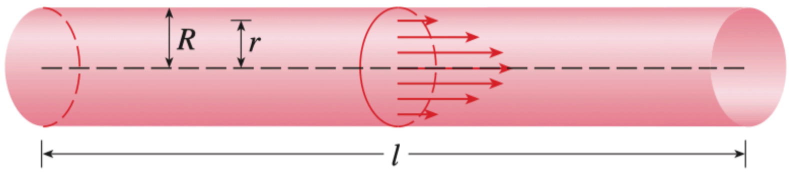
EXAMPLE 9 | **Blood flow** When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius R and length l as illustrated in Figure 6.



Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall. The relationship between v and r is given by the **law of laminar flow** discovered by the French physician Jean-Louis-Marie Poiseuille in 1840. This law states that

$$v = \frac{P}{4\eta l}(R^2 - r^2) \qquad (4)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and l are constant, then v is a function of r with domain $[0, R]$.

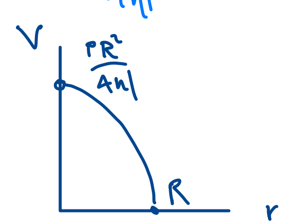
The average rate of change of the velocity as we move from $r = r_1$ outward to $r = r_2$ is given by

$$\frac{\Delta v}{\Delta r} = \frac{v(r_2) - v(r_1)}{r_2 - r_1}$$

and if we let $\Delta r \to 0$, we obtain the **velocity gradient**, that is, the instantaneous rate of change of velocity with respect to r:

$$\text{velocity gradient} = \lim_{\Delta r \to 0} \frac{\Delta v}{\Delta r} = \frac{dv}{dr}$$

$$V = \frac{P}{4\eta l}(R^2 - r^2)$$



گرادیان سرعت $= \frac{dv}{dr} =$

$\eta = 0{,}027$ $\quad l = 2$ cm

$R = 0{,}008$ cm $\quad P = 4000 \frac{\text{dynes}}{\text{cm}^2}$

$r = 20\ \mu m$

$$V = \frac{P}{4\eta l}(R^2 - r^2)$$

$$\frac{dV}{dr} = \frac{P}{4\eta l} \frac{d}{dr}(R^2 - r^2)$$

$$\frac{P}{4\eta l} \; -2r$$

$$= -\frac{2Pr}{4\eta l} = -\frac{4000}{2 \times ,027 \times 2} \qquad \left(-\frac{1000000}{27}\right) r$$

EXAMPLE 8 | **BB** **Gause's logistic model** In an experiment with the protozoan *Paramecium*, the biologist G. F. Gause modeled the protozoan population size with the logistic function

$$P(t) = \frac{61}{1 + 31e^{-0.7944t}}$$

where t is measured in days. According to this model, how fast was the population growing after 8 days?

$$\left.\frac{dP}{dt}\right|_{t=8}$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} = \frac{(1+31e^{-,8t} \times \overset{0}{0}) - 61\frac{d}{dt}(1+31e^{-,8t})}{(1+31e^{0,8t})^2}$$

$$= -\frac{61}{(1+31e^{-,8t})^2} \quad \frac{d}{dt}(1+31e^{-,8t})$$

$f \circ g$ $\quad \boxed{e}\boxed{-,8t}$ $\rightsquigarrow$ $f'(g(x))\,g'(x)$

$$\frac{d}{dt}\, 31e^{-,8t}$$

$g(t) = -,8t \overset{g'}{\rightsquigarrow} -0,8$

$f(t) = e^t \rightsquigarrow e^{t} \to e^{-0,8t}$

$$31\frac{d}{dt}e^{-,8t}$$

$-0,8\,e^{-0,8t}$

$$-\frac{61}{(1+31e^{-,8t})^2} \quad 31 \times -,8e^{-,8t}$$

$\frac{d}{dx}e^x = e^x$

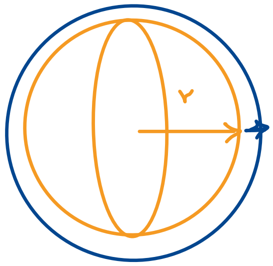
$\frac{d}{dx}e^{ax} = ae^{ax}$

$$= \frac{-61 \times 31 \times -,8e^{-,8t}}{(1+31e^{-,8t})^2}$$

$$= \frac{1512,8\,e^{-0,8t}}{(1+31e^{-0,8t})^2} \quad ✓$$

$$\left.\frac{dP}{dt}\right|_{t=8} = \frac{1512,8\,e^{-6,4}}{(1+31e^{-6,4})^2} = \frac{2,513}{1,105} = 2,27$$

EXAMPLE 14 | **BB** **Growth of a tumor** When the diameter of a spherical tumor is 16 mm it is growing at a rate of 0.4 mm a day. How fast is the volume of the tumor changing at that time?



$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = 4\pi r^2 \Big|_{r = 16\,mm} \cdot 0{,}4 \frac{mm}{day}$$

$$= 409{,}6\pi = 1286{,}14 \frac{mm^3}{day}$$

EXAMPLE 2 | **BB** **Nectar foraging by bumblebees** Many animals forage on resources that are distributed in discrete patches. For example, bumblebees visit many flowers, foraging on nectar from each. The amount of nectar $N(t)$ consumed from any flower increases with the amount of time spent at that flower, but with diminishing returns (see Figure 3). Suppose this function is given by

$$N(t) = \frac{0.3t}{t + 2}$$

where t is measured in seconds and N in milligrams. Suppose also that the time it takes a bee to travel from one flower to the next is 4 seconds.

(a) If a bee spends t seconds at each flower, find an equation for the average amount of nectar consumed per second, from the beginning of a visit to a flower until the beginning of the visit to the next flower.

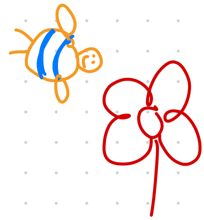
(b) Suppose bumblebees forage on a given flower for an amount of time that maximizes the average rate of energy gain obtained in part (a). What is this optimal foraging time?

$$N(t) = \frac{0{,}3t}{t+2}$$

$$\text{حرکت} = 4\ s$$

a) حداقل مکث در زمان ؟

b) زمان روی هر گل چقدر باشد؟



$$\frac{N}{T} = \frac{N(t)}{t+4} = \frac{\frac{0{,}3t}{t+2}}{t+4}$$

$$= \frac{0{,}3t}{(t+2)(t+4)} = \frac{0{,}3t}{t^2+6t+8} = C(t)$$

$$\rightarrow C'(t) = \frac{d}{dt}\frac{{,}3t}{t^2+6t+8} = \frac{(t^2+6t+8)({,}3) - ({,}3t)(2t+6)}{(t^2+6t+8)^2}$$

$$= \frac{{,}3t^2 + \cancel{1{,}8t} + 2{,}4 - {,}6t^2 - \cancel{1{,}8t}}{(t^2+6t+8)^2}$$

$$C'(t) = \frac{-{,}3t^2 + 2{,}4}{(t^2+6t+8)^2} \stackrel{?}{=} 0$$

$$-{,}3t^2 + 2{,}4 = 0$$

$$-{,}3t^2 = -2{,}4$$

$$t^2 = 8 \rightarrow t_{op} = \sqrt{8} = 2\sqrt{2}$$

EXAMPLE 4 | **BB** Aquatic birds[1] forage underwater and periodically return to the surface to replenish their oxygen stores. Oxygen stores increase with the amount of time spent on the surface but in a diminishing way, according to the model

$$O(t) = \frac{20t}{5 + t}$$

where t is the amount of time spent at the surface (in seconds). Suppose the round-trip travel time to and from the underwater foraging area is T seconds and oxygen is depleted at a constant rate of r mL/s while a bird is underwater. Furthermore, suppose the bird forages until it has just enough oxygen to return to the surface.

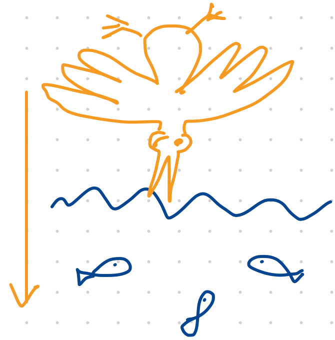
(a) If Q is the fraction of a single dive cycle that the bird spends foraging, find an equation for Q as a function of the surface time t.

(b) If $T = 2$ seconds and $r = 1$ mL/s, find the surface time t that maximizes the fraction of time spent foraging.

$$O(t) = \frac{20t}{5+t}$$

T زمان رفت و برگشت

r mL/s



a $\rightarrow$ $Q = \dfrac{\text{foraging}}{\text{cycle time}}$

$Q(t) =$

b $\rightarrow$ $T = 2\,s$ $\quad r = 1\ mL/s$

Cycle $\rightarrow$ t, T, f $\rightarrow$ $\boxed{\dfrac{O(t) - rT}{r}} = f$

$$Q = \frac{f}{\text{cycle}} = \frac{f}{t+T+f}$$

$$= \frac{\dfrac{O(t)-rT}{r}}{t+T+\dfrac{O(t)-rT}{r}} = \frac{\dfrac{\dfrac{20t}{5+t}-rT}{r}}{t+T+\dfrac{\dfrac{20t}{5+t}-rT}{r}} = \frac{\dfrac{\dfrac{20t}{5+t}-rT}{\cancel{r}}}{\dfrac{tr+\cancel{Tr}+\dfrac{20t}{5+t}-\cancel{rT}}{\cancel{r}}}$$

$$= \frac{\dfrac{20t}{5+t}-rT}{\dfrac{20t}{5+t}+rt} \rightsquigarrow Q(t) = \frac{\dfrac{20t}{5+t}-2}{\dfrac{20t}{5+t}+t} = \frac{\dfrac{20t-10-2t}{5+t}}{\dfrac{20t+5t+t^2}{5+t}} = \frac{18t-10}{25t+t^2}$$

$$\frac{d}{dt}\,\frac{18t-10}{25t+t^2} = \frac{(25t+t^2)(18)-(18t-10)(2t+25)}{(25t+t^2)^2} \overset{?}{=} 0$$

$$(25t+t^2)(18) = (18t-10)(2t+25)$$

$$\cancel{18\cdot 25t} + \cancel{18t^2} = 36t^2 + \cancel{18\cdot 25t} - 20t - 250$$

$$0 = 18t^2 - 20t - 250$$

$$\frac{20 \pm \sqrt{400 - 4\cdot -250\cdot 18}}{36}$$

$t_{op} = 4.32$

EXAMPLE 5 | **BB** **Sustainable harvesting**[2] For many natural fish populations, the net number of new recruits to the population in a given year can be modeled as a function of the existing population size N by an equation of the form

$$R(N) = rN\left(1 - \frac{N}{K}\right)$$

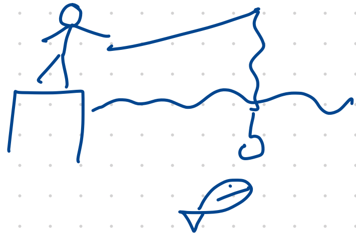
where r and K are positive constants. (K is called the carrying capacity.) The population will increase if the net number of recruits $R(N)$ is positive and it will decrease if $R(N)$ is negative. Thus, because $R(N)$ is positive when $0 < N < K$ and $R(N)$ is negative when $N > K$, we expect the population to stabilize at a constant size of $N = K$.

If the population is subject to harvesting, N will begin to change and once the population has stabilized again, the number of fish harvested each year, which we denote by H, must equal the net recruitment for that year; that is, $R(N) = H$.

(a) Suppose $H = hN$, where h is a measure of the "fishing effort" expended. What is the population size once it has stabilized?

$$R(N) = rN\left(1-\frac{N}{K}\right)$$ رشد جمعیت

$$H(N) = hN$$ ماهیگیری



a) اندازه جمعیت تعادلی

b) $H(h)$ در جمعیت تعادلی

c) ماهیگیری بیشینه

a) $R(N^*) = H(N^*) \rightarrow rN^*\left(1-\frac{N^*}{K}\right) = hN^*$

$$\rightarrow r\left(1-\frac{N^*}{K}\right) = h$$

$$\rightarrow r - r\frac{N^*}{K} = h$$

$$\rightarrow r-h = r\frac{N^*}{K}$$

$$\rightarrow K\frac{(r-h)}{r} = N^* = K\left(1-\frac{h}{r}\right)$$

b) $H = hN^* = h \cdot K\left(1-\frac{h}{r}\right) = K\left(h-\frac{h^2}{r}\right)$

c) $\frac{d}{dh}K\left(h-\frac{h^2}{r}\right) = K\left(1-\frac{2h}{r}\right) \stackrel{?}{=} 0$

$$\rightarrow \left(1-\frac{2h}{r}\right) = 0 \rightarrow \frac{2h}{r} = 1 \rightarrow \boxed{h = \frac{r}{2}}$$

$$\rightarrow K\left(h-\frac{h^2}{r}\right)\Big|_{h=\frac{r}{2}} \rightarrow K\left(\frac{r}{2} - \frac{r^2/4}{r}\right)$$

$$\rightarrow K\left(\frac{r}{2}-\frac{r}{4}\right) = \frac{rK}{4}$$

29. Bird flight paths Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than over land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.

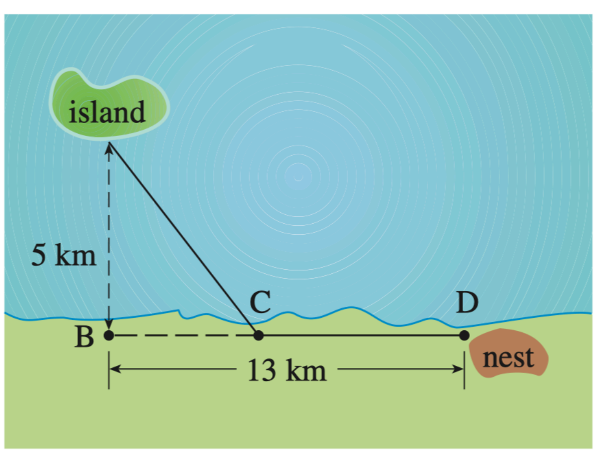
(a) In general, if it takes 1.4 times as much energy to fly over water as it does over land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?

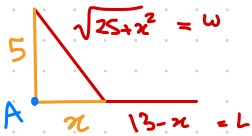
(b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean?

Determine the ratio W/L corresponding to the minimum expenditure of energy.

(c) What should the value of W/L be in order for the bird to fly directly to its nesting area D? What should the value of W/L be for the bird to fly to B and then along the shoreline to D?

(d) If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from B, how many times more energy does it take a bird to fly over water than over land?





$$E(x) = \sqrt{x^2+25} \times 1{,}4 + 13-x$$

$$E'(x) = \frac{d}{dx}\, 1{,}4\sqrt{25+x^2} + \frac{d}{dx}\, 13-x$$

$$= 1{,}4 \frac{d}{dx}\sqrt{25+x^2} - 1$$

$$\hookrightarrow (\sqrt{x})'(25+x^2) \cdot (25+x^2)'$$

$$\frac{1}{2\sqrt{25+x^2}} \cdot 2x = \frac{x}{\sqrt{25+x^2}}$$

$$E'(x) = 1{,}4 \cdot \frac{x}{\sqrt{25+x^2}} - 1$$

$$E'(x) \overset{?}{=} 0 \rightarrow 1{,}4 \frac{x}{\sqrt{25+x^2}} - 1 = 0 \quad x\,?$$

$$1{,}4 \frac{x}{\sqrt{25+x^2}} = 1 \qquad \frac{x}{\sqrt{25+x^2}} = \frac{1}{1{,}4} \qquad 1{,}4\,x = \sqrt{25+x^2} \qquad (1{,}4)^2 x^2 = 25 + x^2$$

$$1{,}96\,x^2 = 25 + x^2 \qquad {,}96\,x^2 = 25 \qquad x^2 = 26{,}04 \qquad \boxed{x = 5{,}1}$$

$$E(x) = W \frac{x}{\sqrt{25+x^2}} - 1 \qquad E'(x) = W \frac{x}{\sqrt{25+x^2}} - 1 \overset{?}{=} 0$$

$$x_{op} = \frac{5}{\sqrt{w^2-1}} \qquad 13 = \frac{5}{\sqrt{w^2-1}} = 13\sqrt{w^2-1} = 5 \qquad \boxed{w \leqslant 1{,}07}$$

$$0 = \frac{5}{\sqrt{w^2-1}} = \boxed{w \rightarrow \infty \quad \text{هیچوقت}}$$

$$4 = \frac{5}{\sqrt{w^2-1}} \qquad \boxed{w = 1{,}6}$$

EXAMPLE 1 | **Predicting a future population** There are currently 5600 trout in a lake and the trout are reproducing at the rate $R(t) = 720e^{0.1t}$ fish/year. However, pollution is killing many of the trout; the proportion that survive after t years is given by $S(t) = e^{-0.2t}$. How many trout will there be in the lake in 10 years?

$N(0) = 5600$

rep

$R(t) = 720\, e^{0{,}1t}$ تولید مثل در بازه زمانی t

survived

$S(t) = e^{-0{,}2t}$ نسبت لاکپشت‌هایی که پس از t سال زنده ماندن

پس از ۱۰ سال چند لاکپشت ?

$N(10) = 5600 \times S(10) + R(1)\, S(9) + R(2)\, S(8) + \cdots$

$$N(10) = 5600 \times S(10) + \int_0^{10} R(t)\, S(10-t)\, dt$$

$$5600 \times e^{-2} + \int_0^{10} 720\, e^{,1t} \cdot e^{-,2(10-t)}$$

$$e^{-2+,2t}$$

$$\boxed{720\, e^{,1t}}\; e^{-2} \cdot e^{,2t}$$

$$5600\, e^{-2} + 720\, e^{-2} \int_0^{10} e^{,3t} \rightsquigarrow \left.\frac{e^{,3t}}{,3}\right]_0^{10}$$

$$5600\, e^{-2} + 720 e^{-2} \times \frac{e^{-3}}{,3} - \frac{e^{0}}{,3} = 6957 \qquad \frac{e^{3}}{,3} - \frac{e^{0}}{,3}$$

Blood Flow

In Example 3.3.9 we discussed the law of laminar flow:

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

which gives the velocity v of blood that flows along a blood vessel with radius R and length l at a distance r from the central axis, where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood (see Figure 1).

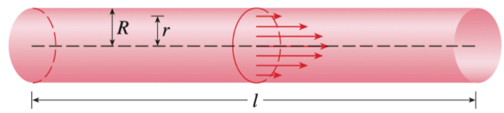


FIGURE 1
Blood flow in an artery

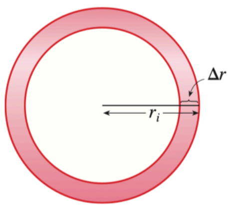


FIGURE 2

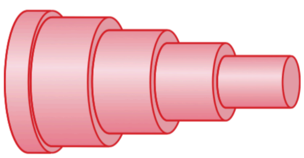
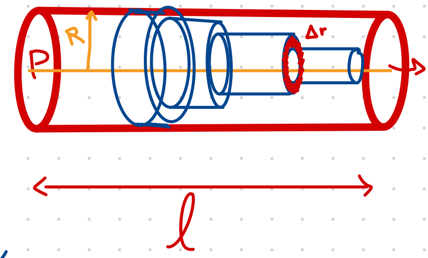


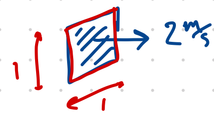
FIGURE 3

جریان لامینار $\rightsquigarrow$ $V = \frac{P}{4\eta l}(R^2 - r^2)$



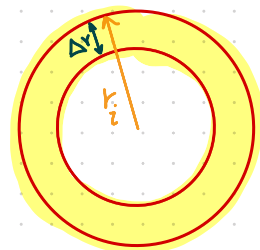
$F =$ جریان flux $=$ مقطع × سرعت

ح بر زمان



$F = 2\,\frac{m^3}{s}$

$$F = \lim_{n\to\infty} \sum \boxed{A_i} \times V_i$$



$$\lim_{n\to\infty} \sum \left(2\pi r_i \Delta r - \cancel{\pi \Delta r^2}\right)\left(\frac{P}{4\eta l}(R^2 - r_i^2)\right)$$

$$A_i = \pi r_i^2 - \pi(r_i - \Delta r)^2$$

$$\int 2\pi r_i \boxed{dr} \left[\frac{P}{4\eta l}(R^2 - r_i^2)\right]$$

$$\pi r_i^2 - \pi\left(r_i^2 - 2r_i\Delta r + \Delta r^2\right)$$

$$\cancel{\pi r_i^2} - \cancel{\pi r_i^2} + \boxed{2r_i\Delta r - \pi\Delta r^2}$$

$$F = \int_0^R 2\pi r_i \left(\frac{P}{4\eta l}(R^2 - r_i^2)\right) dr$$

$$\frac{R^2R^2}{2} - \frac{R^4}{4} \qquad \frac{R^4}{2} - \frac{R^4}{4}$$

$$= \frac{2\pi P}{4\eta l} \int_0^R r\,(R^2 - r^2)\,dr$$

$$\frac{2\pi P}{4\eta l}\,\frac{R^4}{4} \Rightarrow \boxed{F = \frac{\pi P R^4}{8\eta l}}$$

$$= \frac{2\pi P}{4\eta l} \int_0^R (R^2 r - r^3)\,dr = \frac{2\pi P}{4\eta l}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R$$