MA 412: Complex Analysis

Amit Rajaraman

Last updated January 5, 2022

Contents

§1. Introduction

1.1. Some basic definitions

Consider the equation $X^2 + 1 = 0$. Clearly, this equation has no roots over $\mathbb{R}$. Consider the set

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\} = \mathbb{R}^2,$$

and define addition and subtraction over $\mathbb{C}$ as

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b) \cdot (c, d) &= (ac - bd, ad + bc).\end{aligned}$$

It is easy to show that $(\mathbb{C}, +, \cdot)$ is a field with additive identity $(0, 0)$ and multiplicative identity $(1, 0)$. Further observe that $\mathbb{R}$ is a subfield of $\mathbb{C}$ – consider the field homomorphism $\mathbb{R} \to \mathbb{C}$ defined by $a \mapsto (a, 0)$.
Now, we denote $\iota = (0, 1)$, and write (a, b) as $a + b\iota$.

Observe that the equation $X^2 + 1 = 0$ *does* have roots over $\mathbb{C}$ since it can be written as $(X + \iota)(X - \iota)$. For the sake of completeness, we also note that the multiplicative identity of $a + \iota b$ is

$$\frac{1}{a + \iota b} = \frac{a - \iota b}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\iota.$$

When writing $z = a + b\iota$ where $a, b \in \mathbb{R}$, we write $a = \Re z$ (the real part of z) and $b = \Im z$ (the imaginary part of z). We also define the absolute value $|z| = (a^2 + b^2)^{1/2}$ of z, and the *conjugate* $\overline{z} = a - \iota b$ of z. We clearly have

$$\begin{aligned}z\overline{z} &= |z|^2\\ \Re z &= \frac{z + \overline{z}}{2}\\ \Im z &= \frac{z - \overline{z}}{2\iota}.\end{aligned}$$

It is easy to check that

$$\overline{z + w} = \overline{z} + \overline{w} \text{ and } \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

We also have

$$\begin{aligned}\left|\frac{z}{w}\right| &= \frac{|z|}{|w|}\\ |\overline{z}| &= |z|.\end{aligned}$$

Exercise 1.1. Check that the set

$$M = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{R}$$

with matrix addition and multiplication is a field isomorphic to $\mathbb{C}$.

To close out the tedious part of things, we have

$$\begin{aligned}|z + w|^2 &= |z|^2 + |w|^2 + 2\Re(z\overline{w})\\ |z + w| &\le |z| + |w| \qquad (1.1)\end{aligned}$$

Equation (1.1) is referred to as the *triangle inequality*.

1.2. Polar representations and roots

Consider $z = x + \iota y \in \mathbb{C}$. We may then define

$$x = r\cos\theta \qquad y = r\sin\theta,$$

where $|z| = r$ and the angle θ is called the *argument* of z as is denoted $\theta = \arg z$. We typically restrict θ to $(-\pi, \pi]$. We denote $\operatorname{cis}\theta = \cos\theta + \iota\sin\theta$. Therefore, we have

$$z = |z|\operatorname{cis}(\arg z).$$

Observe that rather conveniently,

$$\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2).$$

Therefore, inductively,

$$z_1 z_2 \cdots z_n = \left(\prod_i |z_i|\right) \cdots r_n \operatorname{cis}\left(\sum_i \arg z_i\right).$$

In particular,

$$z^n = r^n \operatorname{cis}(n\theta)$$

for any $n > 0$. If $z \neq 0$ (equivalently, $r \neq 0$), the above holds for all $n \in \mathbb{Z}$.
In the case where $r = 1$, we have

$$(\cos\theta + \iota\sin\theta)^n = \cos(n\theta) + \iota\sin(n\theta) \tag{1.2}$$

Equation (1.2) is referred to as *de Moivre's formula*.

Let us consider the equation $z^n = a$. This equation has n roots of the form

$$z = |a|^{1/n}\operatorname{cis}\left(\frac{2k\pi + \arg z}{n}\right)$$

for $k = 0, 1, \ldots, n-1$.
A *line* in the complex plane is a set of the form

$$L = \{z = a + tb : t \in \mathbb{R}\},$$

for some fixed $a, b \in \mathbb{C}$, where b is a *directional* vector whose absolute value may be assumed to be 1. Since $b \neq 0$, we equivalently have

$$L = \{z : \Im\left(\frac{z-a}{b}\right) = 0\}.$$

We can also define the half-planes

$$H_a = \{z : \Im\left(\frac{z-a}{b}\right) > 0\}$$
$$K_a = \{z : \Im\left(\frac{z-a}{b}\right) < 0\}.$$

Note that $H_a = a + H_0$, where the addition is Minkowski addition:

$$H_a = \{a + z : z \in H_0\}.$$

1.3. The extended plane

Define $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and let $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. We shall show a bijection from $\mathbb{C}_\infty$ to S.
Let $N = (0, 0, 1)$ be the 'north pole' of S, and orient $\mathbb{C}$ (as $\mathbb{R}^2$) in the horizontal plane in a manner such that $\mathbb{C}$ cuts S along the equator. For $z = x + \imath y \in \mathbb{C}$, let us define the corresponding point $Z = (x_1, x_2, x_3) \in S$. We shall draw a line connecting z to N, and let Z be the point of intersection (other than N) of this line with S. Finally, we shall map ∞ to N.
Let us define this more explicitly. The line through N and z is

$$L = \{tN + (1-t)z : t \in \mathbb{R}\}.$$

Then, letting $z = (x, y, 0)$, we have

$$t^2 + (1-t)^2|z|^2 = 1.$$

So,

$$|z|^2 = \frac{1-t^2}{(1-t)^2} = \frac{1+t}{1-t}$$

and

$$t = 1 - \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Therefore, we map z to

$$Z = \left(\frac{2\Re z}{|z|^2+1}, \frac{2\Im z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}. \right) \in S.$$

Based on this, we can define a distance metric between points in $\mathbb{C}_\infty$. For $z, z' \in \mathbb{C}_\infty$ mapping to $Z, Z' \in S$, we let $d(z, z')$ be the Euclidean distance between Z, Z' in $\mathbb{R}^3$. More explicitly,

$$\begin{aligned} d(z,z')^2 &= (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 \\ &= 2 - 2(x_1x_1' + x_2x_2' + x_3x_3') \\ &= \frac{2|z-z'|}{\big((|z|^2+1)(|z'|^2+1)\big)^{1/2}} \end{aligned}$$

when $z, z' \in \mathbb{C}$ and if $z' = \infty$ (so $Z' = (0, 0, 1)$), we have

$$d(z, z') =$$