The KLS Conjecture

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§1. An Isoperimetric Problem

1.1. Introduction

In the context of ball-step, let us look at the mixing time of the chain for a general (not necessarily convex) body. As discussed, it suffices to look at the conductance of the chain, which relates to finding a "cut" of the body of relatively small surface area. For example, in a dumbbell, we could have a cut down the central bottleneck, which would result in a very low conductance.
What about convex bodies? It is seen that hyperplanes are not bad (this is made precise in Theorem 1.10). A natural next question is: could we have some convoluted cut that ends up being a bottleneck? In [KLS95], Kannan, Lovàsz, and Simonovits conjectured that this in fact cannot happen. In particular, they claim that up to a constant factor, hyperplane cuts are in fact the "worst" cuts.
We first formalize this notion of a cut to get an expression similar to that of conductance, discuss some localization lemmata similar to those discussed earlier (to reduce n-dimensional integrals to 1-dimensional integrals), prove an improvement of **??** in Theorems 1.7 and 1.9, and finish off with the statement of the conjecture.
Suppose we have a convex body K and we want to find a surface that divides K into two parts, whose measure is minimum relative to that of the two parts.

Definition 1.1. The *isoperimetric coefficient* of a convex body $K \subseteq \mathbb{R}^n$ is defined as the largest number $\psi = \psi(K)$ such that for any measurable $S \subseteq K$,

$$\psi = \inf_{S \subseteq K} \frac{\mathrm{vol}_{n-1}(\partial S)}{\min\{\mathrm{vol}(S), \mathrm{vol}(K \setminus S)\}}$$

More generally, for any log-concave density p on $\mathbb{R}^n$ (instead of $\mathbb{1}_K$ taken above), we can define ψ_p, the isoperimetric constant for p, as

$$\psi_p = \inf_{S \subseteq \mathbb{R}^n} \frac{p(\partial S)}{\min\{p(S), p(\mathbb{R}^n \setminus S)\}}.$$

In some texts, the above definition is replaced with

$$\psi_p = \inf_{S \subseteq \mathbb{R}^n} \frac{p(\partial S)}{p(S)p(\mathbb{R}^n \setminus S)}.$$

Since both definitions are within a factor of 2 of each other, this does not make much difference in our estimations.

This problem turns out to be very intimately related to that of volume computation we explored in the previous section.
[LS90] bounds the isoperimetric coefficient below by $1/d$, where d is the diameter of the body. Note that this is quite obvious if the separating surface ∂S is a (section of a) hyperplane.
[AK91] gives a more general result where the measure is replaced by that with density equal to any log-concave function and bounds it below by $2/d$. This is in fact as tight as we can get in terms of the diameter and indeed, the bound is attained for a thin long cylinder.
However, the bodies we are interested in (in say, volume computation) tend to have a certain structure to them. In particular, sandwiching makes the bodies somewhat round.
The main result of this section is that for every convex body K,

$$\psi(K) \geq \frac{\ln 2}{M_1(K)},$$

where $M_1(K)$ is the average distance of a point in K from the center of gravity of K.

1.2. Needles and Localization Lemmas

To begin, consider the following motivated by **??**.

Definition 1.2. A *needle* is a segment $[a, b] \in \mathbb{R}^n$ together with a non-negative linear function $\ell : I \to \mathbb{R}^{\geq 0}$ not identically 0. If $N = (I, \ell)$ is a needle and f is an integrable function defined on I, denote

$$\int_N f = \int_0^{|b-a|} f(a+tu)\ell(a+tu)^{n-1}\,\mathrm{d}t,$$

where $u = (b-a)/|b-a|$.

Lemma 1.1. Let f_1, f_2, f_3, f_4 be non-negative continuous functions defined on $\mathbb{R}^n$ and $\alpha, \beta > 0$. The following are equivalent.

- For every convex body K in $\mathbb{R}^n$,
$$\left(\int_K f_1\right)^\alpha \left(\int_K f_2\right)^\beta \leq \left(\int_K f_3\right)^\alpha \left(\int_K f_4\right)^\beta.$$
- For every needle N in $\mathbb{R}^n$,
$$\left(\int_N f_1\right)^\alpha \left(\int_N f_2\right)^\beta \leq \left(\int_N f_3\right)^\alpha \left(\int_N f_4\right)^\beta.$$

Proof. The first implying the second is quite easy to show. For the converse, suppose that the second holds but the first does not.
Adding a sufficiently small quantity to f_3 and f_4, we may further assume that they are (strictly) positive. We may also assume that f_1 and f_2 are positive (Why?). Choose some A such that

$$\left(\frac{\int_K f_1}{\int_K f_3}\right)^\alpha > A > \left(\frac{\int_K f_4}{\int_K f_2}\right)^\beta.$$

Then,

$$\int_K f_1 - A^{1/\alpha} f_3 > 0 \text{ and } \int_K A^{1/\beta} f_2 - f_4 > 0.$$

Using **??**, there is some needle N such that

$$\int_N f_1 - A^{1/\alpha} f_3 > 0 \text{ and } \int_N A^{1/\beta} f_2 - f_4 > 0.$$

This implies that

$$\frac{\left(\int_N f_1\right)^\alpha}{\left(\int_N f_3\right)^\alpha} > A > \frac{\left(\int_N f_4\right)^\beta}{\left(\int_N f_2\right)^\beta},$$

thus proving the claim. ∎

Observe that we can extend this more generally to the case where f_1 and f_2 are upper semicontinuous and f_3 and f_4 are lower semicontinuous by considering an appropriate sequence of continuous functions. In particular, this allows us to restrict ourselves from $\mathbb{R}^n$ to some subset T of $\mathbb{R}^n$ by multiplying the functions with the indicator function $\mathbb{1}_T$ (the functions extend to upper semicontinuous functions if T is closed and lower semicontinuous functions if T is open).

Corollary 1.2. Let T be a bounded open convex set in $\mathbb{R}^n$, g a bounded lower semicontinuous function on T, and h a continuous function on T such that

$$\int_T g > 0 \text{ and } \int_T h = 0.$$

Then there is a needle $N = (I, \ell)$ with $I \subseteq T$ such that

$$\int_N g > 0 \text{ and } \int_N h = 0.$$

Proof. Choose some $0 < \delta < \int_T g$ and let $\varepsilon > 0$. Then,

$$\int_T \left(g - \delta + \frac{1}{\varepsilon}h\right) > 0 \text{ and } \int_T (\varepsilon^2 - h) > 0.$$

Extending these functions to functions on $\mathbb{R}^n$ (multiplying with the indicator function) and using **??**, we get a needle $N_\varepsilon = (I_\varepsilon, \ell_\varepsilon)$ with $I_\varepsilon \subseteq T$ (Why?) such that

$$\int_{N_\varepsilon} \left(g - \delta + \frac{1}{\varepsilon}h\right) > 0 \text{ and } \int_{N_\varepsilon} (\varepsilon^2 - h) > 0. \tag{1.1}$$

Observe that $\int_{N_\varepsilon}(g - \delta + \varepsilon) > 0$.
Taking M as the supremum of g on $\mathbb{R}^n$,

$$-M\varepsilon \int_{N_\varepsilon} 1 < \int_{N_\varepsilon} h < \varepsilon^2 \int_{N_\varepsilon} 1. \tag{1.2}$$

Consider these needles for $\varepsilon = 1/k$ ($k \in \mathbb{N}$). Scaling appropriately, we may assume that the maximum of each linear function $\ell_{1/k}$ is 1. Using the Bolzano-Weierstrass Theorem, there is some subsequence of these needles that converges (in the sense that the endpoints of the $I_{1/k}$ and the $\ell_{1/k}$ converge)[1] to some needle $N = (I, \ell)$. Combining Equations (1.1) and (1.2) implies that N satisfies the required (we get $\int_N (g - \delta) \geq 0$ and $\int_N h = 0$). ∎

While these results are quite nice, exponents of a linear function are not very convenient to deal with. This motivates the following.

1.3. Exponential Needles

Definition 1.3. An *exponential needle* is a segment $[a, b] \in \mathbb{R}^n$ together with a real γ. If $E = (I, \gamma)$ is a needle and f is an integrable function defined on I, denote

$$\int_E f = \int_0^{|b-a|} f(a + tu)e^{\gamma t} \, \mathrm{d}t,$$

where $u = (b - a)/\|b - a\|$.

If we manage to prove our results for an exponential needle instead, it is extremely convenient because taking exponents does not change the underlying structure of the function itself.

Lemma 1.3. Let f_1, f_2, f_3, and f_4 be four non-negative continuous functions defined on an interval $[a, b]$ in $\mathbb{R}$ and $\alpha, \beta > 0$. Then the following are equivalent.

- For every log-concave function F defined on $\mathbb{R}$,

$$\left(\int_a^b F(t)f_1(t)\,\mathrm{d}t\right)^\alpha \left(\int_a^b F(t)f_2(t)\,\mathrm{d}t\right)^\beta \leq \left(\int_a^b F(t)f_3(t)\,\mathrm{d}t\right)^\alpha \left(\int_a^b F(t)f_4(t)\,\mathrm{d}t\right)^\beta.$$

- For every subinterval $[a', b'] \subseteq [a, b]$ and real γ,

$$\left(\int_{a'}^{b'} e^{\gamma t} f_1(t)\,\mathrm{d}t\right)^\alpha \left(\int_{a'}^{b'} e^{\gamma t} f_2(t)\,\mathrm{d}t\right)^\beta \leq \left(\int_{a'}^{b'} e^{\gamma t} f_3(t)\,\mathrm{d}t\right)^\alpha \left(\int_{a'}^{b'} e^{\gamma t} f_4(t)\,\mathrm{d}t\right)^\beta.$$

Proof. The first implying the second is obvious (on setting $F = \mathbb{1}_{[a',b']}e^{\gamma t}$).
Note that if for some $t_0 \in [a, b]$, $f_1(t_0)^\alpha f_2(t_0)^\beta > f_3(t_0)^\alpha f_4(t_0)^\beta$, then both the assertions above fail since we can consider

[1] We can think of an needle $N = ([a, b], \ell)$ as an element $(a, b, \ell(a), \ell(b) - \ell(a)) \in \mathbb{R}^{2n+2}$. In our case, this sequence is bounded because each interval is in the bounded set T and each ℓ is between 0 and 1.

- the log-concave function $e^{-c(t-t_0)^2}$ for a sufficiently large c, or
- a sufficiently small interval containing t_0.

Therefore, we may assume that for all $t \in [a,b]$,

$$f_1(t)^\alpha f_2(t)^\beta \le f_3(t)^\alpha f_4(t)^\beta. \tag{$*$}$$

Suppose the second holds and the first does not for some log-concave function F.
We may assume that $F \neq 0$ (so $F > 0$) on $[a,b]$. Otherwise, we can replace it with its convolution with e^{-ct^2} for a sufficiently large c, which is still log-concave by **??** and would still satisfy the inequality (Why?). We may also assume that $F \geq 1$ on $[a,b]$ by scaling up appropriately. Let $F = e^G$, where G is a non-negative concave function on $[a,b]$.
For each n, define $K_n \subseteq \mathbb{R}^{n+1}$ by

$$K_n = \left\{ (t,x) : t \in [a,b], x \in \mathbb{R}^n, \|x\| \le 1 + \frac{G(t)}{n} \right\}.$$

Let $\hat{f}_i : \mathbb{R}^{n+1} \to \mathbb{R}$ by defined by $\hat{f}_i(t,x) = f_i(t)$.
For sufficiently large n, we have $(1 + G(t)/n)^n \approx e^{G(t)} = F(t)$, so we can write

$$\left(\int_{K_n} \hat{f}_1(t)\, \mathrm{d}t \right)^\alpha \left(\int_{K_n} \hat{f}_2(t)\, \mathrm{d}t \right)^\beta > \left(\int_{K_n} \hat{f}_3(t)\, \mathrm{d}t \right)^\alpha \left(\int_{K_n} \hat{f}_4(t)\, \mathrm{d}t \right)^\beta.$$

Using Lemma 1.1, there exists a needle N_n such that

$$\left(\int_{N_n} \hat{f}_1(t)\, \mathrm{d}t \right)^\alpha \left(\int_{N_n} \hat{f}_2(t)\, \mathrm{d}t \right)^\beta > \left(\int_{N_n} \hat{f}_3(t)\, \mathrm{d}t \right)^\alpha \left(\int_{N_n} \hat{f}_4(t)\, \mathrm{d}t \right)^\beta.$$

If N_n is orthogonal to the t-axis, then ($*$) immediately breaks so we arrive at a contradiction. Otherwise, we may project the needle onto the t-axis to get some $[a_n, b_n] \subseteq [a,b]$ and a linear function ℓ_n such that

$$\left(\int_{a_n}^{b_n} \ell_n(t)^n \hat{f}_1(t)\, \mathrm{d}t \right)^\alpha \left(\int_{a_n}^{b_n} \ell_n(t)^n \hat{f}_2(t)\, \mathrm{d}t \right)^\beta > \left(\int_{a_n}^{b_n} \ell_n(t)^n \hat{f}_3(t)\, \mathrm{d}t \right)^\alpha \left(\int_{a_n}^{b_n} \ell_n(t)^n \hat{f}_4(t)\, \mathrm{d}t \right)^\beta. \tag{1.3}$$

By the Bolzano-Weierstrass Theorem, there is a subsequence such that a_{n_k}, b_{n_k} converge, to say a_0 and b_0. By ($*$), $a_0 < b_0$. Suppose that $\ell_n(a_0) < \ell_n(b_0)$ for infinitely many indices – if not, then exchange a_0 and b_0 in the following argument. Now, let each ℓ_n be normalized such that $\ell_n(b_0) = 1$. Let $\gamma_n = \ell_n(a_0)$ for each n.
For some subsequence, let $\gamma_n \to \gamma$ and $n(1 - \gamma_n) \to \gamma'$, where $0 \le \gamma \le 1$ and $0 \le \gamma' \le \infty$. Henceforth, we restrict ourselves to this subsequence.

- If $\gamma \neq 1$, $\ell_n(t)^n \to 0$ for all $a_0 \le t < b_0$. Dividing Equation (1.3) by $\left(\int_{a_n}^{b_n} \ell_n(t)^n\, \mathrm{d}t \right)^{\alpha+\beta}$ and letting $n \to \infty$, we get
$$f_1(b_0)^\alpha f_2(b_0)^\beta \ge f_3(b_0)^\alpha f_4(b_0)^\beta.$$
If instead of f_3 and f_4 everywhere in the proof above, we instead take $f_3 + \varepsilon$ and $f_4 + \varepsilon$ for a sufficiently small ε, we get a strict inequality above and arrive at a contradiction to ($*$).
- Therefore, $\gamma = 1$. We then have
$$\ell_n(t)^n = \left((1 - (1 - \ell_n(t)))^{1/(1-\ell_n(t))} \right)^{n(1-\ell_n(t))}.$$
The inner expression goes to $1/e$. If $\gamma' = \infty$, then we again get $\ell_n(t)^n \to 0$ for $t < b_0$, so we arrive at a contradiction similar to the first case above. Otherwise, we have
$$\ell_n(t) \to e^{\gamma'(t-b_0)/(b_0-a_0)}.$$

Letting $\gamma'' = \gamma'/(b_0 - a_0)$ and letting $n \to \infty$, we get

$$\left(\int_{a_0}^{b_0} e^{\gamma''(t-b_0)} f_1(t)\,\mathrm{d}t\right)^{\alpha} \left(\int_{a_0}^{b_0} e^{\gamma''(t-b_0)} f_2(t)\,\mathrm{d}t\right)^{\beta} > \left(\int_{a_0}^{b_0} e^{\gamma''(t-b_0)} f_3(t)\,\mathrm{d}t\right)^{\alpha} \left(\int_{a_0}^{b_0} e^{\gamma''(t-b_0)} f_4(t)\,\mathrm{d}t\right)^{\beta}.$$

However, this (after multiplying by $e^{\gamma'' b_0(\alpha+\beta)}$ on either side to remove the b_0 in the exponent) contradicts the original assumption that the opposite inequality holds for any exponential needle, thus completing the proof.

■

The next result is essentially a generalized version of the above lemma, so is relatively straight-forward to prove since we have various tools for localization in our repertoire at this point.

Theorem 1.4. Let f_1, f_2, f_3, and f_4 be non-negative functions on $\mathbb{R}^n$ and $\alpha, \beta > 0$. The following are equivalent.

- For every log-concave function F on $\mathbb{R}^n$ with compact support,

$$\left(\int_{\mathbb{R}^n} F(t) f_1(t)\,\mathrm{d}t\right)^{\alpha} \left(\int_{\mathbb{R}^n} F(t) f_2(t)\,\mathrm{d}t\right)^{\beta} \le \left(\int_{\mathbb{R}^n} F(t) f_3(t)\,\mathrm{d}t\right)^{\alpha} \left(\int_{\mathbb{R}^n} F(t) f_4(t)\,\mathrm{d}t\right)^{\beta}.$$

- For every exponential needle E in $\mathbb{R}^n$,

$$\left(\int_E f_1\right)^{\alpha} \left(\int_E f_2\right)^{\beta} \le \left(\int_E f_3\right)^{\alpha} \left(\int_E f_4\right)^{\beta}.$$

Proof. Going from the first to the second isn't too difficult. Given the exponential needle over $[a, b]$ and constant γ, consider the function F defined by $t \mapsto e^{\gamma\langle t, u\rangle}$, where $u = (b-a)/\|b-a\|$ restricted to some ε-neighbourhood of $[a, b]$. Letting $\varepsilon \to 0$, we get the required.
On the other hand, let the second hold but not the first for some function F. Then applying Lemma 1.1 on the Ff_i, we get some $[a, b]$ and linear function ℓ on $[a, b]$ such that

$$\left(\int_0^{\|b-a\|} f_1(a+tu)F(a+tu)\ell(a+tu)^{n-1}\,\mathrm{d}t\right)^{\alpha} \left(\int_0^{\|b-a\|} f_2(a+tu)F(a+tu)\ell(a+tu)^{n-1}\,\mathrm{d}t\right)^{\beta}$$
$$> \left(\int_0^{\|b-a\|} f_3(a+tu)F(a+tu)\ell(a+tu)^{n-1}\,\mathrm{d}t\right)^{\alpha} \left(\int_0^{\|b-a\|} f_4(a+tu)F(a+tu)\ell(a+tu)^{n-1}\,\mathrm{d}t\right)^{\beta},$$

where u has the usual meaning of $(b-a)/\|b-a\|$.
However, $F\ell^{n-1}$ is log-concave, so by Lemma 1.3, there exists an exponential needle that violates the assumption. ■

1.4. An Example Using the Equivalences

Let K be a convex body and $f : K \to \mathbb{R}$ be integrable. Define its L_p norm by

$$\|f\|_p = \left(\frac{1}{\operatorname{vol} K} \int_K |f(x)|^p\,\mathrm{d}x\right)^{1/p}.$$

It is easy to see that if $0 < p < q$, $\|f\|_p \le \|f\|_q$.

Theorem 1.5. Let $0 < p < q$. There exists a constant $c_{p,q}$ such that for any dimension n, convex body $K \subseteq \mathbb{R}^n$ and linear function $f : K \to \mathbb{R}$,

$$\|f\|_q \le c_{p,q} \|f\|_p$$

Proof. We wish to show that for any K,

$$\left(\int_K |f|^q\right)^{1/q}\left(\int_K 1\right)^{1/p} \leq c_{p,q}\left(\int_K 1\right)^{1/q}\left(\int_K |f|^p\right)^{1/p}.$$

Equivalently, we wish to show that for any exponential needle E,

$$\left(\int_E |f|^q\right)^{1/q}\left(\int_E 1\right)^{1/p} \leq c_{p,q}\left(\int_E 1\right)^{1/q}\left(\int_E |f|^p\right)^{1/p}.$$

That is, we wish to show that for any linear function f, $a, b \in \mathbb{R}$, and real γ,

$$\left(\frac{\int_a^b e^{\gamma t}|f(t)|^q \,\mathrm{d}t}{\int_a^b e^{\gamma t}\,\mathrm{d}t}\right)^{1/q} \leq c_{p,q}\left(\frac{\int_a^b e^{\gamma t}|f(t)|^p \,\mathrm{d}t}{\int_a^b e^{\gamma t}\,\mathrm{d}t}\right)^{1/p},$$

Since f is linear, we may assume without loss of generality that $f(a+tu) = t$ on $[a,b]$ and that $\gamma = 1$; for the general case where $\gamma \neq 0$, we can just substitute appropriately. The cases where $\gamma = 0$ or f is constant on $[a,b]$ are easily shown.

$$\varphi(a,b) = \left(\frac{\int_a^b e^{t}|f(t)|^q \,\mathrm{d}t}{\int_a^b e^{t}\,\mathrm{d}t}\right)^{1/q}\left(\frac{\int_a^b e^{t}|f(t)|^p \,\mathrm{d}t}{\int_a^b e^{t}\,\mathrm{d}t}\right)^{-1/p}.$$

We wish to show that $c_{p,q} = \sup_{a<b}\varphi(a,b)$ is finite. Note that φ is continuous for $a < b$. Further, for any α, $\varphi(a,b) \to 1$ as $a, b \to \alpha$. That is, we may extend the function continuously to $a \leq b$ defining $\varphi(a,a) = 1$.
Now, observe that for fixed a, as $b \to \infty$, $\varphi(a,b) \to 1$.[2] On the other hand, for fixed b and $a \to \infty$, $\varphi(a,b)$ remains bounded. The continuity implies that φ is bounded (and its supremum is finite). ∎

The actual calculation of the supremum above is quite tedious, however.

1.5. Isotropy

The content of this section is closely related to that of volume computation, primarily **??**, which discussed sandwiching.
Given a convex body $K \subseteq \mathbb{R}^n$ and $f : K \to \mathbb{R}^m$, denote by $\mathbf{E}_K(f)$ the average of f over K. That is,

$$\mathbf{E}_K(f) = \frac{1}{\mathrm{vol}(K)}\int_K f(x)\,\mathrm{d}x.$$

Denote by $b(K) = \mathbf{E}_K(x)$ the center of gravity of K, also known as the barycenter of K. If K is clear from context, we often denote it as just b. Denote by $A(K)$ the $n \times n$ matrix of inertia

$$A(K) = \mathbf{E}_K((x-b)(x-b)^\top).$$

Denote by $M_p(K)$ the pth moment of K

$$M_p(K) = \mathbf{E}_K\left(\|x-b\|^p\right).$$

It is seen that $M_2(K)$ is the trace of $A(K)$. Further, the average squared distance between points in K is

$$\frac{1}{\mathrm{vol}(K)^2}\int_K\int_K \|x-y\|^2\,\mathrm{d}x\,\mathrm{d}y = 2M_2(K).$$

As $p \to \infty$, $M_p(K)^{1/p}$ converges to $\sup_{x\in K}\|x-b\|$.

[2] $\int_a^b e^t|f(t)|^p\,\mathrm{d}t$ grows as $e^{-b}b^p$ and $\int_a^b e^t\,\mathrm{d}t$ grows as e^b.

Definition 1.4 (Isotropic)**.** A body K is said to be in *isotropic position* if $b = 0$ and $A(K) = I$, the identity matrix.[3] Similarly, a function $f : \mathbb{R}^n \to [0, \infty)$ is said to be *isotropic* if its covariance matrix is the identity matrix.

Observe that a convex body is in isotropic position iff its indicator function is isotropic.
It may be shown the affine family of a convex body (the set of its image under affine transformations) has a unique body in isotropic position.
First, let us show how isotropic position is related to sandwiching.

Theorem 1.6. If K is in isotropic position, then

$$\sqrt{\frac{n+2}{n}} B_2^n \subseteq K \subseteq \sqrt{n(n+2)} B_2^n.$$

Observe that these inequalities are tight for the regular simplex and also imply the second part of **??**. If K is in isotropic position, then for any unit u,

$$\int_K \langle u, x \rangle^2 \, \mathrm{d}x = \mathrm{vol}(K).$$

Proof.

- Suppose that $\sqrt{(n+2)/n} B_2^n \not\subseteq K$. Choosing our basis appropriately, we may assume that K is contained in the half-space $x_1 > -\sqrt{(n+2)/n}$. Now, we have

  $$\int_K x_1 = 0 \text{ and } \int_K (x_1^2 - 1) = 0.$$

  Using Corollary 1.2 (or rather, an extension of it with a weak inequality on $\int g$), we get some needle $N = ([a, b], \ell)$. We may assume that $[a, b]$ is contained in the x_1 axis, so that

  $$\int_a^b x_1 \ell(x_1)^{n-1} = 0 \text{ and } \int_a^b x_1^2 \ell(x_1)^{n-1} \geq \int_a^b \ell(x_1)^{n-1}.$$

  We have $a > -\sqrt{\frac{n+2}{n}}$. It is easy to see[4] that we may assume that ℓ is decreasing, and thus may suppose that is of the form $t - x$ for some $\lambda \geq b$. We can then manually (and tediously) compute the integrals to arrive at a contradiction.

- Let v be the point in K furthest from 0 (assume that K is closed so this is well-defined). We wish to show that $\|v\| \leq \sqrt{n(n+2)}$. Let $v^\circ = v / \|v\|$ and for each unit u, let $\varphi(u) = \sup\{t \geq 0 : v + tu \in K\}$. Then,

  $$\mathrm{vol}(K) = \int_{\partial B_2^n} \int_0^{\varphi(u)} t^{n-1} \, \mathrm{d}t \, \mathrm{d}u = \int_{\partial B_2^n} \frac{\varphi(u)^n}{n} \, \mathrm{d}u.$$

[3] Some texts use $\mathrm{vol}(K) = 1$ and $A(K) = \lambda_K I$ for some constant λ_K. It remains an open problem as to whether the value of λ_K across convex bodies $K \subseteq \mathbb{R}^n$ is bounded above.

[4] If $|b| > |a|$, then the first equality implies that ℓ cannot be increasing. Otherwise, we can use the second inequality to justify the assumption.

We also have

$$\begin{aligned}
1 &= \frac{1}{\mathrm{vol}(K)} \int_K \langle v^\circ, x\rangle^2 \,\mathrm{d}x \\
&= \frac{1}{\mathrm{vol}(K)} \int_{\partial B_2^n} \int_0^{\varphi(u)} t^{n-1} \langle v^\circ, v + tu\rangle^2 \,\mathrm{d}t \,\mathrm{d}u \\
&= \frac{1}{\mathrm{vol}(K)} \int_{\partial B_2^n} \left(\frac{\varphi(u)^n}{n} \|v\|^2 + 2\frac{\varphi(u)^{n+1}}{n+1} \langle v^\circ, u\rangle + \frac{\varphi(u)^{n+2}}{n+2} \langle v^\circ, u\rangle^2 \right) \mathrm{d}u \\
&= \frac{1}{\mathrm{vol}(K)} \int_{\partial B_2^n} \left(\frac{\varphi(u)^n}{n} \left(\frac{\sqrt{n(n+2)}}{n+1} \|v\| + \sqrt{\frac{n}{n+2}} \varphi(u) \langle v^\circ, u\rangle \right)^2 + \frac{\varphi(u)^n}{n(n+1)^2} \|v\|^2 \right) \mathrm{d}u \\
&= \frac{1}{\mathrm{vol}(K)} \left(\int_{\partial B_2^n} \frac{\varphi(u)^n}{n} \left(\frac{\sqrt{n(n+2)}}{n+1} \|v\| + \sqrt{\frac{n}{n+2}} \varphi(u) \langle v^\circ, u\rangle \right)^2 \mathrm{d}u \right) + \frac{\|v\|^2}{(n+1)^2}
\end{aligned}$$

This gives a bound of $\|v\| \leq n+1$. To get the bound mentioned in the theorem, it remains to bound the integral by a suitable positive quantity. Now, we have

$$\begin{aligned}
0 = b(K) &= \frac{1}{\mathrm{vol}(K)} \int_{\partial B_2^n} \int_0^{\varphi(u)} t^{n-1} (v + tu) \,\mathrm{d}t \,\mathrm{d}u \\
&= \frac{1}{\mathrm{vol}(K)} \int_{\partial B_2^n} \frac{\varphi(u)^n}{n} v + \frac{\varphi(u)^{n+1}}{n+1} u \,\mathrm{d}u \\
&= v + \frac{1}{\mathrm{vol}(K)} \int_{\partial B_2^n} \frac{\varphi(u)^{n+1}}{n+1} u \,\mathrm{d}u.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{\mathrm{vol}(K)} \int_{\partial B_2^n} \frac{\varphi(u)^n}{n} \left(\frac{\sqrt{n(n+2)}}{n+1} \|v\| + \sqrt{\frac{n}{n+2}} \varphi(u) \langle v^\circ, u\rangle \right) \mathrm{d}u& \\
= \left(\frac{\sqrt{n(n+2)}}{n+1} - \frac{n+1}{\sqrt{n(n+2)}} \right) \|v\| &= -\frac{1}{(n+1)\sqrt{n(n+2)}} \|v\|.
\end{aligned}$$

We can then use the Cauchy-Schwarz inequality to get

$$\left(\frac{1}{\mathrm{vol}(K)} \int_{\partial B_2^n} \frac{\varphi(u)^n}{n} \left(\frac{\sqrt{n(n+2)}}{n+1} \|v\| + \sqrt{\frac{n}{n+2}} \varphi(u) \langle v^\circ, u\rangle \right)^2 \mathrm{d}u \right) \cdot 1 \geq \frac{1}{(n+1)^2 n(n+2)} \|v\|^2.$$

That is,

$$1 \geq \left(\frac{1}{(n+1)^2 n(n+2)} + \frac{1}{(n+1)^2} \right) \|v\|^2 = \frac{\|v\|^2}{n(n+2)},$$

proving the result.

■

1.6. The KLS Conjecture

Let us now move on to the main result of this section.

Theorem 1.7. For any convex body K,

$$\psi(K) \geq \frac{\ln 2}{M_1(K)}.$$

In Definition 1.1, let K_3 be the intersection of K with the open $\varepsilon/2$-neighbourhood of ∂S. Further, let $K_1 = S \setminus K_3$ and $K_2 = (K \setminus S) \setminus K_3$. Then, it suffices to prove the following, which is yet another improvement of **????**.

Theorem 1.8. Let K be a convex body and $K = K_1 \cup K_2 \cup K_3$ a decomposition of K into three measurable sets such that $d(K_1, K_2) = \varepsilon > 0$. Then

$$\operatorname{vol}(K_1)\operatorname{vol}(K_2) \leq \frac{M_1(K)}{\varepsilon \ln 2}\operatorname{vol}(K)\operatorname{vol}(K_3).$$

Proof. We may assume that K_1 and K_2 are closed. Assume that $b(K) = 0$. Let f_1, f_2, and f_3 be the indicator functions on K_1, K_2, and K_3 respectively and $f_4(x) = \|x\|/\varepsilon \ln 2$. We then wish to show that

$$\int_K f_1 \int_K f_2 \leq \int_K f_3 \int_K f_4.$$

By Lemma 1.3, it suffices to show that for any exponential needle E,

$$\int_E f_1 \int_E f_2 \leq \int_E f_3 \int_E f_4.$$

Let E be an arbitrary exponential defined by $[a, b]$ and γ. As before, we may assume that $\gamma = 1$ by rescaling appropriately. The case $\gamma = 0$ is taken care of by going to the appropriate limits.
First of all, we may assume that $0 \in [a, b]$. Indeed, otherwise, we can move the body such that 0 goes to the point on $[a, b]$ closest to it initially. Then the integral of f_4 decreases while the others remain the same, so proving it for this case suffices.
So let us restate the problem in the one-dimensional case that we have reduced it to. Let $[a, b]$ be an interval, $u \in [a, b]$ and $[a, b] = J_1 \cup J_2 \cup J_3$ be a decomposition of $[a, b]$ into three measurable sets, where $d(J_1, J_2) \geq \varepsilon > 0$. We wish to show that

$$\int_{J_1} e^t \,\mathrm{d}t \int_{J_2} e^t \,\mathrm{d}t \leq \int_{J_3} e^t \,\mathrm{d}t \int_a^b \frac{|t-u|}{\varepsilon \ln 2} e^t \,\mathrm{d}t.$$

Here, each J_i corresponds to the intersection of K_i with the interval and u corresponds to the position of 0 in $[a, b]$.
Let us first prove the result for the case where J_3 is a single interval. Let $a \leq s < s + \varepsilon \leq b$ (Why does it suffice to prove it for the case where the interval is of length ε?). Then we claim that

$$\int_a^s e^t \,\mathrm{d}t \int_{s+\varepsilon}^b e^t \,\mathrm{d}t \leq \int_s^{s+\varepsilon} e^t \,\mathrm{d}t \int_a^b \frac{|t-u|}{\varepsilon \ln 2} e^t \,\mathrm{d}t.$$

Equivalently,

$$\int_a^s e^t \,\mathrm{d}t \int_\varepsilon^{b-s} e^t \,\mathrm{d}t \leq \int_0^\varepsilon e^t \,\mathrm{d}t \int_a^b \frac{|t-u|}{\varepsilon \ln 2} e^t \,\mathrm{d}t.$$

Now, note that the expression on the left is maximized when $s = (a + b - \varepsilon)/2$ and that on the right is minimized when $u = \ln((e^a + e^b)/2)$. Substituting these values on each side and simplifying, it suffices to show that

$$(e^{(b-a)/2} - e^{\varepsilon/2})^2 \leq \frac{1}{\ln 2} \frac{e^\varepsilon - 1}{\varepsilon} \left(-\ln\left(\frac{e^{a-b} - 1}{2}\right) e^{b-a} + \ln\left(\frac{e^{b-a} - 1}{2}\right) \right).$$

On decreasing ε, the left increases whereas the right decreases. Therefore, it suffices to prove the above in the limit case where $\varepsilon = 0$. Letting $z = e^{(b-a)/2} \geq 1$, we want to prove that

$$\ln 2(z-1)^2 + z^2 \ln\left(\frac{z^{-2}-1}{2}\right) - \ln\left(\frac{z^2-1}{2}\right) \leq 0.$$

This is a computational task and is not too difficult.[5]

For the general case, let $[c_i, d_i]$ be maximal intervals in J_3 for $1 \leq i \leq k$. They are each of length at least ε. Then we get

$$\sum_{i=1}^{k} \int_a^{c_i} e^t \,\mathrm{d}t \int_{d_i}^b e^t \,\mathrm{d}t \leq \int_{J_3} \mathrm{d}t \int_a^b \frac{|t-u|}{\varepsilon \ln 2} e^t \,\mathrm{d}t.$$

We then have

$$\sum_{i=1}^{k} \int_a^{c_i} e^t \,\mathrm{d}t \int_{d_i}^b e^t \,\mathrm{d}t \geq \int_{J_1} e^t \,\mathrm{d}t \int_{J_2} e^t \,\mathrm{d}t,$$

completing the proof. ■

Let K be an arbitrary convex body and for each $x \in K$, let $\chi_K(x)$ denote the longest segment in K that has midpoint x. Let

$$\chi(K) = \frac{1}{\mathrm{vol}(K)} \int_K \chi_K(x).$$

Note that $\chi(K) = \mathrm{diam}(K \cap (2x - K))$.

Theorem 1.9. For any convex body K,

$$\psi(K) \geq \frac{1}{\chi(K)}.$$

Proof. As before, it is equivalent to show that for any decomposition $K = K_1 \cup K_2 \cup K_3$, where $d(K_1, K_2) = \varepsilon > 0$,

$$\mathrm{vol}(K_1)\,\mathrm{vol}(K_2) \leq \frac{1}{\varepsilon}\,\mathrm{vol}(K_3) \int_K \chi_K(x)\,\mathrm{d}x.$$

The proof of this is very similar to the that of the previous theorem. It suffices to show that for any interval $[a, b]$ and any decomposition $[a, b] = J_1 \cup J_2 \cup J_3$ into three measurable sets such that $d(J_1, J_2) \geq \varepsilon$,

$$\int_{J_1} e^t \,\mathrm{d}t \int_{J_2} e^t \,\mathrm{d}t \leq \frac{1}{\varepsilon} \int_{J_3} e^t \,\mathrm{d}t \int_a^b \min\{t-a, b-t\} e^t \,\mathrm{d}t.$$

Similar to earlier, this can be shown without too much difficulty in the case where J_3 is a single interval, and similarly extending it to the general case. ■

The two bounds Theorems 1.7 and 1.9 are not comparable however. For example, Theorem 1.7 gives $\psi(K) = \Omega(n^{-1/2})$ for any body in isotropic position whereas Theorem 1.9 gives $\Omega(1)$ for the isotropic ball and $\Omega(n^{-1})$ for the isotropic simplex.

Theorem 1.10. For any convex body K with covariance matrix A,

$$\psi(K) \leq \frac{10}{\sqrt{\|A\|_{\mathrm{op}}}}.$$

[5] Show that the function f on the left is monotone decreasing and use the fact that $f(1) = 0$.

This is proved using the following result.

Theorem 1.11. Let K be a convex body in $\mathbb{R}^n$ and assume that $b(K) = 0$. Let $u \in \mathbb{R}^n$ have unit norm and $\beta = \mathbf{E}_K(\langle u, x\rangle^2)$. Then

$$\operatorname{vol}(K \cap \{x : \langle u, x\rangle < 0\}) \operatorname{vol}(K \cap \{x : \langle u, x\rangle > 0\}) \geq \frac{1}{10}\sqrt{\beta} \operatorname{vol}(K) \operatorname{vol}_{n-1}(K \cap \{x : \langle u, x\rangle = 0\}).$$

The above can be proved by projecting the body onto the u-axis and considering the resulting log-concave function (using **??**).

Conjecture (KLS Conjecture)**.** There is a constant c (independent of dimension) such that for any log-concave density p on $\mathbb{R}^n$,

$$\psi_p \geq c \cdot \inf_{H \text{ is a halfspace}} \frac{p(\partial H)}{\min\{p(H), p(\mathbb{R}^n \setminus H)\}}.$$

The KLS Conjecture asserts that up to a constant factor, a hyperplane cut is the "worst" cut (involved in the isoperimetric coefficient).

§2. A More Detailed Look

Henceforth, we write $a \gtrsim b$ if there is some constant c (independent of dimension and all parameters under consideration) such that $a \geq cb$.
Generalizing Theorem 1.10 to an arbitrary log-concave density (by a nearly identical proof), it just says that

$$\inf_{H \text{ is a halfspace}} \frac{p(\partial H)}{\min\{p(H), p(\mathbb{R}^n \setminus H)\}} \gtrsim \frac{1}{\sqrt{\|A\|_{\mathrm{op}}}},$$

where A is the covariance matrix of p and $\|A\|_{\mathrm{op}}$ is the largest eigenvalue of A.
In this context, the KLS Conjecture can be restated as follows.

Conjecture (KLS Conjecture (Reformulated))**.** For any log-concave density p with covariance matrix A, $\psi_p \gtrsim \|A\|_{\mathrm{op}}^{-1/2}$. Equivalently, $\psi_p \gtrsim 1$ for any isotropic log-concave density p.

Theorem 1.7 then says that for any isotropic log-concave p, $\psi_p \gtrsim n^{-1/2}$.

Next, we look at a few consequences of the KLS Conjecture.

2.1. The Slicing Conjecture

The slicing conjecture essentially asks whether a convex body of unit volume in $\mathbb{R}^n$ has a hyperplane section whose $(n-1)$-volume is at least some universal constant.

Conjecture (Slicing Conjecture)**.** Any convex body $K \subseteq \mathbb{R}^n$ of volume 1 has at least one hyperplane section H such that

$$\mathrm{vol}_{n-1}(K \cap H) \gtrsim 1.$$

[Bal88] showed that the above is in fact equivalent to asking how much volume is present around the origin. This makes sense because if a large proportion of volume is there around the origin, then no hyperplane will intersect a lot of volume.
Motivated by this intuition, define

Definition 2.1 (Slicing Constant)**.** For any isotropic log-concave density p on $\mathbb{R}^n$, define the *isotropic (slicing) constant* by $L_p = p(0)^{1/n}$.

Conjecture (Slicing Conjecture)**.** For any isotropic log-concave density p on $\mathbb{R}^n$, the slicing constant L_p is $\mathcal{O}(1)$.

2.2. The Thin-Shell Conjecture

Conjecture (Thin-Shell Conjecture)**.** Let p be an isotropic log-concave density. Then

$$\sigma_p^2 \coloneqq \mathbf{E}_{X \sim p}\left[(\|X\| - \sqrt{n})^2\right] \lesssim 1.$$

Equivalently, $\mathbf{Var}_{X \sim p}(\|X\|^2) \lesssim 1$.

The above means that a random point X from a log-concave density lies in a constant width annulus (a thin shell) with constant probability.

It was shown in [EK10] that the Thin-Shell conjecture implies the Slicing conjecture and by Ball that the KLS conjecture implies the Thin-Shell conjecture. That is, we have that $L_p \lesssim \sigma_p \lesssim \psi_p^{-1}$.

2.3. The Poincaré Constant

Definition 2.2. For any isotropic log-concave density p in $\mathbb{R}^n$, define the *Poincaré constant* ζ_p by

$$\zeta_p = \inf_{\text{smooth } g} \frac{\mathbf{E}_p\left[\left\|\nabla g(x)\right\|_2^2\right]}{\mathbf{Var}_p(g(x))}.$$

It may be shown that

$$\zeta_p \sim \psi_p^2,$$

that is, ζ_p is within a constant factor of the square of the isoperimetric constant. Due to this strong relation, we shall in fact use the Poincaré constant in a later proof (of Lemma 3.11) to help bound the isoperimetric constant.

§3. Recent Bounds on the Isoperimetric Constant

3.1. A Look At Stochastic Localization

Most of the progress towards proving the conjecture in recent times has been done using a method known as *stochastic localization*. Recall how in the proof of the localization lemma **??**, which is possibly one of the most powerful tools we have built thus far, we use a bisection argument, where we bisect the body with a hyperplane at each step. In the general setting, this just corresponds to multiplying the current log-concave measure by $\mathbb{1}_H$, where H is a certain half-space.

For now, let us discuss how stochastic localization works out in discrete time. Instead of multiplying by this indicator function, we multiply by an affine functional that is very close to 1. That is, we transform the density $p(x)$ to $(1+\varepsilon\langle x-\mu,\theta\rangle)p(x)$, where μ is the barycenter of the measure and θ is randomly chosen. This is like a reweighting in favour of a certain half-space.
As a result, the resulting measure is a probability measure. Further, this measure remains log-concave (assuming p is log-concave).
This gives a stochastic process (which is discrete time for now) defined by

$$p_0(x) = p(x) \text{ and } p_{t+\Delta t}(x) = (1 + \langle x - \mu_t, \sqrt{\Delta t} Z_t\rangle)p_t(x). \tag{3.1}$$

The $\sqrt{\Delta t}Z_t$ represents $\pm\varepsilon\theta$ and is the random component. Here, μ_t is the barycenter of the measure corresponding to p_t and the Z_t are iid random which are either uniform on the sphere on $\sqrt{n}S^{n-1}$ or standard Gaussians in $\mathbb{R}^n$.
By the averaging property mentioned, the p_t form a martingale (with respect to the filtration with $\mathcal{F}_t = \sigma\{Z_s : 0 \leq s \leq t\}$).
Now, we would like to make this continuous by letting $\Delta t \to 0$. How do we do this? When the Z_t are Gaussian, Equation (3.1) can be rewritten as a stochastic differential equation

$$\mathrm{d}p_t(x) = \langle x - \mu_t, \mathrm{d}W_t\rangle p_t(x), \tag{3.2}$$

where $p_0 = p$ and μ_t, as before, is $\int_{\mathbb{R}^n} x p_t(x)\,\mathrm{d}x$.
Existence and uniqueness for all $t \geq 0$ can be shown using standard means. Moreover, for any time t, p_t is almost surely continuous. If $\mathcal{F}_t$ is the σ-algebra generated by $(W_s)_{0\leq s\leq t}$, then $\mathbf{E}[p_t(x) \mid \mathcal{F}_s] = p_s(x)$ for $s < t$. That is, it is a martingale. The processes are also $\mathcal{F}_t$-adapted.

While the above is the basic idea, the following, slightly more complicated form is what is slightly more handy. Define the stochastic differential equation

$$c_0 = 0 \text{ and } \mathrm{d}c_t = \mathrm{d}W_t + \mu_t\,\mathrm{d}t, \tag{3.3}$$

with p_t and μ_t defined by

$$p_t(x) = \frac{e^{\langle c_t, x\rangle - (t/2)\|x\|^2}p(x)}{\int_{\mathbb{R}^n} e^{\langle c_t, y\rangle - (t/2)\|y\|^2}p(y)\,\mathrm{d}y} \text{ and } \mu_t(x) = \mathbf{E}_{x\sim p_t}[x].$$

Sometimes, to add another method to control the covariance, we add in a *control matrix* C_t to control the covariance matrix A_t of the density p_t at time t. This is incorporated into the previous equations as

$$\begin{gathered}
\mathrm{d}p_t(x) = (x-\mu_t)^\top C_t^{1/2}\,\mathrm{d}W_t p_t(x) \\
c_0 = 0, \quad \mathrm{d}c_t = C_t^{1/2}\,\mathrm{d}W_t + C_t\mu_t\,\mathrm{d}t, \\
B_0 = 0, \quad \mathrm{d}B_t = C_t\,\mathrm{d}t, \\
p_t(x) = \frac{e^{\langle c_t, x\rangle - (1/2)(x^\top B_t x)^2}p(x)}{\int_{\mathbb{R}^n} e^{\langle c_t, y\rangle - (1/2)(y^\top B_t y)^2}p(y)\,\mathrm{d}y}, \quad \mu_t(x) = \mathbf{E}_{x\sim p_t}[x],
\end{gathered} \tag{3.4}$$

where C_t is a Lipschitz function with respect to c_t, μ_t, A_t, and t.
It may be shown that a solution to the above equation exists and is unique (up to almost-sure equivalence).

It is also not too difficult to show using Itô's Lemma that (3.4) implies (3.2).

For the other direction, using Equation (3.2),

$$\begin{aligned}\mathrm{d}\log p_t(x) &= (x-\mu_t)^\top \mathrm{d}W_t - \frac{1}{2}(x-\mu_t)^\top(x-\mu_t)\,\mathrm{d}t \\ &= x^\top(\mathrm{d}W_t + \mu_t\,\mathrm{d}t) - \frac{1}{2}\|x\|^2\,\mathrm{d}t + g(t) \\ &= x^\top\,\mathrm{d}c_t - \frac{1}{2}\|x\|^2\,\mathrm{d}t + \mathrm{d}g(t),\end{aligned}$$

where $\mathrm{d}g(t)$ is independent of x. This explains the appearance of the Gaussian in (3.4) – the above implies that p_t is "more log-concave" than $e^{-t\|x\|^2/2}$.
As the Gaussian factor dominates more and more, the density converges to a Dirac delta "function", where the measure of any subset is 0 or 1. This is stated in Theorem 3.4.

The KLS Conjecture has been proven for Gaussian distributions. More generally, for any distribution whose density is the product of the density of $\mathcal{N}(0,\sigma^2 I)$ and any log-concave function, $\psi_p \gtrsim 1/\sigma$ – this can be proved by normal localization.

To get some sort of bound for ψ_p, we want to bound the covariance matrix (in some meaningful sense of the word bound).
Using Itô's Lemma once more on the covariance matrix, we get

$$\begin{aligned}dA_t &= \int_{\mathbb{R}^n}(x-\mu_t)(x-\mu_t)^\top\cdot(x-\mu_t)^\top\,\mathrm{d}W_t\cdot p_t(x)\,\mathrm{d}x - A_t^2\,\mathrm{d}t \\ &= \mathbf{E}_{x\sim p_t}(x-\mu_t)(x-\mu_t)^\top\cdot(x-\mu_t)^\top\,\mathrm{d}W_t - A_t^2\,\mathrm{d}t. \qquad (3.5)\end{aligned}$$

One important result that we have not mentioned thus far is due to [Mil08], which proves that the *isoperimetric profile* $I_p : [0,1]\to\mathbb{R}^+$, defined by

$$I_p(t) = \inf_{\substack{S\subseteq\mathbb{R}^n \\ p(S)=t}} p(\partial S)$$

is concave. In particular, since $I_p(t) = I_p(1-t)$, it attains its maximum at $1/2$. Concavity implies that $I_p(t)/t$ attains its minimum at $t = 1/2$, and therefore to bound the isoperimetric coefficient (and prove the KLS Conjecture), it suffices to check subsets of measure $1/2$.

With this added information, we desire from stochastic localization that the covariance matrix does not explode. In fact, it turns out that it suffices to show that the measure of a set E of measure $1/2$ (initially) does not change much:

$$\begin{aligned}p(\partial E) &= \mathbf{E}\left[p_t(\partial E)\right] && (p_t \text{ is a martingale}) \\ &\geq \mathbf{E}\left[\frac{1}{2}\left\|B_t^{-1}\right\|_2^{-1/2}\min\{p_t(E), p_t(\mathbb{R}^n\setminus E)\}\right] && (p_t \text{ is more log-concave than the Gaussian}) \\ &\geq \frac{1}{4}\cdot\frac{1}{2}\left\|B_t^{-1}\right\|_2^{-1/2}\Pr\left[\frac{1}{4}\leq p_t(E)\leq\frac{3}{4}\right] && (p_t(E)\geq 0) \\ &= \frac{1}{4}\left\|B_t^{-1}\right\|_2^{-1/2}\Pr\left[\frac{1}{4}\leq p_t(E)\leq\frac{3}{4}\right]\min\{p(E), p(\mathbb{R}^n\setminus E)\}. && (3.6)\end{aligned}$$

Over the next two sections, we give a $n^{-1/4}$ bound on ψ_p, as described in [LV17]. More precisely, we show that $\psi_p \gtrsim \mathrm{Tr}(A^2)^{-1/4}$.

3.2. Towards a $n^{-1/4}$ Bound

Before we begin, define the following for notational convenience.

Definition 3.1. For any stochastic processes x_t and y_t, denote the *quadratic variations* $[x]_t$ and $[x,y]_t$ by

$$[x]_t = \lim_{\|P\|\to 0} \sum_{n=1}^{\infty} (x_{\tau_n} - x_{\tau_{n-1}})^2$$

and

$$[x,y]_t = \lim_{\|P\|\to 0} \sum_{n=1}^{\infty} (x_{\tau_n} - x_{\tau_{n-1}})(y_{\tau_n} - y_{\tau_{n-1}}),$$

where $P = \{0 = \tau_0 \le \tau_1 \le \cdots \uparrow t\}$ is a *stochastic partition* of the non-negative reals and $\|P\| = \max_n(\tau_n - \tau_{n-1})$ is called the *mesh* of P and the limit is defined using convergence in probability.

For example, if x_t and y_t satisfy $\mathrm{d}x_t = \mu(x_t)\,\mathrm{d}t + \sigma(x_t)\,\mathrm{d}W_t$ and $\mathrm{d}y_t = \nu(x_t)\,\mathrm{d}t + \eta(y_t)\,\mathrm{d}t$, then

$$\mathrm{d}[x]_t = \sigma^2(x_t)\,\mathrm{d}t \text{ and } \mathrm{d}[x,y]_t = \sigma(x_s)\eta(y_s)\,\mathrm{d}W_t.$$

The following two results will also come in useful.

Lemma 3.1 (Reflection Principle)**.** Given a Wiener process W_t and $a, t \ge 0$,

$$\Pr\left[\sup_{0\le s\le t} W_s \ge a\right] = 2\Pr\left[W_t \ge a\right].$$

Theorem 3.2 (Dambis, Dubins-Schwarz Theorem)**.** Every continuous local martingale M_t is of the form

$$M_t = M_0 + W_{[M]_t} \text{ for all } t \ge 0,$$

where W_s is a Wiener process.

The first simplest case is when we take the control matrix to just be the identity. That is, the relevant stochastic differential equation is given by (3.3). Denote by A_t the covariance matrix of p_t.

First, we give some "basic estimates". Let us start by bounding the measure of any set of initial measure $1/2$.

Lemma 3.3. For any $E \subseteq \mathbb{R}^n$ with $p(E) = 1/2$ and $t \ge 0$,

$$\Pr\left[\frac{1}{4} \le p_t(E) \le \frac{3}{4}\right] \ge \frac{9}{10} - \Pr\left[\int_0^t \|A_s\|_{\mathrm{op}}\,\mathrm{d}s \ge \frac{1}{64}\right].$$

Proof. Let $g_t = p_t(E)$. Then $\mathrm{d}g_t = \int_E (x-\mu_t)^\top\,\mathrm{d}W_t p_t(x)\,\mathrm{d}x$. We have

$$\begin{aligned}
\mathrm{d}[g]_t &= \left\|\int_E (x-\mu_t)p_t(x)\,\mathrm{d}x\right\|_2^2 \mathrm{d}t \\
&= \max_{\|\zeta\|_2\le 1}\left(\int_E (x-\mu_t)^\top \zeta p_t(x)\,\mathrm{d}x\right)^2 \mathrm{d}t \\
&\le \left(\max_{\|\zeta\|_2\le 1}\int_{\mathbb{R}^n} ((x-\mu_t)^\top\zeta)^2 p_t(x)\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^n} p_t(x)\,\mathrm{d}x\right)\mathrm{d}t \\
&= \max_{\|\zeta\|_2\le 1} \zeta^\top A_t \zeta\,\mathrm{d}t = \|A_t\|_{\mathrm{op}}\,\mathrm{d}t.
\end{aligned}$$

Using Theorem 3.2, let $\tilde{W}_t$ be a Wiener process such that $g_t - g_0$ has the same distribution as $\tilde{W}_{[g]_t}$. Then,

$$\begin{aligned}
\Pr\left[\frac{1}{4} \le g_t \le \frac{3}{4}\right] &= \Pr\left[-\frac{1}{4} \le \tilde{W}_{[g]_t} \le \frac{1}{4}\right] \\
&\ge 1 - \Pr\left[\max_{0 \le s \le 1/64} |\tilde{W}_s| > \frac{1}{4}\right] - \Pr\left[[g]_t > \frac{1}{64}\right] \\
&= 1 - 4\Pr\left[\tilde{W}_{1/64} > \frac{1}{4}\right] - \Pr\left[[g]_t > \frac{1}{64}\right] \\
&\ge \frac{9}{10} - \Pr\left[\int_0^t \|A_s\|_{\mathrm{op}} \,\mathrm{d}s \ge \frac{1}{64}\right].
\end{aligned}$$

In the second-to-last equation, the first two terms simplify on estimates for the concentration of the normal distribution and the second term simplifies on using the earlier bound on $\frac{\mathrm{d}[g]_t}{\mathrm{d}t}$. ■

We now restate the bound on the isoperimetric constant we mentioned earlier for distributions more log-concave than the Gaussian.

Theorem 3.4. Let

$$h(x) = \frac{f(x)e^{-\|x\|^2/2\sigma^2}}{\int f(y)e^{\|y\|^2/2\sigma^2}},$$

where $f : \mathbb{R}^n \to \mathbb{R}^+$ is an integrable log-concave function. Then h is log-concave and $\psi_h \gtrsim 1/\sigma$. That is, for any measurable $S \subseteq \mathbb{R}^n$,

$$\int_{\partial S} h(x)\,\mathrm{d}x \gtrsim \frac{1}{\sigma} \min\left\{\int_S h(x)\,\mathrm{d}x, \int_{\mathbb{R}^n \setminus S} h(x)\,\mathrm{d}x\right\}.$$

Now, let us get to the main estimation of the isoperimetric constant using the above results.

Theorem 3.5. Suppose there is $T > 0$ such that

$$\Pr\left[\int_0^T \|A_s\|_{\mathrm{op}} \,\mathrm{d}s \le \frac{1}{64}\right] \ge \frac{3}{4}.$$

Then $\psi_p \gtrsim T^{1/2}$.

Proof. Let $E \subseteq \mathbb{R}^n$ with $p(E) = 1/2$. We then have

$$\begin{aligned}
\int_{\partial E} p(x)\,\mathrm{d}x &= \mathbf{E}\left[\int_{\partial E} p_T(x)\,\mathrm{d}x\right] && (p_t \text{ is a martingale}) \\
&\gtrsim T^{1/2}\mathbf{E}\left[\min\left\{p_T(E), p_T(\mathbb{R}^n \setminus E)\right\}\right] && (p_T \text{ is more log-concave than the Gaussian}) \\
&\gtrsim T^{1/2} \Pr\left[\frac{1}{4} \le p_T(E) \le \frac{3}{4}\right] \\
&\gtrsim T^{1/2}\left(\frac{9}{10} - \Pr\left[\int_0^T \|A_s\|_{\mathrm{op}} \,\mathrm{d}s \ge \frac{1}{64}\right]\right) && (\text{by Lemma 3.3}) \\
&\gtrsim T^{1/2}.
\end{aligned}$$

■

As mentioned earlier, we now need to control the growth of the covariance matrix A_t, preventing it from exploding and ensuring that the condition in above theorem is satisfied for some large T.

3.3. Controlling A_t

To control the growth of A_t, we use the potential function $\mathrm{Tr}(A_t^2)$.

Lemma 3.6. Given a log-concave distribution p with mean μ and covariance matrix A, for any positive semi-definite matrix C,

$$\left\| \mathbf{E}_{x\sim p}\left[(x-\mu)(x-\mu)^\top C(x-\mu)\right]\right\|_2 \lesssim \|A\|_{\mathrm{op}}^{1/2}\,\mathrm{Tr}\left(A^{1/2}CA^{1/2}\right).$$

Proof. First, consider the case where C is of the form $vv^\top$. Let $w = A^{1/2}v$ and $y = A^{-1/2}(x-\mu)$. Let the distribution of y be $\tilde{p}$ (it is isotropic and log-concave). Then

$$\begin{aligned}
\left\| \mathbf{E}_{x\sim p}\left[(x-\mu)(x-\mu)^\top C(x-\mu)\right]\right\|_2 &= \left\| \mathbf{E}_{y\sim \tilde{p}}\left[A^{1/2}y(y^\top w)^2\right]\right\|_2 \\
&= \max_{\|\zeta\|_2\le 1} \mathbf{E}_{y\sim\tilde{p}}\left[(A^{1/2}y)^\top \zeta (y^\top w)^2\right] \\
&\le \max_{\|\zeta\|_2\le 1} \sqrt{\mathbf{E}_{y\sim\tilde{p}}\left[((A^{1/2}y)^\top\zeta)^2\right]}\sqrt{\mathbf{E}_{y\sim\tilde{p}}\left[(y^\top w)^4\right]} \\
&\lesssim \max_{\|\zeta\|_2\le 1} \sqrt{\zeta^\top A\zeta}\,\mathbf{E}_{y\sim\tilde{p}}\left[(y^\top w)^2\right] && (\tilde{p} \text{ is isotropic}) \\
&= \|A\|_{\mathrm{op}}^{1/2}\,\|w\|_2^2.
\end{aligned}$$

The second-to-last inequality uses a reverse Hölder-like inequality, which says that if q is a log-concave density in $\mathbb{R}^n$ and $k \ge 1$, then

$$\mathbf{E}_{x\sim q}\,\|x\|^k \le (2k)^k\left(\mathbf{E}_{x\sim q}\,\|x\|^2\right)^{k/2}.$$

We use the above with $k = 4$ and $x = y^\top w$, which has a one-dimensional log-concave distribution.
The bound for a general C isn't too difficult to show by writing it as $\sum \lambda_i v_i v_i^\top$ in terms of its eigenvalues $\lambda_i \ge 0$ and eigenvectors v_i. ■

Lemma 3.7. Given a log-concave distribution p with mean μ and covariance matrix C,

$$\mathbf{E}_{x,y\sim p}\left[|\langle x-\mu, y-\mu\rangle|^3\right] \lesssim \mathrm{Tr}(A^2)^{3/2}.$$

Proof. We may assume that $\mu = 0$. For a fixed x, we have

$$\begin{aligned}
\mathbf{E}_{y\sim p}\left[|\langle x,y\rangle|^3\right] &\le \mathbf{E}_{y\sim p}\left[|\langle x,y\rangle|^2\right]^{3/2} \\
&= (x^\top A x)^{3/2} = \left\|A^{1/2}x\right\|^3.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{E}_{x,y\sim p}\left[|\langle x,y\rangle|^3\right] &\le \mathbf{E}_{x\sim p}\left[\left\|A^{1/2}x\right\|^3\right] \\
&\le \mathbf{E}_{x\sim p}\left[\left\|A^{1/2}x\right\|^2\right]^{3/2} \\
&= \mathrm{Tr}(A^2)^{3/2}.
\end{aligned}$$

■

Theorem 3.8. With the previously used notation, there is a universal constant c_1 such that

$$\Pr\left[\max_{t\in[0,T]} \mathrm{Tr}(A_t^2) \ge 8\,\mathrm{Tr}(A_0^2)\right] \le 0.01 \text{ with } T = \frac{c_1}{\sqrt{\mathrm{Tr}(A_0^2)}}.$$

Before we get to the proof, we show that the bound on the isoperimetric constant follows from previously mentioned results.

Corollary 3.9. For any log-concave distribution p with covariance matrix A,

$$\psi_p \gtrsim \operatorname{Tr}(A^2)^{-1/4}.$$

In particular, if p is isotropic log-concave,

$$\psi_p \gtrsim n^{-1/4}.$$

Proof. Theorem 3.8 implies that

$$\Pr\left[\max_{s\in[0,t]} \operatorname{Tr}(A_s^2) \leq 8\operatorname{Tr}(A_0^2)\right] \geq 0.99,$$

where t is equal to the quantity we denoted as T there. Since $\|A_s\|_{\mathrm{op}} \leq \sqrt{\operatorname{Tr}(A_s^2)}$, we have

$$\Pr\left[\max_{s\in[0,t]} \|A_s\|_{\mathrm{op}} \leq \sqrt{8\operatorname{Tr}(A_0^2)}\right] \geq 0.99.$$

Therefore, letting

$$T = \min\left\{c_1, \frac{1}{64\sqrt{8}}\right\}\frac{1}{\sqrt{\operatorname{Tr}(A_0^2)}},$$

we have

$$\Pr\left[\int_0^T \|A_s\|_{\mathrm{op}}\,\mathrm{d}s \leq \frac{1}{64}\right] \geq 0.99,$$

and the result follows on using Theorem 3.5. ∎

Proof of Theorem 3.8. Define the potential function $\Phi_t = \operatorname{Tr}(A_t^2)$.
It is a computational task using Itô's Lemma to check that

$$\mathrm{d}\Phi_t = \left(2\mathbf{E}_{x\sim p_t}(x-\mu_t)^\top A_t(x-\mu_t)(x-\mu_t)^\top\right)\mathrm{d}W_t + \left(\mathbf{E}_{x,y\sim p_t}((x-\mu_t)^\top(y-\mu_t))^3 - 2\operatorname{Tr}(A_t^3)\,\mathrm{d}t\right)\mathrm{d}t.$$

Write this as

$$\mathrm{d}\Phi_t = \delta_t\,\mathrm{d}t + v_t^\top\,\mathrm{d}W_t.$$

By Lemma 3.7, we get

$$\delta_t \lesssim \operatorname{Tr}(A_t^2)^{3/2} = \Phi_t^{3/2}$$

by dropping the negative term in the expression (A_t is positive semi-definite). What about the martingale term? Using Lemma 3.6, we have

$$\|v_t\|_2 = 2\left\|\mathbf{E}_{x\sim p_t}(x-\mu_t)^\top A_t(x-\mu_t)(x-\mu_t)^\top\right\|_2 \lesssim \|A_t\|_{\mathrm{op}}^{1/2}\operatorname{Tr}(A_t^2) \leq \Phi_t^{5/4}.$$

Intuitively, this means that the drift term (the $\mathrm{d}t$ part) grows as $\Phi_t^{3/2}t$ and the martingale term (the $\mathrm{d}W_t$ part) grows as $\Phi_t^{5/4}\sqrt{t}$. So for t up to $\mathcal{O}(\Phi_0^{-1/2})$, the potential Φ_t remains $\mathcal{O}(\Phi_0)$.
To formalize this, define $f(a) = 1/\sqrt{a+\Phi_0}$. Observe that $\Phi_t \geq 8\Phi_0$ if and only if $f(\Phi_t) \geq -1/3\sqrt{\Phi_0}$. We can use Itô's Lemma to get

$$\mathrm{d}f(\Phi_t) = \left(\frac{1}{2}\frac{v_t^\top\,\mathrm{d}W_t}{(\Phi_t+\Phi_0)^{3/2}}\right) + \left(\frac{1}{2}\frac{\delta_t}{(\Phi_t+\Phi_0)^{3/2}} - \frac{3}{8}\frac{\|v_t\|_2^2}{(\Phi_t+\Phi_0)^{5/2}}\right)\mathrm{d}t \leq \mathrm{d}Y_t + C'\,\mathrm{d}t,$$

where C' is a suitable constant and $\mathrm{d}Y_t$ is the martingale term. Observe that

$$\frac{\mathrm{d}[Y]_t}{\mathrm{d}t} = \frac{1}{4}\frac{\|v_t\|_2^2}{(\Phi_t+\Phi_0)^3} \leq \frac{C}{\sqrt{\Phi_0}}$$

for a suitable constant C.
Let $\tilde{W}_t$ be a Wiener process such that $Y_t = \tilde{W}_{[Y]_t}$ (in distribution). Using Lemma 3.1,

$$\begin{aligned}\Pr\left[\max_{t\in[0,T]} Y_t \geq \gamma\right] &\leq \Pr\left[\max_{t\in\left[0,CT/\sqrt{\Phi_0}\right]} \tilde{W}_t \geq \gamma\right] \\ &= \Pr\left[\tilde{W}_{CT/\sqrt{\Phi_0}} \geq \gamma\right] \leq 2\exp\left(-\frac{\gamma^2\sqrt{\Phi_0}}{2CT}\right).\end{aligned}$$

Therefore,

$$\Pr\left[\max_{t\in[0,T]} f(\Phi_t) \geq -\frac{1}{\sqrt{2\Phi_0}} + C'T + \gamma\right] \leq 2\exp\left(-\frac{\gamma^2\sqrt{\Phi_0}}{2CT}\right).$$

Setting $T = 1/256(C+C')\sqrt{\Phi_0}$ and $\gamma = 1/4\sqrt{\Phi_0}$, we get

$$\Pr\left[\max_{t\in[0,T]} f(\Phi_t) \geq -\frac{1}{3\Phi_0}\right] \leq 2\exp(-8).$$

Using our earlier observation about f,

$$\Pr\left[\max_{t\in[0,T]} \Phi_t \geq 8\Phi_0\right] \leq 2\exp(-8) \leq 0.01. \qquad \blacksquare$$

Over the next few sections, we look at the best bound attained (as of the time of writing) of the isoperimetric constant, as described in [Che21].

3.4. An Almost Constant Bound

Here, we prove that there is a constant c such that for any log-concave p in $\mathbb{R}^n$,

$$\psi_p \geq \frac{1}{n^{c\sqrt{\log\log n/\log n}} \|A\|_{\mathrm{op}}^{1/2}}.$$

Rather than the original proof given in [Che21], we give a slightly modified proof from here. Again, we set the control matrix C_t in Equation (3.4) as the identity matrix.
To begin, for any integer $n \geq 1$, define

$$\psi_n = \inf_{\substack{p \text{ is log-concave in } \mathbb{R}^n \\ \text{and has compact support}}} \psi_p \|A\|_{\mathrm{op}}^{1/2}.$$

Using results from [Pao06], we may assume that the log-concave density p has compact support – we can restrict it to a sufficiently large ball losing exponentially small measure in the process. We shall restrict our measure to a ball of radius n^5.
Assume the dimension n is at least 3.
First, like before, let us give a basic estimate like in the last section to bound the isoperimetric constant.

Lemma 3.10. Let p be an isotropic log-concave density. If for some $T > 0$,

$$\mathbf{E}\left[\int_0^T \|A_t\|_{\mathrm{op}}\,\mathrm{d}t\right] \leq \frac{1}{8},$$

then $\psi_p \gtrsim T^{1/2}$.

Proof. Let E be a set of measure initially $1/2$ (under p). As described before, we would like to bound the change in measure of E. We have

$$\int_{\partial E} p(x)\,\mathrm{d}x = \mathbf{E}\left[\int_{\partial E} p_T(x)\,\mathrm{d}x\right] \gtrsim T^{1/2}\mathbf{E}[\min\{p_T(E), 1 - p_T(E)\}] \geq T^{1/2}\mathbf{E}[p_T(E)(1 - p_T(E))].$$

Let $M_t = p_t(E)$. Then

$$\mathrm{d}M_t = \int_E \langle x - \mu_t, \mathrm{d}W_t\rangle p_t(x)\,\mathrm{d}x = \left\langle \int_E (x - \mu_t)p_t(x)\,\mathrm{d}x, \mathrm{d}W_t \right\rangle,$$

so M_t is a martingale. Letting the first expression in the final inner product be v_t, we have

$$\|v_t\| = \sup_{\theta \in S^{n-1}} \int_E \langle x - \mu_t, \theta\rangle p_t(x)\,\mathrm{d}x \leq \sqrt{\sup_{\theta \in S^{n-1}} \int_E \langle x - \mu_t, \theta\rangle^2 p_t(x)\,\mathrm{d}x} = \|A_t\|_{\mathrm{op}}^{1/2}.$$

Using Itô's Lemma,

$$\mathrm{d}(M_t(1 - M_t)) = -\|v_t\|^2\,\mathrm{d}t + (\text{martingale term}).$$

Taking the expectation on eithe side and using the bound,

$$\mathrm{d}\mathbf{E}[M_t(1 - M_t)] \geq -\mathbf{E}\left[\|A_t\|_{\mathrm{op}}\right]\mathrm{d}t.$$

Therefore,

$$\mathbf{E}[M_t(1 - M_t)] \geq M_0(1 - M_0) - \frac{1}{8} = \frac{1}{8},$$

proving the lemma. ∎

Therefore, we wish to control the growth of the spectral norm. To do this, define the potential

$$\Gamma_t = \mathrm{Tr}(A_t^q).$$

It is quite easy to see that $\Gamma_t^{1/q} \geq \|A_t\|_{\mathrm{op}}$.

Lemma 3.11. With the above definition,

$$\mathrm{d}\Gamma_t \leq \frac{2q^2}{t}\Gamma_t\,\mathrm{d}t + (\text{martingale term}) \tag{3.7}$$

and

$$\mathrm{d}\Gamma_t \lesssim q^2\psi_n^{-2}\|A_t\|_{\mathrm{op}}\Gamma_t\,\mathrm{d}t + (\text{martingale term}). \tag{3.8}$$

Proof. Let $\mathrm{d}\Gamma_t = \delta_t\,\mathrm{d}t + (\text{martingale term})$. We wish to show that

$$\delta_t \leq 2q^2\Gamma_t \min\left\{\frac{1}{t}, c\psi_n^{-2}\|A_t\|_{\mathrm{op}}\right\}$$

for a suitable constant c.
Denote the minimum used on the right as κ^{-1}.
Recall the Poincaré constant from Section 2.3. Since p_t is more log-concave than the Gaussian $\exp(-t\|x\|^2/2)$, ζ_{p_t} is at least t. Also, the Poincaré constant is at least $c\psi_n^2/\|A_t\|_{\mathrm{op}}$. So, $\zeta_{p_t} \geq \kappa$.
Recalling (3.5), let $\mathrm{d}(A_t)_{i,j} = \langle \xi_{i,j}, \mathrm{d}W_t\rangle - (A_t^2)_{i,j}\,\mathrm{d}t$, where

$$\xi_{i,j} = \mathbf{E}\left[X_iX_jX\right] = \int_{\mathbb{R}^n} xx_ix_jp_t(x + \mu_t)\,\mathrm{d}x.$$

To bound δ_t, we use the following lemma. Let $0 < \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of A_t. Then for any smooth function f,

$$\mathrm{d}\sum_{i=1}^n f(\lambda_i) = \left(\frac{1}{2} \sum_{i,j=1}^n \left\|\xi_{i,j}\right\|^2 \frac{f'(\lambda_i) - f'(\lambda_j)}{\lambda_i - \lambda_j} - \sum_{i=1}^n \lambda_i^2 f'(\lambda_i) \right) \mathrm{d}t + (\text{martingale term}).$$

If $\lambda_i - \lambda_j = 0$, we interpret the corresponding term as $f''(\lambda_i)$. The $\xi_{i,j}$ are expressed in the basis of eigenvectors of A_t.
Substituting f as $t \mapsto t^q$,

$$\begin{aligned}
\delta_t &= q \left(\frac{1}{2} \sum_{i,j=1}^n \left\|\xi_{i,j}\right\|^2 \frac{\lambda_i^{q-1} - \lambda_j^{q-1}}{\lambda_i - \lambda_j} - \sum_{i=1}^n \lambda_i^{q+1} \right) \\
&\leq q \left(\frac{1}{2} \sum_{i,j=1}^n (q-1) \left\|\xi_{i,j}\right\|^2 \left(\frac{\lambda_i^{q-2} + \lambda_j^{q-2}}{2} \right) - \sum_{i=1}^n \lambda_i^{q+1} \right) \\
&\leq \frac{q(q-1)}{2} \sum_{i,j=1}^n \left\|\xi_{i,j}\right\|^2 \lambda_i^{q-2}.
\end{aligned}$$

Denote

$$\xi_{i,j,k} = \mathbf{E}[X_i X_j X_k], \quad \xi_i = (\xi_{i,j,k})_{j,k=1,\ldots,n} = \mathbf{E}[X_i X X^\top] \in \mathbb{R}^{n \times n}.$$

Observe that $\xi_{i,j} = (\xi_{i,j,k})_{k=1,\ldots,n} \in \mathbb{R}^n$ and further, $\sum_{j=1}^n \left\|\xi_{i,j}\right\|^2 \leq \mathrm{Tr}(\xi_i^2)$. Therefore, it suffices to upper bound

$$\sum_{i=1}^n \lambda_i^{q-2} \,\mathrm{Tr}(\xi_i^2).$$

To control the growth of the individual traces,

$$\begin{aligned}
\mathrm{Tr}(\xi_i^2) &= \mathrm{Tr}\left(\xi_i \mathbf{E}[X_i X X^\top] \right) \\
&= \mathbf{E}\left[X_i \langle \xi_i X, X \rangle \right] \\
&\leq \sqrt{\mathbf{E}[X_i^2]} \sqrt{\mathbf{E}\left[\langle \xi_i X, X \rangle^2 \right]}.
\end{aligned}$$

As we are working in the basis of the eigenvectors of A_t, the first term is upper bounded by $\sqrt{\lambda_i}$.
Using the definition of the Poincaré constant on the function $t \mapsto \langle \xi t, t \rangle$ together with our earlier bound using κ,

$$\mathbf{Var}\langle \xi_i X, X \rangle \leq \frac{1}{\zeta_p} \mathbf{E}\left[\|2\xi_i X\|^2 \right] \leq \frac{4}{\kappa} \mathbf{E}\left[\|\xi_i X\|^2 \right] = \frac{4}{\kappa} \mathrm{Tr}\left(A_t \xi_i^2 \right).$$

Therefore,

$$\mathrm{Tr}(\xi_i^2) \leq 2\sqrt{\frac{\lambda_i}{\kappa}} \sqrt{\sum_{j,k=1}^n \lambda_j \xi_{i,j,k}^2}.$$

Going back to the original expression we care about,

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^{q-2} \operatorname{Tr}(\xi_i^2) &\le \frac{2}{\sqrt{\kappa}} \sum_{i=1}^{n} \lambda_i^{q-(3/2)} \sqrt{\sum_{j,k=1}^{n} \lambda_j \xi_{i,j,k}^2} \\ &\le \frac{2}{\sqrt{\kappa}} \sqrt{\sum_{i=1}^{n} \lambda_i^q} \sqrt{\sum_{i,j,k=1}^{n} \lambda_i^{q-3} \lambda_j \xi_{i,j,k}^2} \\ &\le \frac{2}{\sqrt{\kappa}} \Gamma_t^{1/2} \sqrt{\sum_{i,j,k=1}^{n} \lambda_i^{q-2} \xi_{i,j,k}^2} \\ &\le \frac{2}{\sqrt{\kappa}} \Gamma_t^{1/2} \sqrt{\sum_{i=1}^{n} \lambda_i^{q-2} \operatorname{Tr}(\xi_i^2)},\end{aligned}$$

where the second-to-last inequality follows since $\lambda_i^{q-3}\lambda_j + \lambda_j^{q-3}\lambda_i \le \lambda_i^{q-2} + \lambda_j^{q-2}$. Therefore,

$$\delta_t \le \frac{q(q-1)}{2} \sum_{i=1}^{n} \lambda_i^{q-2} \operatorname{Tr}(\xi_i^2) \le \frac{2q(q-1)}{\kappa} \Gamma_t,$$

completing the proof. ∎

Corollary 3.12. With the above definitions, for $t_2 > t_1 > 0$, we have

$$\mathbf{E}[\Gamma_{t_2}^{1/q}] \le \mathbf{E}[\Gamma_{t_1}^{1/q}] \left(\frac{t_2}{t_1}\right)^{2q}. \tag{3.9}$$

Proof. The function $x \mapsto x^{1/q}$ is concave. Therefore, the Itô term in $\mathrm{d}\Gamma_t^{1/q}$ is negative and we have

$$\mathrm{d}\Gamma_t^{1/q} \le \frac{1}{q} \Gamma_t^{1/q-1} \,\mathrm{d}\Gamma_t \le \frac{2q}{t} \Gamma_t^{1/q} + (\text{martingale term}),$$

where the second inequality follows from (3.7). Taking the expectation on either side to eliminate the martingale term,

$$\mathrm{d}\mathbf{E}[\Gamma_t^{1/q}] \le \frac{2q^2}{t} \mathbf{E}[\Gamma_t^{1/q}] \,\mathrm{d}t,$$

so for $t_2 > t_1 > 0$, we have

$$\mathbf{E}[\Gamma_{t_2}^{1/q}] \le \mathbf{E}[\Gamma_{t_1}^{1/q}] \left(\frac{t_2}{t_1}\right)^{2q}.$$ ∎

Next, we show that our basic estimate Lemma 3.10 is in fact tight up to a log factor.

Corollary 3.13. There exists a sufficiently small constant c such that if $0 < T \le c\psi_n^2/\log n$, then $\mathbf{E}[\|A_T\|_{\mathrm{op}}] \le 3$, and further, $\mathbf{E}[\Gamma_T^{1/q}] \le 3n^{1/q}$ for all $q \ge 1$.

Proof. Recall from earlier that we assume our log-concave density p to be restricted to $n^5 B_2^n$. So, it suffices to show that $\Pr[\|A_t\|_{\mathrm{op}} < 2] \ge 1 - n^{-10}$, since $\|A_T\|_{\mathrm{op}} \le n^{10}$.
Define the stopping time $\tau = \inf\{t \ge 0 : \|A_t\|_{\mathrm{op}} \ge 2\}$ and set $X_t = \Gamma_{\min\{t,\tau\}}$. Using (3.8), we have

$$\mathrm{d}X_t \lesssim q^2 \psi_n^{-2} \|A_t\|_{\mathrm{op}} X_t \,\mathrm{d}t + (\text{martingale term}).$$

Using the definition of X_t to bound $\|A_t\|_{\mathrm{op}}$ by a constant and taking the expectation on either side,

$$\mathrm{d}\mathbf{E}[X_t] \lesssim q^2 \psi_n^{-2} \mathbf{E}[X_t] \,\mathrm{d}t.$$

Setting $q = \lceil 40 \log n \rceil$,

$$\mathbf{E}[X_T] \lesssim n \exp(c \log^2(n) \psi_n^{-2} T) \leq n^2$$

for some constant c and where the last inequality arises from the choice of T. Therefore,

$$n^2 \gtrsim \mathbf{E}[X_T] \geq \Pr\left[\|A_T\|_{\mathrm{op}} > 2\right] \cdot 2^{40 \log n},$$

proving the claim.
The second part of the result is easily proved since $\Gamma_T \leq n \|A_T\|_{\mathrm{op}}^q$. ∎

With the above, we may get to the final proof of the bound on the isoperimetric constant.

Theorem 3.14. There exists a universal constant c' such that

$$\psi_n \gtrsim n^{-c'\sqrt{\log \log n / \log n}}.$$

Proof. Using Corollary 3.13, let $T_0 = c\psi_n^2 / \log n < 1/100$ for sufficiently small c such that for $t \leq T_0$, $\mathbf{E}[\Gamma_t^{1/q}] \leq 3n^{1/q}$. For $t \geq T_0$ and $q \geq 2$,

$$\mathbf{E}[\Gamma_t^{1/q}] \leq \left(\frac{t}{T_0}\right)^{2q} \mathbf{E}[\Gamma_{T_0}^{1/q}] \leq \left(\frac{t}{T_0}\right)^{2q} 3n^{1/q}.$$

So, for any $T_1 > T_0$,

$$\begin{aligned}
\mathbf{E}\left[\int_0^{T_1} \|A_t\|_{\mathrm{op}}\right] &= \int_0^{T_1} \mathbf{E}[\|A_t\|_{\mathrm{op}}] \,\mathrm{d}t \\
&\leq 3T_1 + 3n^{1/q} \int_{T_0}^{T_1} \left(\frac{t}{T_0}\right)^{2q} \mathrm{d}t \\
&\leq \frac{3}{100} + 3\frac{n^{1/q}}{2q+1}\frac{T_1^{2q+1}}{T_0^{2q}} \leq \frac{3}{100} + 3n^{1/q}\frac{T_1^{2q+1}}{T_0^{2q}}.
\end{aligned}$$

How large can we make T_1 while ensuring that the integral we care about (for the purposes of Lemma 3.10) remains less than $1/8$?
We get

$$T_1 \sim n^{-1/q(2q+1)} T_0^{2q/(2q+1)}.$$

This gives a bound of

$$\psi_n \gtrsim T_1^{1/2} \sim n^{-1/2q(2q+1)} T_0^{q/(2q+1)}.$$

By our choice of T_0,

$$\psi_n \gtrsim \psi_n^{2q/(2q+1)} (\log n)^{-q/(2q+1)} n^{-1/2q(2q+1)}.$$

Simplifying,

$$\psi_n \gtrsim \left(\frac{c}{\log n}\right)^q n^{-1/2q}$$

for some universal constant c. Taking q of the order of $\left(\log n / \log \log n\right)^{1/2}$,

$$\psi_n \gtrsim e^{-c'\sqrt{\log n \log \log n}},$$

completing the proof. ∎

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