Computational birational geometry of minimal rational surfaces

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Abstract

The classification of minimal rational surfaces and the birational links between them by Iskovskikh, Manin and others is a well-known subject in the theory of algebraic surfaces. We explain algorithms that realise links of type II between minimal del Pezzo surfaces, one of the major classes of birational links, and we describe briefly how this fits into a large project to implement the results of Iskovskikh's programme in MAGMA.

The theory of minimal rational surfaces and their birational links works over an arbitrary perfect field k. Our interest here is the case $k = \mathbb{Q}$ or a number field, in part because we can compute in these fields, but also because the implementation we present here imposes some conditions on the characteristic. We let $\overline{k}$ denote the algebraic closure of k. When a geometric object A is defined over k, $\overline{A}$ denotes its base change to $\overline{k}$. All algebraic surfaces in this paper are nonsingular. A surface X is *rational* if there is an isomorphism $\overline{X} \xrightarrow{\cong} \mathbb{P}^2_{\overline{k}}$ defined over $\overline{k}$ (but not necessarily over k). This notion is sometimes also called *geometrically rational*, to emphasise the algebraic closure in the statement. A surface X is *minimal* if any birational morphism $X \to Y$ defined over k is an isomorphism.

When $k = \mathbb{C}$ the results of the theory are over a century old. The use of Mori theory and the Sarkisov programme, following Corti [C] and Iskovskikh [I], make our approach here very different to what it would have been 15 years ago. Yet more recent results, such as Hacon and McKernan's approach to the Sarkisov programme, may change it again in the future.

As an introduction, we outline the results of the theory of minimal rational surfaces. This theory is the end result of 150 years of development: from Cayley's computation [Ca] of the 27 lines on a cubic surface around 1850 (in effect, computing the divisor class group); via Castelnuovo's rationality criterion of around 1900 (which determines, over an algebraically closed field, whether or not the surface can be rationally parametrised); Segre's analysis [Se] in 1942 of the nonrationality of a cubic surface in terms of its Picard group, working over an arbitrary field; and Manin [M] and Swinnerton-Dyer's [SD] geometrical analysis of rationality in the 1960s in terms

1

of points and configurations of the 27 lines; and culminating in Iskovskikh's complete analysis in a series of papers during the 1980s. A full account of the modern theory is given in [I]; it is also sketched in the appendix to [C], together with details of the (uni)rationality of these surfaces over the ground field. The book [KSC] gives proofs of several parts of the theory.

We sketch the classical and modern theories in Section 1. Our main technical tools are explained in Section 2, and they are applied in Section 3 to compute the major birational maps of the theory: Geiser and Bertini involutions. In Section 4 we show how our algorithms can be applied to analyse the Cremona group of birational selfmaps of a minimal cubic surface. Section 5 explains how the work described here fits into a broader research programme.

We have implemented the algorithms described below in the computational algebra system MAGMA [BCP]; all code and examples are available at [BKR].

1 Minimal models of rational surfaces

1.1 Classical results over the complex numbers

A complete analysis of this case is in [KSC] Chapters 2 and 3. Over $\mathbb{C}$, a surface is minimal if and only if it does not contain any curves $C \cong \mathbb{P}^1$ with $C^2 = -1$.

The minimal rational surfaces over $\mathbb{C}$ are: $\mathbb{P}^2$, the minimal rational surface scrolls $\mathbb{F}_n$ for $n \geq 2$, and $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, which is often embedded as a quadric $(xy = zt) \subset \mathbb{P}^3$. MAGMA treats scrolls as ambient spaces using multigraded rings:[1] the scroll $\mathbb{F}_n$ has homogeneous coordinate ring $k[u, v, x, y]$ bi-graded by the columns of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & -n \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and irrelevant ideal $(u, v) \cap (x, y)$, which is indicated by the separating vertical line in the matrix. From the point of view of Mori theory, it is crucial to consider each surface with a chosen structure map: for $\mathbb{P}^2$ it is the trivial map to a point, and for the scrolls $\mathbb{F}_n$ with $n \geq 2$ it is their natural map to $\mathbb{P}^1$. The quadric $\mathbb{F}_0$ admits two maps to $\mathbb{P}^1$, the two projections to its factors, and we must choose one of these. Although we often omit the map in our notation, it is illiterate not to have it (or, equivalently, the corresponding choice of extremal ray) in mind.

We describe four classes of *birational links*, or elementary birational maps, between these surfaces (including $\mathbb{F}_1$, although it is not minimal):

   I (blowup) $\mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ in suitable coordinates by $(u, v, w) \mapsto (u, v, w, 1)$.

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[1] This is not in the 2008 export of MAGMA, but will be included in later versions. More generally, any implementation of toric geometry contains surface scrolls among its first examples.

II (elementary transformation) either $\mathbb{F}_i \dashrightarrow \mathbb{F}_{i-1}$ in coordinates by $(u, v, x, y) \mapsto (u, v, x, uy)$, or $\mathbb{F}_i \dashrightarrow \mathbb{F}_{i+1}$ in coordinates by $(u, v, x, y) \mapsto (u, v, ux, y)$.

III (blowdown) $\mathbb{F}_1 \to \mathbb{P}^2$ in suitable coordinates by $(u, v, x, y) \mapsto (uy, vy, x)$.

IV (change factors) the identity map on $\mathbb{F}_0$, but taking the two different projection maps $\mathbb{F}_0 \to \mathbb{P}^1$ on the source and target.

The main result is that any birational map between minimal rational surfaces factorises as a composition of these elementary birational maps.

Theorem 1.1 (Noether–Castelnuovo). *Let X and Y be minimal rational surfaces over $k = \mathbb{C}$ and $\varphi \colon X \dashrightarrow Y$ a birational map between them. Then there are birational links $\varepsilon_1, \ldots, \varepsilon_r$ and an automorphism ψ of Y such that $\varphi = \psi \circ \varepsilon_r \circ \cdots \circ \varepsilon_1$.*

For example, if $X = Y = \mathbb{P}^2$ and $\varphi \colon (x, y, z) \mapsto (1/x, 1/y, 1/z)$ is the standard quadratic Cremona transformation then φ factorises as

$$\mathbb{P}^2 \dashrightarrow \mathbb{F}_1 \dashrightarrow \mathbb{F}_0 \dashrightarrow \mathbb{F}_1 \to \mathbb{P}^2.$$

Indeed, as a composition of maps, up to a linear automorphism of $\mathbb{P}^2$ it is

$$(x, y, z) \mapsto (x, y, z, 1) \mapsto (x, y, z, x) \mapsto$$
$$(x, y, z, xy) = (xz, yz, 1, xy) \mapsto (xz, yz, xy) = (1/y, 1/x, 1/z),$$

corresponding to the sequence: blowup in $(0, 0, 1)$, elementary transform in $(0, 1, 0, 1)$, elementary transform in $(1, 0, 0, 1)$, blowdown of the negative section.

1.2 Minimal rational surfaces over a perfect field

We will be concerned with two classes of surfaces. First, the class of del Pezzo surfaces of Picard rank 1; such surfaces are automatically minimal since any nontrivial (birational) morphism $X \to Y$ decreases the Picard rank. Second, the class of conic bundles over a smooth rational curve. Surfaces in this class are not necessarily minimal; the surface $\mathbb{F}_1$ is not minimal, for instance. Nevertheless, a minimal rational surface over k belongs to one of these two classes.

Minimal del Pezzo surfaces. These are surfaces X with $-K_X$ ample and $\mathrm{Pic}(X) \cong \mathbb{Z}$. In their (pluri-)anticanonical embedding, they are in one of the families listed in Table 1; in this embedding, the divisor of any degree 1 linear section of X is linearly equivalent to $-K_X$. In the table, the entries $X \subset \mathbb{P}^5$, $X \subset \mathbb{P}^6$, $X \subset \mathbb{P}^8$, $X \subset \mathbb{P}^9$ are surfaces of degree d in $\mathbb{P}^d$; these are not our main concern here, so we give only coarse information about their equations (although see Section 5.2 for more results). In degree $d = 5$, the ideal of $X \subset \mathbb{P}^5$ is generated by the five maximal pfaffians of

K_X^2	Description
1	$X_6 \subset \mathbb{P}(1,1,2,3)$
2	$X_4 \subset \mathbb{P}(1,1,1,2)$
3	$X_3 \subset \mathbb{P}^3$
4	$X_{2,2} \subset \mathbb{P}^4$
5	$X \subset \mathbb{P}^5$ a general linear section of $\mathrm{Grass}(2,5) \subset \mathbb{P}^9$
6	$X \subset \mathbb{P}^6$ defined by 9 quadrics
8	$X \subset \mathbb{P}^8$ defined by 20 quadrics
9	$X \subset \mathbb{P}^9$ defined by 27 quadrics

Table 1: Families containing the minimal del Pezzo surfaces. The case $K^2 = 8$ splits into two, according to whether X is isomorphic to a quadric $X \cong X_2 \subset \mathbb{P}^3$ or not. The case $K^2 = 9$ splits into two, according to whether $X \cong \mathbb{P}^2$ or not.

a skew 5×5 matrix of linear forms—that is one interpretation of the role of the Grassmannian.

We emphasise that whether a given surface X in Table 1 actually belongs to the class we are discussing depends on whether $\mathrm{Pic}(X) \cong \mathbb{Z}$; this, in turn, depends on both k and on the defining equations of X, and in general it is a difficult problem. If so then the Picard group is generated by the class of $-K_X$ in all cases except the nonsingular quadric $X_2 \subset \mathbb{P}^3$ (a special case for $K^2 = 8$), when $-\frac{1}{2}K_X$ generates, and $\mathbb{P}^2$ (a special case for $K^2 = 9$), when $-\frac{1}{3}K_X$ generates.

Conic bundles. These are surfaces X that admit a morphism $f\colon X \to C$ to a smooth rational curve C such that $\mathrm{Pic}(X) = f^* \mathrm{Pic}(C) + \mathbb{Z}$ and a general geometric fibre F of f is a conic $F_2 \subset \mathbb{P}^2$. The base curve C may have k-rational points (in which case it is isomorphic to $\mathbb{P}^1$) or not (in which case it is itself isomorphic to a plane conic).

If $q_1, \ldots, q_m \in C$ are the closed points at which f is degenerate—the irregular values of f—then the degree of X is

$$K_X^2 = 8 - \deg(q_1) - \cdots - \deg(q_m).$$

When $C \cong \mathbb{P}^1$, the surface X can be written as a relative anticanonical model:

$$X = (F = 0) \quad \text{in the scroll} \quad \mathbb{F}_{a,b} = \left(\begin{array}{cc|ccc} 1 & 1 & 0 & -a & -b \\ 0 & 0 & 1 & 1 & 1 \end{array} \right)$$

where $0 \le a \le b$ and $d \ge 0$ are all integers, and $F = F(u,v,x,y,z)$, in coordinates u,v,x,y,z on $\mathbb{F}_{a,b}$ of bi-degrees given by the columns of the matrix, is bi-homogeneous of weight $(d,2)$; in particular, F is a quadric in x,y,z. (Although only $d \ge -a$ is required for this linear system to contain irreducible surfaces, such X

has a section whenever $d < 0$ and so is not minimal unless $X \cong \mathbb{F}_n$; compare with Lecture 2 of [R].) In this notation, it is easy to compute

$$K_X^2 = 8 - 3d - 2(a + b),$$

so $\sum \deg(q_i) = 3d + 2(a + b)$. Suitable diagonal surfaces

$$X = (A_d x^2 + B_{d+2a} y^2 + C_{d+2b} z^2 = 0) \subset \mathbb{F}_{a,b}$$

with A, B, C forms in u, v of the indicated degrees give examples with arbitrarily complicated irregular values (the roots of $ABC = 0$ where the remaining quadric is irreducible over k).

1.3 Birational links

The surfaces we consider are examples of two-dimensional Mori fibre spaces, that is, maps $f\colon X \to S$ with X a surface, S a point or a nonsingular curve and f a morphism with connected fibres, $-K_X$ relatively ample and of relative Picard rank 1. The map f is simply the given map for a conic bundle, and it is the trivial map to the point $\operatorname{Spec} k$, denoted also by $\{*\}$ with k implicit, when X is a minimal del Pezzo surface.

A *birational link* between two-dimensional Mori fibre spaces $f\colon X \to S$ and $f'\colon X' \to S'$ is a diagram

$$\begin{array}{ccc} X & \dashrightarrow^{F} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & & S' \end{array} \qquad (1.1)$$

in which F is a birational map arising in one of four ways:
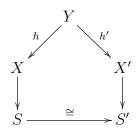
Type I. These are commutative diagrams of the form

$$\begin{array}{ccc} X & \xleftarrow{\ F^{-1}\ } & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

where F^{-1} is the blowup of an irreducible closed point of X. For example, take $X = \mathbb{P}^2$, $S = \{*\}$ (a point), $X' = \mathbb{F}_1$ and $S' = \mathbb{P}^1$.

Type II. In this case there is a surface Y and maps $h\colon Y \to X$ and $h'\colon Y \to X'$ fitting into a commutative diagram:

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle h}\swarrow & & \searrow{\scriptstyle h'} & \\ X & & & & X' \\ \downarrow & & & & \downarrow \\ S & \xrightarrow{\ \cong\ } & & & S' \end{array}$$
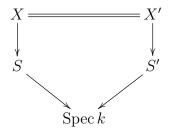
5

The maps h and h' are the blowups of irreducible closed points of X and X' respectively.

Type III. These are inverses of links of type I, so there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ F\ } & X' \\
\downarrow & & \downarrow \\
S & \longrightarrow & S'
\end{array}
$$

in which F is the blowup of an irreducible closed point of X'.

Type IV. Here we have a diagram of the form

$$
\begin{array}{ccc}
X & =\!=\!=\!=\!=\!=\!= & X' \\
\downarrow & & \downarrow \\
S & & S' \\
& \searrow \quad \swarrow & \\
& \operatorname{Spec} k &
\end{array}
$$

in which X and X' are the same surface but the link changes the Mori fibre space structure. For example, take $X = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with $X \to S$ the projection onto the first factor $\mathbb{P}^1$ and $X = X' \to S'$ the projection onto the second factor.

The following central result is the analogue of Theorem 1.1.

Theorem 1.2 (Iskovskikh [I] Theorem 2.5). *Let X and Y be minimal rational surfaces over a perfect field k and $\varphi \colon X \dashrightarrow Y$ a birational map between them. Then there are birational links $\varepsilon_1, \ldots, \varepsilon_r$ and an automorphism ψ of Y, all defined over k, such that $\varphi = \psi \circ \varepsilon_r \circ \cdots \circ \varepsilon_1$.*

1.4 Geiser and Bertini involutions

In addition to the factorisation theorem above, Iskovskikh [I] Theorem 2.6 classifies all the birational links (1.1) that can occur into 41 different classes according to the type of X and X'. In Sections 2–3, we describe the tools central to the implementation of seven of these classes: the Geiser and Bertini involutions on del Pezzo surfaces of degrees 2, 3, 4 and 5. These are all links of type II.

We use the following notation throughout. The degree 1 hyperplane section of $X \subset \mathbb{P}^d$ is denoted by A, and $|nA|$ denotes the linear system of all sections of degree n. We regard elements of $|nA|$ both as homogeneous polynomials on the whole of $\mathbb{P}^d$ and as the zero loci on X they define, and we move freely between these

two descriptions. (When n is large, we may take a vector space complement to those polynomials that lie in the ideal of X—they do not define divisors on X, of course.) A subsystem is denoted by $\mathcal{H} \subset |nA|$; this is some linear subspace of polynomials of degree n. If $P \in X$ is a point, then the space of polynomials of degree n whose zero loci on X vanish to order m at P is denoted by $|nA - mP|$ (or $\mathcal{H}(-mP)$ if restricting attention to a subsystem $\mathcal{H}$).

Geiser involutions

Let $X \subset \mathbb{P}^d$ be a del Pezzo surface of degree $d = 3$, 4 or 5 and $P \in X$ a closed point of degree $d - 2$. The *Geiser involution* $i_P \colon X \dashrightarrow X$ *with centre* P is defined as follows: P spans a linear $\Pi = \mathbb{P}^{d-3}$, and a general hyperplane $H = \mathbb{P}^{d-2}$ containing Π intersects X in d points, the sum of P and an effective 0-cycle Q of degree 2; i_P exchanges the geometric points of Q (whether they are defined over k or not).
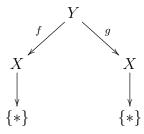
This clearly defines an involution of X and it is straightforward to check that i_P is a rational, and hence birational, map. Moreover, if we suppose that X is minimal, following Iskovskikh the linear system corresponding to i_P is $|(d-1)A - dP|$.

Remark. The minimality condition may be surprising here; the point is that if $(E \subset Y) \to (P \in X)$ is the blowup of $P \in X$ then Riemann–Roch computes

$$\chi\big((d-1)A - dE\big) = d + 1,$$

but we need to prove that this is equal to $h^0\big((d-1)A - dE\big)$, which requires the first cohomology to vanish. Minimality ensures this, although in practice it is often the case for nonminimal examples too. The same remark holds for Bertini involutions.

Now we describe i_P as a link of type II, in the style of Section 1.3. We form the diagram

in which $f \colon Y \to X$ is the blowup of P, the Mori fibre space structure $X \to \{*\}$ on X is the trivial one, mapping to a point, and $g \colon Y \to X$ is a morphism that we define below.

It can be shown that there is a unique effective curve C on X that is of anti-canonical degree $d - 2$ and has multiplicity $d - 1$ at each geometric point $P' \in P$: for example, if $d = 3$ then C is the tangent curve $T_P(X) \cap X$. Let $F \subset Y$ be the strict transform of C. Then the following hold (we omit the proofs): F is a (-1)-curve or a union of conjugate (-1)-curves, its contraction g maps to $\mathbb{P}^d$, and for some choice

7

of g—which is only defined up to an automorphism of $\mathbb{P}^d$ by what we have said so far—we have $g\colon Y \to X \subset \mathbb{P}^d$ and the composite $g \circ f^{-1}$ is equal to $i_P\colon X \dashrightarrow X$.

Bertini involutions

Let $X \subset \mathbb{P}^d$ be a del Pezzo surface of degree $d = 2$, 3, 4 or 5 and $P \in X$ a closed point of degree $d-1$. The *Bertini involution* $j_P\colon X \dashrightarrow X$ *with centre P* is defined as follows: P spans a linear $\Pi = \mathbb{P}^{d-2}$, and a general hyperplane $H = \mathbb{P}^{d-1}$ containing Π intersects X in a nonsingular curve C of genus 1. Moreover, C has a k-rational point Q (the residual point to P in $X \cap \Pi$), so (C, Q) is an elliptic curve. The map j_P acts by -1 in the group law on this elliptic curve.

This clearly defines an involution of X and it can be shown that j_P is a rational, and hence birational, map. Moreover, if we suppose that X is minimal, following Iskovskikh the linear system corresponding to j_P is $|(2d-1)A - 2dP|$. As above, minimality is sufficient, but not necessary, for this linear system to give the right map.

There is an extra detail when $d = 2$: to obtain a map to $\mathbb{P}^3(1, 1, 1, 2)$, we must also compute sections of $|(4d-2)A - 4dP|$ and find an additional section not in the subspace spanned by products from $|(2d-1)A - 2dP|$.

The Bertini involution is described as a link of type II in the same way as the Geiser above; the only difference is that the contracting curve $F \subset Y$, where Y is the blowup of $P \in X$, is now the unique effective curve in $|(2d-2)A - (2d-1)E|$.

Both the Geiser and Bertini involutions are naturally defined algebraically using the pluri-anticanonical model of Y. This description clarifies many of the points we have touched on—for example, it explains why it is natural that the Bertini involution should act by -1 in the group law on the elliptic curve (C, Q) above—and it generalises to higher dimensions. See [CPR] for explanation and many examples.

2   Imposing conditions on functions

This is the main algorithm. Let $X \subset \mathbb{P}^d$ be a surface and $P \in X$ a point (k-rational or not). Let $\mathcal{H} \subset |nA|$ be a linear system on X that is the restriction of some linear system on $\mathbb{P}^d$. We want to compute the linear system $\mathcal{H}(-mP) \subset \mathcal{H}$ of divisors that have multiplicity at least $m \in \mathbb{N}$ at P.

There is one point to note. In solving these equations we compute a system of homogeneous polynomials on the ambient space $\mathbb{P}^d$. Although we can, and do, use these to make a map $\mathbb{P}^d \dashrightarrow \mathbb{P}^d$, we only work with the restriction of this to X—the extension to $\mathbb{P}^d$ is not determined by the theory.

2.1 Basic case: $\mathcal{H} = |nA|$ and $P \in X$ is k-rational

For this discussion, we assume that X is a surface and $P \in X$ a rational point, the case we need for our application. However, the description extends to the blowup of any nonsingular point on a variety of any dimension. It also applies to blowups of higher-dimensional centres $\Gamma \subset X$, as long as one can compute in generic coordinates along Γ.

Normal coordinates along the blowup of P. The idea is to compute the blowup Y of $P \in X$ as a sequence of implicit functions on a blowup patch of the ambient $\mathbb{P}^d$. The functions defining X are denoted $F_1, \ldots, F_r$ (that is, these form a basis for the ideal of X).

Let $c = d - 2$ be the codimension of $X \subset \mathbb{P}^d$, so $c \leq r$. It is easy to describe an affine patch $\varphi \colon \mathbb{A}^d \to \mathbb{P}^d$ of the blowup of $P \in \mathbb{P}^d$. Furthermore, we may assume we have coordinates $u_1, \ldots, u_d$ on $\mathbb{A}^d$ for which the exceptional divisor $E \subset Y$ of the blowup of $P \in X$ is the u_{c+1}-axis ($= u_{d-1}$-axis) and u_d is not a critical direction for the functions F_j: that is, $(\partial F_j / \partial u_d)_{j=1,\ldots,r} \neq 0$ along E. Roughly, we work in coordinates u_{d-1}, u_d on Y in a neighbourhood of the generic point of the exceptional divisor E; precisely, this means working over the function field $k(E) = k(u_{d-1})$ and regarding u_d as a formal power series variable. With these variables as coefficients, the equations defining Y are polynomials in $u_1, \ldots, u_c$, and there is a basis $f_1, \ldots, f_c$ of polynomials of this ideal. We may assume, by using pairwise resultants if necessary, that these polynomials are each univariate in one of the variables: that is, $f_i = f_i(u_i)$ for each $i = 1, \ldots, c$. Now the implicit function theorem guarantees the existence of power series $\varphi_i(u_d)$ over $k(E)$ for which $f_i(\varphi_i) = 0$. These are computable to any given precision; $u_i = \varphi_i$ are the equations of Y, and we can use them to eliminate the variables $u_1, \ldots, u_c$.

Compute $\mathcal{H}$ along the exceptional divisor. Let h be the generic element of $\mathcal{H}$, a homogeneous polynomial of degree n with unknown coefficients; we may assign indeterminates $a_1, \ldots, a_N$ as these coefficients, where N is the number of monomials of degree d on $\mathbb{P}^3$. Pulling h back to the normal coordinates to E on Y that we computed above expresses it as a power series in u_d with coefficients in $k(E)$, say $q_0 + q_1 u_d + q_2 u_d^2 + \cdots$, computed to precision at least u_d^m, with coefficients $q_i = q_i(u_{c+1})$ rational functions in $k(E)$ and in the unknown coefficients $a_1, \ldots, a_N$ of h. The latter only appear linearly in the numerator of each q_i—pulling the expression $a_1 x^d + a_2 x^{d-1} y + \cdots$ back to the normal coordinates evaluates the monomials $x^d, x^{d-1} y, \ldots$ at expressions in those coordinates and then gathers terms together first in powers of u_d and then in powers of u_{c+1}, the variable along E. The order of this power series is the degree of vanishing of $\mathcal{H}$ along E, so sections of $\mathcal{H}(-mP)$ are those polynomials whose coefficients are solutions of the first m coefficients of the power

series, thought of as systems of linear equations in $a_1, \ldots, a_N$.

These systems can be solved using standard computational algebra tools, and a basis for the solution space provides the coefficients of homogeneous polynomials of degree n that base $\mathcal{H}(-mP)$.

2.2 Modifications for more general cases

In general, we do not work with the full linear system $|nA|$ but with some subsystem $\mathcal{H} \subset |nA|$. This makes no essential difference: we use a basis of the sections of $\mathcal{H}$ in place of the monomials of degree n.

The main routine must also impose conditions at points $P \in X$ of higher degree. We may assume that P is irreducible over k; it is enough then to make a finite field extension $k \subset k_1$ that splits P into k_1-geometric points and to apply the algorithm at just one of these. This determines k_1-linear conditions on the coefficients of the linear system, which in turn determine $\deg P$ linear conditions over k.

3 Del Pezzo surfaces of low degree

To construct links of type II on del Pezzo surfaces we must compute Geiser and Bertini involutions given a centre $P \in X$. We concentrate here on surfaces X of degree 3 or 4; we discuss surfaces of degrees 2 and 5 in Section 5 below.

3.1 Applying the algorithm

Suppose $P \in X$ is an irreducible closed point. The calculation is in two steps. If P has degree $d-2$, compute a basis for the sections of $|(d-1)A-dP|$ to make a Geiser involution; if P has degree $d-1$, compute a basis for the sections of $|(2d-1)A-2dP|$ to make a Bertini involution. In either case, this basis has $d+1$ elements and defines a map $\psi\colon \mathbb{P}^d \dashrightarrow \mathbb{P}^d$, which (when restricted to X) is the required involution up to the choice of basis we made for the linear system. But without the correct basis, ψ is unlikely even to map X to itself. There is a unique k-linear automorphism ϑ of $\mathbb{P}^d$ such that $\vartheta \circ \psi$ is the required involution; we must find this 'missing' automorphism ϑ.

To determine ϑ we use the geometric definition of the involution. The aim is to find $d+2$ points of X that span $\mathbb{P}^d$ and whose images can be computed; these points, together with their images, are enough to determine ϑ. As usual, the existence of many k-rational points is not expected, but intersecting X with random linear spaces containing P provides arbitrary numbers of closed points of low degree.

Specifically, in the Geiser case we take general linear spaces of dimension $d-2$ through P and compute the residual intersection with X (using an ideal quotient to remove P from the intersection). Over the closure $\overline{k}$, this consists of two points $q_1, q_2 \in \overline{X}$. We can compute $Q_1 = \psi(q_1)$ and the missing automorphism ϑ must

satisfy $\vartheta(Q_1) = R_1$, where we set $R_1 = q_2$. In the Bertini case we take hyperplanes Π through P and intersect with X. If Π is sufficiently general then $X \cap \Pi$ is nonsingular and we can compute a Weierstrass normal form over k; general fibres of the elliptic involution are defined over k, so they give points of degree 2 on X whose components $q_1, q_2 \in \overline{X}$ are exchanged by the Bertini involution. Again we can compute $Q_1 = \psi(q_1)$ and the missing automorphism ϑ must satisfy $\vartheta(Q_1) = R_1$, where we set $R_1 = q_2$.

It remains to solve for ϑ given $d + 2$ pairs of independent points Q_i, R_i as above.

3.2 Finding the missing automorphism

Let $Q_1, \ldots, Q_{d+2}$ and $R_1, \ldots, R_{d+2}$ be two sequences of points of $\mathbb{P}^d_{\overline{k}}$, each of which spans $\mathbb{P}^d_{\overline{k}}$, and let ϑ be the $\overline{k}$-linear automorphism taking each Q_i to R_i. Standard linear algebra routines compute a matrix representing ϑ, but in our application there are two additional aspects to consider. First, our ϑ is defined over k so the corresponding matrix should have entries in k too; by itself, this is not a problem. Second, in practice we do not work over $\overline{k}$ but over a different extension $k \subset k_i \subset \overline{k}$ for each pair of points Q_i and R_i; this makes it difficult to apply the solution algorithm for linear equations directly. To deal with this we rephrase the algorithm slightly and compute a matrix representing ϑ as follows.

Let M_Q be a matrix whose rows are representatives for $Q_1, \ldots, Q_{d+2}$ and M_R a similar matrix for the R_j. We seek a $(d+1) \times (d+1)$ matrix M for which $M_Q M = M_R$. It is enough to solve this up to scalar multiples of rows; so, letting K_{R_i} be a matrix whose columns are a basis of $\ker R_i$, it is enough to solve the system of equations

$$Q_1 M K_{R_1} = 0, \quad \ldots, \quad Q_{d+2} M K_{R_{d+2}} = 0. \tag{3.1}$$

Each of these imposes d linear conditions on M. Since the equations are independent, the solution space has dimension $(d+1)^2 - d(d+2) = 1$, and the entries of M (itself only defined up to a scalar) are then the coefficients of any nontrivial solution. This does not yet solve the problem: the coefficients appearing in (3.1) still lie in the various fields k_i and it would be expensive to compute a composite field containing them all.

Taking the trace of each of the equations gives a new system

$$\mathrm{Tr}_{k_1/k}(Q_1 M K_{R_1}) = 0, \quad \ldots, \quad \mathrm{Tr}_{k_{d+2}/k}(Q_{d+2} M K_{R_{d+2}}) = 0 \tag{3.2}$$

defined over k. The following lemma is elementary; the main point is to avoid k-linear relations holding between the chosen representatives of the Q_i.

Lemma 3.1. *Let $Q_1, \ldots, Q_{d+2}$ and $R_1, \ldots, R_{d+2}$ be two sequences of points of $\mathbb{P}^d_{\overline{k}}$ as above. For fixed representatives of the Q_i and R_i, consider the systems of linear*

equations (3.1) and (3.2) defined on $V = k^{(d+2)^2}$ as above. Denote the solution sets of these two systems by $W_{(3.1)}$ and $W_{(3.2)}$ respectively.

Then $W_{(3.1)} \subset W_{(3.2)}$ and equality holds for a general choice of representative for each Q_i.

3.3   Degree 4 del Pezzo surfaces

To illustrate our algorithms, we give examples to show how to construct involutions using an implementation in MAGMA; the implementation, together with these examples and their output, is available at [BKR].

First make a nonsingular surface

$$X \colon \left(\begin{array}{l} xy - zt + 2x^2 + s^2 = 0 \\ -x^2 + y^2 - z^2 + t^2 - s^2 = 0 \end{array} \right) \subset \mathbb{P}^4$$

defined over $k = \mathbb{Q}$.

```
> P4<x,y,z,t,s> := ProjectiveSpace(Rationals(),4);
> f := x*y - z*t + 2*x^2 + s^2;
> g := -x^2 + y^2 - z^2 + t^2 - s^2;
> X := Scheme(P4, [f,g]);
```

For a Geiser involution we need a point $P \in X$ of degree 2, which we construct by intersecting X with a particular line:

```
> P := Intersection(X, Scheme(P4, [x,z,s]));
> Degree(P);
2
```

This is all the data needed to construct the Geiser involution centred in P. The map is stored in MAGMA as a composition of simpler maps, so we use `Expand(G)` to see its defining equations of degree $d - 1 = 3$; it is usually costly to do this step and is unnecessary. We also check that the involution really does map X to itself.

```
> G := GeiserInvolution(X, P);
> Expand(G);
Mapping from: Prj: P4 to Prj: P4
with equations :
4/3*x*z^2 + 2/3*x*z*t - 1/3*y*z*t - 1/3*x*t^2 - 1/3*x*s^2
    + 1/3*y*s^2
-2/3*x*z^2 - y*z^2 - 7/3*x*z*t + 2/3*y*z*t + 2/3*x*t^2
    - 1/3*x*s^2 - 2/3*y*s^2
y^2*z + z*t^2 - z*s^2
4*y^2*z - 4*z^3 - y^2*t + 4*z*t^2 - t^3 - 2*z*s^2 + t*s^2
y^2*s + t^2*s - s^3
> G(X) eq X;
true
```

12

For a Bertini involution we need a point $Q \in X$ of degree 3, which we construct as the residual intersection to a 2-plane Π containing a rational point $(0, 1, 1, 0, 0) \in X$. The `Support(Z)` command below computes the k-rational support of Z.

```
> Pi := Scheme(P4, [x+y-z,s]);
> Z := Intersection(X, Pi);
> supp := Support(Z); supp;
{ (0 : 1 : 1 : 0 : 0) }
> Degree(Z);
4
> R := Cluster(Representative(supp));
> Q := Scheme( P4, ColonIdeal(Ideal(Z), Ideal(R)) );
> B := BertiniInvolution(X,Q);
```

Computing the map B takes about half a minute, giving a map defined by polynomials of degree $2d - 1 = 7$; the equations of the map are large, and so here we only show the initial terms.

```
> B;
Mapping from: Prj: P4 to Prj: P4
Composition of Mapping from: Prj: P4 to Prj: P4
with equations :
y^2*z^5 - 301231/288*y^2*z^4*t - 102767/144*x*z^5*t
    - 11755/32*y*z^5*t + 101059/72*z^6*t
    - 4791269/4608*y^2*z^3*t^2 - 2205985/2304*x*z^4*t^2 - ...
> B(X) eq X;
true
```

These maps are indeed involutions and it is not necessary to check this explicitly—although one can check, for instance, that $G \circ G$ is the identity map by using the interpolation routines explained below.

3.4 Examples on a cubic surface

First make the surface $X \colon (x^3 + 2y^3 + 3z^3 + 4t^3 = 0) \subset \mathbb{P}^3$ defined over $k = \mathbb{Q}$.

```
> P3<x,y,z,t> := ProjectiveSpace(Rationals(), 3);
> X := Scheme(P3, x^3 + 2*y^3 + 3*z^3 + 4*t^3);
```

To make a Geiser involution, we need to choose a rational point for its centre. We do not have code for finding rational points—in general, this is an unsolved problem—but in this case we can see some obvious choices: $(1, -1, -1, 1) \in X$, for instance.

```
> p := X ! [1,-1,-1,1];
> G := GeiserInvolution(X,p);
```

For a Bertini involution, we need a point of degree 2. The line $L = (y + z + t = x - z + t = 0)$ meets X in the rational point $(3, 1, 1, -2)$, and the residual intersection to this is an irreducible point of degree 2.

```
> L := Scheme( X, [y+z+t, x-z+t] );
> Z := [ Y : Y in IrreducibleComponents(L) | Degree(Y) eq 2 ][1];
> B := BertiniInvolution(X,Z);
```

(The Bertini calculation takes around 4 seconds; by comparison, the Geiser calculation is instant.) The map B is defined by fairly large polynomials of degree 5.

We can construct another birational selfmap by composing these.

```
> h := Expand(B * G); h;
Mapping from: Prj: P3 to Prj: P3
with equations :
54518131/19784704*x^4*y^6 + 59844679/7419264*x^3*y^7 +
    272174051/29677056*x^2*y^8 + 7725367/1391112*x*y^9 +
    9154681/5564448*y^10 + 82450383/9892352*x^4*y^5*z + ...
```

The equations of h are large polynomials of degree 10 that run over several pages.

4 The group of birational selfmaps of a cubic surface

It is well known, [M] Theorem 38.1 for instance, that Geiser and Bertini involutions (together with the subgroup of linear automorphisms) generate the group $\mathrm{Bir}(X)$ of birational selfmaps of any minimal cubic surface $X = X_3 \subset \mathbb{P}^3$. This also follows from Iskovskikh's classification of links [I] Theorem 2.6: the only elementary links from cubic surfaces are birational selfmaps. (This is not the case in degrees 4 and 5 where factorisations of birational selfmaps into elementary links may pass through other surfaces.)

The proof of Theorem 1.2 works by induction on the degree of the given selfmap φ. The idea is to find a basepoint of degree 1 or 2 that has high multiplicity in curves belonging to the linear system defining φ. Given such a point, a so-called *maximal centre*, the proof precomposes φ by the Geiser or Bertini involution it determines; this decreases the degree and the induction continues. We implement this algorithmic step, demonstrating it here by factorising the map h computed in Section 3.4.

First find the base locus of h. This is done naively, setting the defining equations of h to be zero. The result could be strictly bigger than the base locus, but since we check the multiplicity of base points later this does not matter.

```
> base_h := Scheme(X,DefiningEquations(h));
```

```
> Dimension(base_h);
0
> Degree(base_h);
205
```

We need to identify irreducible components of this base locus, and this could be a
problem if there really were 205 base points. But of course the calculation above
has found a non-reduced scheme, and since we only need to know the base points
set theoretically, we can reduce it before further analysis. Unfortunately, this seems
to be difficult; it takes about 4 minutes.

```
> base_red := ReducedSubscheme(base_h);
> Degree(base_red);
3
```

Since the map h is not linear, there must be a maximal centre in this base locus. Un-
twisting by Bertini involutions is likely to reduce the degree of h more dramatically
than by Geiser involutions, so we look for maximal centres of degree 2 first.

```
> Q := [ Y : Y in IrreducibleComponents(base_red) | Degree(Y) eq 2 ][1];
```

Of course, Q is exactly the enforced base point Z from section 3.4 above, but without
knowing that we need to check that it is a maximal centre.

```
> assert is_maximal_centre(X, Q, h);
```

This takes about a minute: it runs the main algorithm in high degree to check that
Q has multiplicity strictly greater than 10 in the linear system defining h. We could
omit this step since checking that the degree of h is reduced after untwisting is
sufficient.

We must now untwist h by the Bertini involution centred in Q. (Notice the order
of composition in MAGMA: this is $h_1 = h \circ \varepsilon_1$.)

```
> eps1 := BertiniInvolution(X, Q);
> h1 := eps1 * h;
```

There is a practical computational point here: if we expanded out the equations
of h_1, they would have degree 10×5, the product of the degrees of the equations
of ε_1 and h. But untwisting is meant to reduce the degree. Working modulo the
equation of X, the equations of h_1 have a large common factor that can be cancelled,
but it is not clear how to do this calculation.

Instead we use interpolation to find the the correct equations. Bertini involutions
reduce the degree by $4(\text{mult}_Q(h) - \deg(h))$, so even without knowing $\text{mult}_Q(h)$, the
possible degrees for the resulting equations are limited. Given a target degree, the
interpolation evaluates h_1 at many points of X and uses this collection of points and
their images to impose linear conditions on the coefficients of the desired equations.

As usual, we do not have a supply of rational points of X to work with (there may be none), but we can intersect X with random rational lines to get points of degree 3 and use these. If the target degree was too low, the solution space will only include multiples of the defining equation of X. The first time there are other solutions, these will be the coefficients of the map. The function we use below returns a boolean value, which is false unless there is a unique additional solution. If the boolean value is true, the function also returns the equations of h_1. (The `assert bool` statement causes a crash unless the boolean is true.)

```
> bool,eqns := interpolate(X,h1,2);
> assert bool;
```

We rebuild h_1 with the lower-degree equations.

```
> h1 := map< P3 -> P3 | eqns >; h1;
Mapping from: Prj: P3 to Prj: P3
with equations :
x*y + y^2 + 3/2*x*z + 3/2*z^2 + 2*x*t - 2*t^2
1/2*x^2 + 1/2*x*y + 3/2*y*z - 3/2*z^2 + 2*y*t + 2*t^2
1/2*x^2 - y^2 + 1/2*x*z + y*z + 2*z*t + 2*t^2
-1/2*x^2 + y^2 + 3/2*z^2 + 1/2*x*t + y*t + 3/2*z*t
```

Repeat the process with h_1:

```
> base_h1 := Scheme(X, DefiningEquations(h1));
> Dimension(base_h1);
0
> base1_red := ReducedSubscheme(base_h1);
> Degree(base1_red);
1
> r := Representative(Support(base1_red)); r;
(1 : -1 : -1 : 1)
```

So the base locus is a single rational point $(1, -1, -1, 1) \in X$. There is no choice but to untwist by the Geiser involution.

```
> eps2 := GeiserInvolution(X, r);
> h2 := eps2 * h1;
> bool,eqns := interpolate(X,h2,1);
> assert bool;
> h2 := map< P3 -> P3 | eqns >; h2;
Mapping from: Prj: P3 to Prj: P3
with equations : x, y, z, t
```

The resulting map h_2 gives a linear automorphism of X so the factorisation is complete. In this case, it is easy to check that the identity is the only automorphism of X, so the group $\mathrm{Bir}(X)$ is generated by all Geiser and Bertini involutions—what this group is, therefore, has become an arithmetic question involving the low degree points of X and the relations between Geiser and Bertini involutions.

5   Rational surfaces and MAGMA

We describe briefly the broader context of a possible complete implementation of minimal rational surfaces in MAGMA that builds on the algorithms here. It is realistic to expect: to compute the factorisation of rational maps between two-dimensional Mori fibre spaces, the so-called *Sarkisov programme*; to address rationality questions; and to analyse elliptic fibrations. Beck and Schicho have algorithms, implemented in MAGMA, that compute the Picard group of a del Pezzo surface and, if necessary, carry out the minimal model programme in certain circumstances.

5.1   The Sarkisov programme for rational surfaces

We were able to carry out the Sarkisov programme for selfmaps of a cubic surface in Section 4 because in that case other surfaces are not involved. The same is true for del Pezzo surfaces of degrees 1 and 2, but in other cases we need the full classification of links in order to proceed. An implementation of the full Sarkisov programme for factorising birational maps between minimal rational surfaces, following Iskovskikh—so realising Theorem 1.2 explicitly—is feasible, given the following additional components.

The involutions on del Pezzo surfaces of degrees 2 and 5 are essentially the same as those described here. In degree 5 the calculations are currently too slow to be practical. In degree 2 we must work with weighted projective space; this only complicates the calculation slightly. The main additional difficulty is in recognising the surface as an image of the multi-linear system that determines the involution. For example, if $P = (1, 0, 0, 0) \in X \subset \mathbb{P}(1, 1, 1, 2)$ with tangent plane $T_P(X) = (y = 0)$, in coordinates x, y, z, t, then

$$|3A - 4P| = \left\langle y^3, y^2 z, y z^2, y^2 x \right\rangle$$

and $|6A - 8P|$ is spanned by quadratic expressions in these together with $y^4 t$ and $y^3 zt$. These determine a map $X \dashrightarrow \mathbb{P}^5(1^4, 2^2)$ whose image Y is isomorphic to X—but we would need to compute $|-K_Y|$ and $|-2K_Y|$ to make that identification (and then study the geometrical description of the Bertini involution as before to make the right choice of automorphism).

Computing automorphisms of del Pezzo surfaces is straightforward. Since del Pezzo surfaces are embedded by the anticanonical class, any automorphism $X \to X$ extends to an automorphism of the ambient projective space. If this space is $\mathbb{P}^d$ then the only automorphisms are projective linear maps; when $X = X_4 \subset w\mathbb{P}^3 = \mathbb{P}^3(1, 1, 1, 2)$ we must allow quasi-linear maps as follows:

$$x_1 \mapsto f_1(x_1, x_2, x_3), \ldots, \quad x_3 \mapsto f_3(x_1, x_2, x_3), \quad y \mapsto ay + g(x_1, x_2, x_3)$$

where x_1, x_2, x_3, y are the homogeneous coordinates on $w\mathbb{P}^3$, the f_i are linear forms, g is a quadratic form and $a \in k$ does not have a square root in k.

General conic bundles can be embedded in scrolls, even if the base rational curve does not have a rational point. With such a description, one can compute links by explicit equations as we have for del Pezzo surfaces. Cremona and van Hoeij [CvH] give an algorithm for Tsen's theorem (over an extension of k, as necessary), which is one of the essential tools for this.

5.2   Rationality questions for minimal rational surfaces

While we assume that our surfaces X are rational over $\overline{k}$, the question of whether they are also rational over k is well studied. Swinnerton-Dyer [SD] answers the question for a cubic surface in terms of a rational point on X and conditions on the configuration of the 27 lines. In degree 5, Enriques proved that X automatically has a rational point and is rational over k; see [Sh-B]. In degrees ≥ 6, X is rational over k if and only if it has a rational point, and either case can happen. In degrees 8 and 9, de Graaf–Harrison–Pilnikova–Schicho [dG et al] determine whether a del Pezzo surface (not necessarily minimal, and for k a number field) is rational and they compute a parametrisation over k in that case; this is implemented in MAGMA.

5.3   Elliptic fibrations on minimal rational surfaces

Elliptic fibrations birational to minimal rational surfaces are classified in some cases: Dolgachev [D] classifies elliptic fibrations birational to $\mathbb{P}^2$ (the construction of these fibrations dates back to Halphen [H]); Cheltsov [Ch] and Brown–Ryder [BR] analyse elliptic fibrations birational to minimal cubic surfaces; and Cheltsov also analyses the case of del Pezzo surfaces of degrees 1 and 2. A birational map from X to an elliptic fibration can be regarded as a limiting case of a birational map to another Mori fibre space, so the methods of construction and exclusion for the two problems are very similar. The computational aspects of the elliptic fibration problem are fully analysed in the degree 3 case in [BR], and a full MAGMA implementation is given; this is not yet done in the other cases.

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