MA 412: Complex Analysis

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§1. Introduction

1.1. Some basic definitions

Consider the equation $X^2 + 1 = 0$. Clearly, this equation has no roots over $\mathbb{R}$. Consider the set

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} = \mathbb{R}^2,$$

and define addition and subtraction over $\mathbb{C}$ as

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b) \cdot (c, d) &= (ac - bd, ad + bc).\end{aligned}$$

It is easy to show that $(\mathbb{C}, +, \cdot)$ is a field with additive identity $(0, 0)$ and multiplicative identity $(1, 0)$. Further observe that $\mathbb{R}$ is a subfield of $\mathbb{C}$ – consider the field homomorphism $\mathbb{R} \to \mathbb{C}$ defined by $a \mapsto (a, 0)$.
Now, we denote $\iota = (0, 1)$, and write (a, b) as $a + b\iota$.

Observe that the equation $X^2 + 1 = 0$ *does* have roots over $\mathbb{C}$ since it can be written as $(X + \iota)(X - \iota)$. For the sake of completeness, we also note that the multiplicative identity of $a + \iota b$ is

$$\frac{1}{a + \iota b} = \frac{a - \iota b}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\iota.$$

When writing $z = a + b\iota$ where $a, b \in \mathbb{R}$, we write $a = \Re z$ (the real part of z) and $b = \Im z$ (the imaginary part of z). We also define the absolute value $|z| = (a^2 + b^2)^{1/2}$ of z, and the *conjugate* $\overline{z} = a - \iota b$ of z. We clearly have

$$\begin{aligned}z\overline{z} &= |z|^2\\ \Re z &= \frac{z + \overline{z}}{2}\\ \Im z &= \frac{z - \overline{z}}{2\iota}.\end{aligned}$$

It is easy to check that

$$\overline{z + w} = \overline{z} + \overline{w} \text{ and } \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

We also have

$$\begin{aligned}\left|\frac{z}{w}\right| &= \frac{|z|}{|w|}\\ |\overline{z}| &= |z|.\end{aligned}$$

Exercise 1.1. Check that the set

$$M = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R}$$

with matrix addition and multiplication is a field isomorphic to $\mathbb{C}$.

To close out the tedious part of things, we have

$$\begin{aligned}|z + w|^2 &= |z|^2 + |w|^2 + 2\Re(z\overline{w})\\ |z + w| &\leq |z| + |w| \qquad (1.1)\end{aligned}$$

Equation (1.1) is referred to as the *triangle inequality*.

1.2. Polar representations and roots

Consider $z = x + \iota y \in \mathbb{C}$. We may then define

$$x = r\cos\theta \qquad y = r\sin\theta,$$

where $|z| = r$ and the angle θ is called the *argument* of z as is denoted $\theta = \arg z$. We typically restrict θ to $(-\pi, \pi]$. We denote $\operatorname{cis}\theta = \cos\theta + \iota\sin\theta$. Therefore, we have

$$z = |z|\operatorname{cis}(\arg z).$$

Observe that rather conveniently,

$$\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2).$$

Therefore, inductively,

$$z_1 z_2 \cdots z_n = \left(\prod_i |z_i|\right) \cdots r_n \operatorname{cis}\left(\sum_i \arg z_i\right).$$

In particular,

$$z^n = r^n \operatorname{cis}(n\theta)$$

for any $n > 0$. If $z \neq 0$ (equivalently, $r \neq 0$), the above holds for all $n \in \mathbb{Z}$.
In the case where $r = 1$, we have

$$(\cos\theta + \iota\sin\theta)^n = \cos(n\theta) + \iota\sin(n\theta) \tag{1.2}$$

Equation (1.2) is referred to as *de Moivre's formula*.

Let us consider the equation $z^n = a$. This equation has n roots of the form

$$z = |a|^{1/n}\operatorname{cis}\left(\frac{2k\pi + \arg z}{n}\right)$$

for $k = 0, 1, \ldots, n-1$.
A *line* in the complex plane is a set of the form

$$L = \{z = a + tb : t \in \mathbb{R}\},$$

for some fixed $a, b \in \mathbb{C}$, where b is a *directional* vector whose absolute value may be assumed to be 1. Since $b \neq 0$, we equivalently have

$$L = \{z : \Im\left(\frac{z-a}{b}\right) = 0\}.$$

We can also define the half-planes

$$H_a = \{z : \Im\left(\frac{z-a}{b}\right) > 0\}$$
$$K_a = \{z : \Im\left(\frac{z-a}{b}\right) < 0\}.$$

Note that $H_a = a + H_0$, where the addition is Minkowski addition:

$$H_a = \{a + z : z \in H_0\}.$$

1.3. The extended plane

Define $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and let $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. We shall show a bijection from $\mathbb{C}_\infty$ to S.
Let $N = (0, 0, 1)$ be the 'north pole' of S, and orient $\mathbb{C}$ (as $\mathbb{R}^2$) in the horizontal plane in a manner such that $\mathbb{C}$ cuts S along the equator. For $z = x + \iota y \in \mathbb{C}$, let us define the corresponding point $Z = (x_1, x_2, x_3) \in S$. We shall draw a line connecting z to N, and let Z be the point of intersection (other than N) of this line with S. Finally, we shall map ∞ to N.
Let us define this more explicitly. The line through N and z is

$$L = \{tN + (1-t)z : t \in \mathbb{R}\}.$$

Then, letting $z = (x, y, 0)$, we have

$$t^2 + (1-t)^2|z|^2 = 1.$$

So,

$$|z|^2 = \frac{1-t^2}{(1-t)^2} = \frac{1+t}{1-t}$$

and

$$t = 1 - \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Therefore, we map z to

$$Z = \left(\frac{2\Re z}{|z|^2+1}, \frac{2\Im z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}. \right) \in S.$$

Based on this, we can define a distance metric between points in $\mathbb{C}_\infty$. For $z, z' \in \mathbb{C}_\infty$ mapping to $Z, Z' \in S$, we let $d(z, z')$ be the Euclidean distance between Z, Z' in $\mathbb{R}^3$. More explicitly,

$$\begin{aligned} d(z,z')^2 &= (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 \\ &= 2 - 2(x_1x_1' + x_2x_2' + x_3x_3') \\ &= \frac{2|z - z'|}{\left((|z|^2+1)(|z'|^2+1)\right)^{1/2}} \end{aligned}$$

when $z, z' \in \mathbb{C}$ and if $z' = \infty$ (so $Z' = (0, 0, 1)$), we have

$$d(z, z') =$$

This correspondence between points of S and $\mathbb{C}_\infty$ is called the *stereographic projection*.

Exercise 1.2. If P is a plane in $\mathbb{R}^3$ and $\Lambda = P \cap S$ is a circle on S, show that the projection of Λ on $\mathbb{C}$ under the stereographic projection is a circle as well (possibly a circle of infinite radius, namely a line).

1.4. Power series

In this section, we begin discussing convergence of series in $\mathbb{C}$ and related properties.

Definition 1.1. If $a_n \in \mathbb{C}$ for every $n \geq 0$, the series $\sum_{n=0}^\infty a_n$ is said to *converge* to z iff for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{n=0}^m a_n - z \right| < \epsilon$$

for all $m \geq N$.
The series $\sum_{n=0}^\infty a_n$ is said to converge *absolutely* if $\sum_{n=0}^\infty |a_n|$ converges.

Theorem 1.1. $\mathbb{C}$ is complete. That is, every Cauchy sequence in $\mathbb{C}$ is convergent.

Proof. Suppose $\{x_n + \iota y_n\}$ is a Cauchy sequence in $\mathbb{C}$, where $x_n, y_n \in \mathbb{R}$ for each n. We then have the existence of $N \in \mathbb{N}$ such that for all $m, k > N$, $|(x_m - x_k) + \iota(y_m - y_k)| < \epsilon$. Consequently, $|x_m - x_k| < \epsilon$ and $|y_m - y_k| < \epsilon$. However, since $\mathbb{R}$ is complete, this implies that (x_n) and (y_n) are convergent, completing the proof. ■

Theorem 1.2. If $\sum a_n$ converges absolutely, $\sum a_n$ converges.

Proof. Let $\epsilon > 0$, $z_n = \sum_{i=0}^{n} a_i$, and $S_n = \sum_{i=0}^{n} |a_i|$. Because $\mathbb{C}$ is complete, it suffices to show that (z_n) is Cauchy. Since $\sum |a_n|$ is convergent, there exists $N \in \mathbb{N}$ such that $|S_m - S_k| < \epsilon$ for all $m, k > N$. Supposing $m > k$, we have

$$S_m - S_k = \sum_{i=k+1}^{m} |a_i|.$$

So,

$$\begin{aligned} |z_m - z_k| &= \left| \sum_{i=k+1}^{m} a_i \right| \\ &\geq \sum_{i=k+1}^{m} |a_i| < \epsilon, \end{aligned}$$

completing the proof. ■

Exercise 1.3. Show that $\sum_{n=0}^{\infty} z_n$ converges iff $|z| < 1$.

Theorem 1.3. For a given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, define the number of R $(0 \leq R \leq \infty)$ by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Then,

(a) If $|z - a| < R$, the series converges absolutely.

(b) If $|z - a| > R$, the terms of the series become unbounded and the series diverges.

(b) If $0 < r < R$, the series converges uniformly on the set $\{z : |z - a| \leq r\}$.

This R is referred to as the *radius of convergence* of the power series.

Proof.

(a) We assume without loss of generality that $a = 0$. If $|z| < R$, there exists r with $|z| < r < R$. By the definition of R, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{R} - \epsilon < \sup_{k\geq n} |a_k|^{1/k} < \frac{1}{R} + \epsilon$$

for all $n > N$. If we take $\epsilon = 1/r - 1/R$, it follows that $|a_n|^{1/n} < 1/r$ for all $n > N$. That is, for all $n > N$, $|a_n| < 1/r^n$ and so

$$|a_n z^n| < \left(\frac{|z|}{r}\right)^n.$$

Therefore, $\sum_{n=N}^{\infty} a_n z^n$ is dominated by $\sum_{n=N}^{\infty} (|z|/r)^n$. Now however, we can just use the result of Exercise 1.3 to conclude absolute convergence since $|z|/r < 1$.

(b) Let $|z| > R$ and choose r with $|z| > r > R$. For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{R} - \epsilon < \sup_{k \geq n} |a_k|^{1/k} \text{ for all } n > N.$$

Choosing $\epsilon = 1/R - 1/r$,

$$|a_n|^{1/n} > 1/r$$

for infinitely many $n \in \mathbb{N}$. It follows that $|a_n z^n| > (|z|/r)^n$ for infinitely many $n \in \mathbb{N}$. Since $|z|/r > 1$, these terms become unbounded and therefore the series diverges.

(c) Now, suppose $r < R$ and choose ρ such that $r < \rho < R$. Similar to the argument in (a), we get that

$$|a_n| < \frac{1}{\rho^n} \text{ for all } n \geq N.$$

If $|z| \leq r$, $|a_n z^n| \leq (r/\rho)^n$ and $r/\rho < 1$. The Weierstrass M-test then gives that the power series converges uniformly on $\{z : |z| \leq r\}$. ■

It should be noted that we cannot conclude anything when $|z - a| = R$.

Theorem 1.4. If $\sum a_n(z-a)^n$ is a power series with radius of convergence R, then if it exists,

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R.$$

Proof. Again, assume that $a = 0$ and let $\alpha = \lim |a_n/a_{n+1}|$, which we assume exists. Suppose that $|z| < \alpha$ and take $r \in \mathbb{R}$ such that $|z| < r < \alpha$. For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\alpha - \epsilon < \left| \frac{a_n}{a_{n+1}} \right| < \alpha + \epsilon.$$

Taking $\epsilon = \alpha - r$, $|a_n/a_{n+1}| > r$ for all $n \geq N$. Let $B = |a_N| r^N$. Then,

$$a_{N+1} r^{N+1} = |a_{N+1}| r \cdot r^N < |a_N| r^N = B.$$

Similarly, we get that $|a_n| r^n < B$ for all $n \geq N$. Therefore,

$$|a_n z^n| < B \left(\frac{|z|}{r} \right)^n$$

for all $n \geq N$. Thus, the sequence converges absolutely since $|z| < r$.
Since $r < \alpha$ was arbitrary, this implies that $\alpha \leq R$.

On the other hand, if $|z| > \alpha$, take $r \in \mathbb{R}$ such that $|z| > r > \alpha$. Taking $\epsilon = r - \alpha$, we get $N \in \mathbb{N}$ such that

$$\left| \frac{a_n}{a_{n+1}} \right| < r$$

for all $n \geq N$. Letting $B = |a_N| r^N$ again, we once more obtain that $|a_n| r^n > B$ for all $n \geq N$. This gives that

$$|a_n z^n| > B \left(\frac{|z|}{r} \right)^n$$

for all $n \geq N$, and since $|z| > r$, the sequence diverges (we may assume that $B \neq 0$ by making N larger if required to ensure that $a_N \neq 0$ – if this is not possible, the problem is trivial since it means that (a_n) is eventually 0). Since the choice of r was arbitrary, this implies that $R \leq \alpha$, completing the proof. ■

Now, consider the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The radius of convergence of this series is ∞. So, it converges for any complex number z, and convergence is uniform on every compact subset of $\mathbb{C}$.
The above defines a function $\exp : \mathbb{C} \to \mathbb{C}$.
We also denote $e^z = \exp(z)$.

Definition 1.2 (Differentiability)**.** If G is an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$, then f is said to be *differentiable* at a point $a \in G$ if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If it exists, the value of this limit is denoted $f'(a)$ and is called the *derivative* of f at a.

If f is differentiable at each point of G, we say that f is differentiable on G. Note that if f is differentiable on G, then $f' : G \to \mathbb{C}$ is a function. If f' is continuous, f is said to be *continuously differentiable*.

Theorem 1.5. If $f : G \to \mathbb{C}$ is differentiable at a point $a \in G$, f is continuous at a.

Proof. The proof of this is direct:

$$\begin{aligned} \lim_{z \to a} |f(z) - f(a)| &= \left(\lim_{z \to a} \frac{|f(z) - f(a)|}{|z - a|} \right) \cdot \lim_{z \to a} |z - a| \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

∎

Definition 1.3. A function $f : G \to \mathbb{C}$ is said to be *analytic* if f is continuously differentiable on G.

Let f, g be analytic on G and Ω respectively, and suppose that $f(G) \subseteq \Omega$. Then, $g \circ f$ is analytic on G and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z)$$

for all $z \in G$. This is called the *chain rule*.

We shall show later that if f is differentiable then its derivative is continuous, and so f is analytic.

Theorem 1.6. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence $R > 0$. Then

(a) For each $k \geq 1$, the series

$$\sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (z-a)^{n-k}$$

has radius of convergence R.

(b) The function f is infinitely differentiable on $B(a, R)$ (the open ball of radius R centered at a), and further, $f^{(k)}(z)$ is given by the series in (a) for all $k \geq 1$ and $|z - a| < R$.

(c) For $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$.

Proof. Assume that $a = 0$.

(a) Note that it suffices to prove the result for $k = 1$ (Why?). To show this, it is enough to show that

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |na_n|^{1/(n-1)}$$

First, it is not difficult to show that $\lim_{n\to\infty} n^{1/(n-1)} = 1$. It may be shown that for any sequences $(c_n), (d_n)$ in $\mathbb{R}$ where $c_n \geq 0$, if $\lim c_n = c$ and $\limsup d_n = d$, then $\limsup c_n d_n = cd$. Therefore, we are done if we show that $\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/(n-1)}$.

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + z \sum_{n=0}^{\infty} a_{n+1} z^n.$$

Let R' be the radius of convergence of $\sum_{n=0}^{\infty} a_{n+1} z^n$. We want to show that $R' = R$.
If $|z| < R'$, then

$$\sum |a_n z^n| \leq |a_0| + |z| \sum_{n=0}^{\infty} |a_{n+1} z^n| < \infty,$$

so $R' \leq R$. On the other hand, if $|z| < R$ and $z \neq 0$,

$$\sum |a_{n+1} z^n| < \frac{1}{|z|} \left(\sum |a_n z^n| + |a_0| \right) < \infty,$$

so $R \leq R'$ and we are done.

(b) Once again, it suffices to prove the result for $k = 0$. For $|z| < R$ and $g(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$,

$$s_n(z) = \sum_{k=0}^{n} a_k z^k \text{ and } R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k,$$

fix a point $w \in B(0, R)$ and r such that $|w| < r < R$. We wish to show that $f'(w)$ exists and is equal to $g(w)$. Let $\delta > 0$ be arbitrary with $\overline{B(w, \delta)} \subseteq B(0, r)$. Letting $z \in B(w, \delta)$, we have

$$\begin{aligned}
\frac{f(z) - f(w)}{z - w} - g(w) &= \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) + s_n'(w) - g(w) + \frac{R_n(z) - R_n(w)}{z - w} \\
&= \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) + s_n'(w) - g(w) + \frac{1}{z - w} \sum_{k=n+1}^{\infty} a_k (z^k - w^k) \\
&= \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) + s_n'(w) - g(w) + \sum_{k=n+1}^{\infty} a_k \frac{z^k - w^k}{z - w}.
\end{aligned}$$

We have

$$|z^k - w^k| = |z - w||z^{k-1} + z^{k-2} w + \cdots + w^{k-1}| \leq |z - w| k r^{k-1}.$$

Therefore,

$$\left| \frac{R_n(z) - R_n(w)}{z - w} \right| \leq \sum_{k=n+1}^{\infty} |a_k| k r^{k-1}.$$

Since $r < R$, $\sum_{k=1}^{\infty} |a_k| k r^{k-1}$ converges and so for any $\epsilon > 0$, there exists $N_1 \in N$ such that

$$\left| \frac{R_n(z) - R_n(w)}{z - w} \right| < \epsilon/3.$$

Since $\lim s_n'(w) = g(w)$, there exists $N_2 \in \mathbb{N}$ such that

$$|s_n'(w) - g(w)| < \epsilon/3$$

for $n \geq N_2$. Choose $n \geq \max(N_1, N_2)$. Then, there exists $\delta < 0$ such that whenever $0 < |z - w| < \delta$,

$$\left| \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) \right| < \epsilon/3.$$

Putting all these together, we get the desideratum.

(c) This is straightforward using the explicit expression for $f^{(k)}(a)$. ■

If the series $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence $R > 0$, then f is analytic on $B(a, R)$. Therefore, exp is analytic on $\mathbb{C}$.

1.5. Cauchy-Riemann Equations

Let $f : G \to \mathbb{C}$ be analytic and let

$$u(x, y) = \Re(f(x + \iota y)), v(x, y) = \Re(f(x + \iota y))$$

for $x + \iota y \in G$. Let us evaluate the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}.$$

in two different ways.
First, if we let $h \to 0$ through real values, we get

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + \iota \frac{\partial v}{\partial x}(x, y).$$

Along the imaginary axis, we get

$$f'(z) = -\iota \frac{\partial u}{\partial y}(x, y) + \frac{\partial v}{\partial y}(x, y).$$

Therefore,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Supposing that u and v have continuous second derivative (we shall later show that they are infinitely differentiable), we have that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x}.$$

Therefore, since the second derivatives are continuous,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.3}$$

A function u with continuous second partial derivatives satisfying Equation (1.3) is said to be *harmonic*. Similarly, v is also harmonic.

Theorem 1.7. Let u, v be real-valued functions defined on an open connected set (a *region*) G and suppose that they have continuous second partial derivatives. Then, $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + \iota v(z)$ is analytic iff u and v satisfy the Cauchy-Riemann equations.

Proof. We have already shown the forward direction.
For the other direction, let $z = x + \iota y \in G$ and $B(z, r) \subseteq G$. Let $h = s + \iota t \in B(0, r)$. Our goal is to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(z + h) - f(z) - f'(z)h}{h} \right| < \epsilon$$

for all $h \in B(0, \delta)$ for some $f'(z) \in \mathbb{C}$. Note that

$$u(x+s, y+t) - u(x, y) = \big(u(x+s, y+t) - u(x, y+t)\big) + \big(u(x, y+t) - u(x, y)\big)\,.$$

Now, for fixed $t \in (-r, r)$, $s \mapsto u(x+s, y+t)$ is a differentiable function on $(-r, r)$. We apply the mean value theory to conclude that there exist $s_1, t_1 \in (-r, r)$ for each $s + \iota t \in B(0, r)$ such that $|s_1| < |s|$, $|t_1| < |t|$, and

$$\begin{aligned} u(x+s, y+t) - u(x, y+t) &= u_x(x+s_1, y+t)s \\ u(x, y+t) - u(x, y) &= u_y(x, y+t_1)t. \end{aligned}$$

Now, let

$$\varphi(s,t) = \big(u(x+s, y+t) - u(x, y)\big) - \big(u_x(x,y)s + u_y(x,y)t\big)\,.$$

We get that

$$\varphi(s,t) = \big(su_x(x+s_1, y+t) - su_x(x,y)\big) + \big(tu_y(x, y+t_1) - tu_y(x,y)\big)\,.$$

So,

$$\frac{\varphi(s,t)}{s+\iota t} = \frac{s}{s+\iota t}\big(u_x(x+s_1, y+t) - u_x(x,y)\big) + \frac{t}{s+\iota t}\big(u_y(x, y+t_1) - u_y(x,y)\big)$$

and on taking the limit of both sides as $s + \iota t \to 0$, we can use the fact that $|s| \le |s + \iota t|$, $|t| \le |s + \iota t|$, $|s_1| < |s|$, $|t_1| < t$, and the continuity of u_x, u_y, to conclude that

$$\lim_{s+\iota t \to 0} \frac{\varphi(s,t)}{s+\iota t} = 0.$$

Therefore,

$$u(x+s, y+t) - u(x,y) = u_x(x,y)s + u_y(x,y)t + \varphi(s,t).$$

We get a similar equation for v as well, with a function ψ (in place of φ). Combining the two,

$$\begin{aligned} \frac{f(z+s+\iota t) - f(z)}{s+\iota t} &= \frac{u(x+s,y+t) - u(x,y)}{s+\iota t} + \iota\frac{v(x+s,y+t) - v(x,y)}{s+\iota t} \\ &= \frac{su_x(x,y) + tu_y(x,y) + \varphi(s,t) + \iota\big(sv_x(x,y) + tv_y(x,y) + \psi(s,t)\big)}{s+\iota t} \\ &= \frac{u_x(x,y)(s+\iota t) + \iota v_x(x,y)(s+\iota t) + \varphi(s,t) + \iota\psi(s,t)}{s+\iota t}, \end{aligned}$$

where we used Cauchy-Riemann equations in the final step and thus,

$$\lim_{s+\iota t\to 0} \frac{f(z+s+\iota t) - f(z)}{s+\iota t} = u_x(x,y) + \iota v_x(x,y),$$

completing the proof. Since u_x and v_x are continuous, f' is continuous and f is analytic. ∎

A next question is: given some u such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

when does there exist harmonic v such that $u + \iota v$ is analytic? Such a v is referred to as a *harmonic conjugate* of u. It turns out that the answer is not always. Indeed, $u(x,y) = \log((x^2+y^2)^{1/2})$ on $\mathbb{C} \setminus \{0\}$, despite being harmonic, does not have a harmonic conjugate.

Theorem 1.8. Let G be either the entirety of $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is a harmonic function, then u has a harmonic conjugate.

Proof. Let $G = B(0, R)$ for some $0 < R \leq \infty$ and let $u : G \to \mathbb{R}$ be analytic. Define

$$v(x, y) = \int_0^y u_x(x, t)\,\mathrm{d}t + \varphi(x)$$

so that $u_x = v_y$. We shall determine φ such that $v_x = -u_y$. Differentiating with respect to x, we get

$$\begin{aligned} v_x(x, y) &= \int_0^y u_{xx}(x, t)\,\mathrm{d}t + \varphi'(x) \\ &= -\int_0^y u_{yy}(x, t)\,\mathrm{d}t + \varphi'(x) \\ &= -u_y(x, y) + u_y(x, 0) + \varphi'(x). \end{aligned}$$

Therefore, $\varphi'(x) = -u_y(x, 0)$, and the function

$$v(x, y) = \int_0^y u_x(x, t)\,\mathrm{d}t - \int_0^x u_y(s, 0)\,\mathrm{d}s$$

is a harmonic conjugate of u. ■

The above proof requires that the entire segments $[(0, 0), (x, 0)]$ $[(x, 0), (x, y)]$ are contained in G, which is true when we are on a disk.

1.6. Analytic functions as mappings

Consider the two hyperbolas defined by

$$\begin{aligned} x^2 - y^2 &= c \\ 2xy &= d, \end{aligned}$$

where $c, d \neq 0$.
This gives

$$y = \pm\sqrt{\frac{-c \pm \sqrt{d^2 + c^2}}{2}}.$$

Consider the functions

$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy. \end{aligned}$$

The two hyperbolas above are mapped by this $f = u + \imath v$ to the straight lines $u = c$ and $v = d$.

Definition 1.4. A *path* in a region $G \subseteq \mathbb{C}$ is a continuous function $\gamma : [a, b] \to G$ for some interval $[a, b]$ in $\mathbb{R}$. If $\gamma'(t)$ exissts for each $t \in [a, b]$ and $\gamma' : [a, b] \to \mathbb{C}$ is continuous, then γ is

By the existence of γ', we mean that the two-sided limit

$$\lim_{h \to 0} \frac{\gamma(t + h) - \gamma(t)}{h}$$

exists for $t \in (a, b)$ and the right and left sided limits exist for $t = a, b$ respectively. This is equivalent to saying that $\Re\gamma$ and $\Im\gamma$ have derivatives.
Suppose $\gamma : [a, b] \to G$ is a smooth path and for some $t_0 \in (a, b)$, $\gamma'(t_0) \neq 0$. Then, γ has a *tangent line* at the point $z_0 = \gamma(t_0)$. This lines goes through the point z_0 in the direction of the vector $\gamma'(t_0)$, that is, the slope of the line is

$\tan(\arg \gamma'(t_0))$.

If γ_1 and γ_2 are two smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1), \gamma_2'(t_2) \neq 0$, then define the *angle* between the paths γ_1, γ_2 at z_0 to be $\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1))$.

Suppose γ is a smooth path in G and $f : G \to \mathbb{C}$ is analytic. Then, $\sigma = f \circ \gamma$ is also a smooth path and $\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t)$. Further, if t_0 is a fixed point of f with $\gamma(t_0) = z_0$,

$$\arg(\sigma'(t_0)) - \arg(\gamma'(t_0)) = \arg(f'(z_0)).$$

Letting γ_1, γ_2 be smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ with non-zero derivatives at t_1, t_2 respectively, and lt $\sigma_1 = f \circ \gamma_1, \sigma_2 = f \circ \gamma_2$. Further suppose that the two paths γ_1, γ_2 are not tangent to each other at z_0. Then,

$$\arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1)) = \arg(\sigma_2'(t_2)) - \arg(\sigma_1'(t_1)).$$

This says that the angle between two paths are preserved after applying an analytic function to both. A function f satisfying this is said to have the *angle-preserving property.*

Definition 1.5. A function $f : G \to \mathbb{C}$ which has the angle-preserving property and also has

$$\lim_{z \to a} \left| \frac{f(z) - f(a)}{z - a} \right|$$

existing is called a *conformal map.*

It turns out that a function f is a conformal map if and only if it is analytic and $f'(z) \neq 0$ for all z (How?).

Definition 1.6. A mapping of the form

$$S(z) = \frac{az + b}{cz + d}$$

is called a *linear fractional transformation.* If we further have that $ad - bc \neq 0$, then $S(z)$ is called a *Möbius transformation.*

We have

$$S'(z) = \frac{ad - bc}{(cz + d)^2}.$$

If $w = S(z)$, it is relatively simple to show that

$$z = S^{-1}(w) = \frac{dw - b}{-cw + a}.$$

Therefore, the inverse of a Möbius transformation is a Möbius transformation. The composition of two Möbius transformations is a Möbius transformation as well.
Also observe that the coefficiencts a, b, c, d for a given Möbius transformation are not unique since we can multiply them by a constant. We may also extend S to $\mathbb{C}_\infty$ with $S(\infty) = a/c$ and $S(-d/c) = \infty$.

$S(z) = z + a$ is called a *translation*, $S(z) = az$ with $a \neq 0$ is called a *dilation*, $S(z) = e^{\iota\theta} z$ is called a *rotation*, and $S(z) = 1/z$ is called the *inversion.* We shall see later that any Möbius transformation is a composition of these five types of transformations.
What are the fixed points of a Möbius transformation S? $S(z) = z$ gives

$$cz^2 + (a - d)z + b = 0.$$

Therefore, a Möbius transformation has at most two fixed points unless $S(z) = z$ for all $z \in \mathbb{C}_\infty$.

Let $a, b, c \in \mathbb{C}_\infty$ be distinct with $S(a) = \alpha$, $S(b) = \beta$, $S(c) = \gamma$. Let T be another Möbius transformation with $T(a) = \alpha$, $T(b) = \beta$, $T(c) = \gamma$. Then $T^{-1} \circ S$ has three (distinct) fixed points, and therefore $S = T$.
Therefore, any Möbius transformation is uniquely determined by its value at any three distinct points.

Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct. Define $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$S(z) = \begin{cases} \frac{(z-z_3)/(z-z_4)}{(z_2-z_3)(z_2-z_4)}, & z_2, z_3, z_4 \in \mathbb{C}, \\ \frac{z_2-z_4}{z-z_4}, & z_3 = \infty, \\ \frac{z-z_3}{z_2-z_3}, & z_4 = \infty. \end{cases}$$

In any case, $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$, and S is the only transformation having this property.

Definition 1.7. If $z_1 \in \mathbb{C}_\infty$, then (z_1, z_2, z_3, z_4) is referred to as the *cross-ratio* of z_1, z_2, z_3, z_4 and is the image of z_1 under the Möbius transformation described above, which is the unique Möbius transformation taking z_2 to 1, z_3 to 0, and z_4 to ∞.

For example, $(z_2, z_2, z_3, z_4) = 1$ and $(z, 1, 0, \infty) = z$.
If M is any Möbius transformation with $M(w_2) = 1$, $M(w_3) = 0$, $M(w_4) = \infty$, then $M(z) = (z, w_2, w_3, w_4)$ for any $z \in \mathbb{C}_\infty$.

Theorem 1.9. If z_2, z_3, z_4 are distinct points and T is any Möbius transformation, then

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4).$$

Proof. Let $S(z) = (z, z_2, z_3, z_4)$. If $M = ST^{-1}$, then

$$M(T(z_2)) = 1, \quad M(T(z_3)) = 0, \quad M(T(z_4)) = \infty.$$

Therefore, $M = (z, Tz_2, Tz_3, Tz_4)$. That is,

$$ST^{-1}z = (z, Tz_2, Tz_3, Tz_4)$$

for all $z \in \mathbb{C}_\infty$. Setting $z = Tz_1$ yields the required. ∎

Lemma 1.10. If $\{z_2, z_3, z_4\}, \{w_2, w_3, w_4\} \subseteq \mathbb{C}_\infty$, then there exists a unique Möbius transformation S with $Sz_i = w_i$ for each i.

We omit the proof of the above.

Lemma 1.11. Let $\{z_1, z_2, z_3, z_4\} \subseteq \mathbb{C}_\infty$. Then, (z_1, z_2, z_3, z_4) is real iff the four points lie on a circle.

Proof. Define $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $Sz = (z, z_2, z_3, z_4)$. We are done if we show that $S^{-1}(\mathbb{R}_\infty)$ is a circle (since a circle is uniquely determined by three distinct points on it).
Let $S^{-1}(z) = (az + b)/(cz + d)$.

First, let us show that $S^{-1}(\mathbb{R}_\infty) \subseteq \Gamma$ for a circle Γ in $\mathbb{C}_\infty$. Let $w \in S^{-1}(\mathbb{R}_\infty)$. Then, $Sw = \overline{Sw}$ so

$$\frac{aw + b}{cw + d} = \frac{\overline{aw} + \overline{b}}{\overline{cw} + \overline{d}}.$$

This gives that

$$(a\overline{c} - \overline{a}c)|w|^2 + (a\overline{d} - \overline{b}c)w + (b\overline{c} - d\overline{a})\overline{w} + (b\overline{d} - \overline{b}d) = 0. \tag{(*)}$$

If $a\overline{c}$ is real, we get that

$$\Im\left((a\overline{d} - \overline{b}c)w + b\overline{d}\right) = 0,$$

which is a circle through ∞ (a line).
If on the other hand $a\bar{c}$ is not real, then Equation ((*)) becomes

$$2\iota \underbrace{\Im(a\bar{c})}_{\alpha\neq 0} |w|^2 + (a\bar{d} - b\bar{c})w + (b\bar{c} - \bar{a}d)\overline{w} + (b\bar{d} - \bar{b}d) = 0.$$

Dividing by $2\iota\alpha$,

$$|w|^2 + \frac{(a\bar{d} - b\bar{c})w}{2\iota\alpha} + \frac{(b\bar{c} - \bar{a}d)\overline{w}}{2\iota\alpha} + \frac{(b\bar{d} - \bar{b}d)}{2\iota\alpha} = 0.$$

Since α is real,

$$\overline{\frac{(b\bar{c} - \bar{a}d)\overline{w}}{2\iota\alpha}} = \frac{(a\bar{d} - b\bar{c})w}{2\iota\alpha}$$

and

$$\frac{(b\bar{d} - \bar{b}d)}{2\iota\alpha}$$

is real. This gives

$$|w|^2 + \overline{\gamma}w + \gamma\overline{w} - \delta = 0$$

for some $\gamma \in \mathbb{C}, \delta \in \mathbb{R}$. This is equivalent to $|w + \gamma| = (|\gamma|^2 + \delta)^{1/2}$, which is the equation of a circle[1].

Letting $T = S^{-1}$ and Γ be the circle obtained in the previous part of the proof, we must now show that $T(\mathbb{R}_\infty) = \Gamma$. Since $\mathbb{R}_\infty$ is connected and compact and T is a homeomorphism, $T(\mathbb{R}_\infty)$ is a closed arc, say Γ_1, of Γ. If $\Gamma_1 \neq \Gamma$, let z_1, z_2 be the endpoints of this arc. If $T(\infty) = z_3$ which is distinct from z_1, z_2, then $\mathbb{R}_\infty \setminus \{\infty\}$ is connected but $\Gamma_1 \setminus \{z_1\}$ is disconnected, which is a contradiction. So, suppose $T(\infty) = z_1$. Then, $\mathbb{R}_\infty \setminus \{\infty, T^{-1}(z_2)\}$ is disconnected but $\Gamma_1 \setminus \{z_1, z_2\}$ is connected, yielding a contradiction once more and completing the proof. ■

Next, we give a more general version of the above.

Theorem 1.12. A Möbius transformation takes circles to circles.

Note that Lemma 1.11 follows from this since $\mathbb{R}_\infty$ is a circle (of infinite radius) in $\mathbb{C}_\infty$.

Proof. Let Γ be a circle in $\mathbb{C}_\infty$ and S a Möbius transformation. Let z_2, z_3, z_4 be three distinct points on Γ, and set $w_j = Sz_j$ for each j. We claim that $S(\Gamma)$ is the circle Γ' determined by w_2, w_3, w_4. Indeed,

$$(z, z_2, z_3, z_4) = (Sz, w_2, w_3, w_4)$$

for any z, and if $z \in \Gamma$, the LHS is real by Lemma 1.11, and using the same theorem on the RHS completes the proof. ■

Definition 1.8. Let Γ be a circle through z_2, z_3, z_4. The points $z, z^* \in \mathbb{C}_\infty$ are said to be *symmetric* with respect to Γ if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

Remark. The above definition only depends on Γ, not the choice of z_2, z_3, z_4.

[1]it may be checked that $|\gamma|^2 + \delta$ is a positive real by substituting their values