The Minkowski Length of Two-Dimensional Lattice Polytopes

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Abstract

This report looks into the classification of two-dimensional lattice polytopes. It studies the previous method of classification, as well as providing an original Inductive Algorithm for finding polytopes of given Minkowski lengths. The discovery of this algorithm is important for mathematicians who work in Coding Theory, since the lattice points within a polytope can be taken as polynomials. These polynomials are then evaluated at specific points to create a code.

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1 Introduction

Whilst studying toric codes within Coding Theory, one comes across convex lattice polytopes [1][2][3]. These are finite n-dimensional "shapes" which exist in an infinite grid of equally spaced points in a lattice. The lattice points of the polytope P are used to generate a polynomial $f(x_1, x_2, \ldots, x_n)$, whose Newton polytope is $Newt(f) = P$, which can be evaluated in such a way as to create a linear code. This is where the Minkowski length of the polytope is important, since it relates the minimum distance of a toric code to the combinatorics of P.

Working with lattice polytopes is still a very new topic within mathematics. In fact, the classification of all polytopes of Minkowski length 2 was only found within the last two years, there being only 16 of them in total [1].

The process used in [1] to find all polytopes of Minkowski length 2 is very long and generalises poorly. It involves cancelling lattice points individually and then using the remainder to create the polytopes of the length you wish to find. Unfortunately, this process generates many polytopes that are isomorphic to one another, and so is very inefficient.

The aim of my project is to look into finding a more efficient process for classifying polytopes of a given Minkowski length. I will then use this process to find all of the polytopes with Minkowski length 3 (see Section 6).

I will first begin by defining what a lattice polytope is as well as how to calculate its Minkowski length. I will then describe how to classify polytopes using the previous cancellation process and also the polytopes that have been found with this method. Next I will give details of the Inductive Algorithm that I have developed for classifying further polytopes, and finally show how I have used my algorithm to find all polytopes of length 3.

2 Definitions

2.1 Lattice Grid

A n-dimensional lattice L is a free $\mathbb{Z}$-module of rank n. Hence, $L \cong \mathbb{Z}^n$. Since we are looking at polytopes, we need to embed $\mathbb{Z}^n$ into a fixed $\mathbb{R}^n$. To do this, we can pick n $\mathbb{R}$-linearly independent points $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^n$ and a basis $\{e_1, \ldots, e_n\}$ in $\mathbb{Z}^n$. L is then embedded in $\mathbb{R}^n$ via the map:

$$\sigma : \mathbb{Z}^n \longrightarrow \mathbb{R}^n$$
$$e_i \longmapsto v_i$$

For us the (Euclidean) volume of a polytope is not important, so we are free to take $v_i = e_i$ and hence take the natural inclusion of $\mathbb{Z}^n$ in $\mathbb{R}^n$.

We will only be looking at two-dimensional cases throughout this project, therefore $L \cong \mathbb{Z}^2$.

2.2 Lattice Polytopes

A polytope P is the convex hull of a set of points S in $\mathbb{R}^n$ whose convex hull is bounded.

$$P = conv(S) = \{\sum_{s \in S} \lambda_s S \mid \lambda_s \geq 0, \sum_{s \in S} \lambda_s = 1\}$$

That is, a bounded convex set in $\mathbb{R}^n$. The vertices of P, $vert(P)$, consist of those $v \in P$ such that we cannot write $v = \sum_{s \in S} \lambda_s v_s$ with 2 or more $\lambda_s \neq 0$. P is said to be a lattice polytope if $vert(P) \subseteq \mathbb{Z}^n$. In particular, if P is a lattice polytope then P has finitely many vertices $vert(P) = \{v_1, \ldots, v_n\}$ and $P = conv(vert(P))$.

Throughout we will only consider lattice polytopes, and will simply call them "polytopes".

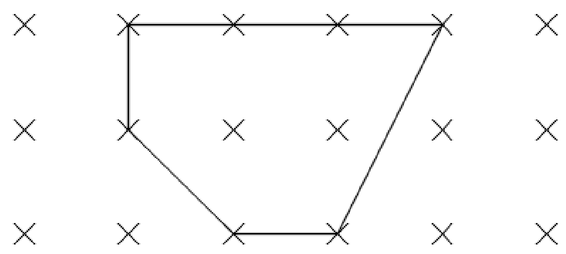


Figure 1: A Lattice Polytope

There are two ways of looking at polytopes:

1. Either as the convex hull of a finite set of vertices of P; or

2. $$P = \bigcap_{finite} H_u$$

   where H_u is a half-space [4]. In the case where P is 2-dimensional, every edge of P corresponds to a half-space, and the intersection of all such half-spaces is equal to P. Looking at P in this way will be relevant in Chapter 5 where I explain the inductive algorithm that I have created to shell around polytopes.

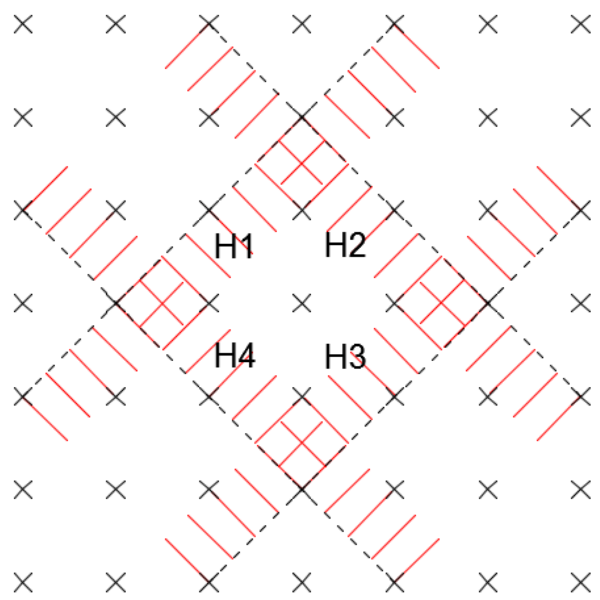


Figure 2: Half-spaces $H_1, \ldots, H_4$ whose intersection is P

2.3 Minkowski Sum

Given two polytopes, P_1 and P_2, we define the Minkowski sum to be equal to the summation of each point of P_1 and P_2. i.e.

$$P_1 + P_2 = \{v_1 + v_2 | v_1 \in P_1, v_2 \in P_2\}$$

This creates a new polytope P_3 with $vert(P_3) \subseteq \{v_1 + v_2 \mid v_1 \in vert(P_1),\ v_2 \in vert(P_2)\}$.

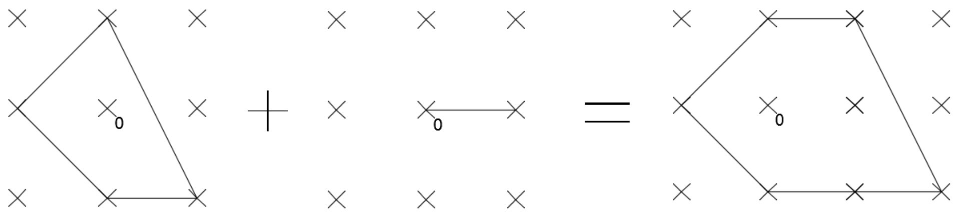


Figure 3: Example of a Minkowski Sum (1)

For the example in Figure 3, the placement of the origin specifically creates the polytope shown in the figure. However, if we were to change the origin, like in Figure 4, the new polytope created would look different; specifically, the location of the origin has changed.

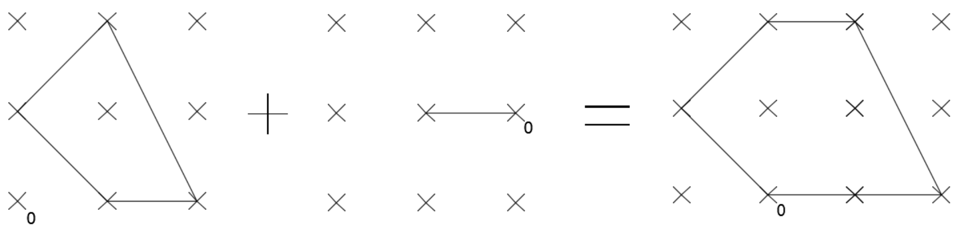


Figure 4: Example of Minkowski Sum (2)

2.4 Minkowski Length

Fix a lattice polytope P, and suppose there exists polytopes Q_i, where $dim(Q_i) > 0$, such that:

$$P \supseteq Q_1 + Q_2 + \ldots + Q_r$$

for some r. The Minkowski length $L(P)$ is defined to be the largest such r for which this sum is possible. That is:

$$L(P) := \max\left(r : Q_1 + Q_2 + \ldots + Q_r \subseteq P\right).$$

Note that it is important that $dim(Q_i) > 0$, otherwise r is unbounded since we can always take Q_i to be a single point.

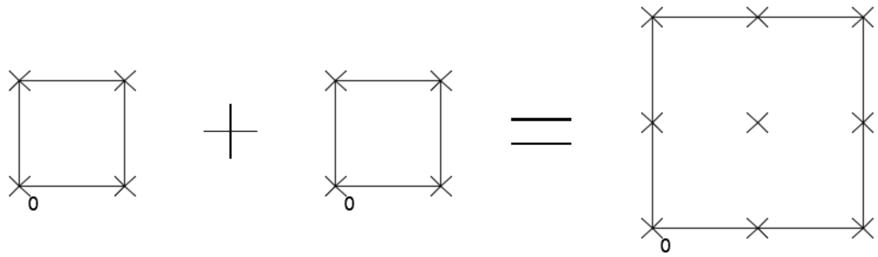


Figure 5: Example of Minkowski length with two-dimensional polytopes

Figure 5 shows an example of two polytopes which sum together to create a larger polytope. We would at first assume that the Minkowski length of this polytope is 2, but this is in fact not the case, as I will now show in the following figures.

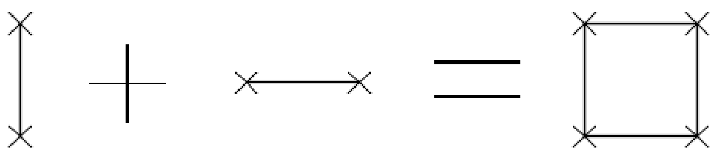


Figure 6: Polytope generated by line segments

For any edge E of Q, we have $P \supseteq Q \supseteq E$, and since E can be written as a sum of line segments of length 1, as shown in Figure 6, without loss of generality we can insist that the Q_i's that calculate our Minkowski length are line segments of length 1. This means that we are looking for the largest number of line segments that can be contained within P.

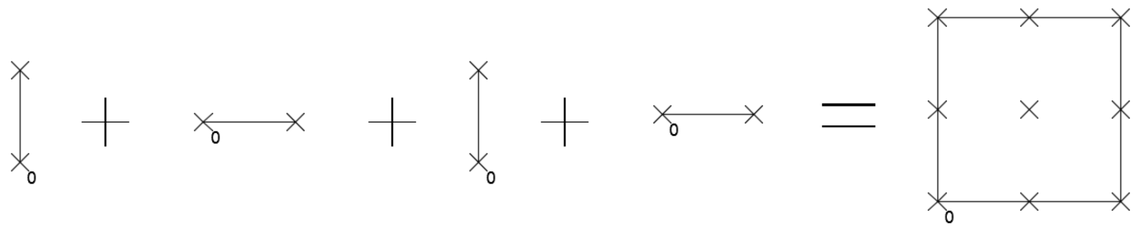


Figure 7: Example of Minkowski length with line segments

In this example, the Minkowski length of our polytope is 4. It is possible that there are multiple ways of including the maximum number of line segments in a polytope, as shown in Figures 8 and 9:

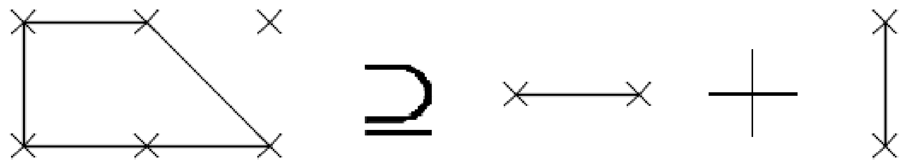


Figure 8: Minkowski length of a Lattice Polytope (1)

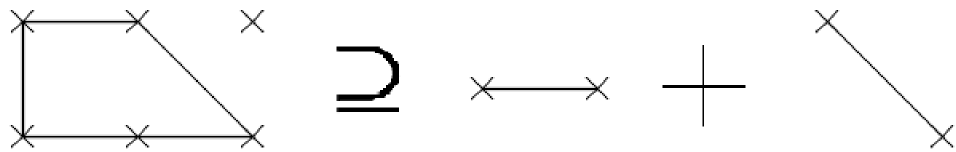


Figure 9: Minkowski length of a Lattice Polytope (2)

That is, there may exist many distinct ways of achieving $L(P)$.

2.5 Isomorphic Polytopes

Let P and Q be two lattice polytopes. $P \cong Q$ if and only if there exists a lattice automorphism:

$$\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$$

such that $\phi(P) = Q$. That is, ϕ is a change of basis of $\mathbb{Z}^2$, or equivalently we may regard *phi* as an invertible 2x2 matrix M with integer coefficients. Hence $det(M) = \pm 1$ and so $M \in GL_2(\mathbb{Z})$.

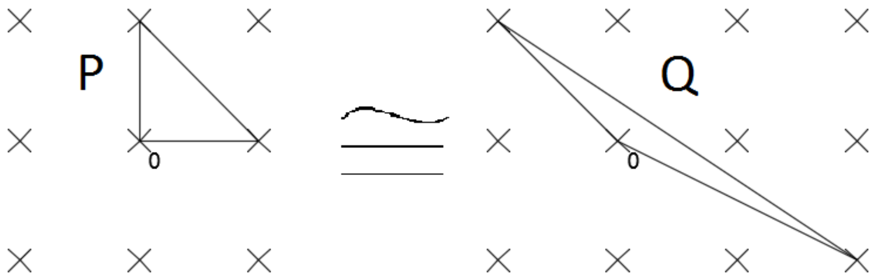


Figure 10: Example of change of basis

Figure 10 shows two polytopes, P and Q. They appear very different but when you apply the change of basis $M = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ at the fixed origin, we see that $PM = Q$ for all points in P and Q. Therefore $P \cong Q$. In the cases where we can apply a change of basis, we only need to refer to one of the isomorphic polytopes.

The placement of the origin is important for finding a change of basis between P and Q. The translation of a polytope can affect where the origin should be place and therefore there would exist no $M \in GL_2(\mathbb{Z})$, even though it may seem obvious that the two polytopes are isomorphic. For example, in Figure 11, one can notice that the shapes are equivalent, however, there is no $M \in GL_2(\mathbb{Z})$ which will map the points in P to the points in Q.

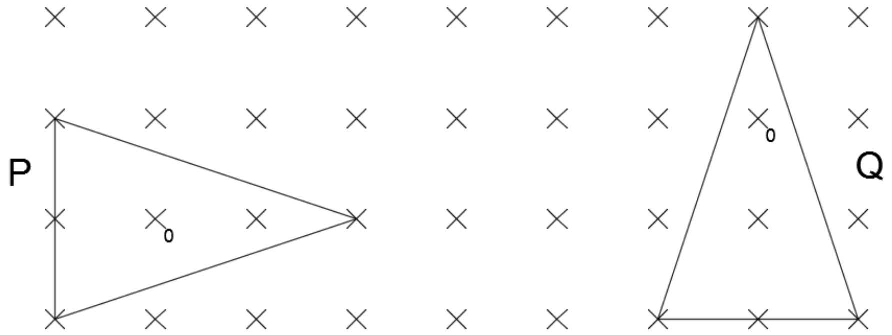


Figure 11: Two shapes that are isomorphic dependant on origin

This is due to the placement of the origin. After translation, $P \cong Q + (0,1)$.

Throughout this paper we are only interested in polytopes up to change of basis and translation, and will often regard any two polytopes that are isomorphic after translation as being equivalent. This is necessary in order to make the challenge of finding polytopes of given Minkowski lengths a finite problem.

2.6 Interior Polytopes

When looking at finding polytopes P, we must look at shelling around $0, 1, 2, \ldots$ interior lattice points. We denote the interior of P by:

$$P^o = P \setminus \partial P$$

These interior lattice points themselves create a polytope, say $P_1 = conv(P^o \cap \mathbb{Z}^2)$, which has its own Minkowski length $s \leq r$, where $L(P) = r$. Again, this new polytope P_1 may contain a polytope $P_2 = conv(P_1^o \cap \mathbb{Z}^2)$ with length $t \leq s \leq r$ and so on until we have a finite sequence of nested polytopes:

$$P \supset P_1 \supset P_2 \supset \ldots \supset P_n$$

where $|P_n^o \cap \mathbb{Z}^2| = 0$.

These polytopes can be of dimension 0 (a single point), of dimension 1 (a straight line) or of dimension 2 (a polygon).

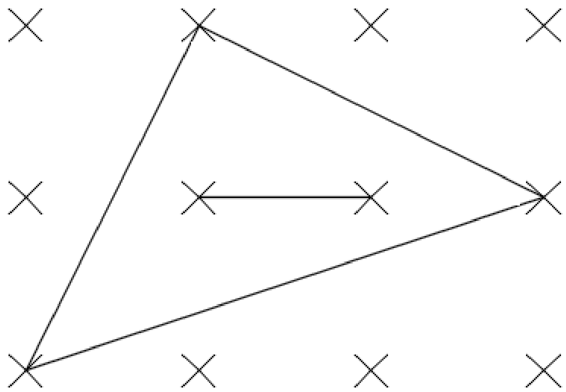


Figure 12: A Polytope that contains two interior points

2.7 Hyperplanes

The algorithm that will be introduced will involve the use of the hyperplanes of the polytopes that we are using. The hyperplanes of our 2-dimensional polytopes are the 1-dimensional lines that are created by the edges of a polytope.

Our aim is to "shell" around a given number of interior points without creating any more in the process. We can "shift" hyperplanes out away from the polytope that we are working with so as to find the set of points that we are able to use in finding the polytopes with the Minkowski length we are looking for. However, we specifically shift the hyperplanes out to one point away from the original polytope. By this I mean to the closest point along each edge, as shown in Figure 13.

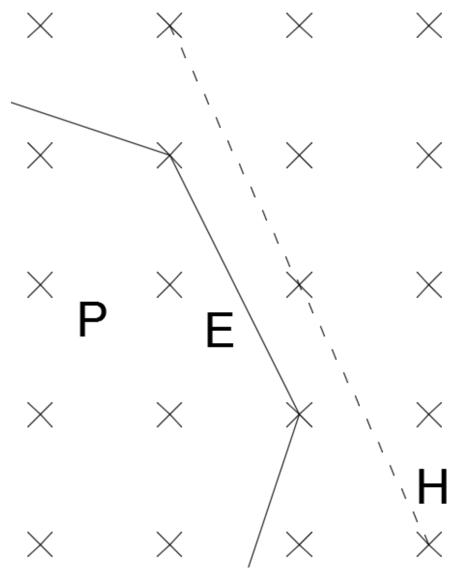


Figure 13: Shifted Hyperplane H relative to an edge E of P

This will not create any more interior points since, by moving the hyperplane to the nearest point to the edge, there will be no points between the original hyperplane and the shifted one.

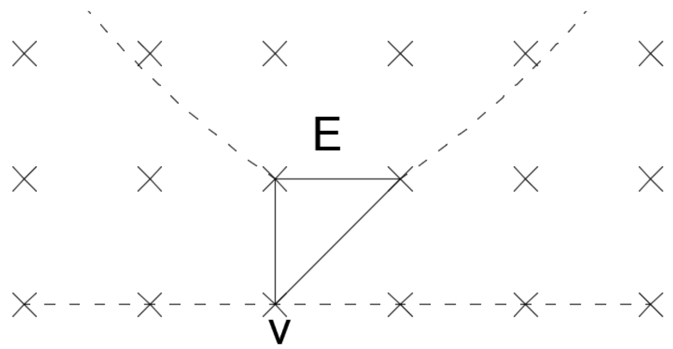


Figure 14: Empty Triangle

Figure 14 shows that, for any edge E on the polytope, the closest point v to that edge will always create an empty triangle and therefore the hyperplane method will never create more interior points and will produce all points that can be used in shelling around an interior polytope.

3 Important Classifications

3.1 Number of Interior Points

Before I can explain my own algorithm for classifying polytopes, there are a few important initial classifications to consider, the first one being the following fact [6][7]:

$$|P^o \cap \mathbb{Z}^2| \begin{cases} = 0 & \Rightarrow \text{Infinite number of polytopes} \\ \neq 0 & \Rightarrow \text{Finite number of polytopes} \end{cases}$$

This means that, up to change of basis and translation, there are only a finite number of polytopes which contain $k \neq 0$ interior lattice points. However, there are infinitely many of which $|P^o \cap \mathbb{Z}^2| = 0$. Therefore, it is important that we can classify these such polytopes so that we are able to classify them for specific Minkowski lengths.

3.2 Polytopes with No Interior Points

There are infinitely many polytopes which consist of having no interior points. However, there is only a finite number of such polytopes for each given length (up to change of basis and translation). We can classify these polytopes by taking two parallel rows of points which are of height 1 away from each other. The height refers to the distance between a and b, defined in the following way:

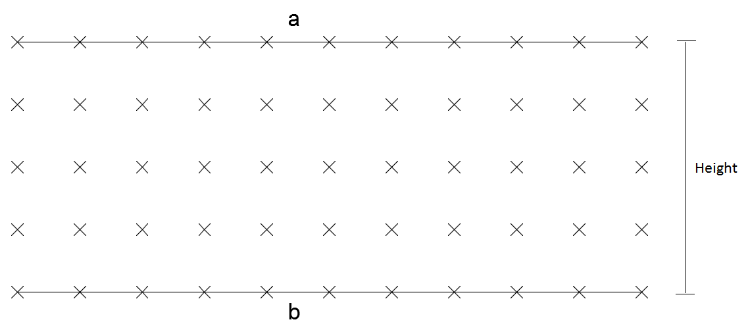


Figure 15: Height = #{Points strictly between a and b} +1

By looking at a height of 1, it is clear that there will never by any interior points created when finding polytopes since the definition of height conclusively proves that there are no points between the top and bottom rows.

All polytopes with no interior can therefore be classified by Figure 16:

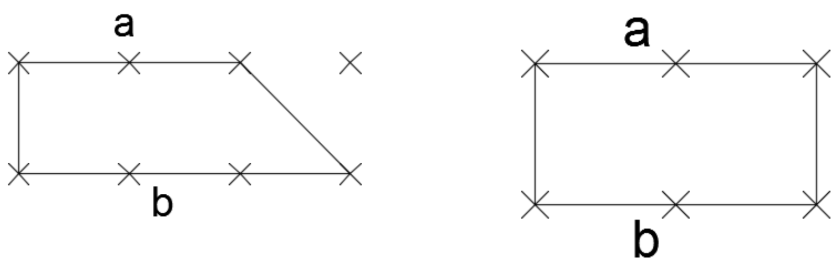


Figure 16: No Interior Points

where the a, b correspond to the parallel rows of height 1.

There are two possible lengths that the line b can take: either $L(P)$ or $L(P) - 1$. These are set in such a way so as not to create any polytopes which are of greater length than $L(P)$. In the former case, the length of a must be strictly less than that of b (i.e. $a < b$) so as to create polytopes of length $L(P)$, and in the latter, a must be equal to b (i.e. $a = b$).

There is one polytope that has no interior points which is an exception to this rule:

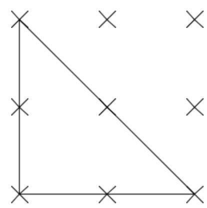


Figure 17: No Interior Points $L(P) = 2$

This polytope is of length $L(P) = 2$ and is the only polytope with no interior points that has the perpendicular edges both of length greater than 1.

3.3 One Interior Point

When the interior polytope is of dimension 0, it means that it is simply just a single lattice point. Shelling around a single interior point is already a known classification [7]. In fact, there are only 16 of such polytopes.

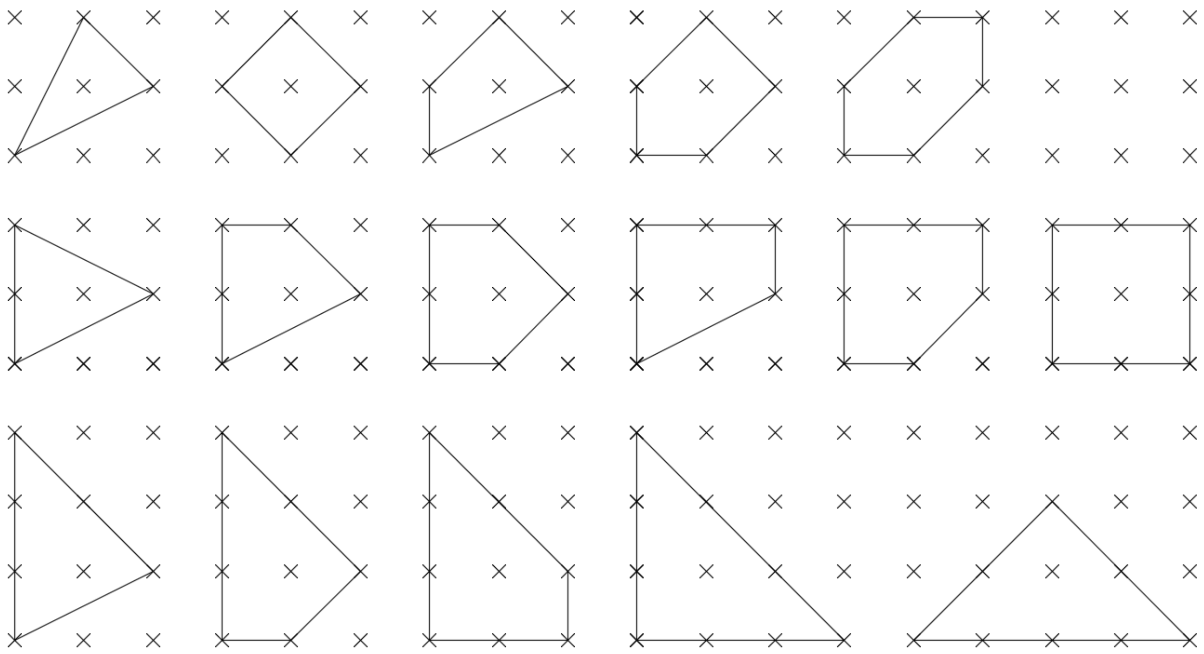


Figure 18: Polytopes with 1 Interior Point

Figure 18 shows all polytopes such that $|P^o \cap \mathbb{Z}^2| = 1$. There has in fact been work into finding polytopes with $|P^o \cap \mathbb{Z}^2| = k$, for small values of k (for $k = 2$ see [8]). However, these results will not be needed for the Inductive Algorithm since I will be looking at being able to shell around interior polytopes P^o such that $dim(conv(P^o \cap \mathbb{Z}^2)) \geq 1$ for $k \geq 2$ only. The only prior knowledge needed for the algorithm is having all polytopes with Minkowski length 1 and 2, which will be shown in Sections 4.2 and 4.3.

4 Former Method of Classification

4.1 The Cancellation Process

Prior to this project, polytopes were found by a Cancellation Process [3]. This involved checking individual points around a given number of interior points P^o to determine whether they can be used in the polytopes of Minkowski length $L(P)$. This process is cumbersome, and finds a large number of equivalent polytopes, which means more work in having to identify such polytopes so as to not include them.

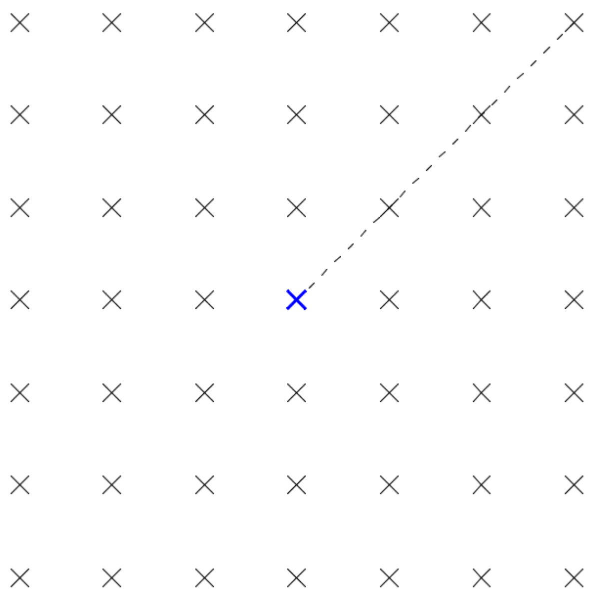


Figure 19: The Cancellation Process (1)

We start by setting the interior points on the lattice grid. From here, we need to look at the points directly around our P^o to see if they can be used. Figure 19 starts an example of using the Cancellation Process in finding polytopes with $L(P) = 1$ and $|P^o \cap \mathbb{Z}^2| = 1$. We begin by setting out a line through the interior point and one of the neighbouring points, as shown. This line will help determine which points that lie along it are within the correct distance to be used.

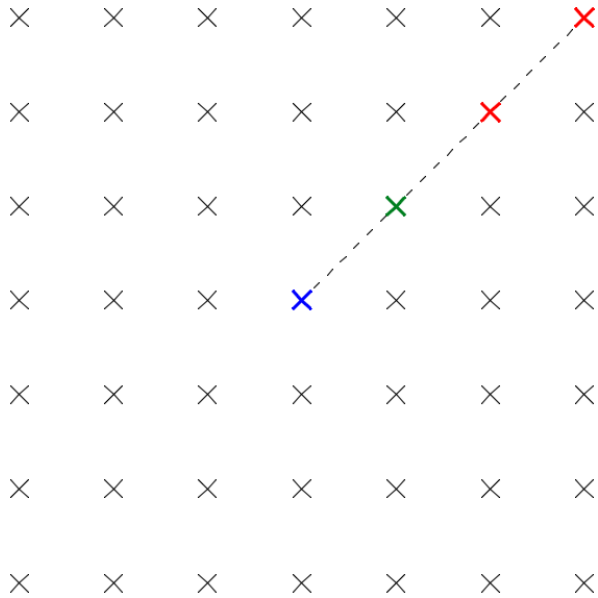


Figure 20: The Cancellation Process (2)

From Figure 20, we see that there is only one point that is of length 1 away from the interior point, and so we can “cancel out” the remaining points along this line.

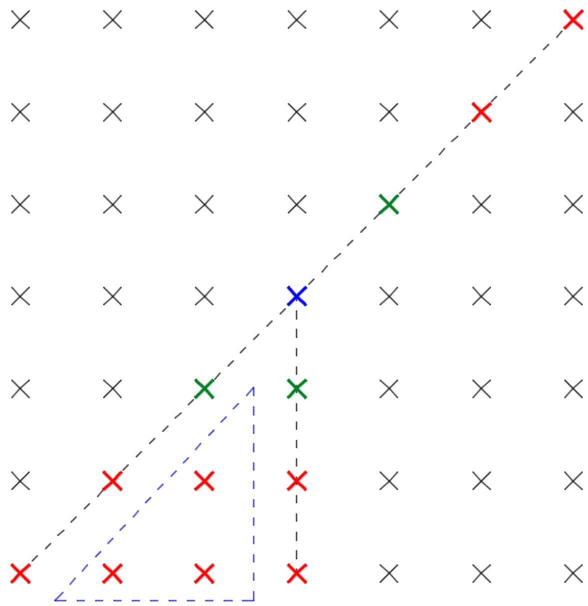


Figure 21: The Cancellation Process (3)

If we do this process for two lines, they create a triangle of points inbetween them, as shown in Figure 21. However, we can notice that most of them can be cancelled out due to being further than distance 1 from the interior point, and the rest, although are at distance 1, if included would create more interiors points. We do not want this and therefore are able to cancel them out as well.

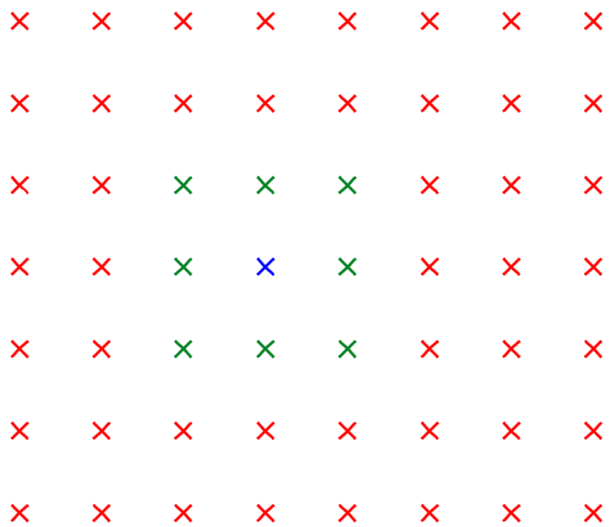


Figure 22: The Cancellation Process (4)

Continuing this process leaves us the possible points that are shown in Figure 22. They are the only possible points that can be used to find polytopes of length 1 with only 1 interior point.

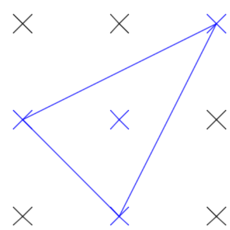


Figure 23: The Cancellation Process (5)

In fact, there is only one polytope of this type, as shown in Figure 23.

4.2 Polytopes with Minkowski Length 1

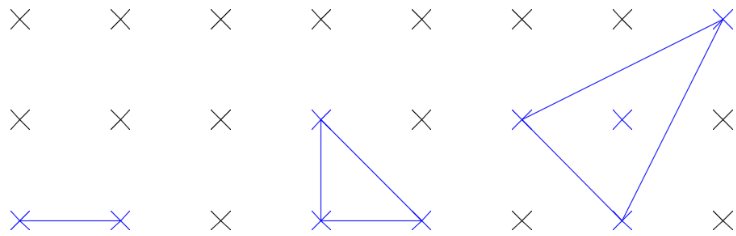


Figure 24: Polytopes with Minkowski length 1

Using the former method of classification, the number of polytopes with Minkowski Length 1 has been found, there being only the three shown above.

This result actually coincides with the following theorem [1]:

Theorem. *Let $Q \subset \mathbb{R}^m$ be strongly indecomposable. Then the number of lattice points in Q is at most 2^m. Moreover, there exist strongly indecomposable polytopes with exactly 2^m lattice points.*

where being strongly indecomposable means that each Q_i which sum together to equal P has length 1.

Since we are working in two-dimensional space, then these indecomposable polytopes should have at most $2^2 = 4$ lattice points, which is shown to be true.

4.3 Polytopes with Minkowski Length 2

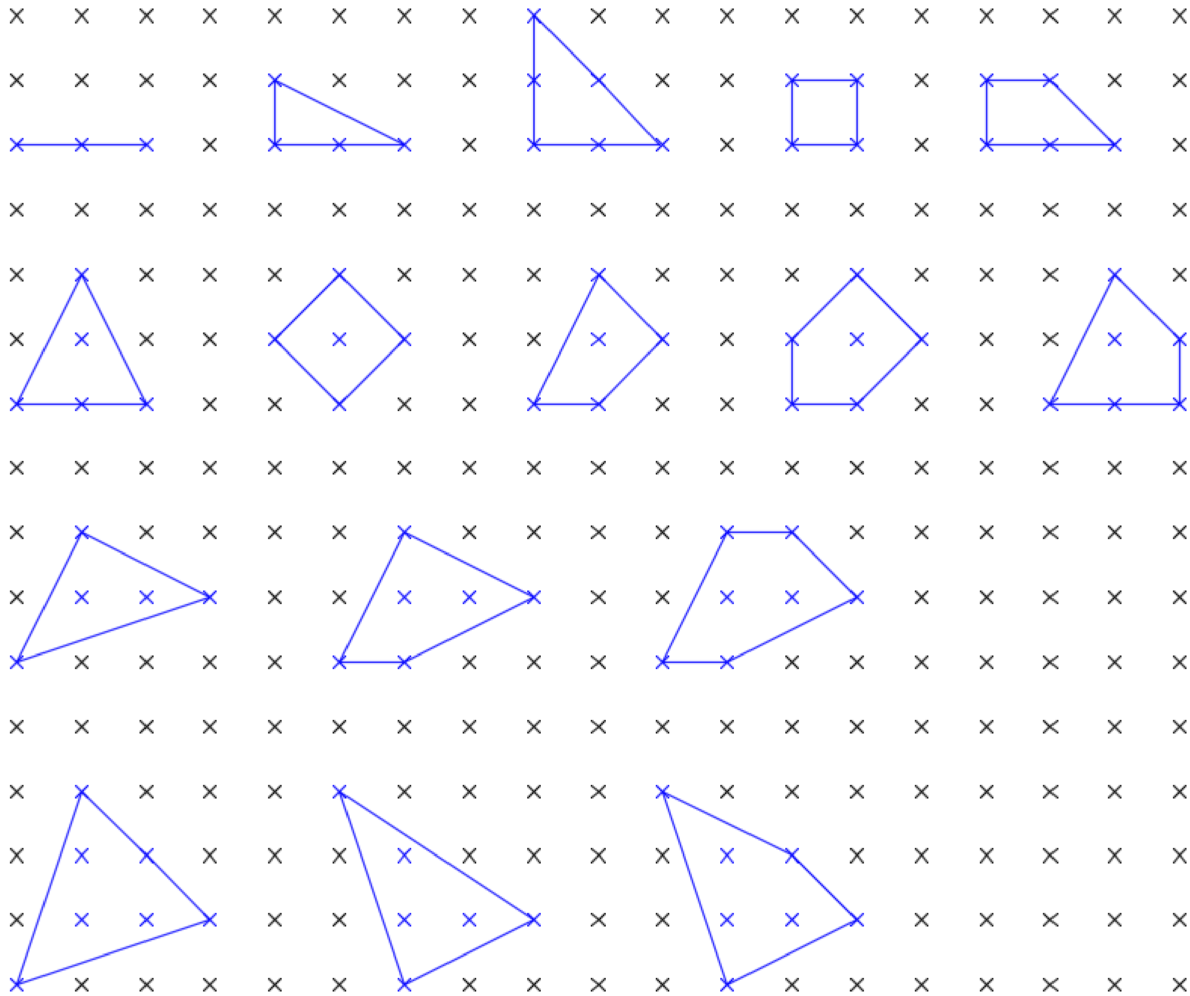


Figure 25: Polytopes with Minkowski length 2

There are precisely 16 polytopes with Minkowski length 2. Again, these were found using the former method of classification [3].

Notice how these polytopes appear to have a maximum number of interior points (i.e. $|P^o \cap \mathbb{Z}^2| \leq 3$).

4.4 Tables of Polytopes

From the information already found on polytopes of length 1 and 2, I constructed these tables, which display how many polytopes there are for each possible total number of points/number of interior points.

|PnZ^2| |P^o n Z^2|	2	3	4	5	6	7
0	1	1	0	0	0	0
1	0	0	1	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0

Figure 26: Table containing the number of polytopes with $L(P) = 1$

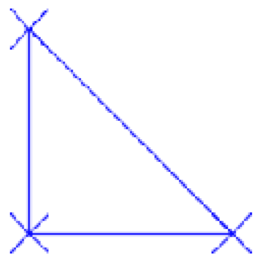
Figure 26 shows the table that is created from all the polytopes with $L(P) = 1$. The columns of the table record the total number of points used to make up a polytope whilst the rows of the table record the number of interior points in a polytope. The sum of all the entries should equal the number of polytopes in total with $L(P) = 1$.

|PnZ^2| |P^o n Z^2|	3	4	5	6	7	8
0	1	2	1	1	0	0
1	0	0	3	2	0	0
2	0	0	1	1	1	0
3	0	0	0	1	2	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0

Figure 27: Table containing the number of polytopes with $L(P) = 2$

Figure 27 show the same information but for $L(P) = 2$. These tables are used in the Inductive Algorithm to help find polytopes with $L(P) \geq 3$. For example, say we wanted to find all polytopes of length 3 with 3 interior points. We would want to shell around all polytopes $L(P) = 1$ and $L(P) = 2$ that are made up of exactly 3 points. Looking at the second column of Figure 26 and the first column of Figure 27 we can see that there are exactly two possible polytopes with 3 points that can be shelled around.

L(P) = 1	
|PnZ^2|	
|P^o n Z^2|	3
0	1
1	0
2	0
3	0
4	0
5	0



L(P) = 2	
|PnZ^2|	
|P^o n Z^2|	3
0	1
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	0

Figure 28: Example of using the tables

5 The Inductive Algorithm

If $|P^o \cap \mathbb{Z}^2| \geq 2$, then the polytope that is created out of these points, Q, will be either one-dimensional or two-dimensional. We must look at both cases individually to determine the approach in shelling around them.

5.1 Case A: Q is one-dimensional

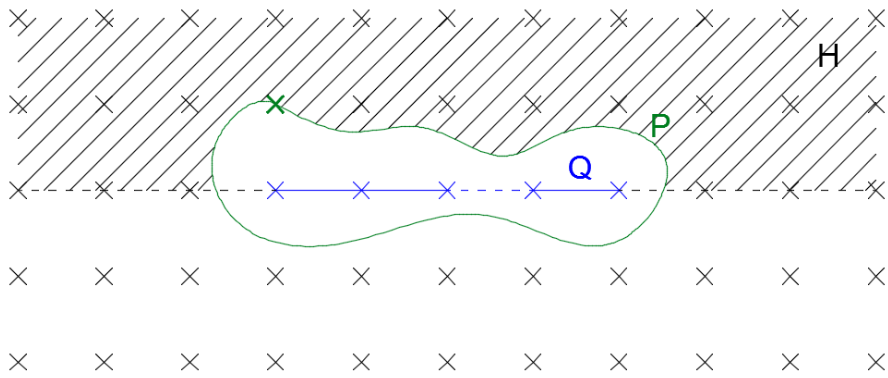


Figure 29: Shelling around one-dimensional interior polytopes

As shown by Figure 29, the polytope P will shell around the interior points in some way such that the polytope is convex and that the line Q splits the lattice grid into 2 halfplanes: H positive and H negative. We must first remove the points that cannot be contained in P.

- **First Case:**

  Consider $P_{\geq 0} := conv(P \cap \mathbb{Z}^2 \cap H)$, with $dim(P) = 2$ and $|P^o_{\geq 0} \cap \mathbb{Z}^2| = 0$. Then, the polytope shape above Q will be one of two possibilities: either the empty triangle of length 2 or a form of trapezium. However, it is clear that we cannot have the former since including the top point will produce more interior points within P.

Figure 30: First Case

Therefore, we have a row of potential lattice points that can be included in P above and below Q (by symmetry) and have therefore cancelled all lattice points beyond these two lines (as shown in Figure 30).

- **Second Case:**

  The row of lattice points to the left of Q are not included in P. This is because we have fixed a vertex of P (shown in Figure 31) and so all points to the left of this vertex have been cancelled out.

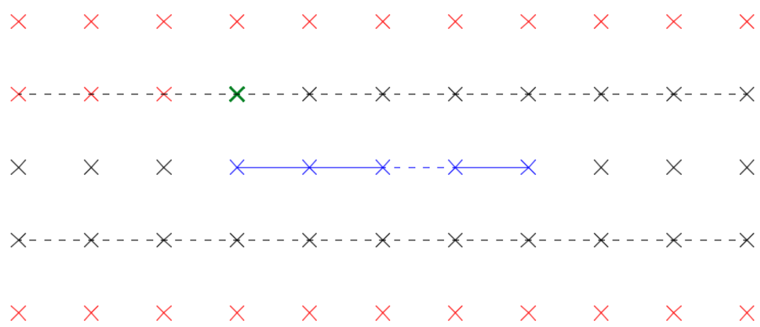


Figure 31: Second Case

- **Third Case:**

  We are able to extend the line Q in either direction by one lattice point. Going further would increase the number of interior points of P, which is not allowed, and therefore we can not include the points further than this.

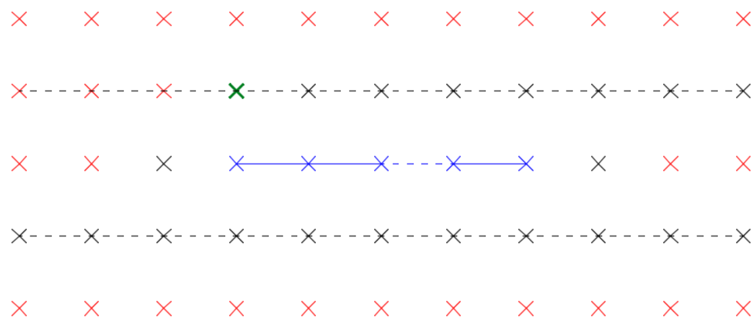


Figure 32: Third Case

Figure 32 now shows all possible points we can use when shelling around Q.

Theorem. *$L(Q)$ cannot be equal to L.*

Proof. $Q = conv(P^o \cap \mathbb{Z}^2)$ i.e. the convex hull of the interior lattice points of P. Therefore I want to shell around Q without creating anymore interior points.

Any points above or below the parallels to Q are not included otherwise they would create more interior points in P. The laws of translation mean that we do not need to include points to the left and right of this newly created rectangle around Q.

If the length of Q was equal to $L(P) = L$, then the two points that extend Q at either end cannot be included otherwise they would create a line of length greater than L, thus $L(P) > L$ which we don't want.

The dotted lines in Figure 33 represent possible edges of P. However, any combination of these lines creates edges of length greater than L, and therefore concludes that Q cannot be of length L.

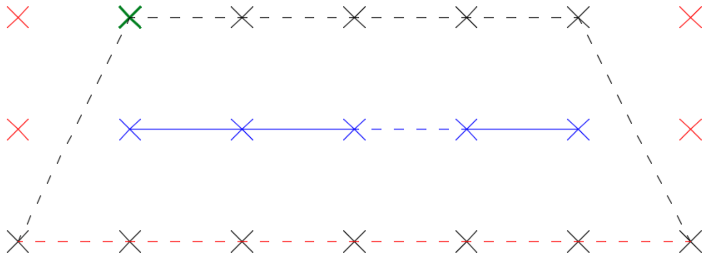


Figure 33: Shelling around line of length L

□

Example: $L(Q) = L - 1$

With the length of Q now equal to $L - 1$, it means that there is an opportunity to extend Q by one point. We must now look at 3 cases: including neither point, including the point to the left of Q, and including the point to the right of Q.

- **Including Neither Point**

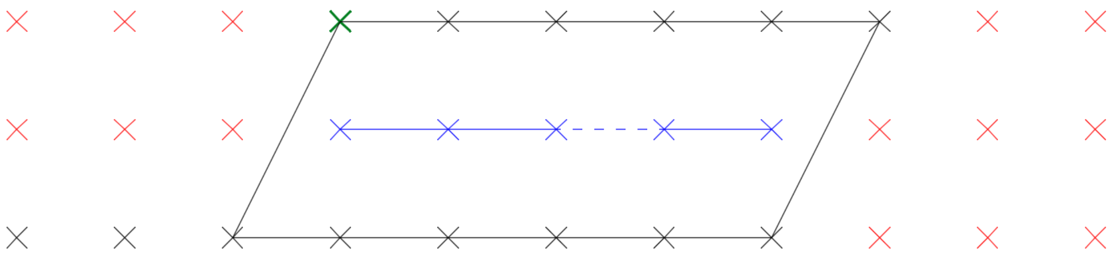


Figure 34: Including Neither Point

The polytope in Figure 34 is all that can be constructed without extending the line Q. However, this polytope has length $L+1$, therefore concluding that no polytopes of length L can shell around a line Q of length $L - 1$ without extending Q by at least one point.

We can rule out the case of extending Q by both ends since this would create a line of length $L + 1$.

- **Including the point to the left of Q**

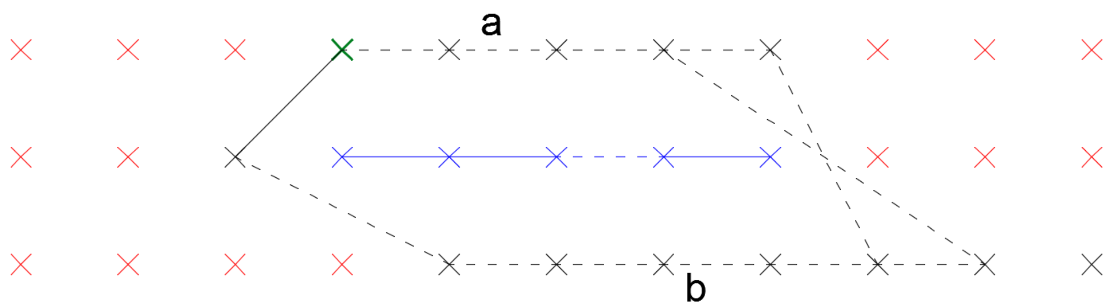


Figure 35: Including Left Point

I have labelled the top and bottom edges a and b, respectively, so as to make the classification simpler to interpret. I start by choosing my fixed point (or origin) to which I will start the process of constructing the edges of my polytope. The point I have chosen, without loss of generality, is the start of my line b.

From here, I simply began choosing different lengths of b that could create a polytope of length L. For instance, I could not have $b = L$ since this would have created a polytope of length $L+1$. With each possible length of b, I input the possible lengths that a could have, thus discovering the following:

$$a \leq L - 1 \text{ and } b \leq L - 1$$

All combinations of a and b will create polytopes of length L as long as they stay within these bounds.

- **Including the point to the right of Q**

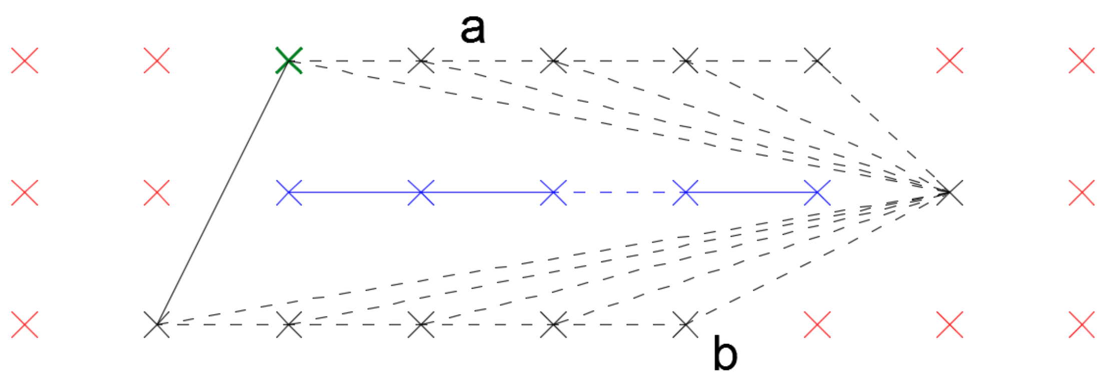


Figure 36: Including Right Point

Again, we shall label the top edge of Figure 36 as a and the bottom edge as b. Following the same procedure as before, we now have that $a \leq b \leq L-1$. We can also see that this set of polytopes includes the previous set, thus we only have to use these polytopes for $L(Q) = L - 1$.

5.2 Case B: Q is two-dimensional

When the interior polytope Q is two-dimensional, we can look at the hyperplanes of Q to determine which lattice points we can use to shell with.

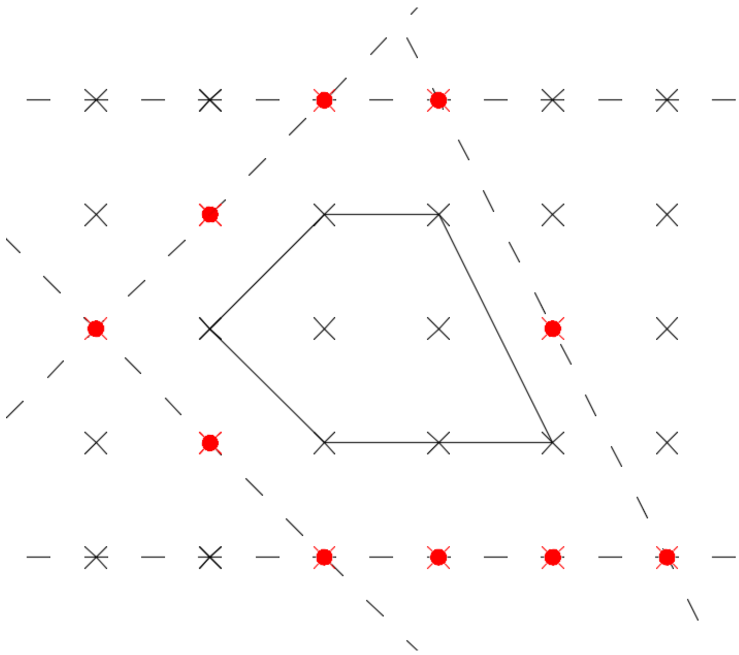


Figure 37: Shifting Hyperplanes of a Polytope

By shifting the hyperplanes of the polytope in Figure 37 by 1, we can use these lattice points to construct our polytope P without fear of accidently including new interior points. This is because when we take any edge of Q and adjoin the two ends to a point along the next row, it creates an empty triangle (i.e. contains no interior points). Since this occurs regardless of the point we choose, we can repeat this for all edges of Q.

We note that shelling has been investigated in the context of toric geometry, seen in [9].

5.3 Shellable Polytopes

For the Inductive Algorithm to be more efficient, it would be good practice to note which of the polytopes that are found are able to be shelled around. This makes the algorithm less tedious, since it would be already known which interior polytopes are able to be shelled around and therefore less work is needed.

A polytope is shellable if the shifted hyperplanes all cross on a lattice point. If any of the shifted hyperplanes do not cross on a lattice point, then we cannot shell around this polytope with any given length. An example of this is given in Figure 38.

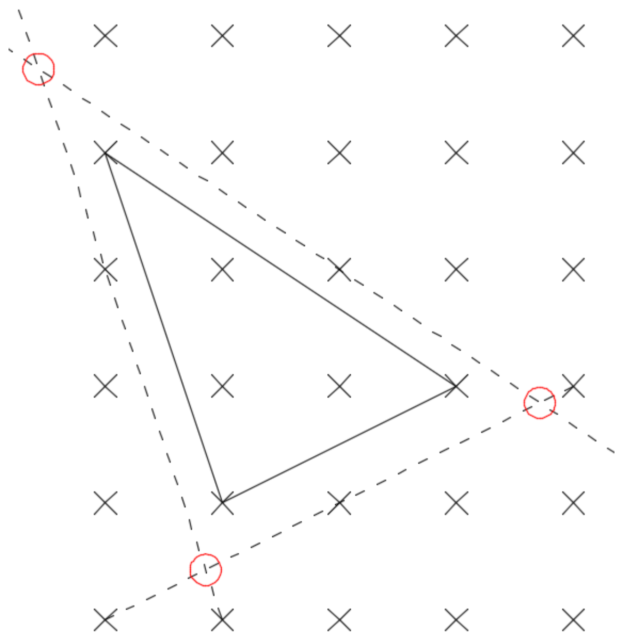


Figure 38: Example of a Polytope which cannot be shelled around

The red circles in Figure 38 draw attention to where the hyperplanes do not meet on a lattice point. For this example, we have that this occurs where all three hyperplanes meet. Since the hyperplanes will always meet where the potential vertex of a polytope would exist, the fact that there is no lattice point here means that we cannot shell around this vertex. Therefore, the interior polytope cannot be shelled around at all.

6 Minkowski Length 3

As part of my dissertation, I have found all polytopes of Minkowski length 3. This section will now show how I have used my Inductive Algorithm to find these polytopes.

6.1 No Interior Points

Since this classification was already known, it was simple to find all polytopes of length 3 that have no interior points. Thus, here are such polytopes:

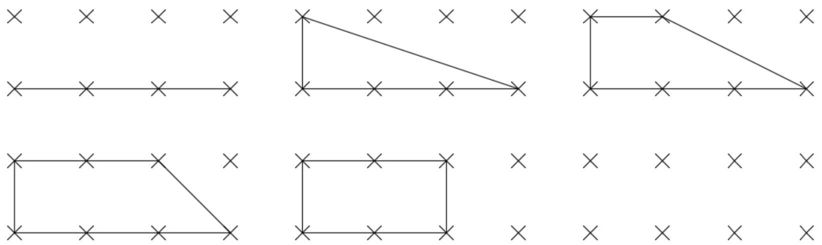


Figure 39: Minkowski Length 3: No Interior Points

6.2 One Interior Point

Again, this classification was already known, and all such polytopes were found previously, hence:

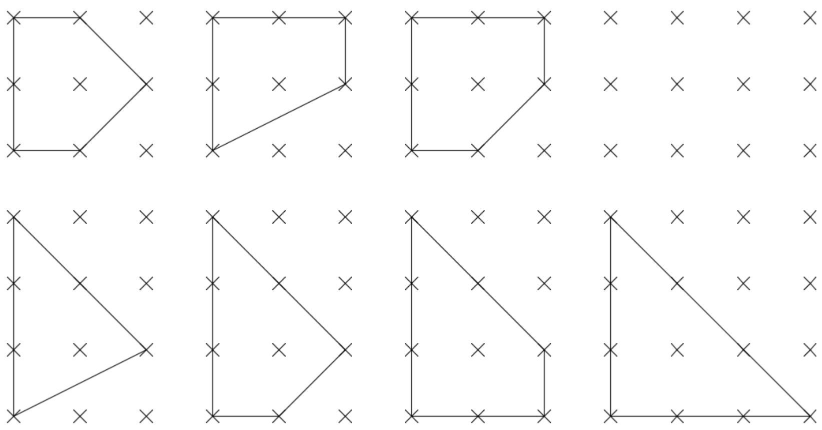


Figure 40: Minkowski Length 3: One Interior Point

6.3 Two Interior Points

Since there is only one way of arranging 2 points, we only need to look at one case:

Figure 41: Two Interior Points

In this instance, our Q is of length $L - 2 = 1$, so we refer to Section 5.1 to determine how we approach this.

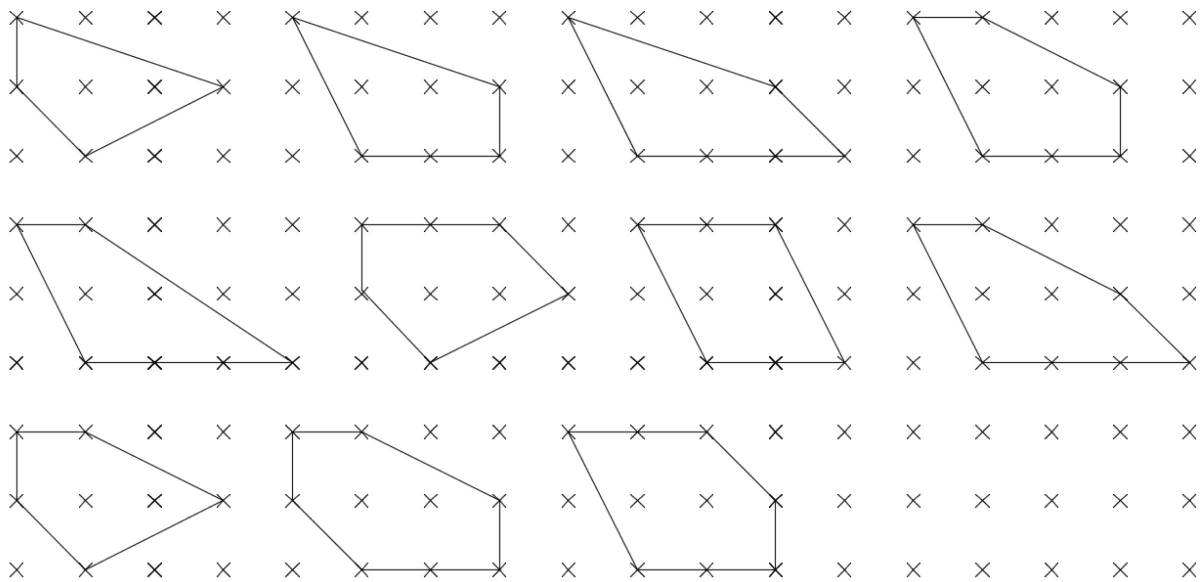


Figure 42: Minkowski Length 3: Two Interior Points

Figure 42 shows all polytopes of length 3 with only 2 interior points. There are 11 in total.

6.4 Three Interior Points

There are now two possible ways of arranging these 3 points:

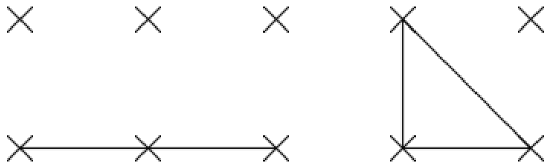


Figure 43: Three Interior Points

The first case is a line of length $L - 1 = 2$ so we know that we can follow the example in Section 5.1.

For the second case, we can use the method from Section 5.2, where we take the hyperplanes of the polytope and shift them out to effectively shell around our interior points.

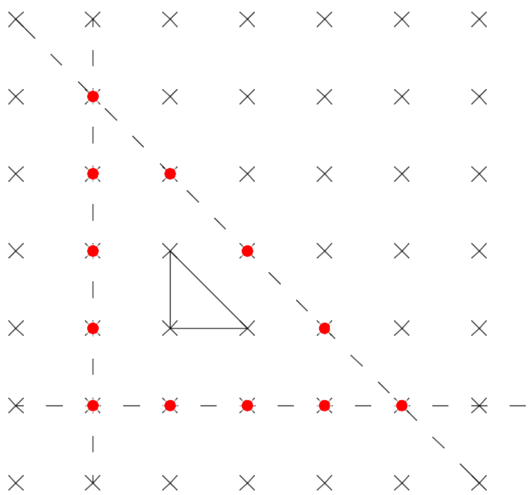


Figure 44: Hyperplane method on three interior points

Using both methods for each interior polytope, I was able to find all polytopes of length 3 with three interior points:

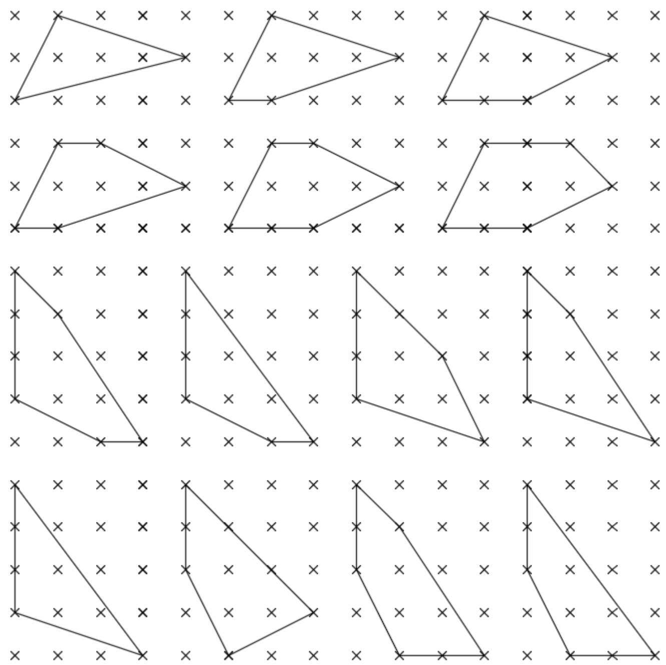


Figure 45: Minkowski Length 3: Three Interior Points (1)

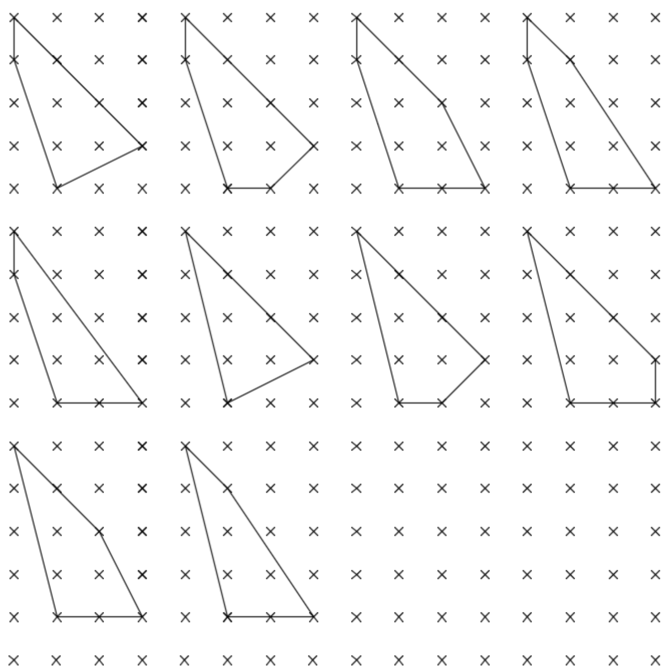


Figure 46: Minkowski Length 3: Three Interior Points (2)

6.5 Four Interior Points

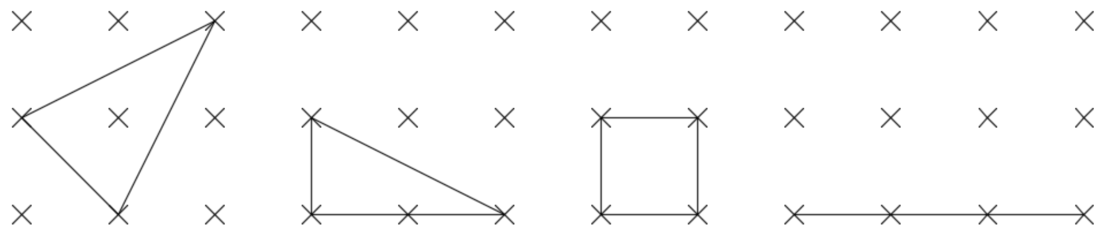


Figure 47: Four Interior Points

From our tables, we find that there are 4 polytopes that are made up of four lattice points that we are able to look at shelling around as shown in Figure 47. However, we need to decide whether they can all be used or whether we can discard any. An obvious example of this is the fourth polytope in the figure. This is a line of length 3, and we have proven in Section 5.1 that we cannot shell a polytope with Minkowski length 3 around a line of length 3, so we can discard this one straight away. After further examination, the other three polytopes are able to be shelled around.

I will look further at the second polytope, since this has an interesting characteristic. This particular interior polytope contains a line of length $L - 1 = 2$. Knowing this, we can refer to our previous work on shelling around one-dimensional polytopes. We know that we must use exactly one of the lattice points either end of the line in order to be able to shell around it (these are shown in blue in Figure 48). Therefore, we have reduced the problem down so that we can look at fewer potential polytopes.

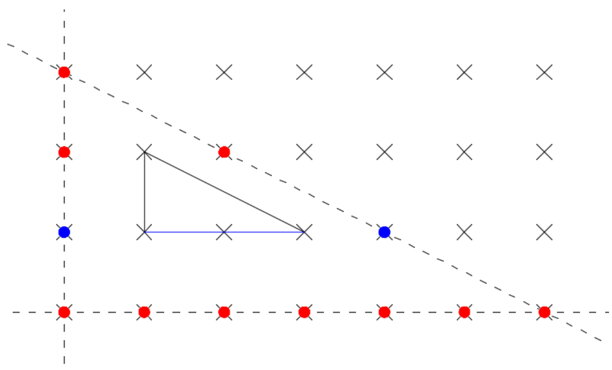


Figure 48: Hyperplane method on four interior points

By using all methods and checking all possibilities, the polytopes of Minkowski length 3 with four interior points are:

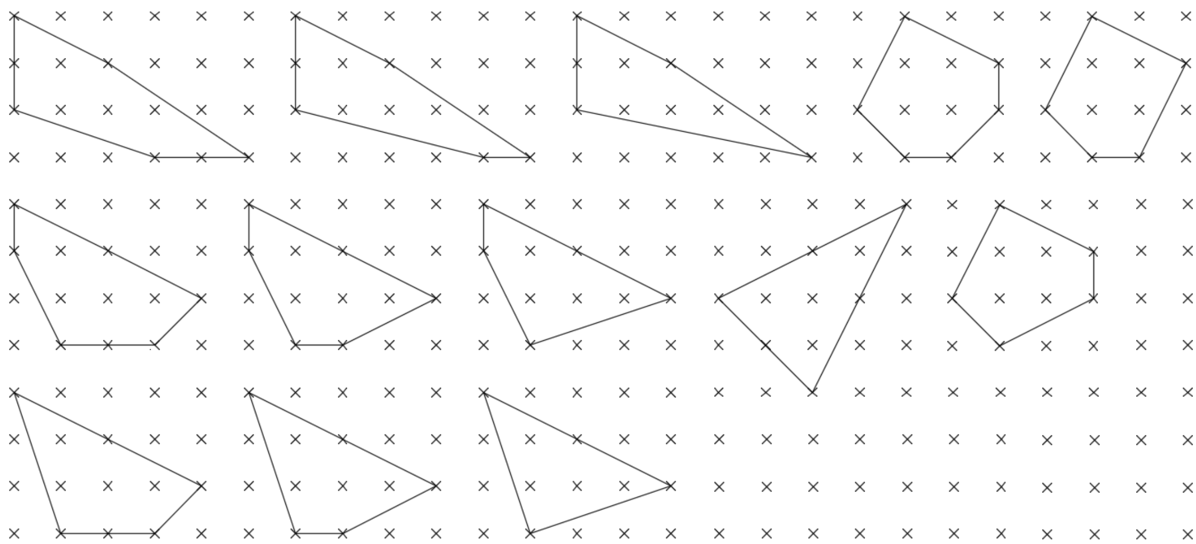


Figure 49: Minkowski Length 3: Four Interior Points

6.6 Five Interior Points

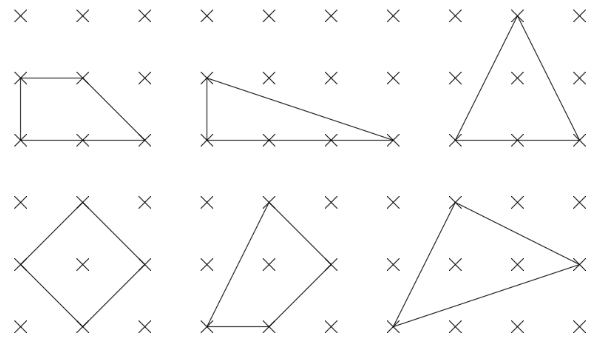


Figure 50: Five Interior Points

Again, we first need to look at which polytopes in Figure 50 can be discarded. We can obviously discard the second polytope from the top row since it contains a line of length 3. The final polytope on the bottom row can also be discarded since this is unable to be shelled around at all, as shown in Figure 51.

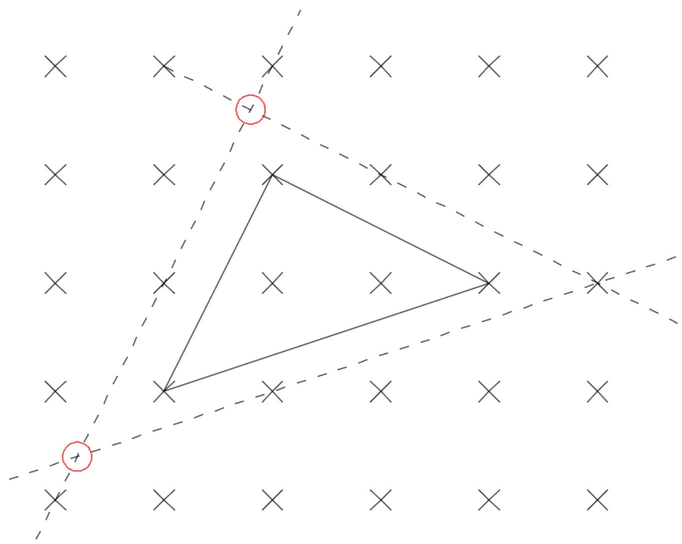


Figure 51: Example of 5 Interior Points that cannot be shelled around

The third and fourth polytopes in Figure 50 (end of top row and beginning of bottom row) can both be discarded also, but they are slightly trickier to see why. Since both have similar reasonings as each other, I will provide details of only the third polytope.

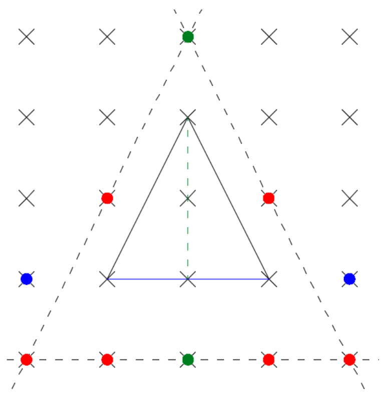


Figure 52: Example of 5 Interior Points that cannot be used

As you can see in Figure 52, this polytope contains 2 lines of length $L - 1 = 2$, so we can again apply the usual method for $L(Q) = L - 1$. However, for the horizontal line, we can see for Figure 52 that the points needed to extend the line by 1 lie outside of the hyperplanes. This means that the line is unable to be shelled around and therefore so is the polytope. A similar reasoning is applied to the other polytope.

Therefore, all polytopes with 5 interior points that are of length 3 are:

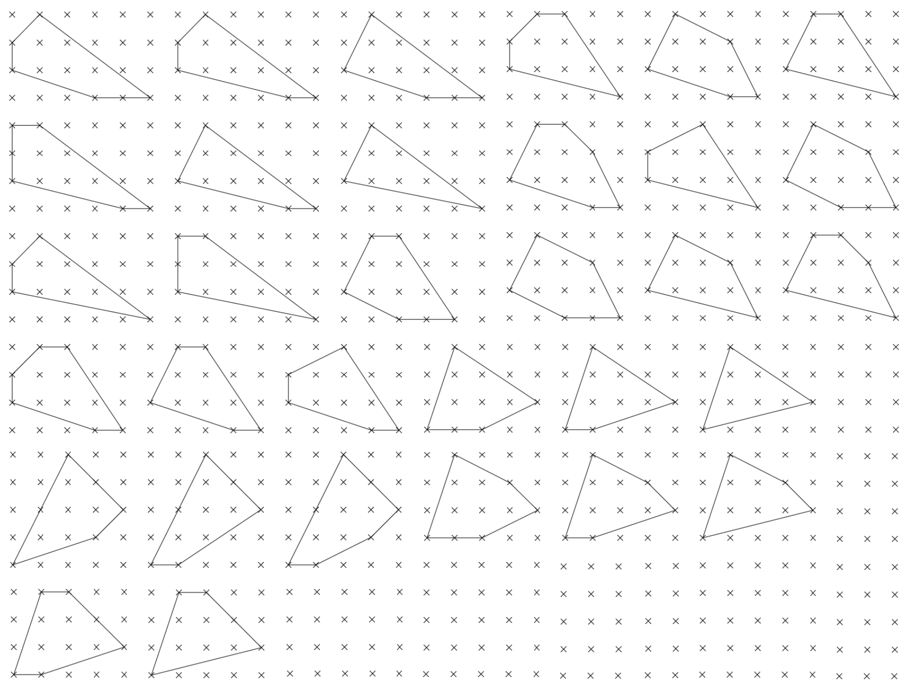


Figure 53: Minkowski Length 3: Five Interior Points

6.7 Six Interior Points

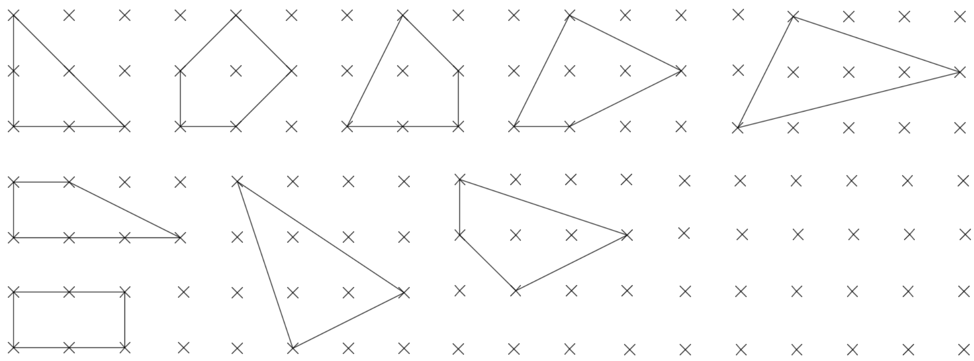


Figure 54: Six Interior Points

There are 9 possible polytopes that we can shell around that have 6 interior points. After discarding the polytopes that contain lines of length 3, and removing the polytopes that cannot be shelled around, we can look at those left to see if we can discard further ones.

We can remove the possibility of another couple of polytopes since they contain multiple lines of length $L - 1 = 2$ in such a way that makes it impossible to shell around those polytopes with Minkowski length 3. For example, we shall look at the following polytope in Figure 55.

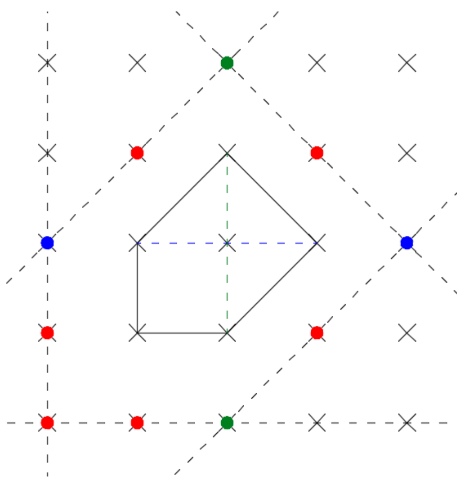


Figure 55: Example of 6 Interior Points unable to be shelled (1)

Both lines of length 2 have been drawn in this figure, along with their corresponding possible end points. We must now look at the different possibilities of which points we can choose, with the polytope to still be able to be shelled around.

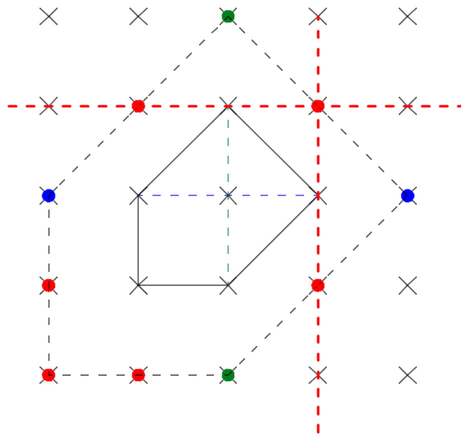


Figure 56: Example of 6 Interior Points unable to be shelled (2)

The extra dotted lines in Figure 56 help us to highlight the fact that there needs to be points so that it is possible to shell around these vertices. From this figure, it is clear that we must choose these two corresponding points for our polytopes. We can therefore remove the other two points and draw in the edges that must now be used.

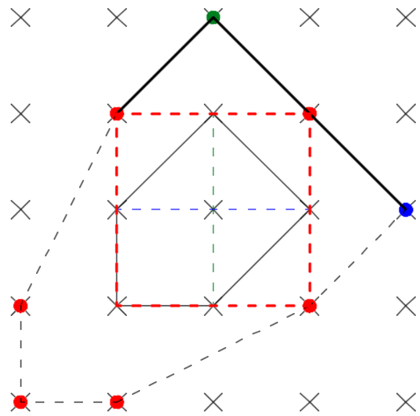


Figure 57: Example of 6 Interior Points unable to be shelled (3)

Since the drawn edge in Figure 57 must be used in any polytopes that are shelled around these six points, we cannot create any polytopes of length 3. This is because there will always be the polytope of length 4 contained within any polytope we create (highlighted by the red square in Figure 57).

Therefore, the polytopes of Minkowski length 3 that I have found are as follows:

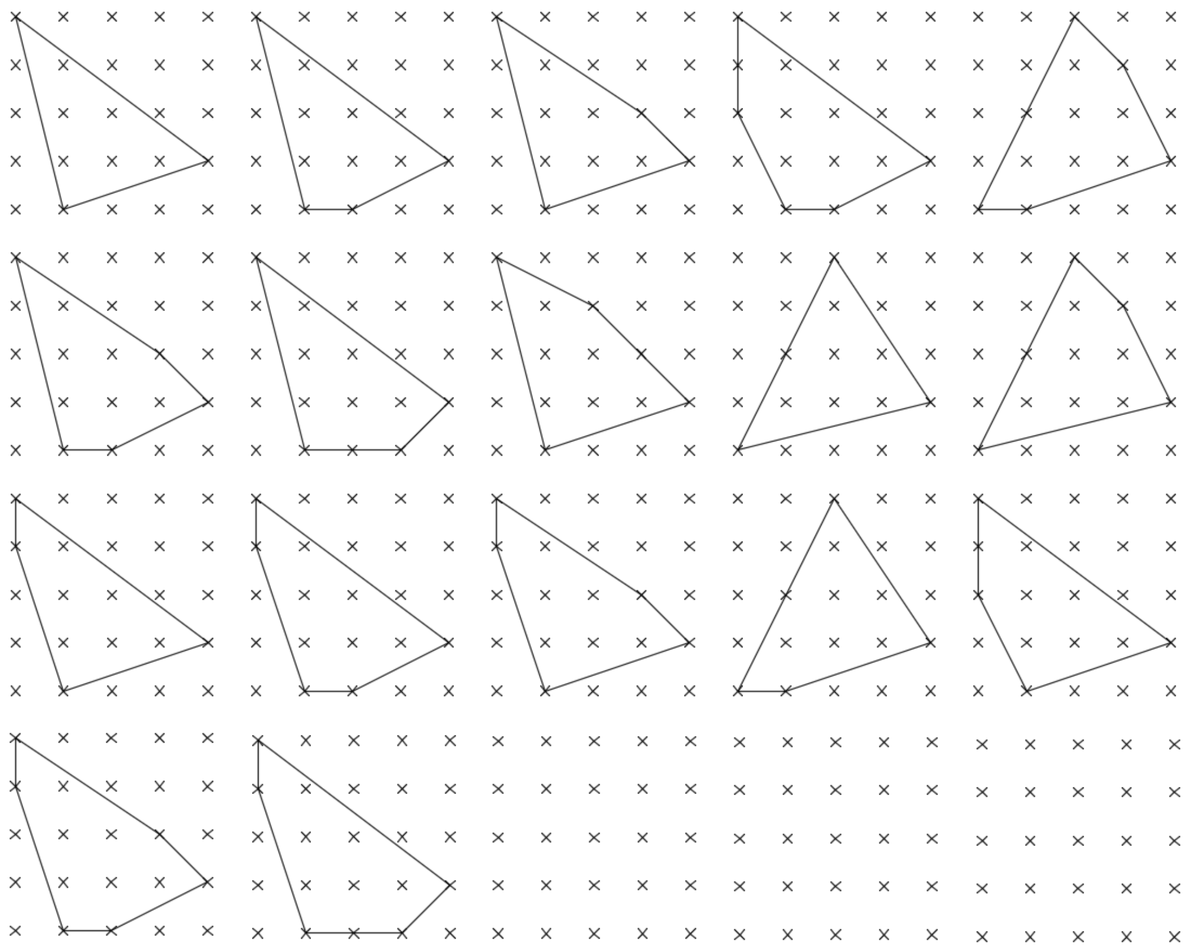


Figure 58: Minkowski Length 3: Six Interior Points

I have now found all 114 polytopes with Minkowski length 3.

There are no further polytopes of length 3 since, when attempting to shell around polytopes $L(Q) \geq 7$ they became unable to be shelled around with Minkowski length 3.

7 Conclusion

The aim of this project was to find all (if not most) of the polytopes with Minkowski length 3. With the use of the Inductive Algorithm from Section 5, all such polytopes have been found and presented in a table, just as Figures 26 and 27.

|PnZ^2| |P^o n Z^2|	4	5	6	7	8	9	10	11	12
0	1	1	2	1	0	0	0	0	0
1	0	0	0	3	2	1	1	0	0
2	0	0	1	2	5	3	0	0	0
3	0	0	1	1	8	10	9	0	0
4	0	0	0	0	1	5	5	2	0
5	0	0	0	0	2	7	11	10	2
6	0	0	0	0	0	1	4	8	4
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0

Figure 59: Table containing the number of polytopes with $L(P) = 3$

Figure 59 displays all 114 polytopes with $L(P) = 3$. In displaying this information in this table, one could continue this work into finding polytopes of length 4, say, and would know which polytopes with $L(P) = 3$ are able to be shelled around (as well as those with $L(P) = 1$ and $L(P) = 2$).

Although this algorithm specifically works on two-dimensional polytopes, I think, with further work, that it can be applied to those of greater dimensions. This is due to that fact that it involves the use of the hyperplanes of the polytopes which implies that the algorithm can be extended to higher dimensions. Also, more work can be applied into finding a better mathematical understanding for what it means for a polytope to be "shellable" so that it can be applied to higher dimensions; here there can be investigation into the connection to toric adjunction theory [9]. There can also be further work into attempting to classify those polytopes, with k interior lattice points, that are not shellable.

With this in mind, coding theorists can use the relationship between the Minkowski length and the minimum distance of toric codes to generate better codes.

References

[1] I. Soprunov *Lattice Polytopes in Coding Theory*

[2] G. Brown and A.M. Kasprzyk *Small Polygons and Toric Codes*

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[5] J. Reeve *On the volume of lattice polyhedra*

[6] J. Lagarias and G. Ziegler *Bound for lattice polytopes containing a fixed number of interior points in a sublattice*

[7] S. Rabinowitz *A census of convex lattice polygons with at most one interior lattice point*

[8] X. Wei and R. Ding *Lattice polygons with two interior lattice points*

[9] S. di Rocco, C. Haase, B. Nill, A. Paffenholz *Polyhedral Adjunction Theory*