Localization Schemes for CHR

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Last updated June 7, 2022

We extensively use the results and terminology of [CE22].

§1. The Discrete Setting

In this section, we analyze mixing time bounds for the pinning procedure of coordinate hit-and-run over the discrete set $[m]^n$. The case $m = 2$ is analyzed in [ALG20, CE22].
We embed the points of $[m]$ $\mathbb{R}^{m-1}$ as the m-simplex, so a point in $[m]^n$ lives in $\mathbb{R}^{(m-1)n}$. Let the vertices of the simplex be $y_1, \ldots, y_m$ (where $\|y_i\|^2 = 1$ for all i). We perform the analysis of each step of the pinning process separately, so it suffices to assume $n = 1$ if we know which coordinate we are going to pin.
Let the measure before pinning be ν_t, and the measure after pinning be ν_{t+1}. For ease of notation, denote $\nu_t(\{k\})$ as α_k. The probability that the coordinate is fixed as $k \in [m]$ is then α_k. Denote the barycenter of the distribution as b_t. Observe that $b_t = \sum \alpha_k y_k$.
The pinning procedure is performed using a linear tilt localization. Suppose that the relevant Z_t is such that it is equal to Z_k with probability $\nu_t(\{k\})$ (this corresponds to the case where the coordinate is pinned as k). There are three properties we desire.

1. $\langle Z_k, y_i - b_t \rangle = -1$ for $i \neq k$.
2. $\langle Z_k, y_k - b_t \rangle = 1/\alpha_k - 1$.
3. $\sum \alpha_k Z_k = 0$.

It turns out that all three properties are satisfied by

$$Z_k = \left(1 - \frac{1}{m}\right) \frac{y_k}{\alpha_k},$$

as is easily checked – this uses the fact that $\alpha_k = \frac{1}{n} + \frac{n-1}{n} \langle b_t, y_k \rangle$.

The general case with any n is similar. Let $C_t = \operatorname{Cov}(Z_t \mid \nu_t)$. It remains now to bound $\left\| C_t^{1/2} \operatorname{Cov}(\nu_t) C_t^{1/2} \right\|_{\text{op}} = \left\| \operatorname{Cov}(\nu_t) C_t \right\|_{\text{op}}$.

References

[ALG20] Nima Anari, Kuikui Liu, and Shayan Oveis Gharan. Spectral independence in high-dimensional expanders and applications to the hardcore model. *CoRR*, abs/2001.00303, 2020.

[CE22] Yuansi Chen and Ronen Eldan. Localization schemes: A framework for proving mixing bounds for markov chains. Mar 2022.