# Connectedness and Compactness

Connectedness

**Def** Let $X$ be a topological space. A separation of $X$ is a pair $U, V$ of disjoint non-empty open subsets of $X$ whose union is $X$. $X$ is said to be connected if it does not have a separation.

**Prop. (3.1)** A space $X$ is connected iff the only subsets of $X$ that are both open and closed ("clopen") are $\emptyset$ and $X$.

        If $U, V$ is a separation, $U$ is clopen.

        If $A \neq \emptyset$ is clopen, $A, X \backslash A$ is a separation. ∎

**Prop (3.2)** If $X$ and $Y$ are homeomorphic, $X$ is connected iff $Y$ is connected.

**Lemma (3.3)** If $Y$ is a subspace of $X$, a separation of $Y$ is pair of disjoint non-empty sets $A, B$ whose union is $Y$, neither of which contains a limit point of the other. The space $Y$ is connected iff there is no separation of $Y$.

This is easily shown since the sets involved in a separation are clopen (in $Y$). $\bar{A} \cap Y = A$

$$A \cap B = \emptyset \rightarrow \bar{A} \cap Y \cap B = \emptyset$$
$$\rightarrow \bar{A} \cap B = \emptyset.$$

The other direction is similarly straightforward. ∎

For example, any topological space with the indiscrete topology is connected.

→ Show that $\mathbb{Q}$ is not connected.

**Lemma.
(3.4)** If the sets $C, D$ form a separation of $X$ and $Y$ is a connected subspace of $Y$, $Y \subseteq C$ or $Y \subseteq D$.

If not, we can write $Y = \underbrace{(Y \cap C) \cup (Y \cap D)}_{\text{open in } Y}$

**Lemma.
(3.5)** A union of connected spaces is connected if their intersection is non-empty.

**Proof.** Let $(A_\alpha)$ be a family of connected subspaces of $X$ and $p \in \bigcap_\alpha A_\alpha$.
We claim that $Y = \bigcup_\alpha A_\alpha$ is connected.
Suppose $C, D$ is a separation of $Y$ and wlog that $p \in C$.
Since $A_\alpha$ is connected and $p \in C$, $A_\alpha \subseteq C$.
Therefore, $Y = \bigcup_\alpha A_\alpha \subseteq C$, contradicting the non-emptiness of $D$. $\blacksquare$

**Theo.
(3.6)** Let $A$ be a connected subspace of $X$. If $A \subseteq B \subseteq \bar{A}$, $B$ is also connected.

(We can add any of the limit points without destroying connectedness)

**Proof** Suppose $C, D$ is a separation of $B$. Assume wlog that $A \subseteq C$. Then $B \subseteq \bar{A} \subseteq \bar{C}$. But $\bar{C} \cap D = \emptyset$, yielding a contradiction and proving the claim. $\blacksquare$

**Theo.
(3.7)** The image of a connected space under a continuous map is connected.

and $f$ is surjective

**Proof** Let $f : X \to Y$ be continuous where $X$ is connected. Suppose $C, D$ is a separation of $Y$. Since $f$ is continuous, $f^{-1}(C)$ and $f^{-1}(D)$ are also open and they form a separation of $X$, resulting in a contradiction. $\blacksquare$

**Theo.** A finite Cartesian product of connected spaces is connected.
**(3.8)** (under either the box or product topo., they are equal)

**Proof.** It suffices to show that if $X$ and $Y$ are connected $X \times Y$ is connected. (Use the fact that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic to $(X_1 \times \cdots \times X_{n-1}) \times X_n$. Fix $x \times y \in X \times Y$. $X \times \{y\}$ is connected (it is homeomorphic to $X$) and so is $\{x\} \times Y$. The result follows on using Theo 3.5. ◻

Show that $\mathbb{R}^\omega$ under the box topology is disconnected.

Hint: Let $A = \{(a_n) : (a_n)$ is bounded $\}$ and
$B = \{(b_n) : (b_n)$ is unbounded $\}$.

Show that $\mathbb{R}^\omega$ under the product topology is connected.

Hint: Show that $\mathbb{R}^\infty$, the set of sequences eventually $0$, is connected and that $\mathbb{R}^\omega = \overline{\mathbb{R}^\infty}$.

$\mathbb{R}^\infty = \bigcup_{n \in \mathbb{N}} \tilde{\mathbb{R}}^n \rightsquigarrow 0$ after first $n$ co-ordinates.

**Theo.** An arbitrary product of connected spaces is connected in the product
**(3.9)** topology.
(the proof is nearly identical to that for $\mathbb{R}^\omega$ above)

**Def.** A simply ordered set $L$ having more than one element is called a
linear continuum if
- $L$ has the least upper bound property.
- if $x < y$ in $L$, there exists $z$ in $L$ such that $x < z < y$.

Clearly, $\mathbb{R}$ is a linear continuum.

**Theo.** If $L$ is a linear continuum, then $L$, intervals in $L$, and rays in $L$,
**(3.10)** are connected

**Theo.** [Intermediate Value Theorem]

**(3.11)** Let $f: X \to Y$ be continuous, where $X$ is a connected space and $Y$ is an ordered set under the ordered topology. If $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, there exists $c \in X$ such that $f(c) = r$.

**Proof.** Suppose otherwise. Then $f(X) \cap (-\infty, r)$ and $f(X) \cap (r, \infty)$ form a separation of $f(X)$. However, the image under $f$ of $X$ is connected, resulting in a contradiction. $\blacksquare$

Lecture 19 - 10/03/21  Path Connectedness

**Def.** Given points $x, y$ of the space $X$, a path from $x$ to $y$ is a continuous function $f: [a,b] \to X$ such that $f(a) = x$ and $f(b) = y$ (for some closed interval $[a,b] \subseteq \mathbb{R}$).
A space $X$ is path-connected if there is a path between any two points in $X$.

**Theo.** Any path-connected space is connected.

**(3.12)**

**Proof.** Suppose otherwise. Let $X$ be path-connected and $f: [a,b] \to X$ be a path in $X$. Let $X = A \cup B$ be a separation of $X$.
Since $[a,b]$ is connected and $f$ is continuous, $f([a,b]) \subseteq A$ or $f([a,b]) \subseteq B$, contradicting path-connectedness (across $A, B$). $\blacksquare$

The converse is not true.
Consider
$$S = \{ x \times \sin(\tfrac{1}{x}) : 0 < x \leq 1 \},$$
known as the topologist's sine curve.
Then $\bar{S} = S \cup (\{0\} \times [-1, 1])$. We claim that $\bar{S}$ is not path-connected.
Let continuous $f: [a,c] \to \bar{S}$ beginning at the origin and ending at some point in $S$. The set
$$\{ t \in [a,c] : f(t) \in \{0\} \times [-1, 1] \}$$
is closed (due to continuity), so it has a largest element $b$.
Then the restriction $f: [b,c] \to \bar{S}$ is a path such that $f(b) \in \{0\} \times [-1,1]$ and $f((a,c]) \subseteq S$.

Wlog, let $[b,c]$ be $[0,1]$ and $f(t) = (x(t), y(t))$. Then
$$x(0) = 0 \text{ and for } t > 0, \; x(t) > 0 \text{ and } y(t) = \sin(1/x(t)).$$

For each $n$, choose a $0 < v < x(1/n)$ such that $y(v) = (-1)^n$. Using the IVT (Theo 3.11), there is a $0 < t_n < 1/n$ such that $x(t_n) = v$. However, then, $t_n \to 0$ but $y(t_n) = (-1)^n$ does not converge, contradicting the continuity of $y$ and proving the claim.

Show that $S$ (and thus $\bar{S}$) is connected, disproving the converse of Theo 3.12

<span style="color:red">**Def.**</span> Given $X$, define an equivalence relation as $x \sim y$ if there exists a connected subspace of $X$ containing both $x$ and $y$. The resulting equivalence classes are called the <span style="color:green">components</span> or <span style="color:green">connected components</span> of $X$.

Component

(Check that it is an equivalence relation)

<span style="color:red">**Theo.**</span>
<span style="color:red">**(3.13)**</span> The components of $X$ are connected disjoint subspaces of $X$ whose union is $X$ such that any non-empty connected subspace of $X$ intersects exactly one of them.

Proof left as exercise.
To show that a component $C$ is connected, fix $x \in C$ and for each $y \in C$, let $C_y \subseteq C$ be a connected subspace containing $C_y$.
$\quad\quad\quad\quad\quad\quad\quad\quad \underset{\text{by the second part}}{\sqsubseteq}$
Then $x \in \bigcap C_y \neq \emptyset$, so $\bigcup C_y = C$ is connected. $\quad\quad$ $\blacksquare$

Similar to connected components, we can define the <span style="color:green">path components</span> of $X$.

Path Component
$\quad\quad\quad\quad$ ($x \sim y$ if there is a path from $x$ to $y$)
$\quad\quad\quad\quad$ (transitivity can be shown using the pasting lemma)

**Theo.** The path components of $X$ are path-connected disjoint subspaces of $X$
**(3.14)** whose union is $X$ such that any non-empty path-connected subspace
of $X$ intersects exactly one of them.

**Corollary.** Any connected component of $X$ is closed.
**(3.15)**

(Use the fact that the closure of a connected space is closed)

It follows that if there are finitely many components, each component
is also open.

It need not be true that path-connected components are closed, however.
Consider the topologist's sine curve $\overline{S}$. Then $S$ is open in $\overline{S}$ (and
not closed) and $\overline{S} \backslash S$ is closed (and not open).

**Def** A space $X$ is said to be <span style="color:green">locally connected</span> at $x \in X$ if for every
neighbourhood $U$ of $x$, there is a connected neighbourhood $V \subseteq U$ of
**Locally**
**Connected** $x$. $X$ is locally connected if it is locally connected at any point
of $X$.
We similarly define <span style="color:green">local path-connected</span>ness.

**Theo.** A space $X$ is locally connected iff for any open $U \subseteq X$, each
**(3.16)** component of $U$ is open in $X$.
**Proof.** Let $X$ be locally connected. $U$ be open in $X$, and $C$ be a component
of $U$. Let $x \in C$. There is then a neighbourhood $V \subseteq U$ of $x$ that
is connected. It follows that $V \subseteq C$ and therefore, $C$ is open.
On the other hand suppose that the components of open sets in $X$
are open. Let $x \in X$ and $U$ a neighbourhood of $x$. We can take
the component of $U$ containing $x$, completing the proof $\blacksquare$

**Theo.** A space X is locally path-connected iff for any open $U \subseteq X$, each
**(3.17)** path component of U is open.

The proof is nearly identical to the previous one.

**Theo.** If X is a topological space, each path component of X lies in a
**(3.18)** component of X. Moreover, if X is locally path-connected, the
components and path components are the same.

**Proof.** The first part is direct since any path component is connected
Let C be a component, $x \in C$, and $P \ni x$ be a path component.
Let X be locally path-connected. Suppose $P \subsetneq C$. Let Q be the
union of all path components other than P that intersect c.
Then $C = P \cup Q$.

Because X is locally path connected, each path component of X is open
in X. In particular, P and Q are open. This contradicts the connectedness
of C, proving the claim.
$\hookrightarrow$ they form a separation of C

**Def.** A collection $\mathcal{A}$ of subsets of X is said to be a covering if
Cover $X = \bigcup_{A \in \mathcal{A}} A$  An open covering is a covering where every subset is
open.
Compact X is said to be compact if any open cover contains a finite subcover.

If Y is a subspace of X and $\mathcal{A}$ is a collection of subsets of X,
$\mathcal{A}$ is said to cover Y if $Y \subseteq \bigcup_{A \in \mathcal{A}} A$.

**Theo.** Let Y be a subspace of X. Then Y is compact iff every covering of Y by
**(3.19)** sets open in X contains a finite subcollection covering y.

**Theo.** Any closed subspace of a compact space is compact.
(3.20)

**Theo.** Every compact subspace of a Hausdorff space is closed.
(3.21)

**Proof.** Let $X$ be Hausdorff and $Y$ a compact subspace. Let $x_0 \in X \setminus Y$.
For each $y \in Y$, choose neighbourhoods $U_y$ of $x_0$ and $V_y$ of $y$ such that $U_y \cap V_y = \emptyset$
Since $Y$ is compact, there exist $y_1, \cdots, y_n$ such that
$$Y \subseteq \bigcup_{1 \le i \le n} V_{y_i} = V.$$

But $Y \cap U \subseteq V \cap U = \emptyset$, where
$$U = \bigcap_{1 \le i \le n} U_{y_i} \text{ is a neighbourhood of } x_0.$$
Therefore, $Y$ is closed. $\blacksquare$

The above need not be true for non-Hausdorff spaces.
(Consider $\mathbb{R}$ under finite complement topology)

**Lemma.** If $Y$ is a compact subspace of the Hausdorff space $X$, and $x_0 \in X \setminus Y$, there
(3.22) are disjoint open sets $U$ and $V$ of $X$ such that $x_0 \in U$ and $Y \subseteq V$.

**Lecture 21** - 12/03/21  More on Compact Spaces

**Theo.** The image of a compact space under a continuous map is compact.
(3.23)

**Theo** Let $f: X \to Y$ be bijective and continuous. If $X$ is compact and $Y$ is
(3.24) Hausdorff, $f$ is a homeomorphism.
(Show that $f$ is a closed map)

**Lemma.** **[Tube Lemma]** Let $X$ and $Y$ be spaces with $Y$ compact. Suppose $x_0 \in X$
**(3.25)** and $N \supseteq \{x_0\} \times Y$ is an open subset of $X \times Y$. Then, there is a
neighbourhood $W$ of $x_0$ in $X$ such that $W \times Y \subseteq N$.

Tube Lemma

$\hookrightarrow$ tube about $x_0 \times Y$.

**Proof.** Let us cover $x_0 \times Y$ with the basis elements $U \times V$ (for the
topology of $X \times Y$) lying in $N$.
Since $\{x_0\} \times Y$ is compact, there is a finite subcover
$$U_1 \times V_1, \ldots, U_n \times V_n$$
We may assume that $x_0 \in U_i$ for each $i$ (Why?). Let
$$W = \bigcap_{1 \leq i \leq n} U_i.$$
Then $W$ is open in $X$ and contains $x_0$.
It is easily shown that $W \times Y$ is covered by the $(U_i \times V_i)$. Since each
$U_i \times V_i \subseteq N$, $W \times Y \subseteq N$. $\blacksquare$


**Theo.** The product of finitely many compact spaces is compact.
**(3.26)**

$\left( \text{It is in fact true for arbitrary products, which we shall see later} \atop \text{in Tychonoff's Theorem} \right)$

**Proof** It suffices to show the result for two spaces. Let $X, Y$ be compact and
$A$ an open subcover of $X \times Y$
For each $x_0 \in X$, $\{x_0\} \times Y$ can be covered by finitely many $A_1, \ldots, A_m \in A$.
Then $\{x_0\} \times Y \subseteq A_1 \cup \cdots \cup A_m = N$, so $N$ contains a tube $W \times Y$
containing $\{x_0\} \times Y$. $W \times Y$ is covered by the $A_i$ $(1 \leq i \leq m)$.
That is, for each $x \in X$, there is a neighbourhood $W_x$ of $x$ such that
$W_x \times Y$ can be covered by finitely many elements.
The collection of all $W_x$ forms an open cover of $X$, so it has
a finite subcover $W_1, W_2, \ldots, W_k$.
Then, $X \times Y \subseteq (W_1 \times Y) \cup (W_2 \times Y) \cup \cdots \cup (W_k \times Y)$.
Each is covered by finitely many elements from $A$, so $X \times Y$ can as well. $\blacksquare$

**Def.** A collection $\mathcal{C}$ of subsets of $X$ is said to have the **finite intersection property** if for any finite $\{C_1, C_2, ..., C_k\} \subseteq \mathcal{C}$,
$$C_1 \cap C_2 \cap \cdots \cap C_k \neq \emptyset.$$

Finite Intersection Property

**Theo.** (3.27) Let $X$ be a topological space $X$ is compact iff for every collection $\mathcal{C}$ of closed subsets of $X$ with the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ is non-empty.

**Proof.** Forward direction

Suppose otherwise. Then $\mathcal{A} = \{X \setminus C : C \in \mathcal{C}\}$ is an open cover so has a finite subcover $A_1, ..., A_m$. But then, $(X \setminus A_1) \cap \cdots \cap (X \setminus A_m) = \emptyset$, contradicting the finite intersection property and proving the result.

Backward direction.

Let $\mathcal{A}$ be an open cover and $\mathcal{C} = \{X \setminus A : A \in \mathcal{A}\}$. Suppose $\mathcal{A}$ does not have a finite subcover. Then $\mathcal{C}$ has the finite intersection property so $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$. This contradicts $\mathcal{A}$ being a cover, proving the result. $\blacksquare$

Let us look at what the compact subspaces of the real line.

**Theo.** (3.28) Let $X$ be a simply ordered set having the least upper bound property. In the order topology, each closed and bounded interval of $X$ is compact.

**Proof.** Let $a < b$ and $\mathcal{A}$ be an open cover of $[a,b]$ in the subspace topology (which is the same as the order topology since $[a,b]$ is compact).
Claim 1. If $x \in [a,b)$, there exists $y > x$ in $[a,b]$ such that $[x,y]$ can be covered by at most two elements of $\mathcal{A}$.
→ If $x$ has an immediate successor, let $y$ be this element. Then $[x,y] = \{x,y\}$, so the claim is obvious.
→ Otherwise, choose $A \in \mathcal{A}$ containing $x$. Since $A$ is open, it contains an interval of the form $[x,c)$. We can then let $y$ be any element of $(x,c)$.

Now, let $C$ be the set of all points $y > a$ such that $[a, y]$ can be covered by finitely many elements of $\mathcal{A}$. (We want to show that $b \in C$.) By the claim, $C \neq \emptyset$. Let $c$ be the least upper bound of $C$.

**Claim 2.** $c \in C$.

 Choose $A \in \mathcal{A}$ containing $c$. $A$ contains an interval of the form $(d, c]$. If $c$ does not have an immediate predecessor (even otherwise, it is similarly shown) Let $z \in C$ such that $z \in (d, c)$ (why does such a $z$ exist?). Then $[a, z)$ can be covered by finitely many elements $A_1, \ldots, A_k$ of $\mathcal{A}$, so it follows that $\{A_1, \ldots, A_k\} \cup \{A\}$ is a finite subset of $\mathcal{A}$ covering $[a, c]$, proving the result.

Finally, let us prove that $b = c$. Suppose otherwise Then there exists $y \in (c, b]$ such that $[c, y]$ can be covered by atmost two elements. This implies that $[a, y]$ can be covered by finitely many elements, contradicting the fact that $c$ is an upper bound of $C$, proving the result.  ∎

<span style="color:red">**Corollary.**</span> Any closed and bounded interval in $\mathbb{R}$ is compact.
<span style="color:red">(3.29)</span>  ↳ w.r.t. Euclidean metric

**Lecture 21** - 17/03/21  <span style="color:teal">Local Compactness</span>

<span style="color:red">**Theo.**</span> A subspace $A$ of $\mathbb{R}^n$ iff it is closed and bounded in the Euclidean
<span style="color:red">(3.30)</span> metric or square metric $\rho$.
<span style="color:red">**Proof**</span> It suffices to consider the metric $\rho$ (Why?).
  Suppose $A \subseteq \mathbb{R}^n$ is compact.
  → Since $\mathbb{R}^n$ is Hausdorff, $A$ is closed.
  → Consider the collection of open sets $\{B_\rho(\mathbf{0}, m) : m \in \mathbb{N}\}$, which forms a covering of $A$. Since it has a finite subcover, it follows that $A \subseteq B_\rho(\mathbf{0}, n)$ for some $n \in \mathbb{N}$ and $A$ is bounded
  Suppose $A \subseteq \mathbb{R}^n$ is closed and bounded. Suppose $\rho(x, y) < M$ for $x, y \in A$. Fix some $x_0 \in A$. Then $A$ is a subset of $\prod_{i=1}^{n} [(x_0)_i - M, (x_0)_i + M]$, which is compact (finite product of compact sets) Since $A$ is closed in this subspace, it is compact.  ∎

**Def.**
*Isolated Point*
If $X$ is a space, $x \in X$ is said to be an <span style="color:green">isolated point</span> of $X$ if $\{x\}$ is open in $X$.

**Theo.**
**(3.31)**
Let $X$ be a non-empty compact Hausdorff space. If $X$ has no isolated points, it is uncountable

**Proof.**
→ We show that given any non-empty open $U \subseteq X$ and $x \in X$, there is a non-empty open $V \subseteq U$ such that $x \notin \overline{V}$.
Choose a $y \in U$ different from $x$ (Why does such a $y$ exist?). Let $W_1$ and $W_2$ be disjoint neighbourhoods of $x$ and $y$. $V = W_2 \cap U$ is the required set — $x \notin \overline{V}$ because $x \in W_1$ and $W_1 \cap V = \emptyset$.

→ We show that given any $f: \mathbb{N} \to X$, $f$ is not surjective.
Let $x_n = f(n)$. Let $V_1 \subseteq X$ be an open set such that $x \in \overline{V_1}$ (choosing $U = X$). In general, given non-empty open $V_{n-1}$, let $V_n \subseteq V_{n-1}$ be a non-empty open set such that $x_n \notin \overline{V_n}$. Then
$$\overline{V_1} \supseteq \overline{V_2} \supseteq \cdots \supseteq \overline{V_n} \supseteq \cdots .$$
Let $V = \bigcap_{n=1}^{\infty} V_n$. Observe that the above collection of subsets has the finite intersection property. Therefore $V$ is non-empty and closed. Let $x \in V$. Then $x \neq x_n$ for any $n$, completing the proof.
$\underset{\in \overline{V_n}}{} \quad \underset{\notin \overline{V_n}}{}$
$\blacksquare$

**Def.**
*Locally Compact*
A space $X$ is said to be <span style="color:green">locally compact</span> at $x$ if there is some compact subspace $C$ of $X$ that contains a neighbourhood of $x$.
If $X$ is locally compact at each $x \in X$, $X$ is said to be locally compact.

Observe that any compact space is locally compact (taking $C = X$).

A slightly more non-trivial example is that $\mathbb{R}$ is locally compact.

If $X$ is Hausdorff and $\mathcal{B}$ forms a basis, then $X$ is locally compact iff for all $x \in X$, there is $B \in \mathcal{B}$ such that $x \in B$ and $\overline{B}$ is compact.

→ Show that $\mathbb{Q}$ is not locally compact.

**Theo.** Let $X$ be a space. $X$ is locally compact and Hausdorff iff there is
**(3.32)** a space $Y$ such that

1. $X$ is a subspace of $Y$.

2. $Y \setminus X$ is a singleton.

3. $Y$ is compact and Hausdorff.

Further, if $Y$ and $Y'$ are two spaces satisfying the above, they are homeomorphic with a homeomorphism whose restriction to $X$ is the identity map.


**Proof** $\to$ We first check uniqueness up to homeomorphism. Let $h: Y \to Y'$ such that $h(x) = x$ for $x \in X$ and the single point $p \in Y$ is mapped to the single point $p' \in Y'$. $h$ is clearly bijective.

Let $U$ be open in $Y$.

- If $p \notin U$, $h(U) = U$ is clearly open. ($X$ is open in $Y$ or $Y'$ since $\{p\}$ and $\{p'\}$ are closed)

- Suppose $p \in U$. Since $C = Y \setminus U$ is closed in $Y$, it is compact in $Y$. Further, it is a compact subspace of $X$. Since $X$ is a subspace of $Y'$, $C$ is a compact subspace of $Y'$. Because $Y'$ is Hausdorff, $C$ is closed in $Y'$ so $h(U) = Y' \setminus C$ is open in $Y'$.

Therefore, $h$ is open. Similarly, $h$ is also continuous so it is a homeomorphism.

$\to$ Suppose $X$ is locally compact and Hausdorff. Take some point $\infty \notin X$. Let $Y = X \cup \{\infty\}$. Give $Y$ a topology as:

i) all sets $U$ open in $X$ are open in $Y$.

ii) all sets of the form $Y \setminus C$ are open in $Y$, where $C$ is a compact subspace of $X$.

Why is this a topology?

- $\emptyset, Y$ are open by i, ii respectively.

- $U_1 \cap U_2$ is open by i

  $(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2)$ is open by ii

  $U_1 \cap (Y \setminus C_1) = U_1 \cap (X \setminus C_1)$ is open by i

- Similarly, an arbitrary union of open sets is open

$\bigcup_\alpha U_\alpha$ is open by i

$\bigcup_\alpha (Y \setminus C_\alpha) = Y \setminus (\bigcap_\alpha C_\alpha) = Y \setminus C$ is open by ii

$\bigcup_\alpha U_\alpha \cup \bigcup_\beta (Y \setminus C_\beta) = U \cup (Y \setminus C) = Y \setminus \underbrace{(C \setminus U)}_{\text{closed}}$ is open

$$C \cap (X \setminus U)$$

Next, let us show that $X$ is a subspace of $Y$. Let $U$ be open in $Y$.

- If $U$ is of type i, $U \cap X$ is open in $X$.
- If $U = Y \setminus C$ is of type ii, $(Y \setminus C) \cap X = X \setminus C$ is open in $X$ because $C$ is closed in $X$.

(it is obvious that for any open $V$ in $X$, there is an open $\underset{\supseteq V}{U \subseteq Y}$ s.t $V = U \cap X$)

Let $\mathcal{A}$ be an open covering of $Y$. $\mathcal{A}$ must contain some open set $(Y \setminus C)$ of type ii ($\infty$ is not in any set of type i).
Consider all the elements of $\mathcal{A}$ other than $Y \setminus C$ and intersect them with $X$. This forms an open cover of $C$.
Because $C$ is compact, finitely many of them cover $C$. The corresponding elements of $\mathcal{A}$ together with $Y \setminus C$ form a finite subcover of $Y$.
Therefore, $Y$ is compact.

Let $x, y \in Y$ with $x \neq y$. If $x, y \in X$, then there are clearly disjoint open sets containing them. Otherwise, suppose $x \in X$ and $y = \infty$. Let $C$ be a compact subspace of $X$ containing a neighbourhood $U$ of $x$. Then $U$ and $Y \setminus C$ are disjoint neighbourhoods of $x$ and $y$.

$\rightarrow$ Let us now prove the converse. We claim that $X$ is locally compact and Hausdorff. Denote by $\infty$ the element of $Y \setminus X$.
$X$ is Hausdorff by Theo. 1.20 (c).
Let $x \in X$. Let $U, V$ be disjoint neighbourhoods of $x$ and $\infty$. Then $(Y \setminus V)$ is closed in $Y$, and thus compact. The required follows since $U \subseteq (Y \setminus V)$.
　　　　　　as a subspace of $X$
　　　　　　since $Y \setminus V \subseteq X$. $\blacksquare$

**Def** · If $Y$ is a compact Hausdorff space and $X$ is a proper subspace of $Y$ whose closure equals $Y$, then $Y$ is said to be a compactification of $X$. If $Y \setminus X$ is a singleton, $Y$ is called a one-point compactification.

Compactification

In Theo 3.32, if $X$ is not compact, $Y$ is its one-point compactification.

→ Show that the one-point compactification of $\mathbb{R}^2_{\wedge}$ (or $\mathbb{C}$) is homeomorphic to $S^2$. $\mathbb{C} \cup \{\infty\}$ is called the Riemann sphere or extended complex plane. If $N$ is the north pole of $S^2$, $S^2 \setminus \{N\} \cong \mathbb{C}$.

(by the stereographic projection)

**Theo.** Let $X$ be a Hausdorff space. Then $X$ is locally compact iff given
**(3.33)** $x \in X$ and a neighbourhood $U$ of $x$, there exists a neighbourhood $V$ of $x$ such that $\overline{V}$ is compact and $\overline{V} \subseteq U$

**Proof.** The backward direction is obvious since $x \in V \subseteq \overline{V}$
To prove the converse, let $U$ open and $x \in U$. Let $Y$ be the one-point compactification of $X$ and $C = Y \setminus U$.
Then $C$ is compact in $Y$. Therefore, there exist open sets $V \ni x$ and $W \supseteq C$ with $\overline{V} \cap W = \emptyset$.
But $\overline{V}$ is compact in $Y$ (and thus $X$), and $\overline{V} \cap C = \emptyset \Rightarrow \overline{V} \subseteq U$ as required. $\blacksquare$

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**Def.** A space $X$ is said to have a countable basis at $x \in X$ if there is a countable collection $\mathcal{B}$ of neighbourhoods of $x$ such that any neighbourhood of $x$ contains an element of $\mathcal{B}$.
A space that has a countable basis at each of its points is said to be first countable.

First countable

For example, any metrizable space is first countable — consider
$$\{ B_d (x, 1/n) : n \in \mathbb{N} \}$$

**Theo.** Let X be a topological space.

**(3.34)** a) Let $A \subseteq X$. If there is a sequence of points in A that converges to $x \in X$, then $x \in \overline{A}$. The converse holds if X is first countable.

b) Let $f: X \to Y$ be continuous. If $(x_n)$ be a sequence of points in X that converges to $x$, then $f(x_n) \to f(x)$. The converse holds if X is first countable.

**Proof.** Recall that we have proved the above for metrizable X by considering $B_d(x, 1/n)$.

Here, just consider
$$B_n = U_1 \cap \cdots \cap U_n \text{ instead,}$$
where $\{U_n\}$ forms a countable basis at $x$.

The proof is relatively straightforward — let $x_n \in B_n \cap A$ for each n.

For (b), show that $f(\overline{A}) \subseteq \overline{f(A)}$. $\square$

**Def** A topological space X is said to be <span style="color:green">second countable</span> if it has a countable basis (for the topology).

Second countable

**Prop.** Any second countable space is first countable.

**(3.35)**

The proof is direct.

For example, $\mathbb{R}$ is second countable — consider $\{ (a,b) : a, b \in \mathbb{Q}\}$.

Similarly, $\mathbb{R}^n$ is second countable as well.

$\mathbb{R}^\omega$ under the product topology is second countable as well.

$$\left\{ \prod_{n \in \mathbb{N}} U_n : U_n = (a,b) \text{ for } a, b \in \mathbb{Q} \text{ for finitely many } n \text{ and } \mathbb{R} \text{ otherwise} \right\}$$

$\hookrightarrow$ Why is this set countable?

→ Show that $\mathbb{R}^\omega$ under the uniform topology is not second countable.

**Theo.** A subspace of a first (resp. second) countable space is first (second)
**(3.36)** countable and a countable product of first (resp second) countable
spaces is first (second) countable.

**Proof** We prove the above for the second countable case. If $X$ has a countable basis
$\mathcal{B}$, then $\{B \cap A : B \in \mathcal{B}\}$ is a countable basis for $A \leq X$ as a subspace.
If $X_i$ has countable basis $\mathcal{B}_i$, then
$$\{\textstyle\prod U_i : U_i \in \mathcal{B}_i \text{ for finitely many } i \text{ and } U_i = X_i \text{ otherwise}\}$$
forms a countable basis of $\prod_i X_i$. ∎

Recall that a subset $A$ of a space $X$ is dense in $X$ iff $\overline{A} = X$.

**Theo** Let $X$ be a second countable space.
**(3.37)** (a) Every open covering of $X$ contains a countable subcover.
(b) $X$ has a countable dense subset.

**Proof.** (a) Let $\mathcal{A}$ be an open covering of $X$. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis
of $X$. For each $n$, if possible, choose an $A_n \supseteq B_n$. Let $\mathcal{A}'$ be the collection
of these $A_n$. It is clearly countable. Further, it covers $X$. Given $x \in X$, let
$A \in \mathcal{A}$ such that $x \in A$ and $B_n \supseteq A$ be a basis element. Then $x \in A_n$, proving the
claim.
(b) For each $B_n$, let $x_n \in B_n$. Let $D = \{x_n : n \in \mathbb{N}\}$. Then $D$ is a countable
dense subset since any basis element of $X$ (and so any open set)
intersects $D$. ∎

**Def.**

Lindelöf space

A space for which every open cover contains a countable subcover is called a **Lindelöf space**.

Separable space

A space having a countable dense subset is said to be **separable**.

**Lecture 23** - 24/03/21    Separation Axioms

Obviously, any compact set is Lindelöf.

→ Show that $\mathbb{R}_\ell$ satisfies all countability axioms except the second.

The product of two Lindelöf spaces need not be Lindelöf.
→ Show that $\mathbb{R}_\ell$ is Lindelöf but $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not.
    Hint: Consider $\mathbb{R}_\ell^2 \setminus \{(x,-x) . x \in \mathbb{R}_\ell\}$.

A subspace of a Lindelöf space need not be Lindelöf either.
→ Consider the ordered square $I_0^2 = [0,1]^2$ (under the dictionary order). Show that $I_0^2$ is compact (and Lindelöf) but the subspace $A = I_0 \times (0,1)$ is not Lindelöf.

**Def.**

Regular

Normal

Suppose that one point sets are closed in X. Then X is said to be **regular** if for each $x \in X$ and closed B disjoint from $x$, there exist disjoint open sets containing $x$ and B.

The space is said to be **normal** if for disjoint closed A,B, there exist disjoint open sets containing A and B.

Observe that any normal space is regular and any regular space is Hausdorff.

**Lemma.**
**(3.38)** Let one-point sets be closed in X.
a) X is regular iff given $x \in X$ and a nbd. U of $x$, there exists a neighbourhood V of $x$ such that $\overline{V} \subseteq U$.
b) X is normal iff given a closed $A \subseteq X$ and open $U \supseteq A$, there is an open set $V \supseteq A$ with $\overline{V} \subseteq U$.

**Proof**
a) (Forward) Let $B = X \setminus U$ be closed. There exist disjoint open V and W containing $x$ and B. Then $\overline{V} \cap B = \emptyset$. Therefore, $\overline{V} \subseteq U$.

(Backward) Let $x \in X$ and $B$ disjoint from $\{x\}$ be closed. Let $U = X \setminus B$. Let $V$ be a nbd. of $x$ such that $\bar{V} \subseteq U$. Then $V$ and $X \setminus \bar{V}$ are disjoint open sets containing $x$ and $B$ respectively. Therefore, $X$ is regular.

The argument for (b) is nearly identical, taking $A$ instead of $\{x\}$. □

**Theo.** A subspace of a regular space is regular. A product of regular spaces is regular.
**(3.39)**

**Proof.** Let $X$ be regular and $Y \subseteq X$. One-point sets are closed in $Y$. Let $x \in Y$ and $B$ a closed subset of $Y$ disjoint from $\{x\}$. Then $\bar{B} \cap Y = B$. Therefore, $x \notin \bar{B}$. Using regularity, let $U, V$ be disjoint open sets containing $x$ and $\bar{B}$. Then $(U \cap Y)$ and $(V \cap Y)$ are disjoint open sets of $Y$ containing $x$ and $B$.

Let $(X_\alpha)$ be a family of regular spaces and $X = \prod_\alpha X_\alpha$. $X$ is Hausdorff, so singletons are closed in $X$. Let $x \in X$ and $U$ be a neighbourhood of $x$. Let $\prod_\alpha U_\alpha$ be a basis element containing $x$ and contained in $U$. For each $\alpha$, let $V_\alpha$ be a nbd. of $x_\alpha$ such that $\bar{V}_\alpha \subseteq U_\alpha$. If $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$. Then $V = \prod_\alpha V_\alpha$ is a neighbourhood of $x$ in $X$. Since $\bar{V} = \prod_\alpha \bar{V}_\alpha$, $x \in \bar{V} \subseteq U$, so $X$ is regular. □
(by Theo 2.9)

$\to$ Show that $\mathbb{R}_K$ is Hausdorff but not regular
Hint: Consider $0$ and $K$.

$\to$ Show that $\mathbb{R}_\ell$ is normal.

**(3.40)**
**Theo.** Any second countable regular space is normal
**Proof.** Let $X$ be regular with countable basis $\mathcal{B}$. Let $A$ and $B$ be disjoint closed subsets of $X$. Each $x \in A$ has a nbd. $U$ disjoint from $B$. Choose a nbd. $V$ of $x$ such that $\bar{V} \subseteq U$. Then, choose an element of $\mathcal{B}$ containing $x$ and contained in $V$.

This gives a countable covering $(U_n)$ of $A$ by open sets whose closures do not intersect $B$. Similarly, choose a countable covering $(V_n)$ of $B$. $\bigcup U_n$ and $\bigcup V_n$ are open sets containing $A$ and $B$, but need not be disjoint. For each $n$, let
$$U_n' = U_n \setminus \bigcup_{i=1}^{n} \bar{V}_i \qquad \text{and} \qquad V_n' = V_n \setminus \bigcup_{i=1}^{n} \bar{U}_i.$$

Each $U_i'$ and $V_i'$ is open. Also, $(U_n')$ covers $A$ because for any $x \in A$, $x \in U_n$ for some $n$ but $x \notin \overline{V_i}$ for $1 \le i \le n$.

The open sets

$$U' = \bigcup U_n' \quad \text{and} \quad V' = \bigcup V_n'$$

are disjoint. It is easy to show that $U'$ and $V'$ are disjoint. ∎

**(3.41)**
**Theo.** Any metrizable space is normal.

**Proof.** Let $X$ be metrizable with metric $d$. Let $A, B$ be disjoint closed subsets of $X$. For each $a \in A$, choose $\varepsilon_a$ such that $B_d(a, \varepsilon_a) \cap B = \emptyset$. Choose $\varepsilon_b$ similarly. Then, let

$$U = \bigcup_{a \in A} B_d(a, \varepsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B_d(b, \varepsilon_b/2).$$

It is easy to show that these are disjoint (they are clearly open and contain $A, B$), completing the proof. ∎

**Theo.** Any compact Hausdorff space is normal.
**(3.42)**

Left as exercise.

**Theo.** [Urysohn Lemma]
**(3.43)** Let $A, B$ be disjoint closed subsets of $X$. $X$ is normal iff for a closed interval $[a,b]$ in the real line, there exists a continuous map $f: X \to [a,b]$ such that $f(x) = a$ for every $a \in A$ and $f(x) = b$ for every $b \in B$.