Toric Varieties and Combinatorics

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ABSTRACT

Toric varieties are varieties that have an n-dimensional algebraic torus $((\mathbb{C}^*)^n)$ embedded in them with the action of the torus on itself extending to the entire variety. Combinatorial computations have made the study of toric varieties easier. In this report the construction of general toric varieties using lattices,toric ideals and semigroup is discussed and combinatorial computations like the divisor class groups, cohomology groups etc of toric varieties is briefly introduced. This helps one to understand their geometry. Resolution of singularities is also studied. The report concludes with the description of Betti Number, Poincare Polynomial, Euler Characteristic and singular homology and cohomology groups with a generalisation for toric varieties.

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I do not claim any originality in this report. I have based my work on the books,articles and internet sources mentioned in my References. Example 3.0.13 ,Example.9.3.2 .,are my work.

1 INTRODUCTION

Toric Varieties were first mentioned in 1970 in Demazure's Paper when he discussed about Cremona Groups. The notion of general toric varieties came into existence much later and thus this topic has been of immense interest to many. It has been used to prove the Ehrhart Reciprocity and the McMullen Conjecture.

The initial motivation was to study torus embeddings. But slowly the researcher realised that by choosing a finite subset S of a character lattice and then producing maps to $\mathbb{C}$ he could genererate a map to $\mathbb{C}^{|S|}$ .The Zariski closure of this map then gives a Toric Variety . Thus, studying toric varieties proved to be easier and so theorems and results were tested on toric varieities before using them in general.

However in order to work with these varieties, he has to develop some technical machinery including some terms and concepts . The main purpose of this report is to introduce one to the basic tools of Toric Geometry and to understand the construction of Toric Varieties. Combinatorial computations such as Homology group are briefly discussed.

The breakdown of the contents of the chapter is as given below:
Chapter 1 deals with Affine and Projective Varieties. Basic concepts related to Affine Varities such as definition of a variety, coordinate ring and its localisation, Zariski Topology is discussed. Normalisation of Affine Varieties is also explained . This is a crucial concept in Toric Geometry since toric varieties in general are normal.
Next, Projective space, homogeneous ideals and weighted projective spaces are briefly introduced. The section ends with a description of the procedure to find the dimension of a projective variety.

In Chapter 2, concepts in Toric Geometry is studied by first describing the various features of a cone and a polytope. Strongly Convex Polyhedral Cones and their minimal generators form the next section in the chapter. The researcher next moves on to Rational subspaces and semigroups associated with rational polyhedral cones. Slowly, the components of an affine toric variety such as a torus, one-parameter subgroup, character group are defined . Construction of affine toric varieties using lattices, toric ideals and using semigroups is explicitly explained. The properties of an affine toric variety such as the torus embedding, the relation between saturation and normality,toric morphisms is dealt with in detail.

Chapter 3 introduces sheaves and abstract varieties. A brief discussion of the properties of varieties such as seperatedness, quasi-compactness, completeness features in this chapter. The last section deals with constructing varieties by gluing affine varieties.

Chapter 4 relates to the construction of general toric varieties . Fans and associating toric varieties to it is explained with examples. One can also give a correspodence between the cones in a fan and the orbit of the torus action in the toric variety associated with the fan. The chapter ends with a brief discussion of proper morphisms.

In Chapter 5, singularities in toric varieties is described and the resolution of the same using various methods such as normalisation,blow-up is discussed. Blowing up is one of the most useful methods to resolve singularities and this is explained in simple terms with an example.

The first glimpse of combinatorial computations pertaining to toric varieties appears in Chapter 6 wherein Prime Divisors, Weil and Cartier Divisors are described.The Divisor Class group and the Picard group for some examples are computed as well.

Chapter 7 deals with Line bundles including examples such as the Tautological bundle and intersection numbers of these bundles and divisors are computed. Sheaves are again studied with the notion of a divisor defining the sheaf .

Chapter 8 is related to Sheaf Cohomology and Cech Cohomology .The construction, properties and computations of the same are explained in detail. The chapter ends with the concept of mixed volume wherein number of common zeros to polynomials is proved using polytopes.

Lattice polytopes are discussed in Chapter 9 . Here the number of lattice points in a polytope are calculated using Ehrhart Theorem for applications like in Pick's Theorem.This also has a Toric Reformulation, which is also briefly discussed.

The report concludes in Chapter 10 with the dicussion of Betti Number, Poincare Polyomial and Euler Characteristic, which describes the geometric nature of the surface in hand.

2 AFFINE AND PROJECTIVE VARIETIES

2.1 Affine Varieties

Definition 2.1.1. [**34**]A ring R is said to be a **Noetherian Ring** if any of the following equivalent conditions are satisfied:

(a) Every ideal in the ring R is finitely generated.

(b) The Ascending Chain Condition for ideals is satisfied.i.e. whenever ideals in R satisfy

$$I_1 \subseteq I_2 \subseteq .... \subseteq I_i \subseteq I_{i+1} \subseteq ....$$

there exists a $k \in \mathbb{N}$ such that $I_k = I_{k+i}$ for all $i \in \mathbb{N}$.

(c) Every nonempty set of ideals of R has a maximal element with respect to set inclusion.

Definition 2.1.2. Given$f_1, f_2, ...f_n$ belongin g to $\mathbb{C}[x_1, x_2, ...x_n]$. Collect the set of points at which all these polynomials vanish . This collection of points is said to be an **Affine Variety** and is denoted by $\mathbb{V}(f_1, f_2, ...f_n)$.

Mathematically it can be written as

$\mathbb{V}(f_1, ...f_s) = \{a \in \mathbb{C}^n | f_1(a) = ... = f_s(a) = 0\}$

Also, given an ideal I in $\mathbb{C}[x_1, x_2, ...x_n]$,one can define a variety in terms of this ideal I. Namely,

$$\mathbb{V}(I) = \left\{a \in \mathbb{C}^n | f(a) = 0 \text{ for all} f \in I\right\}$$

Suppose $I =< f_1, f_2, ...f_n >$ then $\mathbb{V}(I) = \mathbb{V}(f_1, f_2, ...f_n)$ and by Hilbert‘s Basis Theorem, every ideal in a Noetherian Ring is finitely generated and hence every variety can be expressed in the above form when working on a Noetherian Ring.

Conversely, given a variety one can define an ideal consisting of the polynomials that vanish over the points of the variety.i.e.

$$\mathbb{I}(V) = \left\{ f \in \mathbb{C}[x_1, ...x_n] | f(a) = 0 \text{ for all } a \in \mathbb{V} \right\}$$

This correspondence helps us to move back and forth from varieties in $\mathbb{C}^n$ and ideals in $\mathbb{C}[x_1, x_2, ..., x_n]$.This can be made more precise using one of the most important theorem namely, the Hilbert‘s Nullstellensatz Theorem.

Theorem 2.1.3. (Hilbert's Nullstellensatz Theorem) [33]

Given an ideal $I \in \mathbb{C}[x_1, x_2, .x_n]$,$\sqrt{I} = \mathbb{I}(V(I))$ where $\sqrt{I}$ is the radical ideal of I .

Remark. This Theorem helps in obtaining some correspondences which are crucial to move from algebra to geometry and are as below:

(a)There is a one to one correspondence between maximal ideals of $\mathbb{C}[x_1, x_2, ...x_n]$ and points of the variety in $\mathbb{C}^n$ since every maximal ideal in$\mathbb{C}[x_1, x_2, ..., x_n]$ is of the form $\langle x_1 - a_1, x_2 - a_2, ..x_n - a_n \rangle$.

(b)Affine varieties in $\mathbb{C}^n$ can be identified with radical ideals in $\mathbb{C}[x_1, x_2, ...x_n]$

Definition 2.1.4. Coordinate Rings:

Let V be a Variety in $\mathbb{C}^n$ and let $\mathbb{I}(V)$ be the ideal of polynomials that vanish over points in V. The Coordinate ring of the variety is the quotient of polynomials in $\mathbb{C}[x_1, x_2, ..x_n]$ by the polynomials in this ideal.i.e.

$$\mathbb{C}[V] = \mathbb{C}[x_1, x_2, .x_n]/\mathbb{I}(V).$$

This means that two polynomials would be equal on the variety iff their difference is in the ideal.

These coordinate rings play a very major role in the classification of varieties since two affine varieties are isomorphic iff their respective coordinate rings are isomorphic $\mathbb{C} - Algebras$.

An important characterisation of a coordinate ring is the following:

Definition 2.1.5. Spectrum of Rings:

The Spectrum of a Ring A ,(Spec (A)) is the set of all prime ideals of A. Eg: $Spec(\mathbb{Z}) = \{(0), (2), (3)......\}$

We can define a relation between varieties and their coordinate rings as follows: $V = Spec(\mathbb{C}[V])$ where V is a variety in $\mathbb{C}^n$ and $\mathbb{C}[V]$ is the coordinate ring of the variety in $\mathbb{C}[x_1, x_2, ...x_n]$ This is called the affine scheme of the coordinate ring $\mathbb{C}[V]$.

Definition 2.1.6. Topology on Affine Varieties

There are two interesting topologies that can be defined on Affine varieties:

One of the topologies is the Classical Topology. As affine varieties are subsets of $\mathbb{C}^n$, they get the induced topology from $\mathbb{C}^n$ and this is known as the Classical Topology.

Else, these affine varieties can be given the Zariski Topology.Here, if W subset of V, is a subvariety, then V-W ,which consists of points in the variety V that do not vanish at the polynomials defining W, form the Zariski Open Subsets of the variety V . These together form the Zariski Topology.

Definition 2.1.7. Also, given a subset S subset of V, the **Zariski Closure** of S, denoted by $\bar{S}$, is the smallest subvariety of V containing S.

This is obtained by first guessing the zariski closure,say Z, and then checking if a polynomial vanishing on S also vanishes on Z. One can first calculate the vanishing ideal on S and thus, $\mathbb{V}(\mathbb{I}[S]) = Zrcl(S)$

Remark. Some Zariski Open Subsets of an affine variety V are themselves affine varieties.

For example,Given $f \in \mathbb{C}[V] - \{0\}$,

$$V_f = \{a \in V | f(a) \neq 0\} \subset V$$

Definition 2.1.8. An affine variety **V** in $\mathbb{C}^n$ is said to be an **irreducible variety** if it cannot be written as a union of non-trivial subvarieties i.e. $V = V_1 \cup V_2$ where $V_i \neq V$.

Definition 2.1.9. Given an irreducible variety **V**,one can define a **field of rational functions** $\mathbb{C}(V)$ on the integral domain $\mathbb{C}[V]$. This is similar to a field of fractions that can be defined on any integral domain R. So here, consider an element f/g in $\mathbb{C}(V)$. In order for this element to be well defined , g should not vanish on the set on which this element is defined . $g = 0$ defines a subvariety W of the variety**V** and thus, defining f/g on V-W ensures it is a well-defined function. This is called a rational function.

Eg. V=$\mathbb{C}^n$, $\mathbb{C}[V] = \mathbb{C}[x_1, x_2, ..., x_n]$; then $\mathbb{C}(V) = \mathbb{C}(x_1, x_2, ..., x_n)$

Definition 2.1.10. Given an irreducible variety **V** and $f \in \mathbb{C}[V] \neq 0$, then **the localisation of the coordinate ring** $\mathbb{C}[V]$ at f, is

$$\mathbb{C}[V]_f = \{g/(f^n) \in \mathbb{C}(V) | g \in \mathbb{C}[V], n \geqslant 0\}$$

Remark. $\mathbb{V}_f$=Spec($\mathbb{C}(V_f)$)

An important fact about affine varieties is that : Every affine variety **V** can be written as a union of irreducible varieties i.e.

$\mathbf{V} = V_1 \cup V_2 \cup V_3 .... \cup V_r$

where each V_i is irreducible and V_i is not contained in the union of the other irreducible varieties.

Definition 2.1.11. Let R be an integral domain and let K be its field fractions. Then, R is said to be **integrally closed** if,every element in K that is integral in over R, actually belongs to R. An element a in K is said to be integral over R if a is the root of a monic polynomial in R[x].

Example 2.1.12. Any Unique Factorisation Domain is integrally closed.

Definition 2.1.13. A variety V is said to be **a normal variety** if its coordinate ring $\mathbb{C}[V]$ is integrally closed.

Example 2.1.14. $\mathbb{C}^n$ is normal because

$$\mathbb{C}[\mathbb{C}^n] = \mathbb{C}[x_1, x_2, .x_n]$$

is a UFD and hence is integrally closed.

Definition 2.1.15. (**Normalisation of Affine Varieties**): Given an irreducible affine variety V. Consider the coordinate ring of V,$\mathbb{C}[V]$,

Define $\mathbb{C}[V]' = \{a \in \mathbb{C}(V) | a \text{ is integral over } \mathbb{C}[V]\}$

This is called the integral closure of $\mathbb{C}[V]$. Thus V'=Spec($\mathbb{C}[V]$) is the normalisation of the variety V.

Definition 2.1.16. [**19**] There are two equivalent ways of defining the **dimension of a variety**:
(1) The transcendence degree of $\mathbb{C}$(X) , which is the maximal number of algebraically independent elements of $\mathbb{C}$(X).
(2) Maximum number of distinct irreducible subvarieties such that

$$\emptyset \neq V_0 \subset V_1 \subset .... \subset V_n = X.$$

Example 2.1.17. $\mathbb{C}^n; \mathbb{C}^*$; $\mathbb{P}^n$ have dimension n.

Definition 2.1.18. The **Dimension of a variety X at a point** $p \in$ X, denoted by $\dim_p X$, is defined as the maximum of the dimensions of the irreducible components of X that contain the point p.

2.2 Projective Varieties

Definition 2.2.1. Consider the space $\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\sim$ where $\sim$ is the equivalence relation defined as:

$$(x_1, x_2....., x_n) \sim (y_1, y_2, ...y_n)$$

$$\text{iff there exists } \lambda \in \mathbb{C}^* \text{ such that}$$

$$(y_1, y_2, ...y_n) = (\lambda x_1, \lambda x_2, ...\lambda x_n).$$

This space is called the **Projective Space**.

As we vary λ, we get different points on the line through the origin. For every point in $\mathbb{P}^n$, there is a corresponding line passing through the origin . Thus, we get a bijection : $\mathbb{P}^n \simeq \{$ lines through origin in $\mathbb{C}^{n+1}\}$.

Definition 2.2.2. Any point in this projective space is written as $(x_0 : x_1 : ..... : x_n)$ and these are called the **homogeneous coordinates** of a point $p \in \mathbb{P}^n$.

Definition 2.2.3. A **homogeneous polynomial** $f \in \mathbb{C}[x_1, x_2, ..x_n]$ is one whose terms are of the same total degree d.

Remark. Any Polynomial can be written uniquely as the sum of homogenous components.

In order for polynomials to be well-defined on the Projective Space, they have to be homogenous since the points along a line through origin are identified and the polynomials should hence take the same value.

Thus the polynomials which define a projective variety should be homogeneous.

Definition 2.2.4. An ideal I $\subset \mathbb{C}[x_0, ..., .x_n]$, is a **homogeneous ideal** iff for all polynomials $f \in \mathbb{C}[x_0, ..., x_n]$, $f \in$ I iff all the homogenous components of f are present in I.

So if given a homogenous ideal I, one can define a projective variety $\mathbf{V}$(I) and conversely given a projective variety one can define a homogeneous ideal.

In many instances, Projective Geometry is better than Affine Geometry as is seen in the next example.

Example 2.2.5. Given V $\subset \mathbb{C}^n$ and a polynomial map F:V $\to \mathbb{C}^m$. Then $F(V)$ need not be a subvariety of $\mathbb{C}^m$. For example: let V-$\mathbb{V}(xy - 1)$ in $\mathbb{C}^2$. Let F be defined as $F(x, y) = x$. Here $F(V)$ is not a subvariety of $\mathbb{C}$. But in projective space, $F(V)$ is a subvariety.

Definition 2.2.6. Projective Space $\mathbb{P}^n$ can be described as the union of affine spaces. Consider Zariski Open sets $U_i = \mathbb{P}^n - \mathbb{V}(x_i)$. These open sets are called **affine coordinate patches.** Each of these coordinate patches are isomorphic to $\mathbb{C}^n$ and $\mathbb{V}(x_i) \simeq \mathbb{P}^{n-1}$. Thus the projective space is the union of $\mathbb{C}^n$ and $\mathbb{P}^{n-1}$ 'at infinity'.

Definition 2.2.7. Let $q_0, q_1, ...q_n$ belong to $\mathbb{Z}_{\geq 0}$ such that $\gcd(q_0, ..q_n) = 1$. **Weighted projective Space** is defined as

$$\mathbb{P}(q_0, q_1, ..q_n) = (\mathbb{C}^{n+1} - \{0\})/\sim,$$

where $\sim$ is defined as the equivalence relation

$$(x_0, x_1, .x_n) \sim (y_0, ...y_n) \text{ iff } (y_0, y_1, ..y_n) = (\lambda^{q_0}.x_0, ...\lambda^{q_n}.x_n).$$

These integers $q_0, ..., q_n$ are called the **weights** of the projective space.

For the polynomial $f \in \mathbb{C}[x_0, ...x_n]$, x_i is said to have weight q_i and the polynomial f is said to be **weighted homogeneous of degree d** if $f(\lambda_o^q x_0, .., \lambda_n^q.x_n) = \lambda^d f(x_0, x_n)$ This is a way to generalise the projective space $\mathbb{P}^n$.

Example 2.2.8. Working with $\mathbb{P}(1, 2, 3)$ where F: $\mathbb{P}(1, 2, 3) \to \mathbb{P}^3$. Then $x_0^6, x_1^3, x_2^2, x_0x_1x_2, (x_0.x_1)^2$ are homogenous of degree 6.

$$(x_0, x_1, x_2) \to (x_0^6, x_1^3, x_2^2, x_0x_1x_2)$$

is a well defined map.

Procedure to find the dimension of a projective variety[25]: Consider a surjective morphism f:$X \to Y$ and here dim X $\geq$dim Y. Similarly if there are only finite fibres , that is, for all $P \in Y$,$f^{-1}(P)$ is finite , then dim X $\leq$ dim Y. Hence if there is a surjective morphism with finite fibres, then Dim X= Dim Y. Using this , one can determine the dimension of a projective variety as given below.

Consider a projective subvariety $X \subsetneq \mathbb{P}^n$ and a point $P \in \mathbb{P}^n$ but not in X. Let H $\simeq \mathbb{P}^{n-1}$ be a linear subspace that does not contain P. Assume $P = (0 : 0 : .....0 : 1)$ and H=$\{x_n = 0\}$.

Let $\pi : X \rightarrow H$ be a **projection map** such that a point Q $\in$ X is mapped to $\pi(X)$ given by the point of intersection of the line $\bar{PQ}$ with H. This satisfies the properties required by a morphism defined to determine the dimension of X as explained below.

(1) The fibres of this projection should be finite since $\pi^{-1}(Q)$ for a point Q $\in$ H should lie on the line but should not contain P.

(2) Also, we can repeat this process if the projection to $\mathbb{P}^{n-1}$ is not surjective ,by projecting it from $\mathbb{P}^{n-1}$ to $\mathbb{P}^{n-2}$ and so on, till it is surjective.

So once surjectivity is reached, dimension of X can be determined.

3 AFFINE TORIC VARIETIES

The following defintions help one to understand what Affine Toric Varieties are and how they can be constructed.

Definition 3.0.1. Let S $\subset \mathbb{R}^n$ be finite.

A **Convex Polyhedral Cone(CPC)** in $\mathbb{R}^n$ is defined as below:

$$\sigma = \text{Cone(S)} = \left\{\sum_{v \in S} \lambda_v v | \lambda_v \geq 0\right\} \subset \mathbb{R}^n.$$

Here, we say σ is generated by the set S. Also, Cone$\{\emptyset\}$=$\{0\}$.

More About CPCs

If σ is a CPC, then

(a) it is **convex**, that is given x,y $\in \sigma$, then $\lambda x + (1 - \lambda)x \in \sigma$.

(b) it is a **cone** and is thus closed with respect to scalar multiplication. so if $x \in \sigma, \lambda x \in \sigma$.

(c) It is a **closed** subset of $\mathbb{R}^n$.

Some common examples of CPC are the first quadrant of $\mathbb{R}^2$, first octant in $\mathbb{R}^3$. The largest possible CPC is $\mathbb{R}^n$ and the smallest is $\{0\}$.

One can also create Cones from another structure called the Polytopes and they are defined as below:

Definition 3.0.2. A **Polytope P** in $\mathbb{R}^n$ can be defined as :

P=Conv(S)= $\left\{\sum_{v \in S} \lambda_v v | \lambda_v \geq 0 \sum_{v \in S} \lambda_v = 1\right\} \subset \mathbb{R}^n$.

where S $\subset \mathbb{R}^n$ is finite. P is also called **the convex hull** of S.

Remark. A polytope can be lifted to form a Convex Polyhedral Cone by lifting it along one direction. Thus a polytope in $\mathbb{R}^n$ can be lifted to form a Convex Polyhedral Cone in $\mathbb{R}^{n+1}$.

Definition 3.0.3. Dimension of a Convex Polydral cone σ is the dimension of the smallest subspace W containing σ.
W is called the Span of σ.

Theorem 3.0.4. (**CARATHEODORYS THEOREM**) **[17]** *Let σ be a CPC, σ= cone (S). and let $x \in \sigma$. Then, there exists a linearly independent subset T of S such that $x \in$ Cone(T).*

Definition 3.0.5. Convex Polyhedral Cones generated by linearly independent sets are called **Simplicial Cones**.

Remark. Any convex Polyhedral Cone is a finite union of simplicial cones and since these simplicial cones are closed, a CPC being a finite union of closed sets, is thus a closet subset of $\mathbb{R}^n$.

Definition 3.0.6. Suppose σ is a Convex Polyhedral Cone in $\mathbb{R}^n$, its **dual cone** σ^v, is the set defined as follows:

$$\sigma^v = \left\{ u \in \mathbb{R}^{n*} | < u, v > \geq 0 \text{ for all v} \in \sigma \ \right\}$$

Definition 3.0.7. Hyperplane is a surface whose dimension is one less than the ambient space.

Roughly, **half space** is a region of the space that is cut out by a hyperplane.

Example 3.0.8. $\mathbb{R}^n$ where the coordinates are $(x_1, x_2, ..., x_n)$, the hyperplane x_1=3 divides the space into 2 half spaces: $x_1 > 3$ and $x_1 < 3$.

As explained in **[1]** given a linear functional $u \neq 0 \in \mathbb{R}^{*n}$, one can define half spaces with the help of a hyperplane as folows:
Thus, the linear functional is u:$\mathbb{R}^n \rightarrow \mathbb{R}$ .Here the hyperplane is given by Ker(u).

$$H_u = \left\{ v \in \mathbb{R}^n | < u, v >= 0 \right\}$$

This defines the closed half spaces as below:

$$H_u^+ = \left\{ v \in \mathbb{R}^n | < u, v > \geq 0 \right\}$$

$$H_u^- = \left\{ v \in \mathbb{R}^n | < u, v > \leq 0 \right\}$$

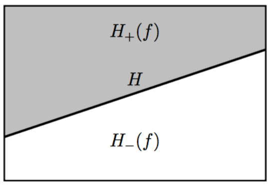

Figure 1: Two halfspaces determined by the hyperplane [**1**]

Definition 3.0.9. The set $\tau = H_u \cap \sigma$, that is elements of $\mathbb{R}^n$ that can be written as a linear combination of elements of S , at which the linear functional u in $\mathbb{R}^{n*}$ vanishes, is called a **face of** σ. σ is considered to be a face of itself. Thus $\tau \subsetneq \sigma$ is called a **proper face** of σ.

One can determine all the different faces of a convex polyhedra cone using $H_u \cap \sigma$,allowing u to vary over nonzero elements of the dual cone σ^v since $\sigma \subset H_u^+$ iff $u \in \sigma^v$.

Since any convex polyhedral cone is the finite union of simplicial cones , they have only finite number of faces.

Lemma 3.0.10. [**17**]

Let σ=Cone(S) be a convex polyhedral cone.

(a)Every face of σ is a convex polyhedral cone.

(b)The intersection of two faces of σ is again a face of sigma.

(c)A face of a face of σ is a face of σ.

Theorem 3.0.11. [**17**]*Given an n-dimensional convex polyhedral cone σ, such that , $\sigma \neq \mathbb{R}^n$. Let the faces $\tau_i = H_{u_i} \cap \sigma$, where $\sigma \in H_u^+$ for i=1,.....s. Then, $\sigma = H_{u_1}^+ \cap .... \cap H_{u_s}^+$ that is, σ is a finite intersection of closed half spaces.*

Proposition 3.0.12. *Let σ in $\mathbb{R}^n$ be a polyhedral cone. Then if $\sigma = H_{u_1}^+ \cap .... \cap H_{u_s}^+$, then*

$$\sigma^v = Cone(u_1, ...u_s)$$

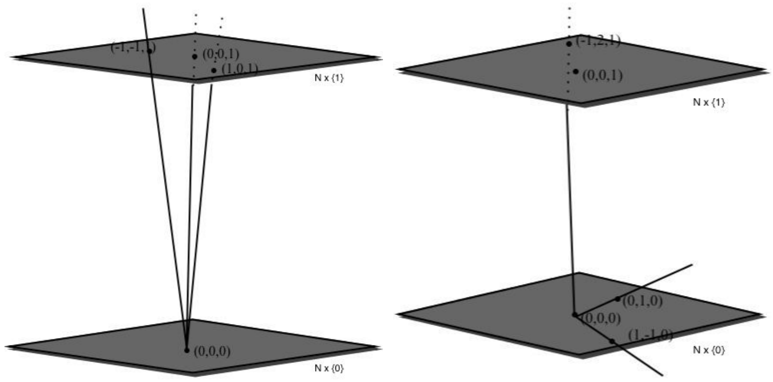

Figure 2: Cone σ and its dual σ^v generated by the inward normals to the faces of σ

Example 3.0.13. Let N=$\mathbb{Z}^2$; $P = \{0, e_1, -e_1 - e_2\} \subseteq N \otimes_{\mathbb{Z}} \mathbb{R} = N_{\mathbb{R}} \cong \mathbb{R}^2$; $N' = N \times \mathbb{Z}$ and thus $N'_{\mathbb{R}} \cong \mathbb{R}^3$. Let

$$i : N \to N'$$

$$i(v) = \{v, 1\}$$

Therefore, i(N)= N $\times\{1\} \subseteq N'$.

Thus, if cone σ=Cone(P) , the dual cone $\sigma^v \subseteq M_{\mathbb{R}}$where M=Hom(N,$\mathbb{Z}$)$\cong$ $\mathbb{Z}^2$ is generated by the inner normals to the faces of the cone. Computing the normals of the faces, one gets σ^v=Cone((1,-1,0); (-1,2,1); (0,1,0)) . The cone and the dual are as illustrated in Figure 2.

Example 3.0.14. [**22**] Consider the cone σ=Cone$(e_1, e_2, e_1 + e_3, e_2 + e_3)$. The Generators of this dual can be computed to be :
σ^v=Cone$(e_1, e_2, e_3, e_1 + e_2 - e_3)$. These are as in the Figure 3.

Definition 3.0.15. Given a convex polyhedral cone σ. τ is called a **facet** of σ if τ is a face of codimension 1.

$$\text{codim}\tau = \text{dim } \sigma \text{ -dim } \tau = 1, \text{ this implies dim}\tau = \text{dim}\sigma \text{ -1}$$

Every proper face $\tau \subset \sigma$, (where σ is as defined in Proposition 2.1.11) is the intersection of the facets of σ containing τ .

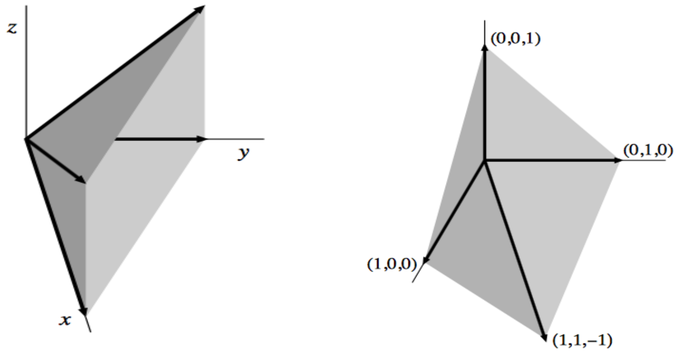

Figure 3: Cone and its dual [**22**]

Theorem 3.0.16. (**FARKAS THEOREM**)[**17**]

If σ is a convex polyhedral Cone, then so is it dual σ^v .

Proof. If $\sigma = \mathbb{R}^n$, then $\sigma^v = \{0\}$ which is a convex poyhedral Cone. Thus, let us assume $\sigma \neq \mathbb{R}^n$. In this case, according to the theorem, σ is a finite intersection of closed half spaces, that is, $\sigma = \cap H^+_{u_i}$ for some $u_i \in \sigma^v$. Then, $(\text{Cone}(u_1, u_2, ..., u_t))^v = \sigma = \cap H^+_{u_i}$ (From the proposition). Thus σ^v= $(\text{Cone}(u_1, ..., u_t))^v)^v = \text{Cone}(u_1, ..., u_t)$ and hence σ^v is a convex polyhedral cone. □

Remark. A finite intersection of closed half spaces is a convex polyhedral cone.

Thus, there are two different and equivalent ways to define a convex polyhedral cone:
(i)they are nonnegative linear combinations of finitely many generators belonging to a set in $\mathbb{R}^n$ (or)
(ii)they are the intersection of finitely many closed half spaces.

Remark. These normals are called **facet normals** and from Example 3.0.13 and 3.0.14, one can observe that the facet normals of σ form the generators of its dual while that of the dual generate σ.

Definition 3.0.17. Given a face τ of a convex polyhedral cone σ, one can define a **Dual Face** of τ as follows:

$$\tau^* = \left\{u \in \sigma^v | < u, v >= 0 \text{ for all } v \in \tau\right\}$$

This can also be defined using the following set:

Define

$$\tau^{\perp} = \{u \in \mathbb{R}^{n*} | < u, v >= 0 \text{ for all } v \in \tau\}$$

Thus τ^*=$\sigma^v \cap \tau^{\perp}$.

Proposition 3.0.18. [17]

Let τ be a face of the convex polyhedral cone σ and let τ^*=$\sigma^v \cap \tau^{\perp}$.

(a) τ^* is a face of σ^v.

(b) the map from τ to τ^* is one-one, onto,inclusion-reversing correspondence between the faces of σ and the faces of σ^v.

An interesting description of the smallest face of σ^v is the following:

the smallest face of σ^v is $\sigma^v \cap (-\sigma^v)$. Since the map from τ to τ^* is inclusion-reversing, if σ is the largest face of σ , then σ^* is the smallest face of σ^v where $\sigma^* = \sigma^v \cap \sigma^{\perp} = \sigma^{\perp}$

Thus, $\sigma^* = \sigma^{\perp} = \sigma^v \cap (-\sigma^v)$. This implies that the two descriptions are equivalent.

3.1 Strongly Convex Polyhedral Cones

Definition 3.1.1. A convex polyhedral cone σ is said to be **strongly convex** if $\sigma \cap (-\sigma) = \{0\}$.

Remark. Simplicial Cones are strongly convex.

Next , in order to understand a cone one needs to determine the generators of the cone and this can be done using edges ,which are defined below.

Definition 3.1.2. An **edge (or ray)** of a convex polyhedral cone is a face of the cone that is one dimensional. When the cone in hand is strongly convex, these edges can be used to give a set of generators,which are infact the **minimal generators**.

Constructing the basis using edges:[**17**]

Given σ to be the strongly convex polyhedral cone with edges $\rho_1, ..., \rho_s$. Choose vertices v_i from $\rho_i - \{0\}$. Let S=$\{v_1, ..., v_s\}$ Then,

(i) σ=Cone(S)

(ii) S is minimal ,that is, if T is any other generating set then for some λ ; $\{\lambda v_1, .\lambda v_s\} \subset$ T. S is called the minimal generating set of σ.

Remark. Let σ be a strongly convex polyhedral cone. Then , σ is simplical iff the minimal generating set has d elements.

Rational Subspaces:

Definition 3.1.3. A **Lattice** N is a Free Abelian group of finite rank i.e. N $\simeq \mathbb{Z}^n$. Given a lattice N, one can define a **dual lattice** to be M=$Hom_{\mathbb{N}}$(N,$\mathbb{Z}$). This also has rank r.

One can construct a vector space $N_{\mathbb{R}}$= N $\otimes_{\mathbb{Z}}\mathbb{R} \simeq \mathbb{R}^n$. The dual space can be identified as $M_{\mathbb{R}}$= M $\otimes_{\mathbb{Z}}\mathbb{R} \simeq \mathbb{R}^{n*}$.

A subspace W $\subset N_{\mathbb{R}}$, is said to be **a rational subspace** if W=$W_0 \otimes_{\mathbb{Z}} \mathbb{R}$ for a sublattice $W_0 \subset$N.

Definition 3.1.4. One can define a polyhedral cone over a vector space $N_{\mathbb{R}}$ obtained using the lattice N.

Let $\sigma \subset N_{\mathbb{R}}$. Then σ is said to be **rational polyhedral cone** if σ =Cone(S) where S $\subset$ N.

Strongly Convex Rational Polyhedral Cones(SCRPC) are widely used when studying toric varieties. Generators of these cones can be determined using their edges as mentioned before. In case of rational rays, say ρ, a unique element generates the semigroup $\rho \cap N$. Thus these generators give a Unique Minimal Generating Set for the cone σ.

Thus if $u_1, u_2, ...u_s \in M$ are the generators of σ^v, then $\sigma = H^+_{u_1} \cap ..... \cap H^+_{u_s}$ where the facet normals $u_i \in M$ are uniquely determined and this is a special feature of rational polyhedral cones.

While dealing with smooth toric varieties, the following plays an important role.

Definition 3.1.5. Consider the minimal generators of the Strongly Rational Polyhedral Cone σ. If these generators form a part of a $\mathbb{Z}$-Basis of N , then σ is said to be a **regular cone**.

Semigroup of a Rational Polyhedral Cone:

If σ is a rational polyhedral cone in $N_{\mathbb{R}}$. Then one can associate the following set with the cone: $S_\sigma = \sigma^v \cap M$. S_σ is a semigroup under addition .

Lemma 3.1.6. (**GORDON'S LEMMA**):[**17**]

Let σ be a rational polyhedral cone in $N_{\mathbb{R}}$. Then $S_\sigma = \sigma^v \cap M$ is a finitely generated semigroup.

Example 3.1.7. Consider the example of the cone and its dual stated in Example 2.1.13. The generators of the semigroup of the cone σ is given by S_σ can be given by the columns of the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Thus the semigroup is the $\mathbb{N}$-linear combinations of the columns of the matrix.

We next try to understand how the semigroup of a cone and that of a face of the cone are related.

Let σ be a rational polyhedral cone in $N_{\mathbb{R}}$ and let τ be a face of the cone. Then $\tau = H_u \cap \sigma$ for some u$\in S_\sigma$ and $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$. So thus, the semigroups of the cone and a face of a cone only differ by a linear change.

3.2 Components of an Affine Toric Variety

We begin to understand toric varieties by first studying affine toric varieties as these are the building blocks of general toric varieties.

Definition 3.2.1. [**22**]Consider the affine variety $(\mathbb{C}^*)^n$. This is a group under the operation of component wise multiplication. The affine variety that is isomorphic to $(\mathbb{C}^*)^n$ is called a **Torus** T.

The objects defined below play a major role in understanding the relation between toric varieties and their defining fans and also helps in recovering the fan from the toric variety.

Definition 3.2.2. One-parameter subgroup[22] of an algebraic torus T is a homomorphism $\lambda : \mathbb{C}^* \to T$. Given a point b=$(b_1, b_2, ..., b_n) \in \mathbb{Z}^n$, one can define a one-parameter subgroup as follows:

$$\lambda^b(t) = (t^{b_1}, t^{b_2}, ..., t^{b_3})$$

All One-parameter subgroups arise this way and thus the set of all such one parameter subgroups, that is, the group Hom$(\mathbb{C}^*, T)$ is a lattice and is isomorphic to $\mathbb{Z}^n$. The one-parameter subgroup associated with a lattice vector $v \in N$ is denoted by λ_v.

Definition 3.2.3. The homomorphism $\chi : T \to \mathbb{C}^*$ is called a **character**. Thus, for example a=$(a_1, a_2, ..., a_n) \in \mathbb{Z}^n$ gives a character as follows:

$$\chi^a(t_1, t_2, ..., t_n) = t_1^{a_1} t_2^{a_2} .... t_n^{a_n}$$

Every character arises this way and thus the **character group** of a torus T, Hom(T, $\mathbb{C}^*$) is isomorphic to $\mathbb{Z}^n$.

The natural identification is that , Hom$(\mathbb{C}^*, T)$=N and Hom(T, $\mathbb{C}^*$)=M. Thus given a toric variety , the lattices associated with the variety, are its One-parameter and Character Groups.

Since Hom$(\mathbb{C}^*, T)$=N and the tensor product N$\otimes_{\mathbb{Z}}\mathbb{C}^* \simeq T$,the torus T is referred to as T_N.

Definition 3.2.4. An irreducible affine variety V is called an **Affine Toric Variety** if it contains the torus T=$(\mathbb{C}^*)^n$ as a Zariski Open subset such that action of T on itself extends to an action of T on the complete variety as an algebraic action $T_N \times V \to V$.

Example 3.2.5. [**22**]$\mathbb{C}^n$ and $(\mathbb{C}^*)^n$ are affine toric varieties.

Example 3.2.6. Consider the variety V=$\mathbf{V}(x^3 - y^2) \in \mathbb{C}^2$. This has the torus

$$C \cap (\mathbb{C}^*) = \{(t^2, t^3)|t \in \mathbb{C}^*\} \simeq \mathbb{C}^*$$

Example 3.2.7. V=$\mathbf{V}(xy - zw) \in \mathbb{C}^4$ and the torus in the variety is

$$\mathrm{V}\cap(\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1})|t_i \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^3$$

3.3 Construction of Affine Toric Varieties

Affine Toric Varieties can be constructed using Lattices, Toric Ideals and Semigroups.

USING LATTICES[**22**] Given an affine toric variety T_N , it has lattices M of characters and N , of One-Parameters. Thus let A=$\{m_1, m_2, ..., m_s\}$ be a finite subset of elements of M. These give characters $\chi^{m_i} : T_N \to \mathbb{C}^*$. Consider the map:

$$\phi_A : T_N \to \mathbb{C}^s$$

$$\phi_A(t) = (\chi^{m_1}(t), \chi^{m_2}(t), ..., \chi^{m_s}(t))$$

Thus given a finite subset A $\subset M$, one can define such a map and the Zariski Closure of this image is defined to be the Affine Toric Variety Y_A.

USING TORIC IDEALS[**22**]Given an affine toric variety Y_A obtained as explained above where $Y_A \in \mathbb{C}^s = \mathbb{C}[x_1, ...x_s]$. The ideal corresponding to this variety can be described as follows:

Consider the map:

$$\bar{\phi_A} : \mathbb{Z}^n \to M$$

which sends the basis vectors $e_1, e_2, ...e_s$ to $m_1, ...m_s$

Let L be the Kernel of this map .This makes the following sequence exact.

$$0 \to L \to \mathbb{Z}^s \to M$$

Let l=$(l_1, l_2, ..., l_s)$ be an element of the Kernel. Since $e_1, e_2, ...e_s$ to $m_1, ...m_s$, this would imply $l_1e_1 + ... + l_se_s = l_1m_1 + ... + l_se_s$ but since l is in the kernel, $l_1m_1 + ... + l_sm_s$=0.

Consider l=$(l_1, l_2, ..., l_s)$ .Define $l_+ = \sum_{l>0} l_ie_i$ and $l_- = -\sum_{l<0} l_ie_i$. Thus, l=$l_+ - l_-$.

Consider the binomial :

$$x^{l_+} - x^{l_-} = \prod_{l_i>0} x_i^{l_i} - \prod_{l_i<0} x_i^{-l_i}$$

This vanishes on the image of ϕ_A and hence on Y_A since it is a Zariski closure of the image.
In fact, one can prove the following very strong statement,

Proposition 3.3.1. *The ideal associated with the affine toric variety Y_A is*

$$\mathbf{I}(Y_A) = < x^{l_+} - x^{l_-} | l \in L > = < x^\alpha - x^\beta | \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L >$$

Example 3.3.2. **[23]** Let the setA=2,1,1$\subset M$. Thus the map $\bar{\phi_A} : \mathbb{Z}^n \to M$ sends

(1,0,0)$\to$ 2
(0,1,0)$\to$ 1
(0,0,1)$\to$ 1

Thus if l$\in Ker(\bar{\phi_A})$,where l=(l_1, l_2, l_3) then, $2l_1 + l_2 + l_3$=0; Some possible elements in the Kernel are (1,-1,-1); (-1,2,0); (1,0,-2); (0,1,-1) etc.

Thus few elements of the toric ideal $I(Y_A)$ are $(x_1 - x_2x_3)$; $(x_2^2 - x_1)$; $(x_1 - x_3^2)$; $(x_2 - x_3)$ etc. In fact one can prove that the ideal

$$I(Y_A) =< x_2^2 - x_1, x_1 - x_3^2 >.$$

Definition 3.3.3. Let $L \subseteq \mathbb{Z}^s$
(1) $I_L =< x^\alpha - x^\beta | \alpha, \beta \in \mathbb{N}^s$ and $\alpha - \beta \in L >$ is called a **lattice ideal**.
(2) If the lattice ideal is prime, then it is called a **toric ideal**.

Since Toric Varieties are defined to be irreducible, the ideal of polynomials vanishing on the variety is prime and hence is a toric ideal as per the above definition.

One can prove that all the Toric Varieties arise from Toric Ideals and thus this proves to be another method of construction.

USING SEMIGROUPS: Using a semigroup one first needs to construct a $\mathbb{C}$-algebra and this would be the coordinate ring of an affine variety by construction . Then finding the spectrum of the $\mathbb{C}$-algebra gives the affine toric variety.

Step 1:**Constructing the Coordinate ring of an Affine Toric Variety**:

In order to construct the coordinate ring of an affine toric variety, consider the subsemigroup of the dual lattice M. One can associate a finitely generated $\mathbb{C}$-Algebra without nilpotent elements , and this is in turn isomorphic to the coordinate ring of an affine variety.

The General construction goes as follows:

Consider this coordinate ring $\mathbb{C}[S]$ as a vector space with S as basis, and for every element u in S , one can associate an element χ^u in the algebra $\mathbb{C}[S]$ .In this manner all the elements of the algebra are generated as linear combinations of the elements χ^u for all $u \in$ S with the product in $\mathbb{C}[S]$ determined by the addition in the exponents.

$$\chi^u . \chi^{u'} := \chi^{u+u'}$$

Thus the semigroup generators in S produce algebra generators for $\mathbb{C}[S]$. This algebra $\mathbb{C}[S]$ is called the **Semigroup Algebra**.

Example 3.3.4. Let the semigroup $S = \mathbb{N} = \mathbb{Z}_{\geq 0}$. The semigroup algebra associated with this is $\mathbb{C}[S] = \mathbb{C}[N] = \mathbb{C}[X]$, that is, the set of polynomials, in one indeterminate. This implies that $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[X, X^{-1}]$. These are referred to as the Laurent Polynomials in one indeterminate.

Since the basis elements of $\mathbb{C}[S]$ depend on S and since from Gordon's Lemma, S is finitely generated,$\mathbb{C}[S]$ is also finitely generated. Also, if S has no torsion elements, that is, there is no nonzero element of finite order, this follows for the semigroup algebra $\mathbb{C}[S]$ too. Hence as stated before, $\mathbb{C}[S]$ is indeed the coordinate ring of an irreducible affine variety V=Spec($\mathbb{C}[S]$).

Generally one can identify the elements of a variety V with the maximal ideals of the corresponding coordinate ring. Here, let us identify the points of the affine variety **V** with the algebra homomorphism between $\mathbb{C}$[V] and $\mathbb{C}$ as follows: Consider the semigroup S and the semialgebra $\mathbb{C}[S]$ associated with it: and let us make the canonical identification between
$Hom_{\mathbb{C}-alg}(\mathbb{C}[S], \mathbb{C}) = Hom_{sgp}(S, \mathbb{C})$,wherein if $x \in$ V, then x can be identified with a semigroup homomorphism $x : S \to \mathbb{C}$ where for a $u \in$ S, $x(u) \in \mathbb{C}$. On the other hand , function χ^u from $\mathbb{C}$[S] to $\mathbb{C}$ corresponding to u in S ,takes the value $x(u) \in C$. So for an $x \in$ V , we have a function between S and $\mathbb{C}$ and also a homomorphism between $\mathbb{C}[S]$ and $\mathbb{C}$, and their values agree.

Example 3.3.5. Let S be semigroup generated by 2 and 3, that is , let S=$\left\{2a + 3b | a, b \in \text{N}\right\}$and thus, $\mathbb{C}[S] = \left\{f = \sum_{i=0}^{<\infty} a_i X^i \in \mathbb{C}[X]; a_1 = 0\right\}$ has all terms that are linear combinations of X^2 and X^3 and vanishing linear terms.

Step 2: Constructing the affine toric variety from the coordinate ring

Given a cone σ and the semigroup S_σ and the associated semigroup algebra $\mathbb{C}[S_\sigma]$, one can define the following: U_σ=Spec($\mathbb{C}[S]$).This is said to be an **affine toric variety corresponding to the cone** σ. Here we use the aforementioned identification ,namely, we identify points in U_σ with homomorphism between S_σ and $\mathbb{C}$.

When dealing with homo between S_σ and $\mathbb{C}$, there are two possible homomorphisms:

(1) All points in S_σ are sent to the unit 1 in $\mathbb{C}$.Let such a function be x_0.

(2) Points that are invertible are sent to 1 and the rest to 0. This is x_σ.

Here this x_σ is considered to be the distinguished point of U_σ .

Laurent Polynomials and Monomial Algebras Consider the zero cone 0:={0} in $N_\mathbb{R}$, the dual of which is the whole of $M_\mathbb{R}$. The associated semigroup S_0 is M. If $(e_1, e_2......, e_n)$ is a basis for N, then $(e_1^*, ..., e_n^*)$ is the dual basis for M which in turn implies that $(\pm e_1^*, .. \pm e_n^*)$ generate M. Thus $A_0 = \mathbb{C}[M] = \mathbb{C}[X^1, X^{-1}, , ..X^n, X^{-n}] = \mathbb{C}[X^1, X^{-1}]$ and this is the algebra of Laurent Polynomials.

Consider a SCRPC σ in $N_\mathbb{R}$, then the semigroup S_σ is a subsemigroup of S_0 and hence the semigroup algebra A_σ is a subset of A_0.Thus, the generators of A_σ are **monomials** and such a subalgebra of $\mathbb{C}[X, X^{-1}]$ is called a **monomial algebra**.

Examples of Affine Toric Varieties

Example 3.3.6. Consider the zero cone σ . The Semigroup corresponding to a zero cone is $S_0 = M$ and $A_0 = \mathbb{C}[S_0] = \mathbb{C}[X, X^{-1}]$, the ring of all Laurent monomials. Consider the affine toric variety corresponding to σ, the zero cone, this is $U_0 = Hom_\mathbb{Z}(M, \mathbb{C}^*) \simeq (C^*)^n$.

This proves that the affine toric variety associated with a zero cone is automatically endowed with the structure of an abelian group, called the **n-dimensional affine complex algebraic torus** which is the algebraic torus defined earlier.

Example 3.3.7. Consider the cone σ spanned by the basis vectors $e_1, e_2, ...e_d$ and let these vectors belong to the set of basis vectors of lattice N. Thus this cone, σ, is regular. Thus the basis vectors of σ^v are $e_1, e_2, ..., e_d, e_{d+1}, \pm e_{d+2}, ... \pm e_n$. Thus these would be the basis vectors of the semigroup S_σ , and thus the coorinate ring, $A_\sigma = [X_1, X_2, ...X_d, X_{d+1}^{\pm 1}, ...X_n^{\pm 1}]$ and thus the variety $U_\sigma = \mathbb{C}^d X(\mathbb{C}^*)^{n-d}$. , the product of affine d-space and an (n-d) algebriac torus.

3.4 Properties of an Affine Toric Variety

•**TORUS ACTION AND TORUS EMBEDDING**:As explained below Example.3.3.4, an n-dimensional affine torus T_N, can be identified with a homomorphism from M $\rightarrow \mathbb{C}^*$. And similarly, points of an affine toric variety that correspond to a SCRPC , is associated with a homomorphism from $S_\sigma \rightarrow$ C. Thus one can restrict the action of elements of the algebraic torus, T_N, to just U_σ, and this in turn gives an action of the torus on affine toric varieties.
Thus,t.x: $S_\sigma \rightarrow$C, acting as , $t.x(u) = t(u).x(u)$, is a semigroup homomorphism and is thus an element of U_σ. Thus this action is an equivariant action and this can be used to give an embedding of the affine torus into the affine toric variety.

Consider the distinguished point x_0 in U_σ and its orbit under the action of T_N. Thus $t.x_0$:$S_\sigma \rightarrow$ C takes a point u in S_σ to t(u) since $x_0(u)$=1 for all u in S_σ. Thus this results in the torus being equivariantly embedded into the affine toric variety U_σ.

•**EQUIVARIANT EMBEDDINGS AND FACES**:This equivariant embedding of the torus in any affine toric variety by the orbit of the distinguished point x_0,can be generalised to the embedding of the toric variety corresponding to a face ,say τ , of a cone σ, into the affine toric variety U_σ.

Consider a face τ of σ . Then the semigroup $S_\sigma \subset S_\tau$. Consider the semigroup homomorphism x:$S_\tau \rightarrow \mathbb{C}$ .This when restricted to S_σ gives a mapping from U_τ to U_σ.

This equivariant embedding plays a major role in generating general toric varieties by gluing affine toric ones.

•**SATURATION** Consider a lattice L and a subsemigroup S of L. Let $u \in$ L .

If for every $k \in \mathbb{Z}$,such that $ku \in$ S , $u \in$ S, then S is saturated.

An important example of a saturated semigroup is the semigroup S_σ associated with a SCRPC σ. Here, since the cone is convex, so is the dual and since the semigroup is a subset of the dual of the cone, it is convex. So consider a $v \in$ M, and a $k \in \mathbb{Z}$, such that $kv \in S_\sigma$, then since $0 \in S_\sigma$, due to convexity $(1/k)(0 + kv) = v \in S_\sigma$ .Hence, S_σ is saturated.

A very interesting remark considering the fact that the subsemigroup S_σ generates M, is the following: Any finitely generated, generating saturated subsemigroup S of M is of the form S_σ for a unique SCRPC σ.

The relation between saturation and normality is given by the following proposition.

Proposition 3.4.1. **[2]** *Let S be a finitely generated subsemigroup. S is saturated iff the algebra $\mathbb{C}[S]$ is normal.*

•TORIC MORPHISMS BETWEEN AFFINE VARIETIES:

Definition 3.4.2. Consider a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^m$ wherein V gets mapped to W. The restriction of this polynomial map gives, f:V $\to$ W and this is called the **morphism** between the 2 varieties V and W.

Proposition 3.4.3. **[2]** *Let V and W be affine varieties . There is a one-one correspondence between the set of morphisms $f{:}V \to W$ and the set of homomorphisms between the coordinate algebras $\phi : \mathbb{C}[W] \to \mathbb{C}[V]$.*

Proof. Given a morphism f between the varieties, consider a $g \in \mathbb{C}[W]$ and composing it with f gives a polynomial map in $\mathbb{C}[V]$ and thus this operation gives a map $f^* : \mathbb{C}[W] \to \mathbb{C}[V]$. Similarly given a map between the coordinate algebras, one can lift it to a map between

$$\mathbb{C}[Y_1, Y_2, ..., Y_m] \to \mathbb{C}[X_1, X_2, ...X_n]$$

and this in turn gives a polynomial map F: $\mathbb{C}^n \to \mathbb{C}^m$. $\square$

•MORPHISMS BETWEEN TORIC VARIETIES:

Here, let $U_{(N,\sigma)}$ and $U_{(N',\sigma')}$ be two affine toric varieties. Let $\phi : N' \to N$ be a homomorphism between the lattices and this maps $N' \cap \sigma'$ to σ . This morphism also gives rise to an induced homomorphism $T_N \to T_{N'}$ between the tori. This lattice hoomorphism then determines a morphism between the toric varieties, $U_{(N,\sigma)} \to U_{(N',\sigma')}$.

4 ABSTRACT VARIETIES

Definition 4.0.1. Let V $\in \mathbb{R}^n$. One can define a set called the **Sheaf of** $\mathbb{C}^\infty$ **functions** on V denoted by $\mathcal{O}_V^\infty$ as follows:

For every open set U in V,

$$\mathcal{O}_V^\infty = \{\ f\text{: U} \to \text{R} \mid f_i \text{ is a } \mathbb{C}^\infty \text{ function}\}.$$

4.1 Sheaves of modules

Definition 4.1.1. Given a topological space X. A **presheaf** $\mathcal{F}$ of abelian groups on X is defined as below:
(a) For every open subset U⊆X, there is an abelian group $\mathcal{F}(U)$.
(b) Given open subsets V⊆U , there is a morphism of abelian groups ρ_{UV} : $\mathcal{F}(U) \to \mathcal{F}(V)$ and this $\mathcal{F}$ is subject to the following conditions:
(0) $\mathcal{F}(\emptyset) = 0$ where $\emptyset$ is the empty set.
(1) $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.
(2) if $W \subseteq V \subseteq U$ are three open subsets , then $\rho_{UW} = \rho_{VW}.\rho_{UV}$.

Definition 4.1.2. Sheaves are pre-sheaves that satisfy certain extra conditions as stated below:
(1) Given an open set U and its covering U_i and suppose a,b$\in \mathcal{F}(U)$ such that $a|_{U_i} = b|_{U_i}$ for every i, then $a = b$.
(2) For an open set U and an open covering of U, $\{U_i\}$,if a section $s_i \in \mathcal{F}(U_i)$ then, for each pair of open sets U_i and U_j the sections will agree on the intersection.i.e. $s_i|_{U_i \cap U_j} = s_j|_{U_j \cap U_i}$.

If $\mathcal{F}$ is a sheaf on X, then $\mathcal{F}(U)$ is called a **section** of the presheaf over the open set U. The above properties imply that the gluing of the sections is done in a unique manner.

Suppose $\mathcal{F}$ is a sheaf and $U \subset X$ is an open subset, then the sections of $\mathcal{F}$ over U ,can be denoted by

$$\mathcal{F} = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F}).$$

Definition 4.1.3. The space X along with the the sheaf $\mathcal{F}$, that is (X,$\mathcal{F}$) Is called a **Ringed space**.

Definition 4.1.4. Structure Sheaf of an Irreducible Affine Variety:

Given an irreducible affine variety V=Spec(R) and given a zariski open subset U of the variety V, one can call a function $\phi : U \to \mathbb{C}$ as **regular** if for every $p \in$ V , there is a polynomial $f_p \in R$ such that $p \in V_{f_p} \in U$ and $\phi|_{V_{f_p}} \in R_{_p}$ where

$$R_f {=} \big\{ g/(f^m) \in \mathbb{C}(V) | g \in R, m \geq 0 \big\}$$

Then, sheaf of $\mathbb{C}$-Algebras is defined as :

$$\mathcal{O}_V(U) = \big\{ \phi : U \to \mathbb{C} | \phi \text{ is a regular function} \big\}$$

Definition 4.1.5. Consider a ringed space (X,$\mathcal{O}_X$) over $\mathbb{C}$.This is an **abstract variety** if for each $p \in$ X ,there is a neighbourhood U such that the restriction (U,$\mathcal{O}_X|_U$) is isomorphic to (V,$\mathcal{O}_V$) for some affine variety V.

Example 4.1.6. [**20**] $\mathbb{P}^n$ is an Abstract variety.
Let us first construct the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$. The field of fractions of $\mathbb{P}^n$ is $\mathbb{C}(\mathbb{P}^n)$. Let U$\subset \mathbb{P}^n$ be Zariski Open ,then one can define the structure sheaf using the above definition. Also, using the concept of affine charts ,each $U_i \simeq \mathbb{C}^n$ and thus $(U_i, \mathcal{O}_{\mathbb{P}^n}|_{U_i}) \simeq (\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$. This proves our claim.

4.2 Properties

Definition 4.2.1. Let X be a variety. X is said to be **separated** if the image of the diagonal map $\Delta : X \to X \times X$ where $\Delta(p) = (p, p)$ for every $p \in X$, is a Zariski closed subset of $X \times X$.

Example 4.2.2. $\mathbb{C}^n$ is separated since the image of the diagonal map is actually equal to $\mathbf{V}(x_1 - y_1, x_2 - y_2, ..., x_n - y_n) \subset \mathbb{C}^n \times \mathbb{C}^n$ and is hence Zariski closed as it is a variety.

Example 4.2.3. Every subvariety of a separated variety is separated.

These two statements above prove that **all affine varieties are separated**. Similarly $\mathbb{P}^n$ is separated and hence **all projective varieties are separated**.

Most varieties in Algebraic Geometry are separated. Hence the term 'varieties' would now mean separated varieties. Non-separated varieties would be 'pre-varieties'.

Definition 4.2.4. A variety is **Quasi-compact** if it is a finite union of affine open subsets. Hence all **affine varieties** are Quasi-compact . So is $\mathbb{P}^n$ since it is the union of affine open subsets $U_i = \mathbb{P}^n - \mathbf{V}(x_i)$.

Varieties would now only mean ' separated quasi-compact abstract variety'.

Definition 4.2.5. A variety is said to be **complete** if the following conditions are satisfied:
(1) X is separated.
(2) X is quasi-compact.
(3) Given a variety Y, the projection map $X \times Y \to Y$ is closed, that is, projection of Zariski closed sets in $X \times Y$ is Zariski closed in Y.

Cartesian Product of Varieties: Given Varieties V and W , and their respective coordinate rings R and S , one can define the Cartesian product as V $\times$ W= Spec(R $\otimes_{\mathbb{C}} S$). However one has to be very careful when defining the topology on the cartesian product as is shown in the example below:
The product topology on $\mathbb{C} \times \mathbb{C}$ is not the same as Zariski Topology on $\mathbb{C}^2$.

Remark. Let $x_0, x_1, ...x_n$ be coordinates on $\mathbb{P}^n$ and $y_0, y_1, ...y_m$ be coordinates on $\mathbb{P}^m$ then V$\subset$ $\mathbb{P}^n \times \mathbb{P}^m$ is Zarsiki closed iff V=$\mathbf{V}(f_1, ...f_s)$ where $f_i = \mathbb{C}[x_0, ...x_n, y_0.., y_m]$ is bihomogenous, that is, homogeneous in both x and y.

4.3 Constructing Varieties from Affine Varieties

One can construct varieties by gluing affine varieties along the Zariski Open subsets using maps that satisfy certain compatibility conditions. These conditions ensure that this process of gluing is an equivalence relation and thus it results in an affine variety.

Example 4.3.1. Consider the affine patches $U_i = \mathbb{P}^n \setminus \mathbf{V}(x_i)$. Each of these U_i is a copy of $\mathbb{C}^n$ with a different set of variables.
Here, $g_{ji} : U_i \to U_j$ where
$g_{ji}(\frac{x_0}{x_i} : \frac{x_1}{x_i} : .... : \frac{x_{i-1}}{x_i} : \frac{x_{i+1}}{x_i} : .... : \frac{x_n}{x_i}) = (\frac{x_0}{x_j} : \frac{x_1}{x_j} : .... : \frac{x_{j-1}}{x_j} : \frac{x_{j+1}}{x_j} : .... : \frac{x_n}{x_j})$

Thus here, $g_{ij} = g_{ji}^{-1}$ and $g_{ki} = g_{kj} \circ g_{ji}$. In this way, we obtain $\mathbb{P}^n$.

Also, any projective variety V $\subseteq \mathbb{P}^n$ can be covered with these open affine patches $V \cap U_i$ and these patches are then glued to give the variety V. Thus all projective varieties are Abstract Varieties.

5 GENERAL TORIC VARIETIES

The objective is to try to understand the construction of general toric varieties by gluing together affine toric varieties. A combinatorial object called the fan is used in this process and this is believed to contain all the information for the gluing.One example is the purely "toric" construction that leads to $\mathbb{P}^n$.

The idea behind gluing is as below:

Let σ and $\sigma^{'}$ be two SCRPC and the intersection of the cones is a face,say τ, that is common for them. The two affine toric varieties U_σ and $U_{\sigma'}$ are then glued along the common affine open subset U_τ.

Definition 5.0.1. Let Δ be a finite, non-empty collection of SCRPC in $N_\mathbb{R}$.This is called a **fan** if:
(1)for every cone σ in Δ, the faces of σ belong to Δ
(2) the intersection of any two cones in Δ is a face common to both.

Given a single non-zero cone σ, one can associate a fan to it,namely,the set $\left\{\tau \in N_\mathbb{R} | \tau \leq \sigma\right\}$ and this is called the **face fan** of σ or an **affine fan** generated by σ.

5.1 Associating a Toric Variety to a Fan

Consider a fan Δ and affine toric varieties U_σ corresponding to the cones of the fan. The disjoint union of these affine toric varieties by gluing them together along their common face, using inclusion maps which give gluing maps, results in a variety X(Δ) and this is the **toric variety associated with the fan** Δ.

Example 5.1.1. IN DIMENSION 1:
N=$\mathbb{Z}$
Possible cones: zero cone, positive half line , negative half line
Affine Toric varieties associated with it: $\mathbb{C}^*$ and $\mathbb{C}$. Non affine Fan: Fan generated by both half lines Toric Variety associated with it: Projective line $\mathbb{P}^1$

Example 5.1.2. IN DIMENSION 2:
N=$\mathbb{Z}^2$
Consider the fan generated by the three 2 dimensional cones:
σ_0=Cone(e_1, e_2); σ_1=Cone$(-e_1 - e_2, e_2)$; σ_3=Cone$(e_1, -e_1 - e_2)$.

Thus ,

$$\mathrm{U}_{\sigma_0}=\mathrm{Spec}(\mathbb{C}(S_{\sigma_0}))\simeq\mathrm{Spec}(\mathbb{C}[x, y])$$

$$\mathrm{U}_{\sigma_1}=\mathrm{Spec}(\mathbb{C}(S_{\sigma_1}))\simeq\mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-1}y])$$

$$\mathrm{U}_{\sigma_2}=\mathrm{Spec}(\mathbb{C}(S_{\sigma_2}))\simeq\mathrm{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]).$$

These are the standard affine patches that cover X and using the gluing maps these can be glued to form $\mathbb{P}^2$. Thus $\mathbb{P}^2$ is recovered as a toric variety.

Example 5.1.3. Construction of $\mathbb{P}^n$ as a Toric Variety: The above method can be generalised to give a common method to construct projective n-space $\mathbb{P}^n$. Let $N_{\mathbb{R}}$=$\mathbb{R}$ and the lattice N=$\mathbb{Z}^n$. This has standard basis $e_1, e_2, ...e_n$. Define

$$e_0 = e_1 + e_2 + ... + e_n$$

Let $\sum$ be the fan constructed from the cones generated by all the possible proper subsets of $e_0, e_1, ..., e_n$. This is a normal fan of the n-simplex Δ_n since the cones in the fan are generated by inner pointed normal vectors. The Toric Variety associated with this fan is $X_{\sum} = \mathbb{P}^n$ since this will give the usual affine open cover as in the earlier example.

Example 5.1.4. IN DIMENSION 3: Let us look at Weighted Projective Spaces:
N=$\mathbb{Z}^n/\mathbb{Z}.(q_0, q_1, ..., q_n)$ and let u_i be the images of the basis vectors in $\mathbb{Z}^{n+1}$.Thus,

$$q_0u_0 + q_1u_1 + ... + q_nu_n = 0$$

holds in N from its definition. Let $\sum$ be the fan that is generated from the cones made from all possible subsets of $u_0, u_1, ..., u_n$.Here if the weights

were all 1, then the above relation is the same as in the previous example and thus $X_{\sum} = \mathbb{P}^n$.
In general, $X_{\sum}$=$\mathbb{P}(q_0, q_1, ..., q_n)$.

Consider $\mathbb{P}(1, 1, 2)$
N=$\mathbb{Z}^3/\mathbb{Z}.(1, 1, 2)$. Here $u_0 + u_1 + 2u_2 = 0$. Thus consider the cones,

$$\sigma\text{=Cone}(u_0, u_1)\text{=Cone}(-u_1 - 2u_2, u_1)$$

and σ^v=Cone$(-u_2, 2u_1 - u_2)$. Here the toric variety associated with the fan is $X_{\sum}$ is the Zariski closure of the image of the map

$$\phi : (\mathbb{C}^*)^2 \to \mathbb{C}^3$$

$$\phi(s, t) = (s^2, st, t^2)$$

Here the image is isomorphic to $\mathbf{V}(xz - y^2)$ and thus

$$\mathbf{V}(xz - y^2) \simeq \mathbb{P}(1, 1, 2)$$

Since $\mathbf{V}(xz - y^2)$ has a singular point at the origin, so does $\mathbb{P}(1, 1, 2)$ in this affine open subset.

5.2 The Orbit-Cone Correspondence

In this section we will deal with the orbits of the torus action T_N acting on the variety $X(\Delta)$ and arrive at the bijective correspondence between the cones in the fan Δ and the T_N orbits in $X(\Delta)$. Let us look at this concept with the help of an example.

Example 5.2.1. **[22]** Consider $\mathbb{P}^2$ and the cones that generate $\mathbb{P}^2$, namely, σ_0=Cone(e_1, e_2); σ_1=Cone$(-e_1 - e_2, e_2)$; σ_2=Cone$(e_1, -e_1 - e_2)$. Here N=$\mathbb{Z}^2$ and thus the torus is $T_N = (\mathbb{C}^*)^2 \subseteq \mathbb{P}^2$ consists of points of the form (1,s,t) where $s, t \neq 0$. Hence, for any point (a,b) $\in \mathbb{Z}^2$, the corresponding curve in the torus and hence in $\mathbb{P}^2$ is

$$\lambda^u(t)=(1,\, t^a, t^b)$$

Analyse the limit as t $\to$ 0. These give the various limit points and depend on the point (a,b). The set of points u which give a particular limit point,equals N $\cap$ Relint(σ). Hence, using these limit points we can find out the structure of the fan containing these cones.

The regions of N correspond to cones of the fan. Considering the definition of $\mathbb{P}^2$ we have 7 distinct T_N-orbits in $\mathbb{P}^2$:

(1,1,1) $\in O_1 = \big\{(x_0, x_1, x_2)|x_i \neq 0$ for all i $\big\}$

(1,1,0) $\in O_2 = \big\{(x_0, x_1, x_2)|x_2 = 0$, and $x_0, x_1 \neq 0\ \big\}$

(1,0,1) $\in O_3 = \big\{(x_0, x_1, x_2)|x_1 = 0$, and $x_0, x_2 \neq 0\ \big\}$

(0,1,1) $\in O_4 = \big\{(x_0, x_1, x_2)|x_0 = 0$, and $x_1, x_2 \neq 0\ \big\}$

(1,0,0) $\in O_5 = \big\{(x_0, x_1, x_2)|x_1 = x_2 = 0$, and $x_0 \neq 0\ \big\}$

(0,1,0) $\in O_6 = \big\{(x_0, x_1, x_2)|x_0 = x_2 = 0$, and $x_1 \neq 0\ \big\}$

(0,0,1) $\in O_7 = \big\{(x_0, x_1, x_2)|x_0 = x_1 = 0$, and $x_2 \neq 0\ \big\}$

Hence every orbit has a unique limit point and this gives the following correpondence :

σ corresponds to O $\leftrightarrow \lim_{t\to 0} \lambda^u(t) \in O$ for all u $\in$ Relint(σ).

Some interesting properties of this are as below:
(1) Let dim $N_{\mathbb{R}} = n$.Then for every cone $\sigma \in \Delta$, dim O(σ)=n-dim(σ).
(2) The affine open subsets U_σ=$\cup_{\tau\leq\sigma} O(\tau)$.

The Orbit of a Cone:

Given X to be a toric variety of a fan Δ in $N_{\mathbb{R}}$. For $\tau \in \Delta$, let N_τ be the sublattice of N which is generated by $\tau \cap N$ and let $T_{N_\tau} = N_\tau \otimes_{\mathbb{Z}} \mathbb{C}^* = T_\tau$.

Consider the action of T_N on x_τ then :

Proposition 5.2.2. *The isotropy subgroup of the T_N-action at x_τ is T_τ.*

This implies that the orbit

$$O_\tau \simeq T_N/T_\tau$$

The important point is that **every orbit of the T_N-action on $X(\Delta)$ is of the form O_τ for some $\tau \in \Delta$.**

Definition 5.2.3. Given a fan Δ and a cone τ in Δ, **the Star of the cone** τ is the set of all cones of Δ which contain τ.

This Star(τ) can be changed to a fan by the following process. Let N_τ be as before and N(τ) = N/N_τ. For each cone $\sigma \in$ Star(τ) ,define

$$\bar{\sigma} = (\sigma + (N_\tau)_{\mathbb{R}})/(N_\tau)_{\mathbb{R}} \subset N(\tau)_{\mathbb{R}}$$

Then the fan associated with it is $\Delta(\tau) = \left\{\bar{\sigma}|\sigma \in Star(\tau)\right\}$.

Definition 5.2.4. The toric variety V(τ)=X($\Delta(\tau)$) is called the **abstract orbit closure** of $\tau \in \Delta$.

The next important component of toric geometry is the **The Zariski Tangent Space** but before defining the same, let us look at some of the necessary definitions for the same.

Given an irreducible variety X.

Definition 5.2.5. A **rational function** on X is a regular function $\phi : U \rightarrow \mathbb{C}$, where U is a non-empty Zariski Open subset.

Definition 5.2.6. Two rational functions are **equivalent** if they agree on some non-empty Zariski Open subset and the set of equivalence classes is $\mathbb{C}(X)$.

Definition 5.2.7. Function defined at a point: Consider a function $\phi \in \mathbb{C}(X).\phi$ is said to be defined at $p \in X$ if there exists a regular function $\phi' : U \rightarrow \mathbb{C}$ such that $p \in U$ and ϕ is equivalent to ϕ'.

Definition 5.2.8. Let p ∈X . We define the **local ring of X at** p as

$$\mathcal{O}_{X,p} = \left\{\phi \in \mathbb{C}(X) | \phi \text{ is defined at p}\right\}$$

Definition 5.2.9. A commutative ring R with unit is called a **local ring** if it has a unique maximal ideal m.

So now we are ready to define a Zariski Tangent Space to a variety X. As in differential calculus, tangent space here is given by the space of tangent vectors in all directions in such a way that it is tangent to X.

Consider a curve defined by the equation F(x,y)=0. In this equation remove all the terms of degree higher than 1. Thus, the linearised polynomial equation is now either 0 or a line. If it is 0, then the tangent space is the whole of $\mathbb{A}^2$(Dimension 2). Else, if it is a line then the tangent space is the affine line(Dimension 1). Now we are ready to give the formal definition of a Zariski Tangent Space to a variety X.

Definition 5.2.10. Let the point $p \in$ the variety X and $m_{X,p}$ be the maximal ideal of the local ring $\mathcal{O}_{X,p}$. Then the **Zariski Tangent Space** is defined as

$$T_p(X)=\mathrm{Hom}_{\mathbb{C}}(m_{X,p}/m^2_{X,p}, \mathbb{C})$$

.

Here, modding out by m^2 refers to the removal of terms of degree greater than 1.

In general, $\dim_{\mathbb{C}} T_p(X) \geq \dim_p X$.

5.3 Proper Morphism

In order to define when a given morphism is proper, we need to define the notions of 'fibred product' and 'base change'.

Definition 5.3.1. Consider the affine varieties X and Y, and the morphisms $X \to S$ and $Y \to S$ .These are called the affine varieties over S and S is called the **'base'**. One can construct the variety X $\times_S Y$ along with morphisms to X and Y which are projections. With the property that for any pair (f,g), f:Z $\rightarrow X$ and g:$\rightarrow Y$ that agree when composed with the respective morphisms to S , there is a unique morphism Z $\rightarrow X \times_S Y$ that makes the diagram commutative.

This variety X $\times_S Y$ along with these projective morphisms to X and Y, is called the **fibred product** of X and Y over S.

Definition 5.3.2. Base change or Base extension: Consider the variety X over S and Y:=S' , a new base variety. The morphism S'$\rightarrow S$ is considered to be the change of base. Then X'=X$\times_S S'$ with the projection on S' as the morphism will now be a variety over S' and this is obtained from X as variety over S by base change.

Definition 5.3.3. A morphism is said to be **closed** if the image of closed sets are closed.

Definition 5.3.4. A morphism $\phi : X \rightarrow Y$ is said to be proper if it **universally closed**, that is, the resulting morphism X':=X $\times_Y Y' \rightarrow Y'$ for every base change Y' $\rightarrow$Y , is closed.

Properness of Toric Morphisms:

Proposition 5.3.5. [**3**] *The Toric Morphism $\phi : X(N', \Delta') \rightarrow X(N, \Delta)$ is proper iff $\phi^{-1}(|\Delta|) = |\Delta'|$ holds.*

Another very interesting example of a proper morphism is obtained from a **subdivision** of fans: Suppose Δ and Δ' are two fans such that $|\Delta| = |\Delta'|$, then $|\Delta'|$ is a subdivision of $|\Delta|$ if each cone in $|\Delta'|$ is contained in some cone in $|\Delta|$. The identity map gives a morphism between the lattices and this in turn gives a morphism between the toric varieties and by Proposition 4.3.5, this would make it a proper morphism.

5.4 Completeness of Toric Varieties

Definition 5.4.1. A Fan Δ in $N_{\mathbb{R}}$ is complete if it covers the whole space. Consider the support of a fan,that is $|\Delta| = \cup_{\sigma\in\Delta}\sigma$. If $|\Delta| = N_{\mathbb{R}}$ holds then the fan is said to be complete.

Proposition 5.4.2. [**3**]*A Toric Variety is complete if and only if its fan is complete*

6 SINGULARITIES IN TORIC VARIETIES

6.1 Description of Toric Singularities

One can describe the type of singularities that any toric variety can have. We would first require the below definitions:

Definition 6.1.1. Given a point $p \in$ a variety X. X is said to be **smooth** or **nonsingular** if $\dim_{\mathbb{C}} T_p(X) = \dim_p X$. The point p is said to be singular if it is not smooth.

A very important method to check the smoothness of an affine variety is the **Jacobian Criterion**.

Theorem 6.1.2. **[26]** *Given $V \subset \mathbb{C}^n$ an affine variety. $\boldsymbol{I}(V)$=$< f_1, f_2, ...f_s >$ and $\dim_p$ V =d. Then V is smooth at p iff the Jacobian Matrix*

$$J_p(f_1, f_2, ...f_s) = (\tfrac{\partial f_i}{\partial x_j}(p))_{1 \leq i \leq s 1 \leq j \leq n}$$

has rank n-d.

Example 6.1.3. V=$\mathbf{V}(xy - zw)$.

$$J_p(xy - zw) = \begin{bmatrix} y & x & -w & -z \end{bmatrix}]$$

and would vanish only at (0,0,0,0) . Hence origin is the only singular point of the variety V.

Example 6.1.4. We have previously proved that $\mathbb{P}(1, 1, 2) \simeq \mathbf{V}(xz - y^2)$ and hence is singular at (0,0,0). This can also be showed using the Jacobian Criterion:

$$J_p(xz - y^2) = \begin{bmatrix} z & -2y & x \end{bmatrix}$$

vanishes only at (0,0,0) and is hence singular at the origin.Also,the rank of the matrix is 1 and since the rank isn't $n - d$=3-1=2, the variety is not smooth.

Definition 6.1.5. Given a variety X. The set of all singular points in X, $X_{sing}=\{p \in X \mid p \text{ is singular }\}$ is called the **singular locus** of X. This Sing(X) is a subvariety of X since it is given by the vanishing locus of the determinant of minors of the Jacobian matrix, and these are polynomials. So Sing(X) is the zero locus of a finite number of polynomials and is hence a closed subset of X.

Definition 6.1.6. Given two varieties X and Y with $p \in X$ and $q \in Y$. p and q are **locally equivalent** if there are Zariski Open sets U and V such that $p \in U \subset X$ and $q \in V \subset Y$ and $U \simeq V$.

Similarly X and Y are **analytically equivalent** at p and q iff the open sets U and V are such that U$\simeq$V as analytic varieties.

This helps since $p \in X$ is smooth iff it is analytically equivalent to $0 \in \mathbb{C}^n$.

G be a finite subgroup of $\mathrm{GL}_n(n, \mathbb{C})$. Let G act on $\mathbb{C}^n$ and $\mathbb{C}^n/G$ is the set of G-Orbits.

Complex Reflections: Complex Reflections are elements of GL(n,$\mathbb{C}$) that fix the hyperplane pointwise.

A finite Subgroup G $\subset$ GL(n,$\mathbb{C}$) is **small** if it contains no nontrivial complex reflections. An interesting remark is that if G_1 and G_2 are small subgroups of GL(n,$\mathbb{C}$)which give analytically equivalent singularities , then G_1 and G_2 are conjuate in GL(n,$\mathbb{C}$).

Definition 6.1.7. This is a type of singularity that is close to being smooth.

Given a point p in a variety X . p is called a **finite quotient singularity** if there is a small subgroup G$\subset$ GL(n,$\mathbb{C}$) such that p is analytically equivalent to $0 \in \mathbb{C}^n/G$. X is **Quasismooth** or is $\mathbb{Q}-$**Smooth** is every $p \in X$ is a finite quotient singularity.

In smooth varieties every point p is smooth and is hence analytically equivalent to $0 \in \mathbb{C}^n$ .So here G is trivial and this is allowed in the above definition and thus, every smooth variety is Quasismooth.

Simplicial Toric Varieties:

Definition 6.1.8. A rational polyhedral Cone $\sigma \in N_{\mathbb{R}}$ is **simplicial** if its minimal generators are linearly independent over $\mathbb{R}$.

Definition 6.1.9. Given a fan Δ , if every cone in Δ is simplicial , then the toric variety X(Δ) is **simplicial**.

A main result concering toric varieties is the following:

Theorem 6.1.10. **[19]** *Given X be a toric variety. Then the following are equivalent:*
(1) X is simplicial
(2) X has finite Abelian Quotient Singularities(i.e. the singularities are analaytically equivalent to 0 $\in \mathbb{C}^n/G$ *where* $G \subset GL(n, \mathbb{C})$ *is Abelian)*
(3) X has finite quotient singularities ,that is, X is quasismooth

Hence consider a toric variety X(Δ) associated with the fan Δ in $N_{\mathbb{R}} \simeq \mathbb{R}^2$. The minimal generators of the rays are primitive , that is ,linearly independent, and can thus be extended to form the basis of N. Thus the cones in the fan are simplicial and so is the toric variety .This implies that X has only finite Quotient Singularities. Consider the variety X' obtained from X by removing the fixed points of the torus action and these are the singularities and thus X' is smooth. But these fixed points which correspond to the 2 dimensional cones are finite in number. Hence ,**all toric varieties have finite number of singularities which are finite quotient**.

But we can describe these further wherein we can prove that this G is in fact cyclic. Let σ be a 2-dimensional SCRPC. If v_1 and v_2 be its minimal generators then since they are primitive , if we choose the basis of N to be e_1 and e_2 then $v_2 = e_2$ and $v_1 = me_1 + le_2$ where gcd(m,l)=1. One can prove that G=$N/N‘ \simeq \mathbb{Z}/m\mathbb{Z}$ and so this would mean $U_\sigma \simeq \mathbb{C}^2/G$. This implies that all toric varieties have at the most **Finite Cyclic Quotient Singularities.**

6.2 Resolution of Toric Singularities

Originally the question in mind was to find a non-singular complete variety X' for every singular algebraic variety X such that they have the same function field (the same set of equivalence classes when considering the regular maps defined on Zariski Open subsets). However this question was later framed in a much easier way. Now, we look for nonsingular complete varieties X' such that there is a **proper** birational map(a rational map that has an inverse)

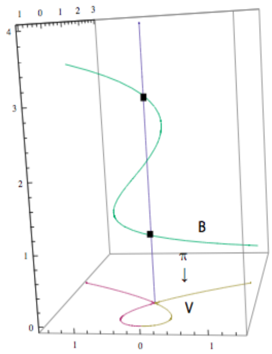

Figure 4: Blowup of V at the origin [**35**]

$X' \to X$. and this condition is required to exclude trivial varieties X' like the subvariety of X containing just the nonsingular points.

Resolution of singularities can be done using various methods. A brief outline of some of them is given below:

(1) **Normalisation**: This method removes all singularities of codimension 1 . But it is not known to work in higher dimensions. This is because there is no extra effect of normalising any variety twice and hence normalising a variety once results in a variety having 'not-so-big' singularities. In the case of curves, normalisation results in removing the singularities since these are codimension 1 and this process kills it. But in surfaces, normalisation results in just isolated points killing all singular curves.

(2)**Blow-Up**: Blowing up a singular point results in pulling the varieties apart in such a way that the singularity goes to codimension 0 and is thus removed. For example, consider two intersecting curves . Blowing up the point of intersection, which is the singular point, results in two non-intersecting curves thus removing the point of singularity.

In the above example, the curve V=$\mathbb{V}(y^2-x^2-x^3)$ was intersecting at the origin and the Blow up at the origin has resulted in the line being extended in such a way that it intersects the vertical line at 2 points since they cross the orgin twice.

Here, the complexity of the singularity is measured and then one has to prove that blowing-up has indeed improved this measure.

Let us consider the case of blowing up the affine space $\mathbb{A}^n$ at a point p. Here the objective is to keep the space almost completely unchanged except at the point p. This point when blowed up is replaced by an entire copy of $\mathbb{P}^{n-1}$ and this is done by associating to every line tangent at p , a point of $\mathbb{P}^{n-1}$.

The blow-up surface B $=\mathbb{A}^n \times \mathbb{P}^{n-1}$ and every point on the surface is given by a pair (q,$\bar{q}$) where q is a point in $\mathbb{A}^n$ and $\bar{q}$ is a line through the origin in $\mathbb{A}^n$. By the change of coordinates , this point p could be considered as the origin O. This blow up also includes a projection map $\pi : B \to \mathbb{A}^n$ where $(q, \bar{q}) \to q$. Hence this is an isomorphism in $\mathbb{A}^n/0$.

The set $\pi^{-1}(O)$ is the set of points at which the blowed up curve intersects the vertical line above O and this occurs as many times as the curve passes through the origin O. This set is called the **Exceptional Divisor of the Blow-up**.**[35]**
There are some very interesting results in the case of $\mathbb{A}^2$.

Lemma 6.2.1. [35] *An irreducible affine variety $V \subset \mathbb{A}^2$ has finitely many singular points.*

Lemma 6.2.2. [35] *If V= $\mathbb{V}(f) \subset \mathbb{A}^2$ is an affine variety , the blowup $B_p(V)$ is non-singular at p.*

Thus, **the singularities of a variety V $\subset \mathbb{A}^2$ can be resolved by blowing up at finitely many points**.

Blowing-up in the case of Toric Varieties: Let σ be an n-dimensional cone generated by the vectors $v_1, ...v_n$. Let $v_0 = v_1 + ...v_n$ and let σ_i=Cone$(v_0, v_1, ..., v_n)$ omitting v_i .These cones generate a fan Δ‘ and this subdivides the “affine” fan generated by σ and the resulting proper morphism between the toric varieties of the subdivided fans is the blow-up of $U_\sigma \simeq \mathbb{C}^n$ at the distinguished point $x_\sigma = 0$.

As mentioned before, singularities can also be resolved using the method of normalisation. An important remark concerning normalised varieties in $\mathbb{A}^2$ is that the set of singular points has codim≥ 2 and thus have dimension 0 and thus, the normalised variety now becomes smooth. So normalisation works for curves.

(3)**Albanese**: Here, consider projective varieties.Using the projection of singular points onto smaller projective spaces , one reaches a variety that has no points. Though this might not be very beneficial , it does solve the problem for curves and makes the case of surfaces much easier.

7 DIVISORS

7.1 Prime Divisors

Definition 7.1.1. [**31**] Given an algebraic variety X. A subvariety Y$\subset$ X of codimension 1 is called a **Prime Divisor** of X.

Definition 7.1.2. A prime divisor Y is **Cartier** if it is locally defined by a single equation.

Example 7.1.3. [**31**] Consider X=$\mathbb{C}$. Here the prime divisors are all the points a$\in$ $\mathbb{C}$. These points are Cartier too since these are given by the vanishing of the polynomials of the form $g(x) = x - a$.

Example 7.1.4. [**31**] X= $\mathbb{V}(y^2 - (x^3 + x^2)) \subset \mathbb{C}^2$. Here the prime divisors are again all the points in X but X is not smooth. There is a node at the origin and thus, Cartier divisors are all the points except 0 since the origin could be given by the equation $x = 0$ but would also result in $x = y^2$=0 and hence would be of multiplicity 2.

Some important remarks regarding these divisors: [**31**] Prime divisors are not closed under deformation. Some example to illustrate this are as below:
(1) Consider the family of curves $\mathrm{X}_t \subset \mathbb{P}^2$ where X_t is given by $xy - z^2t$=0. When deformed from $t = 1$ to $t = 0$, it deforms from a conic to a union of two lines given by $xy = 0$ and this is not a prime ideal. Hence $xy = 0$ does not define an irreducible variety .
(2) Consider the family of curves $xyt - z^2$=0 and deform it from $t = 1$ to $t = 0$. This results in the deformation from a conic to a double line z^2=0 and hence the multiplicity is 2.
This is the reason why we need a divisor which is closed under union and takes multiplicity into account.

7.2 Weil Divisors

Definition 7.2.1. Given a normal variety X, the finite formal sum

$$\mathrm{D}=\textstyle\sum_{i=1}^{s} a_i D_i$$

is called the **Weil Divisor** on the variety X. Here the D_is are distinct irreducible hypersurfaces of X and $a_i \in \mathbb{Z}$. The set of all such Weil divisors is a group under addition and is denoted by **WDiv(X)**. A Weil Divisor is said to be **efficient** if all the coefficients are nonnegative and this is denoted by D$\geq$ 0.

Linearly Equivalent Divisors and the Divisor Class Group:

Definition 7.2.2. Given X to be a normal variety. Two Weil Divisors D_1, $D_2 \in$ WDiv(X) are said to be **linearly equivalent** , written as $D_1 \sim D_2$, if there is an $f \in \mathbb{C}(X)^*$ such that the div(f)= D_1 - D_2 .

Definition 7.2.3. A divisor D $\in$WDiv(X) is a **prinicipal divisor** if D$\sim$ 0 ,that is, there exists $f \in \mathbb{C}(X)$ such that D=div(f) .

Linear equivalence is an equivalence relation and the subgroup of principal divisors is denoted by Div_0(X) and thus the quotient group

$$\mathrm{Cl(X)=WDiv(X)/Div_0(X)}$$

is called the **Divisor Class of X**.This consists of equivalence classes of linearly equivalent divisors.
This Divisor Class Group is also denoted by A_{n-1} (X) where n=dim(X).

Example 7.2.4. [**31**] Consider X$\subset \mathbb{P}^3$. X= $\mathbb{V}(xy - z^2)$ .This is in $\mathbb{P}^3$ and hence it is a smooth cone over the plane w=0. In order to produce linearly equivalent divisors on X , intersect X with various planes H$\subset \mathbb{P}^3$.
(a) H_1: w=0. Here D_1=$H_1 \cap X$ is a smooth plane conic over which X is a cone.
(b) H_2:z=0 . Here, D_2= $H_2 \cap X$ is the union of two planes x=z=0 and y=z=0.
(c) H_3:x=0. This gives the line z^2=0 with multiplicity 2.
Each of these divisors is linearly equivalent pairwise.
For example D_1-D_2= $(\frac{w}{z})_0$

An important theorem in this section is the following:

Theorem 7.2.5. [**27**] *Given a normal affine variety X=Spec(R) , the divisor class of X, Cl(X) is trivial iff R is a Unique Factorisation Domain.*

One can define a Weil Divisor for every f and for every irreducible hypersurface Y in X as below: Given an irreducible variety X where $\mathbb{C}(X)$ is the function field.

Definition 7.2.6. A subvariety Y $\subset$X is called a **hypersurface** if all the irreducible components of Y is of codimension 1 in X.

Definition 7.2.7. If Y $\subset$ X be an irreducible hypersurface .The set

$$\mathcal{O}_{X,Y}=\Big\{ f \in \mathbb{C}(X) | \text{ there exists a non-empty Zariski Open U} \subset Y \text{ such that } f \text{ is defined on U}\Big\}$$

is the **local ring** defined on the irreducible hypersurface Y.

Definition 7.2.8. Given an Integral Domain R with the field of fractions K, then R is said to be a **Discrete Valuation Ring** if there is a surjective function

$$\mathrm{ord}_R : K^* \to \mathbb{Z}$$

such that
(a) $\mathrm{ord}_R(ab)$=$\mathrm{ord}_R(a)$ + $\mathrm{ord}_R(b)$
(b) $\mathrm{ord}_R(a + b) \geq \min(\mathrm{ord}_R(a), \mathrm{ord}_R(b))$
(c) R= $\big\{a \in K^* | \ \mathrm{ord}_R(a) \geq 0 \ \big\} \cup \big\{0\big\}$

The function ord_R is a **valuation** on the field of fractions K and the integral domain R is its **valuation ring**.

Hence given an irreducible hypersurface Y in a normal variety X ,the local ring defined for the hypersurface Y , $\mathcal{O}_{X,Y}$ is a Discrete Valuation Ring and thus the corresponding valuation is given by

$$\mathrm{ord}_Y : \mathbb{C}(X)^* \to \mathbb{Z}$$

Let $f \in \mathbb{C}(X)^*$. f **vanishes to order m** if m=$\mathrm{ord}_Y(f) > 0$ and **has a pole of order m on Y** if m=-$\mathrm{ord}_Y(f) > 0$.

Using this one can define a Weil Divisor for every $f \in \mathbb{C}(X)$ and every irreducible variety Y in X as there is only atmost finitely many hypersurfaces

Y$\subset$X such that the $\text{ord}_Y(f) \neq 0$. Thus one can define the Weil Divisor as below:

$$\text{div}(f) = \textstyle\sum_Y ord_Y(f)Y.$$

Here we can define

$$\text{div}_0(f) = \textstyle\sum_{ord_Y(f)>0} ord_Y(f)Y$$

$$\text{div}_\infty(f) = \textstyle\sum_{ord_Y(f)>0} -ord_Y(f)Y$$

where the former is called the *divisor of zeros of f* and the latter, *divisor of poles of f*.
Next, the objective is to understand when the divisor vanishes.

Proposition 7.2.9. [18] *Given a normal variety X and* $f \in \mathbb{C}(X)^*$. *Then the div(f)* $\geq$ *0 iff f: X*$\rightarrow \mathbb{C}$ *is a morphism ,that is,* $f \in \mathcal{O}_X(X)$.

Also, for a connected complete variety X ,the only morphisms $X \rightarrow \mathbb{C}$ are constant maps. Thus,

Corollary 7.2.10. *Given a normal complete variety X, and* $f \in \mathbb{C}^*$.
Then the div$(f) \geq$ *0 iff* f *is a constant.*

Since div($\frac{1}{f}$)=-div(f) , div(f)=0 iff div(f)$\geq$0 and div($\frac{1}{f}$) ≥ 0.

Corollary 7.2.11. Let X be a normal variety and $f \in \mathbb{C}(X)^*$.
Then div(f)=0 iff $f \in \mathcal{O}_X(X)$.

7.3 Cartier Divisors

Given a Weil Divisor D of a normal variety X, one can restrict the same to a nonempty Zariski Open subset U $\subset$ X . This restriction of D to U is given by

$$\text{D}|_U = \textstyle\sum_{U \cap D_i \neq 0} a_i(U \cap D_i)$$

.

Definition 7.3.1. Given a Weil Divisor D on a normal variety X . Then D is said to be **locally principal** if there is an open cover $\{U_i\}_i \in I$ of X such that $D|_{U_i}$ is principal for every $i \in$I.

Definition 7.3.2. A Locally principal Weil Divisor D is called a **Cartier Divisor**.

This implies that **every principal divisor** is a Cartier Divisor and thus **div**(f) for every $f \in \mathbb{C}(X)^*$ is a Cartier Divisor.

Also, if D_1 and D_2 are two linearly equivalent divisors then D_1 is Cartier iff D_2 is Cartier.

Definition 7.3.3. Since D is a Cartier Divisor, it is locally principal.Hence if it is locally principal relative to an open subset U_i then there exists f_i in $\mathbb{C}(X)^*$ such that $D|_{U_i}$ =div(f_i) . $\{(U_i,f_i)\}_{i\in I}$ is called **local data** for D.

Under certain conditions Weil and Cartier Divisors coincide.

Theorem 7.3.4. [27] *Given a normal variety X such that the local ring $\mathcal{O}_{X,p}$ is a UFD for every point p$\in$ X. Then the group of Weil Divisor on X is isomorphic to the group of Cartier Divisors.*

Corollary 7.3.5. *If the variety X is smooth then the local ring is a UFD and hence by the above theorem , the divisors coincide . Thus,* ***on smooth varieties Weil and Cartier Divisors coincide****.*

Example 7.3.6. [31]

Let X=$\mathbb{C}^n$. Here the local ring is a UFD and thus the Cartier Class group CaCl($\mathbb{C}^n$) and Cl($\mathbb{C}^n$) coincide. If V_1 and V_2 are two Cartier Divisors, then they are given by the vanishing of two polynomials f_1=0 and f_2=0 and hence they will be linearly equivalent since V_1-V_2= $(\frac{f_1}{f_2})_0$. Hence by the **deg** homomorphism, this gives 0 .This implies that all cartier divisors are principal and hence the CaCl is trivial and so is the Divisor Class Group.

Example 7.3.7. Next, consider X=$\mathbb{P}^n$ and this is generated by a hyperplane H$\subset \mathbb{P}^n$. All the prime divisors here are defined by the vanishing of a single homogeneous polynomial . Define the surjective map deg:Cl($\mathbb{P}^n$) $\to \mathbb{Z}$. The Kernel of this map is given by all those linearly equivalent divisors in $\mathbb{P}^n$

whose degree is 0. This is the case if the divisor is principal and is generated by a polynomial of degree 0 . Thus consider two linearly equivalent divisors D_1 and D_2 . This can be written as D=D_1-D_2=$\frac{f}{g}$ wherein f and g have degree d and thus deg(D) is 0 .Thus, the kernel consists of all principal divisors. Principal divisors are linearly equivalent to 0 and hence form the trivial group in Cl($\mathbb{P}^n$) . This makes **deg** an isomorphism. Thus Cl($\mathbb{P}^n$) $\simeq \mathbb{Z}$.

Definition 7.3.8. Given a normal variety X, the set of all Cartier Divisors on X ,denoted by Div(X) form a subgroup of WDiv(X) . Also, since all principal divisors are Cartier, $\text{Div}_0 \subset \text{Div}(X)$. The **Picard Group** of X is defined to be the quotient

$$\text{Pic(X)= Div(X)/Div}_0\text{(X)}$$

Example 7.3.9. From Corollary.6.3.5, since $\mathbb{P}^n$ is smooth, every Weil Divisor is a Cartier Divisor and hence Pic($\mathbb{P}^n$)=Cl($\mathbb{P}^n$)$\simeq \mathbb{Z}$.

Example 7.3.10. Since every prime divisor in the smooth affine space $\mathbb{C}^n$ is principal and hence linearly equivalent to 0 , Pic($\mathbb{C}^n$)= Cl($\mathbb{C}^n$) =0[**22**].

The following are some of the important remarks about invariant divisors which are helpful in understanding Pic(X) and Cl(X) where X is a Toric Variety.

Definition 7.3.11. We say a divisor D=$\sum_{i=1}^{r} a_i D_i$ is **T-invariant** if t(D)=D for every t$\in T$ where we are referring to the action of T on X.

Definition 7.3.12. Suppose D_ρ=V(ρ) is the orbit closure where $\rho \in \Delta$, then,

The **T-invariant Weil Divisors** on X are the divisors $\sum_{\rho\in\Delta} a_\rho D_\rho$.

Thus the group of all T-invariant Weil Divisors is given by $\otimes_{\rho\in\Delta}\mathbb{Z}.D_\rho$.

Definition 7.3.13. Given a lattice N and a dual lattice M. Let $u \in M$, $\rho \in \Delta$ and $v(\rho)$ be the minimal generator of ρ.

Then div(χ^u)=$\sum_{\rho\in\Delta} < u, v(p) > D_\rho$ is a **T-invariant Cartier Divisor** on X.

An interesting result with respect to Toric Varieties: Let U_σ be a toric variety.

Theorem 7.3.14. [18] *Every T-invariant Cartier Divisor D on an affine toric variety U_σ is of the form div(χ^u) for some $u \in M$.*

Recall the theorem that states that when X=Spec(R) ,the Cl(X) is trivial iff R is a UFD. Thus when divisors are restricted to U_0=T_N , they become principal since $\mathbb{C}[U_0]$ is a UFD. Thus, using these principal divisors one can describe an isomorphism between T-invariant divisors and groups of classes of divisors(Weil and Cartier) on X.

Remark. Since on an affine toric variety U_σ, all T-invariant Cartier divisors are of the form, div(χ^u) , the Pic(U_σ)is trivial.

8 LINE BUNDLES AND SHEAVES

8.1 Line Bundles

A **line bundle** basically refers to the generalisation of a line that varies from one point to another in space. The best example of this notion is the tangent line at every point on a curve. A tangent line keeps changing at every point and these lines can be organised into a bundle called the **tangent bundle**. A formal definition of the same is as under.

Definition 8.1.1. [**29**] Let X be the variety. A Vector Bundle of rank n is a space E and a map $\pi : E \to X$ such that:
(1) $\pi^{-1}(X)$ is an n-dimensional Vector Space.
(2) For every x$\in$X , there exists an open set U$\subset X$ such that x$\in$U and a map $\phi_U : U \times k^n \simeq \pi^{-1}(U)$ .This ϕ_U is called the **Local trivialisations**.
(3) $\pi(\phi_U(x, v)) = x$ and $v \to \pi_U(x, v)$ gives an isomorphism between k^n and $\pi^{-1}(x)$.

A vector bundle of rank 1 is a line bundle.

Definition 8.1.2. Given x $\in$X and a line bundle $\pi : L \to X$ then the **fiber** of L over x is L$_x$=$\pi^{-1}(x)$.

Definition 8.1.3. **Fiber bundle** is a space that locally behaves like a product space but globally it has a different type of topological structure.

Formally one can define this with the help of a map $\pi : E \to B$ wherein in small regions on E, this map works like the projection from B$\times$F to B . Here E is called the **Total space of fiber bundle** ;B :**Base Space** and F :**the Fibre**.This is a **Trivial Bundle** if E= B$\times$F.

Definition 8.1.4. Given a fiber bundle $\pi : E \to B$. **Local section** of the fiber bundle is a continuous map $s : U \to E$ where U$\subset B$ is an open set such that $\pi(s(x)) = x$ for every $x \in U$.

Example 8.1.5. [**29**]The simplest example is the **Trivial bundle**. So here the space E is the space X$\times k$ with the map $\pi : X \times k \to k$ .This is the projection onto the first factor. Consider X=$\mathbb{P}^n$. Thus the map $\pi : \mathbb{P}^n \times k \to \mathbb{P}^n$.

Let us now try to determine the sections of the line bundle. This is given by:
$s : \mathbb{P}^n \to \mathbb{P}^n \times k$ such that $\pi(s(x)) = x$. Now rational maps from the projective space $\mathbb{P}^n$ should be such that the numerator and denominator have the same degree. Also, for the map to be well defined on all of $\mathbb{P}^n$ the denominator should not vanish. But since k is an algebraically closed field, and the denominator has a non-zero degree,it should vanish at some point . Thus, this is possible only if the map is constant. Hence,s is a constant map. This implies that all sections of the line bundle associated to $\mathbb{P}^n$ are constant maps.

Example 8.1.6. Tautological Bundle [29] Consider $\mathbb{P}^n$. This is the set of all lines through the origin in k^{n+1}. Hence one can construct a bundle on $\mathbb{P}^n$ by associating to every point x$\in \mathbb{P}^n$ a line in k^{n+1}. Here we shall use transition functions : Consider the affine neighbourhoods that cover $\mathbb{P}^n$.

$$\mathrm{U}_i = \Big\{[z_0 : z_1 : .... : z_n] : z \neq 0\Big\}$$

Local trivialisations are given as per the definition: $\pi^{-1} \simeq U_i \times k$. The correspondence given by the map sending a point $[z_0 : z_1 : .....z_n] \in \mathbb{P}^n$ in the affine patch to a line in k^{n+1} spanned by the point $(z_0, z_1, ..., z_n)$.

Transition maps $\phi_{U_iU_j}$ help in gluing these affine patches ,thus creating a bundle. For example , consider $\phi_{U_0U_1}$ .This will help in gluing U_0 and U_1 by mulitplying by $\frac{z_1}{z_0}$. Thus ,this gluing is very twisted when compared to a trivial bundle.
Consider the global sections of this bundle. Since a global section must be given by $f \in k[z_0/z_1, z_2/z_1, ...z_n/z_1]$ on U_1; $g \in k[z_1/z_0, z_2/z_0, ..., z_n/z_0]$ on U_0 and $f = (z_1/z_0)g$. But this is impossible since this would only be introducing more z_0 in the denominator. Thus there are no nonzero global sections of the tautological bundle.

Another interesting remark is that regarding maps to $\mathbb{P}^n$. This should locally be given as homogeneous polynomials since the map should be well defined on the whole of $\mathbb{P}^n$. Consider a variety X$\subset \mathbb{P}^m$ and a map $F : X \to \mathbb{P}^n$ given by $F = [f_0 : f_1 : .... : f_n]$. Take the corresponding invertible sheaf of global sections f_i. Thus given a map from $\mathbb{P}^n$ we can obtain a line bundle and a collection of generating global sections f_i.
Conversely, given a line bundle $\mathcal{L}$ and a generating section $f_0, f_1, ...f_n$ one can construct a map $F : X \to \mathbb{P}^n$ where $F = [f_0 : f_1 : .... : f_n]$.

Definition 8.1.7. [**10**] Given an algebraic variety X over $\mathbb{C}$ and a line bundle L on X. Let $H^0(X, L)$ be the vector space of global sections of L.
Let V be a finite dimensional subspace of $H^0(X, L)$ and $\mathbb{P}(V)$=(V-0)/$\mathbb{C}^*$, the projective space of lines in V. A **linear system** on X is the pair (L,$\mathbb{P}(V)$).

Definition 8.1.8. Given a linear system (L,$\mathbb{P}(V)$) on an algebraic variety X. The **base locus** of the linear system ,denoted by Bs(L,$\mathbb{P}(V)$) is the set of points $x \in$ X such that the evaluation map $\epsilon_x : V \to L_x$ where $s \in$V is mapped to $s(x) \in L_x$, is the zero map.

Definition 8.1.9. Given a line bundle L on an algebraic variety X. L is **globally generated line bundle** if there exists a finite dimensional subspace V of $H^0(X, L)$ such that the base locus Bs(L,$\mathbb{P}(V)$)=$\emptyset$.

Definition 8.1.10. [**10**] L is a **very ample line bundle** if there is a finite dimensional subspace V of $H^0(X, L)$ such that Bs(L,$\mathbb{P}(V)$)=$\emptyset$ and the canonical map

$$\phi_{(L,\mathbb{P}(V))} : X - Bs(L, \mathbb{P}(V)) \to \mathbb{P}(V^*)$$

is an embedding.

Definition 8.1.11. L is an **ample line bundle** if there exists a positive integer n such that L$^{\otimes n}$ is very ample.

Thus if an algebraic variety cannot be embedded into any projective space , it cannot admit any ample line bundle.

Definition 8.1.12. [**4**] Given a fan Δ . A real-valued function h:$|\Delta| \to \mathbb{R}$ is called a Δ **linear support function** if it is linear on all the cones σ in the fan Δ and takes integer values at the lattice points.

Some interesting remarks:
(1)Over general Complete algebraic varieties (over $\mathbb{C}$), any complete curve is projective.
(2)Any smooth algebraic complete surface is projective.But in dim $\geq$3 there are many smooth algebraic complete singular nonprojective surfaces.
(3)However, there are singular nonprojective algebraic complete surfaces.

In case of toric surfaces this is completely different and the **Projectivity criterion** which states that

A Complete toric variety X with fan Δ is projective iff there exists a strictly convex support function on Δ.

This is helpful in proving the theorem below.

Theorem 8.1.13. [**9**]*Every Complete Toric Surface is Projective.*

Proof. Given a complete Toric surface X that is given by 2 dimensional fan . Choose a cyclic order on the set of 1 dimensional cones . One can then build a strictly convex support function thus making X projective. □

8.2 Intersection Number

Given a complete algebraic variety X, a line bundle L on X, C is a smooth closed curve in X, s be a rational nontrivial section of the restriction of L on C(L$|_C$) . Let the divisor of s be div(s)=$\sum a_i p_i$.Then

Definition 8.2.1. Degree of L$|_C$ is given by **deg(L**$|_C) = \sum a_i$.

Definition 8.2.2. If D is a Cartier Divisor of X, then the **intersection number** of D and C is D.C=deg($\mathcal{O}(D)|_C$).

In general, given a nonsingular variety of dimension n,W_1 and W_2 are irreducible closed subsets of V, Z is an irreducible component of V, then Intersection theory provides multiplicity to Z.

Definition 8.2.3. ([**30**]Let $D_1, D_2, ..., D_n$ be effective divisors. These divisors are said to **intersect properly** at $P \in |D_1| \cap |D_2| \cap ..... \cap |D_n|$ if P is an isolated point.
In this case, the intersection number is given as follows:

$$(D_1, D_2, ..., D_n) = \dim_P \mathcal{O}(P)/(f_1, ...f_n)$$

where f_i are the local equation of the divisors.

Example 8.2.4. **[30]** $Z_1=Y^2 - X^3$; $Z_2 = Y - X^2$ and the point P=(0,0).

$$(Z_1, Z_2)_P = \dim(K[X,Y])/(Y^2 - X^3, Y - X^2) = \dim(K[X])/(X^4 - X^3) = 3$$

Properties of Intersection Number: (1)This is Invariant under Equivalence since if V is a complete variety and $D_n \sim D'_n$ then $(D_1.D_2........D_n) = (D_1.D_2......D'_n)$

(2)Let $D_1, D_2, ..., D_n$ be a set of n divisors.Then there exists n divisors $D'_1, D'_2, ..., D'_n$ such that $(D_1.D_2.....D_n)=(D'_1.D'_2........D'_n)$

With these 2 properties one can determine the intersection numbers of any divisors.

Result. If L is globally generated on X , then the $\deg(L|_C) \geq 0$ where C is any closed irreducible curve.

Definition 8.2.5. Let L be a line bundle on X. L is said to be **nef**(numerically effective) iff $\deg(L|_C) \geq 0$ for any closed irreducible curve C in X. A similar definition can be given for an nef Cartier divisor.

From the previous result, any globally generated line bundle is nef.

8.3 Sheaves

The Sheaf of a Weil Divisor[18]: The objective of this section is to show that a Weil Divisor D determines a sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules on X.

Recall that given a sheaf $\mathcal{F}$ over U$\subset$X , sections of the sheaf are denoted by

$$\mathcal{F}(U) = \Gamma(U, \mathcal{F}) = H^0(U, \mathcal{F}).$$

Thus, given a Zariski open subset U$\subset$ X and a sheaf $\mathcal{O}_X(D)$ that is determined by the Weil Divisor D,one can define the sections of the sheaf as follows:

$$\Gamma(U,\mathcal{F}) = H^0(U,\mathcal{F}) = \{f \in \mathbb{C}(X)^* |\ (\text{div(f)+D})|_U \geq 0\} \cup \{0\}.$$

One can prove that $\mathcal{O}_X(D)$ is a sheaf of $\mathcal{O}_X$ modules.

Given an affine variety X=Spec(R) , one can obtain sheaves of $\mathcal{O}_X$ modules using a standard method. A nonzero element $f \in R$ gives a localisation R$_f$ such that X$_f$=Spec(R$_f$) is the open subset $X \setminus \mathbf{V}(f)$. Suppose M is an R-module, we can obtain an R$_f$-module M$_f$ $= M \otimes_R R_f$. Then there is a unique sheaf $\tilde{M}$ of $\mathcal{O}_X$ modules such that

$$\tilde{M}(X_f) = \mathrm{M}_f.$$

An important result is that linearly equivalent divisors give isomorphic sheaves.

Proposition 8.3.1. **[18]** *If D,E are Weil Divisors such that D $\sim$E, then $\mathcal{O}_X(D)$ and $\mathcal{O}_X(E)$ are isomorphic as sheaves of $\mathcal{O}_X$ modules.*

Proof. Since D and E are linearly equivalent D=E+ div(g) for some $g \in \mathbb{C}(X)^*$. Thus,

$$f \in \Gamma(X,\mathcal{O}_X(D)) \iff div(f)+D \geq 0 \iff div(f)+E+div(g) \geq 0 \iff div(fg)+E \geq 0 \iff fg \in \Gamma(X,\mathcal{O}_X(E))$$

Thus this multiplication by g results in an isomorphism

$$\Gamma(X,\mathcal{O}_X(D)) \simeq \Gamma(X,\mathcal{O}_X(E)).$$

□

This works over any Zariski Open subsets U and this is compatible with the restriction maps as well.

Next, we can show that the R-module $\Gamma(X,\mathcal{O}_X(D))$(the sections of the sheaf) determines the entire sheaf $\mathcal{O}_X(D)$.

Proposition 8.3.2. [18] *Given a normal affine variety X=Spec(R) and D be a Weil Divisor on X. If $g \in R$ is nonzero, then*

$$\Gamma(X_g, \mathcal{O}_X(D)) = \left\{\tfrac{f}{g^m} | f \in \Gamma(X, \mathcal{O}_X(D)), m \geq 0\right\}(1)$$

Since these open sets X_g for $g \in R - \left\{0\right\}$ form a basis for the Zariski Topology of X=Spec(R) , the sheaf $\mathcal{O}_X(D)$ is uniquely determined by its global sections.

The right hand side of equation 1 is the localisation of $\Gamma(X_g, \mathcal{O}_X(D))$ at g. This can be generalised as follows:
Suppose M is a finitely generated R-module ,one can define its localisation at g as M_g then one gets a unique sheaf $\tilde{M}$ on X such that

$$\Gamma(X_g, \tilde{M}) = M_g$$

This generalisation leads us to the following defintion.

Definition 8.3.3. Given $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$ modules on a variety X. This sheaf is said to be **coherent** if there is an affine open cover $\left\{U_i\right\}_{i \in I}$ such that for every $i \in I$ there is an $\mathcal{O}_X(U_i)$ module M_i such that $\mathcal{F}|_{U_i} = \tilde{M}_i$.

From proposition 6.5.2, if D is a Weil Divisor on a normal variety X, then $\mathcal{O}_X(D)$ is coherent.

Definition 8.3.4. Given a variety X and $\mathcal{F}$ be the sheaf of $\mathcal{O}_X$ modules on X. Then the sheaf $\mathcal{F}$ is **invertible** if it is locally trivial i.e. if U_i is a cover of X then $\mathcal{F}|_{U_i} \simeq \mathcal{O}_X|_{U_i}$.

Example 8.3.5. A trivial example of an invertible sheaf is $\mathcal{O}_X$.

A very interesting correspondence is the one between Invertible sheaves and Line Bundles.
This can be proved using the below results.

Proposition 8.3.6. [18] *Given a Weil Divisor D on a normal variety X. $\mathcal{O}_X(D)$ is invertible iff D is a Cartier Divisor.*

Proof. Suppose D is a Cartier Divisor. This implies that D is locally principal and hence D=div(f). Thus , D$\sim$ 0 and hence $\mathcal{O}_X(D) \simeq \mathcal{O}_X(0) = \mathcal{O}_X$(from the Proposition. 8.3.1). Hence, $\mathcal{O}_X(D)$ is invertible.
Going the other way. Suppose $\mathcal{O}_X(D)$ is invertible. This means $\mathcal{O}_X(D)$ is locally isomorphic to $\mathcal{O}_X$. We need to prove D is locally prinicipal. Suppose on an affine open set U, X=Spec(R) and $\mathcal{O}_X \simeq \mathcal{O}_X(D)$ and one can then prove by contradiction that if E=D-div(g) then E=0 and hence D=div(g) hence making D a Cartier Divisor. □

A very interesting definition of the Picard Group can be given in terms of invertible sheaves and the following results are required for the same.

Proposition 8.3.7. *Given Cartier Divisors D and E on a normal variety X. Then D$\sim$ E iff $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$ as $\mathcal{O}_X$ modules.*

Theorem 8.3.8. [27] *Given a normal variety X. Every Invertible sheaf on X is isomorphic to $\mathcal{O}_X(D)$ for some Cartier Divisor D on X.*

An equivalent definition of the Picard Group[18]: Given an invertible sheaf $\mathcal{F}$ its inverse is given by $\mathcal{F}^{-1}$=Hom$_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ and thus

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^{-1} \simeq \mathcal{O}_X.$$

Thus, one can define

$$\text{Pic(X)}=\Big\{ \text{ isomorphism classes of invertible sheaves on X}\Big\}$$

By the above definition, $\mathcal{O}_X$ is the identity isomorphism class.

Using the above results, one gets a surjective homomorphism
Div(X) $\rightarrow$ Pic(X) and the Kernel of the homomorphism is Div$_0$(X).Hence for normal varieties , the two definitions of a Picard Group coincide.

Rank One Reflexive Sheaves: Let D be a Weil Divisor on a normal variety X. If D is not Cartier, then we know that $\mathcal{O}_X(D)$ is not invertible. But one can describe this sheaf using the follwoing definition:

Definition 8.3.9. **[18]** Given a sheaf of $\mathcal{O}_X$ modules ,$\mathcal{F}$ we can define its dual $\mathcal{F}^v$ and here is a canonical map $\mathcal{F} \to \mathcal{F}^{vv}$. $\mathcal{F}$ is said to be reflexive of rank one if:
(1)There is a Zariski Open set U such that $\mathcal{F}|_U$ is trivial.
(2) $\mathcal{F}$ is torsion free.
(3) The map $\mathcal{F} \to \mathcal{F}^{vv}$ is an isomorphism.

A very interesting result concerning Weil Divisors is that one can show upto isomorphism that:

Every reflexive sheaf comes from a Weil Divisor on X.

Thus the divisor class group Cl(X) can be given by the group of isomorphism classes of Rank one reflexive sheaves.

9 COHOMOLOGY OF LINE BUNDLES ON TORIC VARIETIES

9.1 Sheaf Cohomology

Definition 9.1.1. [8] Let $\mathcal{A}$ and $\mathcal{B}$ be abelian groups. A **map of sheaves** is given by the collection of homomorphisms $\{f_U : \mathcal{A}(U) \to \mathcal{B}(U)\}$. Here we let U vary over all possible open sets of X. If $U \subset V$, then $f_U$and f_V commute with the restriction maps in $\mathcal{A}$ and $\mathcal{B}$.

Definition 9.1.2. [8] Given a map $f : \mathcal{A} \to \mathcal{B}$, **Kernel of the map** is the sheaf Ker(f) given by Ker(f)(U)=Ker(f$_U$). The Kernel definition works the same way as in commutative algebra but these represent sections of the sheaf.The restriction maps are those of $\mathcal{A}$.

Definition 9.1.3. [8]For any map g:$X \to Y$, Cokernel ,Coker(g) is the quotient Y/g(X). In case of a map of sheaves, if $f : \mathcal{A} \to \mathcal{B}$ is a map, then the **Cokernel** is defined as follows: a section s of the Coker(f) on $U \subset X$ is given by an open cover (U_α) of U and sections $s_\alpha \in \mathcal{B}(U_\alpha)$ such that given α and β ,

$$(s_\alpha)|_{U_\alpha \cap U_\beta}\text{-}\ (s_\beta)|_{U_\alpha \cap U_\beta} \in f|_{U_{\alpha\beta}}(\mathcal{A}(U_\alpha \cap U_\beta))$$

Thus if the Ker(f) is the zero sheaf then the map $f : \mathcal{A} \to \mathcal{B}$ is injective and if Coker(f) is the zero sheaf , then f is surjective. [8]

Definition 9.1.4. Given sheaves $\mathcal{A}, \mathcal{B}, \mathcal{C}$, a sequence

$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0$$

is said to be an **exact sequence** if $\mathcal{A}$=Ker(g) and $\mathcal{C}$=Coker(f)

Cohomology is roughly a method of linearising objects and since studying linearised objects are much easier, cohomology is a very important concept.

An exact sequence of sheaves on X ,

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \ (*)$$

gives rise to an exact sequence between the global sections

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}).$$

Thus given a sheaf $\mathcal{F}$ on a variety X , one can determine a sheaf cohomology group $H^p(X, \mathcal{F})$. This is done using the exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{I}^0 \xrightarrow{d_0} \mathcal{I}^1 \xrightarrow{d_1} \mathcal{I}^2 \xrightarrow{d_2} .....$$

where $\mathcal{I}^0, \mathcal{I}^1, ...$are injective sheaves. A sheaf $\mathcal{I}$ is said to be **injective** if given a sheaf homomorphism $\mathcal{H} \to^{\alpha} \mathcal{I}$ and an injection $\beta : \mathcal{H} \to \mathcal{G}$ there exists a sheaf homomorphism $\theta : \mathcal{G} \to \mathcal{I}$ making the diagram commutative.

Definition 9.1.5. $\mathcal{I}^{\bullet}$ is called a **complex** if it a sequence of morphisms as defined below:

$$.....\to \mathcal{I}^{i-1} \xrightarrow{d^{i-1}} \mathcal{I}^i \xrightarrow{d^i} \mathcal{I}^{i-1}......$$

where d^i called the **differentials** are such that $d^i.d^{i-1} = 0$. Here $\mathrm{Im}(\mathrm{d}^{i-1}) \subset \mathrm{Ker}(\mathrm{d}^i)$ and equality implies the complex is exact.

Definition 9.1.6. Given a sheaf $\mathcal{F}$ on a variety X , the **resolution** of $\mathcal{F}$ is the complex of sheaves $\mathcal{I}^{\bullet}$ which is nonzero only for indices in $\mathbb{Z}_{\geq 0}$ s.t.

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to .....$$

is exact.

Thus complex of global sections can be given as:

$$\Gamma(X, \mathcal{F}^{\bullet}) : \Gamma(X, \mathcal{I}^0) \xrightarrow{d^0} \Gamma(X, \mathcal{I}^1) \xrightarrow{d^1} \Gamma(X, \mathcal{I}^2) \xrightarrow{d^2} ....$$

Definition 9.1.7. [22]pth sheaf cohomology group of $\mathcal{F}$ is defined to be

$$H^p(X,\mathcal{F})=\text{Ker}(d^p)/\text{im}(d^{p-1}).$$

where for p=0 we define d^{-1} to be the zero map $0 \to \Gamma(X,\mathcal{I}^0)$

Properties of a Cohomology group:

•$H^0(X,\mathcal{F})$ =$\Gamma(X,\mathcal{F})$.

• Given a sheaf homomorphism $\mathcal{F} \to \mathcal{G}$. This induces a homomorphism H$^i(X,\mathcal{F}) \to H^i(X,\mathcal{F})$ between the cohomology groups.

• Given a short exact sequence as in (*) , then these cohomology groups result in long exact sequences:

$$0 \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{H}) \to^{\delta_0} H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) \to$$
$$H^1(X,\mathcal{H}) \to^{\delta_1} ....H^p(X,\mathcal{F}) \to H^p(X,\mathcal{G}) \to H^p(X,\mathcal{H}) \to^{\delta_p} ....$$

9.2 Cech Cohomology

Given a topological space X , a sheaf $\mathcal{A}$ of abelian groups on X and $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ an open cover of X.

Definition 9.2.1. [16] q-Simplex σ of $\mathcal{U}$ is an ordered collection of q+1 sets from $\mathcal{U}$ such that their intersection is non-empty. This intersection is called **support** of σ denoted by $|\sigma|$.

Definition 9.2.2. q-Cochain of $\mathcal{U}$ is defined as below:

$$\hat{C}^q(\mathcal{U},\mathcal{F}) = \bigoplus \mathcal{F}(U_{i_0} \cap ......U_{i_p}) = \bigoplus \mathcal{F}(|\sigma|)$$

and an element of a q-Cochain can be thought to be a function α that associates to every q-Simplex an element of $\mathcal{F}(|\sigma|)$.

Here, we can then define a **differential**

$$\hat{C}^q(\mathcal{U},\mathcal{F}) \xrightarrow{d^p} \hat{C}^{q+1}(\mathcal{U},\mathcal{F})$$

where the differntial $d^p(\alpha)$ operates as below:

Let the index set for the open covers chosen to form the q-Simplex be [l]=$\{i_0, i_1, ...i_q\}$. Then,
$d^p(\alpha)(i_0, i_1, ...i_{q+1}) = \sum_{k=0}^{q+1}(-1)^k\alpha(i_0, i_1, ..., \hat{i_k}.......i_{q+1})|_{U_{i_0}\cap......U_{i_{p+1}}}$

where $\hat{i_k}$ denotes the kth partial boundary which is defined below.

Definition 9.2.3. jth partial boundary: This is the (q-1) simplex obtained by removing jth set from σ.

Definition 9.2.4. [22]Cech Complex: Let $\mathcal{F}$ be q sheaf on X and Let $\mathcal{U} = \{U_i\}_{i\in I}$. Then the Cech Complex is

$$\hat{C}^\bullet(\mathcal{U},\mathcal{F}) : 0 \to \hat{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} \hat{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} \hat{C}^2(\mathcal{U},\mathcal{F}) \xrightarrow{d^2} ....$$

Definition 9.2.5. The **pth Cech Cohomology Group** is

$$\hat{H}^p(\mathcal{U},\mathcal{F}) = H^p(\hat{C}^\bullet(\mathcal{U},\mathcal{F}))=\text{ker}(d^p)/\text{im}(d^{p-1}).$$

Lemma 9.2.6. [27] *For any topological space X, open cover $\mathcal{U}$ of X and $\mathcal{F}$ a sheaf, we have $\hat{H}^0(\mathcal{U},\mathcal{F}) \simeq \Gamma(X,\mathcal{F})$.*

Proof. : $\hat{H}^0(\mathcal{U},\mathcal{F})$=ker(d:$C^0(\mathcal{U},\mathcal{F}) \to C^1(\mathcal{U},\mathcal{F})$) If $\alpha \in C^0$ is given by $\alpha_i \in \mathcal{F}(U_i)$ then the differential function is defined such that for i <j , $d_{ij} = \alpha_j - \alpha_i$. So dα =0 implies that α_i and α_j agree on $U_i \cap U_j$ and since $\mathcal{F}$ is a sheaf, then Ker d= $\Gamma(X,\mathcal{F})$. □

Since $\Gamma(X,\mathcal{F}) = H^0(X,\mathcal{F})$, the Cech- Cohomology group and the cohomology group coincide. But this need not be the case always. Under certain conditions as stated below, this equality is true.

Serre Vanishing for Affine Varieties[22]: Given an affine variety U .Let $\mathcal{F}$ be a Quasicoherent sheaf U. Then $H^p(U,\mathcal{F})$=0 for all p >0.

Thus using the above result, one can compute the cohomology group of a quasicoherent sheaf using any affine open cover. This is explained below:
• Given $\mathcal{U} = U_i$ is an open cover of X. The cohomology of X should thus be derived from the cohomologies of these U_i s and their intersections. Thus the complex, $H^\bullet(X, \mathcal{F})$ can be determined by the cohomology groups $H^\bullet(U_{i_0} \cap U_{i_1} ..... \cap U_{i_{p+1}}, \mathcal{F})$ as we vary over all p.

• Since $\mathcal{U} = \{U_i\}$ is an affine open cover, finite intersection is also affine. Hence, according to the previous result, $H^p(U_{i_0} \cap U_{i_1} ..... \cap U_{i_{p+1}}, \mathcal{F}) = 0$ for all p >0. Thus computing $H^0(U_{i_0} \cap U_{i_1} ..... \cap U_{i_{p+1}}, \mathcal{F})$ would give the Cech complex and hence the sheaf cohomology.

Using the above procedure, one can also arrive at the following result.

Theorem 9.2.7. [27] *Given a variety X and an affine open cover* $\mathcal{U}$=$\{U_i\}$*. Let* $\mathcal{F}$ *be a quasicoherent sheaf. Then :*

$$H^p(\mathcal{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F}).$$

Example 9.2.8. [24] Let us try to compute the cohomology of $\mathcal{O}_{\mathbb{P}^1}$. As discussed before, $H^0(\mathbb{P}^1, \mathcal{O})$ is the set of global sections and here these are constant maps. Thus

$$H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C}$$

Also, since these covers are affine, Serre Vanishing theorem states that $H^i(\mathbb{P}^1, \mathcal{O})$=0 for i > 1. Thus we need to compute $H^1(\mathbb{P}^1, \mathcal{O})$.Consider the affine open covers of $\mathbb{P}^1$, $U_0 = \{x_0 \neq 0\}$ and $U_1 = \{x_1 \neq 0\}$. The Cech complex is given by

$$C^1(\mathcal{O}) = \mathrm{O}(U_0 \cap U_1)$$

$$= \{ \tfrac{f}{x_0^a x_1^b} | \text{ f is homogeneous of degree } a + b \}$$

$$= \langle \tfrac{x_0^m x_1^n}{x_0^a x_1^b} | m + n = a + b \text{ where m,n,a,b} \geq 0 \rangle$$

These can also be given as $U_0 = Spec(\mathbb{C}[x])$ and $U_1 = Spec(\mathbb{C}[x^{-1}])$. and this will then give the differential map as below:

$$\mathbb{C}[x] \bigoplus \mathbb{C}[x^{-1}] \to \mathbb{C}[x, x^{-1}]$$

where $d^0(f(x), g(x^{-1})$=f(x)-g(x^{-1})

and since all polynomials can be written as f(x)-g(x^{-1}), this map is surjective and hence $H^1(\mathbb{P}^1, \mathcal{O})$= Coker($d^0$)=0.

9.3 Mixed Volumes

Definition 9.3.1. **[14]**Given a nonzero Laurent polynomial, $f(x_1, x_2, ..., x_n) \in \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, ..., x_n, x_n^{-1}]$. The **Newton Polytope of** f is defined to be the convex hull of the exponents of the monomials in f. i.e. Suppose $f(x_1, x_2, ...x_n) = \sum_{(i_1,.......,i_n)\in\mathbb{Z}^n} c_{i_1,.....i_n} x_1^{i_1}.....x_n^{i_n}$, then the Newton polynomial $P(f)$ is the convex hull of $(i_1, ..., i_n)$ such that the coefficient $c_{i_1,.....i_n}$ is nonzero.

Example 9.3.2. Given the polynomial

$$y^6 + Q_6(x) = y^6 + a_0 + a_1.x^2 + a_3.x^3 + a_6.x^6$$

The Newton polytope is generated by (0,6) which coresponds to the y^6 term and (0,0);(2,0);(3,0);(6,0) corresponding to the x power terms.

The newton polytope is given in the figure below.

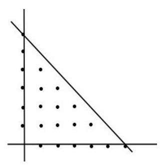

Figure 5: Newton Polytope of $y^6 + a_0 + a_1.x^2 + a_3.x^3 + a_6.x^6$

One can determine the genus of the generic curve using these Newton Polytopes. This is because the **arithmetical genus of a curve** is the number of points with integer coordinates in the interior of the lattice polytope. **([28])**
In the above example the genus of the generic curve $Q_6(x)$ is 10 since the integer points in the lattice are (1,1);(2,1); (3,1); (4,1), (1,2); (2,2); (3,2); (1,3); (2,3); (1,4).

This can be given by a general formula: Genus of a generic curve in n variables with degree m is given by in $\mathbb{C}^2$ is given by (m-1)(m-2)/2.

Remark. The set of all Laurent Polynomials with their respective Newton Polytopes forms a Zariski Open Subset of an affine space.

Definition 9.3.3. [**14**] Given convex polytopes $P_1, P_2, ...P_n$ in $\mathbb{R}^n$ and non-negative real numbers $t_1, ...t_n$, one can define a sum of the polytopes as below:

$$t_1P_1+t_2P_2+......t_nP_n=\{t_1p_1+t_2p_2+......t_np_n|p_1 \in P_1, p_2 \in P_2, ..., p_n \in P_n\}$$

When $t_i = 1$ for all i$\in \{1, 2, ...n\}$ then this sum is called the **Minkowski Sum** $P_1 + P_2...... + P_n$.

Definition 9.3.4. **([14])** The **Volume of this sum**, denoted by vol$(t_1P_1 + t_2P_2 + ...t_nP_n)$ is given by a homogeneous polynomial in t_is and is of degree n. Consider the monomial $t_1t_2...t_n$. The coefficient of this monomial is given by

$$n!V(P_1, ...P_n)$$

where $V(P_1, ...P_n)$ is called the **mixed volume** of these polytopes.

Role of Minkowski Sum[28]: The main reason for studying Minkowski Sum is because the Newton Polytope of a product of polynomials is the Minkowski sum of their respective Newton Polytopes.

Theorem 9.3.5. Bernstein-Kushnirenko's Theorem[14] *Given Laurent Polynomials $f_1, f_2, ..., f_n$ in n variables along with their respective Newton Polytopes $P_1, ..., P_n$. Then the number of zeros of $f_1, f_2, ...f_n$ in $(\mathbb{C}^*)^n$ is*

$$n!V(P_1, ...P_n) \text{ for any } f_1, f_2, ...f_n.$$

As Kushnirenko initially stated the same theorem specifically in the case when the polynomials have identical Newton Polytopes and hence mixed volume is given by the volume of a single such polytope.

This also gives a different version of the **Bezout's Theorem**. This notion is explained in [**28**].
Here if $f \in \mathbb{C}[x_1, x_2, ..., x_n]$ be a polynomial function of degree $\leq d$. The Newton Polynomial $P(f)$ is contained in the simplex $S(d)$ in $\mathbb{R}^n$ since $P(f)$ is the convex hull of its exponents of its n variables and the simplex $S(d)$ is the convex hull of $d+1$ vertices ,namely $(0, 0, ...0), (d, 0, ...0), (0, d, 0, ...0), ...,$ $(0, 0, ..., d)$. Equality between the two occurs for general polynomials f which belong to a nonempty open subset of $\mathcal{S}(P(f))$ where this set consists of the polynomial f and its Newton Polytope $P(f)$. Then,

$$V(S(d_1), ..., S(d_n)) = d_1 d_2.....d_n$$

Thus, the number of common zeros to n polynomials is the product of their degrees.

When monomials of certain degrees $\leq$ m are missing in the given polynomial, the number of roots according to Kushnirenko is smaller than the one given by Bezout's Theorem(m^n) .This is because since some monomials do not appear, some points at infinity become roots of the polynomial . Bezout's Theorem calculates this in the Projective Space and hence takes these roots into consideration but Kushnirenko's doesn't.

Toric Reformulation[**14**] Given convex lattice polytopes $P_1, P_2, ...P_n$ in $\mathbb{R}^n$ with the Minkowski sum denoted by Q.
Given a polynomial $f = \sum c_i.a^i$, then the **Support function** of f is defined as Supp(f)=$\{i|c_i \neq 0\}$.
Thus the support of Q, $h_Q = h_{P_1} + h_{P_2} + ... + h_{P_n}$ and each of these support functions generate a globally generated line bundle L_i .
Define

$$\text{L}=L_1 + L_2 + ...L_n$$

Every 0 homology group $H^0(X, L_i)$ corresponds to the space of Laurent Polynomials whose Newton Polytope is contained in P_i. Thus the Bernstein-Kushnirenko's Theorem can be reformualted for toric varieties as follows:

The number of common zeros of the general sections

$$s_1 \in H^0(X, L_1); s_2 \in H^0(X, L_2)..........; s_n \in H^0(X, L_n)$$

can be given by n!V$(P_1, ...P_n)$ and these zeros are contained in the torus $(\mathbb{C}^*)^n$.

Also, the mixed volume of the polytopes can be interpreted as the mixed degree of the line bundles as below: Consider the line bundle given by $L_1, L_2, ..., L_n$ given by the support functions.
The line bundle $L_1^{\otimes v_1} \otimes L_2^{\otimes v_2}.......... \otimes L_n^{\otimes v_n}$ satisfies

$$\deg(L_1^{\otimes v_1} \otimes L_2^{\otimes v_2}.......... \otimes L_n^{\otimes v_n}) = n!\ V(v_1P_1 + v_2P_2 + ... + v_nP_n).$$

Thus by combining the previous statements, the number of common zeros can be given by the mixed degree of the global sections of these line bundles.

10 THE EHRHART POLYNOMIAL

10.1 Defintions

Consider a lattice polytope. The objective of this section is to find the number of lattice points in all integral multiples of the polytope.

Let us first dissect a polytope into simplices since working over simplices is much easier.

Definition 10.1.1. [7] Given a convex d-polytope P . The **triangulation of P** is T, a finite collection of d-simplices such that
• P=$\cup_{\Delta \in T}\Delta$
• If $\Delta_1, \Delta_2 \in T$,then $\Delta_1 \cap \Delta_2$ is a common face to both Δ_1 and Δ_2.

If the vertices of $\Delta \in T$ are vertices of P for every $\Delta \in T$ then P is said be **triangulated using no new vertices**.

Proposition 10.1.2. [7]**Existence of Triangulations**: *Every Convex Polytope can be triangulated using no new vertices.*

Definition 10.1.3. Consider $\mathcal{K} \subseteq \mathbb{R}^d$ where it is defined as below

$$\mathcal{K} = \Big\{ v + \lambda_1 w_1 + \lambda_2 w_2 + ... + \lambda_m w_m | \lambda_1, \lambda_2, ..., \lambda_m \geq 0 \Big\}$$

Here, v, $w_1, w_2, ..., w_m \in \mathbb{R}^d$ and is such that there exists a hyperplane H satisfying $H \cap \mathcal{K}$=$\big\{$v$\big\}$.This means $\mathcal{K} \setminus \big\{0\big\}$ lies on one side of the hyperplane. This $\mathcal{K}$ is called a **Pointed Cone.**

If these v, $w_1, w_2, ...w_m \in \mathbb{Q}^d$, then it is called a **Rational Cone.**

v is called the **apex** of $\mathcal{K}$ and the w_k , **the generators** of $\mathcal{K}$.

A very important application of cones is **coning over a polytope**. Here, suppose $\mathcal{P}$ is a polytope in $\mathbb{R}^d$ and $v_1, v_2, ..., v_n \in \mathcal{P}$. These vertices can be lifted to vertices in $\mathbb{R}^{d+1}$ by adding a 1 as their last coordinate,i.e.,

$$w_1 = (v_1, 1); w_2 = (v_2, 1); ....; w_n = (v_n, 1)$$

Here Cone($\mathcal{P}$)=$\{\lambda_1 w_1 + ... + \lambda_n w_n | \lambda_1, ...\lambda_n \geq 0\} \in \mathbb{R}^{d+1}$. The origin is the apex of the cone, and the original polytope $\mathcal{P}$ can be obtained by cutting the Cone($\mathcal{P}$) with a hyperplane $x_{d+1} = 1$.

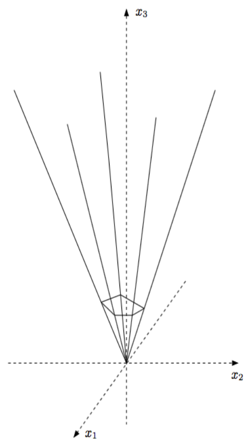

Figure 6: Coning over a polytope $\mathcal{P}$ [**7**]

10.2 Integer-Point Transforms for Rational Cones

Let S$\subset \mathbb{R}^d$ be a set. Consider the lattice points in S. One can try to obtain the information contained in these lattice points using the multivariate generating function defined below. Let S be a Rational Cone or a Polytope and z$\in S$.

$$\sigma_S(z) = \sigma_S(z_1, z_2, ..., z_d) := \sum_{m \in S \cap \mathbb{Z}^d} z^m.$$

This gives all the integer points in S as a sum of the monomials.[**7**]

For example: $\mathcal{K}$=[0,∞] $\sigma_{\mathcal{K}}$(z)= $\sum_{m \in [0,\infty) \cap \mathbb{Z}} z^m = \sum_{m \in \mathbb{Z}_{\geq 0}} z^m = \frac{1}{1-z}$.

Now, the fundamental theorem for counting the number of lattice points in an integer polytope is as below.

Theorem 10.2.1. Ehrhart Theorem[7]: *Given a convex polytope $\mathcal{P}$. $t\mathcal{P}$ is the polytope formed by expanding $\mathcal{P}$ by a factor of t in each dimension. Then there exists a polynomial*

$$L(\mathcal{P},t):=a_d t^d + a_{d-1}t^{d-1} + ... + a_0 \;=\#(t\mathcal{P} \cap \mathbb{Z}^d)$$

and this gives the number of lattice points in the polytope.

Using the generating functions $\sigma_{\mathcal{P}}(z) = \sum_{m\in\mathcal{P}\cap\mathbb{Z}^d} z^m$ one can determine the number of lattice points .This can be done by evaluating the function at z=(1,1,1,.....,1) since

$$\sigma_{\mathcal{P}}(1,1,....1) = \sum_{m\in\mathcal{P}\cap\mathbb{Z}^d} 1^m = \sum_{m\in\mathcal{P}\cap\mathbb{Z}^d} = \#(\mathcal{P}\cap\mathbb{Z}^d)$$

But these generating functions are very useful . As explained before one can form a cone over a polytope and still retrieve the original polytope by cutting with the help of a hyperplane x_{n+1}=1. Similarly , if one cuts the cone with a hyperplane x_{n+1}=2, one gets a twice dilated polytope ,i.e. $2\mathcal{P}$; x_{n+1}=3 gives the polytope $3\mathcal{P}$ and so on.

Here if one computes the integer transform $\sigma_S(z)$, one can obtain points on the polytopes $\mathcal{P}, 2\mathcal{P}, ...$and so on by looking at the terms with $z_{d+1}^1, z_{d+1}^2\cdots$ respectively.

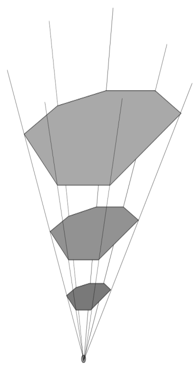

Figure 7: Recovering dilates of $\mathcal{P}$ by cutting with an appropriate hyperplane [7]

Using the generating function, one can prove the Ehrhart theorem for simplices and since every convex integral polytope can be triangulated to simplices, the result will hold for them too.

10.3 Pick's Theorem

There are numerous applications of obtaining the number of lattice points in a lattice. One of it is to obtain more information regarding the polytope such as the area and volume.

Pick's Theorem [32]*Let P be an integral polygon ; I be the number of interior points of P ; B be the boundary points of P. Then,*

$$\text{Area(P)=I+}\frac{B}{2}\text{-1;}$$

Also, one can obtain these measures from the terms of the Ehrhart Polynomial as follows:

$$a_d\text{=Vol}(\mathcal{P}) \text{ ; } a_{d-1} \text{ =Surface area}(\mathcal{P})\text{; } a_0\text{=1 .}$$

11 THE HOMOLOGY GROUP OF TORIC VARIETIES

11.1 Betti Number

Definition 11.1.1. [**13**] Given a nonegative integer q and a topological space X, two abelian groups that one can associate with X is a **singular homology group** $H_q(X)$ and a **cohomology group** $H^q(X)$. These abelian groups have the same rank,called the **Betti Number of X** $b_q(X)$.

Roughly, Betti Number gives the number of times an object(X) can be cut wihtout splitting into pieces.

Example 11.1.2. The Betti Number of a sphere is 0 since cutting it even once splits the sphere into halves.

These Betti Numbers basically describe the features of the object or the topological space including the number of components in the object, number of holes etc in the object.

Given an n-dimesional object there are n+1 Betti Numbers and the Betti numbers of higher index vanish. Each of these numbers represent a particular component of the object in hand. Thus the kth-Betti Number refers to the number of k dimensional components hidden in the object.

Thus,
• the 0th Betti Number b_0 :the number of components that make the object
• the 1st Betti Number b_1: the number of holes in the object
• the 2nd Betti Number b_2: the number of cavities hidden in the object.

Example 11.1.3. Spheres:

b_0=1(since it is made up of a single component)

b_1=0(since there are no holes in a sphere (surface,not solid))

b_2=1(since there is 1 cavity in a sphere ,the inside)

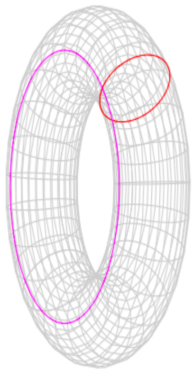

Figure 8: Torus showing the inner and outer circle contributing to the 1st Betti Number [**12**]

Example 11.1.4. Torus:

b_0=1(since it is made up of a single component);

b_1=2(since it has an inner and a outer circle) b_2=1(since there is one cavity, the inside of the torus)

11.2 Poincare Polynomial

Definition 11.2.1. Poincare polynomial is defined to be the generating function of the Betti Numbers and is given as below:

$$P_X(t)=\sum_{q=0}^{n} b_q(X)t^q$$

where $b_q(X)$ is the qth Betti Number of the topological space X.

Example 11.2.2. For a **Sphere** the Poincare Polynomial is given by $1 + x^2$ and for a **Torus** it is given by $1 + 2x + x^2$.

11.3 Euler Characteristic

Definition 11.3.1. Euler Characteristic of the object is the alternated sum of the Betti numbers.

$$\chi(X)=\sum_{q=0}^{n}(-1)^q b_q(X) = P_X(-1)$$

This number describes the structure and shape of a topological space irrespective of how it is bent.A very important result is that

Euler characteristic of a polyhedra is always 2.

Euler Charactersitic can be calculated by splitting the surface into triangles.But no matter how the triangulation of the surface is done, this quantity remains the same.
When given the genus g of the space, the Euler Characteristic is given by

$$\chi(X)= 2\text{-}2g$$

Also, given any surface one can determine the Euler Characteristic using the below formula:

$$\chi(X) = V - E + F$$

by triangulating the surface in a "nice" way i.e. dividing it into triangles in such a way that the intersections are vertices or edges which form the edges of the triangles.

Using the above concepts one can prove the following important theorem concerning smooth toric varieties:

Theorem 11.3.2. (Ehlers)[13] *Given a complete smooth toric variety X of dimension n and defined by the fan Δ. The 2 abelian group $H_*(X)$ and $H^*(X)$ are free and the Poincare Polynomial is as below:*

$$P_X(t) = \sum_{\sigma\in\Delta}(t^2-1)^{n-dim(\sigma)} = \sum_{\sigma\in\Delta} t^{2dim(\sigma)}(1-t^2)^{n-dim(\sigma)}$$

In order to prove the above theorem, the following definitions are important:

Definition 11.3.3. Projective Varieties are inductively built from affine spaces. This can be generalised further as **Cellular Decompositions**.
Given an algebraic variety X , the decomposition of X into finitely many disjoint subsets $C_i (i \in I)$ such that:
(1) Each C_i is a locally closed subvariety of X.
(2) Each C_i is isomorphic to some complex affine space $\mathbb{C}^{n_i}$

These C_i s are called cells of dimension $2n_i$.

Definition 11.3.4. If the cells $C_1, ..., C_r$ are such that each closure $\bar{C_i} \subseteq \cup_{j \leq i} C_j$, then the cellular decomposition is said to be **filtrable**.

Thus a Projective Space can be given a filtered Cellular Decomposition using the affine coordinate patches.

Similarly one can construct a cellular decomposition of a complete smooth toric variety X of dimension n . Let $v \in N$ be a point in the lattice and outside all the hyperplanes spanned by all the $(n-1)$ dimensional cones of the fan Δ defining the toric variety.

One can define a one-parameter subgroup $\lambda_v : \mathbb{C}^* \to T$. The fixed points of this are exactly the T-fixed points x_σ , where $\sigma \in \Delta(n)$.

Define the following set:

$$\mathrm{C}(v,\sigma)=\left\{x \in X \,|\, \lim_{z \to 0} \lambda_v(z)x = x_\sigma\right\}$$

Proposition 11.3.5. : *For a complete smooth toric variety X , these C(v, σ) form the cellular decomposition of X and if X is projective , this decomposition is filtrable.*

One can prove the theorem stating the Poincare Polynomial for a toric variety by using the above proposition and the following concepts.

11.4 Singular Homology and Cohomology groups

Definition 11.4.1. Given a nonnegative integer q. Define :

$$\Delta_q = \left\{(x_0, x_1, ..., x_q) \in \mathbb{R}^{q+1}_{\geq 0} | x_0 + x_1 + x_2 + ... + x_q = 1\right\}$$

This set is called the **standard q-simplex**.One also gets a map

$$\sigma^i : \Delta_{q-1} \to \Delta_q$$

since Δ_q meets the hyperplane (x_i=0) along Δ_{q-1}.

Definition 11.4.2. Given a topological space X, one can define a **singular q-simplex** as a continuous map

$$\sigma : \Delta_q \to X$$

Definition 11.4.3. The group on the set of all singular q-simplices forms an abelian group denoted by $C_q(X)$ and consists of finite sums of singular q-simplices, called **chains**.

A differential map can be defined on these $C_q(X)$ as follows:

$$\delta_q : C_q(X) \to C_{q-1}(X) \text{ where } \delta_q(\sigma) = \textstyle\sum_{i=-0}^{q}(-1)^i \sigma^i$$

Thus, $C_*(X) = (C_q(X), \delta_q)$ is a complex.

Definition 11.4.4. The qth homology group of the complex,$C_*(X)$, is called the **qth singular homology group of X**, $H_q(X)$.

Definition 11.4.5. qth Singular Cohomology Group $H^p(X)$ is the Homology group of the complex $C^*(X)$ which is dual to $C_*(X)$ i.e. $C^*(X)$ =Hom$(C_q(X), \mathbb{Z})$.

The differential $C^q(X) \to C^{q-1}(X)$ is a transpose of

$$\delta_{q-1} : C_{q-1}(X) \to C_q(X)$$

The elements of $C^*(X)$ are called **cochains**.

These two groups, $H^q(X)$ and $H_q(X)$ have the same rank.

Remark. : Given a contractible topological space X. This means the topological space can be shrunk to a point by still staying within the space during the process. In terms of maps, the identity map on X is homotopic to a constant map. One can characterise the two groups as below:

$$H_0(X) \simeq \mathbb{Z} \simeq H^0(X)$$

$$H_q(X) = 0 = H^q(X) \text{ for all q} \geq 1$$

Now, given a space that is built up from smaller subspaces. One can study the homology groups of the bigger space using that of the smaller constituting subspaces.

Definition 11.4.6. [**13**] Given a topological space X and a subspace Y, with an inclusion map $i : Y \to X$ .This defines an injective map $i_* : C_*(Y) \to C^*(X)$ .Define the set as follows:

$$C_q(X, Y) = C_q(X)/i_q C_q(Y)$$

The Homology group of this complex is called **the relative homology groups of the pair(X,Y)** and is denoted by $H_q(X, Y)$.

Using the information about the cellular decomposition of a smooth toric variety and the above mentioned concepts, one can prove Theorm 11.3.2.

12 CONCLUSION

Toric Geometry is a part of Polyhedral Geometry which is closely linked to Algebraic Geometry. In Polyhedral Geometry, cones play the role of affine varieties in algebraic geometry since cones can further be used to form fans which then generate Toric Varieties.

As discussed in the report, one can compute the combinatorial elements of a toric variety like the divisor class group, picard group , cohomology group etc and this reveals a lot of information about the geometric nature of the variety in hand.

In recent times, toric geometry has seen wide number of applications in areas such as mirror symmetry, integer programming, solving systems of polynomial equations ,coding theory and algebraic statistics.

There is a strong interplay between topics in toric geometry and string theory. Physicists believe that world is 4 dimensional -spacetime. But in string theory one requires extra dimensions to understand its behaviour. Some of the ongoing research studies try to understand strings as particles and thus study it by assuming it has large dimensions with the property of compactification wherein it drops down to 4 dimensions. The way the dimensions are compactified don't matter since the properties of the compactified string is the same irrespective of the way of compactification. These compactified spaces form manifolds and these manifolds are said to be symmetric. This field of study is called Mirror Symmetry. Toric Geometry helps in understanding mirror symmetry and this is still a thriving area of research.

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