

SMOOTH TORIC FANO VARIETIES AND THEIR POLYTOPES

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1. INTRODUCTION.

My research is chiefly concerned with classifying toric Fano varieties with certain classes of singularities. For the purposes of this brief talk we shall restrict our attention to smooth varieties. I first hope to outline the connection between toric varieties and combinatorics, for those not already familiar with this construction, and then wish to indicate how the classification of lattice polytopes is connected with the classification of our varieties. Finally, I shall present (all without any proofs) a recent result of Casagrande (c.f. [Cas04]) which gives sharp bounds for the rank of the Picard group of smooth toric Fano varieties.

A substantial body of introductory literature exists on the subject of toric varieties. The standard references are [Ful93], [Ewa96], and [Oda78]. Of equal merit are [BB02] and [Dan78]. For a survey of the current state of research in this field, consult [Cox02]. For more on the relationship between convex lattice polytopes and toric varieties, [Bor00] provides a useful summary.

These notes are for a talk given by the author at GAEL XIII (Géométrie Algébrique en Liberté) in March 2005. I would like to thank the organisers for allowing me to speak at this event.

2. OUR FIRST EXAMPLE: \mathbb{P}^2 .

Before a definition of what it means for an algebraic variety to be a toric variety is given, let us consider the (complex) projective plane \mathbb{P}^2 . We are all familiar with constructing \mathbb{P}^2 by gluing three copies of \mathbb{C}^2 . Namely, if $(z_0 : z_1 : z_2)$ are homogeneous coordinate of \mathbb{P}^2 ,

$$U_i := \left\{ \left(\frac{z_0}{z_i} : \frac{z_1}{z_i} : \frac{z_2}{z_i} \right) \middle| z_i \neq 0 \right\} \cong \mathbb{C}^2, \quad i = 0, 1, 2,$$

give three isomorphic copies of \mathbb{C}^2 . These glue, as illustrated in Figure 1, via maps of the form:

$$\begin{aligned} \phi_{0,1} : U_0 \setminus (z_1/z_0 = 0) &\rightarrow U_1 \setminus (z_0/z_1 = 0) \\ (1 : x : y) &\mapsto (1/x : 1 : y/x). \end{aligned}$$

Associated to each chart U_i is the ring of regular functions, $\mathbb{C}[U_i]$;

$$\mathbb{C}[U_0] = \mathbb{C}[z_1/z_0, z_2/z_0], \quad \mathbb{C}[U_1] = \mathbb{C}[z_0/z_1, z_2/z_1], \quad \mathbb{C}[U_2] = \mathbb{C}[z_0/z_2, z_1/z_2].$$

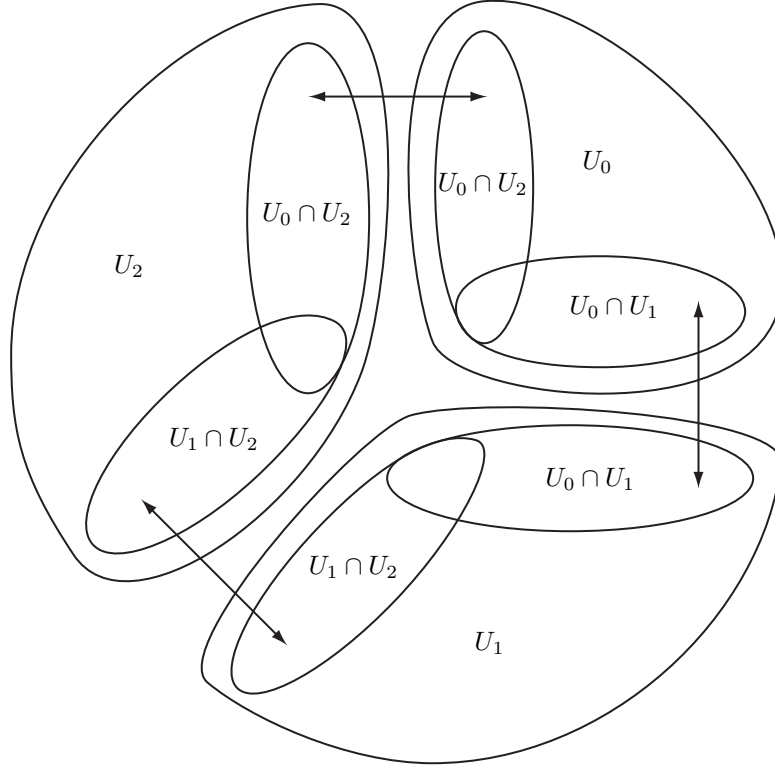


FIGURE 1. \mathbb{P}^2 realised as the gluing of three affine plains U_0, U_1 , and U_2 .

By setting $X := z_1/z_0$ and $Y := z_2/z_0$ we obtain:

$$\mathbb{C}[U_0] = \mathbb{C}[X, Y], \quad \mathbb{C}[U_1] = \mathbb{C}[X^{-1}, X^{-1}Y], \quad \mathbb{C}[U_2] = \mathbb{C}[Y^{-1}, XY^{-1}].$$

Each $\mathbb{C}[U_i]$ is contained within the coordinate ring $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$. This is the coordinate ring associated with the *algebraic torus* $(\mathbb{C}^*)^2 \cong \text{Spec}(\mathbb{C}[X^{\pm 1}, Y^{\pm 1}])$, from which toric varieties derive their name.

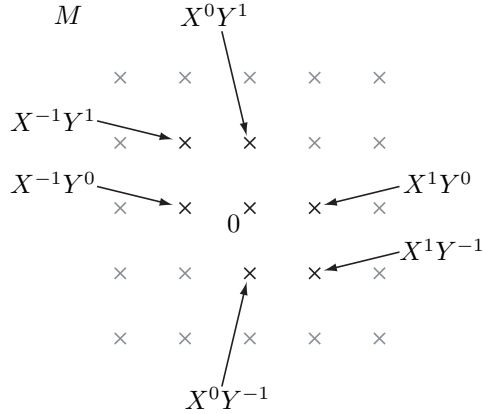


FIGURE 2. The lattice M with the generators of each $\mathbb{C}[U_i]$ indicated.

Let $M \cong \mathbb{Z}^2$ be the lattice of Laurent monomials in X and Y . Thus points in M correspond to monomials $X^m Y^n$ for some $(m, n) \in \mathbb{Z}^2$ (see Figure 2). Let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2$. We regard monomials in the coordinate rings $\mathbb{C}[U_i]$ as lattice points in $M_{\mathbb{R}}$. Thus $\mathbb{C}[U_0]$ corresponds to the cone¹ with generators $\{X, Y\}$ (i.e. the nonnegative quadrant of $M_{\mathbb{R}}$), $\mathbb{C}[U_1]$ corresponds to the cone with generators $\{X^{-1}, X^{-1}Y\}$, and $\mathbb{C}[U_2]$ to that with generators $\{Y^{-1}, XY^{-1}\}$. Thus to each coordinate ring $\mathbb{C}[U_i]$ we have associated a cone in $M_{\mathbb{R}}$, denoted σ_i^{\vee} .

To each cone σ_i^{\vee} we associate its *dual cone*² σ_i . These dual cones will lie in the vector space $N_{\mathbb{R}} \cong \mathbb{R}^2$ obtained from the lattice $N := \text{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^2$. To be precise:

$$\sigma_i := \{v \in N_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } u \in \sigma_i^{\vee}\}.$$

This gives cones σ_0, σ_1 and σ_2 in $N_{\mathbb{R}}$ with generators $\{(1, 0), (0, 1)\}$, $\{(1, 0), (-1, -1)\}$ and $\{(0, 1), (-1, -1)\}$ respectively. The cones σ_i^{\vee} and their dual cones σ_i are depicted in Figure 3.

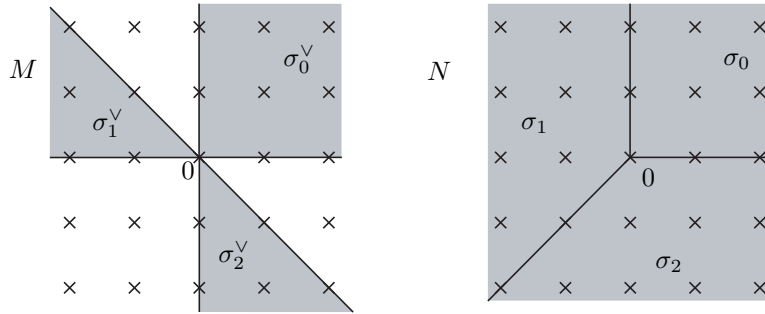


FIGURE 3. The cones of \mathbb{P}^2 in $M_{\mathbb{R}}$ and the dual cones in $N_{\mathbb{R}}$.

Let Δ denote the collection consisting of the two-dimensional cones σ_i ($i = 0, 1, 2$), the three one-dimensional cones generated by $(1, 0)$, $(0, 1)$ and $(-1, -1)$, and the zero-dimensional cone formed by the origin. The resulting cell complex Δ is called the *fan associated with \mathbb{P}^2* .

The structure of the fan Δ reflects the structure of \mathbb{P}^2 . We see three two-dimensional cones, which correspond to the two-dimensional subvarieties U_0, U_1 and U_2 . Each two-dimensional cone σ_i is glued to σ_j along a one-dimensional cone; these correspond to the subvarieties $U_i \cap U_j$, for $i \neq j$. Finally, all the cones are glued together in the zero-dimensional cone given by the origin. This zero-dimensional cone corresponds to the algebraic torus, and “is” the subvariety $U_0 \cap U_1 \cap U_2$.

¹A *cone* (or more precisely a *finitely generated rational polyhedral cone*) in $M_{\mathbb{R}}$ is a set of the form $\left\{ \sum_{i=1}^k \lambda_i u_i \in M_{\mathbb{R}} \mid \lambda_i \geq 0 \right\}$ for some finite collection of elements u_1, \dots, u_k in M .

²The dual to a cone σ is usually denoted by σ^{\vee} . Since it can easily be shown (eg. [Ful93, pg. 9]) that $(\sigma^{\vee})^{\vee} = \sigma$, the expression above conforms to this notation.

3. OUR SECOND EXAMPLE: \mathbb{F}_1 .

The procedure in Section 2 is reversible, and the methods involved generalise to any (suitably defined³) fan. Indeed, the driving force behind the study of toric varieties is the fact that fans and toric varieties are in one-to-one correspondence.

Before we continue, a definition of what it means for a variety to be a toric variety is long overdue.

Definition 3.1. A *toric variety* of dimension n over an algebraically closed field $k = \bar{k}$ is a normal variety X that contains a torus $T \cong (k^*)^n$ as a dense open subset, together with an action $T \times X \rightarrow X$ of T on X that extends the natural action of T on itself.

We now proceed with one final example (see [Ful93, pp. 7–8]), which will be very useful in the sequel. Consider the fan in $N_{\mathbb{R}} \cong \mathbb{R}^2$ shown in Figure 4. Taking the dual of each two-dimensional cone σ_i gives a cone σ_i^\vee in $M_{\mathbb{R}}$. The resulting collection of dual cones is also indicated in Figure 4.

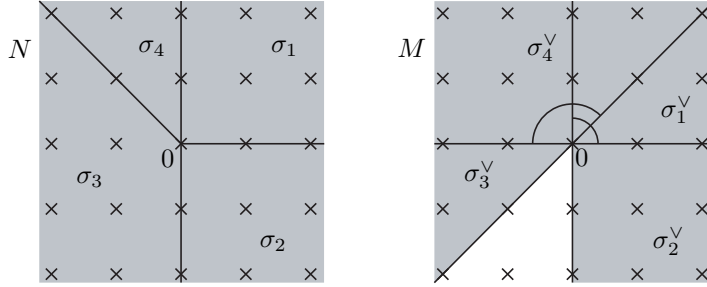


FIGURE 4. The fan in $N_{\mathbb{R}}$ of \mathbb{F}_1 and the dual to this fan in $M_{\mathbb{R}}$.

We obtain four affine varieties, each isomorphic to \mathbb{C}^2 :

$$\begin{aligned} U_1 &= \text{Spec}(\mathbb{C}[X, Y]), & U_2 &= \text{Spec}(\mathbb{C}[X, Y^{-1}]), \\ U_3 &= \text{Spec}(\mathbb{C}[X^{-1}, X^{-1}Y^{-1}]), & U_4 &= \text{Spec}(\mathbb{C}[X^{-1}, XY]). \end{aligned}$$

By considering the relations between the cones σ_i we see that the U_i patch together as follows:

$$\begin{array}{ccccc} U_4 & (x^{-1}, xy) & \longleftrightarrow & (x, y) & U_1 \\ & \updownarrow & & \updownarrow & \\ U_3 & (x^{-1}, x^{-1}y^{-1}) & \longleftrightarrow & (x, y^{-1}) & U_2 \end{array}$$

Observe that this projects to the patching $y \leftrightarrow y^{-1}$ of two copies of \mathbb{C} , which is \mathbb{P}^1 . Similarly for $xy \leftrightarrow x^{-1}y^{-1}$. Hence we see that U_1 and U_2 patch to give $\mathbb{C} \times \mathbb{P}^1$, and also

³A *fan* Δ is a collection of strictly convex cones in $N_{\mathbb{R}}$ (i.e. $\sigma \cap (-\sigma) = \emptyset$) such that each face of a cone in Δ is a cone in Δ , and the intersection of any two cones in Δ is a face of each.

that U_3 and U_4 patch to give $\mathbb{C} \times \mathbb{P}^1$. We recognise the resulting variety to be the rational normal scroll \mathbb{F}_1 .

It is worth noting that the existence of a morphism to \mathbb{P}^1 can be deduced directly from the fan – see Figure 5.

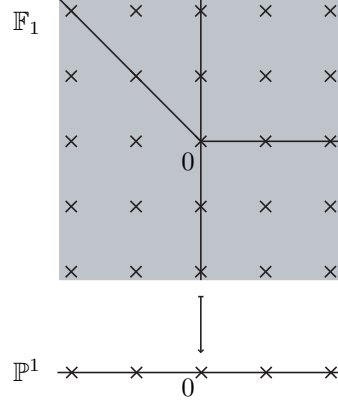


FIGURE 5. A toric morphism $\mathbb{F}_1 \rightarrow \mathbb{P}^1$.

4. THE PROJECTIVITY CRITERION AND SMOOTH TORIC FANO VARIETIES.

We have seen the fans associated to three toric varieties: \mathbb{P}^1 , \mathbb{P}^2 , and \mathbb{F}_1 . In each case it transpires that the fan can be realised by a polytope. Compare the three polytopes depicted in Figure 6 with the corresponding fans. The fan can be obtained from the polytope by taking the collection of cones spanned by the polytope's faces. This is not a coincidence (c.f. [Ewa96, §VII, Theorem 3.11]).

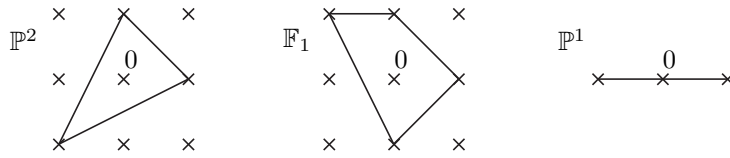


FIGURE 6. Three polytopes defining the fans of three toric varieties.

Theorem 4.1 (The Projectivity Criterion). *A fan $\Delta \subset N_{\mathbb{R}}$ is the set of cones spanned by the faces of a convex polytope P_{Δ} with vertices in N and the origin in its interior if and only if the variety X_{Δ} associated with Δ is projective.*

When X_{Δ} is projective, the corresponding polytope P_{Δ} can give us crucial information about X_{Δ} . In particular it can tell us about the singularities of X_{Δ} .

Definition 4.2. A smooth projective variety with anticanonical divisor $-K$ ample⁴ is called a *smooth Fano variety*.⁵

Theorem 4.3. Let P_Δ be a polytope as described above. Each facet⁶ of P_Δ is the convex hull of a \mathbb{Z} -basis of N if and only if X_Δ is a smooth toric Fano variety.

For brevity we shall call a polytope satisfying the conditions of Theorem 4.3 a *smooth Fano polytope* (c.f. [Ewa96, §V.8 and §VII.8]). Each of the polytopes depicted in Figure 6 is a smooth Fano polytope.

Amongst the huge amount of information which can be gleaned from a smooth Fano polytope P_Δ is the following relationship between the number of vertices of P_Δ and the rank of the Picard group of the variety X_Δ (c.f. [Ewa96, §VII, Theorem 2.16]):

Theorem 4.4. Let X_Δ be a smooth toric Fano variety of dimension n with associated polytope P_Δ . Then:

$$\mathrm{rk} \mathrm{Pic} X_\Delta = |\mathrm{Vert} P_\Delta| - n.$$

5. A CLASSIFICATION.

By classifying all smooth Fano polytopes of dimension n (up to the action of $GL_n(\mathbb{Z})$) one obtains a classification of all smooth toric Fano n -folds (up to isomorphism), due to the result of Theorem 4.3. For small values of n such classifications exists.

In dimension one (i.e. $M_{\mathbb{R}} \cong \mathbb{R}$) it is trivially true that there exists only one smooth Fano polytope – that corresponding to \mathbb{P}^1 . Observe that this polytope has two vertices.

For dimension two the classification is well-known; there are five polytopes (shown in Figure 7) which corresponds to the five toric del Pezzo surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$, \mathbb{P}^2 , the rational normal scroll \mathbb{F}_1 , and \mathbb{P}^2 blown-up at two and at three points. This classification can be obtained via combinatorial methods by considering the polytopes, as in [Ewa96, §V, Theorem 8.2], or by direct consideration of the varieties (c.f. [Wiś02, §5]). Observe that the greatest number of vertices occurring is six.

A complete classification in dimension three was obtained by Batyrev (c.f. [Bat81]) and, independently, by Watanabe and Watanabe (c.f. [WW82]); there are 18 polytopes (see Figure 8), amongst which the maximum number of vertices is eight. A classification in dimension four has proved more difficult, however it appears that there are 124 polytopes (see [Sat00]), with the maximum number of vertices being twelve.

⁴A divisor D is said to be ample if some multiple nD is very ample.

⁵Be warned: In some of the literature, *Fano* is synonymous with *smooth Fano*. One should always be clear concerning the singularities a Fano variety is permitted; this is best indicated with the appropriate adverb.

⁶A *facet* of an n -dimensional polytope is an $(n - 1)$ -dimensional face.

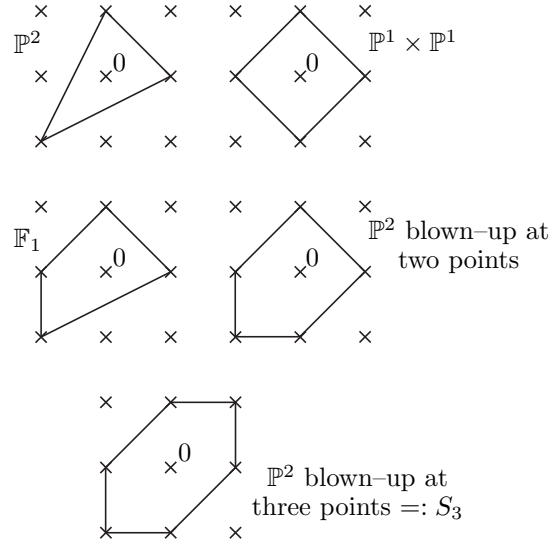


FIGURE 7. The five two-dimensional smooth Fano polytopes.

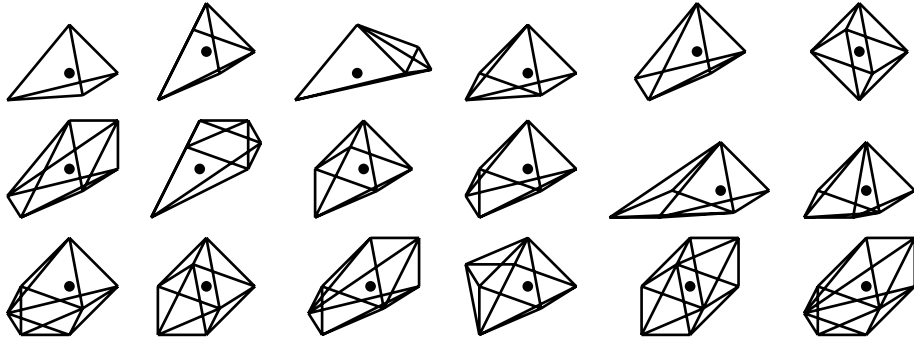


FIGURE 8. The 18 three-dimensional smooth Fano polytopes.

Very little is known in higher dimensions; however one conjecture emerged early on. The maximum number of vertices of a smooth Fano polytope in dimension n is given by:

$$\begin{cases} 3n, & \text{if } n \text{ is even} \\ 3n - 1, & \text{if } n \text{ is odd.} \end{cases}$$

It is easily demonstrated that this conjectured bound is obtained. Let $P := P_{S_3}$ be the polytope in Figure 7 with six vertices. In dimension $n = 2k$ the polytope P^k (i.e. the convex hull of k copies of P , each copy lying in one of the k two-dimensional subspaces of our lattice) is easily seen to be a smooth Fano polytope. It possesses $6k = 3n$ vertices. If our dimension is odd (i.e. $n = 2k + 1$) then we consider the convex hull of P^k and $P_{\mathbb{P}^1}$ (where the polytope associate with \mathbb{P}^1 lies in the remaining one-dimensional subspace of our lattice not spanned by P^k). Again, this polytopes is easily seen to be a smooth Fano polytope, and is has $6k + 2 = 3n - 1$ vertices.

For dimensions less than five, this conjecture holds by the classifications mentioned above. For dimension five the result was proved in [Cas03, Theorem 3.2]. After considerable, and unsuccessful, attention by many people (see [Deb03, pp. 102–105] for more background), this result was finally proved at the end of last year by Casagrande (see [Cas04]). In fact Casagrande’s result holds for a much larger class of Fano polytopes, called *reflexive* Fano polytopes.

Translating back to the language of geometry (via Theorem 4.4), the following holds:

Theorem 5.1. *For any smooth toric Fano variety X of dimension n ,*

$$\mathrm{rk} \mathrm{Pic} X \leq \begin{cases} 2n, & \text{if } n \text{ is even} \\ 2n - 1, & \text{if } n \text{ is odd.} \end{cases}$$

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