THE COMBINATORIAL PICARD GROUP

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These notes are adapted from (Ewald 1996). For more information see also (Fulton 1993, Oda 1978).

Given a fan Δ of cones in $N_{\mathbb{R}}$ we shall denote its set of dual cones by

$$\Delta^\vee := \{\sigma^\vee \mid \sigma \in \Delta\}.$$

To each cone $\sigma \in \Delta$ we assign the commutative semigroup $S_\sigma := \sigma^\vee \cap M$, where $M := \mathrm{Hom}\,(N, \mathbb{Z})$ is the dual lattice of N. We thus obtain a system

$$\mathcal{S}(\Delta) := \{S_\sigma \mid \sigma \in \Delta\}$$

of semigroups assigned to Δ. There exist bijective relations $\Delta \leftrightarrow \Delta^\vee \leftrightarrow \mathcal{S}(\Delta)$.

Lemma 1. *Any n-dimensional cone σ is the vector sum of a cone σ_0 with apex 0 and a linear space U,*

$$\textit{ie. } \sigma = \sigma_0 + U,$$

where $\dim \sigma_0 + \dim U = n$. If in addition σ is a lattice cone, then σ_0 and U can also be chosen to be lattice cones.

Such a U is uniquely determined, however for $\dim U > 0$ we can choose σ_0 in many ways. It suffices to take $U = \sigma \cap (-\sigma)$. We call U the *cospan* of σ and write:

$$U = \mathrm{cospan}\,\sigma.$$

To each $\sigma^\vee \in \Delta^\vee$ we assign $m_\sigma \in M$ such that

$$\text{if } \tau \text{ is a face of } \sigma \text{ then } m_\sigma - m_\tau \in \mathrm{cospan}\,\tau^\vee. \tag{1}$$

We can now replace each sub-semigroup S_σ in $\mathcal{S}(\Delta)$ by its residue class $m_\sigma + S_\sigma$, preserving the inclusion of semigroups in $\mathcal{S}(\Delta)$.

Definition 1. A system $\mathcal{P} := \{m_\sigma + \sigma^\vee\}_{\sigma \in \Delta}$ of translated cones, such that (1) is satisfied, is called a *virtual polytope (with respect to the fan Δ)*.

If Δ is polytopal we may choose $\{m_\sigma\}_{\sigma\in\Delta}$ such that

$$\bigcap_{\sigma \in \Delta} (m_\sigma + \sigma^\vee) =: P$$

is a lattice polytope, and $-P^*$ spans Δ. In such a case, $\mathcal{P}$ and P can be identified. The proofs of the following two lemmas are elementary.

Lemma 2. *The virtual polytopes with respect to the fan Δ are a commutative group $\tilde{\mathcal{G}}$ with respect to addition defined by*

$$\mathcal{P} + \mathcal{P}' := \{m_\sigma + m'_\sigma + \sigma^\vee\}_{\sigma \in \Delta}.$$

The zero element is $\Delta^\vee$.

Lemma 3. *The system $\mathcal{M} := \{m_\sigma + S_\sigma\}_{\sigma\in\Delta}$ of residue classes assigned to the semigroups of $\mathcal{S}(\Delta)$ define a commutative group $\mathcal{G}$ with respect to addition given by*

$$\mathcal{M} + \mathcal{M}' := \{m_\sigma + m'_\sigma + S_\sigma\}_{\sigma \in \Delta}.$$

The zero element is $\mathcal{S}(\Delta)$.

The groups $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are isomorphic.

Many properties of virtual polytopes remain true under translation (applied to all $m_\sigma+\sigma^\vee, \sigma \in \Delta$ simultaneously). Thus we make the following definition:

Definition 2. If $\mathcal{G}$ is the group above, we call $\mathcal{G}/\mathbb{Z}^n$ the *combinatorial Picard group*, $\operatorname{Pic}\Delta$ of Δ. We denote its elements by $\mathcal{P}$.

The Picard group $\operatorname{Pic}\Delta$ is a finitely generated commutative group, and hence (by the fundamental theorem on commutative groups) is equivalent to the direct sum

$$\operatorname{Pic}\Delta \cong \mathbb{Z}^q \oplus \mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_p},$$

where $\mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_p}$ is called the *torsion* of the group, and q is its *Betti number.*

Lemma 4. *For* $\operatorname{Pic}\Delta$ *to be a torsion-free group, it is sufficient that* Δ *contains an* n*-cone* τ*.*

Proof. Suppose $\operatorname{Pic}\Delta$ contains an element of finite order. Then there is a virtual polytope $\mathcal{P} = \{m_\sigma + \sigma^\vee\}_{\sigma\in\Delta}$ and a natural number r such that $r\mathcal{P} = \{rm_\sigma + \sigma^\vee\}_{\sigma\in\Delta}$ can be obtained from $\Delta^\vee$ by adding a lattice vector c. Since τ is n-dimensional, $\{rm_\tau\} = \operatorname{cospan}\tau^\vee$ is a lattice point, and hence $rm_\tau = c$. Since m_τ is also a lattice point c_0, $\mathcal{P} = c_0 + \Delta^\vee$, so that $\mathcal{P}$ represents the zero element of $\operatorname{Pic}\Delta$. □

If Δ contains an n-cone we can calculate $\operatorname{Pic}\Delta$ explicitly. The following two results are taken from (Ewald 1996, pp.171-3).

Theorem 1. *Let* Δ *be a simplicial fan in* $N_\mathbb{R} \cong \mathbb{R}^n$ *which contains at least one* n*-cone, and let* k *be the number of rays of* Δ*. Then*

$$\operatorname{Pic}\Delta \cong \mathbb{Z}^{k-n}.$$

Theorem 2. *Let Delta be a fan in* $N_\mathbb{R} \cong \mathbb{R}^n$ *which contains at least one* n*-cone, and let* $\{\rho_1, \ldots, \rho_k\}$ *be the set of rays of* Δ*. We consider all maximal faces* $\{\sigma_1, \ldots, \sigma_q\}$ *of* Δ *which are not simplex cones, and set, for* $\sigma_i = \rho_{i_1} + \ldots + \rho_{i_s}, i = 1, \ldots, q$,

$$L_{\sigma_i} := \mathcal{L}(d_{i_1}, \ldots, d_{i_s}) \quad \text{(space of linear dependencies)}$$

and

$$L := L_{\sigma_1} + \ldots + L_{\sigma_q}, \qquad \lambda := \dim L.$$

Then

$$\operatorname{Pic}\Delta \cong \mathbb{Z}^{k-n-\lambda}.$$

Definition 3. We call $\mu(\Delta) := k - n - \lambda$ the *combinatorial Picard number* of Δ.

Let Δ be a complete fan. We may choose $m_\tau = m_\sigma$ if τ is a face of $\sigma \in \Delta^{(n)}$, where, of course, condition (1) must be observed. Hence if $\Delta^{(n)} = \{\sigma_1, \ldots, \sigma_q\}$ and $a_i := m_{\sigma_i}, i = 1, \ldots, q$, the cones

$$\left\{a_1 + \sigma_1^\vee, \ldots, a_q + \sigma_q^\vee\right\}$$

determine an element $\mathcal{P} \in \operatorname{Pic}\Delta$, written

$$\mathcal{P} := \left[a_1 + \sigma_1^\vee, \ldots, a_q + \sigma_q^\vee\right].$$

Definition 4. We call P an *associated polytope* of Δ if $\Delta = \Delta(-P)$. That is, if Δ is spanned by $-P^*$, or in other words, if Δ is the fan of normal cones of $-P$.

Lemma 5. *Let* Δ *be complete and polytopal, and let* $-P^*$ *be a spanning polytope of* Δ*, so that* P *is an associated polytope of* Δ*. Then, for* $\operatorname{vert} P = \{a_1, \ldots, a_q\}$,

$$\mathcal{P} = \mathcal{P}(P) = [a_1 + \operatorname{pos}(P - a_1), \ldots, a_q + \operatorname{pos}(P - a_q)]$$

is an element of $\operatorname{Pic}\Delta$ *from which* Δ *can be reconstructed. Thus* $\Delta = \Delta(-P)$ *for* $P = (a_1 + \operatorname{pos}(P - a_1)) \cap \ldots \cap (a_q + \operatorname{pos}(P - a_q)))$.

Any summand of P' *and* P *can also be written in the form*

$$P' = (a_1' + \operatorname{pos}(P - a_1)) \cap \ldots \cap (a_q' + \operatorname{pos}(P - a_q))$$

where the assignment $a_i \mapsto a_i'$ *provides a surjective map* $\chi_{P'} : \operatorname{vert} P \to \operatorname{vert} P'$.

Definition 5. If P' is a lattice summand of an associated polytope P of the polytopal fan Δ, we call

$$\mathcal{P}(P') := \left[a'_1 + \operatorname{pos}(P - a_1), \ldots, a'_q + \operatorname{pos}(P - a_q)\right]$$

a *polytope element* of $\operatorname{Pic}\Delta = \operatorname{Pic}\Delta(-P)$.

Lemma 6. *Let $\Delta = \Delta(-P)$ and let P', P'' be lattice polytopes such that $P = P' + P''$. Then $\mathcal{P}(P) = \mathcal{P}(P') + \mathcal{P}(P'')$.*

In particular for any $r \in \mathbb{N}$, $\mathcal{P}(rP) = r\mathcal{P}(P)$.

The following theorem (Ewald 1996, pp.175-7) enables us to find a finite system of generators of $\operatorname{Pic}\Delta(-P)$, consisting of polytope elements.

Theorem 3. *For any $\mathcal{P} \in \operatorname{Pic}\Delta(-P)$ there exists a lattice polytope P_0 strictly combinatorially isomorphic to P (hence also associated with Δ), and a natural number r such that*

$$\mathcal{P} = \mathcal{P}(P_0) - \mathcal{P}(rP).$$

Definition 6. If $\Delta = \Delta(P)$ is polytopal, we call the group $\tilde{\mathcal{G}}$ the *polytopal group* of Δ.

The following result follows immediately.

Theorem 4. *Let $\Delta = \Delta(P)$ be polytopal.*

(i) *The polytope group $\tilde{\mathcal{G}}$ is the smallest group into which the semigroup of all polytopes strictly combinatorially isomorphic to P can be embedded.*
(ii) $\operatorname{Pic}\Delta$ *can be generated by $f_{n-1}(P) - n - \lambda + 1$ polytope elements strictly isomorphic to P.*

References

Ewald, G. (1996), *Combinatorial Convexity and Algebraic Geometry*, Graduate Texts in Mathematics, Springer.
Fulton, W. (1993), *Introduction to Toric Varieties*, Annals of Mathematics Studies, Princeton University Press.
Oda, T. (1978), *Torus Embeddings and Applications*, Tata Institute Lecture Notes, Springer.