MA 412: Complex Analysis

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§1. Introduction

1.1. Some basic definitions

Consider the equation $X^2 + 1 = 0$. Clearly, this equation has no roots over $\mathbb{R}$. Consider the set

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} = \mathbb{R}^2,$$

and define addition and subtraction over $\mathbb{C}$ as

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b) \cdot (c, d) &= (ac - bd, ad + bc).\end{aligned}$$

It is easy to show that $(\mathbb{C}, +, \cdot)$ is a field with additive identity $(0, 0)$ and multiplicative identity $(1, 0)$. Further observe that $\mathbb{R}$ is a subfield of $\mathbb{C}$ – consider the field homomorphism $\mathbb{R} \to \mathbb{C}$ defined by $a \mapsto (a, 0)$.
Now, we denote $\iota = (0, 1)$, and write (a, b) as $a + b\iota$.

Observe that the equation $X^2 + 1 = 0$ *does* have roots over $\mathbb{C}$ since it can be written as $(X + \iota)(X - \iota)$. For the sake of completeness, we also note that the multiplicative identity of $a + \iota b$ is

$$\frac{1}{a + \iota b} = \frac{a - \iota b}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\iota.$$

When writing $z = a + b\iota$ where $a, b \in \mathbb{R}$, we write $a = \Re z$ (the real part of z) and $b = \Im z$ (the imaginary part of z). We also define the absolute value $|z| = (a^2 + b^2)^{1/2}$ of z, and the *conjugate* $\overline{z} = a - \iota b$ of z. We clearly have

$$\begin{aligned}z\overline{z} &= |z|^2\\ \Re z &= \frac{z + \overline{z}}{2}\\ \Im z &= \frac{z - \overline{z}}{2\iota}.\end{aligned}$$

It is easy to check that

$$\overline{z + w} = \overline{z} + \overline{w} \text{ and } \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

We also have

$$\begin{aligned}\left|\frac{z}{w}\right| &= \frac{|z|}{|w|}\\ |\overline{z}| &= |z|.\end{aligned}$$

Exercise 1.1. Check that the set

$$M = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix} : \alpha, \beta \in \mathbb{R}$$

with matrix addition and multiplication is a field isomorphic to $\mathbb{C}$.

To close out the tedious part of things, we have

$$\begin{aligned}|z + w|^2 &= |z|^2 + |w|^2 + 2\Re(z\overline{w})\\ |z + w| &\leq |z| + |w| \qquad (1.1)\end{aligned}$$

Equation (1.1) is referred to as the *triangle inequality*.

1.2. Polar representations and roots

Consider $z = x + \iota y \in \mathbb{C}$. We may then define

$$x = r\cos\theta \qquad y = r\sin\theta,$$

where $|z| = r$ and the angle θ is called the *argument* of z as is denoted $\theta = \arg z$. We typically restrict θ to $(-\pi, \pi]$. We denote $\operatorname{cis}\theta = \cos\theta + \iota\sin\theta$. Therefore, we have

$$z = |z|\operatorname{cis}(\arg z).$$

Observe that rather conveniently,

$$\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2).$$

Therefore, inductively,

$$z_1 z_2 \cdots z_n = \left(\prod_i |z_i|\right) \cdots r_n \operatorname{cis}\left(\sum_i \arg z_i\right).$$

In particular,

$$z^n = r^n \operatorname{cis}(n\theta)$$

for any $n > 0$. If $z \neq 0$ (equivalently, $r \neq 0$), the above holds for all $n \in \mathbb{Z}$.
In the case where $r = 1$, we have

$$(\cos\theta + \iota\sin\theta)^n = \cos(n\theta) + \iota\sin(n\theta) \tag{1.2}$$

Equation (1.2) is referred to as *de Moivre's formula*.

Let us consider the equation $z^n = a$. This equation has n roots of the form

$$z = |a|^{1/n}\operatorname{cis}\left(\frac{2k\pi + \arg z}{n}\right)$$

for $k = 0, 1, \ldots, n-1$.
A *line* in the complex plane is a set of the form

$$L = \{z = a + tb : t \in \mathbb{R}\},$$

for some fixed $a, b \in \mathbb{C}$, where b is a *directional* vector whose absolute value may be assumed to be 1. Since $b \neq 0$, we equivalently have

$$L = \{z : \Im\left(\frac{z-a}{b}\right) = 0\}.$$

We can also define the half-planes

$$H_a = \{z : \Im\left(\frac{z-a}{b}\right) > 0\}$$
$$K_a = \{z : \Im\left(\frac{z-a}{b}\right) < 0\}.$$

Note that $H_a = a + H_0$, where the addition is Minkowski addition:

$$H_a = \{a + z : z \in H_0\}.$$

1.3. The extended plane

Define $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and let $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. We shall show a bijection from $\mathbb{C}_\infty$ to S.
Let $N = (0, 0, 1)$ be the 'north pole' of S, and orient $\mathbb{C}$ (as $\mathbb{R}^2$) in the horizontal plane in a manner such that $\mathbb{C}$ cuts S along the equator. For $z = x + \iota y \in \mathbb{C}$, let us define the corresponding point $Z = (x_1, x_2, x_3) \in S$. We shall draw a line connecting z to N, and let Z be the point of intersection (other than N) of this line with S. Finally, we shall map ∞ to N.
Let us define this more explicitly. The line through N and z is

$$L = \{tN + (1 - t)z : t \in \mathbb{R}\}.$$

Then, letting $z = (x, y, 0)$, we have

$$t^2 + (1 - t)^2 |z|^2 = 1.$$

So,

$$|z|^2 = \frac{1 - t^2}{(1 - t)^2} = \frac{1 + t}{1 - t}$$

and

$$t = 1 - \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Therefore, we map z to

$$Z = \left(\frac{2\Re z}{|z|^2 + 1}, \frac{2\Im z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}. \right) \in S.$$

Based on this, we can define a distance metric between points in $\mathbb{C}_\infty$. For $z, z' \in \mathbb{C}_\infty$ mapping to $Z, Z' \in S$, we let $d(z, z')$ be the Euclidean distance between Z, Z' in $\mathbb{R}^3$. More explicitly,

$$\begin{aligned} d(z, z')^2 &= (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 \\ &= 2 - 2(x_1 x_1' + x_2 x_2' + x_3 x_3') \\ &= \frac{2|z - z'|}{\left((|z|^2 + 1)(|z'|^2 + 1)\right)^{1/2}} \end{aligned}$$

when $z, z' \in \mathbb{C}$ and if $z' = \infty$ (so $Z' = (0, 0, 1)$), we have

$$d(z, z') =$$

This correspondence between points of S and $\mathbb{C}_\infty$ is called the *stereographic projection*.

Exercise 1.2. If P is a plane in $\mathbb{R}^3$ and $\Lambda = P \cap S$ is a circle on S, show that the projection of Λ on $\mathbb{C}$ under the stereographic projection is a circle as well (possibly a circle of infinite radius, namely a line).

1.4. Power series

In this section, we begin discussing convergence of series in $\mathbb{C}$ and related properties.

Definition 1.1. If $a_n \in \mathbb{C}$ for every $n \geq 0$, the series $\sum_{n=0}^{\infty} a_n$ is said to *converge* to z iff for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{n=0}^{m} a_n - z \right| < \epsilon$$

for all $m \geq N$.
The series $\sum_{n=0}^{\infty} a_n$ is said to converge *absolutely* if $\sum_{n=0}^{\infty} |a_n|$ converges.

Theorem 1.1. $\mathbb{C}$ is complete. That is, every Cauchy sequence in $\mathbb{C}$ is convergent.

Proof. Suppose $\{x_n + \iota y_n\}$ is a Cauchy sequence in $\mathbb{C}$, where $x_n, y_n \in \mathbb{R}$ for each n. We then have the existence of $N \in \mathbb{N}$ such that for all $m, k > N$, $|(x_m - x_k) + \iota(y_m - y_k)| < \epsilon$. Consequently, $|x_m - x_k| < \epsilon$ and $|y_m - y_k| < \epsilon$. However, since $\mathbb{R}$ is complete, this implies that (x_n) and (y_n) are convergent, completing the proof. ∎

Theorem 1.2. If $\sum a_n$ converges absolutely, $\sum a_n$ converges.

Proof. Let $\epsilon > 0$, $z_n = \sum_{i=0}^n a_i$, and $S_n = \sum_{i=0}^n |a_i|$. Because $\mathbb{C}$ is complete, it suffices to show that (z_n) is Cauchy. Since $\sum |a_n|$ is convergent, there exists $N \in \mathbb{N}$ such that $|S_m - S_k| < \epsilon$ for all $m, k > N$. Supposing $m > k$, we have

$$S_m - S_k = \sum_{i=k+1}^{m} |a_i|.$$

So,

$$\begin{aligned} |z_m - z_k| &= \left| \sum_{i=k+1}^{m} a_i \right| \\ &\geq \sum_{i=k+1}^{m} |a_i| < \epsilon, \end{aligned}$$

completing the proof. ∎

Exercise 1.3. Show that $\sum_{n=0}^{\infty} z_n$ converges iff $|z| < 1$.

Theorem 1.3. For a given power series $\sum_{n=0}^{\infty} a_n(z-a)^n$, define the number of R ($0 \leq R \leq \infty$) by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Then,

(a) If $|z - a| < R$, the series converges absolutely.

(b) If $|z - a| > R$, the terms of the series become unbounded and the series diverges.

(b) If $0 < r < R$, the series converges uniformly on the set $\{z : |z - a| \leq r\}$.

This R is referred to as the *radius of convergence* of the power series.

Proof.

(a) We assume without loss of generality that $a = 0$. If $|z| < R$, there exists r with $|z| < r < R$. By the definition of R, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{R} - \epsilon < \sup_{k \geq n} |a_k|^{1/k} < \frac{1}{R} + \epsilon$$

for all $n > N$. If we take $\epsilon = 1/r - 1/R$, it follows that $|a_n|^{1/n} < 1/r$ for all $n > N$. That is, for all $n > N$, $|a_n| < 1/r^n$ and so

$$|a_n z^n| < \left(\frac{|z|}{r}\right)^n.$$

Therefore, $\sum_{n=N}^{\infty} a_n z^n$ is dominated by $\sum_{n=N}^{\infty} (|z|/r)^n$. Now however, we can just use the result of Exercise 1.3 to conclude absolute convergence since $|z|/r < 1$.

(b) Let $|z| > R$ and choose r with $|z| > r > R$. For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{R} - \epsilon < \sup_{k \geq n} |a_k|^{1/k} \text{ for all } n > N.$$

Choosing $\epsilon = 1/R - 1/r$,

$$|a_n|^{1/n} > 1/r$$

for infinitely many $n \in \mathbb{N}$. It follows that $|a_n z^n| > (|z|/r)^n$ for infinitely many $n \in \mathbb{N}$. Since $|z|/r > 1$, these terms become unbounded and therefore the series diverges.

(c) Now, suppose $r < R$ and choose ρ such that $r < \rho < R$. Similar to the argument in (a), we get that

$$|a_n| < \frac{1}{\rho^n} \text{ for all } n \geq N.$$

If $|z| \leq r$, $|a_n z^n| \leq (r/\rho)^n$ and $r/\rho < 1$. The Weierstrass M-test then gives that the power series converges uniformly on $\{z : |z| \leq r\}$. ■

It should be noted that we cannot conclude anything when $|z - a| = R$.

Theorem 1.4. If $\sum a_n(z-a)^n$ is a power series with radius of convergence R, then if it exists,

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = R.$$

Proof. Again, assume that $a = 0$ and let $\alpha = \lim |a_n/a_{n+1}|$, which we assume exists. Suppose that $|z| < \alpha$ and take $r \in \mathbb{R}$ such that $|z| < r < \alpha$. For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\alpha - \epsilon < \left| \frac{a_n}{a_{n+1}} \right| < \alpha + \epsilon.$$

Taking $\epsilon = \alpha - r$, $|a_n/a_{n+1}| > r$ for all $n \geq N$. Let $B = |a_N| r^N$. Then,

$$a_{N+1} r^{N+1} = |a_{N+1}| r \cdot r^N < |a_N| r^N = B.$$

Similarly, we get that $|a_n| r^n < B$ for all $n \geq N$. Therefore,

$$|a_n z^n| < B \left(\frac{|z|}{r} \right)^n$$

for all $n \geq N$. Thus, the sequence converges absolutely since $|z| < r$.
Since $r < \alpha$ was arbitrary, this implies that $\alpha \leq R$.

On the other hand, if $|z| > \alpha$, take $r \in \mathbb{R}$ such that $|z| > r > \alpha$. Taking $\epsilon = r - \alpha$, we get $N \in \mathbb{N}$ such that

$$\left| \frac{a_n}{a_{n+1}} \right| < r$$

for all $n \geq N$. Letting $B = |a_N| r^N$ again, we once more obtain that $|a_n| r^n > B$ for all $n \geq N$. This gives that

$$|a_n z^n| > B \left(\frac{|z|}{r} \right)^n$$

for all $n \geq N$, and since $|z| > r$, the sequence diverges (we may assume that $B \neq 0$ by making N larger if required to ensure that $a_N \neq 0$ – if this is not possible, the problem is trivial since it means that (a_n) is eventually 0). Since the choice of r was arbitrary, this implies that $R \leq \alpha$, completing the proof. ■

Now, consider the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The radius of convergence of this series is ∞. So, it converges for any complex number z, and convergence is uniform on eveyr compact subset of $\mathbb{C}$.
The above defines a function $\exp : \mathbb{C} \to \mathbb{C}$.
We also denote $e^z = \exp(z)$.

Definition 1.2 (Differentiability)**.** If G is an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$, then f is said to be *differentiable* at a point $a \in G$ if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If it exists, the value of this limit is denoted $f'(a)$ and is called the *derivative* of f at a.

If f is differentiable at each point of G, we say that f is differentiable on G. Note that if f is differentiable on G, then $f' : G \to \mathbb{C}$ is a function. If f' is continuous, f is said to be *continuously differentiable.*

Theorem 1.5. If $f : G \to \mathbb{C}$ is differentiable at a point $a \in G$, f is continuous at a.

Proof. The proof of this is direct:

$$\begin{aligned} \lim_{z \to a} |f(z) - f(a)| &= \left(\lim_{z \to a} \frac{|f(z) - f(a)|}{|z-a|} \right) \cdot \lim_{z \to a} |z - a| \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

■

Definition 1.3. A function $f : G \to \mathbb{C}$ is said to be *analytic* if f is continuously differentiable on G.

Let f, g be analytic on G and Ω respectively, and suppose that $f(G) \subseteq \Omega$. Then, $g \circ f$ is analytic on G and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$$

for all $z \in G$. This is called the *chain rule.*

We shall show later that if f is differentiable then its derivative is continuous, and so f is analytic.

Theorem 1.6. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$. Then

(a) For each $k \geq 1$, the series

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z-a)^{n-k}$$

has radius of convergence R.

(b) The function f is infinitely differentiable on $B(a, R)$ (the open ball of radius R centered at a), and further, $f^{(k)}(z)$ is given by the series in (a) for all $k \geq 1$ and $|z - a| < R$.

(c) For $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

Proof. Assume that $a = 0$.

(a) Note that it suffices to prove the result for $k = 1$ (Why?). To show this, it is enough to show that

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |na_n|^{1/(n-1)}$$

First, it is not difficult to show that $\lim_{n\to\infty} n^{1/(n-1)} = 1$. It may be shown that for any sequences $(c_n), (d_n)$ in $\mathbb{R}$ where $c_n \geq 0$, if $\lim c_n = c$ and $\limsup d_n = d$, then $\limsup c_n d_n = cd$. Therefore, we are done if we show that $\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/(n-1)}$.

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + z\sum_{n=0}^{\infty} a_{n+1} z^n.$$

Let R' be the radius of convergence of $\sum_{n=0}^{\infty} a_{n+1} z^n$. We want to show that $R' = R$.
If $|z| < R'$, then

$$\sum |a_n z^n| \leq |a_0| + |z| \sum_{n=0}^{\infty} |a_{n+1} z^n| < \infty,$$

so $R' \leq R$. On the other hand, if $|z| < R$ and $z \neq 0$,

$$\sum |a_{n+1} z^n| < \frac{1}{|z|}\left(\sum |a_n z^n| + |a_0|\right) < \infty,$$

so $R \leq R'$ and we are done.

(b) Once again, it suffices to prove the result for $k = 0$. For $|z| < R$ and $g(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$,

$$s_n(z) = \sum_{k=0}^{n} a_k z^k \text{ and } R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k,$$

fix a point $w \in B(0, R)$ and r such that $|w| < r < R$. We wish to show that $f'(w)$ exists and is equal to $g(w)$. Let $\delta > 0$ be arbitrary with $\overline{B(w, \delta)} \subseteq B(0, r)$. Letting $z \in B(w, \delta)$, we have

$$\begin{aligned}
\frac{f(z) - f(w)}{z - w} - g(w) &= \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) + s_n'(w) - g(w) + \frac{R_n(z) - R_n(w)}{z - w} \\
&= \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) + s_n'(w) - g(w) + \frac{1}{z - w}\sum_{k=n+1}^{\infty} a_k(z^k - w^k) \\
&= \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) + s_n'(w) - g(w) + \sum_{k=n+1}^{\infty} a_k \frac{z^k - w^k}{z - w}.
\end{aligned}$$

We have

$$|z^k - w^k| = |z - w||z^{k-1} + z^{k-2}w + \cdots + w^{k-1}| \leq |z - w| k r^{k-1}.$$

Therefore,

$$\left|\frac{R_n(z) - R_n(w)}{z - w}\right| \leq \sum_{k=n+1}^{\infty} |a_k| k r^{k-1}.$$

Since $r < R$, $\sum_{k=1}^{\infty} |a_k| k r^{k-1}$ converges and so for any $\epsilon > 0$, there exists $N_1 \in N$ such that

$$\left|\frac{R_n(z) - R_n(w)}{z - w}\right| < \epsilon/3.$$

Since $\lim s_n'(w) = g(w)$, there exists $N_2 \in \mathbb{N}$ such that

$$|s_n'(w) - g(w)| < \epsilon/3$$

for $n \geq N_2$. Choose $n \geq \max(N_1, N_2)$. Then, there exists $\delta < 0$ such that whenever $0 < |z - w| < \delta$,

$$\left| \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) \right| < \epsilon/3.$$

Putting all these together, we get the desideratum.

(c) This is straightforward using the explicit expression for $f^{(k)}(a)$. ∎

If the series $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence $R > 0$, then f is analytic on $B(a, R)$. Therefore, exp is analytic on $\mathbb{C}$.