ASSIGNMENT 2 - GMDL

PROBLEM 1

הסטודנט הקליד את 1 ו-2 כמספרים מטיפוס int ולא float לכן מהחלוקה שלהם התקבל 0, לכן זה כאילו ש- $temp \to \infty$, וכפי שלמדנו במקרה הזה ההתפלגות שואפת להתפלגות אחידה.

PROBLEM 2

ניתן לראות שהתוחלת של המכפלה של קודקודים קרובים גדולה יותר מהתוחלת של מכפלת קודקודים רחוקים. הסיבה לכך היא כנראה שהקורלציה בין קודקודים קרובים היא חזקה יותר, לכן יותר סביר שיהיה לקודקודים קרובים ערך זהה מקודקודים רחוקים. כאשר לקודקודים יש ערך זהה המכפלה שלהם היא 1, בזמן שמכפלת קודקודים עם ערך שונה היא 1- מה שמוריד את התוחלת.

רואים שגם עלייה בטמפרטורה מורידה את התוחלת. טמפרטורה גבוהה מחלישה את הקורלציה בין הקודקודים עד כדי כך שכאשר הטמפרטורה אינסופית, ההתפלגות היא אחידה לגמרי ואין אף קורלציה בין הקודקודים. אזי באופן דומה למקרה הקודם, קורלציה חלשה בין הקודקודים מורידה את התוחלת.

PROBLEM 3

השיטה השנייה מביאה תוצאות הרבה יותר מדויקות מהשיטה הראשונה, בעיקר בקירוב של $E\left(X_{(1,1)}X_{(8,8)}\right)$. אני מניח שהשיטה הראשונה פחות מדויקת באופן כללי בגלל ש-25 איטרציות זה לא מספיק על מנת לקבל קירוב משמעותי. גם ממוצע של 10000 קירובים לא שווה כלום אם כל הקירובים הם זבל. הסיבה שהקירוב של $E\left(X_{(1,1)}X_{(8,8)}\right)$ הוא גרוע בפרט בשיטה הראשונה זה בגלל שאין מספיק איטרציות בשביל שקודקודים רחוקים בשריג ישפיעו אחד על השני. $X_{(1,1)}$ ו-$X_{(2,2)}$ קרובים לכן לא צריך הרבה איטרציות בשביל לקבל קירוב מקומי טוב. אך $X_{(8,8)}$ רחוק מ-$X_{(1,1)}$ לכן צריך יותר מ-25 איטרציות בשביל לקרב את ההשפעה של המשתנים האלו אחד על השני.

Copy_of_GMDL212_HW2

April 22, 2021

[ ]:
```python
import numpy as np
import matplotlib.pyplot as plt
```

[ ]:
```python
# Shared code for several exercises.

# These are the temperatures that all the exercises use.
temps = [1.0, 1.5, 2.0]

# Takes a 2D matrix of -1,1 values and a temperature and computes the exp(...)␣
↪part of the ising model i.e. Ztemp * p(lattice).
def ising_exp_part(lattice, temp):
  sum = 1.0
  rows, cols = np.shape(lattice)

  for i in range(rows):
    sum *= ex1_func(lattice[i, :], temp)

  for j in range(cols):
    sum *= ex1_func(lattice[:, j], temp)

  return sum

# Decodes 0<=y<2**width into a row of -1,1's with the given width.
def y2row(y,width=8):
  if not 0<=y<=(2**width)-1:
    raise ValueError(y)
  my_str=np.binary_repr(y,width=width)
  my_list = list(map(int,my_str))
  my_array = np.asarray(my_list)
  my_array[my_array==0]=-1
  row=my_array
  return row
```

[ ]:
```python
# Exercise 1.

def ex1_func(row_s, temp):
  return np.exp((1.0 / temp) * np.dot(row_s[:-1], row_s[1:]))
```

[ ]:
```python
# Exercise 2.

def ex2_func(row_s, row_t, temp):
  return np.exp((1.0 / temp) * np.dot(row_s, row_t))
```

[ ]:
```python
# Exercise 3.

def ztemp2x2(temp):
  R = [-1.0, 1.0]
  sum = 0.0

  for x11 in R:
    for x12 in R:
      for x21 in R:
        for x22 in R:
          lattice = np.array([[x11, x12],
                              [x21, x22]])
          sum += ising_exp_part(lattice, temp)

  return sum

for temp in temps:
  print(f"Temp = {temp} => Ztemp = {ztemp2x2(temp)}")
```

```
Temp = 1.0 => Ztemp = 121.23293134406595
Temp = 1.5 => Ztemp = 40.922799092745386
Temp = 2.0 => Ztemp = 27.04878276433453
```

[ ]:
```python
# Exercise 4.

def ztemp3x3(temp):
  R = [-1.0, 1.0]
  sum = 0.0

  for x11 in R:
    for x12 in R:
      for x13 in R:
        for x21 in R:
          for x22 in R:
            for x23 in R:
              for x31 in R:
                for x32 in R:
                  for x33 in R:
                    lattice = np.array([[x11, x12, x13],
                                        [x21, x22, x23],
                                        [x31, x32, x33]])
                    sum += ising_exp_part(lattice, temp)
```

```python
    return sum

for temp in temps:
    print(f"Temp = {temp} => Ztemp = {ztemp3x3(temp)}")
```

```
Temp = 1.0 => Ztemp = 365645.7491357705
Temp = 1.5 => Ztemp = 10565.421983514263
Temp = 2.0 => Ztemp = 2674.5181230600865
```

[ ]:
```python
# Exercise 5.

def ztemp2x2_improved(temp):
    sum = 0.0

    for y1 in range(4):
        for y2 in range(4):
            y1_row = y2row(y1, 2)
            y2_row = y2row(y2, 2)
            sum += ex1_func(y1_row, temp) * ex1_func(y2_row, temp) * ex2_func(y1_row,␣
↪y2_row, temp)

    return sum

for temp in temps:
    print(f"Temp = {temp} => Ztemp = {ztemp2x2_improved(temp)}")
```

```
Temp = 1.0 => Ztemp = 121.23293134406595
Temp = 1.5 => Ztemp = 40.922799092745386
Temp = 2.0 => Ztemp = 27.048782764334526
```

[ ]:
```python
# Exercise 6.

def ztemp3x3_improved(temp):
    sum = 0.0

    for y1 in range(8):
        for y2 in range(8):
            for y3 in range(8):
                y1_row = y2row(y1, 3)
                y2_row = y2row(y2, 3)
                y3_row = y2row(y3, 3)
                sum += ex1_func(y1_row, temp) * ex1_func(y2_row, temp) *␣
↪ex1_func(y3_row, temp) * ex2_func(y1_row, y2_row, temp) * ex2_func(y2_row,␣
↪y3_row, temp)

    return sum

for temp in temps:
```

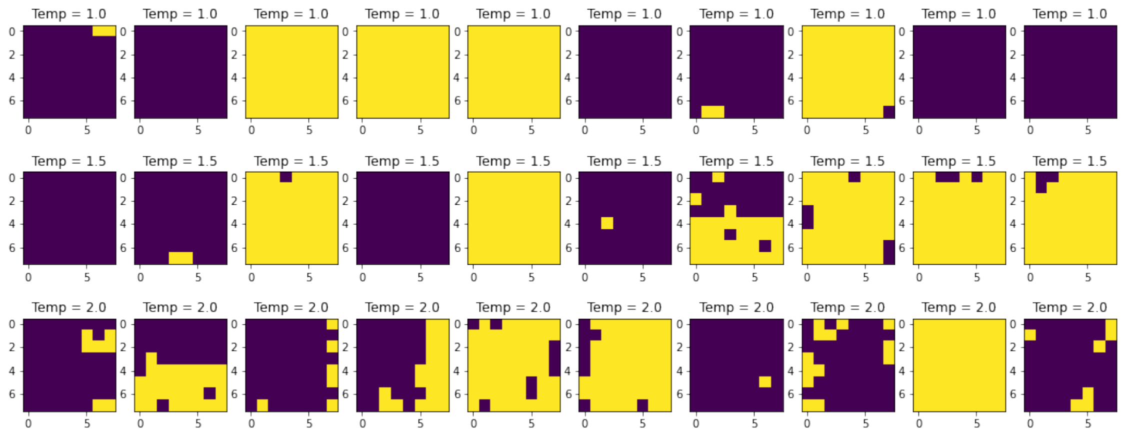
```
print(f"Temp = {temp} => Ztemp = {ztemp3x3_improved(temp)}")
```

```
Temp = 1.0 => Ztemp = 365645.7491357704
Temp = 1.5 => Ztemp = 10565.421983514265
Temp = 2.0 => Ztemp = 2674.518123060087
```

[ ]:
```
# Exercise 7 function definitions.

# Caches a 1D array representation of the G function (ex1_func) and a 2D array␣
↪representation of the function G(yk)*F(yk,yk+1) (f is ex2_func).
# Returns a tuple Gs, GFs.
def compute_Gs_and_GFs(dim, temp):
  value_range = 2**dim
  rows = [y2row(y, dim) for y in range(value_range)]
  Gs = np.array([ex1_func(rows[y], temp) for y in range(value_range)])
  GFs = np.array([[Gs[y] * ex2_func(rows[y], rows[y_next], temp) for y_next in␣
↪range(value_range)] for y in range(value_range)])
  return Gs, GFs

# Computes the T-functions from the assignment for a dim x dim lattice and␣
↪returns them in array form [T0,...,Tdim-1].
def compute_Ts(dim, Gs, GFs):
  value_range = np.shape(Gs)[0]
  Ts = [np.array([np.sum(GFs[:, y]) for y in range(value_range)])]

  for i in range(1, dim - 1):
    Ts.append(np.array([np.sum(Ts[i - 1] * GFs[:, y]) for y in␣
↪range(value_range)]))

  Ts.append(np.sum(Ts[dim - 2] * Gs))
  return Ts

# Gets an array like the one output by compute_Ts and computes an array of the␣
↪form [p0|1,...,pdim-2|dim-1,pdim-1].
def compute_pdfs(Ts, Gs, GFs):
  dim = len(Ts)
  value_range = np.shape(Gs)[0]
  ztemp = Ts[dim - 1]

  pdfs = [0] * dim
  pdfs[dim - 1] = (Ts[dim - 2] * Gs) / ztemp

  for i in reversed(range(1, dim - 1)):
    pdfs[i] = np.array([Ts[i - 1] * GFs[:, y] / Ts[i][y] for y in␣
↪range(value_range)])

  pdfs[0] = np.array([GFs[:, y] / Ts[0][y] for y in range(value_range)])
```

```
  return pdfs

# Packages the previous functions since we're only really interested in keeping␣
 ↪the pdfs in the end.
def compute_pdfs_directly(dim, temp):
  Gs, GFs = compute_Gs_and_GFs(dim, temp)
  Ts = compute_Ts(dim, Gs, GFs)
  return compute_pdfs(Ts, Gs, GFs)

# Gets an array like the one output by compute_pdfs and returns an array of␣
 ↪samples from this probability distribution [y0,...,ydim-1].
def sample(pdfs):
  dim = len(pdfs)
  value_range = np.shape(pdfs[dim - 1])[0]
  sample = [0] * dim
  sample[dim - 1] = np.random.choice(value_range, p=pdfs[dim - 1])

  for i in reversed(range(dim - 1)):
    sample[i] = np.random.choice(value_range, p=pdfs[i][sample[i + 1], :])

  return sample

# Gets a 1D array sample from a markhov chain and transforms it into a 2D␣
 ↪sample from an ising model lattice.
def sample_to_img(sample):
  dim = len(sample)
  img = np.array([y2row(s, dim) for s in sample])
  return img

# Packages the previous functions so you can quickly sample an image which is␣
 ↪what we really care about.
def sample_img(pdfs):
  return sample_to_img(sample(pdfs))
```

[ ]:
```
# Exercise 7 outputs.

dim = 8
sample_count = 10
fig = plt.figure(figsize=(18, 7))

for i in range(len(temps)):
  pdfs = compute_pdfs_directly(dim, temps[i])

  for j in range(sample_count):
    fig.add_subplot(len(temps), sample_count, 1 + i*sample_count + j,␣
 ↪title=f"Temp = {temps[i]}")
    plt.imshow(sample_img(pdfs), interpolation="none", vmin=-1.0, vmax=1.0)
```



[ ]:
```python
# Exercise 8 function definitions.

def make_samples(num_of_samples, dim, temp):
    sampels_arr = [None] * num_of_samples
    pdfs = compute_pdfs_directly(dim, temp)

    for i in range(len(sampels_arr)):
        sampels_arr[i] = sample_img(pdfs)

    return sampels_arr

def make_samples_multi_temps(num_of_samples, dim, temps):
    sampels_array = [None] * len(temps)
    for i in range(len(temps)):
        sampels_array[i] = make_samples(num_of_samples, dim, temps[i])
    return sampels_array
```

[ ]:
```python
# Exercise 8 outputs.

dim = 8
sample_count = 10000
sampels_array = make_samples_multi_temps(sample_count, dim, temps)

avg_11_22 = [0.0, 0.0, 0.0]
avg_11_88 = [0.0, 0.0, 0.0]

for i in range(len(temps)):
    for j in range(sample_count):
        avg_11_22[i] += sampels_array[i][j][0][0] * sampels_array[i][j][1][1]
        avg_11_88[i] += sampels_array[i][j][0][0] * sampels_array[i][j][7][7]
    avg_11_22[i] = avg_11_22[i] / sample_count
    avg_11_88[i] = avg_11_88[i] / sample_count
```

```
for i in range(len(temps)):
    print(f"Temp = {temps[i]} => E'(x11*x22) = {avg_11_22[i]}, E'(x11*x88) =␣
↪{avg_11_88[i]}")
```

```
Temp = 1.0 => E'(x11*x22) = 0.951, E'(x11*x88) = 0.9088
Temp = 1.5 => E'(x11*x22) = 0.7494, E'(x11*x88) = 0.524
Temp = 2.0 => E'(x11*x22) = 0.511, E'(x11*x88) = 0.127
```

[ ]:
```
# Exercise 9 function definitions.

# Returns true iff the indices are in bounds of the matrix.
def in_bounds(matrix, i, j):
  rows, cols = np.shape(matrix)
  return 0 <= i < rows and 0 <= j < cols

# Computes a new more accurate sample for lattice[i, j].
def resample(lattice, temp, i, j):
  neighbor_indices = [(i + 1, j), (i - 1, j), (i, j + 1), (i, j - 1)]
  neighbors = np.array([lattice[i_tag, j_tag] for i_tag, j_tag in␣
↪neighbor_indices if in_bounds(lattice, i_tag, j_tag)])
  probs = np.empty(2)
  probs[0] = np.exp((1.0 / temp) * np.sum(neighbors))
  probs[1] = 1.0 / probs[0]
  probs /= np.sum(probs) # This makes it so probabilities sum to 1.
  lattice[i, j] = np.random.choice([1, -1], p=probs)


# Refreshes the whole lattice with new values that distribute more like the␣
↪true distribution.
def sweep(lattice, temp):
  rows, cols = np.shape(lattice)

  for i in range(rows):
    for j in range(cols):
      resample(lattice, temp, i, j)

# Returns a random dim x dim lattice which approximates the ising model with␣
↪the given amount of sweeps.
def gibbs_sample(dim, temp, sweeps):
  lattice = np.random.randint(low=0, high=2, size=(dim, dim)) * 2 - 1

  for i in range(sweeps):
    sweep(lattice, temp)

  return lattice
```

```python
# Computes the average of n samples from the average of n-1 samples and the new sample.
def add_to_avg(old_avg, sample_index, sample):
  return ((sample_index - 1) * old_avg + sample) / sample_index
```

[1]:
```python
# Exercise 9 method 1.

for temp in temps:
  avg_11_22 = 0.0
  avg_11_88 = 0.0

  for i in range(10000):
    lattice = gibbs_sample(8, temp, 25)
    avg_11_22 = add_to_avg(avg_11_22, i + 1, lattice[0, 0] * lattice[1, 1])
    avg_11_88 = add_to_avg(avg_11_88, i + 1, lattice[0, 0] * lattice[7, 7])

  print(f"Temp = {temp} => E(x11*x22) = {avg_11_22}, E(x11*x88) = {avg_11_88}")
```

```
Temp = 1.0 => E(x11*x22) = 0.931, E(x11*x88) = 5403999999999983
Temp = 1.5 => E(x11*x22) = 0.7315999999999998, E(x11*x88) = 0.3549999999999998
Temp = 2.0 => E(x11*x22) = 0.4958, E(x11*x88) = 0.08999999999999987
```

[2]:
```python
# Exercise 9 method 2.

for temp in temps:
  avg_11_22 = 0.0
  avg_11_88 = 0.0
  lattice = gibbs_sample(8, temp, 100)

  for i in range(25 * 10000 - 100):
    sweep(lattice, temp)
    avg_11_22 = add_to_avg(avg_11_22, i + 1, lattice[0, 0] * lattice[1, 1])
    avg_11_88 = add_to_avg(avg_11_88, i + 1, lattice[0, 0] * lattice[7, 7])

  print(f"Temp = {temp} => E(x11*x22) = {avg_11_22}, E(x11*x88) = {avg_11_88}")
```

```
Temp = 1.0 => E(x11*x22) = 0.951452581032412, E(x11*x88) = 0.9037454981992802
Temp = 1.5 => E(x11*x22) = 0.7670588235294165, E(x11*x88) = 0.5470108043217206
Temp = 2.0 => E(x11*x22) = 0.5055942376950741, E(x11*x88) = 0.1165826330532213
```

```python
[15] # Exercise 10 function definitions.

     # adds Gaussian noise values, sampled i.i.d. from N(0, 2^2) for a given lattice
     def add_noise(lattice):
         rows, cols = np.shape(lattice)
         eta = 2 * np.random.standard_normal(size=(rows, cols))
         return lattice + eta

     def get_indc(x):
         indc = 0
         if x == 1:
             indc = -1
         if x == -1:
             indc = 1
         return indc

     # returns the sum of the neighbors of given cell
     def sum_neighbors(lattice, i, j):
         neighbor_indices = [(i + 1, j), (i - 1, j), (i, j + 1), (i, j - 1)]
         neighbors = np.array([lattice[i_tag, j_tag] for i_tag, j_tag in neighbor_indices if in_bounds(lattice, i_tag, j_tag)])
         return np.sum(neighbors)


     def find_mustly_likelihood(lattice):
         rows, cols = np.shape(lattice)
         lattice_ml = np.zeros((rows, cols))
         for i in range(rows):
             for j in range(cols):
                 if lattice[i, j] > 0:
                     lattice_ml[i, j] = 1
                 else:
                     lattice_ml[i, j] = -1
         return lattice_ml

     def noise_resample(lattice_x, lattice_y, temp, variance, i, j):
         probs = np.empty(2)
         indc = get_indc(lattice_x[i, j])
         probs[0] = np.exp(((1.0 / temp) * sum_neighbors(lattice_x, i, j)) - (
                     (1 / 2 * variance) * (lattice_y[i, j] + indc) * (lattice_y[i, j] + indc)))
         probs[1] = 1.0 / probs[0]
         probs /= np.sum(probs)  # This makes it so probabilities sum to 1.
         return np.random.choice([1, -1], p=probs)

     def find_icm(lattice_x, lattice_y, temp, variance, steps):
         rows, cols = np.shape(lattice_y)
         lattice = np.zeros((rows, cols))

         for n in range(steps):
             for i in range(rows):
                 for j in range(cols):
                     lattice[i, j] = noise_resample(lattice_x, lattice_y, temp, variance, i, j)

         return lattice
```

```
[17] # Exercise 10 output.

size = 100
steps = 50
variance = 4

for temp in temps:
  lattice_x = np.zeros((size, size))
  lattice_n = np.zeros((size, size))
  lattice_ml = np.zeros((size, size))

  # generate size*size sample from Ising-model and adds Gaussian noise i.i.d.
  lattice = gibbs_sample(size, temp, steps)
  lattice_y = add_noise(lattice)

  # generate a sample from the posterior distribution using Gibbs sampling.
  for i in range(size):
    for j in range(size):
      lattice_x[i, j] = lattice_y[i, j]
  for n in range(steps):
    sweep(lattice_x, temp)

  for i in range(size):
    for j in range(size):
      lattice_n[i, j] = lattice_x[i, j]
  lattice_x_new = find_icm(lattice_n, lattice_y, temp, variance, 5)

  lattice_ml = find_mustly_likelihood(lattice_y)

  titles = [ 'Gibbs Sample', 'Sample with Gaussian noise', 'Posterior Gibbs Sampleing', 'ICM', 'Maximum-Likelihood' ]
  lattices = [ lattice, lattice_y, lattice_x, lattice_x_new, lattice_ml ]
  fig, axs = plt.subplots(1, 5, figsize=(18, 7))
  axs[0].set_ylabel(f"Temp = {temp}", fontweight ="bold")

  for ax, title, latt in zip(axs, titles, lattices):
    ax.imshow(latt, interpolation="none", vmin=-1.0, vmax=1.0)
    ax.set_title(title)
    ax.grid(True)

plt.show()
```