Coordinate Hit-and-run

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Given a well-guaranteed membership oracle for K, the problem is to approximately uniformly sample points for K.

The problem

Question

Input: A convex body $K \subseteq \mathbb{R}^n$ with a well-guaranteed membership oracle.
Output: A probability distribution on $\mathbb{R}^n$ that is at total variation distance at most ϵ from the uniform distribution on K.

Denote by π_K the uniform distribution on K.

The approach in broad strokes

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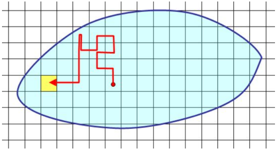

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Definition 2 (Hit-and-run)

Given x_t, we first draw y uniformly at random from $\mathbb{S}^{n-1}$. We then draw x_{t+1} uniformly at random from the set

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Since the scheme is ergodic and reversible, its stationary distribution is the uniform distribution over K.

Coordinate hit-and-run

In the simpler *coordinate hit-and-run* walk (which is the subject of this presentation), we instead draw y u.a.r. from $\{e_1, \ldots, e_n\}$, the standard basis of $\mathbb{R}^n$.

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In other words,

Definition 3 (Coordinate hit-and-run)

For the coordinate hit-and-run (CHR) Markov scheme $\mathcal{C}$, the transition probability density from u to v (with respect to the 1-dimensional Lebesgue measure) is

$$\mathcal{C}_{uv} = \begin{cases} \frac{1}{n|K \cap (u + e_i\mathbb{R})|}, & v \in K \cap (u + e_i\mathbb{R}) \text{ for some } i \in [n], \\ 0, & \text{otherwise.} \end{cases}$$

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The above is also sometimes referred to as the *Gibbs sampler*.

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Definition 4 (Conductance)

Given a Markov scheme P on S, the *ergodic flow* $\Phi_{P,Q}$ with respect to a probability distribution Q on S is

$$\Phi_{P,Q}(A, B) = \int_A P(u, B) Q(\mathrm{d}u)$$

for measurable $A, B \subseteq S$. We also denote $\Phi_{P,Q}(A) = \Phi_{P,Q}(A, S \setminus A)$.

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for measurable $A, B \subseteq S$. We also denote $\Phi_{P,Q}(A) = \Phi_{P,Q}(A, S \setminus A)$. For $0 < s < 1/2$, the *s-conductance* of P with respect to Q is

$$\Phi_s = \inf_{A: s < Q(A) < 1/2} \frac{\Phi_{P,Q}(A)}{Q(A) - s}.$$

The 0-conductance is referred to as merely *conductance*.

Conductance to mixing

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If Q_0 is M-warm with respect to Q, then $H_s \leq Ms$. Setting $s = \epsilon/(2M)$, we get that

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Two bounds

The first bound is due to [NS21], showing that if $B_\infty^n \subseteq K \subseteq R \cdot B_\infty^n$, then

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The second is due to [LV21], showing that if $B_2^n \subseteq K$ and $R_0^2 = \mathbf{E}_{x \sim \pi_K} \|x - b_K\|_2^2$, then

$$\Phi_{\mathcal{C},s} = \Omega\left(\frac{s}{R_0 n^{4.5} \log n}\right).$$

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Overview of the proof

Definition 6 (Gaussian CHR)

Given x_t, we draw i uniformly from $[n]$ and κ from $\mathcal{N}(0, \sigma^2)$. The next point x_{t+1} is defined by

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The original strategy used to prove the convergence of usual hit-and-run was to show that 'close-by' points have 'similar' distributions. Here, we do a multistep version of the same, showing that the distributions become similar after running the walk for several steps.

Some setup for the proof

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Observe that if we drop the $v \in K$ condition in the definition of $\mathcal{G}$, we just get the random walk $\mathcal{H}$ defined by

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Also define the random walk $\mathcal{H}'$ by

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Showing the closeness property for $\mathcal{G}$

To show that close points have similar distributions, we shall first show that

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and then that for close points u, v,

$$d_{\mathrm{TV}}(\mathcal{H}_u^{(\tau)}, \mathcal{H}_v^{(\tau)}) \text{ is small.} \tag{2}$$

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Now, set $\tau = 20n \log n$ and suppose that $\inf_{z \in \partial K} \|v - z\|_\infty > 100\sigma \log n$.

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Now, set $\tau = 20n \log n$ and suppose that $\inf_{z \in \partial K} \|v - z\|_\infty > 100\sigma \log n$. An application of the Chernoff bound (to show that no coordinate is chosen many times) together with a Gaussian tail bound (to show that none of the coordinate changes are too large) shows that with high probability, the point remains in K.

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Indeed, the distance between the two is at most the sum of all $\lambda_{\mathbb{I}}$ over all $\mathbb{I} \in \mathcal{M}_{n,\tau}$ with a zero in them, and this in turn can be made small for large τ. Therefore, it suffices to show that

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Therefore,

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which is small.

Concluding the main proof

Now, consider the multistep Gaussian CHR walk $\mathcal{G}'$, performing $\tau = 20n \log n$ steps of $\mathcal{G}$ at each step.

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Given that $B_\infty \subseteq K$, we may further show that $(1 - 100\sigma \log n)K \subseteq K'$. Consequently,

$$\mathrm{vol}(K') \geq (1 - 100\sigma n \log n)\,\mathrm{vol}(K).$$

This is referred to as the (ϵ, δ, ν) property.

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Observe that for any $x \in T_1', y \in T_2', \|x - y\|_2 > \delta$. Indeed, otherwise,

$$1 - \nu \geq d_{\mathrm{TV}}(\mathcal{G}'(x, \cdot), \mathcal{G}'(y, \cdot)) \geq 1 - \mathcal{G}'(x, S_2) - \mathcal{G}'(y, S_1) > 1 - \nu.$$

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We now use Theorem 2.6 of [LS93].

Theorem 7

Let $\delta > 0$ and $\|\cdot\|_\ell$ be a norm on $\mathbb{R}^n$. Let K be a convex body in $\mathbb{R}^n$ and K_1, K_2 be disjoint measurable subsets of K such that for $u \in K_1$ and $v \in K_2, \|u - v\|_\ell \geq \delta$. Further suppose that $\sup_{x,y \in K} \|x - y\|_\ell = D$.

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$$\pi_K(K \setminus (K_1 \cup K_2)) \geq \frac{2\delta}{D - \delta} \min\{\pi_K(K_1), \pi_K(K_2)\}.$$

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$$\Phi_{\mathcal{G}', \pi_K}(S_1, S_2) \geq \Phi_{\mathcal{G}', \pi_K}(T_1 \setminus T_1', S_2) \geq \frac{\nu}{4}(\pi_K(S_1) - \epsilon).$$

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\end{aligned}$$

This gives a bound on the ϵ-conductance of $\mathcal{G}'$!

Going from $\mathcal{G}'$ to $\mathcal{G}$

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$$\Phi_{\mathcal{G},\pi_K}(S, K \setminus S) \geq \frac{1}{\tau}\Phi_{\mathcal{G}',\pi_K}(S, K \setminus S).$$

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Finally, since the Radon-Nikodym derivative of $\mathcal{C}(x,\cdot)$ with respect to $\mathcal{G}(x,\cdot)$ is bounded below by $\sqrt{2\pi}\sigma/2R$ at any point,

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Substituting $\tau = 20n\log n$, $\nu = 1/4$, $\delta = \sigma$, $\epsilon = s$, $\sigma = 32s/(n\log n)$ and using the fact that $D \leq R\sqrt{n}$, we get

$$\Phi_{\mathcal{C},\pi_K}(S, K \setminus S) \geq \frac{cs^2}{R^2 n^{3.5}(\log n)^3}(\min\{\pi_K(S), \pi_K(K \setminus S)\} - \epsilon),$$

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Overview of the proof

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Similar to how in the earlier proof we used the (ϵ, δ, ν) property to arrive at an isoperimetric inequality, here we use the following theorem to get the same.

The main result

Theorem 9

Let K be a convex body satisfying $B_2^n \subseteq K$ and let $R_0^2 = \mathbf{E}_{x \sim \pi_K} \|x - b_K\|_2^2$. Let $S_1, S_2 \subseteq K$ be axis-disjoint. Then, there exists a universal constant c such that

$$\pi_K(K \setminus (S_1 \cup S_2)) \geq \frac{c\epsilon}{R_0 n^{3.5} \log n}(\min\{\pi_K(S_1), \pi_K(S_2)\} - \epsilon).$$

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To prove this, we consider a grid of cubes, proving an isoperimetric inequality on each of them separately, then combining these together to give a global isoperimetric inequality.

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However, this gives that $|\ell \cap K| > |\ell \cap S_1| + |\ell \cap S_2|$, which is clearly incorrect since the two are equal.

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Before proving Theorem 9, let us show how a bound on the conductance follows from it. Suppose $S_1 \sqcup S_2$ is a partition of K into disjoint measurable subsets. Set

$$S'_i = \left\{ x \in S_i : \mathcal{C}(x, S_{3-i}) < \frac{1}{2n} \right\}.$$

We claim that S'_1 and S'_2 are axis-disjoint. Otherwise, let ℓ be an axis-parallel line intersecting both of them. If $x_i \in \ell \cap S'_i$,

$$\frac{1}{2n} > \mathcal{C}(x_i, S_{3-i}) \geq \frac{1}{n} \frac{|\ell \cap S_i|}{|\ell \cap K|}.$$

However, this gives that $|\ell \cap K| > |\ell \cap S_1| + |\ell \cap S_2|$, which is clearly incorrect since the two are equal.
The remainder of the proof follows near-identically to the final part of the proof of the first bound, except that instead of the (ϵ, δ, ν) property to bound conductance, we use the given theorem regarding axis-disjoint sets.

Isoperimetry on a cube

The following isoperimetric inequality is not too hard to prove and we use it extensively.

Lemma 10

Let C be an axis-aligned cube in $\mathbb{R}^n$. For any axis-disjoint $S_1, S_2 \subseteq C$,

$$\pi_C(C \setminus (S_1 \cup S_2)) \geq \frac{1}{4n \log n} \min\{\pi_C(S_1), \pi_C(S_2)\}.$$

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we also have that

$$\pi_C(C \setminus (S_1 \cup S_2)) \geq \frac{1}{16n \log n} \pi_C(S_1). \tag{3}$$

The lattice

We now move on to the proof of the result. Let S_1, S_2 be axis-disjoint subsets of K. Similar to the first proof, set $K' = (1 - \epsilon/20n)K$, and let $S_i' = S_i \cap K'$. Suppose $\mathrm{vol}(S_1') \leq \mathrm{vol}(S_2')$.

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Consider the lattice $\delta\mathbb{Z}^n$, where $\delta = \epsilon/80n\sqrt{n}$. This value of δ is chosen to ensure that any of the cubes that intersect S_i' are contained completely within K.
Define $\mathcal{C}$ to be the set of hypercubes in the lattice that intersect S_1, $\mathcal{C}_1$ as the set of cubes in $\mathcal{C}$ where S_1 takes up at least $2/3$ the volume of the cube, and $\mathcal{C}_2$ as $\mathcal{C} \setminus \mathcal{C}_1$.

The case where $\mathcal{C}_1$ has many cubes

If there is a significant number of cubes in $\mathcal{C}_1$, that is, $\mathrm{vol}(\mathcal{C}_1 \cap S_1) \geq \mathrm{vol}(S_1)/2$, then we can use the previous isoperimetry on each of the cubes to arrive at an overall isoperimetric inequality.

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$$\pi_K(S_3) \geq \frac{1}{16 n \log n} \sum_{\substack{c \in \mathcal{C}_1 \\ c \cap K' \neq \emptyset}} \pi_K(c \cap S_1)$$

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The idea is as follows. We first show that the cubes adjacent to $\partial\mathcal{C}_2$ not in $\mathcal{C}_2$ have a significant amount of S_3, then use a known isoperimetric inequality to compare the volume of $\mathcal{C}_2$ to that of $\partial\mathcal{C}_2$.

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Combining these two inequalities compares the volume of $\mathcal{C}_2$ (and thus S_1) to that of S_3, which is exactly what we desire.

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Combining these two inequalities compares the volume of $\mathcal{C}_2$ (and thus S_1) to that of S_3, which is exactly what we desire.
Set $\mathcal{C}_2'$ to be the set of cubes in $\mathcal{C}_2$ that intersect K'.

The isoperimetric inequality (1)

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$$\mathrm{vol}(\partial_K(S)) \geq \frac{\log 2}{R_0} \min\{\mathrm{vol}(S), \mathrm{vol}(K \setminus S)\}, \tag{4}$$

where $\partial_K(S)$ is the boundary of S relative to K.

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where $\partial_K(S)$ is the boundary of S relative to K.
In particular, since we have

$$\mathrm{vol}(\mathcal{C}_2' \cap S_1) \leq \mathrm{vol}(\mathcal{C}_2') \leq \frac{3}{2} \mathrm{vol}(S_1 \cap K') \leq \frac{3}{4} \mathrm{vol}(K'),$$

it follows that

The isoperimetric inequality (2)

$$\mathrm{vol}(\partial_{K'}(\mathcal{C}_2')) \geq \frac{\log 2}{R_0} \min\{\mathrm{vol}(\mathcal{C}_2'), \mathrm{vol}(K' \setminus \mathcal{C}_2')\}$$

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If we manage to compare $\mathrm{vol}(S_3)$ and $\mathrm{vol}(\partial_{K'}(\mathcal{C}_2'))$, we are done.

Comparing vol(S_3) and vol($\partial_{K'}(\mathcal{C}_2')$) (1)

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1. If $\mathrm{vol}(c_1 \cap S_1) \leq \mathrm{vol}(c_1)/3$, at least $1/3$ of the $2/3$ fraction earlier (that is not in S_2) is not in S_1 either, and is thus in S_3.

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2. If $\mathrm{vol}(c_1)/3 \leq \mathrm{vol}(c_1 \cap S_1) \leq 2\,\mathrm{vol}(c_1)/3$, we can use (3) to conclude that

$$\mathrm{vol}(c_1 \cap S_3) \geq \frac{1}{16 n \log n}\,\mathrm{vol}(c_1 \cap S_1) \geq \frac{1}{48 n \log n}\,\mathrm{vol}(c_1).$$

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The above says that each cube adjacent to a facet of $\partial_K(\mathcal{C}_2)$ that is not in $\mathcal{C}_2$ must have at least a $1/(48n\log n)$ fraction of S_3.

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That is,

$$\begin{aligned}\mathrm{vol}(S_3) &\geq \sum_{f\in\partial_{K'}(\mathcal{C}_2')} \frac{1}{96n^2\log n}\delta^n \\ &= \frac{\delta}{96n^2\log n} \sum_{f\in\partial_{K'}(\mathcal{C}_2')} \delta^{n-1}\end{aligned}$$

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&= \frac{\delta}{96n^2\log n}\text{vol}(\partial_{K'}(\mathcal{C}_2')).
\end{aligned}$$

Putting the two pieces together

Combining the two equations,

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completing the proof.

Room for improvement

Comparing the two bounds

In general, we always have that

$$\sqrt{n}R_0 \leq R$$

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