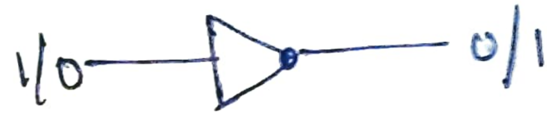
Quantum Circuits

Circuit Model: Sequence of building blocks performing computation : gates.

Single qubit gates

Classical : NOT



Quantum : Quantum theory is unitary →

Quantum gates: Unitary matrix

$(U^{+} U = I)$

I = Identity matrix

Dirac Notation : $U = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$

$= u_{00}|0\rangle\langle 0| + u_{01}|0\rangle\langle 1|$

$+ u_{10}|1\rangle\langle 0|$

$+ u_{11}|1\rangle\langle 1|$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underbrace{|0\rangle\langle 1| + |1\rangle\langle 0|}_{\text{Dirac Notation}}$$

$$\sigma_x |0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

Applying σ_x to qubit $|0\rangle$

$$\sigma_x |1\rangle = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) \cdot |1\rangle$$

$$= |0\rangle\underbrace{\langle 1|1\rangle}_{1} + |1\rangle\underbrace{\langle 0|1\rangle}_{0}$$

$$= |0\rangle$$

$\boxed{\text{Bit Flip}}$ ⇒ Quantum Equivalent NOT Gate.

$$|0\rangle — \boxed{\sigma_x} — |1\rangle$$

Performs rotation around x-axis by π

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\sigma_z|+\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$$

$$\sigma_z|-\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{\sqrt{2}}\Big(|0\rangle\underbrace{\langle 0|0\rangle}_{1} + |1\rangle\underbrace{\langle 1|1\rangle}_{1}\Big)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

$\boxed{\text{Phase flip}}$ → Performs Rotation around z-axis by π

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_x \cdot \sigma_z \rightarrow \text{Bit \& Phase flip}$$

σ_x, σ_y & σ_z → Pauli Matrices.

$$\sigma_i^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Pauli Matrices + I - they form Basis of 2x2 matrices.

<u>Hadamard Gate:</u>

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\big)$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= |+\rangle$$

$$H|1\rangle = |-\rangle$$

<u>Creates Superposition</u>

$$H|+\rangle = |0\rangle \quad H|-\rangle = |1\rangle$$

<u>Used to change Basis</u> (X & Z basis)

Applying H gate then Z measurement = X measurement

S gate: $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ adds 90° to the phase φ

$S|+\rangle = |+i\rangle$

$S|-\rangle = |-i\rangle$

S.H → change Z & Y bases

~~Multipartite Quantum States~~

Multipartite Quantum States

(Use Tensor Product)

$$|a\rangle \otimes |b\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

System A is in state $|1\rangle_A$ and System B is in state $|0\rangle_B$

The total state (bipartite state) is $|10\rangle_{AB}$

$$= |1\rangle_A \otimes |0\rangle_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

2 qubits $2^2 = 4$ bits (Proved)
2^N.

States of this form are called ~~correlated~~ uncorrelated, but also bipartite states, cannot be written as $|\phi\rangle_A \otimes |\psi\rangle_B$. These states are correlated and sometimes even entangled, e.g.

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right).$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

<u>Bell state</u>

Pure states: if correlated they must be entangled.

Two-Qubit Gates

Classical: XOR $X, Y \to \boxed{XOR} \to X \oplus Y$,
→ Irreversible

Quantum Theory $\Rightarrow$ Unitary gates $\Rightarrow$ Reversible.

$$\text{CNOT}: \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

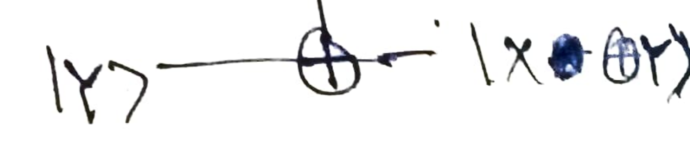
$$\text{CNOT}\,|00\rangle_{XY} = \text{CNOT}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |00\rangle_{XY}$$

$$\text{CNOT}\,|10\rangle_{XY} = |11\rangle_{XY}$$

I/P XY	O/P $X\,(X \oplus Y)$
00	00
01	01
10	11
11	10

$X \oplus Y$

Circuit



CNOT: Controlled NOT

Reversible XOR

Every func. f can be described by a reversible circuit.