RANK TWO WEAK FANO BUNDLES ON DEL PEZZO THREEFOLDS OF PICARD RANK ONE

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1. Weak Fano bundles

In this note, the author would like to sum up joint works with T. Fukuoka and D. Ishikawa, about weak Fano bundles [3, 4]. Throughout this note we work over the complex number field $\mathbb{C}$. We start from the definition of weak Fano bundles.

Definition 1.1. Let X be a smooth projective variety.
(1) X is called *weak Fano* if $-K_X$ is nef and big.
(2) A vector bundle $\mathcal{E}$ on X is called *weak Fano* if $\mathbb{P}_X(\mathcal{E}) = \operatorname{Proj}\operatorname{Sym}^\bullet \mathcal{E}$ is a weak Fano variety.

It is known that a variety which admits a weak Fano bundle is also weak Fano [16, Theorem A]. Fano varieties form an important class of projective varieties from the point of view of the classification of algebraic varieties, and weak Fano varieties can be regarded as a degeneration of Fano manifolds. The classification of weak del Pezzo surfaces, i.e. weak Fano varieties of dimension two, was given by Demazure [2]. In dimension three, it turned out that studying weak Fano manifolds has a contribution to the classification problem of Fano threefolds. Namely, weak Fano threefolds of Picard rank two allowed us to connect two Fano threefolds of Picard rank one in terms of extremal contractions, and this technique is called the 2-ray game method. Using this method, Takeuchi [15] concisely proved the existence of a line on a Mukai 3-fold, which provided a new perspective on the classification of Fano 3-folds. After the work by Takeuchi, classifying weak Fano 3-folds of Picard rank 2 has been considered to be significant and treated in many literatures [1, 8, 10].

Following the big success of the theory of Fano and weak Fano threefolds, it is natural to start studying weak Fano fourfolds. However, the world of Fano and weak Fano fourfolds is much more complicated than the threefold case, which enforce us to concentrate on a certain "nice" class of weak Fano fourfolds as the first step. Then, following the work of Szurek and Wiśniewski [14], it would be interesting to study weak Fano fourfolds that admit a locally trivial $\mathbb{P}^1$-bundle structure, in other words, those are isomorphic to the projectivization of rank two weak Fano bundles. This kind of study was initially started by Yasutake, and he classified rank two weak Fano bundles on $\mathbb{P}^3$ [16, Theorem B].

Let $\mathcal{E}$ be a rank two weak Fano bundle on a weak Fano threefold X. If $Y = \mathbb{P}_X(\mathcal{E})$ has $\rho(Y) = 2$, then $\rho(X) = 1$ and hence X is a Fano threefold. It is well-known that Fano threefolds with $\rho(X) = 1$ have the following classification. Let $i(X)$ be the Fano index of X. When $i(X) = 4$, then $X \simeq \mathbb{P}^3$; $i(X) = 3$ then $X \simeq Q^3$ where Q^3 is the three dimensional quadric; $i(X) = 2$ then $X \simeq X_d$ is a del Pezzo threefold of degree d with $1 \leq d \leq 5$; $i(X) = 1$ then we obtain the class of Mukai threefolds. The complete list of there Fano threefolds can be found in [7, §12.2].

We classify rank two weak Fano bundles on a del Pezzo Fano threefold X of Picard rank one.

2. Key Theorems

Let X be a Fano threefold of Picard rank one. Then there exists an identification $\mathrm{Pic}(X) \simeq \mathbb{Z}$ that sends positive line bundles to positive integers, which allows us to identify the first Chern class with integers. Since the definition of weak Fano bundles is up to line bundle twist, rank two weak Fano bundles on X can be divided into two types according to the evenness and the oddness of their first Chern class. The following theorem gives a description for one of these two types, which works to any Fano threefolds of Picard rank one.

Theorem 2.1 ([3, Theorem 1.7]). *Let $\mathcal{E}$ be a rank two weak Fano bundle on a Fano threefold X of Picard rank one. If $c_1(\mathcal{E}) = c_1(-K_X)$, then $\mathcal{E}$ is globally generated, except when X is a del Pezzo threefold of degree one and $\mathcal{E} \simeq \mathcal{O}_X(H)^{\oplus 2}$, where H is the ample generator of* $\mathrm{Pic}(X)$.

For del Pezzo threefolds of Picard rank one, we can give a complete classification of the other class, i.e. rank two weak Fano bundles with odd $c_1(\mathcal{E})$. Surprisingly, it is turned out that this class contains Fano bundles only. The following theorem is the combination of all our works for rank two weak Fano bundles on del Pezzo Fano threefolds [3, Theorem 1.1] [4, Theorem 1.4, Corollary 4.3] [6, Theorem 1.1].

Theorem 2.2. *Let $\mathcal{E}$ be a rank two weak Fano bundle on a del Pezzo threefold X of Picard rank one. If $c_1(\mathcal{E}) = -1$, then $\mathcal{E}$ is a Fano bundle.*

Note that by [12, Theorem 1.1] it follows that there exists a complete classification of rank two Fano bundles on Fano threefolds with Picard rank one. Therefore, to classify rank two weak Fano bundles on del Pezzo threefolds, we are left to classify bundles with even $c_1(\mathcal{E})$ via the identification $\mathrm{Pic}(X) \simeq \mathbb{Z}$.

3. Classification for del Pezzo threefolds

Throughout this section, let X be a del Pezzo threefold of Picard rank one.

A rank two vector bundle is called split if it is isomorphic to a direct sum of line bundles. A a vector bundle $\mathcal{E}$ on X is stable (resp. semistable) if $\frac{c_1(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} <$ (resp. $\leq$)$\frac{c_1(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$ for every non-trivial subsheaf $0 \neq \mathcal{F} \subsetneq \mathcal{E}$.

Theorem 3.1 ([4, Theorem 1.4]). *Let X_d be a del Pezzo threefold of degree ≤ 2. Then all weak Fano bundles on X_d are split.*

In the following, we concentrate on the case when $c_1(\mathcal{E}) = 0$, because the other case is already covered by Theorem 2.2.

In contrast to Theorem 3.1, the next proposition shows that del Pezzo threefolds of degree ≥ 3 and Picard rank one can admit non-split rank two weak Fano bundles. It is possible to show that any non-split rank two weak Fano bundle is semistable. In addition, if a non-split weak Fano bundle is not stable, then it is described using lines.

Proposition 3.2 ([6, Lemma 3.2]). *Let $X = X_d$ be a del Pezzo threefold of Picard rank one and of degree $3 \leq d \leq 5$. Let $\mathcal{E}$ is a rank two non-split weak Fano bundle* *with $c_1(\mathcal{E}) = 0$. Then $\mathcal{E}$ is not stable if and only if $\mathcal{E}$ fits into a unique non-trivial exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_{l/X} \to 0,$$

where $\mathcal{I}_{l/X}$ is the ideal sheaf of a line $l \subset X$. Conversely, if $\mathcal{E}$ is a vector bundle that fits in the sequence above, then it is weak Fano, but not Fano.

Here the unique non-trivial extension means the one corresponding to a non-trivial element of $\mathrm{Ext}^1(\mathcal{I}_{l/X}, \mathcal{O}) \simeq \mathbb{C}$.

The remaining case, when $\mathcal{E}$ is a stable weak Fano bundle with $c_1(\mathcal{E}) = 0$, can be described in a similar way, but in this case it is related to elliptic curves. Indeed, for generic global section $s \in H^0(\mathcal{E}(1))$, its zero locus $C = V(s)$ is a smooth curve, since $\mathcal{E}(1)$ is globally generated by Theorem 2.1. In addition, there exists an exact sequence

$$0 \to \mathcal{O} \to \mathcal{E}(1) \to \mathcal{I}_{C/X}(2) \to 0,$$

by the Hartshorne-Serre correspondence (see [5, Section 5.1], for example). Applying Riemann-Roch theorem, it follows that C is an elliptic curve of degree $\deg(X) + c_2(\mathcal{E})$. Since $\mathcal{E}(1)$ is globally generated, so is $\mathcal{I}_{C/X}(2)$, which means that C is defined by quadratic equations. Note that $\mathrm{Ext}_1(\mathcal{I}_{C/X}(2), \mathcal{O}) \simeq \mathbb{C}$, and hence a non-trivial extension of $\mathcal{I}_{C/X}(2)$ by $\mathcal{O}_X$ is unique up to isomorphism.

Theorem 3.3 ([3, Theorem 1.1] [4, Section 5.1], [6, Section 3.2.3])**.** *Let $X = X_d$ be a del Pezzo threefold of Picard rank one and of degree $3 \leq d \leq 5$. Let $\mathcal{E}$ be a rank two stable vector bundle on X with $c_1(\mathcal{E}) = 0$. Then $\mathcal{E}$ is weak Fano if and only if the following two conditions are satisfied.*

1. $2 \leq c_2(\mathcal{E}) \leq d - 1$.
2. *$\mathcal{E}$ fits into a unique non-trivial extension*

$$0 \to \mathcal{O}_X \to \mathcal{E}(1) \to \mathcal{I}_{C/X}(2) \to 0,$$

where $\mathcal{I}_{C/X}$ is the ideal sheaf of a smooth elliptic curve $C \subset X$ of degree $d + c_2(\mathcal{E})$ that is defined by quadratic equations.

Furthermore, for any $3 \leq d \leq 5$ and $2 \leq c \leq d - 1$, there is an example of a stable weak Fano bundle on X_d with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = c$.

We remark that all weak Fano bundles as in the above theorem are not Fano.

In the proof of the existence of examples for stable weak Fano bundles, the most difficult case is when $(d, c) = (4, 3)$. In this case, we should prove that any del Pezzo threefold of degree 4 contains a smooth elliptic curve of degree 7 which is generated by quadratic equations. To workout this problem, we prepared the following geometric characterisation for quadratically generated elliptic curves. Recall that a del Pezzo threefold of degree 4 is a smooth intersection of two hyperquadrics in $\mathbb{P}^5$.

Lemma 3.4 ([3, Proposition 5.2])**.** *Let $C \subset \mathbb{P}^5$ be a smooth elliptic curve of degree 7. Then C is generated by quadratic equations if and only if it does not have a trisecant.*

In the construction, we find a reducible curve of arithmetic genus 1 and of degree 7 that doesn't have a trisecant, and then we prove that there exists a smoothing of that curve. Then since the non-existence of trisecants is an open condition, the desired elliptic curve is obtained. See [3, Section 5.3] for more detail.

We finish this section with the following observation. Let $\mathcal{E}$ be a rank two weak Fano bundle on a del Pezzo threefold X of Picard rank one, and assume that $c_1(\mathcal{E}) = 0$ and $\mathcal{E}$ is stable. Then using the exact sequence

$$0 \to \mathcal{O}(-2) \to \mathcal{E}(-1) \to \mathcal{I}_{C/X} \to 0$$

in Theorem 3.3 implies the vanishing $H^1(\mathcal{E}(-1)) = 0$, which is the definition of instanton bundles in the sense of Kuznetsov [9].

Proposition 3.5. *Let $\mathcal{E}$ be a rank two stable weak Fano bundle with $c_1(\mathcal{E}) = 0$ on a del Pezzo threefold X of Picard rank one. Then $\mathcal{E}$ is an instanton bundle.*

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