Geometric Topology

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§1. Singular Homology Theory

Definition 1.1. A *p-simplex* in $\mathbb{R}^n$ is the convex hull of $p+1$ points $\{x_0, \ldots, x_p\}$ such that $\{x_1 - x_0, \ldots, x_p - x_0\}$ forms a linearly indepedent set. The points (x_i) are then called the *vertices* of the simplex. Further, if the vertices of a simplex are given in a specific order, it is called an *ordered simplex*.

The following equivalent formulation is easily proved.

Lemma 1.1. Let $x_0, \ldots, x_p \in \mathbb{R}^n$. The following are equivalent.

- $\{x_1 - x_0, \ldots, x_p - x_0\}$ is a linearly independent set.
- If $\sum_i s_i x_i = \sum_i t_i x_i$ and $\sum_i s_i = \sum_i t_i$, then for each i, $s_i = t_i$.

It follows that any point in a p-simplex can be represented uniquely as a convex combination of the points forming it.
Now, let s be an ordered p-simplex with vertices $x_0, \ldots, x_p$. Define

$$\sigma_p = \{(t_0, \ldots, t_p) \in \mathbb{R}^p : \sum_i t_i = 1 \text{ and for each } i, t_i \geq 0\}$$

and the function $f : \sigma_p \to s$ given by $(t_0, \ldots, t_p) \mapsto \sum_i t_i x_i$. Since f is continuous, and σ_p and s are compact and Hausdorff, f is in fact a homeomorphism.
Viewing σ_p as a simplex with vertices $e_1, \ldots, e_p$, it is known as the *standard p-simplex* with natural ordering.

Definition 1.2. Let X be a topological space. A *singular p-simplex* is a continuous map $\phi : \sigma_p \to X$. If ϕ is a singular p-simplex and $0 \leq i \leq p$ is an integer, define $\partial_i \phi$, the *ith face of* ϕ, by the singular $(p-1)$-simplex

$$\partial_i \phi(t_0, \ldots, t_{p-1}) = \phi(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-1}).$$

If $f : X \to Y$ is a continuous map and ϕ is a singular p-simplex in X, we can define a singular p-simplex in Y by $f_\#(\phi) = (f \circ \phi)$.
Recall that an abelian group G is a *free group* if there exists $A \subseteq G$ such that any $g \in G$ can be uniquely represented as $g = \sum_{x \in A} n_x x$, where $n_x = 0$ for all but finitely many x. In this context, A is said to be a *basis* for G.