Linear Algebra Applications - Discrete Linear Dynamical Systems

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Abstract

Dynamical systems are processes that evolve in time. Through forms of mathematical modelling, the study of dynamical systems is beneficial to comprehend a variety of natural phenomena. In particular, discrete linear dynamical systems are useful to model a scenario as a series of state snapshots. In this paper, we focus on discrete linear dynamical systems in the form of first-order homogenous matrix recurrence relation. We then illustrate the application of these systems through a real-world case study by modelling the spread of a virus.

Contents

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1 Discrete Linear Dynamical Systems

1.1 A matrix recurrence relation

For a square matrix A and an initial state vector x_0, define the first-order homogenous recurrence:

$$x_t = Ax_{t-1}, \quad \text{for } t \geq 1.$$

Iterating yields a sequence of vectors $x_0, Ax_0, A^2x_0, \dots, A^kx_0, \dots$ which represent the evolution of the system starting from x_0. This sequence is called the ***orbit*** of the system starting from x_0.

An important question to raise about this dynamical system is: what is its long-term behaviour? For example, if our system models a particular population of rabbits, it would be worthwhile to observe the population a few months ahead. What happens as $t \to \infty$? Does the system have limiting behaviours?

To compute the elements in our orbit, we require an efficient method to compute A^n for large n, as repeated matrix multiplication is cumbersome.

1.2 Diagonalisation

Recall that it is easy to multiply diagonal matrices (square matrices in which the entries outside the main diagonal are all zero):

$$\begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & & & \\ & b_{22} & & \\ & & \ddots & \\ & & & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & & & \\ & a_{22}b_{22} & & \\ & & \ddots & \\ & & & a_{nn}b_{nn} \end{pmatrix}.$$

This simplicity of multiplying diagonal matrices prompts us to investigate if A is similar to a diagonal matrix; does there exist an invertible matrix P and a diagonal matrix D such that A can be written as PDP^{-1}? If so, then we say that A is ***diagonalisable***. This condition is equivalent to $AP = PD$. Observe that it is easier to computer powers of a diagonalisable matrix:

$$\begin{aligned} A^n &= (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) \\ &= PD\underbrace{(P^{-1}P)}_{I}D\underbrace{(P^{-1}P)}_{I}D\cdots\underbrace{(P^{-1}P)}_{I}DP^{-1} \\ &= PD^nP^{-1}. \end{aligned}$$

Theorem 1.1. *A square matrix is diagonalisable if and only if it has n linearly independent eigenvectors.*

Proof. ($\Longrightarrow$) Suppose that $A \in \mathbb{R}^{n\times n}$ is diagonalisable. Then there exists an invertible matrix $P \in \mathbb{R}^{n\times n}$ and a diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that $AP = PD$.

Let $P = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ where v_i are linearly independent columns, and let $D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Writing $AP = PD$ as columns, we have $\begin{pmatrix} Av_1 & Av_2 & \cdots & Av_n \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{pmatrix}$.

Thus, $Av_i = \lambda_i v_i$ for each $i = 1, \dots n$. Hence, v_i is an eigenvector associated to λ_i.

($\Longleftarrow$) Conversely, suppose that $A \in \mathbb{R}^{n\times n}$ has n linearly independent eigenvectors $v_1, v_2, \dots, v_n$ associated to the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively. Then for each $i = 1, \dots, n$ we have $Av_i = \lambda_i v_i$.

Let $P = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$ where v_i are the columns. Note that P is invertible as the columns are linearly independent. Also, let $D = \operatorname{diag}(\lambda_1, \lambda_2, \cdots \lambda_n)$.

Now the equations $Av_i = \lambda_i v_i$ for each $i = 1, \dots, n$ can be compactly expressed as $AP = PD$, which implies $A = PDP^{-1}$. Hence, A is diagonalisable. ☐

Corollary 1.2. *An eigen-formula for diagonalisable matrices.*

In 1.1, observe that P and D are constructed by the eigenvectors and eigenvalues of A. So if $A \in \mathbb{R}^{n\times n}$ has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$ associated to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}^{-1}.$$

Example 1.3. Diagonalise $A = \begin{pmatrix} 2 & 0 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$, if possible.

We will first find the eigenvalues of A by solving $\det(\lambda I - A) = 0$:

$$\begin{aligned} 0 &= \det \begin{pmatrix} \lambda - 2 & 0 & 0 \\ 2 & \lambda - 1 & -1 \\ -1 & 0 & \lambda + 1 \end{pmatrix} \\ &= (\lambda - 2) \det \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda + 1 \end{pmatrix} \\ &= (\lambda - 2)(\lambda - 1)(\lambda + 1). \end{aligned}$$

Thus the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$.

To find the eigenvectors of A, we solve $(\lambda_i I - A)v_i = 0$ for each $i = 1, 2, 3$.

By row reducing each equation, it can be shown that we have $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$.

The eigenvectors of A are clearly linearly independent, thus by 1.1, A is diagonalisable, and by 1.2, we can write A as PDP^{-1} where $P = \begin{pmatrix} 0 & 3 & 0 \\ 1 & -5 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & & \\ & 2 & \\ & & -1 \end{pmatrix}$.

1.3 Case Study - Problems

1)

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\begin{aligned}
x(k+1) = A(x(k)) &= \begin{pmatrix} 1.03 & 0 & -c \\ 0.1 & 0.5 & c \\ 0 & 0.5 & 0.1 \end{pmatrix} \begin{pmatrix} S_k \\ L_k \\ I_k \end{pmatrix} \\
&= S_k \begin{pmatrix} 1.03 \\ 0.1 \\ 0 \end{pmatrix} + L_k \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \end{pmatrix} + I_k \begin{pmatrix} -c \\ c \\ 0.1 \end{pmatrix} \\
&= \begin{pmatrix} 1.03S_k - cI_k \\ 0.1S_k + 0.5L_k + cI_k \\ 0.5L_k + 0.1I_k \end{pmatrix}.
\end{aligned}
$$

Thus, we have $S_{k+1} = 1.03S_k - cI_k$, $L_{k+1} = 0.1S_k + 0.5L_k + cI_k$ and $I_{k+1} = 0.5L_k + 0.1I_k$.

2) Taking $c = 0.3$ and $x(0) = (3340000, 2100, 1500)^T$,

(a) $x(52) = A^{52}(x(0))$
Computing eigenvalues and eigenvectors is a routine exercise, so we will use numpy (see code extract 2a). It turns out that A is diagonalisable, thus $A^{52} = PD^{52}P^{-1}$, where P and D are the matrices in 1.2. Running the 2a Python code, the population after 52 weeks is 1702386.

2 Code Extracts