Lecture Notes on Electrodynamics

Electricity, magnetism and Electrodynamics

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Undergraduate Lecture Notes

Topics Covered:
Vector Calculus • Electrostatics • problem set

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Contents

1 Vector Calculus

1.1 Scalars and Vectors

A **scalar** is a physical quantity completely specified by a magnitude alone. Examples include temperature, mass, electric potential, and energy.

A **vector** is a physical quantity specified by both magnitude and direction. Examples include displacement, velocity, force, and the electric and magnetic fields.

Definition 1.1. A vector $\mathbf{A}$ in three-dimensional Cartesian coordinates can be written as

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}, \tag{1.1}$$

where A_x, A_y, and A_z are the components of $\mathbf{A}$ along the unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ respectively.

The magnitude of $\mathbf{A}$ is given by

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{1.2}$$

1.2 Vector Algebra

1.2.1 Addition and Subtraction

The sum of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined component-wise:

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}. \tag{1.3}$$

Vector subtraction is defined similarly.

1.2.2 Scalar (Dot) Product

Definition 1.2. The scalar product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta, \tag{1.4}$$

where θ is the angle between the vectors.

In Cartesian coordinates,

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \tag{1.5}$$

1.2.3 Vector (Cross) Product

Definition 1.3. The vector product of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta\, \hat{\mathbf{n}}, \tag{1.6}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$, determined by the right-hand rule.

In component form,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{1.7}$$

1.3 Differential Vector Operators

1.3.1 Gradient

Definition 1.4. The gradient of a scalar field $\phi(x, y, z)$ is defined as

$$\boldsymbol{\nabla}\phi = \hat{\mathbf{x}}\frac{\partial \phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial \phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial \phi}{\partial z}. \tag{1.8}$$

The gradient points in the direction of maximum increase of the scalar field and its magnitude gives the rate of change in that direction.

1.3.2 Divergence

Definition 1.5. The divergence of a vector field $\mathbf{A}$ is defined as

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{1.9}$$

Divergence measures the net outward flux per unit volume from a point.

1.3.3 Curl

Definition 1.6. The curl of a vector field $\mathbf{A}$ is defined as

$$\boldsymbol{\nabla} \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{1.10}$$

The curl represents the local rotational tendency of the vector field.

1.4 Integral Theorems

1.4.1 Gauss's Divergence Theorem

Law 1.1 (Gauss's Divergence Theorem). *Let $\mathbf{A}(\mathbf{r})$ be a continuously differentiable vector field defined throughout a volume V bounded by a closed surface S with outward-directed unit normal $\hat{\mathbf{n}}$. Then the total flux of $\mathbf{A}$ through the surface S is equal to the volume integral of the divergence of $\mathbf{A}$ over V, that is,*

$$\oint_S \mathbf{A} \cdot d\mathbf{a} = \int_V (\boldsymbol{\nabla} \cdot \mathbf{A})\, d\tau, \tag{1.11}$$

where $d\mathbf{a} = \hat{\mathbf{n}}\, da$ is the outward-directed surface area element and $d\tau$ is the volume element.

Gauss's divergence theorem converts a surface integral into a volume integral. Physically, it states that the net outward flux of a vector field through a closed surface equals the total strength of sources contained within the enclosed volume.

1.4.2 Stokes' Theorem

Law 1.2 (Stokes' Theorem). *Let $\mathbf{A}(\mathbf{r})$ be a continuously differentiable vector field defined on an open surface S bounded by a closed curve C. If $\hat{\mathbf{n}}$ is the unit normal to the surface S and the direction of traversal of C is related to $\hat{\mathbf{n}}$ by the right-hand rule, then the line integral of $\mathbf{A}$ around C is equal to the surface integral of the curl of $\mathbf{A}$ over S, that is,*

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{a}. \tag{1.12}$$

Stokes' theorem relates the circulation of a vector field around a closed curve to the flux of its curl through any surface bounded by that curve. Physically, it expresses the idea that local rotational behavior of a field gives rise to macroscopic circulation.

1.5 Important Vector Identities

The following vector identities involving the del operator (∇) are extensively used in electrodynamics, particularly in the formulation and manipulation of Maxwell's equations and in solving boundary-value problems.

1.5.1 First-Order Del Operator Identities

Important Result

$$\boldsymbol{\nabla}(\phi + \psi) = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\psi, \tag{1.13}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{A} + \mathbf{B}) = \boldsymbol{\nabla} \cdot \mathbf{A} + \boldsymbol{\nabla} \cdot \mathbf{B}, \tag{1.14}$$

$$\boldsymbol{\nabla} \times (\mathbf{A} + \mathbf{B}) = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times \mathbf{B}, \tag{1.15}$$

Important Result

$$\boldsymbol{\nabla}(\phi\psi) = \phi\,\boldsymbol{\nabla}\psi + \psi\,\boldsymbol{\nabla}\phi, \tag{1.16}$$
$$\boldsymbol{\nabla}\cdot(\phi\mathbf{A}) = \phi\,\boldsymbol{\nabla}\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\nabla}\phi, \tag{1.17}$$
$$\boldsymbol{\nabla}\times(\phi\mathbf{A}) = \phi\,\boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}\phi\times\mathbf{A}, \tag{1.18}$$

Important Result

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}), \tag{1.19}$$
$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}, \tag{1.20}$$

1.5.2 Identities Involving Curl and Divergence

Important Result

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0, \tag{1.21}$$
$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0, \tag{1.22}$$
$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}, \tag{1.23}$$

These identities impose strong constraints on the possible forms of vector fields and play a fundamental role in Maxwell's equations.

1.5.3 Second-Order Del Operator Identities

Important Result

$$\nabla^2\phi = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi), \tag{1.24}$$
$$\nabla^2\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}), \tag{1.25}$$

Important Result

$$\boldsymbol{\nabla}\left(\nabla^2\phi\right) = \nabla^2(\boldsymbol{\nabla}\phi), \tag{1.26}$$
$$\boldsymbol{\nabla}\times\left(\nabla^2\mathbf{A}\right) = \nabla^2(\boldsymbol{\nabla}\times\mathbf{A}), \tag{1.27}$$
$$\boldsymbol{\nabla}\cdot\left(\nabla^2\mathbf{A}\right) = \nabla^2(\boldsymbol{\nabla}\cdot\mathbf{A}), \tag{1.28}$$

Important Result

$$\nabla^2(\phi\psi) = \phi\,\nabla^2\psi + \psi\,\nabla^2\phi + 2\,\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi, \tag{1.29}$$

1.5.4 Identities Involving Directional Derivatives

Important Result

$$(\mathbf{A}\cdot\nabla)\phi = \mathbf{A}\cdot\boldsymbol{\nabla}\phi, \tag{1.30}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}), \tag{1.31}$$

1.5.5 Useful Special Identities

Important Result

$$\boldsymbol{\nabla}(r) = \hat{\mathbf{r}}, \tag{1.32}$$

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{\mathbf{r}}, \tag{1.33}$$

$$\nabla^2\left(\frac{1}{r}\right) = 0 \quad (r\neq 0), \tag{1.34}$$

$$\boldsymbol{\nabla}\times(\hat{\mathbf{r}}) = 0. \tag{1.35}$$

1.6 Spherical Coordinate System

1.6.1 Definition of Spherical Coordinates

In the spherical coordinate system, the position of a point in space is specified by the coordinates (r,θ,ϕ), where:

- r is the radial distance from the origin,
- θ is the polar angle measured from the positive z-axis $(0\leq\theta\leq\pi)$,
- ϕ is the azimuthal angle measured in the xy-plane from the positive x-axis $(0\leq\phi<2\pi)$.

The transformation between Cartesian and spherical coordinates is given by

$$x = r\sin\theta\cos\phi, \tag{1.36}$$

$$y = r\sin\theta\sin\phi, \tag{1.37}$$

$$z = r\cos\theta. \tag{1.38}$$

Conversely,

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{1.39}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right), \tag{1.40}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{1.41}$$

1.6.2 Unit Vectors in Spherical Coordinates

The spherical coordinate system uses three mutually orthogonal unit vectors:

- $\hat{\mathbf{r}}$ in the radial direction,
- $\hat{\boldsymbol{\theta}}$ in the direction of increasing θ,
- $\hat{\boldsymbol{\phi}}$ in the direction of increasing ϕ.

These unit vectors vary with position and are related to the Cartesian unit vectors by

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \tag{1.42}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \tag{1.43}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}. \tag{1.44}$$

1.6.3 Differential Length, Area, and Volume Elements

In spherical coordinates, the differential displacement vector is

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}. \tag{1.45}$$

The corresponding differential area elements are

$$d\mathbf{a}_r = r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}, \tag{1.46}$$

$$d\mathbf{a}_\theta = r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}}, \tag{1.47}$$

$$d\mathbf{a}_\phi = r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}. \tag{1.48}$$

The differential volume element is

$$d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi. \tag{1.49}$$

1.6.4 Gradient, Divergence, and Curl in Spherical Coordinates

For a scalar field $\phi(r,\theta,\phi)$, the gradient is

$$\boldsymbol{\nabla}\phi = \hat{\mathbf{r}}\frac{\partial\phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\phi}. \tag{1.50}$$

For a vector field $\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$, the divergence is

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}. \tag{1.51}$$

The curl of $\mathbf{A}$ is given by

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}. \tag{1.52}$$

1.6.5 Laplacian in Spherical Coordinates

The Laplacian of a scalar field $\phi(r, \theta, \phi)$ in spherical coordinates is

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2}. \tag{1.53}$$

1.7 Cylindrical Coordinate System

1.7.1 Definition of Cylindrical Coordinates

In the cylindrical coordinate system, a point in space is specified by the coordinates (ρ, ϕ, z), where:

- ρ is the perpendicular distance from the z-axis,
- ϕ is the azimuthal angle measured in the xy-plane from the positive x-axis $(0 \leq \phi < 2\pi)$,
- z is the same coordinate as in Cartesian coordinates.

The transformation between Cartesian and cylindrical coordinates is given by

$$x = \rho\cos\phi, \tag{1.54}$$

$$y = \rho\sin\phi, \tag{1.55}$$

$$z = z. \tag{1.56}$$

Conversely,

$$\rho = \sqrt{x^2 + y^2}, \tag{1.57}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right), \tag{1.58}$$

$$z = z. \tag{1.59}$$

1.7.2 Unit Vectors in Cylindrical Coordinates

The cylindrical coordinate system employs three mutually orthogonal unit vectors:

- $\hat{\boldsymbol{\rho}}$ in the direction of increasing ρ,

- $\hat{\boldsymbol{\phi}}$ in the direction of increasing ϕ,
- $\hat{\mathbf{z}}$ along the positive z-axis.

These unit vectors are related to the Cartesian unit vectors by

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \tag{1.60}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \tag{1.61}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}. \tag{1.62}$$

1.7.3 Differential Length, Area, and Volume Elements

The differential displacement vector in cylindrical coordinates is

$$d\mathbf{l} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}. \tag{1.63}$$

The differential area elements are

$$d\mathbf{a}_\rho = \rho\,d\phi\,dz\;\hat{\boldsymbol{\rho}}, \tag{1.64}$$

$$d\mathbf{a}_\phi = d\rho\,dz\;\hat{\boldsymbol{\phi}}, \tag{1.65}$$

$$d\mathbf{a}_z = \rho\,d\rho\,d\phi\;\hat{\mathbf{z}}. \tag{1.66}$$

The differential volume element is

$$d\tau = \rho\,d\rho\,d\phi\,dz. \tag{1.67}$$

1.7.4 Gradient, Divergence, and Curl in Cylindrical Coordinates

For a scalar field $\phi(\rho, \phi, z)$, the gradient is

$$\boldsymbol{\nabla}\phi = \hat{\boldsymbol{\rho}}\frac{\partial\phi}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial\phi}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial\phi}{\partial z}. \tag{1.68}$$

For a vector field $\mathbf{A} = A_\rho\hat{\boldsymbol{\rho}} + A_\phi\hat{\boldsymbol{\phi}} + A_z\hat{\mathbf{z}}$, the divergence is

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}. \tag{1.69}$$

The curl of $\mathbf{A}$ is given by

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}. \tag{1.70}$$

1.7.5 Laplacian in Cylindrical Coordinates

The Laplacian of a scalar field $\phi(\rho, \phi, z)$ in cylindrical coordinates is

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\phi^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{1.71}$$

Problems

Problem 1.1. For the scalar field

$$\phi(x, y, z) = x^2y + yz^3,$$

find the gradient $\boldsymbol{\nabla}\phi$ at the point $(1, -1, 2)$. *Answer:* $\boldsymbol{\nabla}\phi = -2\hat{\mathbf{x}} + 9\hat{\mathbf{y}} - 12\hat{\mathbf{z}}$

Problem 1.2. Given the vector field

$$\mathbf{A} = x^2\hat{\mathbf{x}} + yz\hat{\mathbf{y}} + z^2\hat{\mathbf{z}},$$

calculate $\boldsymbol{\nabla}\cdot\mathbf{A}$ and $\boldsymbol{\nabla}\times\mathbf{A}$. *Answer:* $\boldsymbol{\nabla}\cdot\mathbf{A} = 2x + z + 2z,\ \boldsymbol{\nabla}\times\mathbf{A} = y\hat{\mathbf{x}}$

Problem 1.3. Show that

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0$$

for the scalar field $\phi = r^2$.

Problem 1.4. Verify that

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0$$

for $\mathbf{A} = yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$.

Problem 1.5. Find $\nabla^2\phi$ for

$$\phi(r) = \frac{1}{r}, \quad r \neq 0.$$

Answer: $\nabla^2\phi = 0$

Problem 1.6. Using spherical coordinates, find

$$\boldsymbol{\nabla}\cdot\left(\frac{1}{r^2}\hat{\mathbf{r}}\right).$$

Answer: 0 *for* $r \neq 0$

Problem 1.7. Evaluate the flux of

$$\mathbf{A} = r\hat{\mathbf{r}}$$

through a sphere of radius R. *Answer:* $4\pi R^3$

Problem 1.8. Using Gauss's theorem, evaluate

$$\oint_S (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})\cdot d\mathbf{a}$$

over a cube of side a centered at the origin. *Answer:* $3a^3$

Problem 1.9. Using Stokes' theorem, evaluate

$$\oint_C \mathbf{A} \cdot d\mathbf{l}$$

for $\mathbf{A} = y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$, where C is a circle of radius R in the xy-plane. *Answer:* $-2\pi R^2$

Problem 1.10. Find the divergence of

$$\mathbf{A} = \rho\hat{\boldsymbol{\rho}}$$

in cylindrical coordinates. *Answer:* $\boldsymbol{\nabla} \cdot \mathbf{A} = 2$

Problem 1.11. Find the curl of

$$\mathbf{A} = \rho^2\hat{\boldsymbol{\phi}}$$

in cylindrical coordinates. *Answer:* $2\rho\,\hat{\mathbf{z}}$

Problem 1.12. Show that if $\phi = \phi(z)$ only, then

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial z^2}.$$

Answer: True

Problem 1.13. For $\mathbf{A} = r^n\hat{\mathbf{r}}$, find $\boldsymbol{\nabla} \cdot \mathbf{A}$ and determine n such that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ for $r \neq 0$. *Answer:* $\boldsymbol{\nabla} \cdot \mathbf{A} = (n+2)r^{n-1}$, $n = -2$

Problem 1.14. Starting from

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0},$$

derive the integral form of Gauss's law. *Answer:* $\oint \mathbf{E} \cdot d\mathbf{a} = \dfrac{Q_{enc}}{\varepsilon_0}$

2 Electrostatics

2.1 Electric Charge

Electric charge is a fundamental property of matter responsible for electrical interactions.

Postulate 2.1. Electric charge is conserved; the total charge of an isolated system remains constant.

Charge is quantized and occurs in integral multiples of the elementary charge $e = 1.602 \times 10^{-19}\,\mathrm{C}$.

2.2 Coulomb's Law

The force between two stationary point charges was first quantified by Coulomb.

Law 2.1 (Coulomb's Law)**.** *The electrostatic force* $\mathbf{F}$ *between two point charges* q_1 *and* q_2 *separated by a distance* r *is given by*

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}, \tag{2.1}$$

where $\hat{\mathbf{r}}$ *is the unit vector from* q_1 *to* q_2 *and* ε_0 *is the permittivity of free space.*

The force acts along the line joining the charges and is repulsive for like charges and attractive for unlike charges.

2.3 Electric Field

2.3.1 Definition of Electric Field

The electric field is defined as the force per unit positive test charge.

Definition 2.1. The electric field $\mathbf{E}$ at a point in space is defined as

$$\mathbf{E} = \lim_{q\to 0}\frac{\mathbf{F}}{q}. \tag{2.2}$$

For a point charge q located at the origin,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{2.3}$$

2.3.2 Principle of Superposition

Since the electrostatic force obeys the principle of superposition, the electric field, being defined as force per unit charge, must also obey the same principle.

Postulate 2.2. The net electric field at any point due to a system of charges is equal to the vector sum of the electric fields produced by the individual charges acting independently.

Electric Field Due to Discrete Charges

For a system of N point charges q_i located at positions $\mathbf{r}_i$, the electric field at a field point $\mathbf{r}$ is given by

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}. \tag{2.4}$$

Each charge produces an electric field as if it were acting alone, and the resultant field is obtained by vector addition.

Electric Field Due to Continuous Charge Distributions

When the charge distribution is continuous, the discrete sum is replaced by an integral. For a volume charge density $\rho(\mathbf{r}')$, the electric field at point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d\tau'. \tag{2.5}$$

Similar expressions can be written for line and surface charge distributions by replacing $\rho \, d\tau$ with $\lambda \, dl$ or $\sigma \, da$, respectively.

Electric Field Due to Line and Surface Charge Distributions

For a *line charge distribution* characterized by a linear charge density $\lambda(\mathbf{r}')$, the electric field at point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dl'. \tag{2.6}$$

For a *surface charge distribution* described by a surface charge density $\sigma(\mathbf{r}')$, the electric field at point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, da'. \tag{2.7}$$

In each case, the vector $\mathbf{r} - \mathbf{r}'$ points from the source charge element to the field point, and the integration extends over the entire charge distribution.

2.4 Electric Flux

Definition 2.2. The electric flux through a surface S is defined as

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a}. \tag{2.8}$$

Electric flux provides a measure of the number of electric field lines passing through a given surface.

Remark 2.1. In electrostatics, the electric field may be viewed as a measure of the density of electric flux in space. Regions of stronger electric field correspond to regions of higher electric flux density.

2.5 Gauss's Law

2.5.1 Statement of Gauss's Law

Law 2.2. *Gauss's law states that the total electric flux emerging from any closed surface is directly proportional to the total electric charge enclosed within that surface. The proportionality constant is the permittivity of free space.*

In compact notation, Gauss's law is expressed as

$$\Phi_E = \frac{Q_{enc}}{\varepsilon_0}, \tag{2.9}$$

where Φ_E *is the total electric flux through the closed surface and* Q_{enc} *is the total charge enclosed.*

Remark 2.2. electric flux measures how much electric field passes through a closed surface, and this flux is determined solely by the charge enclosed inside the surface, independent of the shape or size of the surface.

2.5.2 Integral Form of Gauss's Law

The electric flux through a closed surface S is defined as

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{a}. \tag{2.10}$$

The total charge enclosed within the surface can be written in terms of the volume charge density ρ as

$$Q_{\text{enc}} = \int_V \rho \, d\tau, \tag{2.11}$$

where V is the volume enclosed by the surface S.

Substituting these expressions into the flux–charge relation, Gauss's law in integral form becomes

Law 2.3 (Integral Form of Gauss's law)**.**

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} \int_V \rho \, d\tau. \tag{2.12}$$

This form of Gauss's law is particularly useful for evaluating electric fields in systems possessing spherical, cylindrical, or planar symmetry.

2.5.3 Differential Form of Gauss's Law

Gauss's divergence theorem relates the flux of a vector field through a closed surface to the volume integral of its divergence:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V (\boldsymbol{\nabla} \cdot \mathbf{E})\, d\tau. \tag{2.13}$$

Applying this theorem to the integral form of Gauss's law, we obtain

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{E})\, d\tau = \frac{1}{\varepsilon_0} \int_V \rho\, d\tau. \tag{2.14}$$

Since the above relation must hold for any arbitrary volume V, the integrands must be equal at every point in space. Hence, Gauss's law in differential form is

Law 2.4 (Differential Form of Gauss's Law)**.**

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{2.15}$$

2.6 Applications of Gauss's Law

Gauss's law is most useful when the charge distribution possesses a high degree of symmetry. In such cases, an appropriate Gaussian surface can be chosen so that the electric field is either constant or zero over the surface, allowing straightforward evaluation of the flux integral.

2.6.1 Point Charge (Coulomb's Law)

Consider a point charge q placed at the origin. Due to spherical symmetry, the electric field at any point depends only on the radial distance r and is directed radially outward.

Choose a spherical Gaussian surface of radius r centered at the charge. On this surface, $\mathbf{E}$ has constant magnitude and is normal to the surface everywhere.

The electric flux is

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = E \oint_S da = E(4\pi r^2). \tag{2.16}$$

The charge enclosed is $Q_{\text{enc}} = q$. Applying Gauss's law,

$$E(4\pi r^2) = \frac{q}{\varepsilon_0}. \tag{2.17}$$

Hence,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}, \tag{2.18}$$

which is Coulomb's law in field form.

2.6.2 Electric Field of an Electric Dipole

Consider two point charges $+q$ and $-q$ separated by a distance $2a$, forming an electric dipole.

The electric field at a point $\mathbf{r}$ is obtained by superposition:

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-. \tag{2.19}$$

Using Coulomb's law,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3} - \frac{q(\mathbf{r}+\mathbf{a})}{|\mathbf{r}+\mathbf{a}|^3}\right]. \tag{2.20}$$

For points far from the dipole ($r \gg a$), the electric field reduces to

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right], \tag{2.21}$$

where $\mathbf{p} = 2qa\,\hat{\mathbf{z}}$ is the electric dipole moment.

2.6.3 Infinite Long Straight Wire

Consider an infinitely long straight wire carrying a uniform linear charge density λ. Due to cylindrical symmetry, the electric field is radial and depends only on the distance ρ from the wire.

Choose a cylindrical Gaussian surface of radius ρ and length L, coaxial with the wire.

The electric field is perpendicular to the curved surface and parallel to the end caps. Hence, flux through the end caps is zero.

The total flux is

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = E(2\pi\rho L). \tag{2.22}$$

The enclosed charge is

$$Q_{\text{enc}} = \lambda L. \tag{2.23}$$

Applying Gauss's law,

$$E(2\pi\rho L) = \frac{\lambda L}{\varepsilon_0}. \tag{2.24}$$

Therefore,

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0\rho}\hat{\boldsymbol{\rho}}. \tag{2.25}$$

2.6.4 Infinite Plane Sheet of Charge

Consider an infinite plane sheet with uniform surface charge density σ. The electric field must be perpendicular to the sheet and independent of distance from it.

Choose a cylindrical Gaussian surface (pillbox) of cross-sectional area A, with its flat faces parallel to the sheet.

The flux through the curved surface is zero. The flux through the two flat faces is

$$\Phi_E = EA + EA = 2EA. \tag{2.26}$$

The enclosed charge is

$$Q_{\text{enc}} = \sigma A. \tag{2.27}$$

Applying Gauss's law,

$$2EA = \frac{\sigma A}{\varepsilon_0}. \tag{2.28}$$

Thus,

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}, \tag{2.29}$$

where $\hat{\mathbf{n}}$ is the unit normal to the sheet.

2.6.5 Uniformly Charged Solid Sphere

Consider a solid sphere of radius R with constant volume charge density ρ_0.

Electric Field Outside the Sphere ($r > R$)

Choose a spherical Gaussian surface of radius $r > R$. The total enclosed charge is

$$Q_{\text{enc}} = \rho_0 \frac{4}{3}\pi R^3. \tag{2.30}$$

Applying Gauss's law,

$$E(4\pi r^2) = \frac{1}{\varepsilon_0}\rho_0 \frac{4}{3}\pi R^3. \tag{2.31}$$

Hence,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}, \tag{2.32}$$

where Q is the total charge of the sphere.

Electric Field Inside the Sphere ($r < R$)

For a Gaussian sphere of radius $r < R$, the enclosed charge is

$$Q_{\text{enc}} = \rho_0 \frac{4}{3}\pi r^3. \tag{2.33}$$

Applying Gauss's law,

$$E(4\pi r^2) = \frac{1}{\varepsilon_0}\rho_0 \frac{4}{3}\pi r^3. \tag{2.34}$$

Thus,

$$\mathbf{E} = \frac{\rho_0 r}{3\varepsilon_0}\hat{\mathbf{r}}. \tag{2.35}$$

2.6.6 Uniformly Charged Spherical Shell

Consider a thin spherical shell of radius R carrying uniform surface charge density σ.

Outside the Shell ($r > R$)

The enclosed charge is

$$Q = 4\pi R^2\sigma. \tag{2.36}$$

Applying Gauss's law,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}. \tag{2.37}$$

Inside the Shell ($r < R$)

No charge is enclosed within the Gaussian surface. Hence,

$$\mathbf{E} = 0. \tag{2.38}$$

2.7 Curl of the Electrostatic Field

2.7.1 Curl of the Electric Field Due to a Point Charge

Consider a point charge q placed at the origin. The electric field at a distance r is given by Coulomb's law,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{2.39}$$

This field depends only on the radial coordinate and is directed along $\hat{\mathbf{r}}$. To evaluate its curl, we use the expression for curl in spherical coordinates.

For a purely radial field of the form $\mathbf{E} = E_r(r)\hat{\mathbf{r}}$, the curl vanishes identically:

$$\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0} \qquad (r \neq 0). \tag{2.40}$$

Thus, everywhere in space except at the location of the charge, the electric field due to a point charge is curl-free.

Law 2.5. *In electrostatics, the curl of the electric field vanishes:*

$$\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}. \tag{2.41}$$

This result is independent of the specific charge configuration and holds for all static electric fields.

2.7.2 Conservative nature of Electrostatic field

Theorem 2.1 (Conservativeness of Electrostatic Field)**.** *The electrostatic field* $\mathbf{E}$ *in a region with stationary charges is conservative; that is, the line integral around any closed path is zero:*

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \tag{2.42}$$

Proof. Consider a set of stationary point charges $\{q_i\}$ at positions $\mathbf{r}_i$. The electric field at a point $\mathbf{r}$ is given by the principle of superposition:

$$\mathbf{E}(\mathbf{r}) = \sum_i \frac{1}{4\pi\varepsilon_0}\frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}. \tag{2.43}$$

Curl of a single point charge field.

For a single point charge at the origin:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{2.44}$$

This is a radial field. Using the vector identity for a radial function $f(r)\hat{\mathbf{r}}$, its curl is identically zero:

$$\nabla \times \mathbf{E} = 0 \quad \text{for } r \neq 0. \tag{2.45}$$

Superposition principle.

$$\nabla \times \mathbf{E} = \sum_i \nabla \times \mathbf{E}_i = 0. \tag{2.46}$$

Hence, the electrostatic field due to any configuration of stationary charges is irrotational.
Apply Stokes' theorem.
For any closed curve C bounding a surface S, Stokes' theorem gives

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{a}. \tag{2.47}$$

Since $\nabla \times \mathbf{E} = 0$, the right-hand side vanishes:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \tag{2.48}$$

□

Remark 2.3. The fact that $\mathbf{E}$ is conservative implies that the work done by the electrostatic field on a test charge moving between any two points is independent of the path.

2.8 Scalar Potential

2.8.1 Existence of Scalar Potential

Theorem 2.2. *If the curl of a vector field vanishes in a simply connected region, then the field can be expressed as the gradient of a scalar potential.*

Proof. Since $\nabla \times \mathbf{E} = 0$ in electrostatics, there exists a scalar function V such that

$$\mathbf{E} = -\nabla V. \tag{2.49}$$

The negative sign is chosen by convention so that the electric field points in the direction of decreasing potential. □

2.8.2 Potential Difference

Theorem 2.3 (Fundamental Theorem of Gradient)**.** *For a scalar field $V(\mathbf{r})$, the line integral of its gradient between two points A and B is equal to the difference of the scalar at the endpoints:*

$$\int_A^B \nabla V \cdot d\mathbf{l} = V(B) - V(A). \tag{2.50}$$

Definition 2.3 (Electric Potential Difference)**.** Using $\mathbf{E} = -\nabla V$ in 2.50,

The potential difference between two points A and B is

$$V(B) - V(A) = -\int_A^B \mathbf{E} \cdot d\mathbf{l}. \tag{2.51}$$

Remark 2.4. This derivation shows that the potential difference depends only on the endpoints and is path-independent, a direct consequence of the conservative nature of the electrostatic field.

2.8.3 Potential Due to a Point Charge

Choosing $V(\infty) = 0$, the potential at a distance r from a point charge q is

$$V(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}. \tag{2.52}$$

2.8.4 Potential Due to Multiple Charges

For N point charges q_i located at positions $\mathbf{r}_i$, the potential at $\mathbf{r}$ is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|}. \tag{2.53}$$

For continuous charge distributions:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau', \tag{2.54}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dl', \tag{2.55}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} da'. \tag{2.56}$$

2.8.5 Poisson's and Laplace's Equations

Law 2.6 (Poisson's Equation)**.** *Using* $\mathbf{E} = -\boldsymbol{\nabla} V$ *in Gauss's law, we obtain*

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \implies \boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} V) = -\nabla^2 V = \frac{\rho}{\varepsilon_0}, \tag{2.57}$$

so

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}. \tag{2.58}$$

Remark 2.5. Poisson's equation relates the potential to the charge density in space.

Law 2.7 (Laplace's Equation)**.** *In charge-free regions (*$\rho = 0$*), Poisson's equation reduces to Laplace's equation:*

$$\nabla^2 V = 0. \tag{2.59}$$

2.8.6 Laplace's Equation in Spherical Coordinates

In spherical coordinates (r, θ, ϕ), assuming azimuthal symmetry ($\partial/\partial\phi = 0$), Laplace's equation is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial V}{\partial \theta} \right) = 0. \tag{2.60}$$

Separation of Variables

Assume $V(r,\theta) = R(r)\Theta(\theta)$. Then

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0. \tag{2.61}$$

Setting each part equal to a constant $l(l+1)$, we get:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1), \tag{2.62}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1). \tag{2.63}$$

Above equation is Legendre equation. whose solution is given by legendre polynomial:

$$\Theta(\cos\theta) = P_l(\cos\theta) \tag{2.64}$$

Solution for Radial Part

The radial equation is

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - l(l+1)R = 0, \tag{2.65}$$

whose general solution is

$$R(r) = Ar^l + Br^{-(l+1)}, \tag{2.66}$$

where A and B are constants determined by boundary conditions.

Full Solution

The general solution for the potential is

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right)P_l(\cos\theta), \tag{2.67}$$

where P_l are the Legendre polynomials.

2.9 Uniqueness Theorem in Electrostatics

Theorem 2.4 (Uniqueness Theorem)**.** *In a given volume V, the solution of Poisson's equation*

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \tag{2.68}$$

has unique solution if either

1. *the value of the potential V is specified on the boundary surface S (Dirichlet boundary condition), or*

2. *the normal derivative* $\dfrac{\partial V}{\partial n}$ *is specified on* S *(Neumann boundary condition), together with the total charge in* V*.*

2.9.1 Proof of the Uniqueness Theorem

Proof. Assume that there exist two solutions V_1 and V_2 satisfying Poisson's equation in the volume V and the same boundary conditions on the surface S.

Define their difference

$$\phi = V_1 - V_2. \tag{2.69}$$

Since both V_1 and V_2 satisfy Poisson's equation with the same charge density ρ,

$$\nabla^2\phi = \nabla^2 V_1 - \nabla^2 V_2 = 0. \tag{2.70}$$

Thus, ϕ satisfies Laplace's equation in V.

Consider the volume integral

$$\int_V \phi\nabla^2\phi \, d\tau = 0. \tag{2.71}$$

Using the vector identity

$$\phi\nabla^2\phi = \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\phi) - (\boldsymbol{\nabla}\phi)^2,$$

we obtain

$$\int_V \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\phi)\, d\tau - \int_V (\boldsymbol{\nabla}\phi)^2 \, d\tau = 0. \tag{2.72}$$

Applying Gauss's divergence theorem,

$$\oint_S \phi\boldsymbol{\nabla}\phi\cdot d\mathbf{a} = \int_V (\boldsymbol{\nabla}\phi)^2 \, d\tau. \tag{2.73}$$

For Dirichlet boundary conditions, $V_1 = V_2$ on S, hence $\phi = 0$ on S. Therefore, the surface integral vanishes:

$$\oint_S \phi\boldsymbol{\nabla}\phi\cdot d\mathbf{a} = 0. \tag{2.74}$$

This implies

$$\int_V (\boldsymbol{\nabla}\phi)^2 \, d\tau = 0. \tag{2.75}$$

Since $(\boldsymbol{\nabla}\phi)^2 \geq 0$ everywhere, the only possibility is

$$\boldsymbol{\nabla}\phi = 0 \quad \text{throughout } V. \tag{2.76}$$

Hence ϕ is a constant. But since $\phi = 0$ on the boundary, this constant must be zero:

$$\phi = 0 \quad \Rightarrow \quad V_1 = V_2. \tag{2.77}$$

Thus, the solution is unique. □

Remark 2.6. The uniqueness theorem implies that once the charge distribution and appropriate boundary conditions are specified, the electrostatic potential and electric field are completely determined. Any method that produces a solution satisfying Poisson's equation and the boundary conditions must therefore give the correct physical solution.

Problems

Problem 2.1. Two point charges $q_1 = +2\ \mu C$ and $q_2 = -3\ \mu C$ are separated by $r = 0.5$ m. Calculate the magnitude and direction of the force between them. **Answer:** $F = 0.216$ N, attractive.

Problem 2.2. Three charges, $q_1 = q_2 = q_3 = 1\ \mu C$, are placed at the vertices of an equilateral triangle of side 1 m. Find the net force on q_1. **Hint:** Use vector addition; symmetry helps.

Problem 2.3. A point P is located 0.2 m from a charge $q = 5\ \mu C$. Find the magnitude of $\mathbf{E}$ at P. **Answer:** $E \approx 1.125 \times 10^6$ N/C.

Problem 2.4. A straight wire has linear charge density $\lambda = 2\ \mu C/m$. Find the electric field 2 m away from the wire. **Answer:** $E \approx 1.8 \times 10^5$ N/C, radial.

Problem 2.5. An infinite plane sheet has surface charge density $\sigma = 5\ \mu C/m^2$. Calculate the electric field above the sheet. **Answer:** $E = 2.82 \times 10^5$ N/C, perpendicular.

Problem 2.6. A solid sphere of radius $R = 0.1$ m has uniform charge density $\rho_0 = 1\ \mu C/m^3$. Find the electric field: (a) inside at $r = 0.05$ m, (b) outside at $r = 0.2$ m.
Answer: (a) $E = 2.62 \times 10^{-2}$ N/C, (b) $E = 4.19 \times 10^{-2}$ N/C.

Problem 2.7. A thin spherical shell of radius $R = 0.1$ m carries total charge $Q = 5\ \mu C$. Find $\mathbf{E}$: (a) at $r = 0.05$ m, (b) at $r = 0.2$ m. **Answer:** (a) $E = 0$, (b) $E = 4.5 \times 10^3$ N/C.

Problem 2.8. Find the potential at a point 0.2 m away from a point charge $q = 2\ \mu C$ taking $V(\infty) = 0$. **Answer:** $V = 4.5 \times 10^5$ V.

Problem 2.9. Two point charges, $q_1 = +1\ \mu C$ at $(0, 0, 0)$ and $q_2 = -2\ \mu C$ at $(1, 0, 0)$, calculate potential at point $P = (0.5, 0, 0)$. **Hint:** Superposition, sum contributions algebraically.

Problem 2.10. A dipole consists of charges $\pm q$ separated by $2a$. Find the electric field on the axial line at distance $r \gg a$ from the center. **Answer:** $E = \frac{1}{4\pi\varepsilon_0}\frac{2qa}{r^3}$, along axis.

Problem 2.11. A $+1\ \mu C$ charge is moved from $r_1 = 0.2$ m to $r_2 = 0.4$ m from a $+2\ \mu C$ charge. Find the work done by the electrostatic field. **Answer:** $W = 45\ \mu$J.

Problem 2.12. Three point charges $q_1 = q_2 = q_3 = 1\ \mu C$ are at vertices of an equilateral triangle of side 1 m. Find total electrostatic potential energy. **Answer:** $U \approx 13.5\ \mu$J.

Problem 2.13. A cylinder of radius $R = 0.05$ m has linear charge density $\lambda = 2\ \mu C/m$. Find the flux through a coaxial cylindrical surface of radius $r = 0.1$ m and length $L = 1$ m. **Answer:** $\Phi_E = 2.26 \times 10^{-5}$ C/m^2.

Problem 2.14. A spherical shell of radius $R = 0.1$ m has charge $Q = 5\ \mu C$. Using Gauss's law, calculate the field at $r = 0.05$ m and $r = 0.2$ m. **Answer:** $E_{\text{inside}} = 0$, $E_{\text{outside}} = 4.5 \times 10^3$ N/C.

Problem 2.15. A sphere of radius $R = 0.1$ m has volume charge density $\rho(r) = \rho_0(r/R)$. Find electric field inside as a function of r. **Answer:** $E(r) = \frac{\rho_0 r^2}{4\varepsilon_0 R}\hat{\mathbf{r}}$.