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§1. Definitions

Definition 1.1 (Incidence System)**.** An *incidence system* is a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a set and $\mathcal{L}$ is a set of subsets of $\mathcal{P}$. Elements of $\mathcal{P}$ are called *points* and elements of $\mathcal{L}$ are called *lines*. A line ℓ is said to be *incident* on a point p if $p \in \ell$.

Definition 1.2 (Partial Linear Space)**.** An incidence system $(\mathcal{P}, \mathcal{L})$ is said to be a *partial linear space* if

1. for each $\ell \in \mathcal{L}$, $|\ell| \geq 2$.
2. for distinct $x, y \in \mathcal{P}$, there is at most one $\ell \in \mathcal{L}$ such that $\{x, y\} \subseteq \mathcal{P}$.

Definition 1.3. Given an incidence system $\mathcal{X} = (\mathcal{P}, \mathcal{L})$ and a field $\mathbb{F}$, we define the linear code $\mathcal{C}_{\mathbb{F}}(\mathcal{X})$ over $\mathbb{F}^{\mathcal{P}}$ as follows. Identify each line ℓ with the codeword in $\mathbb{F}^{\mathcal{P}}$ whose xth coordinate is 1 if $x \in \ell$ and 0 otherwise. $\mathcal{C}_{\mathbb{F}}(\mathcal{X})$ is then the space spanned by the codewords corresponding to the lines in $\mathcal{L}$.
If $\mathbb{F} = \mathbb{F}_q$, we sometimes denote the above as $\mathcal{C}_q(\mathcal{X})$.

We also often denote this as $\mathcal{C}_q(\mathcal{L})$ if the point set is clear from context.
The incidence system $\mathcal{X}$ is said to be *trivial* at q if $\mathcal{C}_q(\mathcal{X})$ is all of $\mathbb{F}_q^{\mathcal{P}}$.

Definition 1.4 (Join)**.** Given two partial linear spaces $(\mathcal{P}_1, \mathcal{L}_1)$ and $(\mathcal{P}_2, \mathcal{L}_2)$ with $\mathcal{P}_1 \cap \mathcal{P}_2 = \varnothing$, one can define the *join* of the two partial linear spaces by $(\mathcal{P}_1 \cup \mathcal{P}_2, \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3)$, where

$$\mathcal{L}_3 = \{\{x, y\} : x \in \mathcal{P}_1, y \in \mathcal{P}_2\}.$$

§2. Inamdar's Result

Theorem 2.1. If a PLS $\mathcal{X} = (\mathcal{P}, \mathcal{L})$ is non-trivial at p and has at least $p+1$ lines through every point, then $|\mathcal{P}| \geq 2p$. Moreover, equality holds iff $\mathcal{X}$ is the join of two p-lines.

For the rest of this section, assume that $\mathcal{X} = (\mathcal{P}, \mathcal{L})$ is a partial linear space of the above prescribed format with $\mathcal{P} \leq 2p$. We wish to show that $\mathcal{P} = 2p$. Let $\mathcal{C} = \mathcal{C}_p(\mathcal{X})$
It may be shown that it can be assumed that

1. Each point is incident on exactly $p+1$ lines (throw away extra lines).
2. $\mathcal{C}^\perp$ is one-dimensional. Suppose that it is equal to $\langle w \rangle$ (restrict to the support of the minimum support word in $\mathcal{C}^\perp$).
3. For any line ℓ and any $\ell' \subsetneq \ell$, $\langle w, \ell' \rangle \neq 0$ (do the splitting procedure).

Also consider the colouring of $\mathcal{P}$ wherein each point P is coloured $w(P)$.

Proposition 2.2. If $\mathcal{X}$ has a p-line, it is equal to a join of two p-lines.

Proof. Let $P_1 \cdots P_p$ be a p-line. Since each point has p lines remaining, there must be at least p points other than the P_i, say $(Q_i)_{i=1}^p$. Since $|\mathcal{P}| \leq 2p$, these constitute all the points. Further, since each P_i has p lines to the Q_j, there must be a 2-line $P_i Q_j$ for each $1 \leq i, j \leq p$. Now, suppose that $w(P_1) = 1$. Because $w(P_1) + w(Q_j) = 0$ for all j ($P_1 Q_j$ forms a p-line), $w(Q_j) = -1$ for all j. Each of the Q_i now has one line not accounted for. This line must be contained within the (Q_i). However, due to all of them having the same colour, the size of any such line must be p, completing the proof. ■

Proposition 2.3. If $\mathcal{X}$ has no p-line, the largest line in $\mathcal{X}$ is of size at most $2p/3$.

Proof. Let ℓ be a line with $|\ell| > 2p/3$. Given a $P \in \ell$, let x_P be the number of 2-lines through P. We then have that

$$2p - |\ell| \geq \underbrace{x_P}_{\text{points in 2-lines}} + \underbrace{2(p - x_P)}_{\text{points in} \geq \text{3-lines}},$$

so $x_P \geq \ell$. Observe that if $x_P > (2p - |\ell|)/2$ for all $P \in \ell$, it follows by a pigeonhole argument that any two points P, Q in ℓ have a "2-neighbour" (a point u such that uP and uQ are 2-lines) in common. This is indeed the case because $x_P \geq |\ell| > (2p/3) > (2p - |\ell|)/2$. This in turn implies that $w(P) = w(Q)$, because $w(P) = -w(u) = w(Q)$. Therefore, ℓ is monochromatic, so for any fixed $P \in \ell$,

$$0 = \sum_{P \in \ell} w(P) = |\ell| w(P).$$

As $w(P) \neq 0$, $|\ell| = p$, yielding a contradiction. ■

Define

$$\mathcal{S} = \{S \subseteq \mathcal{P} : \sum_{P \in S} w(P) = 0\}.$$