \subseteq D$.

If not, we can write $Y = \underbrace{(Y \cap C)}_{\text{open in } Y} \cup \underbrace{(Y \cap D)}_{\text{open in } Y}$

Lemma. (3.5) A union of connected spaces is connected if their intersection is non-empty.

Proof. Let (A_α) be a family of connected subspaces of X and $p \in \bigcap_\alpha A_\alpha$. We claim that $Y = \bigcup_\alpha A_\alpha$ is connected. Suppose C, D is a separation of Y and wlog that $p \in C$. Since A_α is connected and $p \in C$, $A_\alpha \subseteq C$. Therefore, $Y = \bigcup_\alpha A_\alpha \subseteq C$, contradicting the non-emptiness of D . \square

Theo. (3.6) Let A be a connected subspace of X . If $A \subseteq B \subseteq \bar{A}$, B is also connected.

(We can add any of the limit points without destroying connectedness)

Proof. Suppose C, D is a separation of B . Assume wlog that $A \subseteq C$. Then $B \subseteq \bar{A} \subseteq \bar{C}$. But $\bar{C} \cap D = \emptyset$, yielding a contradiction and proving the claim. \square

Theo. (3.7) The image of a connected space under a continuous map is connected.

and f is surjective

Proof. Let $f: X \rightarrow Y$ be continuous where X is connected. Suppose C, D is a separation of Y . Since f is continuous, $f^{-1}(C)$ and $f^{-1}(D)$ are also open and they form a separation of X , resulting in a contradiction. \square

Theo. A finite Cartesian product of connected spaces is connected.

(3.8) (under either the box or product topo., they are equal)

Proof It suffices to show that if X and Y are connected $X \times Y$ is connected. (Use the fact that $(x_1, \dots, x_{n-1}) \times x_n$ is homeomorphic to $(x_1, \dots, x_{n-1}) \times x_n$)
Fix $x \times y \in X \times Y$. $X \times \{y\}$ is connected (it is homeomorphic to X) and so is $\{x\} \times Y$. The result follows on using Theo 3.5. \square

Show that \mathbb{R}^ω under the box topology is disconnected.

Hint: Let $A = \{(a_n) : (a_n) \text{ is bounded}\}$ and $B = \{(b_n) : (b_n) \text{ is unbounded}\}$.

Show that \mathbb{R}^ω under the product topology is connected.

Hint: Show that \mathbb{R}^ω , the set of sequences eventually 0, is connected and that $\mathbb{R}^\omega = \overline{\mathbb{R}^\omega}$.

$\mathbb{R}^\omega = \bigcup_{n \in \mathbb{N}} \tilde{\mathbb{R}}^n \hookrightarrow 0 \text{ after first } n \text{ co-ordinates.}$

Theo. An arbitrary product of connected spaces is connected in the product topology.

(3.9) (the proof is nearly identical to that for \mathbb{R}^ω above)

Def. A simply ordered set L having more than one element is called a **linear continuum** if

- L has the least upper bound property.
- if $x < y$ in L , there exists z in L such that $x < z < y$.

Clearly, \mathbb{R} is a linear continuum.

Theo. If L is a linear continuum, then L , intervals in L , and rays in L , are connected

(3.10)

Theo. [Intermediate Value Theorem]

(3.11) Let $f: X \rightarrow Y$ be continuous, where X is a connected space and Y is an ordered set under the ordered topology. If $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, there exists $c \in X$ such that $f(c) = r$.

Proof. Suppose otherwise. Then $f(X) \cap (-\infty, r)$ and $f(X) \cap (r, \infty)$ form a separation of $f(X)$. However, the image under f of X is connected, resulting in a contradiction. \square

Lecture 19 - 10/03/21 Path Connectedness

Def. Given points x, y of the space X , a **path** from x to y is a continuous function $f: [a, b] \rightarrow X$ such that $f(a) = x$ and $f(b) = y$ (for some closed interval $[a, b] \subseteq \mathbb{R}$).

A space X is **path-connected** if there is a path between any two points in X .

Theo. Any path-connected space is connected.

(3.12)

Proof. Suppose otherwise. Let X be path-connected and $f: [a, b] \rightarrow X$ be a path in X . Let $X = A \cup B$ be a separation of X . Since $[a, b]$ is connected and f is continuous, $f([a, b]) \subseteq A$ or $f([a, b]) \subseteq B$, contradicting path-connectedness (across A, B). \square

The converse is not true.

Consider

$$S = \{ x \times \sin(\frac{1}{x}) : 0 < x \leq 1 \},$$

known as the topologist's sine curve.

Then $\bar{S} = S \cup (\{0\} \times [-1, 1])$. We claim that \bar{S} is not path-connected. Let continuous $f: [a, c] \rightarrow \bar{S}$ beginning at the origin and ending at some point in S . The set

$$\{ t \in [a, c] : f(t) \in \{0\} \times [-1, 1] \}$$

is closed (due to continuity), so it has a largest element b .

Then the restriction $f: [b, c] \rightarrow \bar{S}$ is a path such that $f(b) \in \{0\} \times [-1, 1]$ and $f([b, c]) \subseteq S$.

Wlog, let $[b, c]$ be $[0, 1]$ and $f(t) = (x(t), y(t))$. Then $x(0) = 0$ and for $t > 0$, $x(t) > 0$ and $y(t) = \sin(1/x(t))$.

For each n , choose a $0 < u < x(1/n)$ such that $y(u) = (-1)^n$. Using the IVT (Theo 3.11), there is a $0 < t_n < 1/n$ such that $x(t_n) = u$. However, then, $t_n \rightarrow 0$ but $y(t_n) = (-1)^n$ does not converge, contradicting the continuity of y and proving the claim.

Show that S (and thus \bar{S}) is connected, disproving the converse of Theo 3.12

Def. Given X , define an equivalence relation as $x \sim y$ if there exists a connected subspace of X containing both x and y . The resulting equivalence classes are called the **components** or **connected components** of X .

(Check that it is an equivalence relation)

Theo. (3.13) The components of X are connected disjoint subspaces of X whose union is X such that any non-empty connected subspace of X intersects exactly one of them.

Proof left as exercise.

To show that a component C is connected, fix $x \in C$ and for each $y \in C$, let $C_y \subseteq C$ be a connected subspace containing C_y .
by the second part

Then $x \in \bigcap C_y \neq \emptyset$, so $\bigcup C_y = C$ is connected. \square

Similar to connected components, we can define the **path components** of X .

($x \sim y$ if there is a path from x to y)

(transitivity can be shown using the pasting lemma)

Theo. (3.14) The path components of X are path-connected disjoint subspaces of X whose union is X such that any non-empty path-connected subspace of X intersects exactly one of them.

Corollary. (3.15) Any connected component of X is closed.

(Use the fact that the closure of a connected space is closed)

It follows that if there are finitely many components, each component is also open.

It need not be true that path-connected components are closed, however. Consider the topologist's sine curve \bar{S} . Then S is open in \bar{S} (and not closed) and $\bar{S} \setminus S$ is closed (and not open).

Def A space X is said to be **locally connected** at $x \in X$ if for every neighbourhood U of x , there is a connected neighbourhood $V \subseteq U$ of x . X is locally connected if it is locally connected at any point of X .
We similarly define **local path-connectedness**.

Theo. (3.16) A space X is locally connected iff for any open $U \subseteq X$, each component of U is open in X .

Proof. Let X be locally connected, U be open in X , and C be a component of U . Let $x \in C$. There is then a neighbourhood $V \subseteq U$ of x that is connected. It follows that $V \subseteq C$ and therefore, C is open.

On the other hand suppose that the components of open sets in X are open. Let $x \in X$ and U a neighbourhood of x . We can take the component of U containing x , completing the proof

□

Theo. (3.17) A space X is locally path-connected iff for any open $U \subseteq X$, each path component of U is open.

The proof is nearly identical to the previous one.