Maths in LaTeX

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Contents

Chapter 1

Homogeneous function and Euler's theorem

Question: Explain homogeneous functions and derive euler's theorem on homogeneous function.

Solution: A function $f(x,y)$ is said to be homogeneous of degree (or order) n in the variable x and y if it can be expressed in the form $x^n\phi(\frac{y}{x})$ or $y^n\phi(\frac{x}{y})$.

An alternative test for a function f(x,y) to be homogeneous of degree(or order) n is that

$$f(tx,ty) = t^n f(x,y)$$

For example,if $f(x,y) = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$,then

$$f(x,y) = \frac{x\left(1+\dfrac{y}{x}\right)}{\sqrt{x}(1+\sqrt{\frac{y}{x}}} = x^{\frac{1}{2}}\phi\left(\frac{y}{x}\right)$$

$\Rightarrow f(x,y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

$$f(x,y) = \frac{y(1+\dfrac{x}{y})}{\sqrt{y}\left(1+\sqrt{\dfrac{x}{y}}\right)} = y^{\frac{1}{2}}\phi(\frac{x}{y})$$

$\Rightarrow f(x,y)$ is a homogeneous function of degree $\frac{1}{2}$ in x and y.

Euler's theorem on homogeneous function

If u is homogeneous function of degree n in $x$and y, then $\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu.$

Since u is a homogeneous function of degree n in x and y, it can be expressed

$$u = x^n f\left(\frac{y}{x}\right) \tag{1.1}$$

$$\frac{\partial u}{\partial x} = nx^n - 1f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right) \tag{1.2}$$

$$\Rightarrow x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}) - x^n - 1yf'\frac{y}{x}\right) \tag{1.3}$$

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right) . \frac{1}{x} = x^n - 1f'\left(\frac{y}{x}\right) \tag{1.4}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = x^n - 1yf'\left(\frac{y}{x}\right) \tag{1.5}$$

Adding (1.1) and (1.3), we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu$.

Chapter 2

Maxima and Minima

Question: Examine the function $x^3 + y^3 - 3axy$ for maxima and minima.

Solution:Here $f(x, y) = x^3 + y^3 - 3axy$

$$f_x = 3x^2 - 3ay,\ f_y = 3y^2 - 3ax,\ f_{xx} = 6x,\ f_xy = -3a,\ f_{yy} = 6y$$

Now for extreme values $\qquad f_x = f_y = 0$

$\Rightarrow \qquad x^2 - ay = 0$ and $y^2 - ax = 0$

$\Rightarrow \qquad y = \dfrac{x^2}{a}$

$\therefore\ \dfrac{x^4}{a^2} - ax = 0$ or $x(x^2 - a^2) = 0$ or $x = 0,$ a when

$x = 0,\ y = 0$; when $x = a, y = a$

$\therefore$ There are two stationary points $(0, 0)$ and (a, a)

Now $rt - s^2 = 36xy - 9a^2$

At (0,0) $rt - s^2 = -9a^2 < 0$

$\Rightarrow \qquad$ There is no extreme value at $(0, 0)$.

At (a,a) $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

$\Rightarrow \qquad f(x, y)$ has extreme value at (a,a).

Now $r = 6a^2$

If $a > 0$, $r > 0$ so that $f(r, y)$ has minimum value at (a, a).

Minimum value=$f(a, a) = a^3 + a^3 - 3a^3 = -a^3$

Chapter 3

Curve Tracing

Question:Trace the curve $x^3 + y^3 = 3axy$.

Solution: The equation of the curve is
$$x^3 + y^3 - 3axy = 0$$

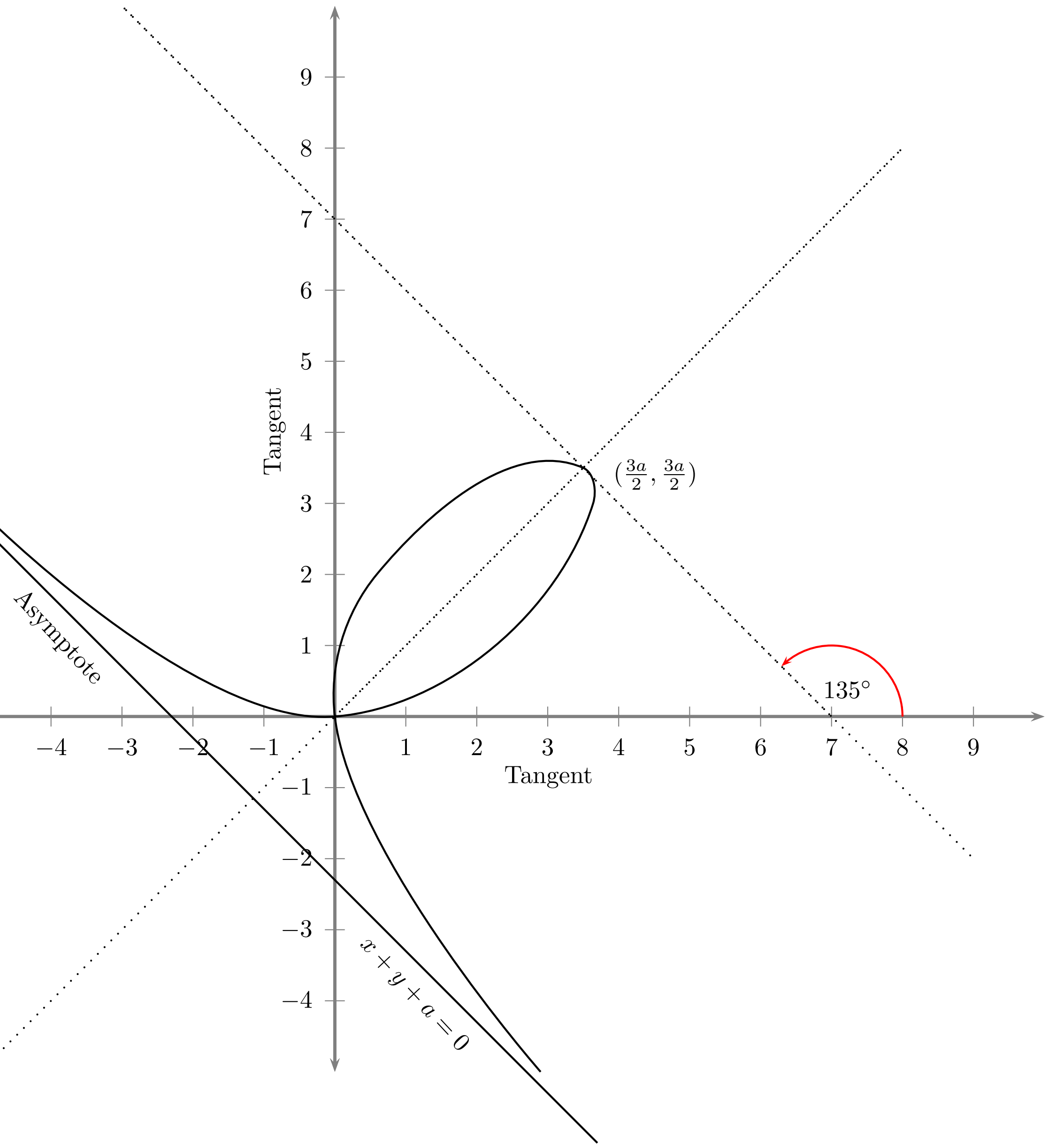
1.**Symmetry**: The curve is neither symmetric about x-axis nor y-axis but about $y = x$.

2.**Origin**:The curve passess through the origin $0, 0$ and the tangents at the origin are given by $3axy = 0$ i.e. $x = 0, y = 0$ i.e. x-axis and y-axis.

3.**Domain and Range**: From above it is clear x and y both cannot be negative $\because$ then L.H.S will be negative but R.H.S will be positive which is impossible $\therefore$ no protion of the curve will lie in 3^{rd} quadrant.

4.**Points of Intersection**: Curve meets x-axis at $(0, 0)$. Curve meets y-axis at $(0, 0)$ $\therefore$ the curve passes only thorugh $(0, 0)$.

Curve intersects $y = x$ where $x^3 + x^3 = 3ax^2$ or $2x^3 = 3ax^2$ or $x = \dfrac{3a}{2}$ $\therefore$ Points of intersection with $y = x$ is $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$.



5.**Tangents**: To find the slope of tangent at $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$ diffrentiate given equation w.r.t x.

$$
\begin{aligned}
3x^2 + 3y^2\frac{\text{dy}}{\text{dx}} =& 3ax\frac{\text{dy}}{\text{dx}} + 3ay \\
(y^2 - ax)\frac{\text{dy}}{\text{dx}} =& ay - x^2 \\
(\frac{\text{dy}}{\text{dx}}) =& \frac{ay - x^2}{y^{-}ax} \\
\frac{\text{dy}}{\text{dx}}_{(\frac{3a}{2},\frac{3a}{2})} =& \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1
\end{aligned}
$$

$\therefore$ At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ slope of tangent = -1

$\therefore$ Tangent makes an angle of 135 with x-axis at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

6.**Asymptotes**: Asymptotes of $x^2 + y^3 = 3axy$ are given by putting $x = 1, y = m$

$$
\begin{aligned}
\phi_3(m) =& 1 + m^3, \phi_3'(m) = 3m^2, \phi_3''(m) = 6m \\
\phi_2(m) =& -3am, \phi_2'(m) = -3a, \\
\phi_3(m) =& 0 \; given \; 1 + m^3 = 0
\end{aligned}
$$

or $(1 + m)(1 - m + m^2) = 0$ only real value of m is -1.
$\therefore$ Asymptote with slope $m = -1$ is $y = -x + c$ where c is given by

$$
\begin{aligned}
c\phi_3'(m) + \phi_2(m) =& 0 \; at \; m = -1 \\
c(3) + (-3a)(-1) =& 0 \\
c(3) + (-3a)(-1) =& 0
\end{aligned}
$$

or $c + a = 0 \therefore c = -a$
$\therefore$ Asymptote is y=-x-a or x+y+a=0

Chapter 4

Projectile Motion

Question To derive the equations for projectile motion, we assume that the projectile is moving along in a vertical plane and that the only force acting on the projectile is the constant force of gravity, which always points straight downward.

Solution We assume that the projectile is lauched from the origin at time t = 0 into the first quadrant with an initial velocity $\vec{v_0}$ .If $\vec{v_0}$ makes an angle α with the horizontal and the initial speed of the projectile is $v_0 = |\vec{v_0}|$, then

$$\vec{v_0} = (v_0 \cos\alpha)\vec{i} + (v_0 \sin\alpha)\vec{j} \text{ and } \vec{r_0} = \vec{0}$$

By Newton's Second Law of Motion $F = m\vec{a}$, so

$$m\vec{a} = (-mg)\vec{j}$$

$$\vec{a} = -g\vec{j}$$

$$\frac{d^2\vec{r}}{dt^2} = -g\vec{j}$$

Integrating twice and using the fact that $\vec{v}(0) = (v_0 \cos\alpha \quad a)\vec{i} + (v_0 \sin\alpha)\vec{j}$ and $\vec{r}(0) = \vec{0}$,we get

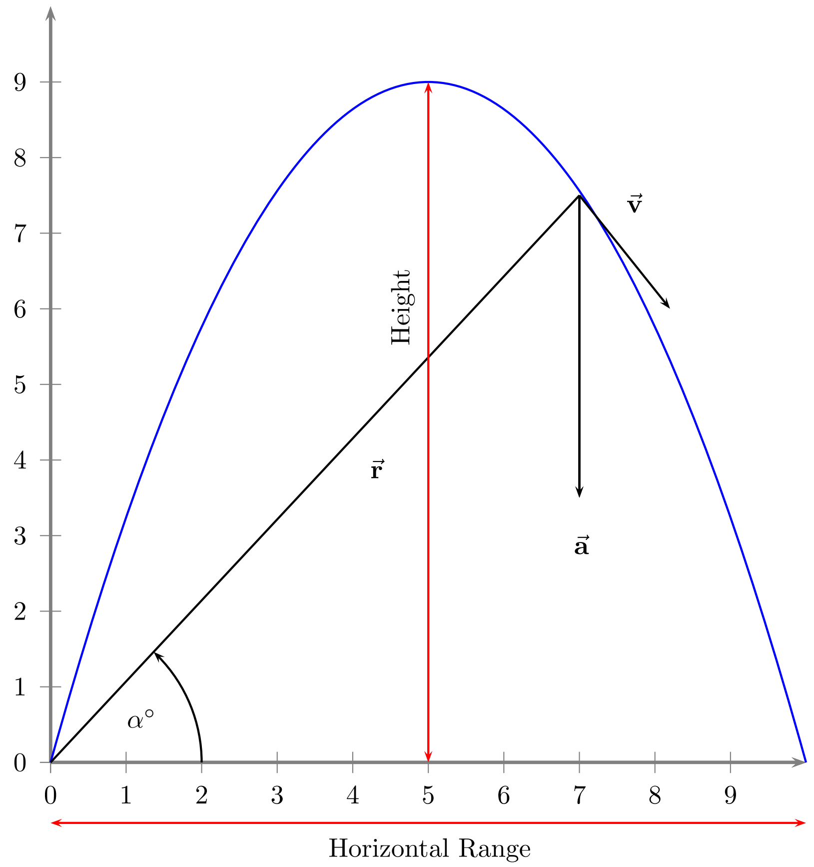
$$\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + \overrightarrow{v_0}t + \overrightarrow{r_0}$$

$$\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + ((v_0 \cos\alpha)\vec{i} + (v_0 \sin\alpha)\vec{j}) + \vec{0}$$

$$\vec{r}(t) = (v_0 \cos\alpha)t\vec{i} + (-\frac{1}{2}gt^2 + (v_0 \sin\alpha)t)\vec{j}$$

Ideal Projectile Motion Equation

$$\vec{r}(t) = (v_0 \cos\alpha)t\vec{i} + (-\frac{1}{2}gt^2 + (v_0 \sin\alpha)t)\vec{j}$$



The angle α is the projectile'slaunch angle.The horizontal and vertical component's of position give the parametric equations

$$x = (v_0 \cos\alpha)t \text{ and } y = -\frac{1}{2}gt^2\vec{j} + (v_0 \sin\alpha)t$$

where x is the distance downrange and y is altitude of the projectile at time t.
Height,Flight Time and Range
The projectile reaches its height point when its vertical velocity is zero, that is, when

$$\frac{dy}{dt} = v_0 \sin\alpha - gt = 0 \text{ and } t = \frac{v_0 \sin\alpha}{g}$$

For this value of time, the altitude of the projectile is

$$y_{max} = v_0 \sin\alpha(\frac{v_0 \sin\alpha}{g}) - \frac{1}{2}g(\frac{v_0 \sin\alpha}{g})^2 = \frac{(v_0 \sin\alpha)^2}{2g}$$

To find when the projectile lands when fired over horizontal ground, we set the vertical component equal to zero and solve for t.

$$-\frac{1}{2}gt^2 + (v_0 \sin\alpha)t = 0$$

$t(-\frac{1}{2}gt + (v_0 \sin\alpha)) = 0$
$t = 0$ or $-\frac{1}{2}gt + (v_0 \sin\alpha) = 0$
$t = 0$ or $\frac{1}{2}gt = (v_0 \sin\alpha)$
$t = 0$ or $t = \frac{2v_0 \sin\alpha}{g}$

To find the projectile's range, we find the value of the horizontal component when $t = \frac{2v_0 \sin\alpha}{g}$

$$x = (v_0 \cos\alpha)t = (v_0 \cos\alpha)(\frac{2v_0 \sin\alpha}{g}) = \frac{v_0^2 2 \sin\alpha \cos\alpha}{g} = \frac{v_0^2 2 \sin 2\alpha}{g}$$

The range is largest when $\sin\alpha = 1$ or when $2\alpha = 90°$, $\alpha = 45°$

Height, Flight Time and Range for Ideal Motion.
For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α:

Maximum Height $y_{max} = v_0 \sin\alpha(\frac{v_0 \sin\alpha}{g}) - \frac{1}{2}g(\frac{v_0 \sin\alpha}{g})^2 = \frac{(v_0 \sin\alpha)^2}{2g}$

Flight Time $t = \frac{2v_0 \sin\alpha}{g}$

Range $x = \frac{v_0^2 2 \sin 2\alpha}{g}$

Chapter 5

Runge-Kutta Method

5.1 Runge-Kutta Method For Simultaneous First Order Equation

Consider the simultaneous equation $\frac{dy}{dx} = f_1(x, y, z)$ with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ .Now starting from (x_0, y_0, z_0) the increments k and l in y and z are given by the following formulae.

$$k_1 = hf_1(x_0, y_0, z_0); \qquad l_1 = hf_2(x_0, y_0, z_0)$$

$$k_2 = hf_1(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}, z_0 + \tfrac{l_1}{2}); \quad l_2 = hf_2(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}, z_0 + \tfrac{l_1}{2})$$

$$k_3 = hf_1(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2}, z_0 + \tfrac{l_2}{2}); \quad l_3 = hf_2(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2}, z_0 + \tfrac{l_2}{2})$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3); \quad l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); \qquad l = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Hence $y_1 = y_0 + k, z_1 = z_0 + l$

To compute $y_2 z_2$ we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

If we consider the the second order Runge-Kutta method, then

$$k_1 = hf_1(x_0, y_0, z_0); \qquad l_1 = hf_2(x_0, y_0, z_0)$$

$$k_2 = hf_1(x_0 + h, y_0 + k_1, z_0 + l_1); \quad l_2 = hf_2(x_0 + h, y_0 + k_1, z_0 + l_1)$$

$$k = \frac{1}{2}(k_1 + k_2); \qquad l = \frac{1}{2}(k_1 + k_2)$$

$\therefore \qquad y_1 = y_0 + k$ and $z_1 = z_0 + l$

5.2 Runge-Kutta Method For Second Order Equation

Consider the second order differential equation

$$\frac{d^2y}{dx^2} = \phi\left[x, y, \frac{dy}{dx}\right]; \qquad y(x_0) = y_0; \qquad y'(x_0) = y'_0$$

Let $\frac{dy}{dx} = z$, then $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Substituting in (1) , we get $\frac{dz}{dx} = \phi\left[x, y, z\right]; \qquad y(x_0) = y_0; \qquad z(x_0) = z_0$

$\therefore$ The problem reduces to solving the simultaneous euation

$$\frac{dy}{dx} = z = f_1(x, y, z)$$

and $\qquad \frac{dz}{dx} = f_2(x, y, z)$ subject to $y(x_0) = y_0; \quad z(x_0) = z_0$

Chapter 6

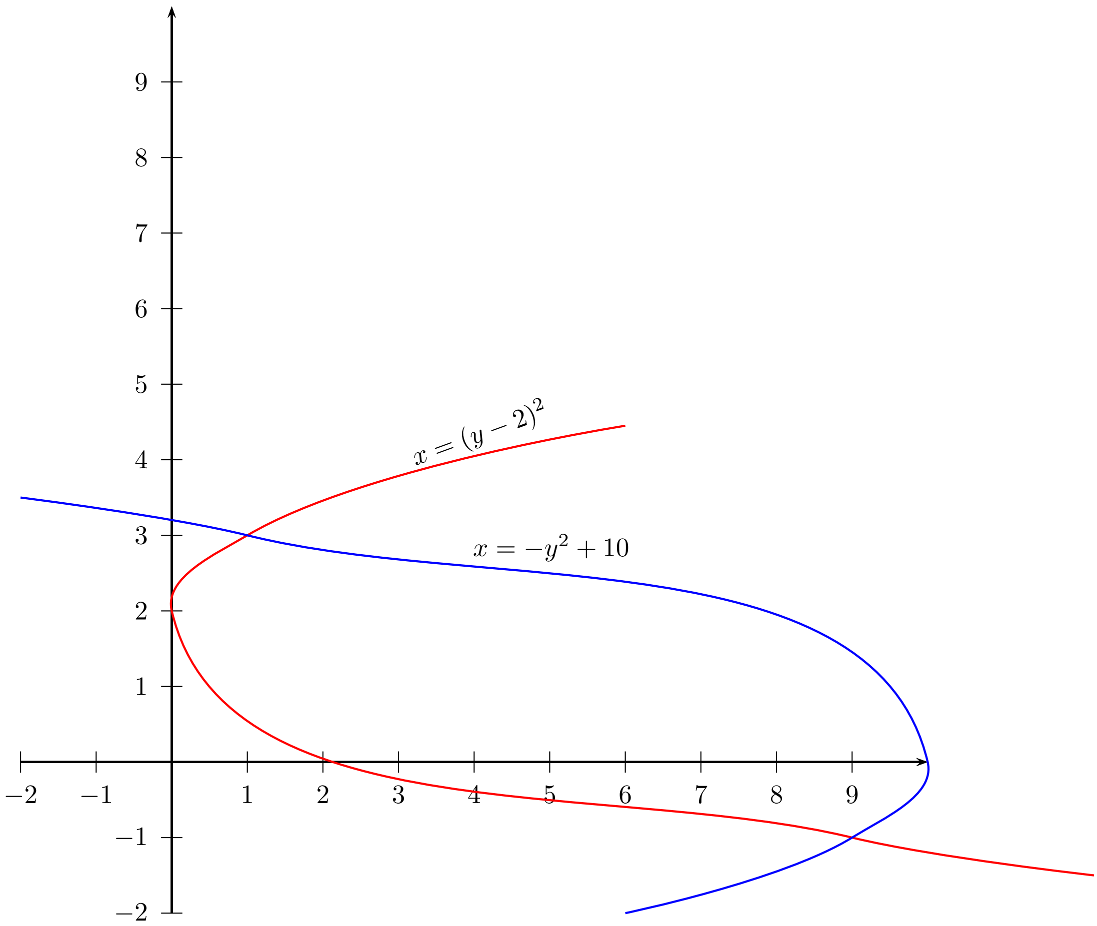
Area Between the Curves

Question:Determine the area of the region bounded by $x = -y^2 + 10$ and $x = (y-2)^2$.

Solution:
Fist we need intersection point

$$\begin{aligned}-y^2+10 =& (y-2)^2\\-y^2+10 =& y^2-4y+4\\0 =& 2y^3-4y-6\\0 =& 2(y+1)(y-3)\end{aligned}$$

The intersection points are $y = -1$ and $y = 3$. Here is sketch of the region.



This is definitely a region where the second area formula will be easier. If we used the first formula there would be three different regions that wed have to look at.

The area in this case is,

$$
\begin{aligned}
A &= \int_c^d (right function) - (left function)\mathrm{dy} \\
&= \int_{-1}^{3} -y^2 + 10 - (y-2)^2 \mathrm{dy} \\
&= \int_{-1}^{3} -2y^2 + 4y + 6\mathrm{dy} \\
&= \left(-\frac{2}{3}y^3 + 2y^2 + 6y\right)\Big|_{-1}^{3} \\
&= \frac{64}{3}
\end{aligned}
$$