Numerical Thinking in Life Sciences

From Probability theory

Amir Zare

Chapter 1: Prologue

I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense.

A mathematician is like a blind man in a dark room looking for a black cat that isn't there.

The fourth pillar of scientific discovery after experimentation, modeling and computation

Two cultures
Prediction vs. Modeling

Logic vs. Inference

Applications
19th Century
1900
1908
1925
1933
1937
1950
1962
1972
1979
1995
2000
2001
2016a
2016b
Mathematics
Computation

Monsieur,

If I undertake to make a point with a single die in eight throws, and if we agree after the money is put at stake, that I shall not cast the first throw, it is necessary by my theory that I take 1/6 of the total sum to he impartial because of the aforesaid first throw.

Monsieur,

1. Impatience has seized me as well as it has you, and although I am still abed, I cannot refrain from telling you that I received your letter in regard to the problem of the points 1 yesterday evening from the hands of M. Carcavi, and that I admire it more than I can tell you. I do not have the leisure to write at length, but, in a word, you have found the two divisions of the points and of the dice with perfect justice. I am thoroughly satisfied as I can no longer doubt that I was wrong, seeing the admirable accord in which I find myself with you.

7. i have no time to send you the proof of a difficult point which astonished M. (de M´er´e) so greatly, for he has ability but he is not a geometer (which is, as you know, a great defect) and he does not even comprehend that a mathematical line is infinitely divisible and he is firmly convinced that it is composed of a finite number of points. I have never been able to get him out of it. If you could do so, it would make him perfect.

He tells me then that he has found an error in the numbers for this reason .

If one undertakes to throw a six with a die, the advantage of undertaking to do it in 4 is as 671 is to 625.

If one undertakes to throw double sixes with two dice the disadvantage of the undertaking is 24.

But nonetheless, 24 is to 36 (which is the number of faces of two dice) 2 as 4 is to 6 (which is the number of faces of one die).

This is what was his great scandal which made him say haughtily that the theorems were not consistent and that arithmetic was demented. But you will easily see the reason by the principles which you have.

Monsieur,

1. Our interchange of blows still continues, and I am well pleased that our thoughts are in such complete adjustment as it seems since they have taken the same direction and followed the same road.

3. For the rest, there is nothing that I will not write you in the future with all frankness. Meditate however, if you find it convenient, on this theorem: The squared powers of 2 augmented by unity 11 are always prime numbers. [That is,] The square of 2 augmented by unity makes 5 which is a prime number;
The square of the square makes 16 which, when unity is added makes 17, a prime number;
The square of 16 makes 256 which, when unity is added, makes 257, a prime number;
The square of 256 makes 65536 which, when unity is added, makes 65537, a prime number;
and so to infinity.
This is a property whose truth I will answer to you. The proof of it is very difficult and I assure you that I have not yet been able to find it fully. I shall not set it for you to find unless I come to the end of it.

No

Applications
19th Century
1900
1908
1925
2016b
2000
2016a
1933
2001
1979
1995
1937
1972
1962
1950
Mathematics
Computation

Karl Pearson (left) wrote that black and indigenous people had "not yet produced a civilization in the least comparable with the Aryan" and that the white man should "completely drive out the inferior race." Francis Galton (right) offered Pearson an endowed Chair of Eugenics at University College London due to his fears that "the race gradually deteriorates, becoming in each successive generation less fit for a high civilisation."

From left to right: Ronald Fisher, Gertrude M. Cox, Frank Yates, and Besse Day at a gathering in Blue Ridge, North Carolina, 1952. Published in Box (1978) and used under the fair use provision, Section 107 of the U.S. Copyright Act.

Chapter 2: Probability

A game of chance

- A game of chance
- Sample space
- Event

- $\Omega = \{s_1, s_2 \ldots, s_n\}$
- Mutually exclusive and homogenous
- $A \subset \Omega$

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

Axioms of probability

- non-zero
- Certainty
- Disjoint events

- $\mathbb{P}(A) \geq 0$
- $\mathbb{P}(\Omega) = 1$
- $\mathbb{P}(A) + \mathbb{P}(B)$

Results of axioms of probability

- Complement

- $\mathbb{P}(A^C) = 1 - \mathbb{P}(A)$
- $\mathbb{P}(A) \leq 1$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Conditional probability

- A certain event has happened
- What is the probability in this new space

- $\Omega_{new} = B$

Conditional probability

Venn diagram of events A and B

Shaded region is $A \cap B$

Shaded region is $A \cup B$

Shaded region is A'

Mutually exclusive events

$$\mathbb{P}(A \mid B) = \frac{|A \cap B|}{|B|}$$

Conditional probability

- Multiplication

- $\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B) \cdot \mathbb{P}(B)$

Bayes' theorom

- Symmetry

- $\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A)$
- $\mathbb{P}(A \mid B) \cdot \mathbb{P}(B) = \mathbb{P}(B \mid A) \cdot \mathbb{P}(A)$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}$$

Law of total probability

- Mutually exclusive and exhaustive chunks of the sample space

- $A_1 \cup A_2 \cup \ldots \cup A_k = \Omega$
- $\mathbb{P}(B) = \mathbb{P}(B \cap A_1) + \ldots + \mathbb{P}(B \cap A_k)$

Law of total probability

$$\mathbb{P}(B) = \sum_{1}^{k} \mathbb{P}(B \mid A_i) \cdot \mathbb{P}(A_i)$$

Independence

- Definition of independence
- Multiplication rule

- $\mathbb{P}(A \mid B) = \mathbb{P}(A)$
- $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

How to systematically update our beliefs

- Prior
- Likelihood
- Evidence
- Posterior

- $\mathbb{P}(H)$
- $\mathbb{P}(D \mid H)$
- $\mathbb{P}(D)$
- $\mathbb{P}(H \mid D)$

$$\mathbb{P}(H_1 \mid D) = \frac{\mathbb{P}(D \mid H_1) \cdot \mathbb{P}(H_1)}{\sum_1^k \mathbb{P}(D \mid H_i) \cdot \mathbb{P}(H_i)}$$

Chapter 3: Random Variables

Random variable

- A mapping from sample space to real number
- $\Omega \rightarrow \mathbb{R}$

Chapter 4: Useful Distributions

Discrete Random Variables

- Uniform
- Bernouli
- Binominal
- Poisson
- Negative Binominal
- Geometric
- Hypergeometric

$$\lim_{n\to\infty} \binom{n}{k} \cdot p^k \cdot (1-p)^{(n-k)}$$

$$n \to \infty$$

$$p \to 0$$

$$np \to \lambda$$

$$\lim_{n\to\infty} (1 - \frac{a}{n})^n = e^{-a}$$

Continous Random Variables

- Uniform
- Normal
- Geometric

$$\binom{n}{\frac{n}{2}+d}(\frac{1}{2})^n \approx \frac{2}{\sqrt{2\pi n}} e^{\frac{-2d^2}{n}}$$

1. There is only one number which gives the distance of the star from the center of the earth, the true distance.

2. All observations are encumbered with errors, due to the observer, the instruments, and the other observational conditions.

3. The observations are distributed symmetrically about the true value; that is the errors are distributed symmetrically about zero.

4. Small errors occur more frequently than large errors.

5. The calculated distance is a function of the direct angular observations such that small adjustments of the observations may result in a large adjustment of the distance.

$$\phi(x) = \frac{m}{2} e^{-m|x|}$$

$$\phi(x) = \frac{1}{2a} \ln\left(\frac{a}{|x|}\right)$$

(Ceres discovered by Piazzi, Jan. 1, 1801)
(Sun)
(orbit of the earth)

$$g(\sqrt{x^2+y^2}) = g(x)g(y)$$

$$\phi(x) \propto e^{-h^2x^2}$$

$$\int_{\mathbb{R}} e^{-x^2} = \sqrt{\pi}$$

$$\phi(x) = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}$$