How Do Properties of a Proof Influence Aesthetic Judgement?

Alex Williams, Alexander Konev, Anna Horrabin, MJ Harman

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School of Mathematics
University of Edinburgh

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Abstract

Within this project there is an exploration of how properties of a proof affect aesthetic judgements of said proof. In addition, there is an extensive literature review on how aesthetics is viewed in the wider context of mathematics and how it can be used to analyse specifically the aesthetics of proofs. A wide range of proofs of the inequality of arithmetic and geometric means (AM-GM) are collected to create a platform which our analysis can be built from. Context is also provided for the reader to aid their understanding of the proofs collected.

Declaration

I declare that this project was composed by ourselves and that the work contained therein is our own, except where explicitly stated otherwise in the text.

(*Alex Williams, Alexander Konev, Anna Horrabin, MJ Harman*)

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Contents

Chapter 1

Introduction

In this project we explore how our aesthetic judgement is impacted by properties of a mathematical proof.

We will begin with a broader exploration of aesthetics philosophy and then apply what we have discussed to mathematics, and more specifically, mathematical proofs. We will end by defining what 'aesthetically pleasing' means in this context, all whilst making use of research surrounding the topic in order to critically evaluate current opinions and views. We do this in the hope of answering our overarching question: 'how do properties of a proof influence aesthetic judgement?'

To illustrate the issues in a concrete way we chose to focus on the inequality of arithmetic and geometric means, known as the AM-GM inequality. Each proof is paired with an aesthetic discussion of the choices we made when formatting. This includes, but is not limited to, decisions relating to layout, length, notation, use of colour, level of instruction, and medium. The proofs are presented in order of what we believe to be the least complex approach to the most complex approach. We believe each formatting decision may influence a proofs beauty and wish to investigate these properties further. We aim to apply the principles of aesthetic philosophy that we investigate in the early sections to our analysis in order to determine if properties of a proof really do influence our judgement.

Aesthetics play a large role in the way we view mathematics. Within early education, aesthetics influences our interests and opinions, which go on to shape our whole educational path (Sinclair, 2004). In a similar sense, aesthetics play a role in mathematicians' judgements which can determine a proof's social acceptance and hence, help or hinder research to come. We will touch on these ideas in more depth throughout this project.

By addressing which properties influence our aesthetic judgement, we aim to investigate a possible metric for an 'aesthetically pleasing' piece of work. There is large scope for future research investigating how aesthetic judgement can be influenced.

Chapter 2

Aesthetics

Analysing questions surrounding the ways aesthetics relate to mathematical proofs requires a robust understanding of the overarching field of aesthetic philosophy. Aesthetics is mostly well-defined, with a wealth of historical discourse surrounding the most prominent questions that have emerged from its study. This section will provide an overview of the field of aesthetics, analysis of the literature surrounding it, and – based on this literature – an outline of which areas of aesthetics will be most useful in analysing mathematical proofs through an aesthetic lens.

Aesthetics is a branch of philosophy that poses and attempts to answer questions of beauty by examining aesthetic values characterised predominantly by judgements of taste. That is to say, different conceptions of what constitutes beauty and aestheticism are derived largely from discourse on perception of attractiveness, or what people enjoy (Burke, 1764).

The field of Aesthetics examines sources of aesthetic experience and judgement, our perception and reaction to external groupings of stimuli (Kelly, 2014). These groupings of stimuli are referred to as aesthetic objects or environments. Aesthetics considers the reactions of the human mind when it engages via the senses with objects such as art, music, poetry, plays or the natural world (Binkley, 1977). Elements of mathematics have been considered as such aesthetic objects, though not as closely as – for example – visual art (Barker, 2009). By using the same processes by which aesthetic philosophers have examined similar aesthetic objects in the past, it is possible to draw meaningful conclusions as to the aesthetic value of mathematical objects, such as proofs.

It must be addressed that there is an informal equivocacy between aesthetics and the philosophy of art, which is in fact only part of the field. Since analysis of art is so consumed by what is considered beautiful, aesthetic judgements play a significant role in determining it's value (Bell, 1913). However, such aesthetic judgements can and are extended to other areas besides art (Barker, 2009). The aim here is to discuss how one might successfully apply such aesthetic judgements to the area of mathematical proofs. This is essentially the task that was taken on centuries ago by philosophers of art and, while some efforts have been made to do the same for mathematics, there is a dearth of consensus and a lack of clear methodology. As such, in this analysis it is pertinent to draw from literature not only the aesthetic discussion of mathematics, but also broader aesthetic discussion.

Aesthetics considers why people enjoy some aesthetic objects and not others, as well as how such objects can affect moods or beliefs (Zangwill, 2021). Within this context one can ask and answer questions about the value, purpose, and quality of mathematical proofs as a whole, individually, and in relation or comparison to one another.

For the purpose of brevity, our conception of the aesthetic is based in the philosophy that comprises the traditional field of Aesthetics. This represents the best process of defining and constructing what this project means by aesthetics, allowing for a more objective representation of what can otherwise defy rigourous qualification. A more colloquial application of the aesthetic quality manifests as a set of subjective principles or values, and an analysis of mathematical proof based on these terms – even clearly defined – would lack academic rigour. Additionally, while other areas of philosophical thought may align with the field, they were excluded unless they explicitly contributed to Aestheticism, for the sake of brevity. This allows adherence to the aims of the project within a reasonable and clearly defined philosophical framework, as well as maximising rigour in the definition of a potentially spurious concept.

Chapter 3

Aesthetics in mathematics

3.1 Research summary

We now turn our attention to mathematical beauty and the aesthetics of mathematics. Mathematical beauty receives a lot of informal attention throughout the mathematics community. Mathematicians may be concerned with airtight logic, but aesthetics play a critical role in their discussions and deliberations. The Pythagoreans of Ancient Greece believed there were connections between mathematics and beauty; Aristotle described how 'the chief forms of beauty are order and symmetry and definiteness, which the mathematical sciences demonstrate in a special degree' (Aristotle (2021) (Book XIII Part 3)). As Breitenbach and Rizza wrote in their 2017 summary, '[mathematical beauty is regarded] as a key motivation driving the formulation of mathematical proofs and even as a criterion for choosing one proof over another'. They also presented the following quote from Herman Weyl as an example of this : 'My work always tried to unite the true with the beautiful, but, when I had to choose one over the other, I usually chose the beautiful' (Breitenbach & Rizza, 2017). Mathematicians value aesthetics highly. Some are now turning their craft towards aesthetics of art itself, as demonstrated by recent proposals to use category theory to interpret underlying structures of disparate media, much as the category theorist draws up the structures underlying mathematical disciplines (Kubota et al., 2017) (Mannone, 2018). Mathematics and the arts are capable of learning from each other thus providing yet more angles from which to view each other and themselves.

Philosophers and mathematicians have been considering beauty in their arguments and creating aesthetic frameworks or lenses through which to do so for centuries. When we discuss an aesthetic lens or an aesthetic framework, we are discussing a certain way of talking about the aesthetic qualities of things or a certain perspective. One such framework is the Platonist. This school is far broader than our context of mathematical aesthetics, or even mathematics as a whole. According to this approach, mathematical beauty is found through 'a particular intellectual insight into fundamental structures of the universe', as described by Breitenbach. Furthermore, Breitenbach cites Chandrasekhar, claiming that mathematicians and natural scientists largely subscribe to this Platonist view (Breitenbach, 2013). And this is perhaps unsurprising. When motivating mathematical learning, educators often lead students to look to an application, to see some sort of natural reflection of the theory we study. The Platonist sees beauty in the intellectual insight into structures themselves, as objects.

Contrasting with this, Breitenbach (2013) and Wang (2019) provide alternative Kantian frameworks. Breitenbach highlights the role that surprise and paradox play in mathematical enquiry, and proposes that the pleasure of mathematics is rather found in the act of mathematical enquiry and how the mind comprehends mathematical results. Wang's work offers some critique of Breitenbach, proposing another application of Kant to aesthetics of proofs. In a sentence, Wang proposes that 'the beauty of a proof is exactly the beauty which mathematicians feel in constructing and apprentices [feel] in studying a proof'.

Mathematicians have their own ideas of course. Paterson's 2013 thesis, *The Aesthetics of Mathematical Proofs*, documents the ideas of three mathematicians: Poincaré, G.H. Hardy and Jerry King. focusing on the first two: Poincaré believed that an aesthetic sensibility was required for the process of doing mathematics to even begin, and this sensibility was intuitive. He believed aesthetic sensibility was essential in the decision-making process (an idea reminiscent of Sinclair's formulation of the generative aesthetic), and a motivator of mathematical enquiry.

Hardy's discussions of aesthetics largely consider utility and reflect the not-quite-facetious opinions of many pure mathematicians. Hardy believed that true mathematics was concerned with things of little to no practical application. To this end he spoke about how some theoretical physicists were in fact mathematicians, due to the abstract nature of their work. As Paterson puts it, Hardy 'drives a wedge between usefulness and aesthetic merit', finding that real mathematics 'must be justified as an art if it can be justified at all'. (Paterson, 2013)

There is no shortage of mathematicians' witticisms on the beauty of mathematics and how this is their motivation in their study. During the research for this project one name in particular came up again and again, and that is Poincaré. In 1930 he described mathematical beauty as a 'real aesthetic feeling that all true mathematicians recognise' (Poincaré, 1952). That particular phrasing, 'all true mathematicians', draws a line clearly defining an in-group and an out-group. Whether intentionally or not Poincaré is making a claim about who is in the community and who is not. This idea reminds us of discussions by Renteln and Dundes in their piece about mathematical folk humor. They examined how mathematical folk humor can both be esoteric and exclusive to outsiders, and exoteric and concerning other 'folks' in academia, such as physicists and engineers (Renteln & Dundes, 2005). Much as a joke about Abelian grapes might signal who has some grasp of abstract algebra, an individual's sense for mathematical beauty (both as something that even exists, and what in particular the individual sees to be beautiful) signals who is a true mathematician, at least by this standard.

Similar lines are drawn in Dreyfus and Eisenberg (1986), where it is noted that undergraduates were not generally able to appreciate an 'aesthetically superior' solution as more attractive than one they had made themselves. We will return to this paper later, but for now we will focus on what Dreyfus and Eisenberg concluded from this research: 'mathematics educators have failed miserably' in the attempt to impart this real aesthetic feeling in their students. In particular, they single out the emphasis on imparting techniques and underlying processes in classrooms, and little more. This sentiment can be found in Sinclair's 2004 investigation, in which she proposes that 'the emphasis that school mathematics places on propositional, logical reasoning' could discourage students from other forms of enquiry, in particular enquiry led by the aesthetic (Sinclair, 2004). This style of teaching could possibly be exacerbating mathematics anxiety, a widely reported phenomenon where doing mathematics (or sometimes just the prospect of doing it) results in manifestations of anxiety (Dowker et al., 2016). The more rigid, strictly logical styles can make students anxious to fail.

Considering many professional mathematicians consider the aesthetic as fundamental to their work, the state of students' sense for mathematical beauty should concern us. And so from here we find our motivation. As undergraduates ourselves – albeit reaching the end of this stage of our education – we wish to explore through this piece some particular aesthetic lenses as they pertain to mathematics. Some lenses might align more easily with our instincts and natural courses of action at this stage in our mathematical development, others might push us further. In order to begin we will need to narrow our search, to particular lenses and to particular mathematical objects.

It should be acknowledged that philosophers do not all accept that mathematical beauty is a meaningful concept (Wang, 2019). Although there have been centuries of mathematicians talking about the beauty they experience in mathematics, these reservations have remained alongside them. While we cannot conclude this debate, talking about mathematics in terms of aesthetics has value even if the concept of mathematical beauty is not as rigourously defined as say, artistic beauty.

Taking a step inwards now, let us refine our search. Sinclair's 2004 paper, *The Roles of the Aesthetic in Mathematical Inquiry* provides some useful language and terminology. The paper seeks to elucidate some ways in which aesthetics could be used to motivate mathematical inquiry, particularly in the classroom and among undergraduates. Sinclair isolates three particular roles of the aesthetic in mathematical inquiry. Firstly the *evaluative aesthetic*, which considers aesthetic qualities mathematical objects may – or may not – have, such as beauty or elegance. Secondly, the *generative aesthetic*, which largely describes the pleasure taken in the process of doing mathematics, and how that directs the mathematician's investigations. Finally, the *motivational aesthetic* concerns the aesthetic responses that incite inquiry and attract practitioners to their chosen fields.

We may note at once the comparison between Sinclair's model of the generative aesthetic and Breitenbach or Wang's Kantian approach. The former describes how aesthetics can direct the mathematician in the process of doing mathematics; the latter two claim that it is the pleasure of actively doing mathematics that lends most to the mathematician's enjoyment. The beauty and enjoyment found in the process of solving a problem or proving a theorem has been disregarded in favour of discussions of more motivational or evaluative grounds. These other aesthetic sub-categories perhaps compare more easily with the Platonist view.

In 2017, Breitenbach and Rizza called a conference in order to assess the state of studies into mathematical beauty. Papers were presented around topics in the aesthetics of mathematics, and as responses to questions that the discipline asks, summarised by Breitenbach and Rizza as follows:

1. What is mathematical beauty?
   - Is it a true aesthetic as in philosophy, or can it be reduced to non-aesthetic qualities? Is beauty a feature of mathematical objects, and what (if anything) distinguishes it from other types of beauty?
2. What is the status of aesthetic judgements in mathematics?
   - Are these judgements made on objective criteria, considering simplicity or symmetry, or are they made on the bases of subjective responses?
3. Can aesthetic considerations play any legitimate role in mathematical or scientific theorising?
   - What is the relation between the beauty of a mathematical statement and its proof, if there is any such relation?
4. Does the phenomenon of aesthetics in mathematics reveal any important analogies between mathematical and artistic practice?
   - How do aesthetic considerations determine what is researched?

As we attempt to discuss how the properties of a proof influence our aesthetic judgement, it would be helpful to consider some of these broader questions. In particular the first three stand out. The first fundamentally asks what this strange world of mathematical beauty even is. The second asks whether aesthetic judgements are based on objects themselves or based on subjective responses to the objects, a theme that will return. The third could prove interesting as a topic of further research; if beauty is as fundamental as some of these mathematicians claim, would it be obscene to think that beauty could be related in some way to truth? The fourth, while not being quite as relevant as the other questions for our work, can be seen in how critique from art criticism has been applied to mathematics. Some mathematicians are even trying to use mathematics to better understand the aesthetics of art.

3.2 Literature review

Much of the literature we found helpful in preparation for this project has been concerned with how the aesthetic can be used to enhance or even motivate mathematics education, in particular Dreyfus and Eisenberg's work, and Sinclair's 2004 paper. These themes also appear in the work of Mannone, who proposed in 2019 how to use modern mathematical techniques to discuss aesthetics of art.

In 1986 Dreyfus and Eisenberg wrote primarily of the findings of an investigation into how mathematics graduate students and trainee teachers in the USA and Israel responded to a set of problems. The students then compared their non-elegant solutions with an elegant one provided by the investigators. Dreyfus and Eisenberg observed that the students on the whole 'failed to grasp [the] aesthetic superiority' of the solution shown to them over their own work. They go on to say that the students 'showed no inner sense of feeling for the cleverness of a solution'. The blame for this was laid upon mathematics educators, positing that the students failed to evaluate the solutions aesthetically because they simply had not been taught to do it. Sinclair agrees with Dreyfus and Eisenberg that there is a discrepancy between the aesthetic evaluations of professionals and students, but stresses that a lack of agreement is not the same as a lack of aesthetic sensibility.

Dreyfus and Eisenberg's conclusions also make some startling claims. There is little justification as to why their solution was aesthetically superior to those provided by the students, which reinforces the idea that only professional mathematicians have true appreciation for the discipline. This is another example of aesthetic appreciation or lack thereof being an indicator of whether you are truly in the group. The concern that students were not learning to consider solutions aesthetically was supported, as they noted that students seemed to focus more on solving the problem and did not garner any additional appreciation upon being shown a more elegant solution. Their conclusion is ultimately that mathematics education failed to teach the skills required to consider mathematics aesthetically and, given the importance we have established that mathematicians place on the beauty of their craft, we are left to conclude that the discipline is not creating new practitioners effectively. How have things changed – if at all – in the past 25 years?

Sinclair's 2004 piece is comprehensive in its scope, although its focus is on pedagogy. As a result, it requires special attention and extensive exploration. We have previously discussed her categorisations of mathematical aesthetics; however there are more themes in this piece that deserve focus. Firstly, Sinclair posits that aesthetics are involved in how we choose what mathematics is canonised and celebrated at large. While non-aesthetic considerations such as utility must play a role, why are we not satisfied with the first proof of a theorem? Why do we search for more, surely the truth is already established? Sinclair does discuss non-aesthetic motivations for undertaking the work of mathematics, including social pressures. Sinclair discusses the case of John Nash, who would 'only work on a problem once he had ascertained that great mathematicians thought it highly important'. However, Sinclair's focus is the aesthetic: we search for proofs more enlightening and beautiful than the last.

At the core of Sinclair's paper is her categorisations of the aesthetics; the evaluative, the generative, and the motivational. The evaluative aesthetic describes the aesthetic dimensions to mathematical objects themselves, as far as such dimensions exist. This aesthetic might consider objects such as theorems or proofs and discuss their 'elegance' or 'beauty'. The generative aesthetic is a more abstract concept but broadly speaking, it concerns pleasure taken in the process of doing mathematics, and how that can steer the course of exploration. We will discuss this a little more in the context of Breitenbach's Kantian framework. To give an example, a symmetry arising in the mathematician's work might inspire them to investigate said symmetry. The aesthetics there are guiding the course of investigation. Finally, the motivational aesthetic describes how mathematicians are pulled towards certain problems and fields by aesthetic considerations. As is the case with any discipline, mathematicians have opinions on other mathematicians and their fields. But what draws an algebraist towards abstracted structures like groups and away from the finicky limits of real analysis? This is one question that the motivational aesthetic could be used to investigate.

Sinclair's work is primarily about mathematics education, so she draws attention to the aesthetic responses of students, how they might differ from those of professionals, and also how a range of aesthetic approaches should be encouraged among students. Unlike Dreyfus and Eisenberg, Sinclair is less dismissive of the students' aesthetic faculties. Whilst she agrees that there is a discrepancy between the experience of students and professionals, she provides alternative explanations. Sinclair draws attention to a conjecture by Brown in 1973 that his students deferred to their less memorable and messier solutions because they 'better encapsulated the students' personal history with the problem', and that they 'wanted to remember the struggle more than the neat end product'. Much as one might knit a scarf and prefer it to one bought in a shop despite being scruffier, the students seem to find pleasure in their work because it was theirs. Sinclair notes that educators tend to consider 'the possibilities of student aesthetic response in the evaluative mode almost exclusively'; however the proposition by Brown implies a more generative aesthetic at play.

Sinclair discusses surprise as a feeling that pleases mathematicians aesthetically. Many great pieces of art hang upon subversion of expectation and it seems mathematicians feel the same way about their work. Sinclair writes that surprise 'makes the mathematician struggle with her expectations and with the limitations of her knowledge'. With the expanding canon of mathematical objects, there is a risk of things becoming less enchanting as they become commonplace, such as Euler's identity ($e^{i\pi} + 1 = 0$), the ubiquity of which halts aesthetic appreciation in some. This again highlights the disparity between the aesthetic appreciation of the student and that of the teacher. When we first meet Euler's identity, we are astonished to see something use e, π and i like this. Learning of the identity which ties together these objects so neatly as a young mathematician is still exhilarating as it is surprising (of course the young mathematician has only just become acquainted with the complex numbers). Perhaps the novelty wears thin, and we learn of even more elegant and surprising results in the same field, such as Cauchy's integral theorem.

While it might seem strange at first to discuss negative feelings in discussions of aesthetics, Sinclair makes a case for the role that frustration can play in aesthetic appreciation of mathematics. She discusses the affective domain via the work of Hofstadter, who 'allowed many emotions to be evoked including tension, curiosity, bewilderment, as well as frustration and loss'. She impresses the need to not discount those negative feelings as they contribute to the 'ultimate excitement and satisfaction'. Hofstadter saw setbacks and struggles in the process of mathematical enquiry as ultimately something to be resolved and worked through, not something to make him abandon his work altogether.

The literature so far leads us to think there might be a lack of consideration for some forms of aesthetic, such as the generative. This leads us to Breitenbach's 2013 paper in which she outlines a Kantian approach to aesthetics in mathematics, particularly proofs. Breitenbach grants that superficially it seems as though Kant's aesthetic philosophy is not applicable to mathematics. The recurring theme of this piece is that a key motivator for the mathematician is surprise, paradox, and a wonderment at the brain's capacity to even understand these concepts. Breitenbach posits that it is this, and not some inherent truth about the beauty of the Universe, that brings the mathematician the most pleasure, in contrast to the majority of professional's opinions for much of the past century.

Recent papers by Kubota et al and Mannone attempted to use category theory to study art. Category theory is a more recent mathematical field dedicated to studying the underlying structures of mathematics itself. These papers were largely outside of our scope both in complexity and in subject matter, but the focus on structure and the idea that a mathematical object is greater than the sum of its parts is worth highlighting. As beautiful as a mathematical object might be by itself, we can find more beauty through its relations to other objects, say in how theorems can come together to prove a new hypothesis, or even in how the pieces in a proof itself come together as a complete argument. Pleasure could be experienced in how the objects relate to each other, in contrast to the beauty they have by themselves.

3.3 Discussion

There are questions which deserve to be asked and considered even if they are beyond the scope of our investigations. One such question is what is mathematical canon, and what determines that some work is elevated while some is forgotten? Sinclair's suggestion is that one factor is the aesthetic. Why save the 'inelegant' when we can instead venerate the beautiful? We also infer from their writing that Dreyfus and Eisenberg have some unspoken yet agreed-upon standard for beautiful solutions the problems they presented to the students, and one that would be agreed upon by professionals at large for that matter, a canonically elegant solution. However, the canon arises because mathematicians (certainly those who are afforded the most attention and respect) choose to focus on some objects and not others. Just because something is different to the agreed-upon professional standard, that does not mean it has no value by itself, as Sinclair shares from Brown. Perhaps by discounting some mathematics we have lost some things of value.

It would be remiss to not explain why some things are omitted from our investigations. We note that a lot of our literature is from a European philosophical lens, the references to Plato and to Kant, the discussion of the Ancient Greeks. Styles of mathematical proofs reflect the historical and artistic context they come from (Mannone, 2018). What mathematics intrigues us and is considered worthy could be influenced by this context. There are other traditions which have not received as much respect and attention from the fields of mathematics and philosophy alike. We have taken our cues from this European philosophical setting, it being the one most familiar and accessible to us. Further work ought to examine other approaches to mathematical aesthetics from other philosophical and mathematical traditions.

Now, we return to mathematics education, the teacher and the student, the in-group and the out-group. The folk humor and aesthetic which indicate belonging to the mathematical community. The primary concerns are that mathematical aesthetics are not being taught to students, and also that this is an oversight as the aesthetic could be used to enhance or even motivate students' learning. While it is certainly true that mathematics educators reference the aesthetic in their instruction, in our experience as students it often takes the form of the visual representation, such as, having a neat argument over a messy one, and writing it up clearly without too many ugly corrections. As Sinclair says, school mathematics places the emphasis on 'propositional, logical reasoning', the correctness of an argument is the factor determining how many marks it would get. This is not to say aesthetic considerations are never discussed, just that they are a garnish. What truly matters in school maths education is the methods and logic. The outcome of this is that many young mathematicians, or potential mathematicians, are not being given the skills to consider mathematical aesthetics, so are not equipped to become fully fledged members of the folk of mathematicians.

It need not remain that way. If mathematical beauty is as Poincaré put it, a 'real aesthetic feeling that all true mathematicians recognise', then we should discuss it more comprehensively in the classroom. As students coming to the end of our mathematical education we hope to take what we have learned and investigate some mathematical objects. If it is true that our judgements as undergraduates are in fact different to those of professionals, then our investigations could prove illuminating to anyone wishing to understand how learning a little something of aesthetics could enhance students' understanding and admiration of mathematics.

If we want to think about mathematics and aesthetic judgement of mathematical objects, we are going to need to think about what aesthetic ideas in particular we want to investigate. Much of the discussion concerns the beauty of the objects themselves insofar as they can possess it. While there is much controversy around the subject of the nature of mathematical objects and so whether they can truly be described as beautiful, practically speaking many mathematicians take it as legitimate description. And we are going to investigate as though that is true, and discuss what properties mathematical entities can have and how those influence our aesthetic appreciation of them.

This first approach will be more intuitive for us, but it would be a wasted opportunity not to investigate some alternatives. And so, we will attempt some discussion through the generative lens also. So, as we move onto our next section we will need to bear in mind that we are after mathematical objects to evaluate, and that we also want to consider the way that the mathematician can derive pleasure through interactions with the object and trying to understand it.

Chapter 4

Proofs in mathematics

4.1 What are proofs?

Now that we have covered aesthetics broadly and related them to mathematics we will refine further to aesthetics in proofs. Before we do this, we will discuss what a proof is, what role they play and their purpose.

A mathematical proof is a line of reasoning which stems from a set of axioms leading to a conclusion via logical steps (Griffiths, 2000). In different contexts proofs hold different purposes. Hersh (1993) suggests that 'convincing' and 'explaining' are the two main roles of a proof. Summarising that in mathematical research it is primarily a tool for convincing. Whilst in an educational context the role of a proof is to explain. 'A good mathematical proof would convince a skeptical mathematician, as well as explain to a naive undergraduate.' For a mathematical statement to be accepted, there must be corresponding proofs. Classically, mathematicians are skeptical of what is true and what is not, so proofs are essential in determining this. A proof is only given 'proof status' after being accepted by other mathematicians. This is often a process spanning generations of criticism and refinement before social acceptance is granted (Manin, 2010).

Mathematics is a field where many strive for absolute certainty. Discoveries rely heavily on building upon current work to be able to advance mathematics. Thus, it is vital that our foundations are solid. The need for proofs and their purpose is apparent.

4.2 Aesthetics of proofs

In this project we will discuss a proof as a mathematical object, and we will define what we mean by this for the sake of this project in this section. A mathematical object is a name given to an abstract mathematical concept such as a number, a set, but can also include structures made up of smaller objects. For example, we can take the objects called vertices and edges and use them to construct objects called graphs or even polygons. Theorems and proofs are objects constructed from other objects and the way they relate to each other. We will consider this as distinct from the written artefact (the ways we express mathematical objects), as this will allow us to be more specific. The mathematical object refers to the abstraction itself. We wish to consider proofs in this sense – as mathematical objects – as it helps us to think of them as objects with qualities such as symmetry or simplicity, which could have aesthetic features. This means that we may be able to approach them with the evaluative aesthetic.

Whether or not a proof is true or not does not necessarily prevent us from looking at proofs as a flow of statements and arguments. Arguments (logical or otherwise) require a flow of premises and conclusions, with logical arguments following sensibly from the previous steps. As we write a proof, we are actively doing mathematics and can be led by aesthetics in the generative form. Similarly, as we read a proof, we should check step by step whether we follow the author where they are leading us, and whether we are convinced. This is also active; we need to reconstruct the arguments in our own minds. So, a collection of proofs could provide an opportunity to discuss the ideas of the generative aesthetic while we construct the list and later as we review it and investigate it further.

These approaches help us skirt the thorny issue of whether proofs can truly be considered *aesthetic* objects. This is a debate vast in scale and it would detract from this project by attempting to conclude it, as with the issue of mathematical beauty. However, as mathematicians do and will continue to consider these objects as having qualities which could be investigated from an aesthetic perspective, there is value in continuing to do so.

Considering different types of proofs could allow us to examine different aesthetic qualities. Proofs can vary in the style of argument, or various types of presentation. Visual proofs (not yet broaching the topic of their validity as proofs) could be used to consider how a proof's abstract structure influences aesthetic appreciation, or even conversely how an application of something abstract to something material could enhance appreciation of it.

There is also the issue of what makes two proofs different. If we wrote out a proof in prose using full English sentences, and then again using symbolic mathematical writing rather than prose, what would the differences between those proofs be? Their arguments would have the same structure and rely on the same ideas. This also ties into the topic of whether the proof itself is beautiful or if it is simply the way it is presented. By considering a range of proofs we might be able to test these problems and propose some solutions.

Breitenbach (2013) discusses mathematical objects and their 'purposiveness', a word Kant uses in discussions of aesthetics. 'Purposiveness' describes having a purpose or being done with purpose. While distinct from utility, the ideas are in proximity to each other. She presents an argument from Kant that a mathematical object's purposiveness is found in how it can be used to solve a myriad of problems in a way that appears 'surprisingly simple'. From this, Kant concludes that what is attributed to mathematical objects is 'an objective purposiveness', as it relates between objects. As such Kant concludes that mathematical objects are not aesthetic objects, as the purposiveness of aesthetic objects is in how they relate to an observer.

While the conclusion might be that mathematical objects are not aesthetic objects, this does not make interaction an un-aesthetic affair. Breitenbach highlights three of Kant's claims concerning the possibility of beauty in mathematics. Firstly that it is because of 'unexpected purposiveness' – an element of surprise – that mathematical objects are considered beautiful. Secondly, that purposiveness indicates a form of perfection rather than beauty, contrary to popular opinion. Thirdly, while mathematical properties or objects are not beautiful by themselves, the 'demonstration of such properties can be the object of aesthetic appreciation'. However, this is not the end of it, and Breitenbach's proposal is that the aesthetic appreciation of mathematics is found in how our minds wrap around these abstract concepts.

Turning now to the Kantian argument of Wang (2019), they critique Breitenbach's focus on 'unexpectedness', stating that this lack of expectation 'does not distinguish beauty from perfection' as Kant would like. Wang lays out the transition between subjective and objective purposiveness in Kant's work and concludes that the 'aesthetic power of judgement represents subjective purposiveness through the mere feeling of pleasure'. Also importantly, Wang says that this pleasure can be found not just in the construction of a proof (the focus of many of the papers we investigated), but also in the study of a proof.

These aesthetic frameworks which discuss the process of the construction of a proof bring us to Cain (2010). This work regards unexpectedness and inevitability, which are two qualities Hardy claimed as properties of beautiful proofs. These qualities seem contradictory, however Cain attempts to resolve them using *deus ex machina*, a concept from literature. Typically this concept refers to an unsatisfying conclusion to a narrative in which forces outwith conclude the story which do not 'fit with the internal framework of the plot'. In the context of a mathematical proof, Cain takes the *deus* to mean an 'unexplained construction or a calculation of elements or a definition of a function which simply 'happens to work''. Just as the *deus* is unsatisfying in literature, so it is in proof too. Avoidance of this device is what Cain considers to be the 'inevitability'.

4.3 Discussion

We have mainly focused on Breitenbach's 2013 paper, but this is not the only position. In fact, Breitenbach would say herself that this perspective is not one shared by the majority of the field, although it is not incompatible with many of the other sources we have read. We wish to consider this perspective because it is novel and intriguing, and so worth introducing at the very least, however much of our investigation will take a more evaluative approach, considering the proofs themselves and what qualities they possess. Dreyfus and Eisenberg spoke of their elegant solutions, and a solution is a demonstration or presentation of properties of mathematical objects. This approach somewhat firmly answers some of the questions raised by Breitenbach and Rizza, stating that mathematical beauty is more to do with our interaction with the objects than the objects themselves.

The main body of our project is a collection of proofs about a particular theorem and a series of aesthetic discussions about these proofs. A paper of a similar subject and style is Stout (1999). Stout uses a few proofs of the Binomial theorem to examine some general aesthetic criteria. In particular, he identifies the following, saying a beautiful proof could: 'make the result it proves immediately apparent', explain 'why the result is true and should be true', be efficient and not use 'more than is necessary', make 'unexpected connection' between different mathematical fields, or 'suggest further development' in the area rather than conclude all investigation. Interestingly, Stout also lists some things that in his understanding do not influence the beauty of a proof. In short, he denies that a proof needs to be useful, or always presented elegantly in order to be beautiful as the beauty is found in the abstract.

As we begin to formulate our tools with which to evaluate proofs, we need to consider what questions matter. We are hoping to consider properties of a proof which enhance aesthetic appreciation, so we do not need to answer the big questions of 'what is mathematical beauty' by the end of this project. We will take a more 'traditional' approach to the proofs we consider at first, considering properties such as simplicity, symmetry, style of argument, and surprise. Additionally we hope to utilise some of these other less conventional lenses in order to gain a fuller understanding of how we can appreciate proofs aesthetically.

We will also think about some sociological phenomena, such as the Expertise Reversal Effect, and we will use our selection of proofs in order to talk through some of these ideas. We believe it is important to talk about these more experimental and psychological frameworks as they lend well to practical investigations, such as surveys, and could be more beneficial to mathematics educators.

While the front of our investigations will be using the more evaluative models as previously discussed, we will also keep the generative and motivational aesthetics in our mind. It would be difficult to determine exactly what the motivation was in the creation of each proof we investigate, but we can consider our motivation in selecting them. Additionally, it is not immediately clear how to discuss the generative aesthetic or the Kantian models presented, as they involve the act of doing mathematics, and that need not directly concern the properties of any given proof. However the relationship between unexpectedness and a proof's construction of properties might allow us a way to approach this.

Returning to the questions posed by Breitenbach and Rizza in response to the conference on aesthetics of mathematics; how could those relate to our exploration of aesthetic qualities of proofs? Let us return to them now.

1. What is mathematical beauty?

    - As we will be discussing what properties can influence our aesthetic evaluation of a mathematical proof, we hope to offer some possible answers to this problem. We do not believe it is necessarily the place of this project to conclude what mathematical beauty truly is, nor would we be able to. Nevertheless we hope to be able to elaborate on some of the possibilities.

2. What is the status of aesthetic judgements in mathematics?

    - Whether the judgements are made based directly upon the properties a proof has or whether they are based on the subjective responses upon interacting with a proof, we can investigate how properties influence aesthetic judgement. If it truly is the latter, then we assume our interactions are still based on the properties the proof or its presentation has, just less directly. We do not need to come to a definitive answer to this question, nor is it within this project's scope to do so. Addressing it however helps us remain focused on the task at hand.

3. Can aesthetic considerations play any legitimate role in mathematical or scientific theorising?

    - We have singled out how the aesthetic could play a role in mathematical education, which is of course a step on the road to mathematical theorising. This is another angle to the 'how' part of our question. Understanding the ways in which properties might influence aesthetic judgement might help motivate mathematical education or even theorising.

4. Does the phenomenon of aesthetics in mathematics reveal any important analogies between mathematical and artistic practice?

    - We will be using some analogous artistic practices to motivate and inform parts of our discussion. It is difficult to avoid discussion of art when it comes to aesthetics since the field was primarily developed in order to discuss it. As for the converse, we have learned a little in our research that there has been research into possible applications of mathematics to art (Kubota et al., 2017) (Mannone, 2018).

In order to refine our toolset, we will need to first know what we need to use them for. In order to investigate the aesthetics of proofs, we will collate proofs of one theorem. Using only one theorem will allow for easier comparisons. Once we have our proofs, we will look at them all individually and analyse them based on some evaluative qualities, such as their simplicity, presentation, and structure. We will outline these qualities as we go. Then we will attempt to draw comparisons between the proofs to see how they interact with each other. Once this process is complete, we will be able to discuss properties that contribute to our appreciation of proofs.

Chapter 5

Collection of proofs

5.1 The AM-GM inequality

Theorem 1. *For any list of n non-negative real numbers $x_1,\ x_2,\ \ldots,\ x_n$,*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$$

Equality holds if and only if $x_1 = \ x_2 = \ldots = x_n$.

Cauchy reformulated the problem as

$$x_1 \cdot x_2 \cdots x_n \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right)^n .$$

E.g. for $n = 2$ we have

$$x_1 x_2 = \left(\frac{x_1 + x_2}{2} \right)^2 - \left(\frac{x_1 - x_2}{2} \right)^2 < \left(\frac{x_1 + x_2}{2} \right)^2 .$$

The AM-GM inequality states that the arithmetic mean of a list of non-negative real numbers is greater than or equal to the geometric mean of the same list of numbers. These means are equal if and only if every number in the list is the same number.

We remind the reader that the arithmetic mean of a list of numbers is the sum of all the numbers in the list divided by the total number of numbers in the list. The geometric mean of the list of non-negative numbers is defined as the n-th root of the product of all the numbers in the list.

5.2 Why we chose the AM-GM inequality

When looking at aesthetics of proofs it was important to focus only on one theorem. From our experience of studying mathematics we know that proofs of seemingly similar-looking statements could require different proof techniques making an aesthetic comparison between them difficult. On the other hand, seemingly different results from different areas of mathematics are often proved by similar methods. These factors would make it difficult to compare the aesthetics of proofs and not the proof techniques or statements under consideration.

We picked the AM-GM inequality over other theorems for several reasons. First, our literature search revealed that there are many different ways the inequality can be proved. From standard proof by induction methods to proofs relying on more advanced concepts and connections with other disciplines, for example, by applying some knowledge of physics, as seen in the thermodynamics proof later on. In addition, it is a theorem that has several different proofs without words or 'picture proofs'.

Second, the AM-GM inequality is a statement that competent mathematicians easily understand. The AM-GM inequality is a very powerful tool for mathematical problem solving from early stages of education. For example in the context of Mathematics Olympiads, it can be found in many books aimed at preparing students to such competitions, for example, Bullen (2003), Bullen et al. (1988), Cvetkovski (2012), and Shirali (2002). Because the inequality is used in the such educational competitions, authors present different proofs relying on different levels of mathematical background. This was helpful when collecting our proofs.

Finally, we decided to pick the AM-GM inequality in particular because there does not currently exist a collection of proofs, that we are aware of, for this theorem. Hence, by gathering a range of different proofs we hope to make an accessible resource for students, teachers and researchers.

5.3 Types of proofs

In our collection we have gathered a variety of types of proofs. From picture proofs to proof by induction and applied proofs by thermodynamics, we consciously chose to cover a wide range in order to aid our aesthetic discussion and comparison.

The first category of proof we see is picture proofs. This raises an opportunity to address the common controversy surrounding classifying pictures as proofs. The foundation of this scepticism stems from a diagrams limitation in handling the notation of infinity. As a diagram represents one case only, many argue that they cannot prove a general theorem. Cellucci (2019) refers to this as the Intuition Argument. Making reference to Russell (1920) to demonstrate this view. 'Diagrams involve an appeal to intuition, and a proof involving an appeal to intuition lacks rigour and hence is defective'. Russel makes a strong claim that 'no appeal to common sense, or 'intuition,' or anything except strict deductive logic, ought to be needed in mathematics after the premises have been laid down.' (Russell, 1920)

In contrast, there is a strong argument made in J.R. Brown's - *An Introduction to the World of Proofs and Pictures* (Brown, 2008), that diagrams can be akin to the more traditional proofs we see throughout this project and the rest of modern mathematics. Arguing that mathematicians look for two things in a proof - evidence and insight. Brown recognises that although picture proofs differ from symbolic proofs, it does not mean that they do not have the ability to provide justification. Brown likens picture proofs to art in his aesthetic discussion, in the sense that they can be simultaneously abstract and concrete. He uses an example of the art of Napoleon, to demonstrate that although the picture is a representation of Napoleon it is a symbol of leadership. He suggests a similar thing happens with proofs, where a diagram is a special case for an arbitrary n but a symbol for every n. Brown recognises that mathematicians must do a 'metaphorical type of seeing' to extend these cases but 'if artists can do it, so can mathematicians'.

We will progress from the standpoint that pictures are an acceptable form of proof for the sake of aesthetic discussion.

A hybrid between symbolic proofs and picture proofs, is our mixed proof. A traditionalist may be more persuaded by the legitimacy of such a proof, in contrast to a complete picture proof. The written portion of the mixed proof could almost entirely serve to prove the theorem, however choosing to include a diagram rather than having the reader construct it in their heads only supports the truth of our proof.

The other identifiable group of proofs we see in our collection is proof by induction. Proof by induction is a widely used type of proof with various independent origins. Induction has many examples including; sum of consecutive natural numbers and The Tower of Hanoi. Mathematicians and educators often use a nice analogy of dominoes or ladders to visualise the process of induction. Showing the base case holds true is like tipping the first domino. If each domino results in its adjacent domino to fall we can conclude all dominoes will fall, i.e. all cases hold true. Or similarly: if you can climb the first step of the ladder you are able to climb all the steps of the ladder.

A general form of proof by induction follows:

$P(n)$ is a mathematical statement for $n \in \mathbb{N}$.

$P(1)$ is true and our base case.

$P(k) \implies P(k+1)$ for all natural numbers k.

Although variations are frequent.

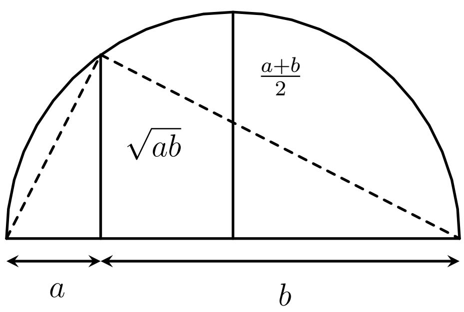
We also see a handful of other proofs including ones that turn to practical application to make their point, these include proof from thermodynamics and Levi's circuit proof.

Jensen's proofs offer the opportunity to discuss self containment in proofs. These proofs rely on Jensen's inequality to be recalled and used. A self-contained proof is one that does not rely on external theorems or lemmas. We will discuss the effect of self-containment on our aesthetic judgement of a proof.

So, we have a collection of proofs that all focus on the same theorem. Why is it that multiple versions of the same proof exist? Why would a mathematician set out to produce a proof that already exists in another form if not for aesthetic purpose? This is a hugely motivational question underpinning this project.

5.4 Proofs and their aesthetic discussions

Visual proof 1



Brown, 2008, page 44.

As we have ordered our proofs in what we believe to be the most simple to least simple, this is a natural place to discuss beauty as simplicity.

Beauty is often associated with simplicity. Even outside the realm of the mathematical world simplicity is often a focus when trying to unravel what makes something aesthetically pleasing. Johann Joachim Winckelmann a renowned art historian and archaeologist from Germany had a clear preference for simplicity. He is famously linked to quotes such as 'unity and simplicity are the two true sources of beauty' and 'noble simplicity and quiet grandeur'. This outlook is often thought to translate into the world of mathematics.

As discussed in McAllister's 'Mathematical beauty and the evolution of the standards of mathematical proof' it is custom for mathematicians to 'regard a proof as beautiful if it conformed to the classical ideals of brevity and simplicity' he goes on to suggest that the reason we find simple proofs so appealing is their ability to 'lend itself to being grasped in single act of mental apprehension.' (McAllister, 2005)

Montano, in Explaining Beauty in Mathematics, states 'the simplicity of a proof is thus a desirable quality, and its complexity an undesirable one.' (Montano, 2014) There is an interesting point in the footnotes of page 36 that questions the fact that since simplicity lends itself to a sounder understanding of the proof, does this influence a subjects reaction and emotion towards it? On the contrary it is said complexity hinders understanding and so could be seen as an obstacle to recognising beauty. This links to beauty as enlightenment which we will discuss alongside our final visual proof.

Testing out this theory, Wells (1990) conducted a study where subjects are asked to evaluate 24 theorems on a scale of 0 to 10 for beauty. He then goes on to investigate the criteria believed to have influenced the rankings. The fourth theme discussed is simplicity and brevity. He found that simplicity played a big role in his subjects aesthetic view of mathematics, with over a quarter of the subjects of this survey scoring an 'extraordinarily simple' theorem as a 7+. (Wells, 1990)

Whilst there is a strong case that simplicity and beauty are correlated we must ask ourselves where does this relationship stop? Wells' study raises the question; 'how simple is too simple?' and 'are easy theorem less beautiful?'. Roger Penrose, a subject of the survey, offer a possible answer. He claims a theorem is most pleasant when done with 'unexpected simplicity.' This therefore still supports the perspective that simplicity influences our aesthetic judgement of beauty, even just to an extent.

In contrast to this perhaps traditional view that simplicity holds at least some weight in our aesthetic judgement of beauty, Inglis and Aberdin make a contrasting statement in their 2014 study that 'beauty and simplicity are almost entirely unrelated in mathematics.' (Inglis & Aberdein, 2014)

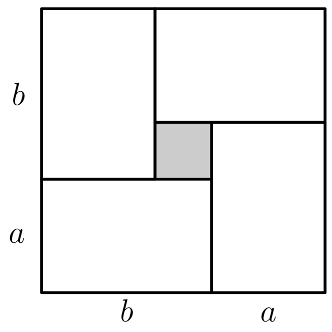
To make such a claim they set out to investigate what they call a 'language game' in mathematics. Looking into when and why mathematicians use the word 'beautiful' in the hope to find an underlying trend.

Inglis and Aberdin reached out to a number of esteemed mathematics departments. The subjects (255 mathematicians: 146 PhD students, 23 postdocs, and 86 faculty members) were asked to recall a recent proof they have seen or worked with. They are then presented with 80 adjective words and asked how well each word described their proof.

After collecting this data, analysing the results and performing statistical tests, details on which can be found in Inglis and Aberdein (2014). There is discussion into the adjectives that correlate most and least with the word 'beautiful.' In contrast to what is often presented as the classical view, 'simple' and 'beautiful' had 'a correlation coefficient not significantly different from zero'. They therefore make the conclusion that simplicity and beauty have no relationship.

Despite this, the overwhelming conclusion from the majority of studies and reports is that simplicity adds to a proof's beauty. More specifically, if a proof is able to strike a good level of simplicity so not to be considered trivial and perhaps even offer an unexpected simpler approach to its counter part, this property will only be an aesthetic benefit.

Visual proof 2



$$(a+b)^2 - (a-b)^2 = 4ab$$
$$\frac{a+b}{2} \geq \sqrt{ab}$$

Brown, 2008, page 43.

Before discussing aesthetics of our next picture proof, note the explanatory use of algebra on the right-hand side. Linking this to our opening discussion of diagrams as proof: is our proof solely visual? Is this an example of a diagram used as intuitive aid to a proof? This will be discussed further when we reach our mixed proof.

Of our three picture proofs it is clear this one holds the most symmetry (rotational symmetry in this case) which may be contributor to its potential beauty. There is no doubt that symmetry is widely viewed as aesthetically pleasing. This is evident in art, faces and nature. We don't believe this to be a surprising statement due to the relevance of this in our daily lives. Various examples of things in nature we often find visually pleasing including flowers, shells and animal patterns - all rife with symmetry. This is also true within mathematics as 'a class of properties that includes simplicity, symmetry, harmony and so forth is often used to explain beauty' (Montano, 2014).

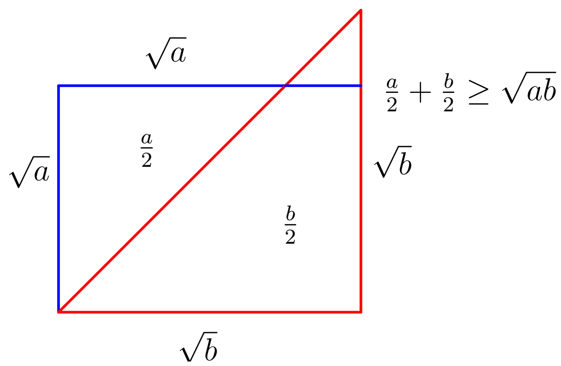
Symmetry provides order and sense to a complex field. It may be seen as a shortcut to understanding as it helps our brains to immediately spot a pattern.

There is something to be said here about subjectivity. 'Mathematics and the Aesthetics', written by Nathalie Sinclair, David Pimm and William Higginson, brings up the idea that mathematicians may loosely be classified as 'classical' and 'romantic'. The described differences between the two in this work is that one has a 'desire for equilibrium' and 'cherishes symmetries' where as the other 'yearns for a lack of balance' and takes 'enjoyment in the breaking of symmetries.' There is mention that this was similarly explored by Freeman Dyson, an American mathematician. Although he named our groups 'unifiers' and 'diversifiers', the premise still holds.

So perhaps symmetry is a good example to demonstrate the subjectivity of aesthetics. From literature, science, and experience, we believe it to be a safe standpoint that the majority have an aesthetic appreciation for symmetry. We therefore believe this property adds to our proofs perceived beauty.

The concept of making an aesthetic judgement was initially derived from judgements of taste; that is, the preference for certain sensory experiences over others (Shelley, 2022). While we talk here mostly about a more sophisticated level of aesthetic reasoning that depends far less on sensory interpretations and more on cognitive processes, it is still prudent to note that sensory experience will inevitably factor in to aesthetic judgements. That said, a picture proof – which has visual balance and weight – provides a visual sensory experience that most would deem superior to prose. This will likely lead to picture proofs being deemed aesthetically pleasing in this quite instinctual, sensory way.

Visual proof 3



In our second picture proof we touched on beauty as enlightenment. Enlightenment is an epistemic concept and is used to describe the moment where you reach full understanding or find clarity where you once did not. Visual proof 3 uses colour to aid understanding and clarity when first faced with this proof. The colour allows us to immediately recognise the two shapes and the two areas we are comparing. The use of colour overcomes any hesitation and accelerate the feeling of enlightenment.

Enlightenment is often achieved when you understand where a new piece of information falls into the jigsaw puzzle that is the world of mathematics. Or as nicely summarised by Rota 'we acknowledge a theorem's beauty when we see how the theorem 'fits' in its place, how it sheds light around itself.' Enlightenment is then a feeling you must experience to be able to appreciate beauty.

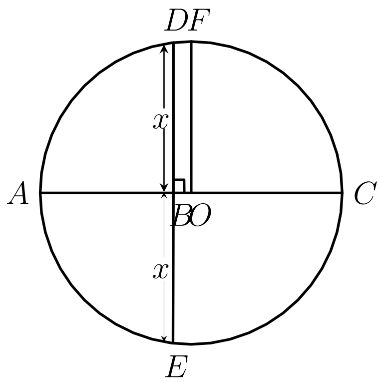
Many believe enlightenment to be the main contributor to mathematical beauty over other qualities. This is true for Rota and can be found in claims such that the term 'mathematical beauty' is only 'devised to avoid facing up to the messy phenomenon of enlightenment.' And whilst we have and will discuss other qualities that contribute to beauty, could it be argued that enlightenment underpins them all? Inglis points out that 'presumably mathematicians perceive simplicity to be a characteristic of beautiful mathematics simply because it is easier to gain enlightenment from a simple proof compared to a complex proof.' (Inglis & Aberdein, 2014)

There seems to be a strong case for enlightenment and its huge impact on our aesthetic judgment, however we believe there are some drawbacks to this criterion. It is obvious that what is enlightening to one may not be enlightening to another. There are many variables between mathematicians that must be considered when listening to opinions on enlightenment. What is their background? What mathematics are they familiar with? What level of mathematics have they studied? Feelings of enlightenment will be heavily influenced by your mathematical background and what is familiar to you. You must be able to understand the logical arguments of a proof to find enlightenment within it. These type of issues make it clear that 'enlightenment is not easily formalized.' (Rota, 2014).

Secondly, as Rota points out, enlightenment admits degrees and concepts admitting degrees are often frowned upon in the mathematical world.

Whilst there are drawbacks to reaching a clear criterion of 'enlightening', it is apparent it does influences our aesthetic judgement. Whilst we note it is a subjective matter, mathematics that lends itself to be enlightening will more likely be preferred and considered more 'aesthetically pleasing'.

Mixed proof



Proof. Observe that $x^2 = AB \cdot BC$, which implies that $x = \sqrt{AB \cdot BC}$.

O is the center of our circle, so $AO = OC$ and $\overline{OF}$ is equal to the arithmetic mean of $\overline{AB}$ and $\overline{BC}$. $\overline{DB}$, or x, is the geometric mean.

Radius $\overline{OF}$ is the length of the largest segment from $\overline{AC}$ to the circle at O, so the geometric mean $\overline{DB}$ can never surpass the AM. Equality is only achieved when $AB = BC = AO = OC$, and both the AM and GM are equivalent to the radius of circle O. □

When considering the aesthetic value of a proof that contains both diagrammatic and written parts it is salient to discuss a few different elements. We will first discuss the proof in regards to judgements that might be made in regard to its qualities and properties as a proof-object, and then aesthetic judgements and insights that might be gained in light of these qualities.

As discussed previously, a mathematical traditionalist may be more persuaded by the legitimacy of such a proof as the above, in contrast to any of the complete picture proofs. Here it is worth mentioning that what we have qualified as picture proofs do still contain prose, in the form of mathematical equations and labels that qualify the lengths of lines and allow them to be compared. What, then, might make this proof more appealing? There is a clear difference in volume and emphasis, with a far more extensive written section in the mixed proof, and an emphatic difference between the focus of the proofs. The picture proofs are diagram-centric, with the visual elements providing the focus, and the written elements supporting the reader through understanding the diagram. By contrast the mixed proof places more weight on the interactions between the elements of the proof, with the written section relying on the diagram to construct an argument, and the diagram relying on genuine mathematical steps taken by the written section to be coherent.

Essentially, the primary differentiating factor here is that the written elements of the picture proofs are purely clarifying – they draw the attention to certain elements of the picture – and since the picture proofs are all about comparing lengths or areas, they make evident the comparison for the reader, whereas the written portion of the mixed proof is explanatory. That is to say, the written section of the mixed proof could almost entirely serve to prove the theorem, and contains features such as mathematical arguments and process. The written section is not self-contained, however, as it refers to a diagram rather than having the reader construct it in their heads.

The mixed proof also contains variations in proof and style that provide what we might call aesthetic texture for a reader. The alternating structure of reading some of the written section, examining its referral to the picture section, and processing the implications before returning to the written element again lends the reading of the proof variation and interest, which may be aesthetically appealing.

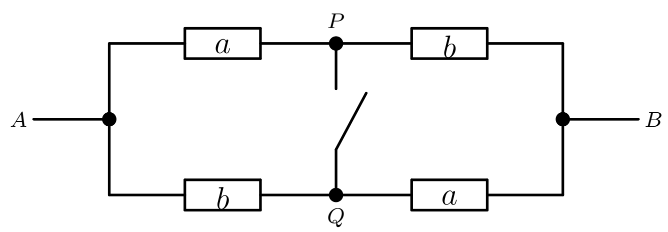
The final quality of the proof as an object that we will discuss is length, as relating to simplicity. The proof could feasibly exists without the diagrammatic element, were there a lengthy verbal construction of the diagram or a similar comparative object. However such a construction would detract from the statements and arguments that actually make up the proof, and would also require the arduous tasks for the reader of parsing out the written construction into a mental image. The diagrammatic element not only allows brevity and simplicity, but also clarity of argument. In particular, it allows the reader to focus more on comprehending the steps of proof, without distraction from elements that do not facilitate the argument. This allows for more clarity of explanation, and thus a stronger potential to be enlightening than a proof that requires more complex internal processes before the reader has a full understanding.

In discussing qualities of the proof as an object, we have touched on some relevant properties outlined in the literature as being key to aesthetic judgements of mathematical objects, such as simplicity and enlightenment.

As has been discussed in preceding sections, simplicity is often a core criterion used by mathematicians when making aesthetic judgements about proofs. Mathematicians value and find appealing efficiency of argument and brevity of style, which together compose the aesthetic quality of simplicity. Not only do the underlying mathematical ideas of the mixed proof contain few and simple steps, but the decision to forgo verbal construction in favour of a diagram makes the proof both more efficient and visually briefer. This means that the proof has a concision and brevity that lends it aesthetic weight. It also potentially makes the proof more enlightening as the reader has to undergo less complex mental processes in order to arrive at an understanding of the proof. This enlightenment as also linked to mathematicians judging a proof as aesthetically pleasing.

One potential challenge to this interpretation is that a briefer proof may omit details, in fact inhibiting the ease of enlightenment. Mathematicians of a certain skill range are happy to have details omitted, as they can confidently and easily construct the omissions in the argument. However some proofs that have brief constructions may do so at the expense of clarity. The first line of argument in the written section of this mixed proof, for example, may present issues to those not familiar with similar mathematical arguments. A proof not easily understood, or that does not lead its reader to understanding clearly, may be considered less aesthetically pleasing.

Levi's circuit proof



Proof. The circuit is made of resistances a and b. Starting with the switch open as shown, each parallel path has resistance $a+b$. Thus the total resistance between A and B is $\frac{a+b}{2}$. Closing the switch, the resistance of a circuit with a short is the same or less than before. We now have two resistors is sequence, each of strength $\left(a^{-1}+b^{-1}\right)^{-1}=\frac{ab}{a+b}$. Thus

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}$$

□

Levi, 2009

The applied nature of this proof is a significant factor in the assessment of aesthetic judgement. As with preceding proofs we will first consider judgements that may be made about the proof as an object, and then consider the aesthetic implications of these judgements.

The proof has a similar structure to the preceding mixed proof, with a diagram that is referred to by a short section of prose. As discussed previously this structure uniquely simplifies the proof, by eliminating wordy constructions and clarifying the proof by referral. The structure also promotes engagement with the steps of the argument, and provides what we have referred to as 'texture' – alternating between written and picture elements – that make the proof more interesting.

The picture part of this proof has a high level of symmetry, a feature that has been linked to positive aesthetic judgements is a range of aesthetic objects, including mathematical objects. Significantly, elements of asymmetry within a relatively symmetrical object may also be more appealing than pure symmetry, due to a subversion of expectations. It may also encourage deeper consideration as a result, which could lead to a fuller understanding of the proof.

One further element of this proof that may cause contention is the final line. Though very clearly equivalent to the statement that the proof is required to prove, the inconclusive nature of the statement that ends the proof – which still requires a few mental steps on the part of the reader before the proof is actually concluded – may be seen as clumsy. We have here a classic conflict between a proof feeling complete, and a proof being simple or brief. The final statement does conclusively prove the theorem, but it may not have the triumphant precision that we might expect from an aesthetically pleasing proof. Mathematicians generally prefer simple steps within proofs to be omitted, but this may outweighed by the desire for a conclusive final line.

Now, a final consideration about the nature of the proof is the applied nature of the argument. The referral to properties of electrical currents and circuitry are unconventional in a mathematical proof, and this may have dual consequences. Some mathematicians are likely to be unimpressed by any reference to context or the physical world, preferring to keep mathematical proof purely abstract. This convention is widely held in many fields of mathematics, particularly in academic circles, and it is common to disdain applied mathematics, or the implication that pure mathematics relies or feeds into it. Many mathematicians may perceive proofs via physics as unwieldy, clumsy, or even invalid. It is also worth noting that in spite of the very simple knowledge of physics required, some mathematicians may not possess this prior knowledge or familiarity and may find the proof inaccessible as a result, in spite of its relative simplicity.

However, there may also be a certain appeal of the applied nature of this proof, even to the pure mathematician. There is something to be said for lightheartedness and an element of humour when it comes to proofs, and this reference to circuits has a whimsy that a mathematician with a certain sense of humour might find appreciable. The mathematical notation and arguments are not unfamiliar, and the unique presentation of the proof might be appealing, and even encourage deeper thought due to subverting common anticipations of what a proof might look like. This 'shock factor' presents a legitimate avenue by which a proof could gain aesthetic value in the eyes of mathematicians, who often appreciate cleverness and creativity. Additionally, mathematicians with a background or knowledge of physics may find this proof appealing.

Ultimately, the discussion here revolves around familiarity, and whether it has a positive or negative effect on perceptions of the aesthetic. There are indications either way: a familiar aesthetic object has the benefit of fulfilling expectations and appearing somehow correct in the eyes of the viewer, whereas an unfamiliar aesthetic object has the advantage of uniqueness and interest. Conversely, a familiar proof may seem boring and rote, or an unfamiliar proof incorrect or unskilled. Familiarity is a quality of aestheticism that can vary based on temperament, as such remaining a rather spurious and subjective qualifier. Nevertheless, it is evident that it has an impact of the aesthetic judgements that mathematicians make, and as such is a point worth consideration.

Proof by induction 1

Proof. For every $a_1, ..., a_n \geq 0$, $\frac{a_1+...+a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$. (1)

For n=1 the statement is true. Suppose the statement holds true for $n-1$. That is:

$\frac{a_1+...+a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 \cdots a_{n-1}}$ (2)

Set $a = a_1$. Define the numbers $b_1, ..., b_{n-1}$ by the formula $b_i = \frac{a_{i+1}}{a}$, $1 \leq i \leq n-1$. AM-GM inequality is then equivalent to:

$a \cdot \frac{1+b_1+...+b_{n-1}}{n} \geq \sqrt[n]{a^n \cdot b_1 \cdots b_{n-1}}$.

We cancel a and multiply by n. We now need to show that

$1 + b_1 + ... + b_{n-1} \geq n \cdot \sqrt[n]{b_1 ... b_{n-1}}$

Manipulating (2) by changing a_i to b_i, $i = 1, ..., n-1$, multiply by $n-1$ and add 1 at both sides:

$1 + b_1 + ... + b_{n-1} \geq 1 + (n-1) \sqrt[n-1]{b_1 \cdots b_{n-1}}$.

holds true, so it suffices to show that

$1 + (n-1) \sqrt[n-1]{b_1 \cdots b_{n-1}} \geq n \sqrt[n]{b_1 \cdots b_{n-1}}$.

Let $x = \sqrt[n(n-1)]{b_1 \cdots b_{n-1}}$.

$1 + (n-1)x^n \geq nx^{n-1}$ by substitution.

Let $f(x) = 1+(n-1)x^n - nx^{n-1}, x \geq 0$. Since $f'(x) > 0$ if $x > 1$, and $f(x) < 0$ if $0 \leq x < 1$ the function has a global minimum at x=1. The value $f(1) = 0$ so $f(x) \geq 0$, inequality holds true for n. □

Gaitanas, 2021

The format of this proof is what we may describe to be a typical proof by induction. For example it follows the expected pattern beginning with 'for n=1' and moving on to show the induction step. We can link this idea back to one previously discussed in our picture proofs, beauty by enlightenment. A proof presented in this familiar fashion may allow enlightenment to be reached easier.

There has been an interesting development in the way we have chosen to present this particular proof. Below we see the original proof directly taken from Gaitanas (2021)

Proof. For all $a_1, ..., a_n \geq 0$ we have,

$\frac{a_1+...+a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$. (1)

We first consider the base cases. The theorem is trivially true for $n = 1$. If $n = 2$, then the equation is equivalent to

$(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$,

which is obviously true.

Suppose now that the theorem holds true for $n - 1$, that is

$\frac{a_1+...+a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 \cdots a_{n-1}}$, (2)

for appropriate choices of $a_1, ..., a_{n-1}$. In order to show that (1) is true for n, we will use the following (crucial) trick: Set $a = a_1$. Then define the numbers $b_1, ..., b_{n-1}$ by the formula $b_i = \frac{a_{i+1}}{a}$, $1 \leq i \leq n - 1$. With this notation the AM-GM inequality is equivalent to

$a \cdot \frac{1+b_1+...+b_{n-1}}{n} \geq \sqrt[n]{a^n \cdot b_1 \cdots b_{n-1}}$.

We can cancel a from both sides and multiply by n. We now only need to show that

$1 + b_1 + ... + b_{n-1} \geq n \cdot \sqrt[n]{b_1...b_{n-1}}$

This can be done by manipulating the induction hypothesis (2) in the following way: We change a_i to b_i, $i = 1, ..., n-1$, multiply by $n-1$, and add 1 to both sides. We obtain that

$1 + b_1 + ... + b_{n-1} \geq 1 + (n - 1) \sqrt[n-1]{b_1...b_{n-1}}$,

holds true, so it suffices to show that

$1 + (n - 1) \sqrt[n-1]{b_1...b_{n-1}} \geq n \sqrt[n]{b_1...b_{n-1}}$

If we set

$x = \sqrt[n(n-1)]{b_1 \cdots b_{n-1}}$

then the last inequality can be written in the from

$1 + (n - 1)x^n \geq nx^{n-1}$

Finally, we let

$f(x) = 1 + (n - 1)x^n - nx^{n-1}, x \geq 0$

and prove the last inequality using the first derivative test. We can see that

$f'(x) = n(n-1)x^{n-2}(x-1)$.

It is an easy task to see that this function has a unique critical point, namely $x = 1$. Since $f'(x) > 0$ if $x > 1$, and $f(x) < 0$ if $0 \leq x < 1$, the function has a global minimum at $x = 1$, the value $f(1) = 0$. This proves that $f(x) \geq 0$ and the inequality holds true for n. This completes the proof.

□

The first striking difference we see between the original version and our final version is the length. We have made the decision to shorten this proof drastically. But what affect does length have on aesthetic judgement? Giaquinto (2016) makes reference on page 63 to a balance that must be struck between length and accessibility. He claims a proof which has the correct blend of both will be preferred to one that has one factor heavily outweighing the other. It is not surprising that many articles and papers agree with this stance. As Rota (2014) points out 'usually, but not always, a proof that is deemed to be beautiful tends to be short.' And so as we made changes this proof, with the aim for it to be a potentially more aesthetically pleasing version of itself, we decided to make it shorter.

To shorten this proof we had to make decisions on what to omit and how to format the remaining information. Firstly, we removed many statements we felt to be redundant (which mainly consisted of descriptive words). For example; we reduced "we first consider the base case. The theorem is trivially true for n=1". to "For n=1 the statement is true". Essentially we removed the vast amount of instructive text. Although this makes our proof shorter and perhaps more concise however this holds the risk of our proof becoming less appealing to some. Consider the expertise reversal effect. This idea tells us that novices learn best from step by step guidance and clear instruction. This is because someone who is new to a topic will not have a well built schema surrounding the field and so will rely on extra support to think critically. On the other side of this, those who are more familiar with a subject, 'the experts', will get the most out of learning tasks with little to no teaching to accompany a task, for example, a proof.

So our decision here may have the desired affect on some audience but perhaps not on others. Since as undergraduates we believe we are still able to follow this proof we concluded to remove such guidance would benefit our potential readers.

Proof using Jensen's inequality

In this section, we provide 2 proofs of the AM-GM inequality using Jensen's inequality. We provide one proof stating Jensen's inequality and one proof without stating Jensen's inequality.

One way of proving the AM-GM inequality is by using Jensen's inequality. There are provided two proofs of this variety that are virtually identical the only differing factor being that one of them is self-contained, as it has the Jensen's inequality stated and proved, whilst the other does not. This raises an issue as to how aesthetically appealing a proof is when using an outside theorem or lemma as it concerns the reader with how useful prior knowledge can be factored in aesthetic value of a proof.

Proof using Jensen's inequality (stating Jensen's inequality)

A thorough discussion of Jensen's inequality is given by Rasheed et al., 2021.

Lemma 1. *Jensen's inequality states: If Ψ is a convex function on the interval $I \subset \mathbb{R}$, where p_i are positive real numbers and $x_i \in I(i = 1, \cdots, n)$, while $P_n = \sum_{i=1}^{n} p_i$. It holds that,*

$$\Psi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i \Psi(x_i)$$

Rasheed et al., 2021

Proof. To use Jensen's inequality we need to apply it to a concave function, as the inequality holds in the opposite direction if applied to a concave function rather than a convex function. Use the concave function $f(x) = \ln(x)$. Hence,

$$\ln\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \geq \frac{1}{n}\sum_{i=1}^{n} \ln(x_i) = \sum_{i=1}^{n} \ln\left(x_i^{\frac{1}{n}}\right) = \ln\left(\prod_{i=1}^{n} x_i^{\frac{1}{n}}\right) = \ln\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right)$$

We can apply the exponential to both sides of the obtained inequality, as it is an increasing function. From this we get,

$$e^{\ln\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)} \geq e^{\ln\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right)}$$

Which implies the AM-GM inequality. $\square$

Jensen, 1906

Proof using Jensen's inequality (not stating Jensen's inequality)

Proof. To use Jensen's inequality we need to apply it to a concave function, as the inequality holds in the opposite direction if applied to a concave function rather than a convex function. Use the concave function $f(x) = \ln(x)$. Hence,

$$\ln\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \geq \frac{1}{n}\sum_{i=1}^{n}\ln(x_i) = \sum_{i=1}^{n}\ln\left(x_i^{\frac{1}{n}}\right) = \ln\left(\prod_{i=1}^{n}x_i^{\frac{1}{n}}\right) = \ln\left(\sqrt[n]{\prod_{i=1}^{n}x_i}\right)$$

We can apply the exponential to both sides of the obtained inequality, as it is an increasing function. From this we get,

$$e^{\ln\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)} \geq e^{\ln\left(\sqrt[n]{\prod_{i=1}^{n}x_i}\right)}$$

Which implies the AM-GM inequality. □

Jensen, 1906

When considering the aesthetic value of a proof that is not self-contained, it is important to consider the reader's prior knowledge of the subject.

Using an outside theorem to prove another theorem is a standard practise in academic mathematics. Throughout higher education of mathematics students they are regularly showed ways that they can prove theorems using a previous theorem usually stated earlier in the course that they are studying. This creates the question of how important prior knowledge is when looking at a new proof. In our case we are using Jensen's inequality to prove the AM-GM inequality.

One could argue that having the outside theorem stated along with its proof not only makes it more understandable for the reader but also makes the proof more aesthetic as all elements of the proof would be visible to the reader. Furthermore, when considering only the aesthetics of this proof having the proof be fully self-contained provides the proof with more continuity as it would flow from one element to the other without the reader needing to stop and think whether they know this prior theorem.

However, is there is an argument for not having the proof self-contained. This stems from the expertise reversal effect (as stated previously in the aesthetic discussion of Proof by induction 1). From this it is also possible to argue that having proof that relies on an outside theorem and is not self-contained is more aesthetic, as it could provide some readers with a better understanding of it. Even though understanding is not strictly aesthetic in a general sense, it is something that is intrinsic to mathematics, hence why understanding of a proof in mathematics can be viewed aesthetic by some if it provides them with the understanding that they need.

Another reason as to why a non-self-contained proof may be viewed more aesthetically is when the outside theorem's proof is wildly different from what will follow it. This in turn can create a clash of styles. For example, in our case the most common way of proving Jensen's inequality is by using expectation values, functions that are primarily used in probability, then it is followed by a proof that has no evident probabilistic functions used. This clash of styles as well as fields of mathematics could confuse the reader as they might receive too much general information and not fully grasp the meaning behind either proofs.

Proof from thermodynamics

Proof. Consider n beakers of water, each kept at a temperature T_i. When all the beakers are mixed, then the final temperature of the system will be equal to the arithmetic mean $T = \sum_i \frac{T_i}{n}$ as a consequence of the first law of thermodynamics (no heat is lost).

Now, consider the change in entropy upon mixing the n beakers. First, changing the temperature from T_i to T, and second, mixing all the beakers. Since all the beakers are then identical after the first step, the second step does not contribute to the entropy change of the system. The entropy change of the first step is given by $\Delta S = C \log \frac{T}{T_i}$, where C is the heat capacity of the water. Then, the total entropy change is given by

$$\Delta S_{tot} = C \sum_i \log(\frac{T}{T_i}) = C \log(\frac{T_n}{\prod_i T_i})$$

By the second law of thermodynamics, this quantity is greater than or equal to zero, and thus we have that

$$T^n \geq \prod_i T_i \Rightarrow T = \sum_i \frac{T_i}{n} \geq \sqrt[n]{\prod_i T_i}$$

with equality iff there is no entropy change, i.e every beaker started off at the same temperature T. [7] □

We will first consider properties of this proof, and then consider how they might affect aesthetic judgements of the proofs.

An initial point that may prove contentious are elements of the notation. The symbols and notation used here should be familiar and legible to mathematicians, however they may be different to what most mathematicians would themselves select if composing the proof. Additions such as the capital delta to refer to change may seem unnecessary in the context of the proof, and the subscript 'tot' completely alien. These notational choices are just slightly foreign to what a pure mathematician would expect – though standard in some applied fields – and this may result in a poor aesthetic judgement of the proof. This comes back to discussions of familiarity in prior sections. Previously we discussed that familiarity or the lack thereof may have both positive and negative effects on aesthetic judgements. However due to generally strong opinions in regards to notation, deviations from typical notational choices are unlikely to be met positively.

The applied nature of the proof is also likely to negatively impact its aesthetic value. As previously discussed in regard to Levi's circuit proof, some mathematicians prefer proofs that do not touch on contextual or applied factors. The thermodynamic approach to proving the theorem is unconventional, not only notationally but also thematically. This applied approach may be viewed by pure mathematicians as invalid or simply unwieldy. The proof depends on laws of thermodynamics, which are potentially far less rigourous than mathematical principles that other proofs are based on. Primarily for this reason, it could be argued that the proof is not even functional.

It is also feasible that some mathematicians may be genuinely thrown off by the unfamiliarity of the argument, and thus find it less enlightening than a more standard proof type such as induction. An applied approach may also be viewed as inefficient, since outlining the context and additional elements takes up space in the proof. From a perspective of valuing simplicity, brevity, and efficiency, relying on an applied analogy about thermodynamics might be perceived as not only crude but also unnecessary, bulky, and overly complicated. Conversely, however, mathematicians might appreciate the novelty and creativity of a proof outside of the norm, and value the alternative perspective. Even in this case, the appreciation is unlikely to translate to a positive aesthetic judgement, since the aspects of the proof that mathematicians may find appealing are not the same as those that they use to make aesthetic judgements. This highlights a key point of the literature on aesthetic judgements: positive feeling towards an aesthetic object does not necessarily translate to perception of that object as having high aesthetic value.

An additional feature of the proof that could prove problematic is the fact that it is not self-contained, relying on external assertions that are not proven within the proof. Though the first and second laws of thermodynamics may be seen as fundamental, the proof is ultimately based on the assumption that they are true. This may be unappealing to mathematicians, and negatively affect their aesthetic view of the proof. This may also negate positive effects of the brevity of the proof, since it is concise only by relying on external works that are not fully encompassed within it.

In favour of the proof, the narrative slant may lend it a level of engagement and interest that more pure mathematical proofs lack. The proof essentially leads the reader through a story, and the practicality could ground the abstract concepts in a reality that is easier to imagine, allowing a deeper understanding of the concept behind the theorem even if it lacks mathematical rigour. This could allow for a higher – or alternative – level of enlightenment, that could be reflected positively in the readers' aesthetic judgements.

Proof using calculus 1

Proof. Consider the following function:

$$f(x) = \ln x - x. \tag{5.1}$$

Find the maximum value of the function by differentiating:

$$f'(x) = \frac{1}{x} - 1 = 0 \iff \frac{1}{x} = 1 \iff 1 = x,$$

Then confirm that this point is the maximum using the second derivative:

$$f''(x) = -\frac{1}{x^2} \implies f''(1) = -\frac{1}{1} = -1.$$

The value of the second derivative at this point is negative, so the maximum of (1.1) is at $x = 1$. The value of (1.1) at this point is found:

$$f(1) = \ln 1 - 1 = -1,$$

$$\implies \ln x - x \leq -1 \iff \ln x \leq x - 1.$$

Let $A = \frac{a_1+a_2+\ldots+a_n}{n}$, the Arithmetic mean. Now we consider $x = \frac{a_i}{A}$:

$$\ln\frac{a_1}{A} \leq \frac{a_1}{A} - 1, \ln\frac{a_2}{A} \leq \frac{a_2}{A} - 1, \ldots, \ln\frac{a_n}{A} \leq \frac{a_n}{A} - 1.$$

Adding up all terms:

$$\ln\frac{a_1}{A} + \ln\frac{a_2}{A} + \ldots + \ln\frac{a_n}{A} \leq \frac{a_1 + a_2 + \ldots + a_n}{A} - n$$

$$\implies \ln\frac{a_1 a_2 \ldots a_n}{A^n} \leq \frac{nA}{A} - n = n - n = 0.$$

Exponentiating both sides:

$$e^{\ln\frac{a_1 a_2 \ldots a_n}{A^n}} \leq e^0 \implies \frac{a_1 a_2 \ldots a_n}{A^n} \leq 1 \implies a_1 a_2 \ldots a_n \leq A^n \implies \sqrt[n]{a_1 a_2 \ldots a_n} \leq A.$$

□

AoPS Online, 2021

This proof is relatively accessible to those who have studied mathematics at A-Level or equivalent stages of education, and uses fairly simple calculus. It has been written out in such a way as to avoid using summation notation and product notation, which we will elaborate more on in the discussion for the next proof. The use of arrows to show implication is an attempt to reduce the wordiness of the proof, although it does result in long strings of expressions on the same line. The arrows give a sense of flow or motion through the proof which could appeal as it takes you through step by step using only simple gestures. This is meant to be an non-intimidating application of ideas an undergraduate student will probably have met before they reach university.

That being said, there is a possibility that the presentation of this proof could be frustrating to a more seasoned student or professional. The steps are often unnecessarily belaboured, for example equalities of $\frac{nA}{A} - n = n - n = 0$. This is inelegant, as it overstates things which the reader could easily see for themselves. This does not allow for any exercise of the mind while reading the proof, so the chance of an aesthetic response driven by coming to understand a piece of mathematics laid in front of the reader is reduced. It is hard to let the generative aesthetic lead inquiry in the reader if the steps are already all filled in by the author. Some of the lines simply feel too long, and so it becomes difficult and even boring to read. Splitting them up helps a little bit to improve the presentation, but we still have the problem of over-explaining. Choices to leave little up to the reader's exploration often result in work that can be messy as it is too full, particularly at these more elementary levels.

There is also the issue of rigour. Several statements move on uncontested in this proof which might satisfy a green student but might not satisfy a ripened one. There is a concern here that what is considered 'rigourous' is not consistently marked among students and professionals, despite rigour being a focus of mathematical education, especially in the early years of undergraduate study. In 2021 Sangwin and Kinnear found that given two induction proofs of 'indistinguishable' simplicity for the same theorem, students often ranked the one which was more formal with the steps presented as instructed to when proving by induction for homework as more rigourous than one which was equally true but less formal (Sangwin & Kinnear, 2021). Once again, a divide between students and professionals. This time, the divide is caused by the aesthetic or presentation (formality verses informality) influencing perception of the rigour.

The writer who found and cited the proof and condensed it has since returned to it and later thought that some steps are over-justified (the expression for $\frac{nA}{A} - n = 0$) and some are under-justified (are we absolutely certain that we have a maximum and not simply a local maximum?). Our relationships to proofs can evolve over time.

Proof using calculus 2

Proof. Consider the following function:

$$f(x) = \ln x - x. \tag{5.2}$$

Find the maximum value of the function by differentiating, and confirm it is the maximum:

$$f'(x) = \frac{1}{x} - 1 = 0 \iff \frac{1}{x} = 1 \iff 1 = x,$$

$$f''(x) = -\frac{1}{x^2} \implies f''(1) = -\frac{1}{1} = -1.$$

The value of the second derivative at this point is negative, so the maximum of (1.2) is at $x = 1$. The value of (1.2) at this point is found:

$$f(1) = \ln 1 - 1 = -1,$$

$$\implies \ln x - x \leq -1 \iff \ln x \leq x - 1.$$

Let $A = \frac{\sum_{i=1}^{n} a_i}{n}$, the Arithmetic mean. Let $x = \frac{a_i}{A}$, and for $\forall i, n \in \mathbb{N}$ and $1 \leq i \leq n$, we have:

$$\ln \frac{a_i}{A} \leq \frac{a_i}{A} - 1$$

Summing terms:

$$\sum_{i=1}^{n} \ln \frac{a_i}{A} \leq \frac{\sum_{i=1}^{n} a_i}{A} - n \implies \ln \frac{\prod_{i=1}^{n} a_i}{A^n} \leq \frac{nA}{A} - n = 0.$$

Exponentiating both sides:

$$e^{\ln \frac{\prod_{i=1}^{n} a_i}{A^n}} \leq e^0 \implies \frac{\prod_{i=1}^{n} a_i}{A^n} \leq 1 \implies \sqrt[n]{\prod_{i=1}^{n} a_i} \leq A.$$

□

AoPS Online, 2021

You do not need to be Cauchy to notice that this proof is more or less the same as the previous one, and that is because by having two proofs which are very similar (in fact identical in terms of reasoning), we can think about small changes which might make a proof more appealing. This version defers to the shorthand notation for summation and products, and also explains less between expressions, assuming that the reader will follow. Additionally, some of the superfluous statements from the previous version have been omitted. It is a relief immediately that no lines are breaking into the margins.

As well as cutting out some of the redundant steps, in this version we have chosen to use the sigma and pi summation and product notation, saving a lot of space. While sometimes this notation can be disagreeable, in contrast to the previous one, the brevity is a relief. Much as a proof is greater than the sum of its parts, sometimes comparing two proofs one after the other can enhance our perceptions of both, even if the changes are cosmetic.

The notation we choose does not merely come down to a surface-level appeal. Two pieces of notation have the same meaning, but one might convey the nature of the object more appropriately. For example, the sigma summation notation. It can be simpler to write a finite sum as simply the first and last terms (ellipses in between), possibly the first couple of terms to avoid ambiguity. This means the same thing as the summation symbol, but it removes the step of interpreting what that symbol means. However, when dealing with infinite sums, or many different sums of things, or sums and products, shorthand will reduce the mess of having lots of terms in your expression. Additionally, we often want to consider an infinite series as an object by itself. The sigma notation comes to be associated with a specific concept: an infinite series which may or may not converge. This goes to infer that can be appealing can be highly contextual.

Proof by induction 2

Proof. For $x > 0$,

$$x^{n+1} - (n+1)x + n = (x-1)^2(x^{n-1} + 2x^{n-2} + \cdots + n) \geq 0$$

Suppose that $x_1, \cdots, x_{n+1} > 0$. Let, $a = \frac{x_1+\cdots+x_{n+1}}{n+1}$, $b = \frac{x_1+\cdots+x_n}{n}$ and, $x = \frac{a}{b}$. We have:

$$\left(\frac{x_1 + \cdots + x_{n+1}}{n+1}\right)^{n+1} - (n+1)\left(\frac{a}{b}\right) + n \geq 0$$

$$\implies a^{n+1} \geq ((n+1)a - nb)b^n$$

$$\implies \left(\frac{x_1 + \cdots + x_{n+1}}{n+1}\right)^{n+1} \geq x_{n+1}\left(\frac{x_1 + \cdots + x_n}{n}\right)^n$$

From this the AM-GM inequality follows by induction on n. □

Hirschhorn, 2007.

When looking at this proof of the AM-GM inequality it is natural to compare it to the other proof by induction that is contained in this project (see Proof by Induction 1). When comparing this to a classic proof by induction, we can see that this is not in a sense a full proof but rather a set up for a proof by induction. This is easy to notice as it lacks a base case, and all the other elements required for a proof to be a proof by induction.

However, when just analysing the aesthetic properties of the given proof, we can establish several elements that affect how aesthetic it is. One of the main elements being the brevity of the proof, as this is a rather short compared to most of the other proofs gathered. As a result, this creates a dialogue as to whether a short proof is more aesthetic than a longer one. From the viewpoint of the reader, one can say that having such a short proof is more aesthetically pleasing as it provides an easier reading experience. Furthermore, a short proof has an element of mathematical elegance as it manages to convey all the necessary information that the reader is required to fully understand what is happening within the proof.

While some may view brevity as being more aesthetic, there is also an argument to be made for longer proofs being more aesthetic, as they generally don't rely on the reader's own knowledge on the subject. This is evident in this proof by induction, as since it is not by standard mathematical terms a proof, but rather a set up for a proof, it heavily relies on the reader being comfortable with proofs by induction, highlighted by the short sentence at the end of the proof. From this we can see that some knowledge about proofs by induction, is not only suggested but required, and if the reader lacks this knowledge it could easily drive them into the viewpoint that the proof itself lacks any aesthetic value as they won't be able to understand why the mathematical intuition of Hirschhorn led him to set up the proof in this manner.

Pólya's proof

Proof. The function $f : x \mapsto e^x$ is strictly convex. Let $g(x)$ be the tangent line to f at (0,1); then $g(x) = x + 1$. Since f is a continuous, differentiable function, it follows that $1 + x \leq e^x$.

Now, let $r_i = \frac{a_i}{\sum_{j=1}^{n} \lambda_j a_j} - 1$ for all integers $1 \leq i \leq n$.

The bounds give that $r_i + 1 \leq e^{r_i}$ which implies $(r_i + 1)^{\lambda_i} \leq e^{\lambda_i r_i}$.

Multiplying by n such inequalities gives that $\prod_{i=1}^{n} (r_i + 1)^{\lambda_i} \leq \prod_{i=1}^{n} e^{\lambda_i r_i}$.

Evaluating both sides we get

$$\prod_{i=1}^{n} (r_i + 1)^{\lambda_i} = \frac{\prod_{i=1}^{n} a_i^{\lambda_i}}{\left(\sum_{j=1}^{n} \lambda_j a_j\right)^{\sum_{j=1}^{n} \lambda_i}} = \frac{\prod_{i=1}^{n} a_i^{\lambda_i}}{\sum_{j=1}^{n} \lambda_j a_j}$$

and,

$$\prod_{i=1}^{n} e^{\lambda_i r_i} = \prod_{i=1}^{n} e^{\left(\frac{a_i \lambda_i}{\sum_{i=1}^{n} \lambda_j a_j} - \lambda_i\right)} = e^{\left(\frac{\sum_{i=1}^{n} a_i \lambda_i}{\sum_{i=1}^{n} \lambda_j a_j} - \sum_{i=1}^{n} \lambda_i\right)} = e^0 = 1$$

Substituting the results for the left and right hand sides

$$\frac{\prod_{i=1}^{n} a_i^{\lambda_i}}{\sum_{j=1}^{n} \lambda_j a_j} \leq 1 \implies \prod_{i=1}^{n} a_i^{\lambda_i} \leq \sum_{j=1}^{n} \lambda_j a_j = \sum_{i=1}^{n} \lambda_i a_i$$

Concluding the proof. $\square$

AoPS Online, 2021

When looking at Pólya's proof of the AM-GM inequality it is evident that this is the least simple out of all the proofs that we have gathered. When analysing this proof, it is important to consider how mathematical complexity impacts a proof's aesthetic value.

There are several aspects that create a sense of complexity for this proof, one that will be analysed more intensely is the notation used in this proof. At a first glance at the proof a reader could find themselves rearing their head if they are not accustomed to summation and product notation as this proof heavily relies on using both. However, when looking at the proof with a more mathematical lens it is easy to see why such notation was used, as when dealing with the functions that are used in the proof it is a more efficient way of writing it down. On the other hand, when looking at the proof through an aesthetic lens there is a viewpoint that having all this notation is unnecessary when ellipses could be used instead.

Now we can explore how the complexity of a proof can affect our aesthetic judgement of said proof. We have already stated how notation can affect the aesthetic judgement, but this is not the only element of Pólya's proof that impacts the complexity of the proof, another being the actual mathematical intuition behind the proof.

When looking at how the proof is structured, we can see that there are several mathematical ideas used that could be viewed by the reader as being too complex for the relative simplicity stated by the AM-GM inequality. One example of this is in the first line of the proof where it assumed that the reader knows what a strictly convex function is. Having made an assumption about the reader, this can already start to impact their aesthetic judgement of the proof as if the reader does not have the prior knowledge required it can hinder their understanding and hence their aesthetic judgement (something already discussed in the aesthetic judgement of the proof using Jensen's inequality).

One other observation that could be made about this proof which can impact the aesthetic judgement of it is the way that it is laid out. Particularly, one could say there is a lack of consistency within the proof. This is seen in the proof when evaluating both sides of the inequality, where rather than the proof following the layout it did before it starts to analyse the functions the layout of the proof suddenly changes. However, this could be viewed as a change made to make it more aesthetic as it allows for the fractions and functions used within those parts of the proof easier to read, perhaps providing the reader with more understanding.

Chapter 6

Conclusion

Using this collection of proofs as a lever for aesthetic discussion allows us to scrutinize the concept of mathematical beauty. We paired each proof with an aesthetic discussion of their individual properties. For example, simplicity, style, symmetry, familiarity, layout, and even the complexity of the mathematics to find that the format of each proof affected how we viewed it as an aesthetic object. By evaluating these properties we hope to provide a sounder understanding to the reader of what positively impacts this judgment and what does not. We have seen and discussed how instruction, or lack thereof, can have conflicting effects depending on ability. Symmetry, colour, and simplicity can be manipulated to improve our mathematical experience. Familiarity and style play a subjective role in our aesthetic judgement.

There is much discussion to be had regarding whether a mathematical object can be beautiful or if it is only our written or verbal presentation of an object that can. We have seen how changing notation can improve some aesthetic appeal of a proof. Nonetheless, these things are quite hard to disconnect from each other; the concept and the representation. It would be very easy to get trapped down this hole if we did not accept that these factors influence each other, and that there are other ways we can look at them. For example, a well-presented proof can be enlightening, and so that ease of understanding can lend to an appreciation of the mathematical object itself. On the other hand, choice omission of some detail could prove a challenge to the reader, forcing them to reckon with the proof in their own mind, and so through this process find pleasure in the act of solving the puzzle. These factors can influence each other.

As mentioned, we have also learned from our readings that due to factors including the Expertise Reversal Effect, professionals and students have different aesthetic responses to mathematics. We do not believe this is an entirely negative thing, rather more perspectives can lend to greater understanding. Students would benefit from these more mature aesthetic perceptions of mathematics, and perhaps educators would benefit from understanding how to motivate the student with their own aesthetic sensibilities.

A deeper understanding of this topic would be beneficial for all levels of mathematicians. For the student, we should be thinking about this as an important part of being a 'true mathematician' in the words of Poincaré. That being said, we must bear in mind that telling people that their aesthetic perceptions are 'inferior' could discourage them and push them away from the field.

As we spent a large part of our project collecting and formatting the proofs we have discussed, we hope our collection of AM-GM inequality proofs holds value. Linking back to our original discussion of what a proof is and their roles in section 1.4.1, a proof exists to convince or explain, depending on the context of research or education. We hope it can find purpose in both settings. For future research into this topic, our collection and corresponding discussions aim to inspire similar investigations or further development of this project.

We also hope it could have a place in educational discussions in the classroom. Introducing aesthetics formally into the classroom encourages students to give thought to different approaches to proofs.

In considering these things seriously, we have deepened our understanding of aesthetics and of what makes proofs beautiful for us. We hope to have demonstrated why it is vital for mathematics education to use the aesthetic as a tool and motivator in and of itself. We learned about aesthetic lenses we had never even considered before, yet made sense with our pre-existing experiences of mathematics, such as the generative aesthetic.

We hope the research that we have done within this project can be further expanded on, either by further analysis or by gathering data on what more mathematicians think counts as a beautiful proof. By gathering experimental data, it would be interesting to see how a mathematician's opinion on what defines an aesthetically pleasing proof as they progress in the field (from undergraduate to post-PhD). Surveys and experiments have been conducted along similar lines before, such as in Sangwin and Kinnear (2021), and in Dreyfus and Eisenberg (1986).

These issues of mathematical aesthetics could perhaps be used in order to find new solutions to the problem of mathematics anxiety. Richardson and Suinn defined mathematics anxiety as 'Mathematics anxiety involves feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems' (Richardson & Suinn, 1972), (Dowker et al., 2016). The 2016 summary by Dowker et al. leads us to believe that it is a prevalent issue. It is important not to reduce mathematics anxiety to only one problem with a narrow handful of solutions, but it could be the neglect of aesthetic considerations in favour of rigourous reasoning and focus on objectivity exacerbates these anxieties. If so, could emphasis on the subjective aesthetic response of the student ease some of the manifestations of mathematics anxiety? Further research could consider this.

We believe it would be instructive to investigate other philosophical traditions outside the ones we examined. For example, there has been much research into the world of Islamic art, a broad tradition known for its mathematical features. This could perhaps tie some of these ideas of mathematics and art together. In any case, broadening the perspectives to other philosophical and mathematical traditions can only enrich our understanding.

As for us, four undergraduates nearing the end of our studies, we can take forward what we have learned and proceed with an enhanced appreciation for the aesthetics of mathematics, and perhaps the world beyond that.

Bibliography

AoPS Online. (2021). *Proof by Calculus*. https://artofproblemsolving.com/wiki/index.php/Proofs_of_AM-GM

Aristotle. (2021). *Metaphysics*. http://classics.mit.edu/Aristotle/metaphysics.html

Barker, J. (2009). Mathematical Beauty. *Sztuka i Filozofia*, (35), 60–74. https://bazhum.muzhp.pl/media//files/Sztuka_i_Filozofia/Sztuka_i_Filozofia-r2009-t35/Sztuka_i_Filozofia-r2009-t35-s60-74/Sztuka_i_Filozofia-r2009-t35-s60-74.pdf

Bell, C. (1913). *Art*. Frederick A. Stokes Co.

Binkley, T. (1977). Piece: Contra Aesthetics. *The Journal of Aesthetics and Art Criticism*, *35*(3), 265–277. http://www.jstor.org/stable/430287

Breitenbach, A. (2013). Beauty in Proofs: Kant on Aesthetics in Mathematics. *European Journal of Philosophy*, *23*(4), 955–977. https://doi.org/10.1111/ejop.12021

Breitenbach, A., & Rizza, D. (2017). Introduction to Special Issue: Aesthetics in Mathematics†. *Philosophia Mathematica*, *26*(2), 153–160. https://doi.org/10.1093/philmat/nkx019

Brown, J. R. (2008). *Philosophy of Mathematics: A Contemporary Introduction to the World of Proofs and Pictures*. Routledge. https://doi.org/10.4324/9780203932964

Bullen, P. S. (2003). *Handbook of Means and Their Inequalities*. Springer Science+Business Media, B.V.

Bullen, P. S., Mitrinović, D. S., & Vasić, P. M. (1988). *Means and their inequalities*. Springer Science+Business Media, B.V.

Burke, E. (1764). *A Philosophical Enquiry Into the Origin of Our Ideas: Of the Sublime and the Beautiful*. Oxford University Press UK.

Cain, A. (2010). Deus ex Machina and the Aesthetics of Proof. *The Mathematical Intelligencer*, *32*, 7–11. https://doi.org/10.1007/s00283-010-9141-z

Cellucci, C. (2019). Diagrams in Mathematics. *Found Sci*, *24*, 583–604. https://doi.org/10.1007/s10699-019-09583-x

Cvetkovski, Z. (2012). *Inequalities Theorems, Techniques and Selected Problems*. Springer.

Dowker, A., Sarkar, A., & Looi, C. Y. (2016). Mathematics Anxiety: What Have We Learned in 60 Years? *Frontiers in Psychology*, *7*. https://doi.org/10.3389/fpsyg.2016.00508

Dreyfus, T., & Eisenberg, T. A. (1986). On the Aesthetics of Mathematical Thought. *For The Learning of Mathematics*, *6*, 2–10.

Gaitanas, K. (2021). The AM-GM Inequality: A Proof to Remember. *Mathematics Magazine*, *94*(4), 302–303. https://doi.org/10.1080/0025570X.2021.1951581

Giaquinto, M. (2016). Mathematical Proofs: The Beautiful and The Explanatory. *Journal of Humanistic Mathematics*, *6*(1), 52–72. https://doi.org/10.5642/jhummath.201601.05

Griffiths, P. A. (2000). Mathematics at the Turn of the Millennium. *The American Mathematical Monthly*, *107*(1), 1–14. https://doi.org/10.1080/00029890.2000.12005154

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, *24*, 389–399. https://doi.org/10.1007/BF01273372

Hirschhorn, M. (2007). The AM-GM Inequality. *Mathematical Intelligencer*, *29*, 7. https://doi.org/10.1007/BF02986168

Inglis, M., & Aberdein, A. (2014). Beauty Is Not Simplicity: An Analysis of Mathematicians' Proof Appraisals. *Philosophia Mathematica*, *23*(1), 87–109. https://doi.org/10.1093/philmat/nku014

Jensen, J. L. W. V. (1906). Sur les fonctions convexes et les inégalités entre les valuers moyennes. *Acta Mathematica*, *30*, 175–193. https://doi.org/10.1007/BF0241857

Kelly, M. (2014). *Encyclopedia of Aesthetics*. Oxford University Press. https://doi.org/10.1093/acref/9780199747108.001.0001

Kubota, A., Hori, H., Naruse, M., & Akiba, F. (2017). A New Kind of Aesthetics —The Mathematical Structure of the Aesthetic. *Philosophies*, *2*(3). https://doi.org/10.3390/philosophies2030014

Levi, M. (2009). *The Mathematical Mechanic: using physical reasoning to solve problems*. Princeton University Press.

Manin, Y. I. (2010). *A Course In Mathematical Logic For Mathematicians*. Springer.

Mannone, M. (2018). cARTegory Theory: Framing Aesthetics of Mathematics. *Journal of Humanistic Mathematics*, *9*. https://doi.org/10.5642/jhummath.201901.16

McAllister, J. W. (2005). Mathematical Beauty and the Evolution of the Standards of Mathematical Proof, 22.

Montano, U. (2014). *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*. Springer. https://doi.org/10.1007/978-3-319-03452-2

Paterson, G. (2013). The aesthetics of mathematical proofs.

Poincaré, H. (1952). *Science and method / translated by francis maitland*. Dover Publications.

Rasheed, T., Butt, S. I., Pečarić, D., Pečarić, J., & Akdemir, A. (2021). Uniform Treatment of Jensen's Inequality by Montgomery Identity. *Journal of Mathematics*, *2021*(3), 1–17. https://doi.org/10.1155/2021/5564647

Renteln, P., & Dundes, A. (2005). Foolproof: A Sampling of Mathematical Folk Humor. *Notices of the American Mathematical Society*, *52*(1), 24–34.

Richardson, F. C., & Suinn, R. M. (1972). The Mathematics Anxiety Rating Scale: Psychometric data. *Journal of Counseling Psychology*, *19*(6), 551.

Rota, G.-C. (2014). *The Phenomenology of Mathematical Beauty - Proof and Progress in Mathematics*. Springer.

Russell, B. (1920). *Introduction to Mathematical Philosophy*. George Allen & Unwin, Ltd.

Sangwin, C. J., & Kinnear, G. (2021). *Investigating insight and rigour as separate constructs in mathematical proof* (Working Paper). EdArXiv. https://doi.org/10.35542/osf.io/egks4

Shelley, J. (2022). The Concept of the Aesthetic. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Spring 2022). Metaphysics Research Lab, Stanford University.

Shirali, S. (2002). *Adventures In Problem Solving*. Sangam Books Ltd.

Sinclair, N. (2004). The Roles of the Aesthetic in Mathematical Inquiry. *Mathematical Thinking and Learning*, *6*, 261–284. https://doi.org/10.1207/s15327833mtl0603_1

Stout, L. N. (1999). Aesthetic Analysis of Proofs of the Binomial Theorem.

Wang, W. (2019). Artistic Proofs: A Kantian Approach to Aesthetics in Mathematics. *Estetika*, *56*(2), 223–243.

Wells, D. (1990). Are These the Most Beautiful? *The Mathematical Intelligencer*, *12*, 37–41.

Zangwill, N. (2021). Aesthetic Judgment. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Winter 2021). Metaphysics Research Lab, Stanford University. https://plato.stanford.edu/archives/win2021/entries/aesthetic-judgment/