M4R Project

Four-dimensional Fano manifolds and reflexive polytopes

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This is my own work except where otherwise stated.

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Abstract
Mirror symmetry is an idea which links Fano manifolds and Laurent polynomials, which is being used to help the classification of Fano manifolds in 4-dimensionsi with the possibility to extend to all dimensions. In this project we look at finding mirror-duals to the 4-dimensional Fano manifolds supported on the 4-dimensional reflexive polytopes.

Contents

1 Introduction

Fano varieties, which take their name from the Italian mathematician Gino Fano (1871-1952), are an important class of algebraic varieties. The classification of the 3 dimensional Fano manifolds was derived quite recently by Mori and Mukai in the 1980s. The classification of the 2 dimensional Fano manifolds has been know for over a century; they are called as the del Pezzo surfaces, named after the mathematician Pasquale del Pezzo (1859-1936).

Classification of Fano manifolds in higher dimensions is still an open problem. In this project I am looking at a method to help classify 4 dimensional Fano manifolds, via mirror-symmetry, which initially came from Golyshev's work [10]. I look into the method laid out by Coates, Corti, Galkin, Golyshev and Kasprzyk [5]. The main work of this project is to look at finding mirror-duals to known 4 dimensional Fano manifolds using the classification of the 4-dimensional reflexive polytopes by Kreuzer and Skarke [12, 13].

I start by looking at the method of classifying the toric Fano Manifolds, using Fano polytopes [2, 4], in section 2. I also introduce a key component for mirror symmetry, the regularized quantum period. Then I go on to introduce Laurent polynomials and polytopes in section 3. These are both very important in mirror symmetry. I then go on to explain a bit about mirror symmetry in section 4.

Section 5 is where most of my own work is. I explain the matching process I will use to try to find mirror-duals. I also introduce the reflexive polytopes, and give some reason why I am using them. I implement the process in the computer algebra package Magma, and explain how I tried to make the programme fast and efficient.

Finally, in section 6, I comment about the results of my programme. I do find quite a few matches, but also realise I only managed to look through a very small proportion of the massive list of 4-dimensional reflexive polytopes. There is a lot of further work to be done from this project, and my efforts were only a brief look into seeing how Laurent polynomials and reflexive polytope can help us classify the 4 dimensional Fano manifolds.

2 Fano Varieties

Given a non-singular projective algebraic variety X, we can form its anticanonical bundle (the inverse of the canonical bundle). This is what we use to define *Fano varieties.*

Definition 2.1. *X is a Fano manifold if its anticanonical bundle is ample (i.e. the base variety of the bundle to a positive power can be embedded into projective space).*

If X is singular we can form the anticanonical bundle after restricting to the non-singular locus of X, or the anticanonical divisor of X. If the antiocanonical divisor is ample then X is a Fano variety.

These are spaces with positive curvature. Some of the simplest Fano varieties projective space, $\mathbb{P}^n$.

Example 2.2. *The one dimensional projective space*

$$\mathbb{P}^1 = \frac{\mathbb{C}^2 \setminus \{0\}}{\sim},$$

where $\sim$ is the equivalence relation

$$(a_1, b_2) \sim (a_2, b_2) \iff \exists \lambda \in \mathbb{C}^* \quad s.t. \quad (a_1, b_1) = \lambda(a_2, b_2).$$

Thus $\mathbb{P}^1$ is the same as lines in $\mathbb{C}^2$ through the origin; also known as the Reimann sphere.

Going up one dimension:

Example 2.3. *Two dimensional projective space is*

$$\mathbb{P}^2 = \frac{\mathbb{C}^3 \setminus \{0\}}{\sim},$$

where

$$(a, b, c) \sim \lambda(a, b, c) \quad \forall \lambda \in \mathbb{C}^*.$$

We can recover the coordinate charts as follows. Let $(a, b, c) \in \mathbb{P}^2$, if $a \neq 0$ then in $\mathbb{P}^2$

$$(a, b, c) = \left(1, \frac{b}{a}, \frac{c}{a}\right) = (1, B, C)$$

for some $B, C \in \mathbb{C}$. We can do a similar thing with $b \neq 0$ and when $c \neq 0$. This defines three charts which covers the whole of $\mathbb{P}^2$

$$\begin{aligned}
\left\{\left(1, \frac{b}{a}, \frac{c}{a}\right) \,:\, a \neq 0\right\} &= \left\{(1, B, C) \in \mathbb{P}^2\right\} \cong \mathbb{C}^2 \\
\left\{\left(\frac{a}{b}, 1, \frac{c}{b}\right) \,:\, b \neq 0\right\} &= \left\{(A, 1, C) \in \mathbb{P}^2\right\} \cong \mathbb{C}^2 \\
\left\{\left(\frac{a}{c}, \frac{b}{c}, 1\right) \,:\, c \neq 0\right\} &= \left\{(A, B, 1) \in \mathbb{P}^2\right\} \cong \mathbb{C}^2.
\end{aligned}$$

Therefore we have that $\mathbb{P}^2$ is three copies of $\mathbb{C}^2$ "glued" together with overlap when two or more of the coordinates are non-zero.

Another example of a two dimensional Fano manifold is:

Example 2.4.

$$\mathbb{P}^1 \times \mathbb{P}^1$$

Lets first consider $\mathbb{P}^1$. Similar to example 2.3 we can recover two coordinate charts. Let $(a, b) \in \mathbb{P}^1$, with $a \neq 0$, then

$$(a, b) = \left(1, \frac{b}{a}\right) = (1, B).$$

If we to a similar thing when $b \neq 0$, then we can get the two coordinate charts

$$\begin{aligned}
\left\{\left(1, \frac{b}{a}\right) \,:\, a \neq 0\right\} &= \left\{(1, B) \in \mathbb{P}^1\right\} \cong \mathbb{C} \\
\left\{\left(\frac{a}{b}, 1\right) \,:\, b \neq 0\right\} &= \left\{(A, 1) \in \mathbb{P}^1\right\} \cong \mathbb{C}.
\end{aligned}$$

These two coordinate charts cover the whole of $\mathbb{P}^1$*.*

If we take a second copy of $\mathbb{P}^1$*, this time with coordinate charts*

$$\left\{\left(1, \frac{b'}{a'}\right) \ : \ a' \neq 0\right\} = \left\{(1, B') \in \mathbb{P}^1\right\} \cong \mathbb{C}$$
$$\left\{\left(\frac{a'}{b'}, 1\right) \ : \ b' \neq 0\right\} = \left\{(A', 1) \in \mathbb{P}^1\right\} \cong \mathbb{C}.$$

Now consider $\mathbb{P}^1 \times \mathbb{P}^1 = \{((a,b),(a',b'))\}$*, with the appropriate equivalence relation. Then we can form 4 coordinate charts, from the combinations of the coordinate charts of the two copies of* $\mathbb{P}^1$*,*

$$\{((A,1),(A',1))\} \cong \mathbb{C}^2$$
$$\{((A,1),(1,B'))\} \cong \mathbb{C}^2$$
$$\{((1,B),(A',1))\} \cong \mathbb{C}^2$$
$$\{((1,B),(1,B'))\} \cong \mathbb{C}^2.$$

This covers the whole of $\mathbb{P}^1 \times \mathbb{P}^1$*, so* $\mathbb{P}^1 \times \mathbb{P}^1$ *can be view as four copies of* $\mathbb{C}^2$*, "glued" together with overlap.*

Fano varieties are used as some of the basic building blocks of the minimal model program, which is a way to classify algebraic varieties, up to birational equivalence. It has been shown that in any given dimension there are finitely many Fano manifolds, so it makes sense to try to classify all of them for a given dimension. The classification is known up to and including dimension 3. Classification in higher dimensions is still an open problem. In dimension 2 there are 10 Fano manifolds, known as the *del Pezzo surfaces*, and in dimension 3 there are 105 Fano manifolds (Mori-Mukai classification).

A useful class of varieties to look at are the *toric varieties*.

Definition 2.5. *A variety* X*, is called toric if it contains a dense algebraic torus* $(\mathbb{C}^\times)^n$*, which acts naturally on* X*.*

This definition is quite abstract and we can get a better understanding using *cones* and *fans*.

Definition 2.6. *Let* $N \cong \mathbb{Z}^n$ *be a lattice and* $N_\mathbb{Q} = N \otimes_\mathbb{Z} \mathbb{Q}$*. Then a strictly-convex rational polyhedron cone, which we will simply call a cone, is a set*

$$\sigma = cone\,\{\rho_1, \ldots, \rho_r\} = \left\{\sum_{i=1}^{r} u_i \rho_i \ : \ u_i \geq 0\right\} \subseteq N_\mathbb{Q},$$

where $\rho_i \in N_\mathbb{Q}$ *and* $r \in \mathbb{Z}_{\geq 1}$*, such that* $\sigma \cap -\sigma = \{0\}$*, which means it contains no one dimensional subspace.*

A fan, Δ*, is a set of cones in* $N_\mathbb{Q}$ *s.t.*

(i) $0 \in \Delta$

(ii) If $\tau \preceq \sigma$ *is a face of* $\sigma \in \Delta$ *then* $\tau \in \Delta$

(iii) If $\sigma_1, \sigma_2 \in \Delta$ *then* $\sigma_1 \cap \sigma_2 \in \Delta$ *and* $\sigma_1 \cap \sigma_2 \preceq \sigma_1, \sigma_2$*.*

Toric varieties can be described in terms of fans in $M_{\mathbb{Q}} = \mathrm{Hom}(N, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$, the dual lattice of N, but taking the dual of the cones.

Definition 2.7. *Given a cone $\sigma \subseteq N_{\mathbb{Q}}$ its dual is*

$$\sigma^{\vee} = \{y \in M_{\mathbb{Q}} \, : \, \langle x, y \rangle \geq 0 \, \forall x \in \sigma\},$$

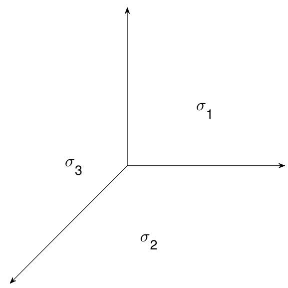
where $\langle , \rangle$ is the inner product.

Lets see an example of this in 2 dimensions.

Example 2.8. *Let*

$$\begin{aligned} \sigma_1 &= cone\{e_1, e_2\} \\ \sigma_2 &= cone\{e_2, -e_1 - e_2\} \\ \sigma_3 &= cone\{e_1, -e_1 - e_2\} \end{aligned}$$

be 3 cones in the 2 dimensional lattice N, with e_1 and e_2 the standard basis vectors. Taking these three we can extend them to a fan Δ, which looks like the following:



Taking duals we get three cones in $M_{\mathbb{Q}}$:

$$\begin{aligned} \sigma_1^{\vee} &= cone\{(1,0), (0,1)\} \\ \sigma_2^{\vee} &= cone\{(1,-1), (0,-1)\} \\ \sigma_3^{\vee} &= cone\{(-1,0), (-1,1)\}. \end{aligned}$$

If we let the point $(a, b) \in M$ be associated to the monomial $X^a Y^b \in \mathbb{C}[X^{\pm}, Y^{\pm}]$, then we can form the semigroup algebras

$$\begin{aligned} \mathbb{C}\left[\sigma_1^{\vee}\right] &= \mathbb{C}\left[X, Y\right] \\ \mathbb{C}\left[\sigma_2^{\vee}\right] &= \mathbb{C}\left[XY^{-1}, Y^{-1}\right] \cong \mathbb{C}\left[x, y\right] \\ \mathbb{C}\left[\sigma_3^{\vee}\right] &= \mathbb{C}\left[X^{-1}, X^{-1}Y\right] \cong \mathbb{C}\left[x', y'\right] \end{aligned}$$

where $x = XY^{-1}$, $y = Y^{-1}$, $x' = X^{-1}$ and $y' = X^{-1}Y$. Then taking spec we get

$$Spec(\mathbb{C}\left[\sigma_i\right]) \cong \mathbb{C}^2,$$

for $i = 1, 2, 3$. Therefore the toric variety described by Δ is three copies of $\mathbb{C}^2$ "glued" together, just like $\mathbb{P}^2$ in example 2.3. In fact they are isomorphic with

$$X \longleftrightarrow \frac{b}{a} \quad and \quad Y \longleftrightarrow \frac{c}{a},$$

with a, b and c as in example 2.3. Then σ_1 corresponds to the chart when $a \neq 0$, σ_2 to the chart when $c \neq 0$ and σ_3 to the chart when $b \neq 0$. With the overlap of the three charts corresponding to boundaries between the cones.

In the example the toric variety described by the fan is Fano and is isomorphic to $\mathbb{P}^2$. This is, in fact, because the fan was generated by a *Fano polytope.*

Definition 2.9. *A lattice polytope, $P \subseteq N_{\mathbb{Q}}$, is the convex hull of a finite subset of the lattice N.*

P is Fano if

(i) dim P = dim N

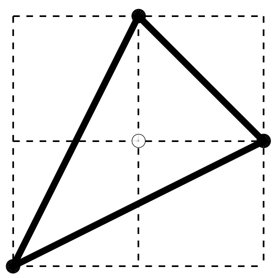
(ii) $0 \in P^{\circ}$, the strict interior of P

(iii) The vertices of P, $v \in \mathcal{V}(P)$, are primitive.

$v = (v_1, \cdots, v_n)$ is primitive if $gcd\{v_1, \cdots, v_n\} = 1$. Which means that the vector from 0 to v crosses no integer points except 0 and v.

In fact a toric variety described by the fan generated by any Fano polytope is Fano. We denote the variety generated by a polytope P, X_P. In our definition property (ii) fits naturally as we are going to forming a fan from the polytope, and 0 is in a fan by definition. Also if a vertex of the polytope is not primitive, we could replace it with a primitive lattice point and get the same fan and therefore the same variety.

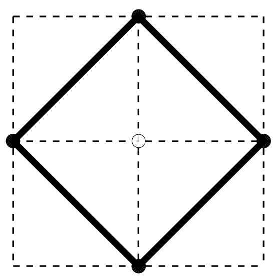
Example 2.10. *Let P be the polytope with $\mathcal{V}(P) = \{(1,0), (0,1), (-1,-1)\}$.*



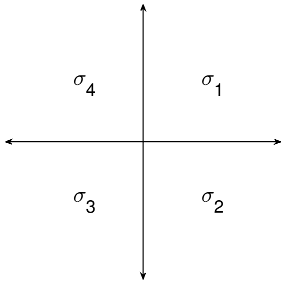
This generates the fan from example 2.8, therefore $X_P \cong \mathbb{P}^2$, which is a Fano variety.

Lets see the Fano polytope for $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 2.11. *Consider the polytope, $P \subseteq N_{\mathbb{Q}}$, with $\mathcal{V}(P) = \{(\pm 1, 0), (0, \pm 1)\}$.*



Forming the fan of this polytope, Δ, gives



where

$$\begin{aligned}
\sigma_1 &= cone\,\{e1, e2\}\\
\sigma_2 &= cone\,\{-e1, e2\}\\
\sigma_3 &= cone\,\{-e1, -e2\}\\
\sigma_4 &= cone\,\{e1, -e2\}
\end{aligned}$$

are cone in N, the two dimension lattice, with standard basis vectors e_1 and e_2. Taking duals we get the cones in $M_{\mathbb{Q}}$:

$$\begin{aligned}
\sigma_1^\vee &= cone\,\{(1,0),(0,1)\}\\
\sigma_2^\vee &= cone\,\{(-1,0),(0,1)\}\\
\sigma_3^\vee &= cone\,\{(-1,0),(0,-1)\}\\
\sigma_4^\vee &= cone\,\{(-1,0),(0,-1)\}\,.
\end{aligned}$$

Associating $(a,b) \in M$ with the monomial X^aY^b, we can form the semigroup algebras

$$\begin{aligned}
\mathbb{C}\left[\sigma_1^\vee\right] &= \mathbb{C}\left[X, Y\right]\\
\mathbb{C}\left[\sigma_2^\vee\right] &= \mathbb{C}\left[X^{-1}, Y\right]\\
\mathbb{C}\left[\sigma_3^\vee\right] &= \mathbb{C}\left[X^{-1}, Y^{-1}\right]\\
\mathbb{C}\left[\sigma_4^\vee\right] &= \mathbb{C}\left[X, Y^{-1}\right].
\end{aligned}$$

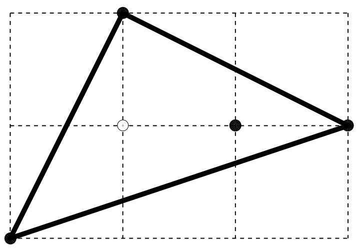
Again we have that $Spec(\mathbb{C}[\sigma_i]) \cong \mathbb{C}^2$, for $i = 1, \ldots, 4$. There is an isomorphism between the toric variety described by Δ and $\mathbb{P}^1 \times \mathbb{P}^1$, with

$$X \longleftrightarrow \frac{b}{a} \quad \textit{and} \quad Y \longleftrightarrow \frac{b'}{a'},$$

where a, b, a' and b' as from example 2.4.

Let look at a non-example of a Fano polytope.

Non-example 2.12. *Consider polytope P, with $\mathcal{V}(P) = \{(1,0),(0,2),(-1,-1)\}$.*



This is not a Fano polytope as $(0,2)$ *is not primitive. This will produce the same fan as the previous example and hence the same variety (i.e.* $X_P \cong \mathbb{P}^2$*, again).*

Example 2.10 and the non-example 2.12 demonstrates why we insist on condition (iii) in definition 2.9, as it gives us uniqueness. In fact toric Fano varieties, up to isomorphism, are in bijection to Fano polytopes, up to $GL(n, \mathbb{Z})$ equivalence (i.e. a change of basis of the underlying lattice N).

We call a Fano polytope smooth if for each facet of the polytope, the vertices in the facet generate the lattice. Then the smooth Fano polytopes generate, and are in bijection with, the smooth toric Fano varieties (toric Fano manifolds). The classification of toric Fano manifolds is then simplified to classifying smooth Fano polytopes. Øbro produced an algorithm which has done this explicitly up to dimension 8[4].

Dimension	1	2	3	4	5	6	7	8
Number of toric Fano manifolds	1	5	18	124	866	7622	72256	749892

As well as the smooth Fano polytopes there are also several classes of singular Fano polytopes. In the most general case a Fano polytope will generate a toric Fano variety with log-terminal singularities. A subset of the Fano polytopes are the canonical polytopes, which are polytopes P such that $P^\circ = \{0\}$ (i.e. the lattice origin is the only point in the strict interior). Canonical polytopes generate toric Fano varieties with canonical singularities. A subset of the canonical polytopes are the reflexive polytopes, which are defined in section 5.1. These generate toric Fano varieties with Gorenstein canonical singularities. A different subset of the canonical polytopes are the terminal polytopes, a polytope P such that $P \cap N = \mathcal{V}(P) \cup \{0\}$ (i.e. the origin and the vertices are the only lattice points in P). Terminal polytopes generate toric Fano varieties with terminal singularities.[11]

Although we can classify the toric Fano manifolds, it still leaves many non-toric Fano manifolds unknown. Up to dimension 3 we know the classification of all Fano manifolds (del Pezzo surfaces in dimension 2 and the Mori-Mukai classification in dimension 3), but in dimension 4 and higher the problem is still open. One method being looked at is using *mirror symmetry*, for which we will need to introduce one more concept about Fano varieties.

Definition 2.13. *For a Fano variety* X*, its regularized quantum period,* $\widehat{G}_X$*, is a generating function for the Gromov-Witten invariants.*[5]

The Gromov-Witten invariants are very hard to compute, and beyond the scope of this work; approximately speaking they count certain curves on X. The quantum period gives detailed information about the *quantum cohomology*. They have all be computed for Fano manifolds up to 3 dimensions, and some of them have been computed in higher dimensions.

It is conjected that Fano manifolds (up to deformation) each have a unique quantum period, $\widehat{G}_X$.

3 Laurent Polynomials

Another idea we need to explain mirror symmetry are Laurent polynomials.

Definition 3.1. *A Laurent polynomial is a element of the ring* $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$.

Laurent polynomials are similar to ordinary polynomials, but we allow negative and positive powers of x_i.

Example 3.2. *An example of a Laurent polynomial in 3 variable is*

$$f = 1 + x_1x_2 - 5\frac{x_1}{x_2x_3} + \frac{x_2^2}{x_1^3} \quad \in \mathbb{C}[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}].$$

It is important to notice that rational function are not necessarily Laurent polynomials.

Non-example 3.3. *Consider*

$$f = \frac{x}{x-1}.$$

This is not a Laurent polynomial, i.e. $f \notin \mathbb{C}[x^{\pm}]$.

Laurent polynomials can be used to form lattice polytopes.

Definition 3.4. *Let* $f \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$, *and let* N *be the* n *dimensional lattice. Then* f *can be written*

$$f = \sum_{i=1}^{r} c_i x_1^{a_{1,i}} \cdots x_n^{a_{n,i}}.$$

The Newton polytope of f *is*

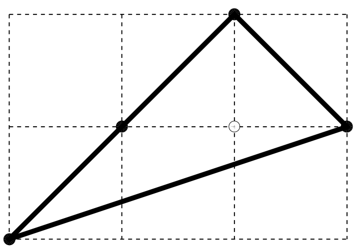
$$Newt(f) = conv\left\{(a_{1,1}, \cdots, a_{n,1}), \cdots, (a_{1,r}, \cdots, a_{n,r})\right\} \subseteq N_{\mathbb{Q}},$$

where $(a_{1,i}, \ldots, a_{n,i}) \in N$, *the convex hull of the set of exponent vectors of the monomials in* f.

Example 3.5. *Let*

$$f = x + y + \frac{1}{x} + \frac{1}{x^2y} \in \mathbb{C}\left[x, y\right],$$

be a Laurent polynomial. This will give the polytope $Newt(f)$



Notice that we get the same polytope if we remove the monomial $\frac{1}{x}$ *and if we change the coefficient of the remaining monomials, as long as they are non-zero.*

This process can also be reversed, we can form a Laurent polynomial from a polytope, although the coefficients of the monomials are not specified. The only requirement is that the monomials corresponding to the vertices have to be non-zero.

If $P = \mathrm{Newt}(f)$, we can then use the polytope to generate a fan and a toric variety X_P. In general the variety will be singular, i.e. not smooth.

Another thing we can form from a Laurent polynomials is its classical period.

Definition 3.6. *The classical period of a Laurent polynomial, f, is*

$$\pi_f(t) = \left(\frac{1}{2\pi i}\right)^n \int_{|x_1|=\cdots=|x_n|=1} \frac{1}{1 - tf(x_1, \ldots, x_n)} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

This is the same as

$$\pi_f(t) = \sum_{i=0}^{\infty} \text{const}(f^i) t^i,$$

where $\text{const}(f^i)$ is the constant coefficient of f^i. This period satisfies a differential equation

$$L_f = \left(\sum_{k=0}^{r} t^k P_k(D)\right) \pi_f = 0,$$

where $D = t\frac{d}{dt}$ and P_k are polynomials in D of degree k; this is called the Picard-Fuchs equation[5].

Example 3.7. *Let*

$$f = x + y + \frac{1}{xy} \in \mathbb{C}[x, y].$$

Then we get the classical period

$$\begin{aligned} \pi_f(t) &= 1 + 6t^3 + 90t^6 + \ldots \\ &= \sum_{k=0}^{\infty} \frac{(3k)!}{(k!)^3} t^{3k}, \end{aligned}$$

and the differential equation

$$L_f = \left[D^2 - 27t^3(D+1)(D+2)\right] \pi_f = 0.$$

4 Mirror Symmetry

Mirror symmetry is an idea which stems from string theory, it links Fano manifolds to Laurent polynomials. We say:

Definition 4.1. *A Laurent polynomial, f, is mirror-dual to a Fano manifold, X, if their periods match, $\pi_f = \widehat{G}_X$.*

We said that the variety generated by $\text{Newt}(f)$ is toric and probably not smooth, whilst Fano manifolds are smooth, and we are interested when they are not toric. Mirror symmetry tells us:

Conjecture 4.2. *If a Laurent polynomial f is mirror-dual to a Fano manifold X then we expect that X_P is a degeneration of X, where $P = \text{Newt}(f)$.*

This has already been used to replicate the classification of 2 and 3 dimensional Fano manifolds. We wish to identify Fano manifolds with their Laurent polynomial duals in 4 dimensions. For known 4 dimensional Fano manifold we will look for Laurent Polynomials to match the period.

Given a Fano variety, if we find a Laurent polynomial which is its mirror dual, then there will be infinitely many more Laurent polynomial which have the same period, so also are the mirror dual. These different Laurent polynomials are expected to be related via a *cluster algebra*. Cluster algebras are constructed from copies of $(\mathbb{C}^*)^n$. Each copy carries a Laurent polynomial, which we can view as a function $(\mathbb{C}^*)^n \longrightarrow \mathbb{C}$, with the same classical period. Therefore we expect to be able to define a cluster algebras with the classical period of any of its Laurent polynomials.

5 Matching Period Sequences

The question we come to now is how do we pick our 4 dimensional Laurent polynomials, $f \in \mathbb{C}\left[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}, x_4^{\pm}\right]$. Experimental evidence suggests that the *reflexive polytopes*, which were introduced as an important tool for mirror symmetry[3], are a good place to start.

5.1 Reflexive Polytopes

Definition 5.1. *Given a polytope $P \subseteq N_{\mathbb{Q}}$ we define its dual polyhedron*

$$P^* = \{y \in M_{\mathbb{Q}} \ : \ \langle x, y\rangle \ \geq -1, \forall x \in P\}.$$

Now we say:

Definition 5.2. *P is a reflexive polytope iff P^* is a lattice polytope.*

In fact, if P has full dimension n and if it contains the origin of the lattice in its interior then we have

$$(P^*)^* = P.$$

Hence we have that P is reflexive iff P^* is reflexive [3].

Lets look at some of the properties of reflexive polytopes.

Proposition 5.3. *The lattice origin is the only interior point in a reflexive polytope. [3]*

This is very useful as it allows us to state the theorem.

Theorem 5.4. *There are finitely many reflexive polytopes, up to lattice isomorphism. [3]*

The reflexive polytopes have been classified up to dimensions 4, by Max Kreuzer and Harald Skarke. When $n = 2$, all polytopes with only the origin in the interior are reflexive; and there are 16 up to isomorphism. In higher dimension this isn't true. The classification does say that there are 4319 reflexive polytopes when $n = 3$, and 473,800,776 when $n = 4$ [12, 13].

As $(P^*)^* = P$ the reflexive polytopes come in pairs. Figure 1 shows all 16 2 dimensional reflexive polytopes, up to isomorphism. The dual pairs are 1 and 16; 2 and 9; 3 and 11; 8 and 4; 10 and 12; 13 and 15; 5, 6, 7 and 15 being their own duals.

It is important to notice that in general the reflexive polytopes do not generate smooth varieties.

Example 5.5. *Consider the variety $\mathbb{P}(1,1,2) = \frac{\mathbb{C}^3 \backslash \{0\}}{(a,b,c)\sim(\lambda a, \lambda b, \lambda^2 c)}$, which is generate by the reflexive polytope P with $\mathcal{V}(P) = \{(1,1), (-1,1), (0,-1)\}$.*

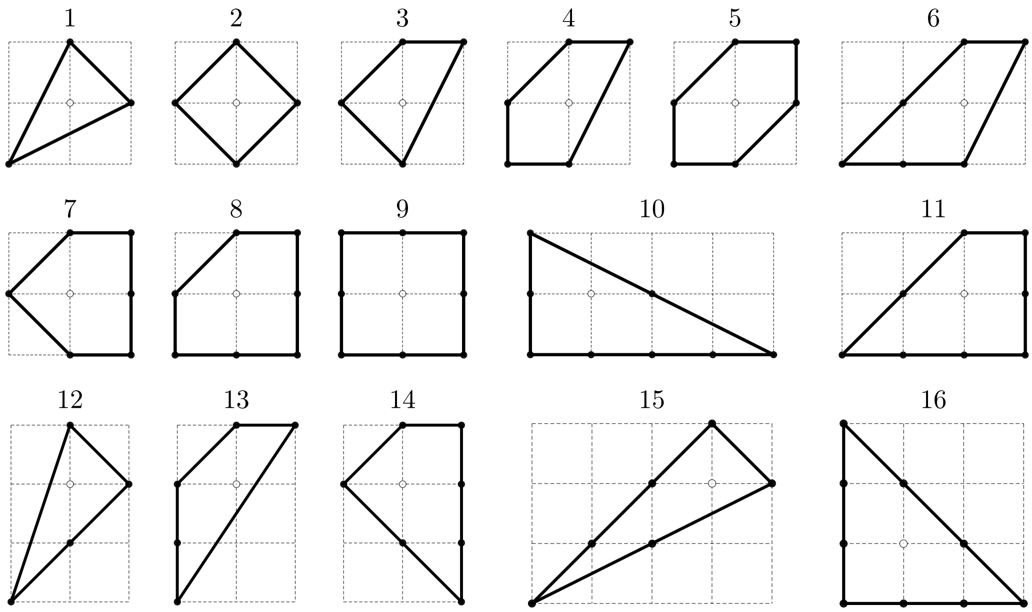
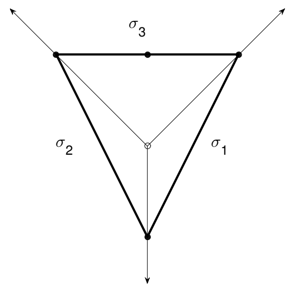


Figure 1: 16 2D Reflexive Polytopes



Here the cones σ_1 and σ_2 are both smooth, but if we look at the dual of cone σ_3 we see that

$$\sigma_3^\vee = cone\{(-1,1),(1,1)\},$$

which is not smooth as the point $(0,1) \in \sigma_3^\vee$ is not $\mathbb{Z}$-generated by $\{(-1,1),(1,1)\}$.

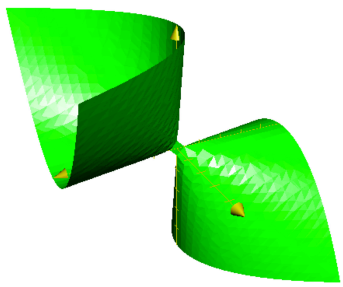
Further

$$\mathbb{C}\left[\sigma_3^\vee\right] = \mathbb{C}\left[XY, Y, X^{-1}Y\right],$$

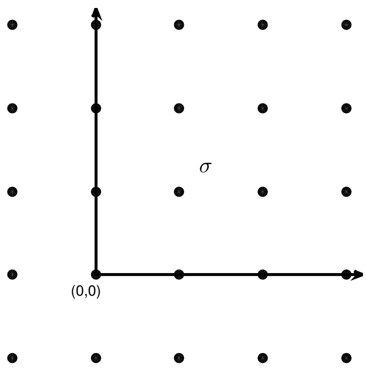
where $(a,b) \in N$ corresponds to the monomial X^aY^b. If we let $x = XY$, $y = Y$ and $z = X^{-1}Y$ then we get

$$\mathbb{C}\left[\sigma_3^\vee\right] = \frac{\mathbb{C}\left[x,y,z\right]}{(xz - y^2)}.$$

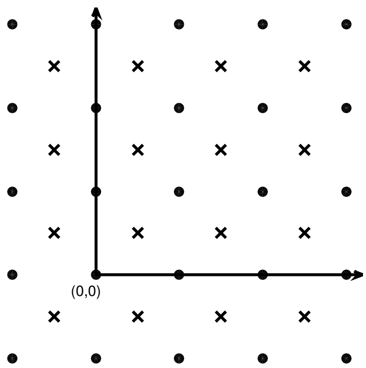
This is called a $\frac{1}{2}(1,1)$ quotient singularity. The singularity is given by the singular point 0 of $xz - y^2 = 0$, which is sketched below.



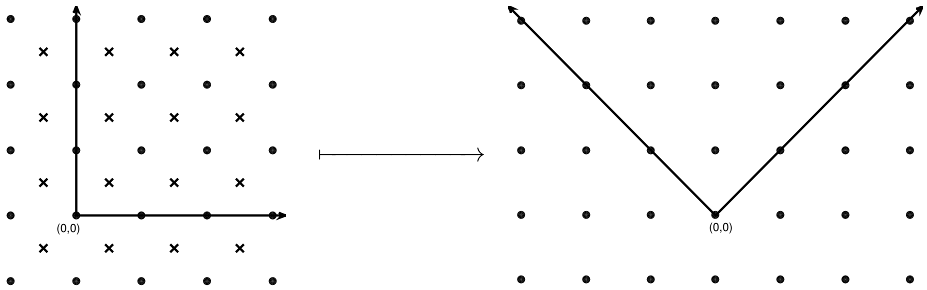
If we consider the lattice N' with a cone, σ, as follows:



If we then add in the sublattice generated by $\frac{1}{2}(1,1)$. Then we get $N' + \mathbb{Z}\frac{1}{2}(1,1)$:



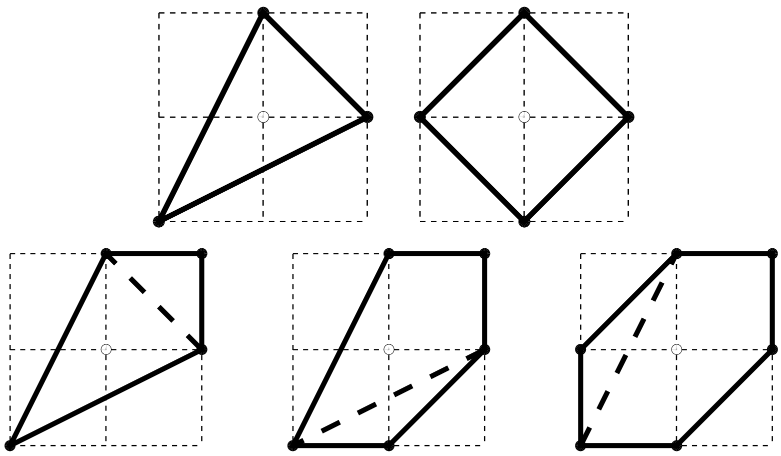
Then we have an isomorphism $N' + \mathbb{Z}\frac{1}{2}(1,1) \cong N$, with



This isomorphism sends $\sigma \in N' + \mathbb{Z}\frac{1}{2}(1,1)$ to $\sigma_3 \in N$. This is why it is called a $\frac{1}{2}(1,1)$ singularity.

Of the 16 2D reflexive polytopes 5 of them produce smooth toric varieties. These are the 5 toric Fano manifolds $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown-up in ≤ 3 points. Blowing up $\mathbb{P}^2$ has a nice description in Fano polytopes; it corresponds to adding a vertex created by adding two of the original vertices. So we get the reflexive polytopes:

$\mathbb{P}^2$ $\qquad$ $\mathbb{P}^1 \times \mathbb{P}^1$



$\mathbb{P}^2$ blown-up in 1 point $\qquad$ $\mathbb{P}^2$ blown-up in 2 points $\qquad$ $\mathbb{P}^2$ blown-up in 3 points

There are a total of 10 Fano manifolds in 2 dimensions, the del Pezzo surfaces. The 5 non-toric Fano manifolds are $\mathbb{P}^2$ blown-up in 4 to 8 points, these are not generated by polytopes, as they are not toric. Of the 10 Fano manifolds, 8 have mirror duals in the reflexive polytopes.

Example 5.6. *Let $f_i \in \mathbb{C}[x, y]$, $i = 1, \ldots, 16$, be the Laurent polynomial of the i-th reflexive polytope from figure 1, when it is mirror dual to a Fano manifold. Then we have:*

$$f_1 = x + y + \frac{1}{xy}$$

$$\pi_{f_1} = (1, 0, 0, 6, 0, 0, 90, 0, 0, 1680, 0, \ldots)$$

$$f_2 = x + y + \frac{1}{x} + \frac{1}{y}$$

$$\pi_{f_2} = (1, 0, 4, 0, 36, 0, 400, 0, 4900, 0, 63504, \ldots)$$

$$f_3 = y + xy + \frac{1}{x} + \frac{1}{y}$$
$$\pi_{f_3} = (1, 0, 2, 6, 6, 60, 110, 420, 1750, 4200, 19152, \dots)$$

$$f_4 = y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}$$
$$\pi_{f_4} = (1, 0, 4, 6, 36, 120, 490, 2100, 8260, 36960, 151704, \dots)$$

$$f_5 = y + xy + \frac{1}{y} + \frac{1}{xy} + \frac{1}{x}$$
$$\pi_{f_5} = (1, 0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, \dots)$$

$$f_6 = y + xy + \frac{1}{y} + \frac{2}{xy} + \frac{1}{x^2y} + \frac{2}{x}$$
$$\pi_{f_6} = (1, 0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, \dots)$$

$$f_7 = y + xy + 2x + \frac{x}{y} + \frac{1}{y} + \frac{1}{x}$$
$$\pi_{f_7} = (1, 0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, \dots)$$

$$f_8 = y + xy + 2x + \frac{x}{y} + \frac{2}{y} + \frac{1}{xy} + \frac{1}{x}$$
$$\pi_{f_8} = (1, 0, 10, 30, 270, 1560, 11350, 77700, 560350, 4040400, 29623860, \dots)$$

$$f_9 = 2y + xy + 2x + \frac{x}{y} + \frac{2}{y} + \frac{1}{xy} + \frac{2}{x} + \frac{y}{x}$$
$$\pi_{f_9} = (1, 0, 20, 96, 1188, 10560, 111440, 1142400, 12154660, 130220160, 1414339920, \dots)$$

$$f_{10} = 2x + \frac{x^3}{y} + 4\frac{x^2}{y} + 6\frac{x}{y} + \frac{4}{y} + \frac{1}{xy} + \frac{2}{x} + \frac{y}{x}$$
$$\pi_{f_{10}} = (1, 0, 20, 96, 1188, 10560, 111440, 1142400, 12154660, 130220160, 1414339920, \dots)$$

$$f_{11} = y + xy + 2x + \frac{x}{y} + \frac{3}{y} + \frac{3}{xy} + \frac{1}{x^2}y + \frac{2}{x}$$
$$\pi_{f_{11}} = (1, 0, 20, 96, 1188, 10560, 111440, 1142400, 12154660, 130220160, 1414339920, \dots)$$

$$f_{12} = y + x + \frac{2}{y} + \frac{1}{xy^2}$$
$$\pi_{f_{12}} = (1, 0, 4, 0, 36, 0, 400, 0, 4900, 0, 63504, \dots)$$

$$f_{13} = y + xy + \frac{1}{xy^2} + \frac{2}{xy} + \frac{1}{x}$$
$$\pi_{f_{13}} = (1, 0, 4, 6, 36, 120, 490, 2100, 8260, 36960, 151704, \dots)$$

$$f_{14} = y + xy + 3x + 3\frac{x}{y} + \frac{x}{y^2} + \frac{2}{y} + \frac{1}{x}$$
$$\pi_{f_{14}} = (1, 0, 10, 30, 270, 1560, 11350, 77700, 560350, 4040400, 29623860, \dots)$$

$$f_{15} = y + x + \frac{2}{xy} + \frac{1}{x^3y^2} + \frac{3}{x^2y} + \frac{3}{x}$$
$$\pi_{f_{15}} = (1, 0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, \dots)$$

$$f_{16} = \frac{y^2}{x} + 3y + 3x + \frac{x^2}{y} + 3\frac{x}{y} + \frac{3}{y} + \frac{1}{xy} + \frac{3}{x} + 3\frac{y}{x}$$
$$\pi_{f_{16}} = (1, 0, 54, 492, 9882, 158760, 2879640, 51982560, 964347930, 18091565520, 343559141604, \dots)$$

As we can see from looking at the 16 classical period, there are only 8 distinct ones, equal to 8 of the quantum periods of del Pezzo surfaces. One interesting observation is that every period sequence appears at least twice except $\pi_{f_{16}}$. If two polytopes have the same classical period, then the area of their duals are the same. The 16th polytope is mirror dual to the 1st polytope in figure 1, which is the smallest reflexive polytope, with all other reflexive polytopes having a larger area. This is one reason why no reflexive polytope shares the same period sequence, $\pi_{f_{16}}$.

In 3 dimensions there are a total of 105 Fano manifolds, of them 18 are toric, and so are generated by a reflexive Fano polytope. A total of 98 of them have mirror-duals in the reflexive polytopes. In fact these 98 Fano manifolds have very-ample anticanonical divisors[6].

As the reflexive polytopes mirror-dual to most of the 2D and most of the 3D Fano varieties, we expect that they will also have mirror-duals to many of the 4D Fano varieties.

5.2 Matching Process

We are given a list of quantum periods of Fano manifolds[1], and we want to match them to Laurent polynomials which generate reflexive polytopes. If we pick a reflexive polytope we can generate a Laurent polynomial with unknown coefficients, we want to see if its possible to pick the coefficients such that its classical period matches a given period sequence. This can be done using ideals and varieties.

Let f be the Laurent polynomial from a polytope, with d unknown coefficients $x_1 \dots x_d$. We can then generate the classical period

$$\pi_f = (\pi_0(x_1, \dots, x_d), \pi_1(x_1, \dots, x_d), \dots),$$

where $\pi_i(x_1, \dots, x_d) = \mathrm{const}(f^i) \in \mathbb{C}[x_1, \dots, x_d]$ is the constant coefficient of f^n.

Let say we are try to match this with the quantum period sequence

$$\widehat{G}_X = (\gamma_0, \gamma_1, \gamma_2, \dots),$$

[1] We will be looking at matching in 4 dimensions. The quantum periods come from [8, 7].

with $\gamma_n \in \mathbb{C}$. We can then form an ideal in $\mathbb{C}[x_1, \ldots, x_d]$

$$\begin{aligned} I &= \langle \pi_0 - \gamma_0, \pi_1 - \gamma_1, \ldots, \pi_r - \gamma_r \rangle \\ &= \left\{ \sum_{i=0}^{r} h_i(\pi_i - \gamma_i) \,:\, h_i \in \mathbb{C}[x_1, \ldots, x_d] \right\}. \end{aligned} \tag{1}$$

We can then use Gröbner basis and elimination theory (see section 5.2.1) to compute the variety

$$\mathbb{V}(I) = \left\{ (x_1, \ldots, x_d) \in \mathbb{C}^d \,:\, \pi_i(x_1, \ldots, x_d) - \gamma_i = 0, \quad \forall i = 0, \ldots, n \right\}.$$

This variety contains the solution for the unknowns, $(x_1, \ldots, x_d)$, such that $\pi_i = \gamma_i$ for $i = 0 \ldots r$. This will contain the solutions such that $\pi_f = \widehat{G}_X$. If we pick a large enough n the variety will be small enough to plug each point of the variety into the unknowns for f, and check its period sequence; or the variety could be empty, which means there is no solution.

When we are in a large dimension or there are many unknowns the computations can take a long time. One way to help speed up the computation will be to restrict the coefficients to integer values and then to use the Chinese remainder theorem. We will pick distinct primes $p_1, \ldots, p_r$ and compute the varieties in $(\mathbb{F}_{p_i})^d$, call these varieties V_i for $i = 1, \ldots, r$. The Chinese remainder theorem then tells us that since $gcd(p_i, p_j) = 1$, $\forall i \neq j$, then for $(c_{i,1}, \ldots, c_{i,d}) \in V_i$ for $i = 1, \ldots, r$ there exists a unique $(c_1, \ldots, c_d) \in (\mathbb{F}_N)^d$, where N is the product of the primes, such that $c_d \equiv c_{d,i} (\text{mod } p_i)$ for $i = 1, \ldots, r$.

Example 5.7. *Lets see the matching process working with the 10th reflexive polytope from figure 1 and using the period sequence $\pi_{f_{10}}$, from example 5.6, as the quantum period I am trying to match it against. Firstly I get the Laurent polynomial with 5 unknowns, (a, b, c, d, e),*

$$f = \frac{y}{x} + \frac{x^3}{y} + \frac{1}{xy} + ax + b\frac{1}{x} + c\frac{1}{y} + d\frac{x}{y} + e\frac{x^2}{y}.$$

Lets work over the primes 3 and 5, so I then compute the two period sequence,

$$\begin{aligned} &\pi_{f,1} = (1, 0, 2ab + 2d, 0, 0, 0, \ldots) \\ &\pi_{f,2} = \left(1, 0, 2ab + 2d, eb + ac, 2ec + 1a^2b^2 + 2a^2 + 4abd + 2b^2 + d^2 + 2, \ldots\right), \end{aligned}$$

each of the entries being in the two polynomial rings $\mathbb{F}_3[a, b, c, d, e]$ and $\mathbb{F}_5[a, b, c, d, e]$ respectively. Taking away the quantum period (modulo 3 and 5) and then forming ideals and varieties I get two varieties $V_1 \subseteq (\mathbb{F}_2)^5$ and $V_2 \subseteq (\mathbb{F}_5)^5$. I then find that I have the two point

$$\begin{aligned} (2, 2, 1, 0, 1) &\in V_1 \\ (2, 2, 4, 1, 4) &\in V_2 \end{aligned}$$

in the varieties. There may be other points in the variety as well depending on how many generators were use for the ideal. Then, by the Chinese remainder theorem, these two points give the unique coefficients $(2, 2, 4, 6, 4)$, as a possible solution so that the periods match. I then substitute these coefficients into the Laurent polynomial and compute the classical period, and check it against the *quantum period. With these coefficients I get f_{10} from example 5.6, so this would give us a mirror-dual Laurent polynomial.*

As we used the primes 3 and 5 the maximum value of any coefficient was 15.

Using the Chinese remainder theorem and working over $(\mathbb{F}_{p_i})^d$ will save time compared to computing varieties in $\mathbb{Z}^d$ when in 4 dimensions. Working modulo $p_1, \ldots, p_r$ means that we will only be able to compute the coefficients modulo $\prod_{i=1}^{r} p_i$ so the choice of primes will be important.

If there are no unknowns the process is a lot more simple. We can compute the the classical period sequence (as there are no unknowns this will be explicit) and compare it directly with the list of the quantum period sequences.

5.2.1 Gröbner Basis and Elimination Theory

Given an ideal, $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, we can find a particularly nice basis called a Gröbner basis (GB, from now on), $G = \{g_1, \ldots, g_r\}$. This is unique for any given ideal and therefore lets us determine if $I = J$ for another ideal J. It's nice as it has properties that lets us easily determine if $f \in I$, and is use to prove the bijection between radical ideals and affine varieties[9, Chapter 4].

Elimination theory is a method of finding the solution to simultaneous equations, using ideals, varieties and GBs. A brief description of the method goes as follows. We form a ideal using the simultaneous equations as generators and then compute the GB w.r.t. lexicographic order. Using this GB we can compute the variety, which will be in a form which will be easy for us to find all the solutions. [9, Chapter 3]

These methods have been implemented on many computer algebra packages, and provide a quick and efficient way to find solutions to simultaneous equations.

5.3 Implementation in Magma

I will implement this matching process in the computer algebra package *Magma*, which has been used for similar tasks before, and has many useful prebuilt functions.

I will write a Magma function which take the dimension of polytope we are working in, the ID of the reflexive polytope and a list of primes; the function will go through the matching process as described above for a list of known quantum period sequences, and return all possible matches. In the function I will assume that all the vertices have coefficient 1, this has been seen to work well in previous studies and fits with them being non-zero.

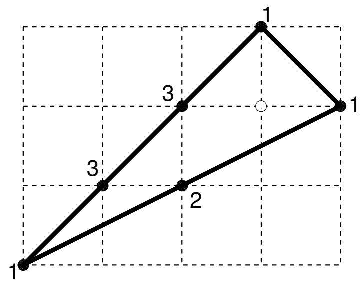
When testing this function on the 2 dimension reflexive polytopes I find that it matches all 16 reflexive polytopes to 8 Fano manifolds. This is what we expect from the theory. From this and also from testing the function on some of the three dimensional reflexive polytopes there are some interesting observations.

5.3.1 Binomial Coefficients

When matching in two and three dimensions we can notice that the coefficients along any of the edges have a particular pattern.

Conjecture 5.8. *The successive coefficients along any edge are successive binomial coefficients.*[1]

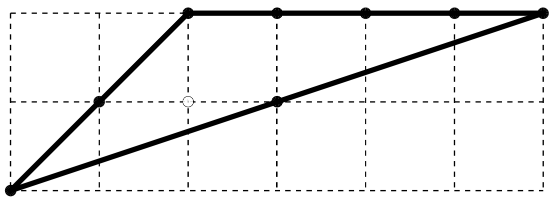
Example 5.9. *Below is a picture of a two dimensional reflexive polytopes, with the coefficients of each of the corresponding monomials when it is mirror-dual to one of the Fano manifolds. Along one edge are the coefficients* $1, 2, 1$ *and along the other edge the coefficients* $1, 3, 3, 1$*; both binomial successive binomial coefficients.*



This has been proven in two and three dimension, but not in four. I will assume this in four dimension, and implement it in my function. This will simplify a lot of the computations as it will decrease the number of unknowns in each computation, and can even simplify problem to the zero unknown case.

Lets look at an example when this seems to go wrong

Example 5.10. *The 13th reflexive polytope (according to Magmas reflexive polytope function) looks like the following.*



According to the conjecture 5.8, we expect the two diagonal edges to have coefficient 2 and the top horizontal edge, including the vertices, to have coefficients (1,4,6,4,1). When I run the matching function I find two sets of coefficients which give a mirror-dual; the one mentioned above and a second with 0s on the diagonal edges and the coefficients (1,0,2,0,1) along the top edge.

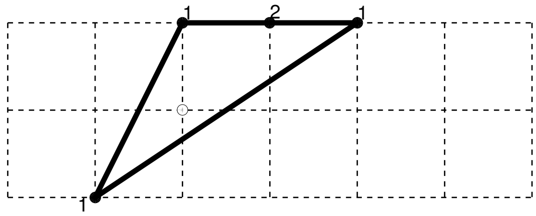
If we write out the Laurent polynomial with this second, unexpected coefficients, we get the following

$$f = x^4y + 2x^2y + y + \frac{1}{x^2y}.$$

Notice how all the monomials are in powers of x^2*, so we can substitute* x^2 *for* x*, to get the following*

$$f = x^2y + 2xy + y + \frac{1}{xy}.$$

This will clearly have the same period sequence and if we take the newton polytope we get:



This is a reflexive polytope, with binomial edge coefficients. Therefore the original polytope with the second, non-binomial edge coefficients, reduces to a reflexive polytope with binomial edge coefficients by scaling the x axis by 2.

One easy check to see if our coefficients are just transformation of another Laurent polynomial for a reflexive polytope is to check that the lattice points corresponding to the non-zero monomials span the whole lattice.

When implementing the binomial edge coefficients it will greatly reduce the chance of this happening, as it will already put non-zero coefficients on some of the unknowns, and together with the vertices these will likely span the lattice.

5.3.2 Non-Bijective Matching

One problem we find is that the matching of reflexive polytopes to the quantum periods of the Fano varieties is not 1-1.

Example 5.11. *Consider the Laurent polynomials*

$$f = xy + y + \frac{1}{y} + \frac{1}{xy}$$
$$g = x^2y + 2xy + y + \frac{1}{xy},$$

which give distinct reflexive polytopes. If we consider their period sequences we find

$$\pi_f = \pi_g = \widehat{G}_X,$$

for the 2D Fano variety $X = \mathbb{P}^1 \times \mathbb{P}^1$.

In fact the 16 2D reflexive polytopes only give 8 distinctive period sequence. In three dimensions there are a particular class of Laurent polynomials, called *Minkowski polynomials*, which are are mirror-dual to the 98 of the 3D Fano varieties with very ample tangent bundles. Of all the reflexive polytopes which generate a Minkowski polynomial, of which there are a few thousand, they only produce 165 distinct period sequences[5]. This suggests that when we look for matches in four dimensions we will probably find many Laurent polynomials are mirror-dual to the same Fano variety.

This property, of several Laurent polynomials being mirror-dual to the same Fano variety, can be explained by *mutations*, which are analogue to birational transformation of 2D Laurent polynomials[2]. These mutations preserve period sequences.

We also find that a single reflexive polytope can have multiple Laurent polynomials with unique periods, such that both periods are mirror-dual to different Fano manifolds. This means that the function should not stop after it has found a match; it should check for matches against all the quantum periods. The exception to this is when there are no unknowns. A Laurent polynomial with no unknowns will always have the same period, so therefore can only be mirror-dual to a single Fano manifold.

5.3.3 Choosing Primes

The matching process implemented in my function requires us to choose a list of prime numbers. This choice is important and it balances the speed of the computation with an upper limit of the coefficients. For a list of primes $(p_1, \ldots, p_r)$, as the matching process uses the Chinese remainder theorem, the final results of the coefficient will have an upperbound at $\prod_{i=1}^{r} p_i$. If a coefficient is larger than this my function will not find it.

Also in general as we increase the dimension of the matching we expect the maximum coefficient to also increase, this could become in issue when working in four dimensions. When the only lattice points are along edges and at the origin this issue is trivial, as we are implementing binomial coefficient along the edges, so there are no unknown coefficients and the maximum coefficient doesn't matter. From experimenting with reflexive polytopes with one unknown I find that the set of primes $(2, 3, 5, 7)$, give a good balance of maximum coefficient, at 210, and most of the computations are reasonably quick. When increasing the number of unknowns we again are likely to encounter larger coefficients, so this may not be large enough.

5.3.4 Number of Generators for the Ideal

The number of generators for the ideal is an important parameter, which will affect the timing of the function significantly. There are two sides to consider; the time it takes to compute each generator, and the size of the variety of the ideal.

When there are no unknowns the problem is quite simple. The quantum periods we have of the Fano manifolds in 4 dimensions contain the first 41 terms. We could just compute the first 41 terms of the Laurent polynomial and compare this with the list of quantum periods. However it takes Magma around 75 seconds to compute the period sequence to 41 terms, if there is no match found this could be a wasted time as most of the time the periods will disagree well before 41 terms. What we will do instead is compute the first 9 terms, if we find a quantum period which agrees with these we will them compute the next 5 terms, and compare them. We will continue to go up in small steps until we reach the end of the list or find a quantum period which agrees on all 41 terms. This we we will often save time by only computing the terms we need.

When there are unknowns, each generator, $\pi_i - \gamma_i$ of the ideal(1), is formed from the classical period of the Laurent polynomial, minus the quantum period (which we are given). If we attempt to compute the generators up to 41 terms it takes far too long, as computing period sequences, especially when there are unknowns in the coefficients, is very expensive computationally. Therefore we try to compute the smallest number of generators, whist trying to make the ideal as large as possible, and therefore the variety as small as possible. To do this we compute the generators one at a time and stop when all the unknown variables have been include in at least one of the generator. This we there is some restriction on each unknown variable, and hopefully this will keep the variety small enough. I also choose to add some extra generators, depending on how many unknown variables there are. If there is one or two unknowns I will add three extra generators, for two unknowns I will add two extra generator and for four unknowns I will add one extra generator. This way I hope to reduced the size of the variety, whilst not spending too much time computing generators. This method was generated by trial and error, and what I found worked best.

6 Results

I will try and match the 4D reflexive polytopes against a list of 796 period sequences of 4D Fano manifolds[2]. I will start by trying to match the more simple reflexive polytopes, with no unknowns (i.e. with lattice points only along edges) or with one unknown. With no unknowns I find that out of 151270 polytopes 5329 have matches, and with one unknown 21334 out of 3000000 have matches. Here is a table showing the period sequences with most matches.

No Unknowns	
Period Sequence Number	Number of Matches
320	37
155, 268	34
249	31
289	30
141 306	29
232	28
134, 145, 358, 489	27

One Unknown	
Period Sequence Number	Number of Matches
706	247
301	187
412	177
429	158
170	157
434	153
312	150

Firstly what is clear from theses tables is the number of matches with one unknowns is significantly higher than with no unknowns. This is not too surprising, as when there is one unknown each reflexive polytope can match to several quantum period sequences. Also in the one unknown reflexive polytopes, the 706th quantum period seems to have significantly more matches than the rest of the period sequences in one unknown.

We also find there are 226 periods with no matches to polytopes with no unknowns and 173 periods with no matches to polytopes with one unknown. Put together there are 103 periods with no matches to the reflexive polytopes with no or one unknown. This could be because there is no matching in the reflexive polytopes for these Fano manifold, as we see in both 2 and 3 dimensions that the reflexive polytopes to not match with all the Fano manifolds. There is also a chance that they still do have mirror-duals in the reflexive polytopes with more unknowns.

7 Conclusion

Through my programme I have found quite a few matches. This has included many different Laurent polynomials being mirror-dual to the same Fano manifold. It would be good to check if these Laurent polynomial, with identical period sequences are mutations of each other. This is conjected to be true.

[2] Found in [8, 7]

The 103 quantum periods for which I found no Laurent polynomial mirror-dual would also be a interesting thing to look into. In 3 dimensions we find there are some Fano manifolds which have no mirror-duals in the reflexive polytopes, but all of theses have ample and not very ample anticanonical divisors. It would be interesting to see if the ones for which I found no match also do not have very ample anticanonicle divisors.

One issue is that I have only checked with reflexive polytopes which give me no unknowns or one unknown. It would be good to expand the search to larger reflexive polytopes, although the matching would take a very long time to compute. Another idea would be to look at ways to speed up the process, either by finding better sets of prime numbers which optimizes the process, or by look for new methods.

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A Magma Code

The following code is use to load the period sequences which are in a file called "period4d.txt", in the working directory.

```
// Loading the period sequence from file
p4dtxt:=Open("period4d.txt","r");
line:=Gets(p4dtxt);
p4d:=[];
while(not IsEof(line)) do
    Append(~p4d,StringToIntegerSequence(line));
    line:=Gets(p4dtxt);
end while;
```

The `LaurentPoly` function takes a polytope as the argument and returns the Laurent polynomial with binomial edge coefficients, and unknowns on the remaining points in the first return argument. It returns the number of unknowns in the second return argument.

```
// Given Fano Polytope P with only one interior point returns the Laurent
// polynomial with coefficient zero at the origin, with binomial coefficients
// along the edges and unknowns for the rest.
// Returns the Laurent poly and the number of unknowns.
LaurentPoly:=function(P)
    error if not IsFano(P), "Argument must be a Fano Polytope";
    error if NumberOfInteriorPoints(P) ne 1,
        "Polytope must have exactly one interior point";
    // Computing number of unknowns
    num:=NumberOfBoundaryPoints(P) - NumberOfEdgePoints(P);
    // Creating sequence of all points
    pts:=Sort(Points(P));
    // Dealing with no unknown case
    if num eq 0 then
        return LaurentPolynomial(pts,BinomialEdgeCoefficients(P)),0;
    end if;
    // Build coefficient ring
    K:=PolynomialRing(Rationals(),num);
    // Make sequence of coefficients
    cs:=ChangeUniverse(BinomialEdgeCoefficients(P),K);
    zeroindex:=Index(pts,Zero(Ambient(P)));
    cs[zeroindex]:=K ! -1;
    i:=1;
    while i le num do
        cs[Index(cs,Zero(K))]:=K.i;
        i+:=1;
    end while;
    cs[zeroindex]:=K ! 0;
    // Return results
    return LaurentPolynomial(pts,cs),num;
end function;
```

The `NoUnknownMatching4D` function takes a 4 dimensional Laurent polynomial with no unknowns and computes it classical period. It then compares it with the quantum periods in the list `p4d`, and returns number of the period which it matches, or 0 if there is no match.

```
// Given a Laurent Polynomial, lpoly, of with no unknowns it will
// compute its period sequence and try match it with list in p2d or p3d if there
// is a match return line number of match, if no match return 0.
NoUnknownMatching4D:=function(lpoly)
    error if not IsLaurent(lpoly), "Argument must be a Laurent Polynomial";
    error if not CoefficientRing(Parent(lpoly)) eq Integers(),
        "lpoly must have no unknowns";
    // Go through p4d checking if it matches
    ps:=PeriodSequence(lpoly,8);
    for perseq in p4d do
        if perseq[1..9] eq ps[1..9] then
            if #ps < 9 then
                ps:=PeriodSequence(lpoly,13);
            end if;
            if perseq[1..14] eq ps[1..14] then
                if #ps < 19 then
                    ps:=PeriodSequence(lpoly,18);
                end if;
                if perseq[1..19] eq ps[1..19] then
                    if #ps < 25 then
                        ps:=PeriodSequence(lpoly,24);
                    end if;
                    if perseq[1..25] eq ps[1..24] then
                        if #ps < 41 then
                            ps:=PeriodSequence(lpoly,40);
                        end if;
                        if perseq[1..41] eq ps[1..41] then
                            return Index(p4d,perseq);
                        end if;
                    end if;
                end if;
            end if;
        end if;
    end if;
    // No matches so return 0
    return 0;
end function;
```

The PeriodSeq function takes 3 arguments, a Laurent polynomial, a number of max terms, and the number of unknowns in the polynomial. It then computes its period sequence, in polynomials of the unknowns. It will compute the sequence until either all unknowns have been used and the extra terms have been computed, or until it reaches the max argument. It returns this period sequence.

```
// Given a Laurent polynomial it returns the period sequence which includes
// all variables of the coefficient ring, with extra terms, if possible, or
// returns period sequence with max number of terms.
// 3 extra terms if less than 3 unknowns, 2 extra if 3 or 4 unknowns, 1 extra
// 5 unknowns, and no extra for 6 or more unknowns. Up to a max number of
// terms, which is greater or equal to 3.
PeriodSeq:=function(lpoly,max_terms,n)
    error if not IsLaurent(lpoly), "Must be a Laurent Polynomial";
    error if max_terms lt 3, "max_terms must be greater than or equal to 3";
    error if n le 0, "Must be positive number of unknowns";
```

```
// Finding Coefficient ring, and its rank
K:=CoefficientRing(Parent(lpoly));
// Finding number of extra terms needed
if n le 2 then
    extra:=3;
elif n le 4 then
    extra:=2;
elif n eq 5 then
    extra:=1;
else
    extra:=0;
end if;
// Setting initial number of terms needed
need:=3;
// Setting initial period sequence
if need+extra ge max_terms then
    periodseq:=PeriodSequence(lpoly,max_terms-1);
else
    periodseq:=PeriodSequence(lpoly,need+extra-1);
end if;
// Checking which variables are included in period sequence
inc:=[false : i in [1..n]];
for i in [1..n] do
    if #Coefficients(periodseq[need],K.i) gt 1 then
        inc[i]:=true;
    end if;
end for;
// Not all variable included
while Index(inc,false) ne 0 do
    // Adding to count of needed terms
    need+:=1;
    // Checking if needed terms greater than max terms
    if need gt max_terms then
        return periodseq;
    end if;
    // Adding more terms to period sequence if reached the end
    if extra eq 0 then
        periodseq:=PeriodSequence(lpoly,need-1);
    elif (need mod extra) eq 0 then
        if need+extra ge max_terms then
            periodseq:=PeriodSequence(lpoly,max_terms-1);
        else
            periodseq:=PeriodSequence(lpoly,need+extra-1);
        end if;
    end if;
    //Checking if variables are included
    for i in [1..n] do
        if not inc[i] then
            if #Coefficients(periodseq[need],K.i) gt 1 then
                inc[i]:=true;
            end if;
        end if;
    end for;
end while;
```

```
    // Returning period sequence
    return periodseq;
end function;
```

The function VarietyCheck4D takes 5 arguments, the Laurent polynomial, the number of unknowns, a sequence of primes, a list of varieties, and a period sequence. It then goes through all combinations of the varieties using the Chinese remainder theorem, and checks it against the period sequence. It returns the coefficients such that the periods match or 0 if it cant find a match.

```
// Given a Laurent Polynomial, lpoly, with n unknowns, a sequence of
// primes, a list of varieties in Z/pZ and a period sequence, perseq;
// return the coefficients of the unknowns such that lpoly gives the
// same period sequence as perseq, with coefficients from the varieties.
// Returns 0 if no such solution exists.
VarietyCheck4D:=function(lpoly,n,primes,var,perseq)
    // Finding Field and Coeff ring of lpoly
    R:=Parent(lpoly);
    K:=CoefficientRing(R);
    // Forming field with integer coeffs
    RI:=FunctionField(Integers(),4);
    // Checking each combination of the varaieties
    for v in CartesianProduct(var) do
        //Using chinese remainder theormn to compute coefficients
        cs:=[ CRT([Integers() |
                   v[i][j] : i in [1..#primes]],primes) : j in [1..n]];
        // Subing these coefficients into lpoly
        phi:=hom<K->RI | cs>;
        psi:=hom<R->RI | phi, [RI.i : i in [1..4]] >;
        f:=psi(lpoly);
        // Checking its period sequence against perseq
        if perseq[1..9] eq PeriodSequence(f,8) then
            if perseq[1..14] eq PeriodSequence(f,13) then
                if perseq[1..19] eq PeriodSequence(f,18) then
                    if perseq[1..25] eq PeriodSequence(f,24) then
                        if perseq[1..41] eq PeriodSequence(f,40) then
                            // Returning matching coefficients
                            return cs;
                        end if;
                    end if;
                end if;
            end if;
        end if;
    end for;
    // No match
    return 0;
end function;
```

The function ChineseMatching4D takes a 4 dimensional Laurent polynomial with unknowns, the number of unknowns, and a list of primes. It then performs the matching process using the Chinese remainder theorem using the list of primes.

```
// Given a Laurent Polynomial, lpoly, with n unknowns, and a sequence
// of primes. Then using the Chinese remainder thm to match period
```

```
// sequence of lpoly with list of periods, p2d.
ChineseMatching4D:=function(lpoly,n,primes)
    error if #primes eq 0, "Must provide non-empty sequence of primes";
    error if not IsLaurent(lpoly), "Polynomial must be a Laurent Polynomial";
    error if n le 0, "Number of unknown coefficients must be greater than zero";
    //Creating empty sequences for solution line number and coefficient values
    sol_line:=[];
    sol_cs:=[];
    // Finding Coefficient ring and Field of lpoly
    R:=Parent(lpoly);
    K:=CoefficientRing(R);
    // Creating lpoly in Z/pZ for p in primes and their period sequence
    lpoly_p:=<>;
    perseq_p:=<>;
    for p in primes do
        // Forming Coefficient ring in Z/pZ and function field
        Kp:=PolynomialRing(FiniteField(p),n);
        Rp:=FunctionField(Kp,4);
        // Setting homomorphisms
        phi:=hom<K->Rp | [Kp.i : i in [1..n] ] >;
        psi:=hom<R->Rp | phi, [Rp.i : i in [1..4]]>;
        // Creating poly with coeffs in Z/pZ and adding to list
        lpoly_temp:=psi(lpoly);
        Append(~lpoly_p,lpoly_temp);
        // Finding period sequence and adding to list
        perseq_temp:=PeriodSeq(lpoly_temp,41,n);
        Append(~perseq_p,perseq_temp);
    end for;
    // Going through p4d and check for matching coefficients
    for perseq in p4d do
        // Computing variety of ideal for each (period sequence - p4d line)
        variety_p:=<>;
        for i in [1..#perseq_p] do
            // Finding generators
            gen:=[perseq_p[i][j]-perseq[j] : j in [3..#perseq_p[i]]];
            // Computing ideal and variety
            I:=Ideal(gen);
            V:=Variety(I);
            // If variety is empty going to next line of p4d
            if IsEmpty(V) then
                continue perseq;
            end if;
            // Adding variety to list
            Append(~variety_p,V);
        end for;
        // Checking the varieties give a solution
        cs:=VarietyCheck4D(lpoly,n,primes,variety_p,perseq);
        // If theres a solution
        if not IsZero(cs) then
            Append(~sol_line,Index(p4d,perseq));
            Append(~sol_cs,cs);
        end if;
    end for;
    // If no solution was found
```

```
    if IsEmpty(sol_line) then
        return -1;
    end if;
    // Returning solution
    return sol_line,sol_cs;
end function;
```

The `Matching4D` function takes the ID of a 4 dimensional reflexive polytope, and a list of primes. It then performs the matching process against the list of quantum periods `p4d`. It returns -1 if there is no match, or a list of the line number which it matches to and a second list of coefficients for the matches.

```
// Given an n, the n-th reflexive polytope it will match it to the
// period sequence in "peiod4d.txt". Using the chinese remainder
// theorem with the given sequence of primes. Returns line number
// of match if no unknowns, or [line numbers], [sequences of values of unknowns],
// Returns 0 for no match and no unknowns, -1 if no match with unknown.
Matching4D:=function(n, primes)
    error if n notin [Integers() | 1..473800776 ],
        "Must pick a n from 1 to 473800776";
    error if #primes eq 0, "Primes must be non-empty sequence";
    //Building the Laurent poly
    lpoly,uks:=LaurentPoly(PolytopeReflexiveFanoDim4(n));
    // Dealing with case when there are no unknowns
    if uks eq 0 then
        return NoUnknownMatching4D(lpoly);
    end if;
    // Dealing with case when there are unknowns
    return ChineseMatching4D(lpoly,uks,primes);
end function;
```

Loading the scrip called “matching_4d.txt”, and running the code on the 11th reflexive polytope and then the 1737th reflexive polytope I get the following:

```
> load "matching_4d.mag";
Loading "matching_4d.mag"
> Matching4D(11,[2,3,5,7]);
20
> Matching4D(1737,[2,3,5,7]);
[ 63 ]
[
    [ 1 ]
]
```

Here I successfully load the script. Then the function output say that the 11th reflexive polytope matches the 20th quantum period sequence. Finally the 1737th reflexive polytope matches the 63 quantum period when the unknown has coefficient 1.