KAWAMATA BOUNDS FOR FANO THREEFOLDS AND THE GRADED RING DATABASE

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Abstract. We give an effective realisation of Kawamata's famous bound for Fano 3-folds. Our result is a list of 39,550 candidate Hilbert series P for Fano 3-folds satisfying the Bogomolov stability inequality. For each P, there may or may not be a Fano 3-fold X that realises P, but even so we give a set of weights that we expect to be among those of a minimal generating set of the anticanonical ring of any such X. This provides a crude first approximation to the ongoing classification of Fano 3-folds. We explain in detail what this data means, the important nuances that the classification already reveals about it to date, and how it relates to a range of different classification programmes.

1. Introduction

The *Graded Ring Database*, or GRDB, is a collection of data relating to polarised varieties and their graded rings. The motivation for such a database is founded on various celebrated finiteness results for Fano varieties [Kaw92, KMMT00, Bir21]. Fano varieties are among the key elementary particles of geometry, and here we are interested specifically in Fano 3-folds over $\mathbb{C}$. Nonsingular Fano 3-folds are classified into 105 families [Isk78, MM82], and are recorded with additional data both in print [IP99] and online [Bel19]. The nonsingular Fanos form a kind of aristocracy, making them a popular topic for certain authors. However, the broad toiling masses of Fano 3-folds with terminal singularities are not to be neglected, and their study offers rich rewards. Several hundred families of them are already known (see §4), with the reasonable expectation of thousands or tens of thousands more yet to be discovered – and orders of magnitude more if we relax the conditions on singularities further. Ultimately, if ever completed, the classification will be a computer database rather than a physical atlas. The GRDB is a first approximation to that database. In the meantime it provides a roadmap for exploring the landscape, and for identifying candidates for construction or elimination.

Two main considerations govern our view of Fano 3-folds (or $\mathbb{Q}$-Fano 3-folds) and the precise terminology we use here. Prime Fano 3-folds (see 1.3) form the strictest class. They have a distinguished place in the birational classification of 3-folds: each one is the endpoint of an MMP in the Mori category [Mor88, (0.3.1)]. On the other hand, the main numerical invariant of a Fano variety that we use is its anticanonical Hilbert series (§2.1) and the ingredients that go into it (§2.2). This is the basic object of the GRDB and it works for a much broader definition allowing canonical singularities (see 1.1). For varieties with worse than canonical singularities this viewpoint breaks down, even for mild log terminal orbifold points. Throughout this paper, we say *3-fold* to mean a normal variety of dimension 3 over $\mathbb{C}$.

Definition 1.1. A *Fano 3-fold with canonical singularities* is a projective 3-fold X with at worst canonical singularities, for which the anticanonical class $-K_X$ is ample. We usually abbreviate 'Fano 3-fold with canonical singularities' to 'Fano 3-fold'. By definition, the *genus* of X is $g_X := h^0(X, -K_X) - 2$.

In [KMMT00] this class is called *canonical* $\mathbb{Q}$*-Fano 3-folds* and it is shown to be bounded (ibid. 1.3). The point of the GRDB is to realise this bound explicitly under stated conditions. Our main result is a list $\mathcal{F}_3$ of rational functions (defined in 2.9) that satisfy necessary conditions to be the Hilbert series of a Fano 3-fold (polarised by $-K_X$ as in §2.2), subject to an additional numerical condition.

Theorem 1.2. *Let X be a Fano 3-fold satisfying $-K_X^3 \leq -3K_X c_2$. Then its Hilbert series $P_X(t)$ is one of the 39,550 rational functions in the list $\mathcal{F}_3$.*

The condition $-K_X^3 \leq -3K_X c_2$ is natural, as we discuss further below.. Our use of it is motivated by the proof of Kawamata's boundedness theorem [Kaw92], one of the earliest for 3-folds, and one which seems well suited to explicit classification in a range of contexts.

Definition 1.3 (Prime Fano 3-fold)**.** The *Mori category* consists of 3-folds X with $\mathbb{Q}$-factorial terminal singularities. We say a 3-fold X in the Mori category is a *prime Fano 3-fold* if $-K_X$ is ample, and the Weil divisor class group $\operatorname{Cl} X$ has rank $\rho_X = 1$.

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Here we intend $\mathbb{Q}$-factorial in the algebraic sense: the local ring $\mathcal{O}_{X,P}$ at every $P \in X$ is $\mathbb{Q}$-factorial, or equivalently, any algebraic Weil divisor on X has a multiple that is Cartier. The complex analytic variety X^{an} may not be $\mathbb{Q}$-factorial in the local or etale analytic setting: as an example, a cubic or quartic 3-fold hypersurface $X \subset \mathbb{P}^4$ with a single ordinary node is $\mathbb{Q}$-factorial, although the node is (analytic, etale or formal)-locally isomorphic to $xy = zt$.

On the other hand, we do not insist that $-K_X$ generates the class group $\mathrm{Cl}\,X$: we allow $-K \in \mathrm{Cl}\,X$ to be divisible (see 2.1 below for a discussion of index), and $\mathrm{Cl}\,X$ may have a finite torsion group.

Prime Fano 3-folds remain unclassified. The proof of [Kaw92, Prop 1] determines inequalities on prime Fano 3-folds of the form $-K_X^3 \le -\kappa K_X c_2(X)$ for certain values of $\kappa > 0$. These in turn impose numerical constraints on the Hilbert series P_X of X. A Fano 3-fold X is Bogomolov *semistable* if the sheaf $\mathcal{E} = (\Omega^1_X)^{\vee\vee}$ is semistable. Kawamata shows that for prime semistable X, the inequality above holds with $\kappa = 3$ as in 1.2; it follows at once from the Bogomolov–Miyaoka–Yau inequality $c_1^2 \le 3c_2$ applied to $\mathcal{E}$ restricted to suitable $S \in |-RK_X|$ for large divisible R.

More recently Liu and Liu [LL25b] prove that the strict inequality $-K_X^3 < -3K_Xc_2(X)$ holds for any prime Fano 3-fold with terminal singularities (cf. [Suz24]), and so 1.2 applies in this case. Only 208 of the 39,550 fail this strict inequality.

Corollary 1.4. *If X is a prime Fano 3-fold, then $P_X \in \mathcal{F}_3$.*

In fact, $\mathcal{F}_3$ contains almost all Fano 3-folds we know. Most series in $\mathcal{F}_3$ are not the Hilbert series of a prime Fano 3-fold: the main issue is to begin to understand which ones are, which ones are are realised by more general classes of Fano 3-fold, and which ones are not realised by Fano 3-folds at all.

Relaxing $\kappa = 3$ to $\kappa = 4$ accommodates most of Kawamata's conditions even in the non-semistable case (see 2.7). Imposing only that weaker condition gives a larger list $\mathcal{F}_4$ of 52,646 rational functions that contains $\mathcal{F}_3$ (see 2.10). Liu–Liu [LL25a] also prove that if X is a Fano 3-fold with $\rho_X = 1$ which is nonsingular in codimension 2, then it satisfies the strict inequality $-K_X^3 < -4K_Xc_2$. Only 87 of the 52,646 fail this strict inequality.

Corollary 1.5. *If X is a Fano 3-fold with isolated canonical singularities and $\rho_X = 1$, then $P_X \in \mathcal{F}_4$.*

Definition 1.6 (cf. 2.9)**.** The *Fano 3-fold database* is the pair of lists $\mathcal{F}_3 \subset \mathcal{F}_4$. We say that a Fano 3-fold X is in $\mathcal{F}_3$ when $P_X \in \mathcal{F}_3$. The Fano 3-fold database may be accessed online [BK09] and the raw data is available at [BK22] under a CC0 licence [CC0], with code to generate it at [BK09].

The GRDB also includes the data of all toric Fano 3-folds, so it is easy to derive the following.

Proposition 1.7 (§4.5 and Figure 6)**.** *The 674,688 toric Fano 3-folds realise 5772 different Hilbert series. If X is a toric Fano 3-fold, then $P_X \in \mathcal{F}_4$, and 5760 of them line in $\mathcal{F}_3$.*

We emphasise that some unknown, and possibly significant, fraction of the elements of $\mathcal{F}_3$ cannot be realised as the Hilbert series of a Fano 3-fold; see §5.3 for a recapitulation of this point and §5.1 for several other possible confusions. Moreover, we do not know a single example of a prime Fano 3-fold not in $\mathcal{F}_3$, though we do know a small number of examples with mild canonical singularities in $\mathcal{F}_4 \setminus \mathcal{F}_3$, such as the weighted projective space $\mathbb{P}(1, 1, 3, 5)$.

We explain the proof of 1.2 in §2.4. This may be thought of as folklore, and similar in spirit to Prokhorov's degree bound computation [Pro07, (1.2)]. Nevertheless the steps of turning Kawamata's theorem into a concrete list are important in light of the bounds on Fano varieties more generally. The main results of this paper supplement 1.2 as follows:

(1) For each $P \in \mathcal{F}_3$, we estimate the smallest anticanonical embedding that a Fano 3-fold X with Hilbert series $P_X = P$ could have (§3), and we use those to present $\mathcal{F}_3$ as a geographical map (Figure 1). This geography, and its meaning as the basis for a programme of classification, is the main result of this paper.
(2) We identify some known classifications within the Fano 3-fold database (§4) and compare with some well-known results (§3.3).
(3) We highlight possible misunderstandings of the Fano 3-fold database (§5.1, §5.3).
(4) The numbers of cases $\#\mathcal{F}_3 = 39{,}550$ and $\#\mathcal{F}_4 = 52{,}646$ arise from elementary combinatorial considerations (2.10).

In outline, §2 constructs the lists $\mathcal{F}_3$ and $\mathcal{F}_4$, §3 explains the estimates of codimension that lead to Figure 1, §4 describes a number of constructions and comparisons with known lists of Fano 3-folds, and finally §5 summarises the GRDB together with important warnings about its meaning.

Codimension \ Genus	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
1		54	32	6	2	1																																		
2	1	45	29	6	2	1	1																																	
3		26	29	8	3	2	1	1																																
4	3	60	54	15	6	3	2	1	1																															
5	1	58	63	21	8	6	3	2	1	1																														
6	4	80	98	35	15	8	6	3	2	1	1																													
7		81	116	49	21	15	8	6	3	2	1	1																												
8	6	128	182	79	35	21	15	8	6	3	2	1	1																											
9	3	107	208	109	49	35	21	15	8	6	3	2	1	1																										
10	10	194	314	171	79	49	35	21	15	8	6	3	2	1	1																									
11	5	168	378	236	109	79	49	35	21	15	8	6	3	2	1	1																								
12	7	244	538	353	171	109	79	49	35	21	15	8	6	3	2	1	1																							
13	8	254	638	485	236	171	109	79	49	35	21	15	8	6	3	2	1	1																						
14	11	377	879	698	353	236	171	109	79	49	35	21	15	8	6	3	2	1	1																					
15	10	349	889	870	480	351	236	171	109	79	49	35	21	15	8	6	3	2	1	1																				
16	15	509	802	998	675	473	349	235	171	109	79	49	35	21	15	8	6	3	2	1	1																			
17	21	501	480	740	786	637	456	342	233	171	109	79	49	35	21	15	8	6	3	2	1	1																		
18	17	604	284	318	773	658	571	427	331	228	169	109	79	49	35	21	15	8	6	3	2	1	1																	
19	23	465	103	43	401	516	488	474	380	308	221	167	108	79	49	35	21	15	8	6	3	2	1	1																
20	31	220	44	4	106	159	267	299	345	308	272	204	160	106	79	49	35	21	15	8	6	3	2	1	1															
21	32	3	5		8	24	42	101	137	202	213	213	176	149	101	77	49	35	21	15	8	6	3	2	1	1														
22	27		1		1	1	3	7	28	43	88	117	141	135	128	94	75	48	35	21	15	8	6	3	2	1	1													
23	13							1	1	5	7	25	43	71	84	98	78	68	46	35	21	15	8	6	3	2	1	1												
24	10										1	1	5	10	23	36	59	54	57	41	33	21	15	8	6	3	2	1	1											
25	4													1	1	4	10	25	27	39	35	31	20	15	8	6	3	2	1	1										
26	2																1	1	7	9	20	23	24	18	15	8	6	3	2	1	1									
27																				1	1	6	10	15	13	13	8	6	3	2	1	1								
28																							1	2	4	8	11	7	6	3	2	1	1							
29																										1	2	6	5	6	3	2	1	1						
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FIGURE 1. Number of semistable Fano 3-fold Hilbert series in $\mathcal{F}_3$, listed by genus and estimated minimal codimension.

We put some emphasis on toric Fano 3-folds throughout, in part to profit from simultaneous use of the Fano 3-fold database and the toric Fano classification in the GRDB. We indicate in §4.5 some different ways that this interdisciplinary approach may yet be exploited.

The systematic study of the graded rings of all Fano 3-folds together in this way was initiated by Miles Reid; see [ABR02], which alongside [IF00, Alt98] was the starting point for the GRDB. The GRDB has been available since then in different forms [BK09, BK22], though without the documentation and detailed analysis this paper provides. The need for this analysis has become more important recently, as a series of developments promise to expose large new parts of the classification:

- The use of formats by Takagi [Tak21a, Tak21b, Tak23, Tak24a, Tak24b] and Coughlan–Ducat [CD20] extends the cases of [BKQ18, BKZ19] (§4.6) and opens up mid-range codimension to similar analysis;
- New Kawamata–Miyaoka inequalities [LL25a] provide sharper data in broader contexts (compare Corollaries 1.4 and 1.5) and also higher dimension.
- New variations of Papadakis and Reid's Tom and Jerry unprojection techniques, originally exploited in [BKR12], provide constructions for many more families [Pet22b, BCD25, BCD26, Tay];
- The Fanosearch programme [CCG$^+$13, BCC+22] (§4.8) attacks classification via Mirror Symmetry, exploiting certain types of degeneration; thus as a first approximation the raw GRDB list is supplemented by matching toric Fano 3-folds, which are classified (§4.5).

This paper makes the GRDB precise: it provides the theorems that underpin the data and its construction, it relates the data to a range of classical results, and by locating a class of toric varieties within the data it provides a first step in identifying certain degenerations. There is a quick startup guide in §5, which also summarises the essential warnings for its use.

The beauty and enduring interest of Fano classification lies in the individual 3-folds and deformation families we meet, rather than the bureaucracy of compartmentalising them. A map is only a map: the actual adventures happen out in the field, and the real value of the geography in Figure 1 is to identify hundreds of wonderful places to explore: compare [BKR12, CHNP13, Sü14, DNFF17, CD20, Heu25, Tak24a].

2. Building the Fano 3-fold database

We recall standard material related to the plurigenus formula and use it to assemble the data that proves 1.2 (cf. [ABR02, §4]).

2.1. Fano 3-folds. Any Fano 3-fold X comes with an intrinsic embedding $X \subset w\mathbb{P}$ in weighted projective space (up to automorphisms of $w\mathbb{P}$) as follows. The *graded ring* of X is

$$R(X, -K_X) = \bigoplus_{m\in\mathbb{N}} H^0(X, -mK_X)$$

and $X \cong \operatorname{Proj} R(X, -K_X)$ by ampleness. Any choice $f_0, \ldots, f_n$ of minimal homogeneous generators for $R(X, -K_X)$ has the same collection of weights $\{a_0, \ldots, a_n\}$, where $a_i = \deg f_i$, and we may suppose $a_0 \leq \cdots \leq a_n$. (Note the conventional abuse of notation: $\{a_0, \ldots, a_n\}$ is a list with possible repetitions, and similarly $\{a_0, \ldots, a_n\} \setminus \{a_3, a_7\}$ is the list obtained by removing one instance of each of a_3 and a_7.) Thus any choice of minimal generating set determines a projectively normal embedding

$$X \subset \mathbb{P}(a_0, \ldots, a_n)$$

which we refer to as the *anticanonical embedding of* X. (This is not the same as the image $\Phi_{-K_X}(X)$ under the linear system $|-K_X|$ unless $-K_X$ is very ample, which is rarely the case here.)

The *Hilbert series* P_X of a Fano 3-fold X is simply that of $R(X, -K_X)$, the formal power series

$$P_X = P_X(t) = \sum_{m\in\mathbb{N}} h^0(X, -mK_X)t^m.$$

A simple but essential theme throughout this paper is that the Hilbert series P_X of a Fano 3-fold X does not determine the weights a_i of its anticanonical embedding.

2.2. **Numerical data of a Fano 3-fold.** We write $\frac{1}{r}(a_1, a_2, a_3)$ to denote the singularity germ of the origin in $\mathbb{C}^3/\boldsymbol{\mu}_r$, where $\boldsymbol{\mu}_r$ is the multiplicative group of rth roots of unity, and $\varepsilon \in \boldsymbol{\mu}_r$ acts on coordinates by $(\varepsilon^{a_1}x_1, \varepsilon^{a_2}x_2, \varepsilon^{a_3}x_3)$. Any Fano 3-fold has a *basket of singularities* $\mathcal{B}$, which is a collection of terminal quotient singularities $\frac{1}{r}(1, a, -a)$ (possibly including repeats, where again we use set notation on this understanding). Following [Rei87, §§8–10], in general $\mathcal{B}$ is derived locally from the singularities of X by crepant blowup and Q-smoothing. When X has terminal singularities and lies in weighted projective space as a quasismooth variety, then X is an orbifold and $\mathcal{B}$ is exactly the collection of singularities of X.

The *genus* $g = h^0(X, -K_X) - 2 \geq -2$ and basket $\mathcal{B}$ together determine the degree by the formula

$$-K_X^3 = 2g - 2 + \sum_{\mathcal{B}} \frac{b(r-b)}{r} \tag{2.A}$$

where for each element $\frac{1}{r}(1, a, -a)$ of the basket $\mathcal{B}$, we define $0 < b < r$ by $ab \equiv 1 \mod r$.

Although they are not used for the initial construction of the GRDB, there are various different divisorial indices defined for a Fano 3-fold X that we consider later.

Definition 2.1. The *Gorenstein index* or *singularity index* of X is

$$i_X = \min\{n \in \mathbb{Z}_{>0} \mid nK_X \text{ is Cartier}\}.$$

The *Fano index* is $f_X = r/i_X \in \mathbb{Q}$ where $r > 0$ is the largest integer for which $-i_XK_X \equiv rA$ for a Cartier divisor A. There are further types of Fano index:

$$q_X = \max\{q > 0 \mid -K_X \sim qA, \text{ where } A \text{ is a } \mathbb{Q}\text{-Cartier Weil divisor}\}$$
$$q_{\mathbb{Q}X} = \max\{q > 0 \mid -K_X \sim_{\mathbb{Q}} qA, \text{ where } A \text{ is a } \mathbb{Q}\text{-Cartier Weil divisor}\}.$$

These are natural generalisations of the divisibility of the anticanonical divisor of a nonsingular Fano 3-fold in its Picard group. For a Fano 3-fold, both q and $q_{\mathbb{Q}}$ are positive integers, and they are equal if $\mathrm{Cl}(X)$ has no torsion [Pro10].

2.3. **Effective Kawamata bounds for prime Fano 3-folds.** We construct the set of genus–basket pairs $(g, \mathcal{B})$ that satisfy the constraints of [Kaw92].

2.3.1. *Possible baskets.* Controlling the possible baskets of Fano 3-folds can be done much more generally than the prime Fano case. Recall that a basket $\mathcal{B}$ is a set with possible repeats.

Definition 2.2. We define a set of baskets $\mathbb{B} = \left\{\mathcal{B} \mid \sum_{\mathcal{B}}(r - \frac{1}{r}) < 24\right\}$ where the sum is taken over $\frac{1}{r}(1, a, -a) \in \mathcal{B}$. It is easy to check that $\#\mathbb{B} = 8314$.

The calculation of 8314 is elementary. For any $\mathcal{B} \in \mathbb{B}$, the defining bound of 24 implies first that $r \leq 24$ for each $\frac{1}{r}(1, a, -a) \in \mathcal{B}$, and moreover that $\#\mathcal{B} \leq 15$, since each $r - (1/r) \geq 3/2$. Therefore there are only finitely many possible collections of indices r appearing in $\mathcal{B}$. The only terminal quotient singularity with $r = 2$ is $\frac{1}{2}(1, 1, 1)$, and for each index $r > 2$, there are $\phi(r)/2$ distinct terminal quotient singularities, namely $\frac{1}{r}(1, a, -a)$ for $1 \leq a < r/2$ coprime to r. Thus for each index r that appears in $\mathcal{B}$, there are only finitely many singularities of index r that may occur. The bound 8314 comes by enumerating all possible baskets satisfying these conditions (by computer, for example).

Theorem 2.3 ([Kaw86, Lemma 2.2, 2.3], [Rei87, (10.3)])**.** *Let X be a projective 3-fold with canonical singularities. Then*

$$24\chi(\mathcal{O}_X) = -K_Xc_2(X) + \sum_{\mathcal{B}} \left(r - \frac{1}{r}\right) \tag{2.B}$$

where the sum is over the singularities $\frac{1}{r}(1, a, -a)$ of the basket $\mathcal{B}$ of X.

When X is a Fano 3-fold, $\chi(\mathcal{O}_X) = 1$ and (2.B) simplifies.

Lemma 2.4. *If X is a Fano 3-fold with canonical singularities which satisfies $-K_Xc_2(X) > 0$, then the basket $\mathcal{B}$ of X is in $\mathbb{B}$.*

Proof. The condition $-K_Xc_2(X) > 0$ together with (2.B) implies that $\sum r - (1/r) < 24$, where the sum is taken over the basket $\mathcal{B}$. □

Remark 2.5. The condition $-K_Xc_2(X) > 0$ holds for prime Fano 3-folds by 2.7. Far more generally, any weak Fano 3-fold (that is, $-K_X$ is only required to be nef and big) with terminal singularities satisfies $-K_Xc_2(X) \geq 0$ by [KMMT00, 1.2(1)]. It is easy to check that relaxing the inequality provides (coincidentally) 24 additional cases with $\sum(r - \frac{1}{r}) = 24$, such as $16 \times \frac{1}{2}(1, 1, 1)$ and $5 \times \frac{1}{5}(1, 2, 3)$.

2.3.2. *Genus bounds.* Controlling the possible values for the genus g_X for each basket, uses the bounds deteremined by [Kaw92].

Proposition 2.6. *Let X be a Fano 3-fold with basket $\mathcal{B}$ and genus g that satisfies*

$$-K_X^3 \le \kappa(-K_X c_2(X)) \tag{2.C}$$

for some $\kappa > 0$. Then $g_{\min} \le g \le g_{\max}$ where

$$g_{\min} = \max\left\{-2, \left\lfloor \tfrac{1}{2}\left(2 - \sum_{\mathcal{B}} \tfrac{b(r-b)}{r}\right)\right\rfloor + 1\right\}$$

and

$$g_{\max} = \left\lfloor \tfrac{1}{2}\left(2 - \sum_{\mathcal{B}} \tfrac{b(r-b)}{r} + \kappa\left(24 - \sum_{\mathcal{B}}\left(r - \tfrac{1}{r}\right)\right)\right)\right\rfloor \tag{2.D}$$

where each sum is over $\frac{1}{r}(1,a,-a) \in \mathcal{B}$ and b is defined by $ab \equiv 1 \mod r$, and $\kappa > 0$ in (2.D) is as determined in 2.7.

Proof. The lower bound is simply the condition that $-K_X^3 > 0$ in (2.A). For the upper bound, substituting (2.A) and (2.B) into (2.C) gives

$$2g - 2 + \sum_{\mathcal{B}} \tfrac{b(r-b)}{r} \le \kappa\left(24 - \sum_{\mathcal{B}}\left(r - \tfrac{1}{r}\right)\right)$$

and the upper bound follows. $\square$

Theorem 2.7 (Kawamata [Kaw92])**.** *Let X be a prime Fano 3-fold. Then*

$$-K_X^3 \le \kappa(-K_X c_2(X))$$

for some real number $\kappa > 0$. In particular, $-K_X c_2(X) > 0$.

If X is semistable, then the formula (2.D) holds with $\kappa = 3$. If $(\Omega_X^1)^{\vee\vee}$ has a rank 2 maximal destabilising subsheaf, then the formula holds with $\kappa = 4$. If $(\Omega_X^1)^{\vee\vee}$ has a rank 1 maximal destabilising subsheaf, then it holds with possibly larger $\kappa > 0$.

2.4. **Proof of 1.2.** The Hilbert series P_X is equivalent to the data of genus–basket pair $(g, \mathcal{B})$ by the following Fletcher–Reid plurigenus formula together with (2.A) and the independence of basket contributions [Fle89, 4.2]. We could instead use the integral orbifold Riemann–Roch formulas of [BRZ13], and for human calculation they are much preferred.

Theorem 2.8 ([Fle87, Theorem 2.5], [Rei87, (10.3)])**.** *Let X be a Fano 3-fold with canonical singularities and basket $\mathcal{B}$. Then*

$$P_X(t) = \frac{1+t}{(1-t)^2} - \frac{t(1+t)}{(1-t)^4}\,\frac{K_X^3}{2} - \sum_{\mathcal{B}}\left(\frac{1}{(1-t)(1-t^r)}\sum_{i=1}^{r-1}\frac{\overline{bi}(r-\overline{bi})t^i}{2r}\right) \tag{2.E}$$

where for each element $\frac{1}{r}(1,a,-a)$ of the basket $\mathcal{B}$, we fix any integer b that satisfies $ab \equiv 1 \mod r$, and set $\overline{bi} \in \{0,1,\dots,r-1\}$ to be the least residue of (bi) modulo r.

Definition 2.9. For any $g \ge -2$ and $\mathcal{B} \in \mathbb{B}$, the right-hand side of (2.E) determines a rational function that we denote $P_{g,\mathcal{B}}$, where (2.A) expresses the $-K_X^3$ factor in terms of $g, \mathcal{B}$. We may expand $P_{g,\mathcal{B}} = \sum P_n t^n$ as a power series, and conversely given such a power series we may recover $g, \mathcal{B}$ and the expression (2.E) as a rational function. For any $\kappa > 0$, define

$$\mathcal{F}_\kappa = \{(g, \mathcal{B}, P_{g,\mathcal{B}}) \mid \mathcal{B} \in \mathbb{B} \text{ and } g_{\min} \le g \le g_{\max}, \text{ and if } P_1 > 0 \text{ then } P_n > 0 \text{ for all } n \ge 1\}$$

where $g_{\min}$ and $g_{\max}$ are defined for $\mathcal{B}$ by (2.D) with the given κ. We treat $\mathcal{F}_\kappa$ as a set of genus–basket pairs, or equivalently as a set of rational functions they determine, and we write $(g, \mathcal{B}) \in \mathcal{F}_\kappa$ or $P_{g,\mathcal{B}} \in \mathcal{F}_\kappa$ accordingly.

In the special case $\kappa = 3$, we write $\mathcal{F}_3 = \mathcal{F}_3$.

It follows from everything so far that if X is a Fano 3-fold with genus g and basket $\mathcal{B}$ that satisfies (2.C) for $\kappa > 0$, then $P_X = P_{g,\mathcal{B}} \in \mathcal{F}_\kappa$. The condition on P_n holds for any such P_X: there cannot be a nonzero section $x \in H^0(X, -K_X)$ with $x^n = 0$ for some $n > 1$.

Proposition 2.10. $\#\mathcal{F}_3 = 39{,}550$ *and* $\#\mathcal{F}_4 = 52{,}646$.

Proof. Lemma 2.4 provides exactly 8314 possible baskets. Of these, (2.D) calculates $g_{\max} \geq -2$ in 7492 cases for $\kappa = 3$ and in 7683 cases for $\kappa = 4$. Assembling pairs $(g, \mathcal{B})$ with $\mathcal{B}$ one of these baskets and $g_{\min} \leq g \leq g_{\max}$ bounded by 2.6 gives 39,558 cases for $\kappa = 3$ and 52,654 cases for $\kappa = 4$.

For each genus–basket pair $(g, \mathcal{B})$, expand the rational function $P_{g,\mathcal{B}}$ as a power series. In eight cases, this power series starts

$$1 + t + t^2 + \cdots + t^n + O(t^{n+2}),$$

with either $n = 2$ or $n = 4$, where the t^{n+1} term has coefficient zero. We discard these eight series at this stage, to leave 39,550 series in $\mathcal{F}_3$ and 52,646 series in $\mathcal{F}_4$. □

Computer code to enumerate the Hilbert series of 2.10, either in the Go-language [Goo17] or independently for the Magma system [BCP97], is available at [BK09].

3. The geography of Fano 3-folds

For each rational function $P \in \mathcal{F}_4$, the GRDB gives a collection of weights $a_0, \ldots, a_n$ for which the product

$$P \cdot \prod_{i=0}^{n} (1 - t^{a_i}) = 1 - \sum_{d_i} t^{d_i} + \sum_{e_i} t^{e_i} - \cdots \pm t^k. \tag{3.A}$$

is a polynomial. This polynomial is called the *Hilbert numerator of* P_X, and clearly it depends on the choice of weights a_i. The point is that if there really is a Fano 3-fold X embedded as

$$X \subset \mathbb{P}^n(a_0, \ldots, a_n) \tag{3.B}$$

then the expression (3.A) is related to the equation degrees d_i, syzygy degrees e_i and adjunction number k of the defining equations; see [Rei02, 3.6]. To be suggestive and provocative (and usually helpful), GRDB presents each genus–basket pair $(g, \mathcal{B})$ in the form (3.B), for weights chosen to suit the corresponding series $P_{g,\mathcal{B}}$. One must be aware that there could be many different apparently 'good' choices of weights, and it is important to understand how the GRDB weights are chosen, since, as we explain in §5.1, there are many traps to fall into when interpreting them.

The process to assign weights to Hilbert series P in the GRDB follows 4 steps: identify the known classification of weighted complete intersections §3.1.1; apply simple numerics to identify the weights of necessary generators for any graded ring with given P §3.1; additional generators may be needed to polarise the singularities of $\mathcal{B}$ §3.1.3; finally an inductive Ansatz based on projection gives the weights in high codimension §3.2. Which steps are used most varies with genus, as we explain below, but in any case the weights provided by any step are consistent with those from all other steps.

3.1. Hilbert series in low codimension.

An embedding $X \subset \mathbb{P}(a_0, \ldots, a_n)$ is *nondegenerate* if there are no quasi-linear equations in the defining ideal of X. Anticanonical embeddings of Fano 3-folds are nondegenerate, as they are determined by the choice of a minimal generating set.

3.1.1. *The famous* 95 *and the result of Chen–Chen–Chen.* The famous 95 weighted hypersurfaces of Reid [Rei80, (4.5)], Johnson–Kollàr [JK01, BK16], and others realise the codimension 1 (top) row of Figure 1. This classification of hypersurfaces is well established: if a prime Fano 3-fold is anticanonically embedded as a hypersurface in weighted projective space, then it is in one of the 95 families. The converse is false: a variety may have the Hilbert series of one of the 95 without being a hypersurface, such as a complete intersection $X_{2,4} \subset \mathbb{P}(1^5, 2)$ with non-quasi-linear quadric; cf. other degenerations in [Bro07, Table 2].

In similar vein, Iano-Fletcher, [IF00, (16.7) Table 6] lists 85 families of Fano 3-folds in codimension 2, realising the codimension 2 (second) row of Figure 1. Chen–Chen–Chen [CCC10] prove that this list is complete in the following sense: if $X \subset \mathbb{P}(a_0, \ldots, a_5)$ is a codimension 2 complete intersection prime Fano 3-fold, then either it is in one of Iano-Fletcher's 85 families, or it is a degeneration of one of the famous 95. Again the converse is false: for example, [Bro06, Table 3] lists degenerations of 13 of the 85 families that lie in codimension 3 (described as K3 surfaces, but each extends to a Fano 3-fold with an additional variable of degree 1).

3.1.2. *The graded ring game.* Suppose $X \subset \mathbb{P} = \mathbb{P}(1^{m_1}, 2^{m_2}, \dots)$ is a variety in a nondegenerate projectively normal embedding. Such X has a Hilbert series $P_X(t) = \sum_{i\geq 0} n_i t^i$, where $n_i = h^0(X, \mathcal{O}_X(i))$. In general, knowing $P_X(t)$ alone is not enough to determine the m_j, but it can provide estimates, even without information about the equations of X.

The following 'graded ring game' is standard. Given a series $P_0 = 1 + n_1 t + n_2 t^2 + \cdots$, we attempt to build some X as above, polarised by $\mathcal{O}_X(1)$, with $P_X = P_0$. By the assumptions, we must start with $m_1 = n_1$. Write $P_1 = (1-t)^{m_1} P_0 = 1 + n_2' t^2 + \cdots$. If $n_2' \geq 0$, then there are necessarily at least n_2' variables in weight 2, so we set $m_2 = n_2'$. (Of course we may add more variables of weight 2, but each one would necessitate an equation of weight 2, and at least for a complete intersection that is general in its deformation family that would be excluded by nondegeneracy.)

Continuing, write $P_2 = (1-t^2)^{m_2} P_1 = 1 + n_3'' t^3 + \cdots$. If now $n_3'' \geq 0$, then there are necessarily at least n_3'' variables in weight 3, so again we set $m_3 = n_3''$ and we consider $P_3 = (1-t^3)^{m_3} P_2$. At some stage $n_{r+1}^{(r)} < 0$ and the game ends: we can no longer infer necessarily additional generators in degree $r+1$. Interpreting the Hilbert numerator according to (3.A) we may now attempt to construct $X \subset \mathbb{P}(1^{m_1}, 2^{m_2}, \dots, r^{m_r})$ using suitably general polynomials of degrees d_i, and check whether $P_X = P_0$ and whichever properties – irreducible, quasismooth, Fano, and so on – that we require do in fact hold.

As we saw, at any stage there may be more generators in the degree we are considering, but there cannot be fewer. Indeed, at the ith stage

$$P_0 = \left(\tfrac{1}{(1-t)^{m_1}\cdots(1-t^i)^{m_i}}\right) \cdot \left(1 + n_{i+1}^{(i)} t^{i+1} + \text{ higher order terms}\right)$$

where the first factor is the Hilbert series of $\mathbb{P}(1^{m_1}, \dots, i^{m_i})$ (with no relations), so in degree $i+1$

$$n_{i+1}^{(i)} = n_{i+1} - N_i$$

where N_i is the maximum possible span of lower degree terms; if $n_{i+1}^{(i)} > 0$ then the ring needs that many additional generators in degree $i+1$.

When X is a complete intersection, this approach with a small twist determines the unique weights and degrees; cf. [IF00, §18]. To illustrate the twist, the genus 5 hyperelliptic curve $C_{2,6} \subset \mathbb{P}(1^3, 3)$ has Hilbert series $P = 1 + 3t + 5t^2 + 8t^3 + \cdots$, so the first step is to consider $(1-t)^3 P = 1 - t^2 + t^3 - t^5$. The game is complete, but clearly there is no variety defined by a single quadric in $\mathbb{P}^2$ with a linear syzygy. Speculatively looking ahead for the next positive coefficient suggests considering a variable of weight 3, which recovers the numerical data of C as $(1-t^3)(1-t)^3 P = 1 - t^2 - t^6 + t^8$. In practice, this phenomenon is rare, and when it does arise the solution is as simple as this.

Even when $X \subset \mathbb{P}$ is not a complete intersection, this graded ring game provides a lower bound for the weights m_i of a minimal generating set, and so provides both a lower bound on the actual codimension of X and a necessary subset of the weights.

Although this seems to give rather a lot away, when applied to the series $P_{g,\mathcal{B}} \in \mathcal{F}_3$ with low genus or small basket with small singularity indices it works well. For example, with

$$g = -1 \quad \text{and} \quad \mathcal{B} = \left\{\tfrac{1}{2}(1,1,1), \tfrac{1}{3}(1,1,2), \tfrac{1}{5}(1,1,4), \tfrac{1}{5}(1,2,3), \tfrac{1}{7}(1,1,6)\right\}$$

this identifies necessary weights $1,5,6,7,8,9,10$ and Hilbert numerator $1 - t^{16} - t^{17} - t^{18} - t^{19} - t^{20} + t^{25} + t^{26} + t^{27} + t^{28} + t^{29} - t^{45}$, and it is easy to write down a Fano 3-fold $X_{16\dots20} \subset \mathbb{P}(1,5,6,7,8,9,10)$.

3.1.3. *Polarising baskets.* We also add weights to realise the singularity basket correctly. For example, $g = 2$, $\mathcal{B} = \left\{\tfrac{1}{2}(1,1,1), \tfrac{1}{3}(1,1,2)\right\}$ determines a series $P(t)$ that satisfies

$$(1-t)^4(1-t^2)^2 P(t) = 1 - 3t^4 + 3t^6 - 2t^7 + 3t^9 - 3t^{10} + \cdots .$$

Thus the graded ring game only demands generators in degrees $1,1,1,1,2,2$, whereas there must be some ambient orbifold locus with stabiliser $\boldsymbol{\mu}_3$ to accommodate the index 3 orbifold point. That could be achieved with a generator in degree 3, or a combination of generators whose degrees have 3 as their greatest common divisor. Here the simplest thing works:

$$(1-t)^4(1-t^2)^2(1-t^3)P(t) = 1 - t^3 - 3t^4 + 3t^6 + t^7 - t^{10}$$

suggests a variety $X \subset \mathbb{P}(1,1,1,1,2,2,3)$ in codimension 3 defined by five Pfaffians of degrees $3,4,4,4$ and 5. (Notice that the equation of degree 5 is *masked* in the Hilbert numerator by a syzygy of degree 5: the Hilbert numerator is 'really' $1 - t^3 - 3t^4 - t^5 + t^5 + 3t^6 + t^7 - t^{10}$. Knowing that codimension 3 Gorenstein ideals have an odd number of generators defined as Pfaffians is extra information that comes from the Buchsbaum–Eisenbud theorem [BE82, CR02].) It is now easy to construct such a Fano 3-fold.

Other ways to introduce index 3 points, such as introducing weights 6 and 9, may work but increase codimension. In practice, by making this check in descending order of singularity index, such questions are avoided, since additional high-degree variables may already polarise lower-degree singularities.

By this stage, for each Hilbert series we have a collection of generator weights that are enough to span low-degree pieces of the graded ring and to polarise the singularities. This already determines the GRDB weights for codimension ≤ 3, but is not always correct in high codimension. We fix this next.

3.2. **Numerical unprojection Ansatz and weights.** Type I Gorenstein unprojection [PR04,KM83] is a technique that takes as input a pair of Gorenstein schemes $D \subset X$, with $D \subset X$ of codimension 1, and returns a new Gorenstein scheme Y. In applications to projective geometry, it often corresponds to a birational contraction of D to a point of Y, and that is how we wish to apply it.

3.2.1. *The Type I unprojection Ansatz.* We describe a model case of Kustin–Miller unprojection following Papadakis–Reid [PR04, 2.4–9]. Consider the following hypothetical input-output process:

Input: Let $X \subset \mathbb{P}(a_0, \dots, a_n)$ be a Fano 3-fold with terminal singularities in its anticanonical embedding with basket $\mathcal{B}_X$. Suppose $D \subset X$ for some coordinate plane $D = \mathbb{P}(a_i, a_j, a_k)$ with weights $(a_i, a_j, a_k) = (1, a, b)$ in some order and $\gcd(a, b) = 1$, and suppose moreover that X is quasismooth away from finitely many nodes $\Sigma \subset D$.

Output: A quasismooth Fano 3-fold $Y \subset \mathbb{P}(a_0, \dots, a_n, r)$ in its anticanonical embedding, with $r = a + b$, such that:

(1) Y contains the point $P = (0 : \ldots : 0 : 1)$ and $P \in X$ is a terminal quotient singularity $\frac{1}{r}(1, a, b)$.
(2) Y has the same genus as X. Moreover, if $a+b+c > d$ for every $c, d \in \{a_0, \dots, a_n\} \setminus \{a_i, a_j, a_k\}$, then the equations of Y have no quasi-linear terms.
(3) The Gorenstein projection from $P \in Y$ is a birational map $Y \dashrightarrow X$ that factorises into birational morphisms as follows:

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

where $Z \to Y$ is the contraction of the birational transform $D \subset Z$ to $P \in Y$ (which, viewed from Y, is the Kawamata blowup of $P \in Y$), and $Z \to X$ is the small D-ample resolution of the nodes of X (the contraction of finitely many flopping curves, viewed from Z).
(4) The basket $\mathcal{B}_Y$ of Y satisfies

$$\mathcal{B}_X \cup \left\{\tfrac{1}{r}(1, a, b)\right\} = \mathcal{B}_Y \cup \left\{\tfrac{1}{a}(1, b, -b), \tfrac{1}{b}(1, a, -a)\right\} \tag{3.C}$$

where $\frac{1}{a}(1, b, -b)$ is omitted if $a = 1$, and analogously if $b = 1$.
(5) The Hilbert series of Y is

$$P_Y = P_X + \tfrac{t^{a+b+1}}{(1-t)(1-t^a)(1-t^b)(1-t^r)}.$$

In many cases this process is a theorem; see [BKR12, 3.2] for example. Indeed the setup $D \subset X$ satisfies the conditions for Kustin–Miller unprojection [PR16, 2.4], giving a new variable s of degree $r = k_X - k_D = -1 - (-1 - a - b) = a + b$ and additional equations involving s of the form $sf_i = g_i$, where f_i form a basis of the ideal I_D in the coordinate ring of X. One can see using a free resolution of the coordinate ring $\mathbb{C}[Y]$ over $\mathbb{C}[\mathbb{P}(a_0, \dots, r)]$ ([Pap04]) that $\mathcal{O}(-K_Y) = \mathcal{O}(1)$, and the numerics of (1), (2), (4), and (5) follow (cf. [PR16, 2.7–9]). Then given Y, the Kawamata blowup of $P \in Y$ is a weak Fano, and it has an anticanonical model $Z \to X'$. More complicated situations can arise – see (4.B), where $Z \to X$ makes both a crepant divisorial contraction and a disjoint flopping contraction – but since the assumption of nodes here already establishes that contracting the flopping curves result in a Fano, there can be no further contraction in the given situation.

The key point here is that we do not use the setup above as a theorem to be applied, rather we turn it around to act as an Ansatz, as follows.

Ansatz 3.1 (Type I unprojection)**.** Suppose that a genus–basket pair $(g, \mathcal{B}_Y) \in \mathcal{F}_4$ is not among the $95 + 85$ cases assigned weights in codimension 1 or 2 by §3.1.1, and that $(g, \mathcal{B}_X)$ is another genus–basket pair (with matching genus) for some X which satisfies (3.C) with $r = a + b$ and a, b coprime. If the weights $(a_0, \dots, a_n)$ of X listed in GRDB contain $(1, a, b)$ as a sublist, then we insist that the weights of Y in GRDB are $(a_0, \dots, a_n, r)$.

3.2.2. *Type I consistency.* Ansatz 3.1 works as an inductive procedure from low to high codimension and is simple to arrange in any particular case. The main point is that this operation is well defined over the whole Fano 3-fold database and is consistent with the necessary weights identified in §3.1; the proof is a computer check on the collection of all weights.

Lemma 3.2. *Suppose $(g, \mathcal{B}_Y) \in \mathcal{F}_3$ admits the Type I unprojection relation 3.1 for two different pairs $(g, \mathcal{B}_{X_1})$ and $(g, \mathcal{B}_{X_2})$ (with respect to different sums $r_i = a_i + b_i$). Then the weights for $(g, \mathcal{B}_Y)$ determined by 3.1 contain the minimal weights of §3.1 and are independent of which X_i pair is used.*

To give some idea of the potency of this result, only 1087 of the 39,370 genus–basket pairs not among the $95 + 85$ cases in codimension ≤ 2 do not satisfy the Type I projection numerics for $(g, \mathcal{B}_Y)$ in 3.1.

But to be clear: the claim is not that for each of these $(g, \mathcal{B}_Y)$ pairs we may find a particular $D \subset X$ that satisfies the conditions specified as input to a Type I unprojection above: it is only that the numerics of the weights are consistent with its existence. There is no promise that we will be able to make unprojections in accordance with 3.1 in every case to realise most of the Hilbert series (compare §4.7 for an attempt to tackle this stronger claim).

Remark 3.3. In fact, more is true. There is a class of higher Gorenstein projections, referred to as Type II_n for $n \geq 1$; see [Pap08, Tay]. These may also be used to describe weights for genus–basket pairs based on the same relation (3.C) in the case when one of the polarising weights $c \in \{1, a, b\}$ does not lie among the weights of X, but instead the list with c replaced by some multiple $(n+1)c$ does (for $n \geq 1$ minimal). In this case, the unprojection adjoins $n + 1$ variables of weights $r, r + c, \ldots, r + nc$.

For example, $\frac{1}{7}(1, 2, 5) \in Y \subset \mathbb{P}(2, 3, 4, 5, 6, 7, 7, 8, 9)$ satisfies these numerics for Type II_2. That suggests a construction of Y by unprojection of a suitable divisor $D \subset X_{12,14} \subset \mathbb{P}(2, 3, 4, 5, 6, 7)$; this is carried out in [Tay].

Making a similar Ansatz for Type II_n unprojection results in an analogous consistency result to 3.2; cf. [Bro07, 3.4]. This does not change the results of the Type I constructions, but does help in very low genus: Gorenstein projection preserves genus, and when $g = -2$, there are no Type I projections at all. For example, 735 of the 52,646 genus–basket pairs do not admit a numerical Type I projection, but do admit a numerical Type II_1 projection (and, in fact, all lie in $\mathcal{F}_3$); a further 159 admit a Type II_2, then a further 68 with Type II_3, 24 with Type II_4 and so on with diminishing returns.

3.3. **Numerical corollaries.** The crude classification by 1.2 already contains enough information to provide approximations to various strong and sharp theorems by elementary and easy means. For example, recall Prokhorov's sharp bound on the degree.

Theorem 3.4 (Prokhorov [Pro07])**.** *If X is a prime Fano 3-fold which is not Gorenstein, then $-K_X^3 \leq 62\frac{1}{2}$, and this bound is realised only by $X = \mathbb{P}(1^3, 2)$.*

In the semistable case, Theorem 1.2 recovers the weaker bound $-K_X^3 \leq 72$ at once. Applying [Kar09, Kar15] sharpens this to $-K_X^3 \leq 66\frac{1}{2}$. The next case to exclude is $(g, \mathcal{B}) = (34, \{\frac{1}{2}(1, 1, 1)\})$, which we see (§4.2.1) is populated by the blowup $\mathrm{Bl}_p\,\mathbb{P}(1, 1, 3, 5)$ at a smooth point P, a Fano 3-fold with strictly canonical singularities. The numerical methods alone cannot say more: instead, Prokhorov's stronger geometric Sarkisov methods show this case cannot be realised by a prime Fano 3-fold.

Another example is the following result of [CCC10], proving [IF00, Conjecture 18.19(2)], which bounds the codimension of Fano complete intersections.

Corollary 3.5 ([CCC10, Theorem 1])**.** *If X is a prime Fano 3-fold whose anticanonical embedding $X \subset w\mathbb{P}^n$ is a complete intersection in weighted projective space, then $n \leq 6$.*

The proof is difficult and subtle, but in the semistable case this follows again from 1.2 by inspection of the numerators of all Hilbert series in the database. Indeed, for a complete intersection, the process of determining generic weights and numerator (§3.1) is well defined. Conversely, given weights, the numerator determines a minimal set of degrees for equations. In most cases, this immediately rules out a complete intersection, as there are too many equations – and adding additional weights does not alter that. Of the remaining, there are cases of apparent complete intersections, but by equations whose degrees are too small to accommodate the high-degree variables: therefore to be terminal there must be further equations, and again complete intersection is ruled out.

4. Populating the Fano 3-fold database

We locate some established classes of Fano 3-folds within the Fano 3-fold database $\mathcal{F}_3 \subset \mathcal{F}_4$. We consider Fano 3-folds that are: §4.1 nonsingular, §4.2 Gorenstein, §4.3 Gorenstein index 2, §4.4 higher index, §4.5 toric, §4.6 in Gorenstein formats, §4.7 unprojections, and finally in §4.8 those arising in the fanosearch program as qG-smoothings of toric varieties. This gives a much more detailed understanding of Figure 1, and pinpoints places where the next questions lie. We give cases of $P \in \mathcal{F}_3$ which have no matching prime Fano 3-fold (see §4.1, 4.3) and others where there are multiple deformation families (cf. Figure 3), but this analysis is far from complete. More generally, we know only a single element of $\mathcal{F}_3$ for which it is proven that there is no matching Fano 3-fold with canonical singularities (see §4.2), and we give examples of such Fano 3-folds whose Hilbert series do not appear in $\mathcal{F}_3$ (see §4.5).

4.1. **Nonsingular Fano 3-folds.** The classification of 105 families of nonsingular Fano 3-folds [Isk78, Isk78, MM82] is a cornerstone of higher-dimensional birational geometry. It is listed in [IP99, Table 12.2] and online at [Bel19]. Each of the 105 families lies in its anticanonical embedding along the leading diagonal of Figure 1: Figure 2 and its explanation shows how. We should expect similarly subtle behaviour in hundreds of places throughout Figure 1, so it is useful to absorb this familiar case first.

g	1-n		2-n				3-n				4-n	ρ-n	$\#\mathcal{T}$
2	1												0
3	2				1_{11}								1
4	3					2						10-1	7
5	4	11			3_{12}								23
6	5		4^*										54
7	6				5_{13}	6				1		9-1	135
8	7			$7^\dagger$		8				2			207
9	8	12			10_{14}	9							314
10	9				11_{13}		3^*		4_{18}			8-1	373
11			12^*	$13^\dagger$	14_{15}		5^*						416
12	10		15^*		16_{14}				$6_{25,33}$				413
13		13	$17^{*\dagger}$			18	$7^*_{32}, 8^*_{24}$				1^*	7-1	413
14			19^*_{14}		20_{15}				$9_{36}, 10_{29}$		13^*		348
15				$21^\dagger$			$12^*_{27,33}$	$11^{*\dagger}_{25}$			2_{31}	5-1	344
16			22^*_{15}	$23^\dagger$		24			13_{32}		$3^*_{17,28}$	6-1	274
17		14	25^*				$15^*_{29,31}$	$14^\dagger_{36}$			$4^{30}_{18/9}, 5^{31}_{21/8}$		234
18				$26^\dagger_{15}$				$16^\dagger_{27,32}$			6^*_{25}		179
19							17^*		$18_{29,30/3}$		$7_{24,28}$	5-2/3	151
20			27^*				21^*	$19^\dagger$	$20_{31/2}$		8^*_{31}		117
21		15	28^*	$29^\dagger$			22^*_{36}				$9_{25/6/8,30}$		87
22							24^*_{32}	$23^\dagger_{30/1}$			10^*_{28}		66
23									25_{33}		$11_{28,31}$		40
24			30^*	$31^\dagger$				$26^{*\dagger}$			12_{30}		42
25						32	28^*			27			27
26								$29^\dagger, 30^\dagger_{33}$					18
27										31			8
28		16	33^*			34							13
29			35^*										9
30													4
31													2
32						36							2
33		17											5
			${}^*_{17}\mathbb{P}^3$	${}^\dagger_{16}X_2$			${}^*_{34}\mathbb{P}^1 \times \mathbb{P}^2$	${}^\dagger_{35}\widehat{\mathbb{P}^3}$			${}^*_{27}(\mathbb{P}^1)^3$		

FIGURE 2. The 105 families of nonsingular Fano 3-folds, listed as ρ-n for the nth variety of Picard rank ρ, with a row for each genus $g = 2, \dots, 33$.

Figure 2 uses the numbering convention of [Bel19] (which adapts [IP99, MM82]) to list the 105 families: ρ-n denotes the n-th family of 3-folds with Picard rank ρ. The table is indexed in columns by Picard rank $\rho = 1, \dots, 10$ across the top (the sparse cases $\rho = 5$–10 are compressed into a single column). The entries of the table are the family numbers n (in full-size font) for the n-th family with that ρ; these entries often have sub/superscript decorations that we will come to. Each column is indexed by genus $g_X = 2, \dots, 33$ down the side, so that each row lists all families of a given genus g. Since for a nonsingular Fano 3-fold X the genus g_X determines $-K_X^3$ and P_X, one may imagine this table lying sideways along the leading diagonal of Figure 1. For example, the nonsingular Hilbert series for $g = 13$ is realised by the 7 families 1-13, 2-17, 2-18, 3-7, 3-8, 4-1 and 7-1, and in Figure 1 these all lie at the single position genus 13 and codimension 11 on the leading diagonal.

The decorations and positioning within each column indicate additional information. Fano manifolds X with $\rho_X = 1$, the first column, are divided with those of Fano index $f_X = 1$ on the left-hand side of column 1-n, and $f_X \geq 2$ on the right-hand side. When $\rho \geq 2$, many of these Fano 3-folds are constructed by extremal extractions from other nonsingular Fano 3-folds. The decorations on entries of the table include some of these extremal divisorial contractions by writing entries as ρ-n_m, abbreviated to n_m in the table, to indicate a map from members of family ρ-n to members of $(\rho-1)$-m. Some codomains are very common, and we indicate these by the following special notation:

(1) 2-$n^* \equiv$ 2-n_{17} means map to 1-17 $= \mathbb{P}^3$
(2) 2-$n^\dagger \equiv$ 2-n_{16} means map to 1-16 $= X_2 \subset \mathbb{P}^4$
(3) 3-$n^* \equiv$ 3-n_{34} means map to 2-34 $= \mathbb{P}^1 \times \mathbb{P}^2$
(4) 3-$n^\dagger \equiv$ 3-n_{35} means map to 2-35 $= \mathrm{Bl}_P\, \mathbb{P}^3$
(5) 4-$n^* \equiv$ 4-n_{27} means map to 3-27 $= \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Columns $\rho = 2, 3$ are arranged to give a brief indication of the extremal contraction data, with varieties which admit maps to the common target listed on the left-hand side of each column, with those which only admit other morphisms to the right of centre, and finally those with no birational contractions to another nonsingular Fano 3-fold down the right-hand side; most of the latter are products or double covers, and their extremal rays are Mori fibrations. For example, the diagram

$$\begin{tikzcd}
 & 3\text{-}15 \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{P}^1 \times \mathbb{P}^2 = 2\text{-}34 & 2\text{-}29 \arrow[dr] & 2\text{-}31 \arrow[d] \\
 & & 1\text{-}16 = (X_2 \subset \mathbb{P}^4)
\end{tikzcd}$$

indicates sequences of extremal contractions from 3-15 that one can read from Figure 2; see [Bel19] for full details. Finally, the right-hand column $\#\mathcal{T}$ lists the number of toric models with Gorenstein canonical, but not necessarily $\mathbb{Q}$-factorial, singularities by genus; see §4.5.

Generalising from smooth to Gorenstein terminal Fano 3-folds does not increase the number of families: any Gorenstein terminal Fano 3-fold may be smoothed [Nam97] so it appears in Figure 2. Moreover the Picard rank does not change on smoothing (see [JR11], which uses this to determine smoothing families). In contrast, there are Fano 3-folds with canonical Gorenstein singularities that do realise other families in the leading diagonal of Figure 1, as we discuss next.

4.2. **Gorenstein Fano 3-folds.** When X is a Fano 3-fold with canonical singularities that has $-K_X$ Cartier, then its singularities are Gorenstein and its basket is empty. Therefore its Hilbert series P_X behaves as though X has no singularities at all, and so again lies on the leading diagonal of Figure 1.

There is no classification of Fano 3-folds with Gorenstein canonical singularities (though [CPS05] settles the hyperelliptic and trigonal cases), but there are precise results for extreme cases, which we describe now in relation to the GRDB geography. Recall that at the pointy right-hand end of Figure 1 the number of Hilbert series in $\mathcal{F}_3$ by genus and codimension is:

Codim \ Genus	33	34	35	36	37
31	1				
32	1	1			
33		1	1		
34				1	
35					1

(4.A)

Prokhorov [Pro05, 1.5] proves that the largest possible genus for a Fano 3-fold with $-K_X$ Cartier is $g = 37$ with degree $-K_X^3 = 2g - 2 = 72$ (strengthening Cheltsov's result $-K_X^3 \leq 184$ [Che99]). Moreover Prokhorov proves that $g = 37$ is realised only by $\mathbb{P}(1,1,4,6)$ and $\mathbb{P}(1,1,1,3)$. That completely clears up the final column.

Cheltsov and Karzhemanov extend this to Gorenstein 3-folds in genus $g \geq 33$.

Theorem 4.1 ([Kar09]). *Let Z be a Fano 3-fold with Gorenstein canonical singularities and $34 \le g \le 36$.*

(1) If $g = 36$ then Z is the image of $\widetilde{Z} = \mathrm{Bl}_p\, \mathbb{P}(1,1,4,6)$ after contracting the flopping line $\widetilde{\Gamma} \subset \widetilde{Z}$, where $p \in \Gamma = \mathbb{P}(4,6)$ is an index 2 point, Bl_p is the discrepancy 1 divisorial extraction at p, and $\widetilde{\Gamma}$ is the birational transform of Γ in $\widetilde{Z}$.
(2) The case $g = 35$ is not possible.
(3) If $g = 34$ then Z is the anticanonical image of the projectivised bundle $U = \mathrm{Proj}_{\mathbb{P}^1} \mathcal{O} \oplus \mathcal{O}(2) \oplus \mathcal{O}(5)$.

The point p in (1) is not a toric 0-stratum, but for a local toric model of Bl_p blow up the toric cone $\langle (0,1,0),(1,0,1),(1,0,-1)\rangle$ at $(1,1,0)$ (rather than the usual discrepancy 2 blowup at $(2,1,0)$).

4.2.1. *Genus 34.* We consider Karzhemanov's degree 66 example Z in more detail. The weighted projective space $X = \mathbb{P}(1^2,3,5)$ is a Fano 3-fold with a terminal singularity $\frac{1}{3}(1,1,2)$, a canonical singularity $\frac{1}{5}(1,1,3)$, genus 34 and degree $-K^3 = 66\frac{2}{3}$. Its Hilbert series lies in $\mathcal{F}_4$ but not $\mathcal{F}_3$, just outside the region of Figure 1. It has anticanonical embedding $X \subset \mathbb{P}(1^{36},2,3)$ in the blank position $(34,34)$ in (4.A). It fits into a diagram of toric varieties and maps:

$$\begin{tikzcd}
 & & W \arrow[dl] \arrow[dr] & & & \\
U \arrow[d] & Y \arrow[l, dashed, "\text{flip}"'] \arrow[dr] & & V \arrow[dl] \arrow[dr] \arrow[rr, dashed, "\text{flop}"] & & V^+ \arrow[dl] \\
\mathbb{P}^1 & & X & & Z &
\end{tikzcd}$$

where $Y \to X$ is the Kawamata $\frac{1}{3}(1,1,2)$ blowup of $P_3 \in X$ (the index 3 point), $Y \dashrightarrow U$ is the $(5,2,-1,-1)$ (canonical) flip to Karzhemanov's bundle $U \to \mathbb{P}^1$, $V \to X$ is the $\frac{2}{3}$-discrepancy $\frac{1}{3}(2,2,1)$ blowup of $P_3 \in X$ and $V \dashrightarrow V^+$ is the $(5,1,-3,-3)$ (canonical) flop with base Z, while finally W is the resolution of $P_3 \in X$, which itself admits two flops to divisorial extractions of U and V^+ respectively. In this picture

$$\big(X \subset \mathbb{P}(1^{36},2,3)\big) \dashrightarrow \big(Y \subset \mathbb{P}(1^{36},2)\big) \dashrightarrow \big(Z \subset \mathbb{P}^{35}\big) \tag{4.B}$$

is a sequence of projections between Fano 3-folds of degrees $66\frac{2}{3}$, $66\frac{1}{2}$ and 66, where $\rho_X = \rho_Z = 1$ and $\rho_Y = 2$, and X and Y are $\mathbb{Q}$-factorial while Z is not.

Loosely speaking, we see how knowing Karzhemanov's example provides other Fano 3-folds by birational contractions, while, from the other end, knowing $\mathbb{P}(1^2,3,5)$ provides other Fano 3-folds by birational blow ups; we discuss this further in §4.5.4.

4.2.2. *Genus 33.* Karzhemanov [Kar15] also classifies the case of degree 64: there is $\mathbb{P}^3$ and 6 other cases, all with strictly canonical singularities. The GRDB matches 5 toric cases, and we illustrate with a beautiful non-$\mathbb{Q}$-factorial example

$$\Phi_{-K_X} \colon X = \mathrm{TorVar}_{\left(\begin{smallmatrix}5\\6\end{smallmatrix}\right)} \begin{pmatrix} 1 & 1 & 5 & 3 & 0 \\ 0 & 1 & 6 & 4 & 1 \end{pmatrix} \subset \mathbb{P}^{34}.$$

(The notation $\mathrm{TorVar}_v\, M_{2\times r}$ denotes the toric variety $\mathbb{C}^r /\!\!/_v (\mathbb{C}^*)^2$, where $(\mathbb{C}^*)^2$ acts by weights that are the rows of M; see [BCZ04, §A].) This is the base of a $(6,1,-2,-5)$ (canonical) flop

$$\begin{tikzcd}
 & U \arrow[dl] \arrow[dr] \arrow[rr, dashed, "\text{flop}"] & & U^+ \arrow[dl] \arrow[dr] & \\
\mathbb{P}(1^2,4,6) & & X & & \mathbb{P}(1^2,3,5)
\end{tikzcd}$$

where $U \to \mathbb{P}(1^2,4,6)$ is the blowup of a smooth point and $U^+ \to \mathbb{P}(1^2,3,5)$ is a weighted $\frac{1}{3}(1,2,4)$-blowup of the index 3 point; that is,

$$\text{blowup } (-1,-4,-6) \text{ in the cone } \langle (1,0,0),(0,0,1),(-1,-3,-5)\rangle\,.$$

(The numerics come from the observation that $(6,1,0,-2,-5)$, $(-1,0,1,1,1)$ and $(4,1,2,0,-3)$ lie in the span of the rows of the defining matrix of X.)

4.3. **Index 2 singularities.** The list $\mathcal{F}_4$ contains 360 pairs $(g,\mathcal{B})$, where $\mathcal{B}=\{N\times\frac{1}{2}(1,1,1)\}$ with $N\geq 1$ of which 272 lie in $\mathcal{F}_3$. Restricting to $g\geq 2$, these numbers reduce to 325 and 238 respectively. Sano [San95, San96] and Campana–Flenner [CF93] classify such terminal Fano 3-folds X under the assumption Fano index $f_X\geq 1$. For non-empty baskets $\{N\times\frac{1}{2}(1,1,1)\}$ and $f_X>1$, these are [San96] (with their anticanonical embeddings mapping to the right)

$$\begin{array}{rcll}
\mathbb{P}(1^2,2^2,3)\supset X_6 & \hookrightarrow & \mathbb{P}(1^8,2^3) & (N=3,\ g=6,\ \text{codim } 7)\\
\mathbb{P}(1^3,2^2)\supset X_4 & \hookrightarrow & \mathbb{P}(1^{16},2^2) & (N=2,\ g=14,\ \text{codim } 14)\\
\mathbb{P}(1^4,2)\supset X_3 & \hookrightarrow & \mathbb{P}(1^{23},2) & (N=1,\ g=21,\ \text{codim } 20)\\
\mathbb{P}(1^3,2) & \hookrightarrow & \mathbb{P}(1^{34},2) & (N=1,\ g=32,\ \text{codim } 31)
\end{array}$$

and when $f_X=1$ there are 14 cases of nonsingular Fano 3-folds, for which particular members admit $\boldsymbol{\mu}_2$ quotients [San95, 1.1], which are also in rather high anticanonical codimension.

The remaining cases for index 2 baskets satisfy $i_X=2$ and $f_X=1/2$. Takagi [Tak02] classifies prime Fano 3-folds with such genus–basket pairs under these conditions with $g\geq 2$. The result is precisely 35 families matching 23 elements of $\mathcal{F}_3$. They are presented in Tables 1–5 of [Tak02], with the individual families numbered 1.1, 1.2, …, 5.5. They are listed in Figure 3, ranging from Family 3.1, $X_5\subset\mathbb{P}(1^4,2)$ to Family 1.14, $X\subset\mathbb{P}(1^{10},2^2)$ in codimension 8, with Type I projections going up the columns. (The division into 2 cases in codimension 4 is the starting point for [BKR12].)

Codimension \ Genus	2	3	4	5	6	7	8
1	3.1						
2	3.2	5.2					
3	2.1	5.3	4.3				
4	2.2, 3.3	4.1, 5.1	1.3, 4.4	1.4			
5	2.3, 3.4, 5.1	4.2, 5.5	1.2, 1.3, 4.5	1.5, 1.6	1.9, 1.10		
6	2.4		4.6	1.7, 1.8	1.11	1.12	
7			4.7				1.13, 4.8
8							1.14

FIGURE 3. Families of Tables $n.m$ of [Tak02] as they appear in Figure 1, arranged by genus g and codimension c. The generic member of each family is embedded as $X\subset\mathbb{P}(1^{g+2},2^N)$ with basket $\mathcal{B}=\{N\times\frac{1}{2}(1,1,1)\}$ where $N=c-g+2$.

The comparison with the geography in Figure 1 is striking. The 272 pairs $(g,\mathcal{B})\in\mathcal{F}_3$ are spread over most of the table, away from the top diagonal line of Gorenstein pairs $(g,\emptyset)$, and Takagi's result shows that most are not realised by prime Fano 3-folds. However, we see in §4.5 that many of the remaining $272-23=249$ pairs are realised by more general Fano 3-folds, and the complete classification of Fano 3-folds of Gorenstein index 2 remains unknown.

4.4. **Higher Fano index.** Among all Fano 3-folds X, there are some that have divisible canonical class, and we indicate the Hilbert series of those in Figure 4. The notion of divisibility we consider here is the following: $-K_X=\iota A$ for a maximal integer $\iota\geq 2$ and ample integral Weil divisor A. (Note that ι is quite different from the Gorenstein index i_X and the Fano index f_X.) The graded ring $R(X,A)=\oplus_{m\geq 0}H^0(X,mA)$ is Gorenstein as $H^0(X,-K_X)\subset R(X,A)$ [GW78, 5.1.9]. Suzuki [Suz04, BS07a, BS07b] carries out the analysis to find a set of possible baskets $\mathcal{B}_\iota$ for each ι, with additional genus information when $\iota\leq 2$. Again we impose the stability condition $-K_X^3\leq -3K_Xc_2$. The numbers of baskets (or basket–genus pairs for $\iota\leq 2$) per index $\iota\geq 1$ is:

ι	1	2	3	4	5	6	7	8	9	11	13	17	19
$\#\mathcal{B}_\iota$	39550	1413	181	82	34	6	12	4	2	3	2	1	1

There are no prime Fano 3-folds of indices $\iota=12$, 14, 15, 16 or 18 by [Suz04], and by applying more sophisticated techniques [Pro10] also proves that there is no prime Fano 3-fold of index 10.

Codimension \ Genus	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
1																																								
2		1	3		1																																			
3								1																																
4			2		1																																			
5			5	2		2																																		
6		5	7	5	1				1																															
7			4			2						1																												
8		6	12	8	2	1	3		2	1																														
9			7		1	4				2																														
10		10	15	13	11		4	1	1				1																											
11		4	14	1	1	4		5		2						1																								
12		6	18	20	12	1	8		1		5		1																											
13		7	14	3	2	15		6		3				3																										
14		10	31	50	23	2	10	1	11	1	3	1	2				2																							
15		11	28	6	3	25	4	12		7	1	5	1	2						1																				
16		12	24	64	37	1	25	2	12		7	1	1		3		1																							
17		18	11	3	4	44	3	16	1	17	2	8	1	4				2																						
18		17	12	44	67	5	42	1	22	3	9	1	8	2	5		2	1	1		1																			
19		13	5		2	40	3	47		21		14		3		5		3						1																
20		2			14	1	38	2	26	1	25	2	16		7	1	2		3	1	1			1																
21						9		16	1	24	3	19	2	19		6		4				2																		
22									8		14	3	13	1	12		8		4	1	2				2															
23										1		7	1	9	1	11	1	4		8		2		1				1												
24													1		6		8	1	7	1	4		3		3															
25																		8	1	5	1	4				2														
26																			1		4		4		1		1		1		1									
27																						1	1	3	2	3						1								
28																									1		3		2											
29																														2										
30																													1			1	1							
31																																			1	2				
32																																		1						
33																																								
34																																								
35																																								1

FIGURE 4. Number of Hilbert series of semistable Fano 3-folds of index $\iota \geq 2$ listed by genus and estimated minimal codimension.

The point here is that if $-K_X = \iota A$, then the A-embedded model $\operatorname{Proj} R(X, A)$ is usually much simpler that the anticanonical ring. The numbers in low A-embedding codimension are:

ι	2	3	4	5	6	7	8	9	11	13	17	19	total
codim 0	0	0	1	1	0	1	0	0	1	1	1	1	7
codim 1	8	7	2	5	1	4	3	2	2	1			35
codim 2	26	6	7	1									40

(4.C)

Example 4.2. The codimension 0 row of (4.C) contains the 7 weighted projective spaces

	$\mathbb{P}^3$	$\mathbb{P}(1^3, 2)$	$\mathbb{P}(1^2, 2, 3)$	$\mathbb{P}(1, 2, 3, 5)$	$\mathbb{P}(1, 3, 4, 5)$	$\mathbb{P}(2, 3, 5, 7)$	$\mathbb{P}(3, 4, 5, 7)$
$-K^3$	64	125/2	343/6	1331/30	2197/60	4913/210	6859/420
g	33	32	29	22	18	11	7
cod	31	31	30	27	25	22	20
W	1^{35}	$1^{34}, 2$	$1^{31}, 2^2, 3$	$1^{24}, 2^4, 3^2, 5$	$1^{20}, 2^4, 3^3, 4, 5$	$1^{13}, 2^5, 3^4, 4^2, 5, 7$	$1^9, 2^7, 3^4, 4, 5^2, 7$

and these embed anticanonically in $\mathbb{P}(W)$ as the model given in GRDB.

Example 4.3. Many of the A-hypersurfaces embed in a bigger space under their anticanonical ι-Veronese embedding than GRDB suggests. For example, $X_{10} \subset \mathbb{P}(1^2, 2, 3, 5)$ has $\iota = 2$, so $-K_X = 2A$ and the anticanonical embedding is

$$X \xrightarrow{\cong} \Phi_{2A}(X) \subset \mathbb{P}(1^4, 2^2, 3^3, 4)$$

in codimension 6. Its ($-K_X$-polarised) Hilbert series, however, admits a simpler model $Y_{4,4} \subset \mathbb{P}(1^4, 2, 3)$, and this is the one given in GRDB.

There is a small twist in the GRDB numerics in this case. For varieties with $\iota \geq 2$, Gorenstein projection does not work in the same way as the $\iota = 1$ case (the image of projection has non-isolated singularities). Therefore rather than using projection to identify weights, the GRDB compares with the K3 database [Bro07]: a degree ι slice $S \in |\iota A|$ has the numerical properties of a K3 surface polarised by $A_{|S}$, so a choice of weights is determined by [Bro07]; then including a variable of degree ι (noting that it may then be eliminated by an equation of degree ι) gives a choice of weights for X.

4.5. **Toric Fano 3-folds.** Kasprzyk [Kas10a] classifies toric Fano 3-folds with canonical singularities as a list $\mathcal{T}_{\mathrm{can}}$ of 674,688 lattice polytopes; this classification is available from [Kas10b]. These varieties are all $\mathbb{Q}$-Gorenstein, and 12,190 of them are $\mathbb{Q}$-factorial.

The GRDB contains $\mathcal{T}_{\mathrm{can}}$, and it can be analysed online [BK09]. More importantly, the GRDB connects $\mathcal{T}_{\mathrm{can}}$ with the Fano 3-fold database: each polytope $\Delta \in \mathcal{T}_{\mathrm{can}}$ has a Hilbert series $P_\Delta \in \mathcal{F}_4$, all but 12 of which lie in $\mathcal{F}_3$. These absent 12 all have isolated, $\mathbb{Q}$-factorial, strictly canonical singularities, and are either one of the 6 weighted projective spaces

$$\mathbb{P}(1, 1, 3, 5),\ \mathbb{P}(1, 2, 5, 7),\ \mathbb{P}(1, 3, 7, 10),\ \mathbb{P}(1, 3, 7, 11),\ \mathbb{P}(1, 5, 7, 13),\ \mathbb{P}(3, 5, 11, 19)$$

or a rank 2 blowup of one of these. The GRDB links Δ to P_Δ, and conversely for any $P \in \mathcal{F}_3$ reports all those Δ with $P_\Delta = P$. This matching is significant in different ways, as we discuss next.

4.5.1. *Location in Geography.* The toric Fano 3-folds in $\mathcal{T}_{\mathrm{can}}$ populate large areas of Figure 1. Figure 6 is the submap of Hilbert series that are realised by at least one toric Fano 3-fold.

Only 8 of the toric cases are prime Fano 3-folds: the 7 weighted projective spaces of 4.2 together with the magical fake weighted projective space of degree 64/5 and genus 5

$$\mathbb{P}^3/\boldsymbol{\mu}_5(1, 2, 3, 4) \hookrightarrow \mathbb{P}(1^7, 2^8, 3^4, 5^4)$$

in codimension 19. Including these, there are 634 terminal cases, of which 233 are terminal $\mathbb{Q}$-factorial.

For some Hilbert series there are many matching toric 3-folds, and this is recorded in the GRDB, with an idea of the multiplicities in Figure 7. It seems amazing to us that two different polytopes can contain the same number of lattice points at all dilations – but of course this is exactly what happens whenever two polytopes are mutation equivalent [ACGK12].

There are 4319 varieties $X \in \mathcal{T}_{\mathrm{can}}$ that have $-K_X$ Cartier (cf. [KS98]); of these, 194 are $\mathbb{Q}$-factorial. The number of these is listed by genus in the column $\#\mathcal{T}$ of Figure 2. We have not checked whether any satisfy Petracci's non-smoothability condition [Pet20, 1.1], nor whether any lie at the intersection of multiple smooth families (see §4.5.3).

4.5.2. *Toric degenerations.* Some approaches to or applications of Mirror Symmetry require toric degenerations of Fano 3-folds [CI16, CCGK16]. The link between lists $\mathcal{F}_3$ and $\mathcal{T}_{\text{can}}$ summarised in Figure 6 is a necessary condition for a Fano 3-fold X with given Hilbert series to have a toric Fano 3-fold degeneration X_0. This numerical condition is not sufficient, as it does not determine whether X and X_0 lie in the same deformation family, for example.

It would be natural to extend this correspondence either to include more general toric varieties, or reducible varieties composed of toric varieties glued along toric strata.

4.5.3. *Deformation of toric varieties and intersections of families.* Following Altmann's local analysis [Alt97, Alt00], a lot is known about how toric varieties deform. For toric Fano 3-folds with isolated singularities, global deformations surject onto local deformations [Pet22, 2.3], so understanding the deformation theory of singularities on toric Fano 3-folds is a powerful tool.

Example 4.4. The first element $X_1 \in \mathcal{T}_{\text{can}}$ has the Hilbert series of some $X \subset \mathbb{P}(1^7, 2^4)$ with a basket $4 \times \frac{1}{2}(1,1,1)$. However, X_1 is not quasismooth: it has six Gorenstein facets that are the cone on the del Pezzo surface of degree 6 and four cones of type $\frac{1}{2}(1,1,1)$. Each of these del Pezzo cone singularities has two smoothing components locally, so since deformations of toric Fano 3-folds surject onto local deformations, there are at least seven distinct quasi-smoothing components that contain different small deformations of X_1; cf. 4.6 below which suggests there are at least 24 families.

Extending this to Gorenstein-index-2 toric Fano 3-folds that also have isolated cone over del Pezzo degree 6 singularities gives three more examples with two distinct quasi-smoothing families:

$\mathcal{T}_{\text{can}}(\text{id})$	$\#\frac{1}{2}$	$\#\text{dP}_6$	$\mathcal{F}_3(\text{id})$	$X \subset w\mathbb{P}$	g	codim
1	4	6	27334	$\mathbb{P}(1^7, 2^4)$	5	7
254482	3	1	38250	$\mathbb{P}(1^{17}, 2^3)$	15	16
254485	6	1	36639	$\mathbb{P}(1^{13}, 2^6)$	11	15
254810	3	1	38935	$\mathbb{P}(1^{20}, 2^3)$	18	19

4.5.4. *High codimension representatives and cascades.* Many of the weights in GRDB are constructed inductively by considering a single projection, but varieties frequently arise in sequences, or *cascades*, of projections: famously $X = \Phi_{-K}(\mathbb{P}^2) \subset \mathbb{P}^9$ has sequences of projections from points (which in this case are blowups) that recover elements of most families of del Pezzo surfaces; see [RS03] for extensions. It seems typical that the ends of such cascades are simpler to describe than the middles: things like toric varieties live at the top, while hypersurfaces live at the bottom. Thus any $X \in \mathcal{T}_{\text{can}}$ of high codimension for its genus may be a good candidate for the head of a cascade.

For example, in $g = 8$, the highest codimension $\mathbb{Q}$-factorial terminal Fano 3-fold is

$$\mathcal{T}_{\text{can}}(544385)\colon \mathbb{P}^1 \times \mathbb{P}^2/\tfrac{1}{3}(0,1,0,1,2) \subset \mathbb{P}(1^{10}, 2^6, 3^6)$$

of Picard rank 2 in codimension 18, with $6 \times \frac{1}{3}(1,1,2)$ singularities at the 6 index 3 toric 0-strata. The Fano polytope is the simplicial decomposition on vertices

$$(1,0,0), (0,1,0), (-1,-1,0), (1,2,3), (-1,-2,-3)$$

with six index 3 cones meeting at a central '$\mathbb{P}^2$' triangular equator with a cycle of three northern cones and three southern cones with Sym_3 symmetry. The Kawamata blowup of any one of the index 3 points is equivalent to any other, and gives the first projection. There are four ways to project from a pair of index 3 points, depending on adjacency; in the case of the blowup of a northern cone and the adjacent southern cone, the equator becomes a flopping curve that is contracted to an ordinary node. Continuing with different configurations of projections, we end up at the projection from all index 3 points, which is a variety

$$Y \subset \mathbb{P}(1^{10}, 2^6)$$

in codimension 12 with $6 \times \frac{1}{2}(1,1,1)$ singularities and 3 nodes. This variety admits a quasi-smoothing, so is a Gorenstein-index-2 Fano 3-fold that does not appear in Takagi's classification Figure 3, as it has $\rho_Y > 1$. Further projections from index 2 points give more Gorenstein-index-2 varieties that extend Figure 3 in genus 8 to Fano 3-folds that are not prime Fano.

The highest-codimension toric Fano 3-folds with terminal singularities by genus – that is, the toric candidates for the top of terminal cascades – are listed in Figure 5.

Beyond toric, [BHHN16, BHHN17] initiates the analysis of low complexity Fano varieties, with the classification of Picard rank 1, $\mathbb{Q}$-factorial terminal Fano 3-folds of complexity 1. As in 4.3, these

$\mathcal{T}_{\text{can}}$(id)	g	ρ_X	$\mathcal{F}_3$(id)	$X \subset w\mathbb{P}$	$\mathcal{B}$	codim	
547383	5	1	29211	$\mathbb{P}(1^7, 2^8, 3^4, 5^4)$	$4 \times \frac{2}{5}$	19	$\mathbb{P}^3/\frac{1}{5}(1,2,3,4)$
547379	7	1	32734	$\mathbb{P}(1^9, 2^7, 3^4, 4, 5^2, 7)$	$\frac{1}{3}, \frac{1}{4}, \frac{2}{5}, \frac{3}{7}$	20	$\mathbb{P}(3,4,5,7)$
544385	8	2	33967	$\mathbb{P}(1^{10}, 2^6, 3^6)$	$6 \times \frac{1}{3}$	18	
547380	11	1	36623	$\mathbb{P}(1^{13}, 2^5, 3^4, 4^2, 5, 7)$	$\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{3}{7}$	22	$\mathbb{P}(2,3,5,7)$
430483	12	4	36948	$\mathbb{P}(1^{14}, 2^3, 3^3)$	$3 \times \frac{1}{3}$	16	
520102	13	3	37585	$\mathbb{P}(1^{15}, 2^4, 3^3, 4)$	$\frac{1}{2}, 2 \times \frac{1}{3}, \frac{1}{4}$	19	
544370	14	2	38020	$\mathbb{P}(1^{16}, 2^4, 3^3, 4, 5)$	$\frac{1}{3}, \frac{1}{4}, \frac{2}{5}$	21	
544376	15	2	38404	$\mathbb{P}(1^{17}, 2^5, 3^2, 5)$	$2 \times \frac{1}{2}, \frac{1}{3}, \frac{2}{5}$	21	
430473	16	4	38533	$\mathbb{P}(1^{18}, 2^3, 3)$	$2 \times \frac{1}{2}, \frac{1}{3}$	18	
520107	17	3	38760	$\mathbb{P}(1^{19}, 2^3, 3)$	$2 \times \frac{1}{2}, \frac{1}{3}$	19	
520148	17	2	38760	same as 520107			
547382	18	1	39006	$\mathbb{P}(1^{20}, 2^4, 3^3, 4, 5)$	$\frac{1}{3}, \frac{1}{4}, \frac{2}{5}$	25	$\mathbb{P}(1,3,4,5)$
520124	19	3	39052	$\mathbb{P}(1^{21}, 2^2, 3)$	$\frac{1}{2}, \frac{1}{3}$	20	
520103	20	3	39192	$\mathbb{P}(1^{22}, 2^3, 3)$	$2 \times \frac{1}{2}, \frac{1}{3}$	22	
544394	21	1	39278	$\mathbb{P}(1^{23}, 2^3, 3)$	$2 \times \frac{1}{2}, \frac{1}{3}$	23	$X_4 \subset \mathbb{P}(1^2, 2^2, 3)$
547381	22	1	39368	$\mathbb{P}(1^{24}, 2^4, 3^2, 5)$	$\frac{1}{2}, \frac{1}{3}, \frac{2}{5}$	27	$\mathbb{P}(1,2,3,5)$
520128	24	3	39416	$\mathbb{P}(1^{26}, 2)$	$\frac{1}{2}$	23	
520131	24	3	39416	same as 520128			
544383	25	2	39457	$\mathbb{P}(1^{27}, 2^2, 3)$	$\frac{1}{2}, \frac{1}{3}$	26	
544389	26	2	39476	$\mathbb{P}(1^{28}, 2)$	$\frac{1}{2}$	25	
544388	28	2	39510	$\mathbb{P}(1^{30}, 2)$	$\frac{1}{2}$	27	
547384	29	1	39526	$\mathbb{P}(1^{31}, 2^2, 3)$	$\frac{1}{2}, \frac{1}{3}$	30	$\mathbb{P}(1^2,2,3)$
547385	32	1	39541	$\mathbb{P}(1^{34}, 2)$	$\frac{1}{2}$	31	$\mathbb{P}(1^3,2)$

FIGURE 5. High-codimension non-Gorenstein $\mathbb{Q}$-factorial terminal toric Fano 3-folds.

are almost all hypersurfaces with high-codimension anticanonical embedding, where the complexity condition enforces very particular trinomial equations.

4.6. **Formats and low codimension.** All 95 + 85 Hilbert series in $\mathcal{F}_4$ whose GRDB model is in codimension 1 or 2 actually lie in $\mathcal{F}_3$ and may be constructed by hand as complete intersections as proposed; these are the first two rows of Figure 1. These varieties all have Picard rank 1 by the Lefschetz hyperplane theorem (cf. [CPR00, 3.5]).

The same is true of all 70 Hilbert series with codimension 3 models, which occupy the third row of Figure 1. In that case, only $X_{2,2,2} \subset \mathbb{P}^6$ is a complete intersection. The remaining 69 cases are cut out by the five maximal Pfaffians of a skew 5×5 matrix. Corti and Reid [CR02], following Grojnowski, explain this as a pullback from a weighted Grassmannian $w\mathrm{Grass}(2,5)$ in a precise sense, which informally we may treat as saying that the Plücker embedding $\mathrm{Grass}(2,5) \subset \mathbb{P}^9$ is described by the Pfaffians of a generic skew 5×5 matrix of linear forms, and we may specialise these forms as we please, taking care with homogeneity. Again, these varieties have Picard rank 1 by [BF20, 3].

This idea leads to the general idea of ‘format’ [BKZ19], where the equations (and syzygies, and indeed the whole minimal free resolution) of a ‘key variety’ (that is, of any variety you like) are used as a model for the equations of other varieties by graded pullback.

This idea is implemented in several places; [CR02, QS11, BKZ19, CD20, Tak21a], for example. One point that arises is that the Picard rank should be inherited from the format, and so it is possible to target Fano 3-folds of different rank.

Example 4.5. In [BKQ18, 1.2], the variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ in its Segre embedding is used as a key variety to model some Fano 3-folds in codimension 4 that have Picard rank 2. That analysis constructs examples of deformation families in different codimension for the same Hilbert series. For example, $\mathcal{F}_3(548)$ is presented in GRDB as

$$X \subset \mathbb{P}(1,3,4,5,6,7,10)$$

a codimension 3 Pfaffian that is easy to construct, but there is another family

$$X' \subset \mathbb{P}(1,3,4,5,6,7,9,10)$$

Codimension / Genus	−2…2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
1		1																																		
2		1	1																																	
3		2	1	1																																
4		3	2	1	1																															
5		6	3	2	1	1																														
6		7	6	3	2	1	1																													
7		12	8	6	3	2	1	1																												
8		15	15	8	6	3	2	1	1																											
9		23	20	15	8	6	3	2	1	1																										
10		28	32	21	15	8	6	3	2	1	1																									
11		40	43	35	21	15	8	6	3	2	1	1																								
12	1	45	65	47	35	21	15	8	6	3	2	1	1																							
13		59	86	71	48	35	21	15	8	6	3	2	1	1																						
14		54	126	92	72	48	35	21	15	8	6	3	2	1	1																					
15		51	148	120	89	74	48	35	21	15	8	6	3	2	1	1																				
16		33	161	127	107	93	72	46	35	21	15	8	6	3	2	1	1																			
17		15	100	102	84	128	87	69	46	35	21	15	8	6	3	2	1	1																		
18		3	33	37	32	131	112	84	66	42	33	21	15	8	6	3	2	1	1																	
19		1	4	10	2	122	88	95	61	54	33	33	20	15	8	6	3	2	1	1																
20			1	1		49	52	58	29	32	21	28	25	21	15	8	6	3	2	1	1															
21						11	10	17	2	11	3	9	11	28	18	14	7	6	3	2	1	1														
22						1		2		1			1	13	25	14	13	8	6	3	2	1	1													
23														3	14	15	14	15	8	6	3	2	1	1												
24															6	1	9	17	13	6	6	3	2	1	1											
25															1		3	17	10	6	7	5	3	2	1	1										
26																		5	5	1	12	6	6	2	2	1	1									
27																			1		12	7	6	1	2	2	1	1								
28																					1	2	3		1	2	2	1	1							
29																												2	1	1						
30																												1	1	1	1					
31																														1	1	1				
32																														1			1			
33																																	1			
34																																				
35																																				1

Figure 6. The 5760 Hilbert series of (canonical) toric Fano 3-folds that lie in $\mathcal{F}_3$, listed by genus and codimension. (Note that 12 other toric Fano Hilbert series lie in $\mathcal{F}_4$ outside $\mathcal{F}_3$.)

which arises from unprojection of a degeneration. One may suspect that such Fano 3-folds are degenerations of the Pfaffian model, but this is not the case: quasismooth members of the two families have different Picard rank $h^{1,1}(X)$ and Euler characteristic e_X, so do not lie in a common flat family.

Different formats cover everything in codimension ≤ 3, much in codimension 4 (see [CD20, 5.3], where a cluster variety format describes certain subfamilies of deformations), and some in codimension 5. but although there are examples in higher codimension, they seem to realise only a small part of the classification there – see [QS11, 4.4, 5.2], where flag variety formats recover classical nonsingular Fano 3-folds but no other element of $\mathcal{F}_3$ has suitable Hilbert series, or [CD20, 5.8], where a codimension 6 format realises no Fano 3-folds. (Of course, such failures could be because there are few Fano 3-folds in high codimension – it seems unlikely, but we simply do not know.)

4.7. **Codimension 4: unprojection.** The unprojection Ansatz in §3.2.1 can sometimes be realised: given a coordinate plane $D = \mathbb{P}(a_i, a_j, a_k) \subset \mathbb{P}(a_0, \dots, a_n)$, one may be able to construct a Fano 3-fold X that contains D with X quasismooth away from finitely many nodes on D. In this model case, the Type I unprojection constructs a quasismooth Fano 3-fold $Y \subset \mathbb{P}(a_0, \dots, a_n, r)$ (the mild numerical conditions of §3.2 are only to exclude quasilinear equations in the ideal of Y).

This is carried out in [BKR12] in the case $X \subset w\mathbb{P}^6$ lies in codimension 3, so that $Y \subset w\mathbb{P}^7$ lies in codimension 4, with further cases in [BCD26]. Type II unprojections are considered in [Rei02, Pap08, Tay], which construct other cases in codimension 4. Using this and §4.6, for every Hilbert series of $\mathcal{F}_3$ whose GRDB model is in codimension 4, there is a construction of a variety that matches that model, and in the majority of cases there are 2 or more distinct deformation families.

4.8. **The Fanosearch programme.** Coates, Corti, Galkin, Golyshev, Kasprzyk [CCG+13] and others, following ideas of Golyshev [Gol07], provide an alternative approach to the Fano classification problem. The idea is that, via Mirror Symmetry, Fano classification can be rephrased as a fundamentally combinatorial problem of identifying suitable Laurent polynomials whose periods generate solutions to Picard–Fuchs equations on the other side of the mirror. Although the combinatorial problems seem hard, and the required mirror theorems are not wholly in place, [CCGK16] confirms that the two sides of the mirror agree in the smooth case, and in doing so provides a host of tools for constructing Fano varieties and passing through the mirror. The GRDB is a key tool: see e.g. [CHKP, Heu25, BCC+22].

Example 4.6. Let P be the Fano polytope whose spanning fan gives rise to $X_1 \in \mathcal{T}_{\mathrm{can}}$. Using the terminology of [CKPT21], P supports eleven rigid maximally mutable Laurent polynomials (rigid MMLPs), up to automorphisms of P. The periods of these rigid MMLPs give solutions to eleven distinct Picard–Fuchs equations; compare this with Example 4.4, where prima facie we see seven deformation families. By [CKPT21, Conjecture 5.1] we expect each of these eleven rigid MMLPs to correspond to a deformation of X_1 to a terminal locally toric Fano with $4 \times \frac{1}{2}(1,1,1)$ singularities. This expectation agrees with the output of Ilten's Macaulay2 package [Ilt12]. More generally, considering those $X \in \mathcal{T}_{\mathrm{can}}$ with Hilbert series equal to that of X_1 gives a total of 24 rigid MMLPs, up to mutation, corresponding to 24 distinct Picard–Fuchs equations and hence, conjecturally, at least 24 deformation families of terminal Fano 3-folds with basket $4 \times \frac{1}{2}(1,1,1)$.

5. Synopsis

5.1. **Review of guiding examples.** The Fano 3-fold database consists of two lists of Hilbert series $\mathcal{F}_3 \subset \mathcal{F}_4$ (defined in 2.9) together with the estimated weights $X \subset \mathbb{P}(a_0, \dots, a_n)$ that the GRDB assigns to each one (§3.1–3.2). It is intended as a first coarse approximation to the classification of Fano 3-folds. However, it is certainly nowhere near to a final classification. The following remarks and examples are intended as quick reminders to help avoid misunderstandings.

(1) Overview of the Fano 3-fold database:
   (a) We distinguish between Fano 3-folds (with canonical singularities; Definition 1.1) and the much smaller class of prime Fano 3-folds (outcomes of the Minimal Model Program; Definition 1.3).
   (b) The set $\mathcal{F}_3$ contains all rational functions that satisfy necessary conditions to be the Hilbert series of a Fano 3-fold X with $-K_X^3 \leq -3K_X c_2$. The geography of Figure 1 is of $\mathcal{F}_3$ only.
   (c) By [Kaw92], the inequality in (1b) is satisfied by semistable prime Fano 3-folds (§2). A larger set $\mathcal{F}_4 \supset \mathcal{F}_3$ allows in principle for some strictly non-semistable prime Fano 3-folds (2.7), although we do not know any examples of these.
   (d) A series $P \in \mathcal{F}_3$ may be realised by many deformation families (Figures 2, 3) or by none (4.1(2)): $\mathcal{F}_3$ does not count the number of deformation families. This is a basic part of the classification

problem, and it is fully understood only in the case of nonsingular Fano 3-folds and some specific cases with only $\frac{1}{2}(1,1,1)$ singularities.

(2) Existence and non existence:
   (a) Proven cases of prime Fano 3-folds are sparse: the nonsingular Fano 3-folds of Picard rank 1 (§4.1), Takagi's Gorenstein-index-2 classification (§4.3), and a range of cases in low anticanonical codimension (§4.6) or of high Fano index (§4.4).
   (b) There is no reason why the Hilbert series of more general Fano 3-folds should appear in $\mathcal{F}_3$ or $\mathcal{F}_4$, yet every Fano 3-fold we know does in fact appear on these lists.
   (c) Many locations in Figure 1 are realised by a Fano 3-fold but not by a prime Fano 3-fold.
   (d) We expect that many of the high codimension, lower genus Hilbert series are not realised by any Fano 3-fold. However, we only know one place in Figure 1 where this is proven: Karzhemanov's nonexistence result for genus 35 (§4.2).

(3) The estimated anticanonical embedding $X \subset \mathbb{P}(a_0, \dots, a_n)$:
   (a) The Hilbert series P_X of a Fano 3-fold X does not determine the weights $a_0, \dots, a_n$ of its anticanonical embedding (4.5).
   (b) The embedding weights given to each $P \in \mathcal{F}_4$ in the GRDB are only a suggestion. They are derived from known examples in low codimension (§3.1.1), an analysis of conjectured Gorenstein projections (§3.2) and, in harder cases, an analysis of singularities or the linear systems on possible K3 sections. Although the weights are often right, there is no reason why your X should be embedded in this way.
   (c) Even if the general member of a deformation family is $X \subset \mathbb{P}(a_0, \dots, a_n)$, there are likely to be degenerations in higher codimension (§3.1.1).
   (d) It can happen that $P \in \mathcal{F}_3$ is realised by two or more distinct deformation families whose general members embed in different codimensions (4.5).
   (e) Some higher index Fano 3-folds lie in higher codimension than GRDB suggests (4.3).

5.2. **Nonexistence and other challenges.** The GRDB is simply one way of assembling and presenting the vast amount of data associated to the classification of Fano 3-folds, and as such it naturally invites more questions than it answers. A selection of topics:

(1) Find $P \in \mathcal{F}_3$ not realised by a Fano 3-fold, or not realised by a prime Fano 3-fold.
(2) Can one show that each model for $P \in \mathcal{F}_3$ in codimension 5 or 6 is realised by a Fano 3-fold using similar birational methods as in codimension 4?
(3) There are toric Fano 3-folds in high codimension: can projection from these realise Fano 3-folds for sequences of Hilbert series?
(4) Fano 3-folds with $|-K_X|$ empty are rare. The GRDB has 264 semistable Hilbert series with linear coefficient zero (the left-hand column $g = -2$ of Figure 1). The first are Iano–Fletcher's example $X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$ and the three families in codimension 4, studied by [AR00, Pap08, Tay], though they remain to be fully understood. The next candidate lies in codimension 6, namely $X \subset \mathbb{P}(2,3,4^2,5^2,6^2,7^2)$, and has no Type I or Type II projection attack.
(5) The GRDB is constructed with Gorenstein projection in mind, and the Fano 3-fold database includes that data; click [BK09]. Sarkisov links provide another connection between Fano 3-folds, and projection is often the first step of a Sarkisov link. Classification attacks often exploit such links, [Tak89] for example, but does it even make sense to describe a web of Sarkisov links overlaying the GRDB? (cf. [BBK22].)
(6) The bounds of [Kaw92] work over any field k of characteristic 0, not necessarily algebraically closed. Thus, for example, the GRDB makes sense over $k = \mathbb{Q}$ (the Type I Ansatz 3.1 is defined over $\mathbb{Q}$, but that is not clear for more complicated unprojection constructions), and the generic fibres of relatively 3-dimensional Mori fibre spaces also have relative Hilbert series in the Fano 3-fold database. Compare the general conditions and classifications of [CFST16, CFST18].

5.3. **Closing repetition of the main warning.** It is easy to mistake the Fano 3-fold database for a classification of prime Fano 3-folds. It is **not** that classification: that classification does not yet exist. It is instead the classification of genus–basket pairs that satisfy certain conditions of geometric origin, or equivalently it is a list of the rational functions they determine by the plurigenus formula.

The confusion arises in part because each rational function is presented as though it is the Hilbert series of a Fano 3-fold $X \subset \mathbb{P}(a_0, \dots, a_n)$ embedded by its total anticanonical ring in weighted projective space with given weights. These weights are sometimes accurate, and in all known examples they appear at least as a subset of the generators, but they are not proven to be necessary and should be treated only as a convenient informed estimate. There is no claim that a Fano 3-fold exists with this data, nor that a particular one you may be considering is necessarily embedded anticanonically as indicated here. Moreover, a single Hilbert series may be realised by more than one family of varieties, and these multitudes may lie in different ambient weighted projective spaces.

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Codimension \ Genus	−2 ⋯ 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
1		1																																		
2		1	7																																	
3		3	10	23																																
4		8	55	50	54																															
5		32	162	307	182	135																														
6		76	584	921	927	422	207																													
7		231	1468	3083	2705	1980	745	314																												
8		490	3598	6537	7282	4971	2954	1009	373																											
9		1025	6084	12835	13094	10904	6230	3372	1121	416																										
10		1493	9308	17594	20444	15942	11726	6365	3366	1142	413																									
11		1897	10196	21656	22519	19992	14512	10109	5439	3068	1046	413																								
12	7	1739	9926	17859	21825	18423	15498	10568	7540	4321	2498	904	348																							
13		1302	7133	13379	15659	14817	12117	10204	7041	5492	3292	2017	744	334																						
14		637	4727	7202	9911	9294	8810	7628	5982	4467	3816	2417	1529	622	274																					
15		262	2317	3599	4356	5580	5063	5694	3699	3602	2912	2535	1652	1272	486	234																				
16		72	989	1210	1600	2818	3139	3581	2291	2133	2188	1885	1672	1398	950	386	179																			
17		21	253	389	369	1763	1656	2235	1155	1335	1131	1274	1080	1491	953	657	274	151																		
18		3	51	80	64	751	895	1022	518	532	517	644	601	938	986	668	455	186	117																	
19		1	4	16	2	276	302	390	164	194	135	254	241	621	690	645	412	360	147	87																
20			1	1		68	81	98	36	47	33	53	78	285	557	441	364	300	225	91	66															
21						13	10	21	2	11	3	10	12	106	264	252	187	323	168	120	72	40														
22						1		2		1			1	20	119	85	96	199	185	103	110	32	42													
23														3	32	28	42	182	103	72	80	61	27	27												
24															9	1	13	81	68	29	68	29	27	11	18											
25															1		3	34	17	11	30	26	19	13	16	8										
26																		5	6	1	25	15	18	3	15	5	13									
27																			1		17	10	10	1	5	4	3	9								
28																					2	2	3		1	2	3	2	4							
29																												2	4	2						
30																												1	2	1	2					
31																														1	1	5				
32																														1			1			
33																																	1			
34																																				
35																																				2

Figure 7. Number of (canonical) toric Fano 3-folds that satisfy the semistable numerical condition, listed by genus and codimension.