Bagchi's Conjecture

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§1. Definitions

Definition 1.1 (Incidence System)**.** An *incidence system* is a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a set and $\mathcal{L}$ is a set of subsets of $\mathcal{P}$. Elements of $\mathcal{P}$ are called *points* and elements of $\mathcal{L}$ are called *lines*. A line ℓ is said to be *incident* on a point p if $p \in \ell$.

Definition 1.2 (Partial Linear Space)**.** An incidence system $(\mathcal{P}, \mathcal{L})$ is said to be a *partial linear space* if

1. for each $\ell \in \mathcal{L}$, $|\ell| \geq 2$.
2. for distinct $x, y \in \mathcal{P}$, there is at most one $\ell \in \mathcal{L}$ such that $\{x, y\} \subseteq \mathcal{P}$.

An incidence system is said to be a *linear space* if in the second condition above, the "at most one" is replaced with "exactly one".

Definition 1.3. Given an incidence system $\mathcal{X} = (\mathcal{P}, \mathcal{L})$ and a field $\mathbb{F}$, we define the linear code $\mathcal{C}_{\mathbb{F}}(\mathcal{X})$ over $\mathbb{F}^{\mathcal{P}}$ as follows. Identify each line ℓ with the codeword in $\mathbb{F}^{\mathcal{P}}$ whose xth coordinate is 1 if $x \in \ell$ and 0 otherwise. $\mathcal{C}_{\mathbb{F}}(\mathcal{X})$ is then the space spanned by the codewords corresponding to the lines in $\mathcal{L}$.
If $\mathbb{F} = \mathbb{F}_q$, we sometimes denote the above as $\mathcal{C}_q(\mathcal{X})$.

We also often denote this as $\mathcal{C}_q(\mathcal{L})$ if the point set is clear from context.
The incidence system $\mathcal{X}$ is said to be *trivial* at q if $\mathcal{C}_q(\mathcal{X})$ is all of $\mathbb{F}_q^{\mathcal{P}}$.

Definition 1.4 (Join)**.** Given two partial linear spaces $(\mathcal{P}_1, \mathcal{L}_1)$ and $(\mathcal{P}_2, \mathcal{L}_2)$ with $\mathcal{P}_1 \cap \mathcal{P}_2 = \varnothing$, one can define the *join* of the two partial linear spaces by $(\mathcal{P}_1 \cup \mathcal{P}_2, \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3)$, where

$$\mathcal{L}_3 = \{\{x, y\} : x \in \mathcal{P}_1, y \in \mathcal{P}_2\}.$$

§2. Inamdar's Result

Theorem 2.1. If a PLS $\mathcal{X} = (\mathcal{P}, \mathcal{L})$ is non-trivial at p and has at least $p+1$ lines through every point, then $|\mathcal{P}| \geq 2p$. Moreover, equality holds iff $\mathcal{X}$ is the join of two p-lines.

For the rest of this section, assume that $\mathcal{X} = (\mathcal{P}, \mathcal{L})$ is a partial linear space of the above prescribed format with $\mathcal{P} \leq 2p$. We wish to show that $\mathcal{P} = 2p$. Let $\mathcal{C} = \mathcal{C}_p(\mathcal{X})$.
It may be shown that it can be assumed that

1. Each point is incident on exactly $p + 1$ lines (throw away extra lines).
2. $\mathcal{C}^\perp$ is one-dimensional. Suppose that it is equal to $\langle w \rangle$ (restrict to the support of the minimum support word in $\mathcal{C}^\perp$).
3. For any line ℓ and any $\ell' \subsetneq \ell$, $\langle w, \ell' \rangle \neq 0$ (do the splitting procedure).

Also consider the colouring of $\mathcal{P}$ wherein each point P is coloured $w(P)$.

Proposition 2.2. If $\mathcal{X}$ has a p-line, it is equal to a join of two p-lines.

Proof. Let $P_1 \cdots P_p$ be a p-line. Since each point has p lines remaining, there must be at least p points other than the P_i, say $(Q_i)_{i=1}^p$. Since $|\mathcal{P}| \leq 2p$, these constitute all the points. Further, since each P_i has p lines to the Q_j, there must be a 2-line P_iQ_j for each $1 \leq i, j \leq p$. Now, suppose that $w(P_1) = 1$. Because $w(P_1) + w(Q_j) = 0$ for all j (P_1Q_j forms a p-line), $w(Q_j) = -1$ for all j. Each of the Q_i now has one line not accounted for. This line must be contained within the (Q_i). However, due to all of them having the same colour, the size of any such line must be p, completing the proof. ■

Proposition 2.3. If $\mathcal{X}$ has no p-line, the largest line in $\mathcal{X}$ is of size at most $2p/3$.

Proof. Let ℓ be a line with $|\ell| > 2p/3$. Given a $P \in \ell$, let x_P be the number of 2-lines through P. We then have that

$$2p - |\ell| \geq \underbrace{x_P}_{\text{points in 2-lines}} + \underbrace{2(p - x_P)}_{\text{points in} \geq 3\text{-lines}},$$

so $x_P \geq \ell$. Observe that if $x_P > (2p - |\ell|)/2$ for all $P \in \ell$, it follows by a pigeonhole argument that any two points P, Q in ℓ have a "2-neighbour" (a point u such that uP and uQ are 2-lines) in common. This is indeed the case because $x_P \geq |\ell| > (2p/3) > (2p - |\ell|)/2$. This in turn implies that $w(P) = w(Q)$, because $w(P) = -w(u) = w(Q)$. Therefore, ℓ is monochromatic, so for any fixed $P \in \ell$,

$$0 = \sum_{P \in \ell} w(P) = |\ell| w(P).$$

As $w(P) \neq 0$, $|\ell| = p$, yielding a contradiction. ■

As a result, we are done if we manage to show that $\mathcal{X}$ has a line of size greater than $2p/3$.

Now, let us restrict ourselves to the case where $\mathcal{X}$ is a linear space with exactly $p+1$ lines through each point. Let C be the $m \times n$ incidence matrix of $\mathcal{X}$, where $m = |\mathcal{L}|$ and $n = |\mathcal{P}|$. Observe that $(C^tC)_{ij} = pI + J$, where J is the all 1s matrix. As a result, its characteristic equation is just $(X - (p+n))(X-p)^{n-1}$. Recall that the characteristic matrix of CC^t and C^tC only differ by a power of X. That is, the characteristic equation of CC^t is $X^{m-n}(X-(p+n))(X-p)^{n-1}$. Over $\mathbb{F}_p$, this becomes $X^{m-1}(X-n)$.

Suppose that y is an eigenvector with eigenvalue 0. That is, $CC^ty = 0$. A question to ask is: is C^ty always 0? If it is not, we can explicitly get a vector in $\mathcal{C}^\perp$.

Proposition 2.4. There exists y in the kernel of CC^t such that $C^ty \neq 0$ (in $\mathbb{F}_p$).

Proof. Indeed, note that the kernel of CC^t is of dimension $m-1$, whereas the kernel of C^t is of dimension at most $m-n$. It follows that the kernel of C^t is a proper subset of CC^t. ■

Definition 2.1 (Irreducible Matrix)**.** A matrix $A = (a_{ij})_{n\times n}$ is said to be *irreducible* if the (directed) graph G_A with vertex set $[n]$ and edge from i to j iff $a_{ij} \neq 0$ is strongly connected.

Observe that CC^t is irreducible! Indeed, given any lines ℓ, ℓ', there is a line ℓ'' through some (any) $x \in \ell$ and $y \in \ell'$.

Theorem 2.5 (Perron-Frobenius Theorem)**.** Let $A = (a_{ij})_{n\times n}$ be a non-negative irreducible matrix. Then,

1. There exists a positive eigenvalue λ such that any other eigenvalue is of absolute value at most λ. Further, there is a number h called the *period* such that $\lambda\omega_h^k\lambda$ is an eigenvalue for every $0 \leq k < h$. These are the only eigenvalues with absolute value λ.
2. λ is a simple root of the characteristic polynomial. As a result, its corresponding eigenspace is one-dimensional.
3. λ has an eigenvector v with all positive entries. Any eigenvector with all positive entries is in the span of v.

Since CC^t is irreducible and non-negative with maximum eigenvalue $n+p$, there is an eigenvector v of $n+p$ with all positive entries.
For example, we now have that

$$\begin{aligned}
m(n+p) &= (n+p)\mathbf{1}^\top\mathbf{1} \\
&\geq \mathbf{1}^\top CC^\top\mathbf{1} \\
&= \sum_{\ell,\ell'\in\mathcal{L}} |\ell \cap \ell'| \\
&= \sum_{\ell\in\mathcal{L}} (p+1)|\ell| && \text{(each point on ℓ has $p+1$ lines through it)} \\
&= (p+1)\cdot n(p+1) = n(p+1)^2
\end{aligned}$$

Since $p+2 < n \leq 2p$,

$$m \geq \frac{n(p+1)^2}{n+p} > \frac{(p+1)^2}{3}.$$

The system has many lines! Can we use this to show that either there is a “large” line somewhere or the code is trivial?
Can we draw stronger conclusions by using another vector in place of $\mathbf{1}$? It seems that the ideal choice is to assign a larger weight to larger lines. What about the vector defined by $v_\ell = |\ell|$?
Recall that

$$\sum_{\ell\in\mathcal{L}} |\ell| = n(p+1).$$

Proposition 2.6. The following bound is true.

$$\sum_{\ell \in \mathcal{L}} |\ell|^2 \geq n(n+p).$$

Note that the bound is tight in the size $2p$ equality case!

Proof. Consider the vector v with $v_\ell = |\ell|$. Then,

$$\begin{aligned}
\sum_{\ell \in \mathcal{L}} |\ell|^2 &= v_\ell^t v_\ell \\
&\geq \frac{1}{n+p} v_\ell^t C C^t v_\ell \\
&= \frac{1}{n+p} \left(\sum_{\ell \in \mathcal{L}} v_\ell^2 |\ell| + \sum_{\substack{\ell, \ell' \in \mathcal{L} \\ \ell \neq \ell' \\ \ell \cap \ell' \neq \varnothing}} v_\ell v_{\ell'} \right).
\end{aligned}$$

Let us compute the second quantity.

$$\begin{aligned}
\sum_{\substack{\ell, \ell' \in \mathcal{L} \\ \ell \neq \ell' \\ \ell \cap \ell' \neq \varnothing}} v_\ell v_{\ell'} &= \sum_{P \in \mathcal{P}} \sum_{\substack{\ell, \ell' \ni P \\ \ell \neq \ell'}} |\ell||\ell'| \\
&= \sum_{P \in \mathcal{P}} \left(\sum_{\ell, \ell' \ni P} |\ell||\ell'| - \sum_{\ell \ni P} |\ell^2| \right) \\
&= \sum_{P \in \mathcal{P}} \left(\left(\sum_{\ell \in P} |\ell| \right)^2 - \sum_{\ell \ni P} |\ell^2| \right).
\end{aligned}$$

Observe that in the first summation (which is squared), because the incidence system is a linear space, there is precisely one line $\ell \ni P$ passing through each $Q \neq P$, and all $p+1$ lines pass through P. Therefore, the inner summation is just $(n-1) + (p+1) = n + p$. So,

$$\begin{aligned}
\sum_{\substack{\ell, \ell' \in \mathcal{L} \\ \ell \neq \ell' \\ \ell \cap \ell' \neq \varnothing}} v_\ell v_{\ell'} &= \sum_{P \in \mathcal{P}} \left((n+p)^2 - \sum_{\ell \ni P} |\ell|^2 \right) \\
&= n(n+p)^2 - \sum_{P \in \mathcal{P}} \sum_{\ell \ni P} |\ell|^2 \\
&= n(n+p)^2 - \sum_{\ell \in \mathcal{L}} \sum_{P \in \ell} |\ell|^2 \\
&= n(n+p)^2 - \sum_{\ell \in \mathcal{L}} |\ell|^3.
\end{aligned}$$

Therefore, we get that

$$\sum_{\ell \in \mathcal{L}} |\ell|^2 \geq n(n+p).$$

∎

Consider the graph G with vertex set $\mathcal{L}$, and $u \leftrightarrow v$ iff $u \cap v \neq \varnothing$. What do our conditions on the linear space correspond to?

1. Each point has exactly $p+1$ lines through it: any maximal clique in G is of size $p+1$.
2. There is a line passing through any two points (linearity): any two maximal cliques have a single point in common.

We wish to count the number of (maximal) cliques in the graph.
Note that each vertex ℓ has degree $|\ell|$. As a result, the graph has $\frac{1}{2}n(p+1)$ edges. Consider the random variable X that chooses a vertex with probability proportional to its degree (the measure of ℓ is $|\ell|/n(p+1)$). Note that this corresponds to the stationary distribution of a standard random walk on the graph!
Further, we have that

$$\mathbb{E}d(v) \geq \frac{p+n}{p+1}.$$