Mirror symmetric Gamma conjecture for Fano and Calabi-Yau manifolds

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Abstract

The mirror symmetric Gamma conjecture roughly speaking says that the Gamma class of a manifold determines the asymptotics of (exponential) periods of the mirror. We recast the method in [22] in a more general context and show that the mirror symmetric Gamma conjecture for a Fano manifold F implies, via Laplace transformation, that for the total space K_F of the canonical bundle or for anticanonical sections in F. More generally, we discuss the mirror symmetric Gamma conjecture for the total space of a sum of anti-nef line bundles over F or for nef complete intersections in F.

1 Overview

In this paper we compare the 'mirror symmetric' Gamma conjectures for Fano and Calabi-Yau manifolds, which roughly speaking say the following.

(1) For a Fano manifold F and its Laurent polynomial mirror $W(x)$, we have

$$\int_{(\mathbb{R}_{>0})^n} e^{-W(x)/z}\frac{dx_1\cdots dx_n}{x_1\cdots x_n} \sim \int_F z^{c_1}\widehat{\Gamma}_F \quad \text{as } z\to +0$$

(2) For a Calabi-Yau manifold Y with Kähler form in the class $\tau\in H^2(Y)$ and its mirror Calabi-Yau manifold Z_τ with a holomorphic volume form Ω_τ, we have

$$\int_{C_\tau}\Omega_\tau \sim \int_Y e^{\tau}\widehat{\Gamma}_Y \quad \text{as } \tau \text{ approaches the large radius limit}$$

for some integral cycles C_τ.

For a Fano manifold F, we can construct a Calabi-Yau manifold by either taking the total space K_F of the canonical bundle or taking an anticanonical section $Y\subset F$. We study the relationship between the mirror symmetric Gamma conjectures in these cases.

More precisely, if a Laurent polynomial mirror W of F satisfies the mirror symmetric Gamma conjecture, we show that a certain relative period of the pair $((\mathbb{C}^\times)^n, W^{-1}(-1))$ satisfies the Gamma conjecture for the local Calabi-Yau K_F and a period of the fiber $W^{-1}(1)$ satisfies the Gamma conjecture for an anticanonical Calabi-Yau $Y\subset F$. The main results will be stated in Theorems 4.6 and 6.6 below.

2 Non-mirror-symmetric Gamma conjecture for Fano manifolds in a nutshell

Non-mirror-symmetric Gamma conjecture for Fano manifolds [9] roughly speaking says that one can extract topological information (the Gamma class) of a Fano manifold F by counting rational curves in F. Recall that the Gamma class $\widehat{\Gamma}_F$ is a characteristic class defined by

$$\begin{aligned}\widehat{\Gamma}_F &= \Gamma(1+\delta_1)\cdots\Gamma(1+\delta_n)\\ &= \exp\left(-\gamma c_1 + \sum_{k=2}^{\infty}(-1)^k\zeta(k)(k-1)!\operatorname{ch}_k(TF)\right)\end{aligned}$$

where $c_1 = c_1(TF)$, $\delta_1,\dots,\delta_n$ are the Chern roots of TF so that $c(TF) = (1+\delta_1)\cdots(1+\delta_n)$, $\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n}\frac{1}{k} - \log n)$ is the Euler constant and $\zeta(k)$ is the Riemann zeta function. Let $J_F(\tau,z)$ denote the Givental J-function [13] of F:

$$J_F(\tau,z) = e^{\tau/z}\left(1 + \sum_{d\in H_2(F,\mathbb{Z}),d\neq 0} e^{\tau\cdot d}\sum_i \left\langle \frac{\phi^i}{z(z-\psi)}\right\rangle_{0,1,d}\phi_i\right)$$

where $\tau\in H^2(F)$ and $\{\phi_i\}$ and $\{\phi^i\}$ are dual bases of $H^*(F)$ such that $\int_F \phi_i\cup\phi^j = \delta_i^j$. This is a generating function of genus-zero one-point descendant Gromov-Witten invariants. The *Gamma conjecture I* [9, §3] says that we should have

$$[\widehat{\Gamma}_F] = \lim_{t\to+\infty}[J_F(c_1\log t, 1)] \tag{2.1}$$

in the *projective space* $\mathbb{P}(H^*(F))$ of cohomology, where the square bracket $[\cdots]$ means a point in the projective space. Note that the right-hand side is determined only by counting rational curves (Gromov-Witten invariants) in F whereas the left-hand side contains topological information of F.

2.2. The J-function $J_F(c_1\log t, 1)$ is a solution to the quantum differential equation which has irregular singularities at $t=\infty$; the above limit (2.1) detects the most dominant component of the solution J_F as $t\to+\infty$. When the Gamma conjecture I holds, the Gamma class also arises as a row of the connection matrix between fundamental solutions around regular and irregular singularities of the quantum connection.

3 Mirror symmetric Gamma conjecture for Fano manifolds

Mirror symmetry for Fano manifolds (see Givental [11]) gives oscillatory integral representations of solutions (e.g. the J-function) to the quantum differential equation. Such integral representations are useful for the asymptotic analysis and for proving the Gamma conjecture. We discuss a mirror symmetry version of the Gamma conjecture (as studied in [21, 22, 10, 1, 23]) which would lead to a (partial) proof of the original (non-mirror-symmetric) Gamma conjecture.

3.1. Suppose that an n-dimensional Fano manifold F is mirror to a Laurent polynomial $W = W(x_1,\dots,x_n)$ in n-variables. Conjecturally, we obtain W by counting holomorphic discs with boundary on a Lagrangian torus in F [16, 5, 2]. We assume that

(A1) the Newton polytope of W contains the origin in its interior; and

(A2) all non-vanishing coefficients of W are positive real.

Under these assumptions (A1)-(A2), $W|_{(\mathbb{R}_{>0})^n}$ is a strictly convex function with respect to the coordinates $(\log x_1, \dots, \log x_n)$ and attains a global minimum at a unique point. *Mirror symmetric Gamma conjecture* is the equality

$$\int_{(\mathbb{R}_{>0})^n} e^{-W/z} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} = \int_F (z^{c_1} z^{\deg/2} J_F(0, -z)) \cup \widehat{\Gamma}_F \tag{3.2}$$

for $z > 0$, where $z^{c_1} = e^{c_1 \log z}$ and 'deg' is regarded as an endomorphism of $H^*(F)$ that sends ϕ_i to $\deg(\phi_i)\phi_i$. Note that the above assumptions ensure the convergence of the integral in the left-hand side, and that the left-hand side has the asymptotics $\sim e^{-T/z}$ as $z \to +0$, where $T = \min_{x \in (\mathbb{R}_{>0})^n} W(x)$. In this equation the parameter $\tau \in H^2(F)$ in the J-function is zero. More generally, we conjecture that there is a family of Laurent polynomials W_τ depending on $\tau \in H^2(F)$ such that

$$\int_{(\mathbb{R}_{>0})^n} e^{-W_\tau/z} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} = \int_F (z^{c_1} z^{\deg/2} J_F(\tau, -z)) \cup \widehat{\Gamma}_F \tag{3.3}$$

whenever τ is a real class. Here we assume that $W = W_\tau$ satisfies (A1)-(A2) for all real $\tau \in H^*(F, \mathbb{R})$. These formulae are known to be true for Fano toric manifolds or Fano complete intersections in them [21, 22] and the original Gamma conjecture for those spaces follows[1] from such formulae, see [10, Theorem 6.3, Theorem 7.5].

3.4 Remark. As remarked above, mirror symmetry predicts that the oscillatory integrals and the J-function satisfy the same differential equation. Therefore, the essential content of the Gamma conjecture (3.3) is contained in the asymptotic behaviour:

$$\int_{(\mathbb{R}_{>0})^n} e^{-W_\tau/z} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \sim \int_F z^{c_1} e^{-\tau} \cup \widehat{\Gamma}_F \qquad \text{as } z \to \infty.$$

3.5 Remark. We can consider more general cycles and integrands in this conjecture. When a (typically non-compact) integration cycle $C_V \subset (\mathbb{C}^\times)^n$ is mirror to a coherent sheaf V on F (in the sense of homological mirror symmetry) and an n-form $\check{\varphi}_\tau e^{-W_\tau/z} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$ (with $\check{\varphi}_\tau$ a Laurent polynomial in $x_1, \dots, x_n$) is mirror to a class $\varphi \in H^*(F)$, we should have

$$\int_{C_V} \check{\varphi}_\tau e^{-W_\tau/z} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} = \int_F (z^{c_1} z^{\deg/2} \mathbb{J}_F(\tau, -z)\varphi) \cup \widehat{\Gamma}_F \operatorname{Ch}(V)$$

where $\mathbb{J}_F(\tau, z) \in \operatorname{End}(H^*(F))$ is a matrix extension of the J-function (which gives a fundamental solution of the quantum connection)

$$\mathbb{J}_F(\tau, z)\varphi = e^{\tau/z} \left(\varphi + \sum_{d \in H_2(F,\mathbb{Z}), d \neq 0} e^{\tau \cdot d} \sum_{j=0}^{N} \left\langle \varphi, \frac{\phi^j}{z - \psi} \right\rangle_{0,2,d} \phi_j \right)$$

and $\operatorname{Ch}(V) := (2\pi \mathtt{i})^{\deg/2} \operatorname{ch}(V) = \sum_{k \geq 0} (2\pi \mathtt{i})^k \operatorname{ch}_k(V)$ is a modified Chern character. The formulae (3.2)-(3.3) are special cases of this equality when $\varphi = 1$ and $V = \mathcal{O}$ because $\mathbb{J}_F(\tau, z)1 = J_F(\tau, z)$.

[1] To be more precise, the proof of the Gamma conjecture in [10] is conditional on the assumption that these spaces satisfy a mirror analogue of Conjecture $\mathcal{O}$.

4 Gamma conjecture for Calabi-Yau manifolds

The original Gamma conjecture for Fano manifolds was formulated without a reference to mirrors, but Gamma conjecture for Calabi-Yau manifolds (we treat in this paper) involves mirror symmetry. The Gamma conjecture in the Calabi-Yau case roughly speaking says that *the vanishing cycle of the mirror family (at the singularity nearest the large complex structure limit) corresponds to the Gamma class on the symplectic side.* This would explain a motivic origin of the ζ-values appearing in the Gamma class. Closely related observations have been made by many people since the beginning of mirror symmetry [4, 19, 25, 17, 8, 18, 14].

4.1. A version of the Gamma conjecture for Calabi-Yau manifolds presented in [1, Conjecture A], which originates from Hosono's conjecture [18, Conjecture 2.2], says the following. Let Y be a Calabi-Yau manifold and let Z_τ be a family of mirror Calabi-Yau manifolds parametrized by $\tau \in H^2(Y)$ equipped with a holomorphic volume form Ω_τ. When Ω_τ is suitably normalized and a cycle $C_\tau \subset Z_\tau$ is mirror to a coherent sheaf V on Y in the sense of homological mirror symmetry, we expect the following asymptotics of periods:

$$\int_{C_\tau \subset Z_\tau} \Omega_\tau \sim \int_Y e^{-\tau} \widehat{\Gamma}_Y \operatorname{Ch}(V)$$

as $\operatorname{Re}(\tau \cdot d) \to -\infty$ for all non-zero effective curve classes $d \in H_2(Y, \mathbb{Z})$, where $\operatorname{Ch}(V) = (2\pi \mathtt{i})^{\deg/2} \operatorname{ch}(V)$. The limit $\operatorname{Re}(\tau \cdot d) \to -\infty$ is known as the *large radius limit.* When V is the structure sheaf $\mathcal{O}_Y$, we expect that C_τ is a vanishing cycle at a singularity "closest" to the large complex structure limit[2].

4.2. The main goal of this paper is to derive the Calabi-Yau Gamma conjecture from the Fano case. For a Fano manifold F, we consider a Calabi-Yau manifold given either as the total space K_F of the canonical bundle or as a smooth anticanonical section $Y \subset F$. It is expected that the mirror of K_F is given by relative periods of the pair $((\mathbb{C}^\times)^n, W_\tau^{-1}(-1))$ and that the mirror of Y is given by periods of (a compactification of) $W_\tau^{-1}(1)$.

The mirror symmetric Gamma conjecture for local Calabi-Yau manifolds K_F (with F Fano toric manifolds) was recently studied by Wang [27] using the Gross-Siebert mirrors. Wang considered the case where the cycle C_τ is mirror to the structure sheaf of a curve. The Gamma conjecture in the local case is also closely related to the *Apéry extensions* studied by Golyshev, Kerr and Sasaki [15].

4.3. Let I_{K_F} and I_Y denote the I-functions [6] of K_F and Y respectively.

$$I_{K_F}(\tau, z) = e^{\tau/z} \sum_{d \in H_2(F,\mathbb{Z})} e^{\tau \cdot d} J_d(z) \prod_{k=0}^{c_1 \cdot d - 1} (-c_1 - kz) \tag{4.4}$$

$$I_Y(\tau, z) = e^{\tau/z} \sum_{d \in H_2(F,\mathbb{Z})} e^{\tau \cdot d} J_d(z) \prod_{k=1}^{c_1 \cdot d} (c_1 + kz) \tag{4.5}$$

where $\tau \in H^2(F)$ and we expand $J_F(\tau, z) = e^{\tau/z} \sum_{d \in H_2(F,\mathbb{Z})} e^{\tau \cdot d} J_d(z)$. We regard I_{K_F} as an $H^*(K_F)$-valued function and I_Y as an $H^*(Y)$-valued function. The quantum Lefschetz

[2] However, the notion of the "closeness" here is unclear: in many examples, we can choose the closest singularity with respect to a given coordinate.

theorem of Coates-Givental [6] implies that these I-functions are related to the J-functions of K_F and Y via a change of variables (mirror transformation), and therefore compute certain Gromov-Witten invariants of K_F or Y.

4.6 Theorem. *Let $\tau \in H^2(F, \mathbb{R})$ and suppose that we have a Laurent polynomial W_τ satisfying* (A1)-(A2) *and the mirror symmetric Gamma conjecture* (3.3) *at τ. Let $s > 0$ be a sufficiently large positive number. Then we have*

$$(2\pi\mathtt{i}) \int_{B_{\tau,s}} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} = \int_{K_F} I_{K_F}(\tau - c_1 \log(-s), -1) \cup \widehat{\Gamma}_{K_F} \operatorname{Ch}(\mathcal{O}_F) \tag{4.7}$$

where $B_{\tau,s} = (\mathbb{R}_{>0})^n \cap \{W_\tau(x) \leq s\}$ is a relative n-cycle with boundary in $W_\tau^{-1}(s)$ and we choose the branch $\operatorname{Im}(\log(-s)) = -\pi$. We also have

$$s \int_{C_{\tau,s}} \frac{d\log x_1 \cdots d\log x_n}{dW_\tau} = \int_Y I_Y(\tau - c_1 \log s, -1) \cup \widehat{\Gamma}_Y \tag{4.8}$$

where $C_{\tau,s} = (\mathbb{R}_{>0})^n \cap W_\tau^{-1}(s)$ is an $(n-1)$-cycle in $W_\tau^{-1}(s)$.

4.9 Remark. $C_{\tau,s}$ is the vanishing cycle at $s = T_\tau := \min_{x \in (\mathbb{R}_{>0})^n} W_\tau(x)$. In this theorem, s needs to be (at least) greater than T_τ (otherwise the cycles $B_{\tau,s}, C_{\tau,s}$ are empty).

4.10 Remark. $\mathcal{O}_F$ means the structure sheaf of the zero-section $F \subset K_F$. The ($2\pi\mathtt{i}$-modified) Chern character $\operatorname{Ch}(\mathcal{O}_F)$ lies in the compactly supported cohomology of K_F, and hence the integral in the right-hand side of (4.7) makes sense.

4.11 Remark. On the right-hand side of (4.7)-(4.8), the argument z in the I-function is specialized to -1. Note however that $z^{c_1(TX)} z^{\deg/2} I_X(\tau, -z) = I_X(\tau, -1)$ for $X = K_F$ or Y.

4.12 Remark. Suppose that we have a family $\{W_\tau\}_{\tau \in H^2(F,\mathbb{R})}$ of Laurent polynomial mirrors satisfying the mirror symmetric Gamma conjecture. Let $B_{\tau,-1}$ denote the parallel translate of $B_{\tau,s}$ along an anti-clockwise path in the s-plane (increasing $\arg s$ by π). It is a relative cycle with boundary in $\{W_\tau(x) + 1 = 0\}$. Then Theorem 4.6 implies the asymptotics

$$(2\pi\mathtt{i}) \int_{B_{\tau,-1}} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \sim \int_{K_F} e^{-\tau} \widehat{\Gamma}_{K_F} \operatorname{Ch}(\mathcal{O}_F)$$
$$\int_{C_{\tau,1}} \frac{d\log x_1 \cdots d\log x_n}{dW_\tau} \sim \int_Y e^{-\tau} \widehat{\Gamma}_Y$$

as $\operatorname{Re}(\tau \cdot d) \to -\infty$ for all non-zero effective curve classes $d \in H_2(F, \mathbb{Z})$.

4.13 Example. As we mentioned earlier, the mirror symmetric Gamma conjecture holds for a Fano toric variety. Let F be a Fano toric variety and let $D_1, \ldots, D_c$ be all prime toric divisors of F. We write $b_j \in \mathbb{Z}^n$ for the primitive vector of the 1-dimensional cone of the fan for F corresponding to D_j. The J-function of F is given by [12]

$$J_F(\tau, z) = e^{\tau/z} \sum_{d \in H_2(F,\mathbb{Z})} e^{\tau \cdot d} \prod_{j=1}^{c} \frac{\prod_{k=-\infty}^{0} D_j + kz}{\prod_{k=-\infty}^{D_j \cdot d} D_j + kz}.$$

The Laurent polynomial $W(x) = \sum_{j=1}^c e^{-\lambda_j} x^{b_j}$ satisfies the mirror symmetric Gamma conjecture (3.3) at $\tau = -\sum_{j=1}^c \lambda_j D_j$ [21, Theorem 4.14]. By applying the first half of Theorem 4.6 to F, we obtain (when $\sum_{j=1}^c \lambda_j D_j$ is sufficiently ample)

$$\operatorname{vol}\left\{ t \in \mathbb{R}^n : \sum_{j=1}^c e^{-\lambda_j + \langle t, b_j\rangle} \le 1 \right\} = \int_F \sum_{d \in H_2(F,\mathbb{Z})} e^{\sum_{j=1}^c \lambda_j (D_j - D_j \cdot d)} \frac{\prod_{j=1}^c D_j \Gamma(D_j - D_j \cdot d)}{\Gamma(c_1 + 1 - c_1 \cdot d)}$$

where vol stands for the Euclidean volume and $c_1 = D_1 + \cdots + D_c$. The leading asymptotics at the large-radius limit $\sum_{j=1}^c \lambda_j D_j \to \infty$ gives the classical Duistermaat-Heckman formula

$$\operatorname{vol}(P_\lambda) = \int_F e^{\sum_{j=1}^c \lambda_j D_j}$$

where $P_\lambda = \{t \in \mathbb{R}^n : \langle t, b_j\rangle \le \lambda_j, j = 1, \dots, c\}$ is the moment polytope of F with respect to a symplectic form in the class $\sum_{j=1}^c \lambda_j D_j$.

5 Laplace transformation and a proof of Theorem 4.6

We assume the mirror symmetric Gamma conjecture (3.3). Let $\mathcal{I}(z)$ denote the quantity in (3.3) and set $t = 1/z$:

(5.1)
$$\begin{aligned} \mathcal{I}(1/t) &= \int_F (t^{-c_1} t^{-\deg/2} J_F(\tau, -1/t)) \cup \widehat{\Gamma}_F \\ &= \int_{(\mathbb{R}_{>0})^n} e^{-tW_\tau} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}. \end{aligned}$$

We calculate the Laplace transform of $\mathcal{I}(1/t)$

$$\widehat{\mathcal{I}}(s) = \int_0^\infty e^{st} \mathcal{I}(1/t) dt$$

for negative real $s < 0$. This integral is convergent because $\mathcal{I}(t)$ is $O((\log t)^n)$ as $t \to 0$ by the first formula of (5.1) (see also (5.3) below) and $\mathcal{I}(t)$ has exponential decay as $t \to +\infty$ by the second formula of (5.1). Using the first formula of (5.1), we obtain the following.

5.2 Proposition (cf. [22, Proposition 5.1])**.** *For a sufficiently negative $s \ll 0$, we have*

$$\widehat{\mathcal{I}}(s) = \frac{1}{-s} \int_F \widetilde{I}_Y(\tau - c_1 \log s, -1) \cup e^{-\pi \mathtt{i} c_1} \Gamma(1 - c_1) \widehat{\Gamma}_F$$

where $\widetilde{I}_Y(\tau, z)$ is defined by the same formula (4.5) *as $I_Y(\tau, z)$ but is regarded as a function taking values in $H^*(F)$* (*instead of in $H^*(Y)$*) *and we choose the branch* $\operatorname{Im} \log s = \pi$

Proof. We expand $J_F(\tau, z) = e^{\tau/z} \sum_{d \in H_2(F,\mathbb{Z})} e^{\tau \cdot d} J_d(z)$. The homogeneity of Gromov-Witten invariants implies that $J_d(z) = \sum_i J_{d,i} z^{-c_1 \cdot d - \deg \phi_i/2} \phi_i$ for some $J_{d,i} \in \mathbb{C}$. This shows:

(5.3)
$$t^{-c_1} t^{-\deg/2} J_F(\tau, -1/t) = e^{-\tau} \sum_{d \in H_2(F,\mathbb{Z})} e^{\tau \cdot d} J_d(-1) t^{-c_1 + c_1 \cdot d}.$$

We compute the Laplace transform termwise, using the formula

$$\begin{aligned}\int_0^\infty e^{st} t^{-c_1+c_1\cdot d} dt &= (-s)^{c_1 - c_1\cdot d - 1}\Gamma(1 + c_1\cdot d - c_1) \\ &= \frac{1}{-s}(-s)^{c_1} s^{-c_1\cdot d}\Gamma(1-c_1)\prod_{k=1}^{c_1\cdot d}(k - c_1).\end{aligned}$$

Then we arrive at the right-hand side of the proposition. It suffices to show that we can interchange the sum and the integral for $s \ll 0$. The convergence of the small quantum product of the Fano manifold F implies the estimate (see [20, Lemma 4.1])

$$|J_{d,i}| \le C_1 \frac{C_2^{c_1\cdot d}}{(c_1\cdot d)!} \tag{5.4}$$

for some $C_1, C_2 > 0$. Hence a partial sum has the estimate

$$\left| e^{-\tau} \sum_{d\in H_2(F,\mathbb{Z}), c_1\cdot d\le N} e^{\tau\cdot d} J_d(-1) t^{-c_1+c_1\cdot d}\right| \le C_3(\tau) e^{C_4(\tau)t} \quad (\forall N > 0)$$

for some $C_3(\tau), C_4(\tau) > 0$ and a norm $|\cdot|$ on $H^*(F)$. Therefore the sum and the integral can be interchanged when $s < -C_4(\tau)$ by the dominated convergence theorem. □

Next using the second formula of (5.1), we obtain the following.

5.5 Proposition (cf. proof of [22, Proposition 5.9]). *For $s < 0$, we have*

$$\widehat{\mathcal{I}}(s) = \int_{(\mathbb{R}_{>0})^n} \frac{1}{W_\tau(x) - s} \frac{dx_1\cdots dx_n}{x_1\cdots x_n}$$

Moreover, the right-hand side makes sense as an analytic function of $s \in \mathbb{C}\setminus[T_\tau,\infty)$ where $T_\tau = \min_{x\in(\mathbb{R}_{>0})^n}(W_\tau(x)) > 0$.

Proof. We have

$$\widehat{\mathcal{I}}(s) = \int_0^\infty e^{st} dt \int_{(\mathbb{R}_{>0})^n} e^{-tW_\tau} \frac{dx_1\cdots dx_n}{x_1\cdots x_n}$$

Since the integrand $e^{st}e^{-tW_\tau}$ is positive, we can use the Fubini theorem and perform the integration in t first. This gives the expression in the proposition. The integral on the right-hand side converges for all $s \in \mathbb{C}\setminus[T_\tau,\infty)$ because of the estimate

$$\left|\frac{1}{W_\tau(x) - s}\right| \le \left(\sup_{u\ge T_\tau}\left|\frac{u+1}{u-s}\right|\right)\frac{1}{W_\tau(x)+1} \qquad x\in(\mathbb{R}_{>0})^n \tag{5.6}$$

and the fact that it converges at $s = -1$. □

5.7. The l-loop Banana Feynman amplitude [26, 3, 24] is of the form $\widehat{\mathcal{I}}(s)$ in the above proposition where we choose F to be a degree $(1,\ldots,1)$ hypersurface in $(\mathbb{P}^1)^{l+1}$.

Proof of Theorem 4.6. Choose a sufficiently large positive number $M \gg 0$. We consider the integral of $\widehat{\mathcal{I}}(\tau, s)$ in s along a path which starts from $s = -M$ and is contained in the region $\mathbb{C} \setminus [T_\tau, \infty)$.

$$\mathcal{M}(s) := \int_{-M}^{s} \widehat{\mathcal{I}}(s')ds'$$

Using the formula in Proposition 5.5 and the Fubini theorem[3], we integrate out s' to find

$$\mathcal{M}(s) = \int_{(\mathbb{R}_{>0})^n} \log\left(\frac{W_\tau(x) + M}{W_\tau(x) - s}\right) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}.$$

The function $\mathcal{M}(s)$ is analytic for $s \in \mathbb{C} \setminus [T_\tau, \infty)$. We compute the 'jump' of this function across the branch cut $[T_\tau, \infty)$. For $f(s) = \log((W_\tau(x) + M)/(W_\tau(x) - s))$, we have

$$\lim_{\epsilon \to +0} (f(s + \mathtt{i}\epsilon) - f(s - \mathtt{i}\epsilon)) = \begin{cases} 2\pi\mathtt{i} & \text{if } W_\tau(x) < s \\ 0 & \text{if } W_\tau(x) > s \end{cases}$$

Moreover, $|f(s+\mathtt{i}\epsilon) - f(s-\mathtt{i}\epsilon)|$ is bounded by an integrable function independent of $\epsilon \in (0, 1)$, i.e. when $0 < \epsilon < 1$,

$$|f(s + \mathtt{i}\epsilon) - f(s - \mathtt{i}\epsilon)| = \left|\log \frac{W_\tau(x) - s + \mathtt{i}\epsilon}{W_\tau(x) - s - \mathtt{i}\epsilon}\right| \le \begin{cases} 2\pi & \text{if } W_\tau(x) \le s + 1 \\ 2/(W_\tau(x) - s) & \text{if } W_\tau(x) \ge s + 1 \end{cases}$$

The function on the right-hand side is bounded by a constant multiple of $1/(W_\tau(x) + 1)$ on $(\mathbb{R}_{>0})^n$, and hence is integrable with respect to the measure $\frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$. Therefore the jump of $\mathcal{M}(s)$ is given by

$$\lim_{\epsilon \to +0} (\mathcal{M}(s + \mathtt{i}\epsilon) - \mathcal{M}(s - \mathtt{i}\epsilon)) = 2\pi\mathtt{i} \int_{(\mathbb{R}_{>0})^n \cap \{W_\tau(x) \le s\}} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}. \tag{5.8}$$

for $s > T_\tau$. This equals the left-hand side of (4.7). On the other hand, Proposition 5.2 gives a convergent power series expansion of $\widehat{\mathcal{I}}(s)$ around $s = \infty$. By integrating the power series termwise, we get for $|s| \gg 0$,

$$\mathcal{M}(s) = -\int_F \sum_{d \in H_2(F,\mathbb{Z})} e^{-\tau + \tau \cdot d} \widetilde{I}_d(-1) \frac{s^{c_1 - c_1 \cdot d} - (-M)^{c_1 - c_1 \cdot d}}{c_1 - c_1 \cdot d} \cup e^{-\pi \mathtt{i} c_1} \Gamma(1 - c_1) \widehat{\Gamma}_F$$

where we write $\widetilde{I}_Y(\tau, z) = e^{\tau/z} \sum_{d \in H_2(F,\mathbb{Z})} e^{\tau \cdot d} \widetilde{I}_d(z)$. Here the branch of $s^a = \exp(a \log s)$ is chosen so that $\operatorname{Im} \log s \in (0, 2\pi)$. From this we compute, for $s \gg 0$,

$$\begin{aligned}
&\lim_{\epsilon \to +0} (\mathcal{M}(s + \mathtt{i}\epsilon) - \mathcal{M}(s - \mathtt{i}\epsilon)) \\
&= -\int_F \sum_{d \in H_2(F,\mathbb{Z})} e^{-\tau + \tau \cdot d} \widetilde{I}_d(-1) \frac{(1 - e^{2\pi \mathtt{i} c_1}) s^{c_1 - c_1 \cdot d}}{c_1 - c_1 \cdot d} \cup e^{-\pi \mathtt{i} c_1} \Gamma(1 - c_1) \widehat{\Gamma}_F \\
&= \int_F \sum_{d \in H_2(F,\mathbb{Z})} e^{-\tau + \tau \cdot d} J_d(-1) \left(\prod_{k=0}^{c_1 \cdot d - 1} (c_1 - k)\right) \frac{1 - e^{2\pi \mathtt{i} c_1}}{-c_1} s^{c_1 - c_1 \cdot d} e^{-\pi \mathtt{i} c_1} \Gamma(1 - c_1) \widehat{\Gamma}_F \\
&= \int_F I_{K_F}(\tau - c_1 \log(-s), -1) \cup \Gamma(1 - c_1) \widehat{\Gamma}_F \frac{1 - e^{2\pi \mathtt{i} c_1}}{-c_1}
\end{aligned} \tag{5.9}$$

[3]The Fubini theorem applies because $s \mapsto \int_{(\mathbb{R}_{>0})^n} \frac{1}{|W_\tau(x) - s|} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}$ is continuous on $\mathbb{C} \setminus [T_\tau, \infty)$ (by the estimate (5.6)), and therefore integrable along any compact path in $\mathbb{C} \setminus [T_\tau, \infty)$.

where we choose the branch $\operatorname{Im}\log(-s) = -\pi$. The last line equals the right-hand side of (4.7). Comparing this with (5.8), we obtain (4.7).

The equality (4.8) can be obtained from (4.7) by differentiating in s. It is obvious that the left-hand side of (4.8) multiplied by $2\pi\mathtt{i}/s$ is the derivative of the left-hand side of (4.7) Using the expression (5.9) for the right-hand side of (4.7), we get that the derivative of the right-hand side of (4.7) is

$$s^{-1}\int_F \sum_{d\in H_2(F,\mathbb{Z})} e^{-\tau+\tau\cdot d}\widetilde{I}_d(-1)s^{c_1-c_1\cdot d}(e^{\pi\mathtt{i}c_1}-e^{-\pi\mathtt{i}c_1})\Gamma(1-c_1)\widehat{\Gamma}_F.$$

Using $\widehat{\Gamma}_Y = \widehat{\Gamma}_F/\Gamma(1+c_1) = \widehat{\Gamma}_F\Gamma(1-c_1)\frac{e^{\pi\mathtt{i}c_1}-e^{-\pi\mathtt{i}c_1}}{2\pi\mathtt{i}c_1}$, we see that this equals the left-hand side of (4.8) multiplied by $2\pi\mathtt{i}/s$. $\square$

6 Generalizations to a sum of line bundles/complete intersections

The above method can be easily generalized to a sum of line bundles over a Fano manifold F or complete intersections in F. Consider the decomposition (the so-called "nef partition")

$$c_1 = v_0 + v_1 + \cdots + v_c$$

of the first Chern class $c_1 = c_1(TF)$ such that each $v_i \in H^2(F,\mathbb{Z})$ is nef. Suppose that we have a line bundle $\mathcal{V}_i \to F$ with $c_1(\mathcal{V}_i) = v_i$ for $i = 1,\dots,c$. In this section, we consider the Gamma conjecture for the total space $\mathcal{V}^\vee = \mathcal{V}_1^\vee \oplus \cdots \oplus \mathcal{V}_c^\vee$ of the sum of the anti-nef line bundles $\mathcal{V}_1^\vee,\dots,\mathcal{V}_c^\vee$ or a smooth complete intersection $Y \subset F$ cut out by a transversal section of $\mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_c$. Note that the first Chern classes of these spaces are given by the nef class v_0. When $v_0 = 0$, these spaces are numerically Calabi-Yau.

6.1. Let $W^{(0)},\dots,W^{(c)}$ be Laurent polynomial whose coefficients are positive real. We consider the family of Laurent polynomials

$$W_r = W^{(0)} + r_1 W^{(1)} + \cdots + r_c W^{(c)} \tag{6.2}$$

parametrized by $r = (r_1,\dots,r_c) \in (\mathbb{R}_{>0})^c$ and assume that the origin is contained in the interior of the Newton polytope of W_r; then W_r satisfies the assumptions (A1)-(A2). Suppose that the mirror symmetric Gamma conjecture (3.3) holds for W_r via the identification of parameters

$$\tau = \tau(r) = \tau_0 + v_1 \log r_1 + \cdots + v_c \log r_c$$

for some $\tau_0 \in H^2(F)$, that is, we have

$$\int_{(\mathbb{R}_{>0})^n} e^{-W_r/z}\frac{dx_1\cdots dx_n}{x_1\cdots x_n} = \int_F (z^{c_1}z^{\deg/2}J_F(\tau,-z)) \cup \widehat{\Gamma}_F \tag{6.3}$$

with $\tau = \tau(r)$. Such a family of Laurent polynomials would arise as the potential function of holomorphic discs, see Remark 6.10 below. We also expect that $W^{(0)} = 0$ when $v_0 = 0$, but we do not need to assume this in the following discussion.

6.4. In this situation, we expect the following mirror correspondence:

$$\mathcal{V}^\vee \longleftrightarrow \text{relative (exponential) periods of } \Big((\mathbb{C}^\times)^n, (W^{(1)})^{-1}(-1/r_1) \cup \cdots \cup (W^{(c)})^{-1}(-1/r_c)\Big)$$

$$Y \longleftrightarrow \text{(exponential) periods of } (W^{(1)})^{-1}(1/r_1) \cap \cdots \cap (W^{(c)})^{-1}(1/r_c)$$

where 'exponential periods' mean integrals of algebraic forms multiplied by $e^{-W^{(0)}/z}$; they are usual periods when $W^{(0)} = 0$.

6.5. Introduce the I-functions [6] for $\mathcal{V}^\vee$ and Y as follows:

$$I_{\mathcal{V}^\vee}(\tau, z) = e^{\tau/z} \sum_{d \in H_2(F,\mathbb{Z})} e^{\tau \cdot d} J_d(z) \prod_{i=1}^{c} \prod_{k=0}^{v_i \cdot d - 1} (-v_i - kz)$$

$$I_Y(\tau, z) = e^{\tau/z} \sum_{d \in H_2(F,\mathbb{Z})} e^{\tau \cdot d} J_d(z) \prod_{i=1}^{c} \prod_{k=1}^{v_i \cdot d} (v_i + kz)$$

where the J-function of F is expanded as $J_F(\tau, z) = e^{\tau/z} \sum_{d \in H_2(F,\mathbb{Z})} e^{\tau \cdot d} J_d(z)$ as before. We regard $I_{\mathcal{V}^\vee}(\tau, z)$ as $H^*(\mathcal{V}^\vee)$-valued function and $I_Y(\tau, z)$ as $H^*(Y)$-valued function. These I-functions are related to the respective J-functions via mirror transformation and therefore compute certain Gromov-Witten invariants of $\mathcal{V}^\vee$ or Y.

6.6 Theorem. *Let W_r be a family of Laurent polynomials as in* (6.2) *satisfying the mirror symmetric Gamma conjecture* (6.3). *Let $s_1, \ldots, s_c > 0$ be sufficiently large positive numbers. We have*

$$(2\pi \mathtt{i})^c \int_{B_s} e^{-W^{(0)}/z} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} = \int_{\mathcal{V}^\vee} \Big(z^{v_0} z^{\frac{\deg}{2}} I_{\mathcal{V}^\vee}(\tau_0 - v \log(-s), -z) \Big) \cup \widehat{\Gamma}_{\mathcal{V}^\vee} \operatorname{Ch}(\mathcal{O}_F)$$

where $v \log(-s) = \sum_{i=1}^c v_i \log(-s_i)$, $B_s = (\mathbb{R}_{>0})^n \cap \bigcap_{i=1}^c \{W^{(i)}(x) \le s_i\}$ is a relative n-cycle and we choose the branch $\operatorname{Im}(\log(-s_i)) = -\pi$. *We also have*

$$\left(\prod_{i=1}^c s_i\right) \int_{C_s} e^{-W^{(0)}/z} \frac{d\log x_1 \cdots d\log x_n}{dW^{(1)} \cdots dW^{(c)}} = \int_Y \Big(z^{v_0} z^{\frac{\deg}{2}} I_Y(\tau_0 - v \log s, -z) \Big) \cup \widehat{\Gamma}_Y,$$

where $v \log s = \sum_{i=1}^c v_i \log s_i$ and $C_s = (\mathbb{R}_{>0})^n \cap \bigcap_{i=1}^c (W^{(i)})^{-1}(s_i)$ is an $(n-c)$-cycle.

6.7 Remark. For a generic $s \in (\mathbb{R}_{>0})^c$, $C_s \subset (\mathbb{R}_{>0})^n$ is either empty or a smooth complete intersection of codimension c. Therefore the measure $d\log x_1 \cdots d\log x_n / (dW^{(1)} \cdots dW^{(c)})$ on C_s makes sense.

Proof of Theorem 6.6. The proof is parallel to that of Theorem 4.6, so we only give an outline. We consider the Laplace transform of the quantity (6.3) with respect to $r_1, \ldots, r_c$:

$$\widehat{\mathcal{I}}(s, z) := \int_{[0,\infty)^c} e^{\sum_{i=1}^c r_i s_i / z} \mathcal{I}(\tau(r), z)\, dr_1 \cdots dr_c$$

with $s_i < 0$ and $z > 0$, where $\mathcal{I}(\tau(r), z)$ denotes the quantity in (6.3). Using the left-hand side of (6.3) and integrating r_i out, we find

$$\widehat{\mathcal{I}}(s, z) = z^c \int_{(\mathbb{R}_{>0})^n} \frac{e^{-W^{(0)}/z}}{(W^{(1)} - s_1) \cdots (W^{(c)} - s_c)} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}. \tag{6.8}$$

On the other hand, by using the right-hand side of (6.3) and computing the Laplace transform termwise, we obtain

$$(6.9)\qquad \widehat{\mathcal{I}}(s,z) = \frac{z^c}{\prod_{i=1}^c(-s_i)}\int_F \left(z^{v_0} z^{\deg/2}\widetilde{I}_Y(\tau_0 - v\log s, z)\right)\cup\left(\prod_{i=1}^c e^{-\pi\mathtt{i}v_i}\Gamma(1-v_i)\right)\widehat{\Gamma}_F$$

where $\widetilde{I}_Y(\tau,z)$ is defined by the same formula as $I_Y(\tau,z)$ but is regarded as a function taking values in $H^*(F)$ and we choose the branch $\mathrm{Im}(\log s_i)=\pi$. The interchange of the sum and the integral can be justified when $s_j \ll 0$ by the estimate (5.4) as before; this simultaneously shows the convergence of the Laplace transformation for $s_i \ll 0$. The integral representation (6.8) then shows that $\widehat{\mathcal{I}}(s,z)$ can be analytically continued to a holomorphic function for $s\in(\mathbb{C}\setminus[0,\infty))^c$.

We choose a sufficiently large $M\gg 0$ and consider the integral

$$\mathcal{M}(s,z) = \int_{-M}^{s_1} ds_1' \int_{-M}^{s_2} ds_2' \cdots \int_{-M}^{s_c} ds_c' \,\widehat{\mathcal{I}}(s',z)$$

where the path of integration is contained in the region $(\mathbb{C}\setminus[0,\infty))^c$. Using (6.8), we find that

$$\mathcal{M}(s,z) = z^c\int_{(\mathbb{R}_{>0})^n}\left(\prod_{i=1}^c \log\left(\frac{W^{(i)}+M}{W^{(i)}-s_i}\right)\right) e^{-W^{(0)}/z}\frac{dx_1\cdots dx_n}{x_1\cdots x_n}$$

which is analytic in $s\in(\mathbb{C}\setminus[0,\infty))^c$. We compute iteratively the jump of $\mathcal{M}(\tau,s,z)$ across the branch cut $s_j\in[0,\infty)$ for $j=1,\dots,c$. Writing $\Delta_j f(s_1,\dots,s_c) = \lim_{\epsilon\to+0}(f(s_1,\dots,s_j+\mathtt{i}\epsilon,\dots,s_c) - f(s_1,\dots,s_j-\mathtt{i}\epsilon,\dots,s_c))$ for a function f defined on $(\mathbb{C}\setminus[0,\infty))^c$, we have

$$(\Delta_1\cdots\Delta_c\mathcal{M})(s,z) = (2\pi\mathtt{i}z)^c\int_{B_s} e^{-W^{(0)}/z}\frac{dx_1\cdots dx_n}{x_1\cdots x_n}.$$

Calculating the same jump using the power series expansion (6.9), we obtain the first formula of the proposition. The second formula follows from the first by differentiating it in $s_1,\dots,s_c$. □

6.10 Remark. The Laurent polynomial mirror W_r as in (6.2) should arise from disc counting as follows [16, 5, 2]. Suppose that we have simple normal crossing divisors $D_0, D_1, \dots, D_c$ such that $-K_F = D_0+D_1+\cdots+D_c$ with $v_i=[D_i]$ and that we have a special Lagrangian torus fibration $F\setminus\bigcup_{i=0}^c D_i\to B$. We consider holomorphic discs of Maslov index two with boundaries on a Lagrangian torus fiber $L\subset F$. The potential function is given by counting such holomorphic discs:

$$W_r = \sum_{\substack{\beta\in H_2(F,L)\\ \mu(\beta)=2}} n_\beta x^{\partial\beta}\prod_{i=0}^c r_i^{D_i\cdot\beta}$$

where n_β is the open Gromov-Witten invariant, a virtual count of stable holomorphic discs with boundary in L of class $\beta\in H_2(F,L)$. The factor $\prod_{i=0}^c r_i^{D_i\cdot\beta}$ can be interpreted as the exponentiated area $e^{\int_\beta\tau}$ of a disc with $\tau=\sum_{i=0}^c v_i\log r_i$. The Maslov index $\mu(\beta)$ is given by $\mu(\beta)=2\beta\cdot(D_0+\cdots+D_c)$ and hence $\mu(\beta)=2$ implies that there exists $i\in\{0,1,\dots,c\}$ such that $D_i\cdot\beta=1$ and $D_j\cdot\beta=0$ for all $j\neq i$. This gives a function of the form (6.2).

6.11 Example. Let F be a Fano toric variety as in Example 4.13. With notation as there, suppose that we have a partition $\{D_1, \dots, D_c\} = U \sqcup V_1 \sqcup \cdots \sqcup V_l$ such that $u = \sum_{D_i \in U} D_i$ and $v_j = \sum_{D_i \in V_j} D_i$ $(1 \le j \le l)$ are nef. Let $F' \subset F$ be a smooth Fano hypersurface in the class u. Applying the second half of Theorem 6.6 to $\mathcal{V} = \mathcal{O}(u)$, we obtain the mirror symmetric Gamma conjecture for F'

$$\int_{Z \cap (\mathbb{R}_{>0})^n} e^{-W'/z} \frac{d\log x_1 \wedge \cdots \wedge d\log x_n}{d(\sum_{D_i \in U} e^{-\lambda_i} x^{b_i})} = \int_{F'} (z^{c_1(F')} z^{\deg/2} I_{F'}(\tau, -z)) \cup \widehat{\Gamma}_{F'}$$

for $\tau = -\sum_{i=1}^{c} \lambda_i D_i$, where we set $W' = e^{-\lambda_{k+1}} x^{b_{k+1}} + \cdots + e^{-\lambda_c} x^{b_c}$ and $Z = \{x \in (\mathbb{C}^\times)^n : e^{-\lambda_1} x^{b_1} + \cdots + e^{-\lambda_k} x^{b_k} = 1\}$. This result already appeared in [22, Theorem 5.7].

We further apply the first half of Theorem 6.6 to F' and the nef partition $c_1(F') = v_1 + \cdots + v_l$. The function $W' : Z \to \mathbb{C}$ is not (apparently) a Laurent polynomial. In most cases, however, we can find a torus chart $(\mathbb{C}^\times)^{n-1} \subset Z$ such that $Z \cap (\mathbb{R}_{>0})^n$ corresponds to $(\mathbb{R}_{>0})^{n-1}$ and that W is a Laurent polynomial on that chart, see e.g. [7, §2]. Also, as mentioned before, $I_{F'}$ and the J-function $J_{F'}$ coincide after a change of variables, but they are not necessarily the same. Nevertheless, we can easily check that the argument in the proof of Theorem 6.6 applies equally to this situation where the mirror is not necessarily a Laurent polynomial and the J-function is replaced with the I-function. We obtain the following equality:

$$\int_B \frac{d\log x_1 \cdots d\log x_n}{d(\sum_{D_i \in U} e^{-\lambda_i} x^{b_i})} = \int_F \sum_{d \in H_2(F,\mathbb{Z})} e^{-\tau + \tau \cdot d} \frac{\prod_{i=1}^{c} D_i \Gamma(D_i - D_i \cdot d)}{\Gamma(u - u \cdot d) \prod_{i=1}^{l} \Gamma(v_i + 1 - v_i \cdot d)}$$

where $B = \{x \in Z \cap (\mathbb{R}_{>0})^n : \sum_{D_i \in V_j} e^{-\lambda_i} x^{b_i} \le 1, j = 1, \dots, l\}$ and $\tau = -\sum_{i=1}^{c} \lambda_i D_i$.

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