Connectedness and Compactness

Lecture 18 - 05/03/2021 Connectedness

Def. Let X be a topological space. A *separation* of X is a pair U, V of disjoint non-empty open subsets of X whose union is X. X is said to be *connected* if it does not have a separation.

Prop. (3.1) A space X is connected iff the only subsets of X that are both open and closed ("clopen") are $\emptyset$ and X.

If U, V is a separation, U is clopen.

If $A \neq \emptyset$ is clopen, $A, X \setminus A$ is a separation. $\square$

Prop (3.2) If X and Y are homeomorphic, X is connected iff Y is connected.

Lemma (3.3) If Y is a subspace of X, a separation of Y is pair of disjoint non-empty sets A, B whose union is Y, neither of which contains a limit point of the other. The space Y is connected iff there is no separation of Y.

This is easily shown since the sets involved in a separation are clopen (in Y). $\bar{A} \cap Y = A$

$$A \cap B = \emptyset \rightarrow \bar{A} \cap Y \cap B = \emptyset$$
$$\rightarrow \bar{A} \cap B = \emptyset.$$

The other direction is similarly straightforward. $\square$

For example, any topological space with the indiscrete topology is connected.

→ Show that $\mathbb{Q}$ is not connected.

Lemma. (3.4) If the sets C, D form a separation of X and Y is a connected subspace of Y, $Y \subseteq C$ or $Y \subseteq D$.

If not, we can write $Y = \underbrace{(Y \cap C) \cup (Y \cap D)}_{\text{open in } Y}$

Lemma. (3.5) A union of connected spaces is connected if their intersection is non-empty.

Proof. Let (A_α) be a family of connected subspaces of X and $p \in \bigcap_\alpha A_\alpha$. We claim that $Y = \bigcup_\alpha A_\alpha$ is connected.

Suppose C, D is a separation of Y and wlog that $p \in C$.

Since A_α is connected and $p \in C$, $A_\alpha \subseteq C$.

Therefore, $Y = \bigcup_\alpha A_\alpha \subseteq C$, contradicting the non-emptiness of D. $\square$

Theo. (3.6) Let A be a connected subspace of X If $A \subseteq B \subseteq \overline{A}$, B is also connected.

(We can add any of the limit points without destroying connectedness)

Proof Suppose C, D is a separation of B. Assume wlog that $A \subseteq C$. Then $B \subseteq \overline{A} \subseteq \overline{C}$. But $\overline{C} \cap D = \emptyset$, yielding a contradiction and proving the claim. $\square$

Theo. (3.7) The image of a connected space under a continuous map is connected.

Proof Let $f: X \to Y$ be continuous where X is connected and f is surjective. Suppose C, D is a separation of Y. Since f is continuous, $f^{-1}(C)$ and $f^{-1}(D)$ are also open and they form a separation of X, resulting in a contradiction. $\square$

Theo. (3.8) A finite Cartesian product of connected spaces is connected. (under either the box or product topo., they are equal)

Proof It suffices to show that if X and Y are connected $X \times Y$ is connected. (Use the fact that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic to $(X_1 \times \cdots \times X_{n-1}) \times X_n$)
Fix $x \times y \in X \times Y$. $X \times \{y\}$ is connected (it is homeomorphic to X) and so is $\{x\} \times Y$. The result follows on using Theo 3.5. $\square$

Show that $\mathbb{R}^{\omega}$ under the box topology is disconnected.

Hint: Let $A = \{(a_n) : (a_n) \text{ is bounded}\}$ and
$B = \{(b_n) : (b_n) \text{ is unbounded}\}$.

Show that $\mathbb{R}^{\omega}$ under the product topology is connected.

Hint: Show that $\mathbb{R}^{\infty}$, the set of sequences eventually 0, is connected and that $\mathbb{R}^{\omega} = \overline{\mathbb{R}^{\infty}}$.

$\mathbb{R}^{\infty} = \bigcup_{n \in \mathbb{N}} \tilde{\mathbb{R}}^n$ $\hookrightarrow$ 0 after first n co-ordinates.

Theo. (3.9) An arbitrary product of connected spaces is connected in the product topology.
(the proof is nearly identical to that for $\mathbb{R}^{\omega}$ above)

Def. A simply ordered set L having more than one element is called a linear continuum if

- L has the least upper bound property.
- if $x < y$ in L, there exists z in L such that $x < z < y$.

Clearly, $\mathbb{R}$ is a linear continuum.

Theo. (3.10) If L is a linear continuum, then L, intervals in L, and rays in L, are connected.

Theo. [Intermediate Value Theorem]
(3.11) Let $f: X \to Y$ be continuous, where X is a connected space and Y is an ordered set under the ordered topology. If $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, there exists $c \in X$ such that $f(c) = r$.

Proof. Suppose otherwise. Then $f(X) \cap (-\infty, r)$ and $f(X) \cap (r, \infty)$ form a separation of $f(X)$. However, the image under f of X is connected, resulting in a contradiction. $\square$

Lecture 19 - 10/03/21 Path Connectedness

Def. Given points x, y of the space X, a path from x to y is a continuous function $f: [a,b] \to X$ such that $f(a) = x$ and $f(b) = y$ (for some closed interval $[a,b] \subseteq \mathbb{R}$).
A space X is path-connected if there is a path between any two points in X.

Theo. Any path-connected space is connected.
(3.12)

Proof. Suppose otherwise. Let X be path-connected and $f: [a,b] \to X$ be a path in X. Let $X = A \cup B$ be a separation of X.
Since $[a,b]$ is connected and f is continuous, $f([a,b]) \subseteq A$ or $f([a,b]) \subseteq B$, contradicting path-connectedness (across A, B). $\square$

The converse is not true.
Consider
$$S = \{ x \times \sin\left(\tfrac{1}{x}\right) : 0 < x \leq 1 \},$$
known as the topologist's sine curve.
Then $\bar{S} = S \cup (\{0\} \times [-1,1])$. We claim that $\bar{S}$ is not path-connected.
Let continuous $f: [a,c] \to \bar{S}$ beginning at the origin and ending at some point in S. The set
$$\{ t \in [a,c] : f(t) \in \{0\} \times [-1,1] \}$$
is closed (due to continuity), so it has a largest element b.
Then the restriction $f: [b,c] \to \bar{S}$ is a path such that $f(b) \in \{0\} \times [-1,1]$ and $f((a,c]) \subseteq S$.

Wlog, let $[b,c]$ be $[0,1]$ and $f(t) = (x(t), y(t))$. Then
$x(0) = 0$ and for $t > 0$, $x(t) > 0$ and $y(t) = \sin(1/x(t))$.

For each n, choose a $0 < u < x(1/n)$ such that $y(u) = (-1)^n$. Using the IVT (Theo 3.11), there is a $0 < t_n < 1/n$ such that $x(t_n) = u$. However, then, $t_n \to 0$ but $y(t_n) = (-1)^n$ does not converge, contradicting the continuity of y and proving the claim.
Show that S (and thus $\bar{S}$) is connected, disproving the converse of Theo 3.12

Def. Given X, define an equivalence relation as $x \sim y$ if there exists a connected subspace of X containing both x and y. The resulting equivalence classes are called the **components** or **connected components** of X.

(Check that it is an equivalence relation)

Theo. (3.13) The components of X are connected disjoint subspaces of X whose union is X such that any non-empty connected subspace of X intersects exactly one of them.

Proof left as exercise.
To show that a component C is connected, fix $x \in C$ and for each $y \in C$, let $C_y \subseteq C$ be a connected subspace containing C_y.
by the second part
Then $x \in \bigcap C_y \neq \emptyset$, so $\bigcup C_y = C$ is connected. $\square$

Similar to connected components, we can define the **path components** of X.
($x \sim y$ if there is a path from x to y)
(transitivity can be shown using the pasting lemma)

Theo. (3.14) The path components of X are path-connected disjoint subspaces of X whose union is X such that any non-empty path-connected subspace of X intersects exactly one of them.

Corollary. (3.15) Any connected component of X is closed.

(Use the fact that the closure of a connected space is closed)

It follows that if there are finitely many components, each component is also open.

It need not be true that path-connected components are closed, however. Consider the topologist's sine curve $\bar{S}$. Then S is open in $\bar{S}$ (and not closed) and $\bar{S} \setminus S$ is closed (and not open).

Def A space X is said to be locally connected at $x \in X$ if for every neighbourhood U of x, there is a connected neighbourhood $V \subseteq U$ of x. X is locally connected if it is locally connected at any point of X.
We similarly define local path-connectedness.

Theo. (3.16) A space X is locally connected iff for any open $U \subseteq X$, each component of U is open in X.

Proof. Let X be locally connected, U be open in X, and C be a component of U Let $x \in C$. There is then a neighbourhood $V \subseteq U$ of x that is connected. It follows that $V \subseteq C$ and therefore, C is open.
On the other hand suppose that the components of open sets in X are open. Let $x \in X$ and U a neighbourhood of x. We can take the component of U containing x, completing the proof $\square$

Theo. (3.17) A space X is locally path-connected iff for any open $U \subseteq X$, each path component of U is open.

The proof is nearly identical to the previous one.

<u>Lecture 20</u> - 10/03/21 Introduction to Compactness

Theo. (3.18) If X is a topological space, each path component of X lies in a component of X. Moreover, if X is locally path-connected, the components and path components are the same.

Proof. The first part is direct since any path component is connected.
Let C be a component, $x \in C$, and $P \ni x$ be a path component.
Let X be locally path-connected. Suppose $P \subsetneq C$. Let Q be the union of all path components other than P that intersect C.
Then $C = P \cup Q$.
Because X is locally path connected, each path component of X is open in X. In particular, P and Q are open. This contradicts the connectedness of C, proving the claim.
↳ they form a separation of C

Def. A collection $\mathcal{A}$ of subsets of X is said to be a covering if $X = \bigcup_{A \in \mathcal{A}} A$. An open covering is a covering where every subset is open.
X is said to be compact if any open cover contains a finite subcover.

If Y is a subspace of X and $\mathcal{A}$ is a collection of subsets of X, $\mathcal{A}$ is said to cover Y if $Y \subseteq \bigcup_{A \in \mathcal{A}} A$.

Theo. (3.19) Let Y be a subspace of X. Then Y is compact iff every covering of Y by sets open in X contains a finite subcollection covering Y.

Theo. (3.20) Any closed subspace of a compact space is compact.

Hint Consider the open cover $\mathcal{A} \cup \{X \setminus Y\}$

Theo. (3.21) Every compact subspace of a Hausdorff space is closed.

Proof. Let X be Hausdorff and Y a compact subspace. Let $x_0 \in X \setminus Y$. For each $y \in Y$, choose neighbourhoods U_y of x_0 and V_y of y such that $U_y \cap V_y = \emptyset$
Since Y is compact, there exist $y_1, \dots, y_n$ such that
$$Y \subseteq \bigcup_{1 \le i \le n} V_{y_i} = V.$$

But $Y \cap U \subseteq V \cap U = \emptyset$, where
$$U = \bigcap_{1 \le i \le n} U_{y_i} \text{ is a neighbourhood of } x_0.$$
Therefore, Y is closed. $\square$

The above need not be true for non-Hausdorff spaces.
(Consider $\mathbb{R}$ under finite complement topology)

Lemma. (3.22) If Y is a compact subspace of the Hausdorff space X, and $x_0 \in X \setminus Y$, there are disjoint open sets U and V of X such that $x_0 \in U$ and $Y \subseteq V$.

Lecture 21 - 12/03/21 More on Compact Spaces

Theo. (3.23) The image of a compact space under a continuous map is compact.

Theo (3.24) Let $f: X \to Y$ be bijective and continuous. If X is compact and Y is Hausdorff, f is a homeomorphism.
(Show that f is a closed map)

Lemma (3.25) [Tube Lemma] Let X and Y be spaces with Y compact. Suppose $x_0 \in X$ and $N \supseteq \{x_0\} \times Y$ is an open subset of $X \times Y$. Then, there is a neighbourhood W of x_0 in X such that $W \times Y \subseteq N$.
↳ tube about $x_0 \times Y$.

Proof Let us cover $x_0 \times Y$ with the basis elements $U \times V$ (for the topology of $X \times Y$) lying in N.
Since $\{x_0\} \times Y$ is compact, there is a finite subcover

$$U_1 \times V_1, \ldots, U_n \times V_n$$

We may assume that $x_0 \in U_i$ for each i (Why?). Let

$$W = \bigcap_{1 \le i \le n} U_i.$$

Then W is open in X and contains x_0.
It is easily shown that $W \times Y$ is covered by the $(U_i \times V_i)$. Since each $U_i \times V_i \subseteq N$, $W \times Y \subseteq N$. ∎

Theo. (3.26) The product of finitely many compact spaces is compact.

(It is in fact true for arbitrary products, which we shall see later in Tychonoff's Theorem)

Proof It suffices to show the result for two spaces. Let X, Y be compact and A an open subcover of $X \times Y$
For each $x_0 \in X$, $\{x_0\} \times Y$ can be covered by finitely many $A_1, \ldots, A_m \in A$.
Then $\{x_0\} \times Y \subseteq A_1 \cup \cdots \cup A_m = N$, so N contains a tube $W \times Y$ containing $\{x_0\} \times Y$. $W \times Y$ is covered by the A_i $(1 \le i \le m)$.
That is, for each $x \in X$, there is a neighbourhood W_x of x such that $W_x \times Y$ can be covered by finitely many elements.
The collection of all W_x forms an open cover of X, so it has a finite subcover $W_1, W_2, \ldots, W_k$.
Then, $X \times Y \subseteq (W_1 \times Y) \cup (W_2 \times Y) \cup \cdots \cup (W_k \times Y)$.
Each is covered by finitely many elements from A, so $X \times Y$ can as well. ∎

Def. A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property if for any finite $\{C_1, C_2, \ldots, C_k\} \subseteq \mathcal{C}$,

$$C_1 \cap C_2 \cap \cdots \cap C_k \neq \varnothing.$$

Theo. (3.27) Let X be a topological space X is compact iff for every collection $\mathcal{C}$ of closed subsets of X with the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ is non-empty.

Proof. Forward direction

Suppose otherwise. Then $\mathcal{A} = \{X \setminus C : C \in \mathcal{C}\}$ is an open cover so has a finite subcover $A_1, \ldots, A_m$. But then, $(X \setminus A_1) \cap \cdots \cap (X \setminus A_m) = \phi$, contradicting the finite intersection property and proving the result.

Backward direction.

Let $\mathcal{A}$ be an open cover and $\mathcal{C} = \{X \setminus A : A \in \mathcal{A}\}$. Suppose $\mathcal{A}$ does not have a finite subcover. Then $\mathcal{C}$ has the finite intersection property so $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. This contradicts $\mathcal{A}$ being a cover, proving the result. $\square$

Let us look at what the compact subspaces of the real line.

Theo. (3.28) Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval of X is compact.

↳ $[a,b]$, not rays

Proof. Let $a<b$ and $\mathcal{A}$ be an open cover of $[a,b]$ in the subspace topology (which is the same as the order topology since $[a,b]$ is compact).

Claim 1. If $x \in [a,b)$, there exists $y > x$ in $[a,b]$ such that $[x,y]$ can be covered by at most two elements of $\mathcal{A}$.

- → If x has an immediate successor, let y be this element. Then $[x,y] = \{x,y\}$, so the claim is obvious.
- → Otherwise, choose $A \in \mathcal{A}$ containing x. Since A is open, it contains an interval of the form $[x,c)$. We can then let y be any element of (x,c).

Now, let C be the set of all points $y > a$ such that $[a, y]$ can be covered by finitely many elements of $\mathcal{A}$. (We want to show that $b \in C$.)
By the claim, $C \neq \emptyset$. Let c be the least upper bound of C.

Claim 2. $c \in C$.

Choose $A \in \mathcal{A}$ containing c. A contains an interval of the form $(d, c]$.
If c does not have an immediate predecessor (even otherwise, it is similarly shown)
Let $z \in C$ such that $z \in (d, c)$ (Why does such a z exist?).
Then $[a, z)$ can be covered by finitely many elements $A_1, \ldots, A_k$ of $\mathcal{A}$, so it follows that $\{A_1, \ldots, A_k\} \cup \{A\}$ is a finite subset of $\mathcal{A}$ covering $[a, c]$, proving the result.

Finally, let us prove that $b = c$. Suppose otherwise Then there exists $y \in (c, b]$ such that $[c, y]$ can be covered by atmost two elements. This implies that $[a, y]$ can be covered by finitely many elements, contradicting the fact that c is an upper bound of C, proving the result.

$\square$

Corollary (3.29). Any closed interval in $\mathbb{R}$ is compact.