A SURVEY ON TORIC DEGENERATIONS OF PROJECTIVE VARIETIES

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Abstract. In this survey we summarize the constructions of toric degenerations obtained from valuations and Gröbner theory and describe in which sense they are equivalent. We show how adapted bases can be used to generalize the classical Newton polytope to what we call a $\mathbb{B}$-Newton polytope. The $\mathbb{B}$-Newton polytope determines the Newton–Okounkov polytopes of all Khovanskii-finite valuations sharing the adapted standard monomial basis $\mathbb{B}$.

1. Introduction

Toric varieties are popular objects in algebraic geometry due to a dictionary between their geometric properties (e.g. dimension, degree) and properties of associated combinatorial objects (e.g. fans, polytopes). This dictionary can be extended from toric varieties to varieties admitting a *toric degeneration*. A toric degeneration is a (flat) family of varieties that share many properties with each other (e.g. dimension, degree, Hilbert-polynomial). This family contains the variety we are interested in, a toric variety, so properties of our variety can be read from the combinatorics of the toric variety.

The study of toric degenerations has various applications in pure and applied mathematics, e.g. in probability, statistics, and mathematical biology. Tailored to the variety of interest, it is a great challenge to decide which toric degeneration has the desired properties. The task is therefore to study and compare possible constructions.

Besides its applications in classical algebraic geometry, toric degenerations have proven to be useful in several other subjects such as

- Symplectic geometry [NNU12, HK14, FLP18, HP18, HK15, Kav19],
- Newton–Okounkov bodies [LM09, KK12, KL17],
- Representation theory [GL96, Cal02, AB05, KM05, HJL$^+$09, FFL11, Kav15, FFL17],
- Mirror symmetry [Giv97, BCFKvS00, Bat04, GS11, FOOO11, ACGK12, Nis15],
- Cluster algebras [GHKK18, BFF$^+$18, RW19, BFMN20],
- Numerical and computational algebraic geometry [CM97, BLMM17, BSW20]
- Algebraic statistics [KMS15, Ber17]

The above list and the included citations are far from being complete as the subject is broad and new applications are discovered on a regular basis. I apologize if I have missed your favorite paper using toric degenerations and I would be happy to receive emails with hints to more exiting applications.

The aim of this survey is to describe two main constructions of toric degenerations and how they are related. In particular, we focus on the constructions from valuations which go back to Anderson [And13] and those from Gröbner theory or the tropicalization of an ideal. In practice the *bridges* connecting one construction to another are particularly useful as each approach has its own benefits and shortcomings.

To be more precise consider a projective variety X. A **toric degeneration of** X is a flat morphism $\phi : \mathfrak{X} \to \mathbb{A}^1$ that trivializes away from the fibre over $0 \in \mathbb{A}^1$

$$\begin{tikzcd}
\mathfrak{X} \setminus \phi^{-1}(0) \arrow[rr, "\sim"] \arrow[dr, "\phi"'] & & X \times \mathbb{A}^1 \setminus \{0\} \arrow[dl] \\
& \mathbb{A}^1 \setminus \{0\} &
\end{tikzcd}$$

We will work with T-equivariant toric degenerations, that is we assume that the action on the central fibre is an extension of the torus action on X.

Outline: We summarize the background on valuations in §2.1 and on Gröbner theory and tropicalization of ideals in §2.2. In §3 we explain the equivalence of the constructions of toric degenerations from valuations and from the tropicalization of an ideal [KM19, Bos21b]. In §3.3 we consider more general *algebraic toric degenerations* and under which circumstances they can be obtained as the toric degenerations from a valuation [KMM17]. In §4 we explain the importance of adapted bases for toric degenerations. In particular, in §4.1 we show how adapted bases give rise to $\mathbb{B}$-Newton polytopes that project to all Newton–Okounkov polytopes of valuations that share an adapted basis. In §4.2 we recall the definition of wall-crossing formulas for Newton–Okounkov polytopes [EH20]. We review this notion from a more geometric point of view in the context of flat families incorporating various toric degenerations in §5, [BMNC21]. In §6 we elaborate on the example of the Grassmannian $\mathrm{Gr}_3(\mathbb{C}^6)$.

Acknowledgements: I would like to thank the editors of the proceeding for the opportunity to contribute with a survey on my favorite topic. The ideas of §4.1 arose in the context of Newton–Okounkov bodies for cluster varieties in collaboration with Man-Wai Cheung, Timothy Magee and Alfredo Nájera Chávez. I am grateful for all the inspiring discussions over the course of our collaboration.

2. Preliminaries

In this section we introduce the main tools used in this article to construct toric degenerations. In particular we review background on valuations and Newton–Okounkov bodies as well as background on initial ideals, Gröbner theory and the tropicalization of an ideal. We use the maximum convention throughout the paper which might imply slight differences in the definitions (mostly just a sign) in comparison to the original articles cited.

2.1. **Valuations.** Let k be an algebraically closed field of characteristic zero. Throughout the paper we denote by A a **positively (multi-)graded algebra and domain**, that is $A = \bigoplus_{w\in\mathbb{Z}^m_{\geq 0}} A_w$. Let Γ be an abelian group that is totally ordered by $<$. By a **(Krull) valuation** on A we mean a map $\nu : A \setminus \{0\} \to \Gamma$ that satisfies for all $a, b \in A$ and $c \in k$

$$\nu(ab) = \nu(a) + \nu(b), \quad \nu(a+b) \leq \max\{\nu(a), \nu(b)\}, \quad \nu(ca) = \nu(a).$$

If ν only satisfies $\nu(ab) \leq \nu(a) + \nu(b)$ it is called a **quasivaluation**. Notice that the image of a valuation ν carries the structure of an additive semigroup. It is therefore called the **value semigroup** of ν and we denote it by $S(A, \nu)$. The **rank** of ν is defined as the rank of the group completion of its semigroup inside Γ, ν is said to have **full rank** if its rank coincides with the Krull dimension of A. Every valuation induces a filtration on A with filtered pieces for $\gamma \in \Gamma$ defined by

$$\mathcal{F}_{\nu,\gamma} := \{a \in A : \nu(a) \geq \gamma\} \quad (\text{resp. } \mathcal{F}_{\nu,>\gamma} := \{a \in A : \nu(a) > \gamma\}).$$

The **associated graded algebra** is $\mathrm{gr}_\nu(A) := \bigoplus_{\gamma\in\Gamma} \mathcal{F}_{\nu,\gamma}/\mathcal{F}_{\nu,>\gamma}$. There is a natural quotient map of vector spaces from A to $\mathrm{gr}_\nu(A)$ given by sending $f \in A$ to $\mathcal{F}_{\nu,\nu(f)}/\mathcal{F}_{\nu,>\nu(f)}$, denote its image by $\hat{f} \in \mathrm{gr}_\nu(A)$. Note that $\nu(fg) = \nu(f) + \nu(g)$ implies that $\widehat{fg} = \hat{f}\hat{g}$. If the quotients $F_{\nu,\gamma}/\mathcal{F}_{\nu,>\gamma}$ are at most one-dimensional, then we say ν has **one-dimensional leaves**. This property is desirable as it gives an identification

$$\mathrm{gr}_\nu(A) \to k[S(A, \nu)], \quad \text{given by} \quad \hat{f}_\gamma \mapsto \nu(f_\gamma),$$

where $\hat{f}_\gamma \in \mathcal{F}_{\nu,\gamma}/\mathcal{F}_{\nu,>\gamma}$ is a generator and $f_\gamma \in A$ lies in the preimage of $\hat{f}_\gamma$ under the quotient map $\hat{}: A \to \mathrm{gr}_\nu(A)$. It is a consequence of Abhyankar's inequality that full-rank valuations have one-dimensional leaves.

An important definition is the notion of a **Khovanskii basis** for a valuation ν: that is a subset B of A whose image in $\mathrm{gr}_\nu(A)$ is an algebra generating set. It is not hard to see that if B is a Khovanskii basis for ν then the set $\{\nu(b) : b \in B\}$ generates the value semigroup [KM19, Lemma 2.10].

A valuation is called **homogeneous** if it respects the grading on A, more precisely if $f \in A$ has homogeneous presentation $\sum_i f_i$ then $\nu(f) = \max\{\nu(f_i)\}$. A valuation is **fully homogeneous** if $\nu(f) = (\deg(f), \nu'(f))$, that is $S(A, \nu) \subset \mathbb{Z}^m_{\geq 0} \times \Gamma'$. Any homogeneous valuation is obtained from a fully homogeneous one by composing with an isomorphism of semigroups [IW20, Remark 2.6]. So when studying homogeneous valuation we may without loss of generality assume they are fully homogeneous.

Given a fully homogeneous valuation $\nu : A \setminus \{0\} \to \mathbb{Z}^m_{\geq 0} \times \Gamma'$ we define its **Newton–Okounkov cone**

$$C(A,\nu) := \overline{\mathrm{cone}(S(A,\nu))} = \overline{\mathrm{cone}(\nu(f) : f \in A)}, \tag{1}$$

where the closure (in the Euclidean topology) is taken inside $(\mathbb{Z}^m_{\geq 0} \times \Gamma') \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\Gamma'_{\mathbb{R}} = \Gamma' \otimes_{\mathbb{Z}} \mathbb{R}$. The **Newton–Okounkov body** of ν is then defined as the intersection

$$\Delta(A,\nu) := C(A,\nu) \cap \{(1,\dots,1)\} \times \Gamma'_{\mathbb{R}}, \tag{2}$$

where $(1,\dots,1)$ denotes the element whose entries are all one in $\mathbb{Z}^m_{\geq 0}$. The definition was introduced independently by Lazarsfeld–Mustata [LM09] and Kaveh–Khovanskii [KK12] who based their work on a construction of Okounkov [Oko98]. Newton–Okounkov bodies far generalize Newton polytopes of polynomials and carry a lot of information about the algebra A or the (weighted)projective variety $X = \mathrm{Proj}(A)$[1].

Theorem 2.1 (Corollary 3.2 [KK12])**.** *Let $X = Proj(A)$ and $\nu : A \setminus \{0\} \to \mathbb{Z}^d$ be a full-rank homogeneous valuation. Then the dimension q of $\Delta(A,\nu)$ coincides with the dimension of X and moreover, the q-dimensional integral volume of $\Delta(A,\nu)$ multiplied by $q!/\mathrm{ind}(S(A,\nu))$ is the degree of X, where $\mathrm{ind}(S(A,\nu))$ refers to the index of the sublattice spanned by $S(A,\nu)$ inside $\mathbb{Z}^d$.*

In general the Newton–Okounkov body of a valuation need not be bounded nor polyhedral, but they are convex. Computing them is in general challenging, but much simplified when the valuation posses a finite Khovanskii basis as we will see in what follows.

2.1.1. *Khovanskii-finite valuations.* A **Khovanskii-finite valuation** is a homogeneous (Krull) valuation of full rank whose value semigroup is finitely generated. In particular, Khovanskii-finite valuations have finite Khovanskii bases. The concept was introduced and studied in great detail by Ilten and Wrobel in [IW20].

The existence of Khovanskii bases has computational advantages. Given a Khovanskii basis $\{b_1,\dots,b_n\}$ for ν we may represent $\mathrm{gr}_\nu(A)$ as a quotient of a polynomial ring $S := k[x_1,\dots,x_n]$. Define

$$\pi_\nu : k[x_1,\dots,x_n] \to \mathrm{gr}_\nu(A) \quad \text{by} \quad x_i \mapsto b_i.$$

Then $I_\nu := \ker(\pi_\nu)$ gives $S/I_\nu \cong \mathrm{gr}_\nu(A)$.

We say the value semigroup $S(A,\nu) \subset (\Gamma,<)$ is **minimum well-ordered** if every subset of $S(A,\nu)$ has a unique minimal element with respect to $<$. In this case by [KM19, Proposition 2.13] the following version of the subduction algorithm terminates in finite time.

Algorithm 2.2. *Let A be positively graded algebra and domain, $\nu : A \setminus \{0\} \to (\Gamma,<)$ full-rank homogeneous Khovanskii-finite valuation with minimum well-ordered $S(A,\nu)$ with $\{b_1,\dots,b_n\}$ a Khovanskii basis.*

Input: $f \in A \setminus \{0\}$;
Output: *a polynomial expression of f in terms of $\{b_1,\dots,b_n\}$.*

1. *As $\bar{b}_1,\dots,\bar{b}_n$ generate $\mathrm{gr}_\nu(A)$ we may find a polynomial expression for $\bar{f}$ in terms of $\bar{b}_1,\dots,\bar{b}_n$: $\bar{f} = p(\bar{b}_1,\dots,\bar{b}_n)$, here $p \in \pi_\nu^{-1}(\bar{f})$.*
2. *We distinguish two cases*
   - a. *If $f = p(b_1,\dots,b_n)$ then* **output** *p;*
   - b. *If $\nu(f - p(b_1,\dots,b_n)) < \nu(f)$ replace f by $f - p(b_1,\dots,b_n)$ and go back to Step 1.*

In particular, every Khovanskii basis is a generating set for the algebra.

[1] Recall, that the projective spectrum of the $\mathbb{Z}^m_{\geq 0}$-graded polynomial ring whose generators have degrees $d_i e_i$, $\{e_1,\dots,e_m\}$ being the standard basis of $\mathbb{Z}^m$, is the weighted projective space $\mathbb{P}(d_1,\dots,d_m)$. In particular, if A is multigraded, $\mathrm{Proj}(A)$ can be seen as a subvariety of a weighted projective space. For details we refer to [Dol82].

2.2. **Initial ideals and tropicalization.** Our second tool box for toric degenerations comes from Gröbner theory. For more detailed information we refer to [HH11, CLO15, Eis13, Stu96].

For $m \in \mathbb{Z}_{\geq 0}^n$ we write $x^m := x_1^{m_1} \cdots x_n^{m_n} \in k[x_1, \dots, x_n]$. A total order on the set of monomials in $S := k[x_1, \dots, x_n]$ is a **term order** if it satisfies:

$$(i)\ 1 < x^m\ \forall m \in \mathbb{Z}_{\geq 0}^n \setminus \{0\} \quad \text{and} \quad (ii)\ x^a < x^b \Rightarrow x^{a+c} < x^{b+c}\ \forall a, b, c \in \mathbb{Z}_{\geq 0}^n.$$

The **leading term** of an element $f = \sum c_a x^a \in S$ with respect to a term order $<$ is $\mathrm{in}_<(f) = c_b x^b := \max_<\{c_a x^a : c_a \neq 0\}$, where c_b is called the **leading coefficient** and x^b is called the **leading monomial**. For an ideal in $I \subset S$ we define its **initial ideal with respect to** $<$ as

$$\mathrm{in}_<(I) := (\mathrm{in}_<(f) : f \in I).$$

The initial ideal is finitely generated and a generating set G of I that satisfies $(\mathrm{in}_<(g) : g \in G) = \mathrm{in}_<(I)$ is called a **Gröbner basis of I with respect to** $<$. Every ideal possesses only a finite number of distinct initial ideals [Stu96, Theorem 1.2]. It has been shown by Mora and Robbiano that the initial ideals can be organized in a polyhedral fan [MR88]. To see how, we need the notion of initial ideals with respect to weight vectors: fix $w \in \mathbb{R}^n$, we call it a **weight vector** and define the **initial form** of an element $f = \sum c_a x^a$ with respect to w as

$$\mathrm{in}_w(f) = \sum_{b:\ w \cdot b = \max\{w \cdot a : c_a \neq 0\}} c_b x^b.$$

Notice that depending on w and f the initial form $\mathrm{in}_w(f)$ is not necessarily just one term. Similarly, we define the **initial ideal** of I **with respect to** w as $\mathrm{in}_w(I) := (\mathrm{in}_w(f) : f \in I)$. For any weight vector w we may define the **homogenization** of I in $k[x_1, \dots, x_n, t]$: for a single element $f = \sum c_a x^a$ we set

$$f^{h;w} := \sum c_a x^a t^{\max\{w \cdot b : c_b \neq 0\} - w \cdot a}.$$

Similarly, for the ideal I we define $I^{h;w} := (f^{h;w} : f \in I)$. The homogenization of I is a family of deformations of I and the quotient algebra $A^{h;w} := k[x_1, \dots, x_n, t]/I^{h;w}$ is a free $k[t]$-module [Eis13, §15.8]. Let $A^w := S/\mathrm{in}_w(I)$. The degeneration of $\mathrm{Spec}(A)$ to $\mathrm{Spec}(A^w)$ defined by $\mathrm{Spec}(A^{h;w})$ is called a **Gröbner degeneration**.

Given the ideal I any term order can be **represented** by a weight vector in $w \in \mathbb{Z}_{>0}^n$ (see, e.g. [HH11, Lemma 3.1.1]), that is $\mathrm{in}_w(I) = \mathrm{in}_<(I)$. Conversely, a weight vector w belongs to the **Gröbner region** $\mathrm{GR}(I)$ if there exists a term order $<$ such that $\mathrm{in}_<(\mathrm{in}_w(I)) = \mathrm{in}_<(I)$. The Gröbner region carries a fan structure, called the **Gröbner fan** $\mathrm{GF}(I)$ that was discovered by Mora and Robbiano in [MR88]. Two weight vectors $v, w \in \mathbb{R}^n$ lie in the relative interior of a cone C, denoted by $v, w \in C^\circ$, if and only if $\mathrm{in}_v(I) = \mathrm{in}_w(I)$. The maximal dimensional cones in $\mathrm{GF}(I)$ correspond to monomial initial ideals associated with term orders on S. These are particularly useful as they induce vector space bases for the quotient algebra $A = S/I$: we call a monomial x^a that is *not* contained in $\mathrm{in}_<(I)$ a **standard monomial**. The set $\mathbb{B}_< := \{\bar{x}^a \in A : x^a \notin \mathrm{in}_<(I)\}$ is a vector space basis of A called a **standard monomial basis**. In fact, if $w \in C$ for some maximal cone $C \in \mathrm{GF}(I)$ associated to $<$, then $\mathbb{B}_<$ is a basis for the free $k[t]$-module $A^{h;w}$, see e.g. [Eis13, Proof of Theorem 15.17].

The Gröbner fan has an interesting subfan that will lead us back to valuations on A: we define the **tropicalization** of I:

$$\mathrm{Trop}(I) := \{w \in \mathrm{GR}(I) : \mathrm{in}_w(I) \text{ does not contain monomials}\}.$$

The dimension of $\mathrm{Trop}(I)$ coincides with the Krulldimension of A [EKL06]. We may in fact reduce our attention to homogeneous ideals according to [BJS+07, Lemma 4]. For homogeneous ideals we have $\mathrm{GR}(I) = \mathbb{R}^n$ by [Stu96, Proposition 1.12]. In this case the tropicalization is a pure fan whose dimension coincides with the Krull-dimension of A. Moreover, $\mathrm{Trop}(I)$ and $\mathrm{GF}(I)$ have a linear subspace $\mathcal{L}_I$, called the **lineality space** which consists of elements $w \in \mathbb{R}^n$ such that $\mathrm{in}_w(I) = I$. More precisely we have the following straight forward Lemma:

Lemma 2.3. *Let I be a (multi-)homogeneous ideal inside S with respect to a $\mathbb{Z}_{\geq 0}^m$-grading given by $\deg(x_{i_j}) = e_i$ where $S = k[x_{i_j} : 1 \leq i \leq m, 1 \leq j \leq k_i]$ for some k_i that satisfy $k_1 + \cdots + k_m = n$ and $\{e_i : 1 \leq i \leq m\}$ is the standard basis of $\mathbb{Z}^m$. Then for $1 \leq i \leq m$ we have*

$$\ell_i := (0, \dots, 0, 1, \dots, 1, 0 \dots, 0) \in \mathcal{L}_I \tag{3}$$

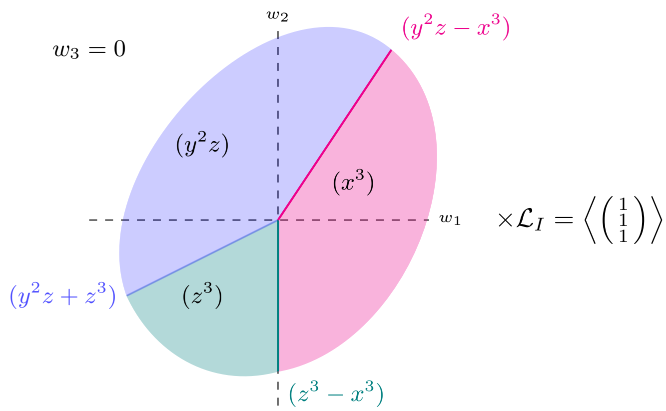


FIGURE 1. The Gröbner fan of $I = (y^2z - x^3 + z^3) \subset \mathbb{C}[x, y, z]$ modulo $\mathcal{L}_I$, its one-skeleton is $\mathrm{Trop}(I)$, and all initial ideals.

where the 1*'s appear in the positions* $i_1, \ldots, i_{k_i}$*.*

Among the maximal cones of $\mathrm{Trop}(I)$ we may look for **prime cones** whose associated initial ideal is binomial and prime. Hence, their vanishing sets define toric varieties. In particular, any Gröbner degeneration associated to a weight vector in the interior of a maximal prime cone is in fact a **toric degeneration**.

Example 2.4. *Consider* $I = (y^2z - x^3 + z^3) \subset \mathbb{C}[x, y, z]$*. The lineality space* $\mathcal{L}_I$ *is one dimensional generated by* $(1, 1, 1)^T$*. We draw the Gröbner fan* $\mathrm{GF}(I)$ *modulo* $\mathcal{L}_I$ *inside the hyperplane* $\{(w_1, w_2, w_3)^T \in \mathbb{R}^3 : w_3 = 0\}$ *in Figure 1. The one skeleton whose maximal cones correspond to the rays in the above picture is the tropicalization of* I*.*

2.2.1. *Initial ideals with respect to weighting matrices.* Before moving on to the next section we need a bit more background on a slight generalization of initial ideals: a *higher dimensional analogue of Gröbner theory* (see e.g. [FR16]).

Recall that $S = k[x_1, \ldots, x_n]$ and consider as before $f = \sum c_a x^a \in S$. We call a matrix $M \in \mathbb{Q}^{d\times n}$ a **weighting matrix** and together with a linear order $\prec$ on $\mathbb{Z}^d$ we define the **initial form** of f with respect to M as

$$\mathrm{in}_M(f) := \mathrm{in}_{M,\prec}(f) := \sum_{b:\ Mb=\max_\prec\{Ma: c_a\neq 0\}} c_b x^b.$$

As before we define the **initial ideal** of an ideal $I \subset S$ **with respect to** M **(and** $\prec$**)** as $\mathrm{in}_{M;\prec}(I) := (\mathrm{in}_M(f) : f \in I)$. To simplify notation we will drop the linear order from the index and simply assume that we have fixed it once and for all. The Gröbner region also has a higher dimensional analogue: the $d^{\mathbf{th}}$ **Gröbner region** is denoted $\mathrm{GR}^d(I)$ and defined as the set of all weighting matrices $M \in \mathbb{Q}^{d\times n}$ such that there exists a term order $<$ on S with $\mathrm{in}_<(I) = \mathrm{in}_<(\mathrm{in}_M(I))$. Given $<$ let $C^d_< \subset \mathrm{GR}^d(I)$ be the set of all M satisfying the previous relation. We may also define equivalence classes of weighting matrices by setting $C_M := \{M' \in \mathrm{GR}^d(I) : \mathrm{in}_M(I) = \mathrm{in}_{M'}(I)\}$. In the higher dimensional case several features of Gröbner theory still hold, among these the existence of standard monomial bases. For example, $\mathrm{GR}^d(I)$ always contains the positive orthant $\mathbb{Q}^{d\times n}_{\geq 0}$ and if I is homogeneous we have $\mathrm{GR}^d(I) = \mathbb{Q}^{d\times n}$ (see [KM19, Lemma 8.7] but be aware that the authors are using the minimum convention which introduces a sign).

We may use weighting matrices to define quasivaluations as follows. Consider the quotient map $\pi : S \to S/I =: A$ and denote by $\bar{f}$ the coset of f in the quotient. For $f = \sum c_a x^a \in S$ set $\tilde{\nu}_M(f) := \max_\prec\{Ma : c_a \neq 0\}$. This defines a valuation $\tilde{\nu}_M : S \setminus \{0\} \to \mathbb{Z}^d$. By [KM19, Lemma 3.2] there exists a quasivaluation $\nu_M : A \setminus \{0\} \to (\mathbb{Z}^d, \prec)$ given for $\bar{f} \in A$ by

$$\nu_M(\bar{f}) = \min_\prec\{\tilde{\nu}_M(f) : f \in \bar{f}\}$$

called the **quasivaluation with weighting matrix** M. Its associated graded algebra, denoted $\mathrm{gr}_M(A)$, satisfies $\mathrm{gr}_M(A) \cong S/\mathrm{in}_M(I)$. In particular, this isomorphism gives us standard monomial bases for $\mathrm{gr}_M(A)$: let $<$ be a term order with $M \in C^d_<$. Then $\mathbb{B}_<$ is a vector space basis for $\mathrm{gr}_M(A)$. Moreover, we may use $\mathbb{B}_<$ to compute the values of ν_M: for $\bar f \in A$ let $\bar f = \sum_{\bar x^b \in \mathbb{B}_<} c_b \bar x^n$ be its expression in $\mathbb{B}_<$. Then

$$\nu_M(\pi(f)) = \max_{\prec}\{Mb : c_b \neq 0\}.$$

We explore standard monomial bases and their influence on valuations further in §4.

3. Valuations, tropicalization and toric degenerations

In this section we merge the concepts of Khovanskii-finite valuations and the tropicalization of an ideal. This section is based on results in [KM19] and [Bos21b].

3.1. **Valuations from tropicalization.** The main aim of Kaveh and Manon in [KM19] is to establish a connection between the toric degenerations from prime cones in a tropicalization to toric degenerations obtained from Newton–Okounkov polytopes. It relies on the quasivaluations with weighting matrices introduced above.

As before, let $I \subset S$ be a homogeneous ideal. Suppose there exists a maximal prime cone $\tau \in \mathrm{Trop}(I)$ and choose a basis $r_1, \dots, r_d \in \mathbb{Q}^n$ for the real vector space spanned by τ. The quotient $\tau/\mathcal{L}_I$ is a strongly convex cone (see e.g. [BMNC21, Lemma 2.13]) so we may take a maximal linearly independent set of cosets of primitive ray generators of $\tau/\mathcal{L}_I$. Together with a basis of the lineality space this will be our choice for $\mathbf{r} := \{r_1, \dots, r_d\}$. In particular, we set $r_i = \ell_i$ for $1 \le i \le m$, see Lemma 2.3. Define

$$M_{\mathbf{r}} := (r_{ij})_{1\le i\le d, 1\le j\le n}$$

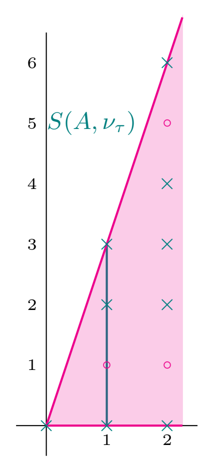
where r_{ij} is the j^{th} entry in r_i, so the r_i are the rows of $M_{\mathbf{r}}$.

Proposition 3.1 (Proposition 4.2 and 4.6 in [KM19])**.** *If τ is a maximal prime cone in* $\mathrm{Trop}(I)$ *then quasivaluation with weighting matrix $M_{\mathbf{r}}$ is in fact a full rank valuation with one-dimensional leaves. Its value semigroup is generated by the images of $\bar x_1, \dots, \bar x_n$ and it only depends on τ, not on our choice of* $\mathbf{r}$.

Given the Proposition we adopt our notation and set $M_\tau := M_{\mathbf{r}}$ and $\nu_\tau := \nu_{M_\tau}$. We obtain the following corollary about the associated Newton–Okounkov polytopes:

Corollary 3.2 (Corollary 4.7 in [KM19])**.** *The Newton–Okounkov body of the valuation ν_τ is a convex polytope whose vertices are $\nu_\tau(\bar x_1), \dots, \nu_\tau(\bar x_n)$, which are exactly the columns of M_τ. Moreover, up to linear isomorphism $\Delta(A, \nu_\tau)$ only depends on τ.*

Notice that for I homogeneous with respect to the standard grading and $r_1 = (1, \dots, 1)$ as above the Corollary implies that (up to linear isomorphism) we have $\Delta(A, \nu) \subset \{1\} \times \mathbb{R}^{d-1}$.



Example 3.3. *Consider $I = (y^2z - x^3 + z^3) \subset \mathbb{C}[x, y, z]$ from Example 2.4 above. The initial ideal $(y^2z - x^3)$ corresponding to the ray $(2, 3, 0)^T$ in Figure 1 is a maximal prime cone τ in* $\mathrm{Trop}(I)$. *The associated ray matrix is $M_\tau = \left(\begin{smallmatrix}1&1&1\\2&3&0\end{smallmatrix}\right)$. Let $A = \mathbb{C}[x, y, z]/I$, then the valuation $\nu_\tau : A \setminus \{0\} \to \mathbb{Z}_{\ge 0} \times \mathbb{Z}$ satisfies $\nu_\tau(\bar x) = (1, 2), \nu_\tau(\bar y) = (1, 3)$ and $\nu_\tau(\bar z) = (1, 0)$. In particular, the value semigroup of ν_τ is generated by these three elements and depicted on the right.*

Here $\times$ denotes the lattice points in $S(A, \nu_\tau)$ and $\circ$ denotes lattice points not contained in $S(A, \nu_\tau)$. The shaded region is the Newton–Okounkov cone $C(A, \nu_\tau)$ and the line segment connecting $(1, 0)^T$ and $(1, 3)^T$ is the Newton–Okounkov polytope $\Delta(A, \nu_\tau)$. Note that $S(A, \nu_\tau)$ is not saturated: for example, $(2, 2)^T$ is in $S(A, \nu_\tau)$, but $(1, 1)^T$ is not.

3.2. **Tropicalization from valuations.** Fix a Khovanskii-finite valuation $\nu : A \setminus \{0\} \to \mathbb{Z}^d$ on a positively graded algebra and domain A. We may assume without loss of generality that $\dim_{\mathrm{Krull}}(A) = d$ (if this was not the case we may apply [BG09, Proposition 2.17(e)] as the image of ν is in fact a monoid whose only unit is 0). Moreover, we may assume that the underlying total order on $\mathbb{Z}^d$ is the lexicographic order (if this was not the case we may follow Mora and Robbiano [MR88] and represent the order by a $d \times s$ matrix M such that our order coincides with the lexicographic order on $M\mathbb{Z}^d$). Choose a finite generating set $a_1, \dots, a_n$ for the value semigroup $\nu(A \setminus \{0\})$ and let M_ν be $d \times n$ matrix whose columns are $a_1, \dots, a_n$. Notice that

$$\mathrm{rank}(M_\nu) = \dim(\mathrm{im}(M_\nu) = \dim(\mathrm{span}_{\mathbb{Z}}(a_1, \dots, a_n)) = \dim(\mathrm{cone}(a_1, \dots, a_n)) = \mathrm{rank}(\nu).$$

In particular, M_ν is of full rank.

Lemma 3.4. *Given the generators $a_1, \dots, a_n$ of the value semigroup $S(A, \nu)$ choose $b_1, \dots, b_n \in A$ with $\nu(b_i) = a_i$. Then the set $\{b_1, \dots, b_n\}$ is a Khovanskii basis.*

Proof. As $k[S(A, \nu)] \cong \mathrm{gr}_v(A)$ the elements $a_1, \dots, a_n$ form a set of algebra generators for $\mathrm{gr}_v(A)$. □

Notice further that for dimension reasons the Khovsankii basis $\{b_1, \dots, b_n\}$ is a generating set for A as ν is full-rank. Hence, we may use it to obtain a presentation of A. Define

$$\pi : S := k[x_1, \dots, x_n] \to A, \quad \text{by} \quad x_i \mapsto b_i.$$

Notice that $b_1, \dots, b_n$ not necessarily are of degree one in A. To have a *graded* presentation of A we endow the polynomial ring with a (not necessarily standard) grading given by $\deg(x_i) := \deg(b_i)$. Then $I := \ker(\pi)$ is a homogeneous ideal and $S/I \cong A$. Our valuation ν is intimately related with the tropicalization of the ideal I.

Theorem 3.5 ([Bos21b])**.** *Let $\nu : A \setminus \{0\} \to \mathbb{Z}^d$ be a full rank valuation with finitely generated value semigroup and let $S/I \cong A$ be the presentation induced by a Khovanskii basis $b_1, \dots, b_n$.*

Then there exists a maximal prime cone $\tau \in \mathrm{Trop}(I)$ such that $\nu = \nu_\tau$ and

$$k[S(A, \nu)] \cong \mathrm{gr}_\nu(A) \cong S/\,\mathrm{in}_\tau(I).$$

Proof. Notice that $M := M_\nu$ the weighting matrix of ν is of full rank $d \leq n$ as ν is of full rank. Then by [Bos21b, Theorem 1] the initial ideal $\mathrm{in}_M(I)$ is prime as the value semigroup $S(A, \nu)$ is generated by $\nu(b_1), \dots, \nu(b_n)$. Here we use the total order on $\mathbb{Z}^d$ given by ν. We may restrict our attention to the case of usual initial ideals as by [Bos21b, Lemma 3] there exists a weight vector $w \in \mathbb{Z}^n$ such that $\mathrm{in}_w(I) = \mathrm{in}_M(I)$. It is left to show that

(1) $w \in \mathrm{Trop}(J)$;
(2) $k[S(A, \nu)] \cong S/\,\mathrm{in}_\tau(J)$, where $w \in \tau^\circ$.

The first item follows from [Bos21b, Corollary 3]. For the second consider the quasivaluation ν_M. By Proposition 2.1 ν_M is a valuation and by [Bos21b, Proposition 1] it satisfies $\nu = \nu_M$. Further, by [Bos21b, Equation 3.3] we have

$$S/\,\mathrm{in}_w(I) \cong \mathrm{gr}_\nu(A) \cong k[S(A, \nu)],$$

which finishes the proof. □

One nice feature of this connection is that it may be used to characterize when a toric degeneration is a Gröbner degeneration. We explore this direction further in the following subsection.

Theorem 3.5 depends on a choice of Khovanskii basis, so naturally one may ask how strong this dependence is. If we change the Khovanskii basis, the presentation of A changes and so does the tropicalization.

Proposition 3.6. *Assume $\mathcal{B} := \{b_1, \dots, b_n\}$ and $\mathcal{B}' := \{b'_1, \dots, b'_n\}$ are two Khovsankii bases of (A, ν). Let I and I' be the ideals presenting A and let τ and τ' be the cones in the corresponding tropicalizations from Theorem 3.5. Then there exists an ideal $\tilde{I} \subset k[y_1, \dots, y_m]$ for some $m \geq n$ presenting A and projections $p : \mathbb{R}^m \to \mathbb{R}^n$ and $p' : \mathbb{R}^m \to \mathbb{R}^n$ such that for a maximal prime cone $\tilde{\tau} \in \mathrm{Trop}(\tilde{I})$ we have $p(\tilde{\tau}) = \tau$ and $p'(\tilde{\tau}) = \tau'$.*

Proof. We have two presentations of A:

$$\pi : S \to A, \pi(x_i) = b_i \quad \text{and} \quad \pi' : S \to A, \pi'(x_i) = b_i'$$

given by $\ker(\pi) = I$ and $\ker(\pi') = I'$. To see how the two tropicalizations $\mathrm{Trop}(I)$ and $\mathrm{Trop}(I')$ are related we proceed recursively and introduce another presentations of A given by $\mathcal{B} \cup \mathcal{B}'$. For simplicity assume $b_i = b_i'$ for all $i < n$. Consider $\tilde{\pi} : k[x_1, \dots, x_{n+1}] \to A$ given by $x_i \mapsto b_i, x_{n+1} \mapsto b_n'$ and let $\tilde{I} := \ker(\tilde{\pi})$. Let $p : k[x_1, \dots, x_{n+1}] \to k[x_1, \dots, x_n]$ and $p' : k[x_1, \dots, x_{n-1}, x_{n+1}]$ be the natural projections. By construction we have $I \subset p(\tilde{I})$ and $I' \subset p'(\tilde{I})$. Let $\tilde{\tau} \in \mathrm{Trop}(\tilde{I})$ be the maximal prime cone given by Theorem 3.5. Then the corresponding projections p and p' from $\mathbb{R}^{n+1} \to \mathbb{R}^n$ have the desired properties. $\square$

Proposition 3.6 invites us to change our point of view: suppose we have two Khovanskii-finite valuations ν and ν' on A with two different Khovanskii bases $\mathcal{B}$ and $\mathcal{B}'$. We may use the proof of Proposition 3.6 to construct a tropicalization that contains simultaneously prime cones corresponding to ν and ν'. This idea is closely related to a procedure in [BLMM17] for constructing new prime cones by changing the presentation of A ([BLMM17, Procedure 1]). It was shown in [IW20] that for non Khovanskii-finite valuations the above mentioned procedure does not terminate. We further elaborate on these ideas in the example of the Grassmannian $\mathrm{Gr}_3(\mathbb{C}^6)$ in §6.

3.3. **Which toric degenerations are Gröbner?** Theorem 3.5 shows that toric degenerations induced by Khovanskii-finite valuations equivalently arise as Gröbner degenerations from the tropicalization of an adequate ideal. Naturally we may extend the question and ask which toric degenerations are Gröbner degenerations.

Recall the definition of a toric degeneration from §1. Assume X is a projective variety and we have $X = \mathrm{Proj}(A)$. Given $\nu : A \setminus \{0\} \to \mathbb{Z}^d$ a Khovanskii-finite valuation, $\mathrm{gr}_\nu(A)$ is a $\mathbb{Z}^d$-graded toric algebra. Anderson showed how to construct a toric degenerations of X given ν and we briefly recall his construction [And13]. In order to associate a (Noetherian) Rees algebra to ν that deforms A to $\mathrm{gr}_\nu(A)$ we apply a standard trick [Bay82, Proposition 1.8] to change from the $\mathbb{Z}^d$-grading on $\mathrm{gr}_\nu(A)$ to a $\mathbb{Z}$-grading:

Lemma 3.7. *Let F be a finite subset of $\mathbb{Z}^d$. Then there exists an* **order preserving projection** *$e : \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ such that for all $m, n \in F$ we have $m < n$ (in $\mathbb{Z}^d$) implies $e(m) < e(n)$ (in $\mathbb{Z}$).*

In our setting the set F is induced by a Gröbner basis. Consider a presentation of $A = S/I$ given by a Khovanskii-basis for ν and let $\tau \in \mathrm{Trop}(I)$ be the maximal prime cone corresponding to ν. Then choose a maximal cone $C \in \mathrm{GF}(I)$ who contains τ as a face and fix a Gröbner basis $g_1, \dots, g_s$ for I with respect to the monomial initial ideal $\mathrm{in}_C(I)$. We may assume without loss of generality that ν is the valuation associated to the matrix M whose rows are representatives of the rays of τ. If this is not the case by Corollary 3.2 there is a linear isomorphism that maps $S(A, \nu)$ to $S(A, \nu_M)$. The valuation ν_M has the advantage that it compatible with $\tilde{\nu}_M : S \setminus \{0\} \to \mathbb{Z}^d$ defined as above by

$$\tilde{\nu}_M(f) = \max_{\prec} \left\{ Ma : f = \sum c_a \mathbf{x}^a, c_a \neq 0 \right\}$$

where $\prec$ is the total order on $\mathbb{Z}^d$. For every element of the Gröbner basis g we have an expression

$$g = \sum_{a:\ Ma = \nu_M(g)} c_a \mathbf{x}^a + \sum_{b:\ Mb \prec \nu_M(g)} c_b \mathbf{x}^b,$$

where in particular $Ma \succ Mb$. The elements Ma, Mb for all g in the Gröbner basis constitute the finite set F of Lemma 3.7 that determines the order preserving projection $e : \mathbb{Z}^d \to \mathbb{Z}$. It induces a $\mathbb{Z}$-filtration of A with filtered pieces

$$\mathcal{F}_{\nu,i} := \{ f \in A : e(\nu(f)) \geq i \}.$$

The associated graded algebra coincides with $\mathrm{gr}_\nu(A)$ by construction. We define the **Rees algebra of** ν as

$$\mathcal{R}_{\nu,A} := \bigoplus_{i \geq 0} t^i \mathcal{F}_{\nu,i} \subset A[t].$$

Proposition 3.8 (Proposition 5.1 in [And13]). *The Rees algebra $\mathcal{R}_{\nu,A}$ is a flat $k[t]$-algebra with $\mathcal{R}_{\nu,A}/(t) \cong \mathrm{gr}_\nu(A)$ and $\mathcal{R}_{\nu,A}[t^{-1}] \cong A[t,t^{-1}]$ as $k[t]$-modules.*

Moreover, the isomorphisms on the above proposition hold for the *graded* algebras (remember that ν is homogeneous). In particular, $\phi : \mathrm{Proj}(\mathcal{R}_{\nu,A}) \to \mathbb{A}^1$ is a toric degeneration of X with special fibre $\phi^{-1}(0) = X_0 = \mathrm{Proj}(\mathrm{gr}_\nu(A))$. While X_0 is normal if and only if $S(A,\nu)$ is saturated, Anderson shows that its normalization is the projective toric variety associated with the Newton–Okounkov polytope $\Delta(A,\nu)$.

Given the example of a toric degeneration induced by a valuation we may formulate an *algebraic* definition of toric degeneration. We summarize the definiton and its relation to valuations in the following result of Kaveh, Manon and Murata:

Theorem 3.9 (Theorem 1.11 in [KMM17]). *Let A be a positively graded domain and let $\mathcal{R}$ be a finitely generated positively graded $k[t]$-module and domain with the following properties:*

- *$\mathcal{R}[t^{-1}] \cong A[t,t^{-1}]$ as $k[t]$-modules and graded algebras;*
- *the algebra $\mathcal{R}/(t)$ is a graded semigroup algebra $k[S]$ where $S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d$;*
- *the standard k^*-action on $k[t]$ extends to an action on $\mathcal{R}$ respecting its grading, moreover this $k[t]$-action acts through $(k^*)^d$ on the semigroup algebra $k[S]$.*

Then there is a full-rank valuation $\nu : A \setminus \{0\} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}^d$ such that $S = S(A,\nu)$.

We call a toric degeneration of $\mathrm{Proj}(A)$ induced by an algebra $\mathcal{R}$ as in Theorem 3.9 an **algebraic toric degeneration**. We obtain the following Corollary by combining Theorem 3.9 and Theorem 3.5:

Corollary 3.10. *Every algebraic toric degeneration is induced by a valuation and can be realized as a Gröbner degeneration associated with a maximal prime cone in the tropicalization of an apropriate ideal.*

4. Adapted bases and wall-crossing formulas

Suppose $A = \bigoplus_{j \geq 0} A_j$ is a graded algebra, for example the section ring of a line bundle, and it is equipped with a vector space basis $\mathbb{B}$. We assume that the basis is **graded**, i.e. basis elements are homogeneous and a homogeneous element f of degree i is a linear combination of basis elements of degree i. Recall that given a valuation $\nu : A \setminus \{0\} \to \Gamma$ the basis $\mathbb{B}$ is **adapted to** ν if for every $\gamma \in \Gamma$ the set $\mathbb{B} \cap \mathcal{F}_{\nu,\gamma}$ is a vector space basis of the filtered piece $\mathcal{F}_{\nu,\gamma} \subset A$. Reversely we say that ν **is adapted to** $\mathbb{B}$. In particular, if ν has one-dimensional leaves we have a bijection of sets

$$\mathbb{B} \quad \leftrightarrow \quad S(A,\nu).$$

In this section we explore the consequences. Assume the basis is **parametrized by lattice points**, so we have an assignment of $b \mapsto m(b) \in \mathbb{Z}^n$ for all $b \in \mathbb{B}$. In fact we may find an adapted basis with a parametrization by lattice points for every Khovanskii-finite valuation. Consider $\nu : A \setminus \{0\} \to \mathbb{Z}^d$ with Khovanskii basis $b_1, \dots, b_n$ and let $S/I \cong A$ and $\tau \in \mathrm{Trop}(I)$ be the presentation of A and the maximal prime cone in $\mathrm{Trop}(I)$ as in Theorem 3.5. By definition $\mathrm{Trop}(I)$ is a subfan of the Gröbner fan $\mathrm{GF}(I)$ and τ is a face of at least one maximal cone in $\mathrm{GF}(I)$. Maximal cones in $\mathrm{GF}(I)$ are in correspondence with monomial initial ideals of I as defined above. For a maximal cone C we denote by $\mathrm{in}_C(I) \subset S$ the corresponding monomial ideal. In particular, the set

$$\mathbb{B}_C := \{\mathbf{x}^m : \mathbf{x}^m \notin \mathrm{in}_C(I)\}$$

is a vector space basis for all quotients $A_w := S/\mathrm{in}_w(I)$ with $w \in C$ called a **standard monomial basis**. Notice that for $w = 0$ the quotient $A_w = A$ and for $w \in \tau^\circ$ we have $A_w \cong \mathrm{gr}_\nu(A)$. Hence, $\mathbb{B}_C$ is an adapted basis for ν. The assignment

$$\mathbf{x}^m \mapsto m \in \mathbb{Z}^n$$

is a parametrization by lattice points. Recall that every monomial ideal has a unique set of monomial generators (see e.g. [HH11, Proposition 1.1.6]). Let $\mathbf{x}^{g_1}, \dots, \mathbf{x}^{g_t}$ be this generating set. Then $\mathbf{x}^m \in \mathrm{in}_C(I)$ if and only if there exists i such that $\mathbf{x}^{g_i}$ divides $\mathbf{x}^m$. This translates to $m \notin \bigcup_{i=1}^t g_t + \mathbb{Z}^n_{\geq 0}$ for elements of the standard monomial basis, where + denotes the Minkowski sum. So we have bijections of sets

$$S(A,\nu) \quad \leftrightarrow \quad \mathbb{B}_C \quad \leftrightarrow \quad \mathbb{Z}^n_{\geq 0} \setminus \bigcup_{i=1}^{t} g_i + \mathbb{Z}^n_{\geq 0}$$

Hence, we obtain the following corollary:

Corollary 4.1. *Every Khovanskii-finite valuation has an adapted basis parametrized by lattice points.*

Example 4.2. *Consider as above $I = (y^2z - x^3 + z^3) \subset \mathbb{C}[x,y,z]$ and the maximal prime cone $\tau \in \mathrm{Trop}(I)$ spanned by the ray $(2,3,0)^T$ modulo $\mathcal{L}_I$. Inside $\mathrm{GF}(I)$ the cone τ is adjacent to two maximal cones: one of them has associated initial monomial ideal (x^3) (see Figure 1). Let C be this maximal cone. Then $\mathbb{B}_C = \{x^a y^b z^c : a < 3\}$ is the set of all monomials in $\mathbb{C}[x,y,z]$ that are not divisible by x^3, hence they are not in $\mathrm{in}_C(I)$.*

4.1. **Polytopes from adapted bases.** Fix a Khovanskii-finite valuation $\nu : A \setminus \{0\} \to \mathbb{Z}^d$ and an associated adapted standard monomial basis $\mathbb{B} := \mathbb{B}_C$ with the parametrization given above. For an element $f \in A$ let $f = \sum_{\mathbf{x}^m \in \mathbb{B}} c_m \mathbf{x}^m$ be its linear extension in $\mathbb{B}$. Let M be the matrix whose rows $r_1, \dots, r_m$ are representatives of primitive ray generators of C. In particular, let $r_1, \dots, r_s$ be the generators of the lineality space. Define $\mathrm{supp}_{\mathbb{B}}(f) := \{m \in \mathbb{Z}^n : c_m \neq 0\} \subset \mathbb{Z}^n$ and the **$\mathbb{B}$-Newton polytope** of f by

$$\mathrm{New}_{\mathbb{B}}(f) := \mathrm{conv}\left(Ma : a \in \mathrm{supp}_{\mathbb{B}}(f)\right) \subset \mathbb{R}^m.$$

Notice that $\mathrm{New}_{\mathbb{B}}(f)$ depends on our choice of ray generators for C. We will slightly abuse notation and not include M in the index. We think of the $\mathrm{New}_{\mathbb{B}}(f)$ as placeholder for a valuation adapted to $\mathbb{B}$ and we define a placeholder for the Newton–Okounkov body of ν:

$$\Delta_{\mathbb{B}}(A) := \mathrm{conv}\left(\bigcup_{j \geq 1} \left\{ \frac{1}{j} \mathrm{New}_{\mathbb{B}}(f) : f \in A_j \right\}\right) \subset \mathbb{R}^m.$$

Recall that the **Newton–Okounkov body** of a valuation $\nu : A \setminus \{0\} \to \mathbb{Z}^d$ is defined as

$$\Delta(A,\nu) := \mathrm{conv}\left(\bigcup_{j \geq 1} \left\{ \frac{\nu(f)}{j} : f \in A_j \right\}\right).$$

If ν is fully homogeneous, i.e. of form $\nu : A \setminus \{0\} \to \mathbb{Z}^m_{\geq 0} \times \Gamma'$ given by $\nu(f) = (\deg(f), \nu(f))$ then the above definition coincides with the one in Equation (2). Any valuation constructed from a maximal prime cone τ in the tropicalization of an ideal as in §3.1 is by construction fully homogeneous. Recall that the first m rows of M_τ are the elements ℓ_i from Lemma 2.3. Let M_ℓ be the submatrix with rows $\ell_1, \dots, \ell_m$. Denote by $\mathrm{pr} : \mathbb{Z}^m_{\geq 0} \times \Gamma' \to \mathbb{Z}^m_{\geq 0}$ the projection. Then for any element $f \in A$ we have

$$\mathrm{pr}(\nu_M(f)) = M_\ell b = \deg(f)$$

where b is such that $M_\tau b = \max_{<}\{M_\tau a : f = \sum c_a \mathbf{x}^a, c_a \neq 0\}$. Moreover, in this context the bijection between the basis $\mathbb{B} = \mathbb{B}_C$ and the valuation ν_{M_τ} is given explicitly by

$$\mathbb{B} \to S(A, \nu_M), \quad \mathbf{x}^a \mapsto M_\tau a$$

Theorem 4.3. *Let $C \in \mathrm{GF}(I)$ be a maximal cone that contains the maximal prime cones $\tau_1, \dots, \tau_q \in \mathrm{Trop}(I)$ as d-dimensional face with associated Khovanskii-finite valuations $\nu_i : A \setminus \{0\} \to \mathbb{Z}^d$. Assume additionally that $\mathrm{in}_C(I)$ does not contain any variables. Then there exist projections $p_i : \mathbb{R}^n \to \mathbb{R}^d$ for $1 \leq i \leq q$ such that*

$$p_i(\Delta_{\mathbb{B}}(A)) = \Delta(A, \nu_i).$$

Proof. Let M be the matrix whose rows $r_1, \dots, r_{n_C}$ are either generators of the lineality space $\mathcal{L}_I$ or representatives of primitive ray generators for $C/\mathcal{L}_I$. Notice that $n_C \geq n$ as C is a maximal cone with equality if and only if $C/\mathcal{L}_I$ is simplicial. For every cone τ_i choose a collection of rows $r_1, \dots, r_s, r_{i_1}, \dots, r_{i_{d-s}}$ (where $r_1, \dots, r_s$ are the generators of the lineality space), that correspond to rays spanning the same real vector space as τ_i:

$$\langle r_1, \dots, r_s, r_{i_1}, \dots, r_{i_{d-s}} \rangle_{\mathbb{R}} = \langle \tau_i \rangle_{\mathbb{R}}.$$

Denote the matrix whose rows are $r_1,\dots,r_s,r_{i_1},\dots,r_{i_{d-s}}$ by M_i and define $p_i:\mathbb{R}^{n_C}\to\mathbb{R}^d$ as the projection onto the coordinates $1,\dots,s,i_1,\dots,i_{d-s}$. Recall that $\Delta(A,\nu_i)$ without loss of generality by Corollary 3.2 is the convex hull of the columns of the matrix M_i. We verify $p_i(\Delta_{\mathbb{B}}(A))=\Delta(A,\nu_i)$ pointwise by tracing the elements of $\mathbb{B}$ through both constructions. As $\mathbb{B}$ is in bijection with $S(A,\nu_i)$ the claim follows. Consider $\mathbf{x}^a\in\mathbb{B}$, then $\mathrm{New}_{\mathbb{B}}(\mathbf{x}^a)=Ma$. In $\Delta_{\mathbb{B}}(A)$ the element $\mathbf{x}^a$ corresponds to the point $\frac{1}{a_1+\cdots+a_n}Ma$ and

$$p_i\left(\frac{1}{a_1+\cdots+a_n}Ma\right)=\frac{1}{a_1+\cdots+a_n}M_ia=\frac{1}{a_1+\cdots+a_n}\nu_i(\mathbf{x}^a)\in\Delta(A,\nu_i),$$

so $p_i(\Delta_{\mathbb{B}}(A))\subset\Delta(A,\nu_i)$. To show equality it suffices to verify that the vertices of $\Delta(A,\nu_i)$ are contained in $p_i(\Delta_{\mathbb{B}}(A))$. By the additional assumption that $\mathrm{in}_C(I)$ does not contain any variables we know that $x_1,\dots,x_n\in\mathbb{B}$. The computation above applied to a variable $x_j=\mathbf{x}^a$ yields:

$$p_i\left(\frac{1}{a_1+\cdots+a_n}Ma\right)=M_ie_j=M_{ij},$$

where M_{ij} is the jth column of M_i and a vertex of $\Delta(A,\nu_i)$ by Corolary 3.2. □

Example 4.4. *We continue with the Example 4.2. For the maximal cone C we choose the ray matrix:*

$$M_C=\begin{pmatrix}1&1&1\\2&3&0\\1&0&1\end{pmatrix}.$$

Notice that $(1,0,1)\mod\mathcal{L}_I=(0,-1,0)\mod\mathcal{L}_I$ which corresponds to the teal ray in Figure 1. Let $\mathbb{B}=\mathbb{B}_C$ and $A=\mathbb{C}[x,y,z]/I$. Then

$$\Delta_{\mathbb{B}}(A)=\mathrm{conv}\left(\frac{1}{a+b+c}\left(\begin{smallmatrix}a+b+c\\2a+3b\\a+c\end{smallmatrix}\right):\begin{matrix}a+b+c\geq 1,a<3\\a,b,c\in\mathbb{Z}_{\geq 0}\end{matrix}\right)=\mathrm{conv}\left(\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\3\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)\right).$$

Let p_1 be the projection away from the third coordinate in $\mathbb{R}^3$, then

$$p_1(\Delta_{\mathbb{B}}(A))=\mathrm{conv}\left(\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)=\Delta(A,\nu_\tau).$$

Where $\tau\in\mathrm{Trop}(I)$ is the maximal prime cone spanned by $(2,3,0)^T\mod\mathcal{L}_I$ as in Example 3.3.

4.2. **Wall-crossing formulas.** The Newton–Okounkov polytopes associated to the faces $\tau_1,\dots,\tau_q$ of C are related by piecewise-linear maps called **wall-crossing formulas** that were introduced by Escobar and Harada in [EH20]. We briefly review their construction.

Assume that τ_1 and τ_2 are two adjacent faces of the maximal cone $C\in\mathrm{GF}(I)$, so that $\tau:=\tau_1\cap\tau_2$ is a facet of both. Then we may choose the ray matrices M_1 and M_2 such that they agree in all rows but the last one. Let $M_{1,2}$ be the matrix with the $d-1$ rows that M_1 and M_2 have in common. In particular, we have two full-rank homogeneous valuations $\nu_1:=\nu_{\tau_1},\nu_2:=\nu_{\tau_2}:A\setminus\{0\}\to\mathbb{Z}^m_{\geq0}\times\mathbb{Z}^{d-m}$ and one homogeneous valuation $\nu_{1,2}:A\setminus\{0\}\to\mathbb{Z}^m_{\geq 0}\times\mathbb{Z}^{d-m-1}$ of almost full rank, that is rank $d-1$. We denote by $p_{[d-m-1]}:\mathbb{R}^{d-m}\to\mathbb{R}^{d-m-1}$ the projection onto the first $d-m-1$ coordinates. By construction (and Corollary 3.2) we have the following relation between the associated Newton–Okounkov polytopes

$$\begin{tikzcd}
\Delta(A,\nu_1) \arrow[dr, "p_{[d-m-1]}"'] & & \Delta(A,\nu_2) \arrow[dl, "p_{[d-m-1]}"] \\
& \Delta(A,\nu_{1,2}) &
\end{tikzcd}$$

In particular there exist piecewise linear maps $\varphi_i:\Delta(A,\nu_{1,2})\to\mathbb{R}$ and $\psi_i:\Delta(A,\nu_{1,2})\to\mathbb{R}$ for $i\in\{1,2\}$ such that

$$\Delta(A,\nu_i)=\left\{(\mathbf{1},v,z)\in\{\mathbf{1}\}\times\mathbb{R}^{d-m-1}\times\mathbb{R}:\begin{matrix}(\mathbf{1},v)\in\Delta(A,\nu_{1,2})\\\varphi_i(\mathbf{1},v)\leq z\leq\psi_i(\mathbf{1},v)\end{matrix}\right\},\tag{4}$$

where $\mathbf{1}=(1,\dots,1)\in\mathbb{R}^m$. By [EH20, Theorem 3.4] there exists a constant $\kappa>0$ such that for all $(1,v)\in\Delta(A,\nu_{1,2})$:

$$\kappa(\psi_1(\mathbf{1},v)-\varphi_1(\mathbf{1},v))=\psi_2(\mathbf{1},v)-\varphi_2(\mathbf{1},v)$$

We define the piecewise linear wall-crossing maps

$$
\begin{aligned}
S_{12} &: \mathbb{R}^d \to \mathbb{R}^d \text{ given by } (\mathbf{1}, v, z) \mapsto (\mathbf{1}, v, \kappa(z - \varphi_1(\mathbf{1}, v)) + \varphi_2(\mathbf{1}, \mathbf{v})) \\
F_{12} &: \mathbb{R}^d \to \mathbb{R}^d \text{ given by } (\mathbf{1}, v, z) \mapsto (\mathbf{1}, v, \kappa(\varphi_1(\mathbf{1}, v) - z) + \psi_2(\mathbf{1}, \mathbf{v})).
\end{aligned} \tag{5}
$$

The map S_{12} is called the **shift** and the map F_{12} is called the **flip**.

Theorem 4.5 (Theorem 2.7 in [EH20])**.** *Let I be a (multi-)homogeneous ideal in S and C a maximal cone in $\mathrm{GF}(I)$ such that there exist two maximal prime cones $\tau_1, \tau_2 \subset C \cap \mathrm{Trop}(I)$ that share a common facet $\tau = \tau_1 \cap \tau_2$. Let ν_1, ν_2 and $\nu_{1,2}$ be the associated homogeneous valuations. Then for $\Phi_{12} \in \{F_{12}, S_{12}\}$ the associated Newton–Okounkov polytopes are related by*

$$
\begin{tikzcd}
\Delta(A,\nu_1) \arrow[rr, "\Phi_{12}"] \arrow[dr, "p_{[d-m-1]}"'] & & \Delta(A,\nu_2) \arrow[dl, "p_{[d-m-1]}"] \\
& \Delta(A,\nu_{1,2}) &
\end{tikzcd}
$$

and the Euclidean lengths of the fibers of $p_{[d-m-1]}$ are equal.

Example 4.6. *In our running example the maximal cone $C \in \mathrm{GF}(I)$ has two maximal prime cones in $\mathrm{Trop}(I)$ as facets. Let $\tau_1 \in \mathrm{Trop}(I)$ be the cone generated by $(2,1,1)^T \mod \mathcal{L}_I$ (teal in Figure 1) and τ_2 be the cone generated by $(2,3,0)^T \mod \mathcal{L}_I$. Then*

$$
\Delta(A,\nu_1) = \mathrm{conv}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}\right) \quad \text{and} \quad \Delta(A,\nu_2) = \mathrm{conv}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right).
$$

The Newton–Okounkov polytope $\Delta(A,\nu_{1,2})$ is simply the point $\{1\} \in \mathbb{R}$. Hence, the piecewise linear functions φ_i, ψ_i are constants:

$$
\begin{aligned}
\Delta(A,\nu_1) &= \{(1,z) : \varphi_1(1) := 0 \le z \le 3 =: \psi_1(1)\}, \\
\Delta(A,\nu_2) &= \{(1,z) : \varphi_2(1) := 0 \le z \le 1 =: \psi_2(1)\}.
\end{aligned}
$$

The global constant can be computed from the volume of the Newton–Okounkov polytopes (with respect to the ambient subspace where they are full-dimensional poyltopes). We have that $\kappa_1 \mathrm{vol}(\Delta(A,\nu_1)) = \kappa_2 \mathrm{vol}(\Delta(A,\nu_2)) = \deg(y^2z - x^3 + z^3)$ and $\kappa = |\kappa_1/\kappa_2| = \frac{1}{3}$, so

$$
S_{12} : (1,z) \mapsto \left(1, \frac{z}{3}\right), \quad F_{12} : (1,z) \mapsto \left(1, 1 - \frac{z}{3}\right).
$$

5. Families of Gröbner degenerations

In this section, we recall the main construction of the paper [BMNC21]. It gives a multi-parameter flat family associated to a maximal cone $C \in \mathrm{GF}(I)$ where $I \subset S$ is a homogeneous ideal. We will see that this algebraic construction is closely related to the polyhedral objects from the previous section.

Let A be the quotient S/I. Recall the classical construction of a Gröbner degenerations associated to w a weight vector $w \in C^\circ$ from §2.2 defined by the quotient $S[t]/I^{h;w}$. The ideal $I^{h;w}$ defines the flat family $\mathrm{Spec}(S[t]/I^{h;w}) \to \mathrm{Spec}(k[t])$ whose fiber over the closed point (t) is isomorphic to $\mathrm{Spec}(A^w)$, where $A^w := S/\mathrm{in}_w(I)$ and the fiber over any non-zero closed point $(t-c)$ is isomorphic to $\mathrm{Spec}(A)$. Both, the construction of $I^{h;w}$ and [Eis13, Theorem 15.17] hold for arbitrary cones in $\mathrm{GF}(I)$. In what follows, for simplicity, we focus on maximal cones as the generalization to lower dimensional ones is straight forward.

To generalize the construction of $I^{h;w}$ we fix vectors $r_1, \dots, r_{n_C} \in C$ such that $\{\overline{r}_1, \dots, \overline{r}_{n_C}\}$ is the set of primitive ray generators for $\overline{C}$, which is possible due to [BMNC21, Lemma 2.13]. Let M be the $(n_C \times n)$-matrix whose rows are $r_1, \dots, r_{n_C}$. Additionally, we write $<$ for a monomial term order compatible with C and denote by G the associated reduced Gröbner basis.

Definition 5.1. *For $f = \sum_{\alpha \in \mathbb{Z}^n_{\ge 0}} c_\alpha \mathbf{x}^\alpha \in I$ set $\mu_M(f) := (\max_{c_\alpha \neq 0}\{r_i \cdot \alpha\})_{i=1,\dots,n_C} \in \mathbb{Z}^{n_C \times 1}$, hence $\mu_M(f)$ as a column vector with n_C entries. Define the* **lift of** *f as the polynomial $\tilde{f}_M \in S[t_1, \dots, t_{n_C}]$*

given by the following formula

$$\tilde{f}_M := \tilde{f}_M(\mathbf{t},\mathbf{x}) := f(\mathbf{t}^{-M\cdot e_1}x_1,\dots,\mathbf{t}^{-M\cdot e_n}x_n)\mathbf{t}^{\mu_M(f)} = \sum_{\alpha\in\mathbb{Z}^n_{\geq 0}} c_\alpha \mathbf{x}^\alpha \mathbf{t}^{-M\cdot\alpha+\mu_M(f)}.$$

Similarly, we define the **lifted ideal** *as* $\tilde{I}_M := \left\langle \tilde{f}_M : f\in I\right\rangle \subset S[t_1,\dots,t_{n_C}]$ *and the* **lifted algebra** *as the quotient*

$$\tilde{A}_M := S[t_1,\dots,t_{n_C}]/\tilde{I}_M. \tag{6}$$

Although by construction the lifted algebra depends on the choice of ray matrix M it can be shown that different choices yield the same algebra [BMNC21, Corollary 3.10]. Another useful result about the lifted ideal $\tilde{I}_M$ is an explicit construction of a Gröbner basis. On $S[t_1,\dots,t_{n_C}]$ we consider the following term order induced by the term order $<$ on S corresponding to C:

$$\mathbf{x}^\alpha\mathbf{t}^\lambda \ll \mathbf{x}^\beta\mathbf{t}^\mu \quad \text{if and only if} \quad \text{(i) } \mathbf{x}^\alpha < \mathbf{x}^\beta \quad \text{or} \quad \text{(ii) } \mathbf{x}^\alpha = \mathbf{x}^\beta \text{ and } \mathbf{t}^\lambda <_{\text{lex}} \mathbf{t}^\mu. \tag{7}$$

Then by [BMNC21, Proposition 3.9] the lifts of the elements of a Gröbner basis G of I with respect to $<$ form a Gröbner basis for $\tilde{I}_M$ with respect to $\ll$. The main result is proven using this Gröbner basis and the standard monomial basis $\mathbb{B}_C$ of A:

Theorem 5.2 (Theorem 3.14 in [BMNC21])**.** *Let I be a homogeneous ideal in S, $A = S/I$, C a maximal cone in $\mathrm{GF}(I)$ with associated term order $<$ and M an $(m\times n)$-matrix whose rows are representatives of the primitive ray generators of $\overline{C}\subset\mathbb{R}^n/\mathcal{L}(I)$. Then:*

(i) The algebra $\tilde{A}_M$ is a free S-module with basis $\mathbb{B}=\mathbb{B}_C$, the standard monomial basis of A with respect to $\mathrm{in}_C(I)$. In particular, we have a flat family

$$\begin{tikzcd} \mathrm{Proj}(\tilde{A}_M) \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{P}^{n-1}\times\mathbb{A}^{n_C} \arrow[dl, two heads] \\ \mathbb{A}^{n_C} & \end{tikzcd}$$

(ii) For every face τ of C there exists $\mathbf{a}_\tau\in\mathbb{A}^{n_C}$ such that $\pi^{-1}(\mathbf{a}_\tau) = \mathrm{Proj}(S/\mathrm{in}_\tau(I))$. In particular, generic fibers are isomorphic to $\mathrm{Proj}(A)$ and there exist special fibers for every proper face $\tau\subset C$.

Example 5.3. *In our running example $I = (f := y^2z - x^3 + z^3)$ the generator f itself forms a Gröbner basis for every maximal cone in $\mathrm{GF}(I)$ (which is true more generally for hypersurfaces). Consider the maximal cone $C\in\mathrm{GF}(I)$ with its ray matrix M_C from Example 4.2. For this construction we may omit the rays of M_C coming from the lineality space. So that $M = \left(\begin{smallmatrix} 2 & 3 & 0 \\ 1 & 0 & 1\end{smallmatrix}\right)$ Then $\mu_M(f) = \binom{6}{3}$ and*

$$\tilde{f}_M = f(xt_0^{-1}t_1^{-2}t_2^{-1}, yt_0^{-1}t_1^{-3}, zt_0^{-1}t_2^{-1})t_0^3t_1^6t_2^3 = y^2zt_2^2 - x^3 + z^3t_1^6.$$

Recall the two maximal prime cones $\tau_1,\tau_2\in\mathrm{Trop}(I)$ from Example 4.6. We have

$$\tilde{f}_M|_{(t_1,t_2)=(0,1)} = y^2z - x^3 = \mathrm{in}_{\tau_1}(f) \quad \textit{and} \quad \tilde{f}_M|_{(t_1,t_2)=(1,0)} = -x^3 + z^3 = \mathrm{in}_{\tau_2}(f).$$

Notice that Theorem 4.3 may be interpreted as a polyhedral version of Theorem 5.2. Let $C\in\mathrm{GF}(I)$ be the maximal cone and let τ_1,τ_2 be faces of C that are maximal prime cones in $\mathrm{Trop}(I)$. Denote $X_1 := \mathrm{Proj}(A_{\tau_1})$ and $X_2 := \mathrm{Proj}(A_{\tau_2})$ and let ν_1 and ν_2 be the associated valuations. Recall that $\Delta(A,\nu_1)$ resp. $\Delta(A,\nu_2)$ is the polytope of the normalization of X_1 resp. X_2. We have the following diagrams

$$\begin{tikzcd} X_1 \arrow[r, hook] \arrow[d] & \mathrm{Proj}(\tilde{A}_M) \arrow[d, "\pi"] & X_2 \arrow[l, hook'] \arrow[d] \\ \{\mathbf{a}_{\tau_1}\} \arrow[r, hook] & \mathbb{A}^{n_C} & \{\mathbf{a}_{\tau_2}\} \arrow[l, hook'] \end{tikzcd} \qquad \begin{tikzcd} \Delta(A,\nu_1) & & \Delta(A,\nu_2) \\ & \Delta_{\mathbb{B}}(A)\subset\mathbb{R}^{n_C} \arrow[ul, two heads, "p_1"] \arrow[ur, two heads, "p_1"'] & \end{tikzcd}$$

If τ_1 and τ_2 are adjacent in the sense that $\tau := \tau_1 \cap \tau_2$ is a common facet of both we additionally have access to the wall-crossing formulas F_{12} and S_{12} defined in Equation (5). Let $\Phi_{12} \in \{F_{12}, S_{12}\}$:

$$\begin{tikzcd}
 & \Delta_{\mathbb{B}}(A) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\Delta(A,\nu_1) \arrow[rr, "\Phi_{12}"'] \arrow[dr, "p_{[d-m-1]}"'] & & \Delta(A,\nu_2) \arrow[dl, "p_{[d-m-1]}"] \\
 & \Delta(A,\nu_{1,2}) &
\end{tikzcd} \tag{8}$$

The upper triangle of the diagram is not necessarily commutative as the following example shows:

Example 5.4. *Let τ_1, τ_2 and C be as in the previous examples. We have previously computed the polytopes $\Delta_{\mathbb{B}}(A) = \operatorname{conv}\left(\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\3\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)\right)$, $\Delta(A,\nu_1) = \operatorname{conv}\left(\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\3\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right)$, and $\Delta(A,\nu_2) = \operatorname{conv}\left(\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right)$, where all the integral points are written in the order of the variables x, y, z that induce them. Recall the maps S_{12} and F_{12} from Example 4.6. We have*

$$S_{12} : \begin{pmatrix}1\\2\end{pmatrix} \mapsto \begin{pmatrix}1\\2/3\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix} \mapsto \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix},$$
$$F_{12} : \begin{pmatrix}1\\2\end{pmatrix} \mapsto \begin{pmatrix}1\\1/3\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\1\end{pmatrix}.$$

So in this case neither F_{12} nor S_{12} make the upper triangle in (8) commute.

However, the bijections $\mathbb{B} \leftrightarrow S(A,\nu_i)$ for $i \in \{1,2\}$ induce a map $A_{12} : S(A,\nu_1) \to S(A,\nu_2)$ that makes the upper triangle in (8) commutative. This map is called the **algebraic wall-crossing** in [EH20] and it is simply the composition of the bijections. In special cases the algebraic wall-crossing coincides with the flip map. An example is given in [EH20, §5] where the authors show this is the case for the Grassmannian of planes. Moreover, in this case the flip map is the *Fock–Goncharov tropicalization* (see e.g. [GHK15, Remark 2.3]) of Fomin–Zelevinsky's *cluster mutation* (see e.g. [FZ02]) as is shown in [BMNC21, §4.6].

Example 5.5. *The algebraic wall-crossing $A_{12} : S(A,\nu_1) \to S(A,\nu_2)$ is different from both of them. It is simply a bijection of sets induced by the bijections of $\mathbb{B}$ with $S(A,\nu_1)$ and $S(A,\nu_2)$. In particular, we have*

$$A_{12} : \begin{pmatrix}1\\2\end{pmatrix} \mapsto \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}1\\1\end{pmatrix}.$$

6. Example: the Grassmannian $\mathrm{Gr}_3(\mathbb{C}^6)$

In this section we apply the Proposition 3.6 to the Grassmannian $\mathrm{Gr}_3(\mathbb{C}^6)$, or more precise to its homogeneous coordinate ring with respect to the Plücker embedding. The **Plücker algebra** $A_{3,6}$ is quotient of the polynomial ring

$$S := \mathbb{C}[p_{ijk} : 1 \le i < j < k \le 6]$$

by the **Plücker ideal** $I_{3,6}$ that is generated minimally by the following 34 Plücker relations:

$p_{256}p_{346} - p_{246}p_{356} + p_{236}p_{456}$	$p_{156}p_{346} - p_{146}p_{356} + p_{136}p_{456}$	$p_{256}p_{345} - p_{245}p_{356} + p_{235}p_{456}$
$p_{246}p_{345} - p_{245}p_{346} + p_{234}p_{456}$	$p_{236}p_{345} - p_{235}p_{346} + p_{234}p_{356}$	$p_{156}p_{345} - p_{145}p_{356} + p_{135}p_{456}$
$p_{146}p_{345} - p_{145}p_{346} + p_{134}p_{456}$	$p_{136}p_{345} - p_{135}p_{346} + p_{134}p_{356}$	$p_{156}p_{246} - p_{146}p_{256} + p_{126}p_{456}$
$p_{236}p_{245} - p_{235}p_{246} + p_{234}p_{256}$	$p_{156}p_{245} - p_{145}p_{256} + p_{125}p_{456}$	$p_{146}p_{245} - p_{145}p_{246} + p_{124}p_{456}$
$p_{126}p_{245} - p_{125}p_{246} + p_{124}p_{256}$	$p_{156}p_{236} - p_{136}p_{256} + p_{126}p_{356}$	$p_{146}p_{236} - p_{136}p_{246} + p_{126}p_{346}$
$p_{156}p_{235} - p_{135}p_{256} + p_{125}p_{356}$	$p_{145}p_{235} - p_{135}p_{245} + p_{125}p_{345}$	$p_{136}p_{235} - p_{135}p_{236} + p_{123}p_{356}$
$p_{126}p_{235} - p_{125}p_{236} + p_{123}p_{256}$	$p_{146}p_{234} - p_{134}p_{246} + p_{124}p_{346}$	$p_{145}p_{234} - p_{134}p_{245} + p_{124}p_{345}$
$p_{136}p_{234} - p_{134}p_{236} + p_{123}p_{346}$	$p_{135}p_{234} - p_{134}p_{235} + p_{123}p_{345}$	$p_{126}p_{234} - p_{124}p_{236} + p_{123}p_{246}$
$p_{125}p_{234} - p_{124}p_{235} + p_{123}p_{245}$	$p_{136}p_{145} - p_{135}p_{146} + p_{134}p_{156}$	$p_{126}p_{145} - p_{125}p_{146} + p_{124}p_{156}$
$p_{126}p_{135} - p_{125}p_{136} + p_{123}p_{156}$	$p_{126}p_{134} - p_{124}p_{136} + p_{123}p_{146}$	$p_{125}p_{134} - p_{124}p_{135} + p_{123}p_{145}$

$$p_{126}p_{345} - p_{125}p_{346} + p_{124}p_{356} - p_{123}p_{456} \quad p_{136}p_{245} - p_{135}p_{246} + p_{134}p_{256} + p_{123}p_{456}$$
$$p_{146}p_{235} - p_{135}p_{246} + p_{125}p_{346} + p_{123}p_{456} \quad p_{156}p_{234} - p_{134}p_{256} + p_{124}p_{356} - p_{123}p_{456}$$
$$p_{145}p_{236} - p_{135}p_{246} + p_{125}p_{346} + p_{156}p_{234}$$

The tropicalization of $I_{3,6}$ was computed back in 2004 by Speyer and Sturmfels who found out that there are several maximal prime cones in $\mathrm{Trop}(I_{3,6})$, [SS04, §5]. We briefly summarize their findings.

Theorem 6.1. *The tropical Grassmannian* $\mathrm{Trop}(I_{3,6}) \subset \mathbb{R}^{20}/\mathcal{L}_{I_{3,6}}$ *is a four-dimensional fan with 1005 maximal cones, 990 of which are prime. The 65 rays consist of*

(1) 20 standard basis elements e_{ijk}, $1 \leq i < j < k \leq 6$*;*
(2) 15 vectors of form $f_{ij} = \sum_{k \notin \{i,j\}} e_{ijk}$, $1 \leq i < j \leq 6$*;*
(3) 30 vectors associated with 15 tripartitions $\{\{i_1, i_2\}, \{i_3, i_4\,\{i_5, i_6\}\}$ *of* $[6]$ *each defining two rays* $g_{i_1i_2i_3i_4i_5i_6} := f_{i_5i_6} + e_{i_3i_4i_5} + e_{i_3i_4i_6}$ *and* $g_{i_1i_2i_5i_6i_3i_4} := f_{i_3i_4} + e_{i_3i_5i_6} + e_{i_4i_5i_6}$.

The symmetric group action on the index sets of Plücker coordinates induces a symmetry on $\mathrm{Trop}(I_{3,6})$*. The maximal cones are grouped in seven orbits, six of which consist of prime cones:*

EEEE there are 30 simplicial prime cones of type $\{e_{123}, e_{145}, e_{246}, e_{356}\}$*;*
EEFF1 there are 90 simplicial prime cones of type $\{e_{123}, e_{456}, f_{56}, f_{12}\}$*;*
EEFF2 there are 90 simplicial prime cones of type $\{e_{125}, e_{345}, f_{12}, f_{34}\}$*;*
EFFG there are 180 simplicial prime cones of type $\{e_{345}, f_{34}, f_{12}, g_{123456}\}$*;*
EEEG there are 240 simplicial prime cones of type $\{e_{126}, e_{134}, e_{356}, g_{125634}\}$*;*
EEFG there are 360 simplicial prime cones of type $\{e_{234}, e_{125}, f_{34}, g_{125634}\}$*;*
FFFGG there are 15 non-simplicial non-prime cones of type $\{f_{56}, f_{34}, f_{12}, g_{123456}, g_{125634}\}$.

Many families of (full rank homogeneous) valuations are known for $A_{3,6}$ (see e.g. [Bos21a, FFL17, MS18]) and whenever the Plücker coordinates form a Khovanskii basis there is a unique maximal prime cone in $\mathrm{Trop}(I_{3,6})$ associated with it by Theorem 3.5. It is however not true that all *known* valuations on $A_{3,6}$ share the Plücker coordinates as a Khovanskii basis. For an example, you may want to consider [RW19, §9] where Rietsch and Williams exihibit an example of a valuation induced by a *plabic graph*. This example can be generalized to higher Grassmannians, see [Bos21b, §5]. For more details, including background on how to obtain valuations from plabic graphs we refer the reader to the mentioned references.

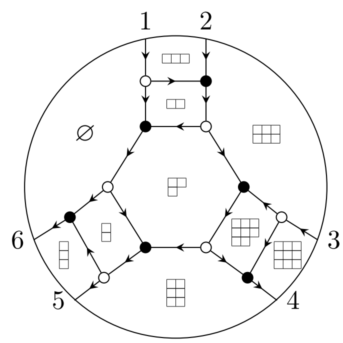
Example 6.2. *Let* $\nu_G : A_{3,6} \to \mathbb{Z}^9$ *be the Rietsch–Williams valuation associated with the plabic graph on the left in Table 1. The values of Plücker coordinates under* ν_G *can be found in Table 1. The valuation is Khovanskii-finite, but the Plücker coordinates do not form a Khovanskii basis. Among the vertices of* $\Delta(A_{3,6}, \nu_G)$ *there is one non integral of form*

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 1, \frac{1}{2}\right).$$

It is obtained from the ray of the Newton–Okounkov cone generated by $\nu_G(\bar{p}_{124}\bar{p}_{356} - \bar{p}_{123}\bar{p}_{456})$.

The aim of this section is to illustrate how Proposition 3.6 can be applied to find an appropriate tropicalization where the Khovanskii-finite valuation ν_G appears as associated with a maximal prime cone.

6.1. **Cluster embedding.** The Grassmannian $\mathrm{Gr}_3(\mathbb{C}^6)$ has a cluster structure that was first exhibited by Scott in [Sco06]. More precisely, the algebra $A_{3,6}$ is a cluster algebra which roughly means that is can be constructed recursively from particular sets of maximal algebraically independent elements, called *seed* by a combinatorial procedure called *mutation* [FZ02]. The elements of the seeds are called the *cluster variables*. There are 22 of them for $A_{3,6}$, 20 of which are Plücker coordinates together with two more generators of degree two denoted by X and Y that are binomials in Plücker coordinates. In particular, Y agrees with the element $\bar{p}_{124}\bar{p}_{356} - \bar{p}_{123}\bar{p}_{456}$ from Example 6.2.



$\bar{p}_{123}$	0	0	0	0	0	0	0	0	0
$\bar{p}_{124}$	0	0	1	0	0	0	0	0	0
$\bar{p}_{125}$	0	0	1	1	0	0	0	1	0
$\bar{p}_{126}$	0	0	1	1	1	0	0	1	0
$\bar{p}_{134}$	0	1	1	0	0	0	0	1	0
$\bar{p}_{135}$	0	1	1	1	0	0	0	1	0
$\bar{p}_{136}$	0	1	1	1	1	0	0	1	0
$\bar{p}_{145}$	0	1	2	1	0	0	0	1	0
$\bar{p}_{146}$	0	1	2	1	1	0	0	1	0
$\bar{p}_{156}$	0	1	2	2	0	0	1	2	1
$\bar{p}_{234}$	1	1	1	0	0	0	0	1	0
$\bar{p}_{235}$	1	1	1	1	0	0	0	1	0
$\bar{p}_{236}$	1	1	1	1	1	0	0	1	0
$\bar{p}_{245}$	1	1	2	1	0	0	0	1	0
$\bar{p}_{246}$	1	1	2	1	1	0	0	1	0
$\bar{p}_{256}$	1	1	2	2	1	0	1	2	1
$\bar{p}_{345}$	1	2	2	1	0	1	1	2	0
$\bar{p}_{346}$	1	2	2	1	1	1	1	2	0
$\bar{p}_{356}$	1	2	2	2	1	1	1	2	1
$\bar{p}_{456}$	1	2	3	2	1	1	1	2	1

TABLE 1. the plabic graph G for $\mathrm{Gr}_3(\mathbb{C}^6)$ for which $\Delta(A_{3,6}, \nu_G)$ is not integral (see [RW19, §9]) and the images of Plücker coordinates under the valuation $\mathfrak{v}_G$ as in Example 6.2.

We change the representation of $A_{3,6}$ to be $\mathbb{C}[p_{ijk}, X, Y : 1 \leq i < j < k \leq 6]/J_{3,6}$, where the ideal $J_{3,6}$ is minimally generated by the following 37 quadratic polynomials:

$$\begin{array}{lll}
p_{145}p_{236} - p_{123}p_{456} - X, & p_{124}p_{356} - p_{123}p_{456} - Y, & p_{136}p_{245} - p_{126}p_{345} - X, \\
p_{125}p_{346} - p_{126}p_{345} - Y, & p_{146}p_{235} - p_{156}p_{234} - X, & p_{134}p_{256} - p_{156}p_{234} - Y, \\
p_{246}p_{356} - p_{346}p_{256} - p_{236}p_{456}, & p_{245}p_{356} - p_{345}p_{256} - p_{235}p_{456}, & p_{146}p_{356} - p_{346}p_{156} - p_{136}p_{456}, \\
p_{145}p_{356} - p_{345}p_{156} - p_{135}p_{456}, & p_{245}p_{346} - p_{345}p_{246} - p_{234}p_{456}, & p_{235}p_{346} - p_{345}p_{236} - p_{234}p_{356}, \\
p_{145}p_{346} - p_{345}p_{146} - p_{134}p_{456}, & p_{135}p_{346} - p_{345}p_{136} - p_{134}p_{356}, & p_{146}p_{256} - p_{246}p_{156} - p_{126}p_{456}, \\
p_{145}p_{256} - p_{245}p_{156} - p_{125}p_{456}, & p_{136}p_{256} - p_{236}p_{156} - p_{126}p_{356}, & p_{135}p_{256} - p_{235}p_{156} - p_{125}p_{356}, \\
p_{235}p_{246} - p_{245}p_{236} - p_{234}p_{256}, & p_{145}p_{246} - p_{245}p_{146} - p_{124}p_{456}, & p_{136}p_{246} - p_{236}p_{146} - p_{126}p_{346}, \\
p_{134}p_{246} - p_{234}p_{146} - p_{124}p_{346}, & p_{125}p_{246} - p_{245}p_{126} - p_{124}p_{256}, & p_{134}p_{245} - p_{234}p_{145} - p_{124}p_{345}, \\
p_{135}p_{245} - p_{235}p_{145} - p_{125}p_{345}, & p_{135}p_{236} - p_{235}p_{136} - p_{123}p_{356}, & p_{134}p_{236} - p_{234}p_{136} - p_{123}p_{346}, \\
p_{125}p_{236} - p_{235}p_{126} - p_{123}p_{256}, & p_{124}p_{236} - p_{234}p_{126} - p_{123}p_{246}, & p_{134}p_{235} - p_{234}p_{135} - p_{123}p_{345}, \\
p_{124}p_{235} - p_{234}p_{125} - p_{123}p_{245}, & p_{135}p_{146} - p_{145}p_{136} - p_{134}p_{156}, & p_{125}p_{146} - p_{145}p_{126} - p_{124}p_{156}, \\
p_{125}p_{136} - p_{135}p_{126} - p_{123}p_{156}, & p_{124}p_{136} - p_{134}p_{126} - p_{123}p_{146}, & p_{124}p_{135} - p_{134}p_{125} - p_{123}p_{145},
\end{array}$$

$$p_{135}p_{246} - p_{156}p_{234} - Y - p_{123}p_{456} - X - p_{126}p_{345}.$$

The tropicalization of the ideal $J_{3,6}$ is not known completely, but the intersection of the tropicalization with a specific maximal cone in its Gröbner fan was computed in [BMNC21, §4.4]. We summarize their findings.

Let $\{e_{123}, \dots, e_{456}, e_x, e_y\}$ denote the standard basis of $\mathbb{R}^{22}$ we define (by slight abuse of notation) the elements f_{ij} and g_{abcdef} in $\mathbb{R}^{22}$ as the same linear combinations of standard basis elements as in $\mathbb{R}^20$. Then the lineality space $\mathcal{L}_{J_{3,6}}$ is six-dimensional and spanned by

$$E_i := \sum_{k,j \neq i} e_{ijk} + e_x + e_y.$$

The ideal $J_{3,6}$ is invariant under the action of the group $\Sigma := \langle (123456), (16)(25)(34) \rangle \subset S_6$. In particular, this action translates to an action on $\mathrm{Trop}(J_{3,6})$.

Theorem 6.3 (§4.4 in [BMNC21])**.** *There is a distinguished maximal simplicial cone $C \in \mathrm{GF}(J_{3,6})$ that is invariant under the action of Σ with ray generators*

(1) 6 of form $e_{i,i+1,i+2}$ for $i \in \mathbb{Z}_6$;
(2) 6 of form $f_{i,i+1} + \begin{cases} e_y & i \text{ odd} \\ e_x & i \text{ evem} \end{cases}$;
(3) 2 of form $g_{123456} + e_y$ and $g_{456123} + e_x$;
(4) 2 of form $g_{654321} + e_y$ and $g_{321654} + e_x$.

All 16 rays are also rays of $\mathrm{Trop}(J_{3,6})$ (more precisely of its totally positive part). The intersection $C \cap \mathrm{Trop}(J_{3,6})$ contains 50 maximal simplicial cones of $\mathrm{Trop}(J_{3,6})$ that are all prime. Specifically we find

(1) 6 in the Σ-orbits of $\{e_{123}, e_{156}, f_{23} + e_x, f_{56} + e_y\}$ projecting onto cones of type EEFF1;
(2) 12 in the Σ-orbits of $\{e_{123}, e_{456}, f_{12} + e_y, f_{56} + e_y\}$ and $\{f_{34} + e_y, f_{16} + e_x, e_{156}, e_{234}\}$ projecting onto cones of type EEFF2;
(3) 12 in the Σ-orbits of $\{e_{123}, g_{321654} + e_x, f_{23} + e_x, f_{45} + e_x\}$ and $\{f_{34} + e_y, g_{123456} + e_y, f_{12} + e_y, e_{345}\}$ projecting onto cones of type EFFG;
(4) 4 in the Σ-orbit of $\{e_{123}, g_{123456} + e_y, e_{156}, e_{345}\}$ projecting onto cones of type EEEG;
(5) 12 in the Σ-orbits of $\{g_{456123} + e_x, e_{456}, f_{23} + e_x, e_{234}\}$ and $\{g_{654321} + e_y, f_{56} + e_y, e_{456}, e_{234}\}$ projecting onto cones of type EEFG;
(6) 4 in the Σ-orbit of $\{f_{45} + e_x, f_{16} + e_x, f_{23} + e_x, g_{321654} + e_x\}$ projecting onto FFFG type pyramids inside the maximal cones of bipyramid type FFFGG.

Here the projections refer to the projection $\mathbb{R}^{22} \supset \mathrm{Trop}(J_{3,6}) \to \mathrm{Trop}(I_{3,6}) \subset \mathbb{R}^{20}$.

The ideal $J_{3,6}$ has the following advantage over the ideal $I_{3,6}$:

Corollary 6.4. *The tropicalization $\mathrm{Trop}(J_{3,6})$ contains maximal prime cones associated with all Khovanskii-finite valuations on $A_{3,6}$ for which $\{p_{123}, \dots, p_{456}, X, Y\}$ is a Khovanskii basis. In particular, this includes all valuations associated with plabic graphs for $\mathrm{Gr}_3(\mathbb{C}^6)$.*

In particular, in $\mathrm{Trop}(J_{3,6})$ we find a maximal prime cone associated with the valuation from Example 6.2. It is identified with the cone whose rays are $\{f_{45} + e_x, f_{16} + e_x, f_{23} + e_x, g_{456123} + e_x\}$.

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