Munkres Solutions

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Contents

§2. Topological Spaces and Continuous Functions

2.13. Basis for a Topology

Exercise 2.13.1. Let X be a topological space and $A \subseteq X$. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subseteq A$. Show that A is open in X.

> **Solution**
>
> For each $x \in A$, denote by U_x an open subset of A that contains A. Then $A = \bigcup_{x \in A} U_x$. However, an arbitrary union of open sets is open and thus, so is A.

Exercise 2.13.5. Show that if $\mathcal{A}$ is a basis for a topology on X, the topology generated by $\mathcal{A}$ equals the intersection of all topologies that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

> **Solution**
>
> Let $\mathcal{T}$ be the topology generated by $\mathcal{A}$ and $\mathcal{T}'$ be a topology that contains $\mathcal{A}$. Let $U \in \mathcal{T}$. Then $U = \bigcup_{i \in I} B_i$ for some $(B_i)_{i \in I}$ in $\mathcal{A}$. However, each B_i is also in $\mathcal{T}'$. Since an arbitrary union of open sets is open, $U \in \mathcal{T}'$ as well. Therefore, $\mathcal{T} \subseteq \mathcal{T}'$, proving the result. The solution for the case where $\mathcal{A}$ is a subbasis is very similar and so omitted.

2.16. The Subspace Topology

Exercise 2.16.1. Show that if Y is a subspace of X and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

> **Solution**
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> The topology A inherits as a subspace of X is
>
> $$\begin{aligned} \mathcal{T} &= \{U \cap A : U \text{ open in } X\} \\ &= \{(U \cap Y) \cap A : U \text{ open in } X\} \\ &= \{V \cap A : V \text{ open in } Y\}, \end{aligned}$$
>
> which is the topology it inherits as a subspace of Y.

Exercise 2.16.2. If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X.

> **Solution**
>
> It is easily seen that $\mathcal{T}'_Y$ is finer than $\mathcal{T}_Y$. We further see that it need not be strictly finer by considering the example $X = \{a, b, c\}$, $Y = \{a, b\}$, $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$, and $\mathcal{T}'$ as the discrete topology on X.

Exercise 2.16.3. Consider $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following is open in Y? Which are open in $\mathbb{R}$?

$$\begin{aligned} A &= \left\{x : \frac{1}{2} < x < 1\right\} \\ B &= \left\{x : \frac{1}{2} < x \leq 1\right\} \\ C &= \left\{x : \frac{1}{2} \leq x < 1\right\} \\ D &= \left\{x : \frac{1}{2} \leq x \leq 1\right\} \\ E &= \left\{x : 0 < x < 1 \text{ and } 1/x \notin \mathbb{Z}^+\right\} \end{aligned}$$

Solution

A and B are open in Y and only A is open in $\mathbb{R}$. This is reasonably straightforward to prove.
C is not open in Y (and so not $\mathbb{R}$ either) because there is no basis element U of the order topology such that $1/2 \in U \subseteq C$. A similar argument holds for D as well.
E is open in both $\mathbb{R}$ and Y because it can be written as a union of basis elements

$$E = \bigcap_{n \in \mathbb{Z}^+} \left(\frac{1}{n+1}, \frac{1}{n} \right).$$

Exercise 2.16.4. A map $f : X \to Y$ is said to be an open map if for every open U of X, $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open.

Solution

We shall only show that π_1 is open, the other case is nearly identical. Let

$$U = \bigcup_{i \in I} U_i \times V_i$$

be open in $X \times Y$ for some indexing set I, where each U_i and V_i are open in X and Y respectively. Then,

$$\pi_1(U) = \pi_1 \left(\bigcup_{i \in I} U_i \times V_i = \bigcup_{i \in I} \pi_1(U_i \times V_i) = \bigcup_{i \in I} U_i \right)$$

is open in X.

2.17. Closed Sets and Limit Points

Exercise 2.17.1. Let $\mathcal{C}$ be a collection of subsets of set X. Suppose that $\varnothing$ and X are in $\mathcal{C}$ and that finite unions and arbitrary intersections of elements of $\mathcal{C}$ are in C. Show that the collection $\mathcal{T} = \{X \setminus C : C \in \mathcal{C}\}$ is a topology on X.

Solution

Let $(U_i)_{i \in I}$ be in $\mathcal{T}$ with $U_i = X \setminus C_i$ for each i. Then

$$\bigcup_{i \in I} U_i = X \setminus \bigcap_{i \in I} C_i = X \setminus C \in \mathcal{T}$$

for some $C \in \mathcal{C}$. Closure under finite intersections is shown similarly. We trivially have $\varnothing, X \in \mathcal{T}$ because $X, \varnothing \in \mathcal{C}$.

Exercise 2.17.2. Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Solution

Let U be open in X such that $Y \setminus A = U \cap Y$. Then, we can write A as $X \setminus ((X \setminus Y) \cup U)$. Since $X \setminus Y$ and U are open in X, A is closed in X.

Exercise 2.17.3. Show that if A is closed in X and B is closed in Y, $A \times B$ is closed in $X \times Y$.

Solution

Observe that
$$(X \times Y) \setminus (A \times B) = ((X \setminus A) \times (Y \setminus B)) \cup ((X \setminus A) \times Y) \cup (A \times (Y \setminus B)).$$
Since each of the sets on the right are open in $X \times Y$, $A \times B$ is closed.

Exercise 2.17.4. Show that if U is open in X and A is closed in X, $U \setminus A$ is open in X and $A \setminus U$ is closed in Y.

Solution

This is easily seen on writing $U \setminus A = U \cap (X \setminus A)$ and $A \setminus U = A \cap (X \setminus U)$.

Exercise 2.17.19. If $A \subseteq X$, define the boundary of A by
$$\operatorname{Bd} A = \overline{A} \cap \overline{X \setminus A}.$$

(a) Show that A° and $\operatorname{Bd} A$ are disjoint, and $\overline{A} = A^\circ \cup \operatorname{Bd} A$.

(b) Show that $\operatorname{Bd} A = \varnothing$ iff A is both open and closed.

(c) Show that U is open iff $\operatorname{Bd} U = \overline{U} \setminus U$.

(d) If U is open, is it true that $U = \overline{U}^\circ$? Justify your answer.

Solution

(a) Let $x \in A \setminus A^\circ$. Then for any open $U \ni x$, $U \not\subseteq A$ (otherwise, $A^\circ \cup U \supsetneq A^\circ$ is open and contained in A). That is, $U \cap (X \setminus A) \neq \varnothing$. However, this implies that $x \in \overline{X \setminus A}$, that is, $A \setminus A^\circ \subseteq \overline{X \setminus A}$. Therefore,
$$\begin{aligned} \overline{A} \setminus A^\circ = (\overline{A} \setminus A) \cup (A \setminus A^\circ) &\subseteq \overline{X \setminus A} \\ \overline{A} &\subseteq A^\circ \cup \overline{X \setminus A} \\ &= \overline{A} \cap (A^\circ \cup \overline{X \setminus A}) \\ &= A^\circ \cup (\overline{A} \cup \overline{X \setminus A}) = A^\circ \cup \operatorname{Bd} A. \end{aligned}$$

(b) If A is not closed, $\overline{A} \supsetneq A$ intersects $X \setminus A \subseteq \overline{X \setminus A}$, contradicting $\operatorname{Bd} A = \varnothing$. Similarly, $X \setminus A$ is closed as well, so A is both open and closed.
The other direction is similarly straightforward.

(c) If U is open, $X \setminus U$ is closed so $\operatorname{Bd} U = \overline{U} \cap (X \setminus U) = \overline{U} \setminus U$.
On the other hand, if $\overline{U} \cap (X \setminus U) = \overline{U} \cap \overline{X \setminus U}$, $X \setminus U$ must be closed. Indeed, otherwise, $\overline{X \setminus U} \setminus (X \setminus U) \subseteq U \subseteq \overline{U}$, contradicting the equality.

(d) No, this is not the case. Consider the open set $U = (1,2) \cup (2,3) \subseteq \mathbb{R}$. Then $\overline{U}^\circ = (1,3)$.

2.18. Continuous Functions

Exercise 2.18.1. Suppose that $f : X \to Y$ is continuous. If x is a limit point of $A \subseteq X$, is $f(x)$ necessarily a limit point of $f(A)$?

Solution

No. Let $f : \mathbb{R} \to \mathbb{R}$ be the zero function, $A = \{1/n : n \in \mathbb{N}\}$, and $x = 0$.

Exercise 2.18.8. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x : f(x) \leq g(x)\}$ is closed in X.

(b) Let $h : X \to Y$ be the function
$$h(x) = \min\{f(x), g(x)\}.$$
Show that h is continuous.

Solution

(a) It suffices to show that $V = \{x : f(x) > g(x)\}$ is open in X. Let $x \in V$. Since Y is Hausdorff, there are open sets U_1, U_2 such that $f(x) \in U_1$, $g(x) \in U_2$, and $U_1 \cap U_2 = \varnothing$.
We can consider a basis element $B_1 \subseteq U_1$ that contains $f(x)$ in lieu of U_1. That is, we may suppose wlog that U_1 and U_2 are disjoint basis elements. Further, we may assume that U_1 is of the form (a, ∞) (if it is of the form (a, c) instead, we can replace it with (a, ∞)). Due to the disjointness assumption, this means that we can consider U_1 to be of the form (a, ∞) and U_2 of the form $(-\infty, b)$ such that for any $c \in U_1, d \in U_2$, $c > d$
Now, let $U = f^{-1}(U_1) \cap g^{-1}(U_2) \ni x$. f and g are continuous so U is open. Further, for any $y \in U$, $f(y) > g(y)$ (by the above assumption), that is, $U \subseteq V$.
It follows that V is open (for any $x \in V$, there is an open $V \subseteq U$ such that $x \in V$).

(b) Let $U_1 = \{x : f(x) \geq g(x)\}$ and $U_2 = \{x : f(x) \leq g(x)\}$. By (a), U_1 and U_2 are both closed. Since g is continuous on U_1, f is continuous on U_2, and $f = g$ on $U_1 \cap U_2$, we can use the pasting lemma to conclude that h is continuous ($h(x) = g(x)$ on U_1 and $f(x)$ on U_2).

2.19. The Product Topology

Exercise 2.19.4. Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic to $X_1 \times \cdots \times X_n$.

Solution

Let the two topological spaces above be denoted by X and Y respectively. Consider the map $f : X \to Y$ with

$$f(x) = (\pi_1(\pi_1(x)), \ldots, \pi_{n-1}(\pi_1(x)), \pi_2(x)).$$

We claim that f is a homomorphism.
If U_i is open in X_i for each i, then $f^{-1}(U_1 \times \cdots \times U_n) = (U_1 \times \cdots \times U_{n-1}) \times U_n$ is open in X ($U_1 \times \cdots \times U_{n-1}$ is open in $X_1 \times \cdots \times X_{n-1}$ and U_n is open in X_n). Therefore, f is continuous.
On the other hand, if U is open in $X_1 \times \cdots \times X_{n-1}$ and U_n is open in X_n, then $f(U \times U_n) = U_1 \times \cdots \times U_n$, where each U_i is open in X_i by Exercise 2.16.4.