RANK TWO WEAK FANO BUNDLES ON FANO THREEFOLDS

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1. Weak Fano bundles

In this note, the author would like to sum up joint works with T. Fukuoka and D. Ishikawa, about weak Fano bundles [1, 2, 3]. Throughout this note we work over the complex number field $\mathbb{C}$. We start from the definition of weak Fano bundles.

Definition 1.1. Let X be a smooth projective variety.
(1) X is called *weak Fano* if $-K_X$ is nef and big.
(2) A vector bundle $\mathcal{E}$ on X is called *weak Fano* if $\mathbb{P}_X(\mathcal{E}) = \operatorname{Proj}\operatorname{Sym}^{\bullet}\mathcal{E}$ is a weak Fano variety.

It is known that a variety which admits weak Fano bundle is also weak Fano [9]. Fano varieties forms an important class of projective varieties from the point of view of the classification of algebraic varieties, and weak Fano varieties can be regard as a degeneration of Fano manifolds. However studying weak Fano manifolds still has a big contribution to the classification problem. For example, studying weak Fano threefolds revealed a new aspects of the classification of Fano threefolds. Namely, weak Fano threefolds allowed to connect different classes of Fano threefolds in terms of the technique so called 2-ray game [8].

Following the big success of the theory of Fano and weak Fano threefolds, it is natural to start studying weak Fano fourfolds. However, the world of Fano and weak Fano fourfolds is much more complicated than the threefold case, so we should concentrate on a certain "nice" class of weak Fano fourfolds as the first step. Then, following the work of Szurek and Wiśniewski [7], it would be interesting to study weak Fano fourfolds that admit a locally trivial $\mathbb{P}^1$-bundle structure, in other words, those are isomorphic to the projectivization of rank two weak Fano bundles. This kind of study is initially stated by Yasutake, and he classified rank two weak Fano bundles on $\mathbb{P}^3$ [9].

Let $\mathcal{E}$ be a rank two weak Fano bundle on a threefold X. If $Y = \mathbb{P}_X(\mathcal{E})$ has $\rho(Y) = 2$, then $\rho(X) = 1$ and hence X is a Fano threefold. It is well-known that Fano threefolds of $\rho(X) = 1$ have the following classification. Let $i(X)$ be the Fano index of X.

$i(X) = 4$ $\mathbb{P}^3$ three dimensional projective space.
$i(X) = 3$ Q^3 three dimensional quadric hypersurface.
$i(X) = 2$ X_d del Pezzo threefold of degree d, where $1 \leq d \leq 5$.
$i(X) = 1$ This class is called Mukai threefolds and also can be classified.

Our work classified rank two weak Fano bundles on Fano threefold X of $\rho(X) = 1$ and $i(X) \geq 2$.

2. Key Theorems

Since we consider rank two bundles $\mathcal{E}$, and since the definition of weak Fano bundles is up to line bundle twist, rank two weak Fano bundles are divided into two types according to the evenness and the oddness of their first Chern class. The following theorem give a very good description for one of these two types.

Theorem 2.1 ([1]). *Let $\mathcal{E}$ be a rank two weak Fano bundle on a Fano threefold X of Picard rank one. If $c_1(\mathcal{E}) = c_1(-K_X)$, then $\mathcal{E}$ is globally generated.*

For the other type, we can give a complete classification. Surprisingly, it's revealed that this type contains Fano bundles only.

Theorem 2.2. *Let $\mathcal{E}$ be a rank two weak Fano bundle on a Fano threefold X of Picard rank one. If the difference $c_1(\mathcal{E}) - c_1(-K_X)$ is an odd integer via the identification $\mathrm{Pic}(X) \simeq \mathbb{Z}$, then $\mathcal{E}$ is a Fano bundle.*

This theorem is proved by combining all existing works for rank two weak Fano bundles on Fano threefolds [1, 2, 4, 9] and our work in progress [3].

Note that there exists a complete classification for rank two Fano bundles by Munõz, Occhetta, and Solá Conde [6]. Therefore, to classify weak Fano bundles, we can concentrate on the first case.

3. Classification for del Pezzo threefolds

A rank two vector bundle is called split if it is isomorphic to a direct sum of line bundles.

Theorem 3.1 ([2]). *Let X_d be a del Pezzo threefold of degree ≤ 2. Then all weak Fano bundles on X are split.*

For del Pezzo threefolds of degree ≥ 3, we can find non-split weak Fano bundles. In the following, we concentrate on the case when $c_1(\mathcal{E}) = 0$, because the other case is already covered by Theorem 2.2. It's possible to show that any non-split vector bundle is semistable, and if it's not stable, then it's related to the geometry of lines.

Proposition 3.2 ([4]). *Let $X = X_d$ be a del Pezzo threefold of Picard rank one and of degree $3 \leq d \leq 5$. If $\mathcal{E}$ is a rank two non-split weak Fano bundle with $c_1(\mathcal{E}) = 0$ that is not stable, then $\mathcal{E}$ fits into a unique non-trivial exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_{l/X} \to 0,$$

where $\mathcal{I}_{l/X}$ is the ideal sheaf of a line $l \subset X$. Conversely, if $\mathcal{E}$ is a vector bundle that fits in the sequence above, then it's weak Fano, but not Fano.

Here the unique non-trivial extension means the one corresponds to a non-trivial element of $\mathrm{Ext}^1(\mathcal{I}_{l/X}, \mathcal{O}) \simeq \mathbb{C}$.

The remaining case, when $\mathcal{E}$ is stable weak Fano bundle with $c_1(\mathcal{E}) = 0$, it can be described in a similar way, but in this case it's related to elliptic curves. Recall that $\mathcal{E}(1)$ is globally generated by Theorem 2.1. Thus for generic global section $s \in H^0(\mathcal{E}(1))$, one can show by elementary calculations that it's zero locus $C = V(s)$ is a smooth elliptic curve of degree $\deg(X) + c_2(\mathcal{E})$. In addition, since $\mathcal{E}(1)$ fits into the exact sequence

$$0 \to \mathcal{O} \to \mathcal{E}(1) \to \mathcal{I}_{C/X}(2) \to 0,$$

this elliptic curve C is generated by quadratic equations. Note that $\mathrm{Ext}_1(\mathcal{I}_{C/X}(2), \mathcal{O}) \simeq \mathbb{C}$, and hence the non-trivial extension above is unique up to isomorphism.

Theorem 3.3 ([1, 2, 4])**.** *Let $X = X_d$ be a del Pezzo threefold of Picard rank one and of degree $3 \leq d \leq 5$. Let $\mathcal{E}$ is a rank two stable vector bundle on X with $c_1(\mathcal{E}) = 0$. Then $\mathcal{E}$ is weak Fano if and only if the following two conditions are satisfied.*

(1) $2 \leq c_2(\mathcal{E}) \leq d - 1$.

(2) *$\mathcal{E}$ fits into a unique non-trivial exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E}(1) \to \mathcal{I}_{C/X}(2) \to 0,$$

where $\mathcal{I}_{C/X}$ is the ideal sheaf of a smooth elliptic curve $C \subset X$ of degree $d + c_2(\mathcal{E})$ which is defined by quadratic equations.

Furthermore, for any $3 \leq d \leq 5$ and $2 \leq c \leq d - 1$, there is an example of a stable weak Fano bundle on X_d with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = c$.

We remark that all weak Fano bundles as in the theorem are not Fano.

In the proof of the existence of examples for stable weak Fano bundles, the most difficult case is when $(d, c) = (4, 3)$. In this case, we should prove that any del Pezzo threefold of degree 4 contains a smooth elliptic curve of degree 7 which is generated by quadratic equations. To workout this problem, we prepared the following geometric characterisation for quadratically generated elliptic curves. Recall that a del Pezzo threefold of degree 4 is a smooth intersection of two hyperquadrics in $\mathbb{P}^5$.

Lemma 3.4 ([1])**.** *Let $C \subset \mathbb{P}^5$ be a smooth elliptic curve of degree 7. Then C is generated by quadratic equations if and only if it doesn't have a trisecant.*

In the construction, we find a reducible curve of arithmetic genus 1 and of degree 7 that doesn't have a trisecant, and then we prove that there exists a smoothing of that curve. Then since the non-existence of trisecants is an open condition, the desired elliptic curve is obtained. Please see [1] for more detail.

We finish this section with the following remark, which is immediately follows from Theorem 2.1

Proposition 3.5. *Let $\mathcal{E}$ be a rank two stable weak Fano bundle with $c_1(\mathcal{E}) = 0$ on a del Pezzo threefold of Picard rank one. Then $\mathcal{E}$ is an instanton bundle in the sense of Kuznetsov* [5]*.*

4. Further study

4.1. **Moduli.** Studying moduli space is an important part of the classification. Let $M_{d,a,c}$ be the moduli of rank two weak Fano bundles on del Pezzo threefold of degree d with $(c_1, c_2) = (a, c)$. We studied this moduli $M_{d,a,c}$ when $d = 5$ [2], and the case $d = 3$ is known by previous works about instanton bundles. The most difficult case is, again, when $(d, a, c) = (4, 0, 3)$. In this case we only know that $M_{4,0,3} \neq \emptyset$ by Theorem 3.3.

4.2. **Second contraction.** If $\mathcal{E}$ is a weak Fano bundle which is not Fano, then $Y = \mathbb{P}(\mathcal{E})$ has a K_Y-trivial contraction. It's very interesting to study the explicit geometry of this contraction for each examples in our classification. We worked out some easy cases in our work in progress, but there are still many open cases. We hope to address to this problem in the future.

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