PROJECTING FANOS IN THE MIRROR

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Abstract. A new structure connecting toric degenerations of smooth Fano threefolds by projections was introduced by I. Cheltsov *et al.* in "Birational geometry via moduli spaces". Using Mirror Symmetry, these connections were transferred to the side of Landau–Ginzburg models, and a nice way to connect the Picard rank one Fano threefolds was described. We apply this approach to all smooth Fano threefolds, connecting their degenerations by toric basic links. In particular, we find many Gorenstein toric degenerations of the smooth Fano threefolds we consider. We implement mutations in this framework too. It turns out that appropriately chosen toric degenerations of the Fanos are connected by toric basic links from a few roots. We interpret the relations we found in terms of Mirror Symmetry.

To the master of birational geometry, Professor Yuri Prokhorov.

1. Introduction

One of the central topics of research in birational geometry are Fano varieties: varieties with ample anticanonical bundle. They play a crucial role in the Minimal Model Program and present a rich geometric picture. Fano varieties are central in Mirror Symmetry; many constructions of mirror duality are either Calabi–Yau manifolds or are for Fano varieties.

The classification problem for smooth Fano varieties goes back to the XIX century. Riemann showed that the only Fano curve is a projective line $\mathbb{P}^1$. Pasquale del Pezzo classified the smooth Fano surfaces (now called *del Pezzo* surfaces in his honour). He showed that these surfaces (with very ample anticanonical class) are non-degenerate surfaces of degree n in $\mathbb{P}^n$; now we also add two trigonal examples of degrees two and one. These surfaces have degree at most nine and form an irreducible family in each degree, with the exception of degree eight where there are two irreducible families. The modern (that is, classical, not pre-classical) description of del Pezzo surfaces is as blow-ups of general-enough points on $\mathbb{P}^2$, together with a quadric surface. That is, they are projections of an anticanonically embedded $\mathbb{P}^2 \subset \mathbb{P}^9$ from general-enough points (again together with a quadric). If we choose any points as centres of projection we obtain singular del Pezzo surfaces because, in this case, the projection may contract lines through the point; however we arrive at the same family, so smooth del Pezzo surfaces can be obtained as smoothings of projected singular ones.

Classification of Fano threefolds is more tricky. It was initiated by Gino Fano and developed later by Iskovskikh [47, 48] (the modern definition of Fano varieties is due to Iskovskikh, who named them after Fano). Soon afterwards, Mori and Mukai, using Iskovskikh's approach and the Minimal Model Program, classified all smooth Fano threefolds [62]; there are 105 families (the final family, accidentally overlooked in the original classification, was found in 2002 [63]). There is currently no classification known in higher dimensions, however Kollár–Miyaoka–Mori show that there are a finite number of families of Fano varieties in any given dimension. It is expected that even in dimension four the number of families of Fano varieties is very large.

Unlike the two-dimensional case, there is no structure in the list of Fano threefolds (see [50]) systematically relating one with each other. An approach to obtaining such a structure is described in [16]. Briefly, the idea is as follows: similarly to the two-dimensional case, one hopes to relate all Fano varieties to some "specific" varieties (not necessary smooth); a class of simple relations between these new varieties should include projections from singular points, tangent spaces to smooth points, lines, and conics.

To place the problem on the combinatorial level, we choose toric Fano varieties as the "specific" varieties. We call the simple projections between toric varieties *toric basic links*. One can describe the needed projections in terms of the spanning polytopes. An example of a nice subtree in the projections tree relating Picard rank one Fano varieties (see Figure 1) is found in [16]. Moreover, one can add

Taiwanese Journal of Mathematics, **29**(6) (2025), 1411–1494, doi:10.11650/tjm/250704

mutations to this picture (see §4); that is, deformations from one toric degeneration to another. In this paper we study projections systematically for all Fano threefolds. In particular, we prove the following. Given a toric variety T let us call a projection in an anticanonical embedding from tangent space to invariant smooth point, invariant cDV point, or an invariant smooth line an *F-projection*.

Theorem 1.1 (Theorem 8.1)**.** *Given any smooth Fano threefold X, there exists a Gorenstein toric degeneration of X that can be obtained by a sequence of mutations and F-projections from a toric degeneration of one of fifteen smooth Fano threefolds (see Table 2). The directed graph connecting all Fano varieties with very ample anticanonical class via the projections and mutation is presented in Table 3. Each of the toric degenerations we use can be equipped with a toric Landau–Ginzburg model.*

Theorem 1.2 (Theorem 8.3)**.**
(i) For any smooth Fano threefold with very ample anticanonical class there exists a choice of Gorenstein toric degeneration such that all these degenerations are connected by sequences of F-projections. The directed graph of such projections can be chosen as a union of fifteen trees with roots shown in Table 2. The directed graph connecting all Fano varieties with very ample anticanonical class via projection and mutation is presented in Figure B. Each of the toric degenerations can be equipped with a toric Landau–Ginzburg model.
(ii) For any smooth Fano threefold with very ample anticanonical class there exists a choice of toric degeneration such that all these degenerations are connected by sequences of projections in the anticanonical embedding with toric centres which are either tangent spaces to smooth points, or cDV points, or smooth lines, or smooth conics. The directed (sub)graph of such projections connecting degenerations of all smooth Fano threefolds can be chosen to have five roots which are: $\mathbb{P}^3$, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$, the quadric threefold, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^3$ blown-up in a line. The directed graph connecting all Fano varieties via the projections and mutations is presented in Figure C. Each of the toric degenerations can be equipped with a toric Landau–Ginzburg model.

Note that $\mathbb{P}^3$ blown-up in a line is not a projection of $\mathbb{P}^3$ since a line in anticanonical embedding has degree four.

As the theorems suggest, the toric degenerations providing basic links correspond to toric Landau–Ginzburg models (that is, Laurent polynomials related to toric degenerations and representing mirrors; see below), and the theorems show that two Fano varieties are related by a toric basic link if their toric Landau–Ginzburg models are closely related too.

The idea of presenting Landau–Ginzburg models dual to Fano varieties, or dual to varieties close to Fano, as Laurent polynomials goes back to Givental. In [37] he suggested a Landau–Ginzburg model for a smooth toric variety as a complex torus with a complex-valued function (a *superpotential*) represented by a Laurent polynomial with support equal to the spanning polytope of the toric variety. His construction was generalised to varieties admitting nice toric degenerations; in this case one associates the Laurent polynomial with the fan of the toric degeneration of the Fano variety. This idea, in turn, goes back to Batyrev–Borisov's approach [9] to mirror duality for toric varieties as duality of the corresponding polytopes and good deformational behaviour of Gromov–Witten invariants (see [8] and references therein). In this spirit Laurent presentations of Landau–Ginzburg models for Grassmannians were constructed by Eguchi–Hori–Xiong [34], and for Grassmannians and partial flag varieties by Batyrev–Ciocan-Fontanine–Kim–van Straten [10, 11].

The crucial part of the duality between Fano varieties and Landau–Ginzburg models in this approach is an identification of (a part of) Gromov–Witten theory for the Fano varieties and periods of the dual one-dimensional family. If the total space of the family is a complex torus, then the Landau–Ginzburg model, as we mentioned above, is (in some basis) represented by a Laurent polynomial. In this case the main period of the family is a generating series for constant terms of powers of Laurent polynomials, see §3. In [37] Givental proves the coincidence of the two series for smooth toric varieties; the same was done in [10, 11] for Grassmannians and for partial flag varieties.

In [37] Givental suggested an approach to constructing Landau–Ginzburg models for (almost) Fano complete intersections in toric varieties. This approach was generalized to complete intersections in Grassmannians and (partial) flag varieties in [10, 11], see also [13]. The output of Givental's construction is not a complex torus with a function but a complete intersection in a torus with a function. These Landau–Ginzburg models satisfy the Gromov–Witten period condition via Quantum Lefshetz Theorem, which enables one to pass from Gromov–Witten theory of a variety to Gromov–Witten theory of an ample (Fano) hypersurface.

The natural idea is to realise birationally Landau–Ginzburg models as Laurent polynomials. This was done in [21, 46, 69–71] for smooth Fano threefolds and complete intersections in projective spaces. In the case of Fano complete intersections (Picard rank one) it was shown that the resulting Laurent polynomials are related to toric degenerations. This led to the idea that Laudau–Ginzburg models presented as Laurent polynomials should be assigned with toric degenerations used in the generalisation of Givental's approach. Another concept is the Compactification Principle which says that a correctly chosen Laurent Landau–Ginzburg model admits a fiberwise compactification to a family of compact Calabi–Yau varieties such that this family satisfy (an algebraic part of) Homological Mirror Symmetry. These two concepts, together with the initial concept of Gromov–Witten period coincidence, form the central ideas on which the present paper is based. The above ideas were initiated by Golyshev, who has suggested a program of finding (toric) Landau–Ginzburg model by guessing Laurent polynomials having prescribed constant terms and, hence, periods. The first results were obtained for smooth rank one Fano threefolds in [69, 70]; later in [46] the proof of their toricity was completed. These results clarified the connection with toric degenerations to (possibly very singular) toric varieties. The specific simple form of found Laurent polynomials leads to binomial principle suggested in [70]. This principle states that coefficients on facets of the Newton polytope of the Laurent polynomial correspond to binomial coefficients of a power of a sum of independent variables. Surprisingly this principle covers most of (but not all) smooth Fano threefolds: most of them have toric degenerations with cDV singularities, which means that integral points of the Newton polytope for Laurent polynomial are the origin and points lying on edges. This gives an algorithm for finding Landau–Ginzburg models.

Binomial principle was generalised to the Minkowski principle in [20]. It relates coefficients of the Laurent polynomial with Minkowski decompositions of facets of its Newton polytope into particular elementary summands. Moreover, all canonical polytopes that are Newton polytopes of Minkowski Laurent polynomials were found and all J-series of smooth Fano threefolds were computed. This gives, for any smooth Fano threefold, a Laurent polynomial (which is not unique) satisfying the period condition. In this paper we establish toric degeneration condition as well; see §5.

Theorem 1.3. *Let T be a Gorenstein toric variety appeared in Theorems 1.1 and 1.2. Then T is a toric degeneration of the corresponding Fano threefold.*

Remark 1.4. The assertion of Theorem 1.3 holds for much larger class of Gorenstein toric varieties; see Appendix B.

As we have mentioned, a Laurent polynomial, as a mirror dual to Fano, is not unique. However its Calabi–Yau compactification is unique. Indeed, under mild natural conditions, this holds for rank one Fano threefolds [32, 46]. This means that Landau–Ginzburg models are birational over the base field $\mathbb{C}(x)$. In other words, corresponding Laurent polynomials differ by mutations. It is proven in [2] that all Laurent polynomials with support on reflexive polytopes that produce the same period differ by (a sequence of) mutations. So they have a common log Calabi–Yau compactification. This suggests the last, at the moment, and the strongest concept of assigning a Laurent Landau–Ginzburg model to a reflexive polytope (or, more generally, to any Fano polytope): the rigid maximally mutable Laurent polynomials [22, 27].

The above findings can be interpreted categorically. We propose the following:

Conjecture 1.5. *The moduli spaces of Landau–Ginzburg models (defined in [29]) for directed graphs of Fano varieties from Theorem 1.2 are contained in each other with the top Landau–Ginzburg models contained in the ones obtained by projections.*

In such a way the behaviour of Landau–Ginzburg models for three-dimensional Fano varieties is very similar to the behaviour of Landau–Ginzburg models of del Pezzo surfaces. We lift this conjecture to further categorical levels. As a consequence of the connection of curve complexes and stability conditions it was noticed in [30] that stability conditions should behave well in families. Later on, the following theorem was proven by Haiden–Katzarkov–Kontsevich–Pandit [42]. Below we use the definition of stability conditions given by Bridgeland. We give the most general version of the statement. After that we will explain the connection with our situation. In what follows we give a categorical description of a family of hyperplane sections. We use the language of comonads. Here the category $\mathcal{C}_{\text{special}}$ is the analogue of a singular hyperplane section and the category $\mathcal{C}_{\text{general}}$ is the analogue of a general section. The category $\mathcal{C}_0$ is the global family.

Theorem 1.6. *Consider the following data:*

(i) *a category $\mathcal{C}_{\text{special}}$ which is an $(\infty, 1)$-category;*
(ii) *a stability condition on $\mathcal{C}_{\text{special}}$;*
(iii) *a comonad T on $\mathcal{C}_{\text{special}}$ such that $Cone(T \to Id) = [2]$.*

Let $\mathcal{C}_0$ be a category of comodules corresponding to $\mathcal{C}_{\text{special}}$ and T. There is a functor $\mathcal{C}_{\text{special}} \to \mathcal{C}_0$. We define $\mathcal{C}_{\text{gen}} = \mathcal{C}_0/\mathcal{C}_{\text{special}}$. In the situation above there exists a stability condition on $\mathcal{C}_{\text{gen}}$ such that its central charge and its phase are lift from a central charge and a phase on $\mathcal{C}_{\text{special}}$.

Applied to our situation the above theorem suggests the following:

(i) Stability conditions of Fano varieties can be obtained from stability conditions of (singular) toric varieties. Indeed, the latter have exceptional collections and moduli spaces of stability conditions are easier to understand.
(ii) Stability conditions of Calabi–Yau varieties can be obtained from stability conditions of Fano manifolds via Tyurin degenerations.

Combining these facts with the finding of the present paper suggests:

Conjecture 1.7. *The moduli spaces of stability conditions (defined by Bridgeland) for directed graphs of Fano varieties in Theorem 1.2 are contained in each other with the top moduli spaces of stability conditions contained in the ones obtained by projections.*

The above observations suggest that obtaining, via degenerations, stability conditions for one of three-dimensional Fano varieties leads to computing stability conditions for all of them. In a similar fashion we propose that Apery constants (defined in [38]) for all these Fano varieties are connected with each other. We expect that similar behaviour of Fano varieties extends to high dimensions.

The paper is organised as follows. In §2 we recall results from [16] and define toric basic links relating Fano threefolds. In §3 we define the toric Landau–Ginzburg models associated with toric Fano threefolds. Toric basic links can be interpreted as their transformations. In §4 we define mutations between toric Landau–Ginzburg models; that is, relative birational transformations between them. They correspond to deformations of toric degenerations of given Fano threefolds and they can be implemented to the toric basic links graph. In §5 we study toric degenerations of Fano threefolds. In §6 we describe the directed graph of reflexive polytopes. In §7 we describe the algorithm we use to compute the projections graph. In §8 we compute the directed (sub)graph of projections relating smooth Fano threefolds; roots of the graphs are several particular Fano threefolds. Finally in the Appendices we present the data which is the output of our construction. In Appendix A we present the appropriate projection directed (sub)graphs. In Appendices B and C we present Gorenstein toric degenerations of Fano threefolds and some related data.

1.1. **Notation.** Smooth del Pezzo threefolds, that is, smooth Fano threefolds of index two, we denote by V_d, where d is the degree with respect to a generator of the Picard group; the single exception is the quadric, which we denote by Q. Fano threefold of Picard rank one, index one, and degree d, we denote by X_d. The remaining Fano threefolds we denote by $X_{k\text{–}n}$, where k is the Picard rank and n is its number according to [50]. When $k = 4$ the numbers n differ from the identifiers in Mori–Mukai's original classification [62] due to the 'missing' rank four Fano $X_{4\text{–}2}$ [63], which has been placed in the appropriate position within the list.

To any reflexive polytope $P \subset N_{\mathbb{Q}}$ we associate the Gorenstein toric Fano variety X_P whose fan Σ in the lattice N is generated by taking the cones over the faces of P. We call Σ the *spanning fan* of P. The moment polytope is denoted by either $P^* \subset M_{\mathbb{Q}}$ or by $\Delta \subset M_{\mathbb{Q}}$, and is dual to P. We identify Gorenstein toric Fano varieties by the corresponding reflexive polytope P, numbered from 1 to 4319 by the *Reflexive ID* as given by the online Graded Ring Database [15]. This agrees with the order of the output from the software PALP [60], developed by Kreuzer–Skarke for their classification [58], and with the databases used by the computational algebra systems Magma and Sage, the only complication being whether the numbering starts from 1 (as we do here) or from 0 (as done in, for example, Sage).

The Laurent polynomial associated to a Fano threefold of Picard rank k and number m we denote by $f_{k\text{–}m}$. The toric variety whose spanning fan is generated by the Newton polytope of $f_{k\text{–}m}$ we denote by $F_{k\text{–}m}$. When appropriate, we may refer to the period period sequence $\pi_f(t)$ of a Laurent polynomial f by its *Minkowski ID*, an integer from 1 to 165. The Minkowski IDs are defined in [2, Appendix A], used in [21], and can be looked-up online at [15].

1.2. **The use of computer algebra and databases.** Because of the large number of 3-dimensional Gorenstein toric Fano varieties, several results are derived with the help of computational algebra software and databases of classifications such as [15]. Computer-assisted rigorous proofs play an increasingly important role as we move from surfaces to threefolds, and will become an essential mathematical technique if we ever hope to progress to the systematic study of fourfolds and Kreuzer–Skarke's massive classification [59] of 473 800 776 reflexive polytopes in dimension four.

We highlight our use of computer algebra. In §5, step (i), we make use of Magma [14] in order to compute additional toric models, besides those arising from the Minkowski polynomials as classified in [2]. In §5, step (iii), we use Macaulay2 [39] to compute the dimension of the tangent space for each of the Gorenstein toric Fano threefolds. The software Topcom [74] is used in §5.2 to search for reflexive polytopes with appropriate boundary triangulations. Finally, in §8 we make use of several computer programs that rely on Palp [60] in order to build and manipulate the relevant projection graphs, as well as to further explore the effects of using different combinations of allowed projections and mutations. We emphasise that although any particular example can be worked by hand, the number of cases under consideration means that the only practical way to ensure accuracy is to employ the use of a computer.

1.3. **Acknowledgements.** We are extreamly grateful to Nathan Ilten for his help with the calculations in §5. Without his generous contributions, this paper would not have been possible. Alexander Kasprzyk was supported by EPSRC Fellowship EP/N022513/1. Ludmil Katzarkov was supported by Simons Investigators award — HMS, by NSF grant Theory of Atoms, by the Simons Foundation, grant SFI-MPS-T-Institutes-00007697, and the Ministry of Education and Science of the Republic of Bulgaria, contract DO1-239/10.12.2024. He has been also funded within the framework of the HSE University Basic Research Program. The work of Victor Przyjalkowski was performed at the Steklov International Mathematical Center and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-15-2025-303). He was supported by the Theoretical Physics and Mathematics Advancement Foundation "BASIS."

2. Basic links

Definition 2.1. A Fano variety X is called *Gorenstein* if its anticanonical class $-K_X$ is a Cartier divisor. Such a variety is called *canonical* (or is said to have *canonical singularities*) if for any resolution $\pi\colon \widetilde{X} \to X$ the relative canonical class $K_X - \pi^*K_{\widetilde{X}}$ is an effective divisor.

Definition 2.2. A threefold singularity P (not necessary isolated) is called *cDV* if its general transversal section has a du Val singularity at P.

The only canonical Gorenstein Fano curve is $\mathbb{P}^1$. Canonical Gorenstein Fano surfaces are called del Pezzo surfaces; they are given by the quadric surface, $\mathbb{P}^2$, and the blow-up of $\mathbb{P}^2$ in at most eight points in general position, along with the degenerations of these smooth surfaces. Canonical Gorenstein Fano threefolds are not yet classified, although partial results can be found in [17, 51, 52, 64, 68]. Smooth Fano threefolds were classified by Iskovskikh [47, 48] and Mori–Mukai [62, 63]: there are 105 deformation classes, of which 98 have very ample anticanonical divisor $-K_X$.

Let $\varphi_{|-K_X|}\colon X \to \mathbb{P}^{g+1}$ be a map given by $|-K_X|$. Then one of the following occurs:

(i) $\varphi_{|-K_X|}$ is not a morphism, that is $\mathrm{Bs}\,|-K_X| \neq \varnothing$, and all such X are found in [51];
(ii) $\varphi_{|-K_X|}$ is a morphism but not an embedding, the threefold X is called *hyperelliptic*, and all such X are found in [17];
(iii) $\varphi_{|-K_X|}$ is an embedding and $\varphi_{|-K_X|}(X)$ is not an intersection of quadrics — then the threefold X is called *trigonal*, and all such X are found in [17];

(iv) $\varphi_{|-K_X|}(X)$ is an intersection of quadrics.

In particular an anticanonically embedded Fano variety is either trigonal or an intersection of quadrics. The varieties that cannot be anticanonically embedded are classified: only few of them can be smoothed. In this paper we focus on anticanonically embedded threefolds.

2.1. **The del Pezzo surfaces.** In the 1880s Pasquale del Pezzo considered the surfaces of degree n in $\mathbb{P}^n$; in other words, surfaces with a very ample anticanonical class. The modern definition of *del Pezzo surfaces* as ones with ample anticanonical class adds blow-ups of $\mathbb{P}^2$ in seven and eight general points to del Pezzo's initial list of surfaces.

When classifying Fano varieties, we in fact classify their moduli spaces or components of their deformation spaces. This means that it is natural to consider degenerations of Fano varieties. Given one point on each moduli space together with its deformation information one can reconstruct a classification of Fano varieties. For del Pezzo surfaces this means that we allow birational transformations related to points in both general and non-general position. Of course, in this case we may get singular surfaces. Following the work of del Pezzo, let us consider anticanonically embedded surfaces. A blow-up of a general point is simply a projection from this point. A similar transformation related to a non-general point is a blow-up of this point followed by taking the anticanonical model. The anticanonical map contracts -2-curves that can appear after the blow-up, i.e. strict transforms of the lines passing through the original point. Thus these transformations are nothing but projections in the anticanonical embedding.

Projections from smooth points relate (possibly non-general) anticanonically embedded del Pezzo surfaces, starting at either $\mathbb{P}^2 = S_9 \subset \mathbb{P}^9$ or the quadric Q, and finishing with the cubic $S_3 \subset \mathbb{P}^3$:

$$\begin{tikzcd} & & Q \arrow[d, dashed, "\pi_{8'}"] & & & & \\ \mathbb{P}^2 = S_9 \arrow[r, dashed, "\pi_9"] & S_8 \arrow[r, dashed, "\pi_8"] & S_7 \arrow[r, dashed, "\pi_7"] & S_6 \arrow[r, dashed, "\pi_6"] & S_5 \arrow[r, dashed, "\pi_5"] & S_4 \arrow[r, dashed, "\pi_4"] & S_3. \end{tikzcd}$$

Our projections terminated on a cubic: further projections are non-birational. The reason for this is that cubic surface is trigonal, and blowing up a point produces a hyperelliptic surface whose anticanonical map is a double covering. Thus we want to consider birational transformations

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\alpha"'] \arrow[dr, "\varphi_{|-K_{\widetilde{X}}|}"] & \\ X \arrow[rr, dashed, "\pi_i"'] & & X', \end{tikzcd}$$

where α is a blow-up, such that:

(i) the map $\varphi_{|-K_{\widetilde{X}}|}$ is birational;
(ii) the variety X' is Fano.

Condition (i) is satisfied by being X an intersection of quadrics; in this case, $\widetilde{X}$ is at most trigonal. Via a careful choice of centres of the blow-ups, condition (ii) can also be satisfied. Considering projections in the surface case, we need to include smooth points in the set of admissible centres; *a posteriori* we see that this is sufficient.

In order to simplify the model surface picture, we make one final choice: the particular points of the deformation spaces of del Pezzo surfaces that we want to relate. We wish to make use of the toric del Pezzo surfaces. Our main motivation for this choice comes from our desire to exploit the relation, via toric Landau–Ginzburg models, with mirror symmetry; the relative ease of working with toric varieties; and the classifications of toric varieties. In this case, we insist that the centres of blow-ups should also be toric points. Thus we obtain to the following definition:

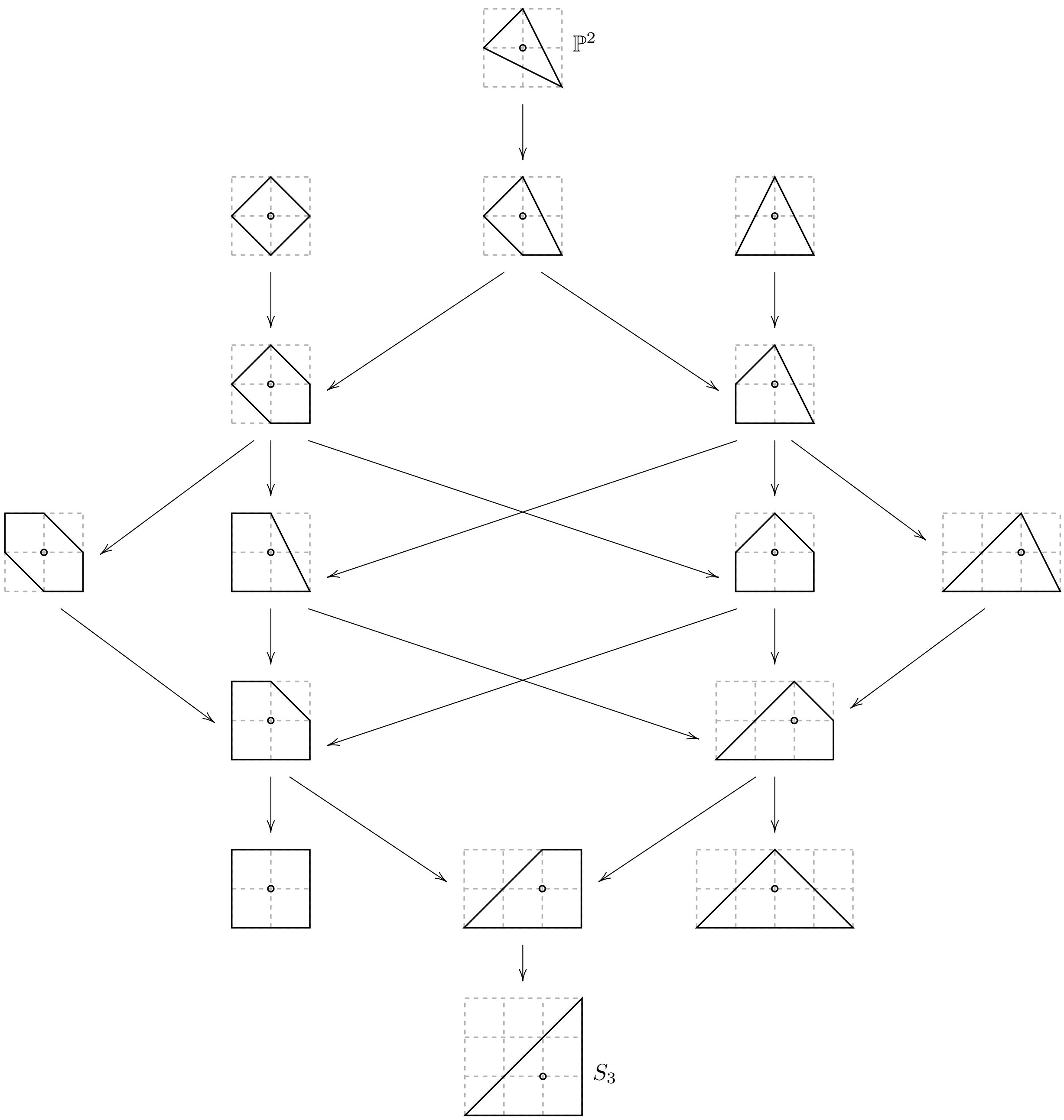
Definition 2.3. A diagram

$$\begin{tikzcd} & \widetilde{S}_i \arrow[dl, "\alpha_i"'] \arrow[dr, "\varphi_{|-K_{\widetilde{S}_i}|}"] & \\ S_i \arrow[rr, dashed, "\pi_i"'] & & S_{i-1}, \end{tikzcd}$$

where α_i is a blow-up of a smooth point and $i \geq 4$ is called a *basic link* between del Pezzo surfaces. If the varieties and the centre of the blow-up are toric then the basic link is called *toric*.

There are sixteen toric del Pezzo surfaces, corresponding to the sixteen reflexive polygons [7, 73]. All possible toric basic links between them are drawn in Figure 1. Any chain of toric basic links from $\mathbb{P}^2$ to the toric cubic, plus an appendix with the quadric, gives us the classification of del Pezzo surfaces in the sense discussed above.

FIGURE 1. The toric del Pezzo directed graph, starting with $\mathbb{P}^2$ at the top of the diagram, and working down through basic links to the toric cubic S_3 at the bottom of the diagram.



2.2. **Threefold case.** The situation described in §2.1 changes dramatically when moving to three dimensions. Iskovskih [47, 48] and Mori–Mukai [62, 63] classified the smooth Fano threefolds around 1980, a century after del Pezzo's work in two dimensions[1]. It had already observed by Iskovskih–Manin [49] that, unlike the two-dimensional case, not all of the smooth Fano threefolds are rational. Thus, if the basic links considered are required to be birational, there is no hope of producing a direct analogue of Figure 1 for smooth three-dimensional Fano varieties. This can be rectified by considering not the smooth Fano variety X itself, but the toric degenerations of X (and corresponding basic links). When X is very ample, these degenerations are toric Fano threefolds with Gorenstein singularities. Reid has shown [75, Corollary 3.6] that Gorenstein toric varieties have at worst canonical singularities, and Batyrev has shown [6] that the Gorenstein toric Fano varieties of dimension n are equivalent, in a precise sense arising from the combinatorics of toric geometry, to the n-dimensional reflexive polytopes.

Definition 2.4. Let $N \cong \mathbb{Z}^n$ be a lattice of rank n, and let $P \subset N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ be a convex lattice polytope of maximum dimension. That is, the vertices $\operatorname{vert}(P)$ of P are points in N, and the dimension of the smallest affine subspace containing P is equal to the rank of N. We say that P is *reflexive* if the *dual* (or *polar*) polyhedron

$$P^* := \{u \in M_{\mathbb{Q}} \mid u(v) \geq -1 \text{ for all } v \in P\}, \qquad \text{where } M := \operatorname{Hom}(N, \mathbb{Z}),$$

is a lattice polytope in M.

The three-dimensional reflexive polytopes were classified by Kreuzer–Skarke [58]; up to $\mathrm{GL}_3(\mathbb{Z})$-equivalence there are 4319 cases. We will blur the distinction between a reflexive polytope $P \subset N_{\mathbb{Q}}$ and the corresponding toric Fano threefold X_P, and both will often be referred to by their Reflexive ID [15].

We now need to produce a generalisation of the two-dimensional basic links in Definition 2.3 that would naturally connect the Gorenstein toric Fano threefolds. Let X be such a threefold. Following [16, §2.5], let Z be one of:

(i) a smooth point of X;
(ii) a terminal cDV point of X;
(iii) a line on X not passing through any non-cDV points.

Let $\alpha : \tilde{X} \to X$ be the blow-up of the ideal sheaf of the subvariety $Z \subset X$, and let $\beta : \tilde{X} \to X'$ be the morphism defined by $|-K_{\tilde{X}}|$, making β an embedding. Then (see [16, Lemma 2.2]):

Proposition 2.5. *The morphism β is birational and X' is a Fano threefold with Gorenstein singularities.*

Definition 2.6. Using the morphisms α and β as defined above, consider the commutative diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\pi"'] & & X' \end{tikzcd}$$

We call $\pi : X \dashrightarrow X'$ a *basic link* between the threefolds X and X'. We denote individual types of basic links as:

(i) Π_{p} if Z is a smooth point;
(ii) Π_{dp} (or Π_{o}, or Π_{cDV}) if Z is a double point (or, respectively, an ordinary double point, or a non-ordinary double point);
(iii) Π_{l} if Z is a line.

If all the varieties in question are toric, we call π a *toric basic link*.

In all of these cases, π can be naturally seen as a projection of X: if Z is a smooth point, then π is the projection from the projective tangent space of X at Z, and in all other cases it is the projection of X from Z itself. We call X the *root* of the projection π. Under certain additional assumptions, it is possible to similarly define basic links for Z being a higher-degree curve. For example, we let Π_{c} denote the basic link in the case when $Z \subset X$ is a conic curve; see [16] for the definition. Since these more-general basic links are less natural, we deliberately try to avoid them in our main calculation. We will refer to projections from points and lines as the *allowed* projections.

[1]To explore this classification, see Belmans' beautiful website Fanography [12].

Example 2.7 ([16, §2.6])**.** Similarly to the two-dimensional case, we can begin with $\mathbb{P}^3$ (or Q^3) and start applying toric basic links to it and to the varieties we get as a result. We would expect to obtain, up to degeneration, all (or almost all) the very ample smooth Fano threefolds. Since the basic links can be seen as projections, we can formulate this as a directed graph with vertices corresponding to smooth Fano threefolds, up to degeneration, and arrows corresponding to the projections from a degeneration of one variety to that of another. For example, one can see a small piece of such a graph in Figure 2.

Figure 2. The Fano snake

$$\begin{tikzcd}
 & & Q^3 \arrow[d, dashed, "\Pi_{\mathrm{p}}"] & & & & & & \\
\mathbb{P}^3 \arrow[d, dashed, "\Pi_{\mathrm{p}}"] & & X_{2\text{–}30} \arrow[d, dashed, "\Pi_{\mathrm{l}}"] & & & & & & \\
X_{2\text{–}35} \arrow[d, dashed, "\Pi_{\mathrm{p}}"] & & X_{3\text{–}23} \arrow[d, dashed, "\Pi_{\mathrm{o}}"] & & & & & & \\
X_{2\text{–}32} \arrow[r, dashed, "\Pi_{\mathrm{c}}"] \arrow[d, dashed, "\Pi_{\mathrm{p}}"] & X_{3\text{–}24} \arrow[r, dashed, "\Pi_{\mathrm{o}}"] & X_{4\text{–}10} \arrow[r, dashed, "\Pi_{\mathrm{c}}"] & X_{4\text{–}7} \arrow[r, dashed, "\Pi_{\mathrm{c}}"] & X_{3\text{–}12} \arrow[r, dashed, "\Pi_{\mathrm{o}}"] & X_{3\text{–}10} \arrow[r, dashed, "\Pi_{\mathrm{o}}"] & X_{4\text{–}1} \arrow[r, dashed, "\Pi_{\mathrm{o}}"] & V_{22} \arrow[r, dashed, "\Pi_{\mathrm{o}}"] & X_{2\text{–}13} \arrow[d, dashed, "\Pi_{\mathrm{o}}"] \\
B_5 \arrow[d, dashed, "\Pi_{\mathrm{p}}"] & & & & & & & & V_{18} \arrow[d, dashed, "\Pi_{\mathrm{cDV}}"] \\
B_4 \arrow[d, dashed, "\Pi_{\mathrm{p}}"] & & V_4 & V_6 \arrow[l, dashed, "\Pi_{\mathrm{cDV}}"'] & V_8 \arrow[l, dashed, "\Pi_{\mathrm{cDV}}"'] & V_{10} \arrow[l, dashed, "\Pi_{\mathrm{cDV}}"'] & V_{12} \arrow[l, dashed, "\Pi_{\mathrm{o}}"'] & V_{14} \arrow[l, dashed, "\Pi_{\mathrm{o}}"'] & V_{16} \arrow[l, dashed, "\Pi_{\mathrm{o}}"'] \\
B_3 \arrow[d, dashed, "\Pi_{\mathrm{p}}"] & & & & & & & & \\
B_2 & & & & & & & &
\end{tikzcd}$$

3. Toric Landau–Ginzburg models

In this section we define a toric Landau–Ginzburg model, the main object of our study. For details and examples see [5,8,20,21,44,54,69,70,72] and references therein.

Let X be a smooth Fano variety of dimension n and Picard rank ρ. Fix a basis $\{H_1, \ldots, H_\rho\}$ in $H^2(X)$ so that for any $i \in [\rho]$ and any curve β in the Kähler cone K of X one has $H_i \cdot \beta \geq 0$. Introduce formal variables $q_i := q^{\tau_i}$ for each $i \in [\rho]$, and for any $\beta \in H_2(X)$ define

$$q^\beta := q^{\sum \tau_i (H_i \cdot \beta)}.$$

Consider the Novikov ring $\mathbb{C}_q$, i.e. a group ring for $H_2(X)$. We treat it as a ring of polynomials over $\mathbb{C}$ in formal variables q^β, with relations $q^{\beta_1} q^{\beta_2} = q^{\beta_1+\beta_2}$. Notice that for any $\beta \in K$ the monomial q^β has non-negative degrees in the q_i.

Let the number

$$\langle \tau_a \gamma \rangle_\beta, \qquad \text{where } a \in \mathbb{Z}_{\geq 0}, \gamma \in H^*(X), \beta \in K,$$

be a one-pointed Gromov–Witten invariant with descendants for X; see [61, VI-2.1]. Let $\mathbf{1}$ be the fundamental class of X. The series

$$I_0^X(q_1, \ldots, q_\rho) = 1 + \sum_{\beta \in K} \langle \tau_{-K_X \cdot \beta - 2} \mathbf{1} \rangle_{-K_X \cdot \beta} \cdot q^\beta$$

is called the *constant term of I-series* (or the *constant term of Givental's J-series*) for X, and the series

$$\widetilde{I}_0^X(q_1,\ldots,q_\rho)=1+\sum_{\beta\in K}(-K_X\cdot\beta)!\langle\tau_{-K_X\cdot\beta-2}\mathbf{1}\rangle_{-K_X\cdot\beta}\cdot q^\beta$$

is called the *constant term of regularised I-series* for X. Given a divisor $H=\sum\alpha_i H_i$, one can restrict these series to a direction corresponding to the given divisor by setting $\tau_i=\alpha_i\tau$ and $t=q^\tau$. Thus one can define the restriction of the constant term of regularised I-series to the anticanonical direction, referred to as the *regularised quantum period.* This has the form

$$\widetilde{I}^X(t)=1+a_1t+a_2t^2+\cdots.$$

Definition 3.1. A *toric Landau–Ginzburg model* is a Laurent polynomial $f\in\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm}]$ which satisfies three conditions:

(i) *(Period condition)* The constant term of f^k is a_k, for each $k\in\mathbb{Z}_{>0}$;
(ii) *(Calabi–Yau condition)* Any fibre of $f\colon(\mathbb{C}^\times)^n\to\mathbb{C}$ has trivial dualising sheaf;
(iii) *(Toric condition)* There exists an embedded degeneration $X\rightsquigarrow T$ to a toric variety T whose fan is equal to the spanning fan of $\mathrm{Newt}(f)$, the Newton polytope of f.

Given a Laurent polynomial $f\in\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$, the *classical period* of f is given by:

$$\pi_f(t)=\left(\frac{1}{2\pi i}\right)^n\int_{|x_1|=\ldots=|x_n|=1}\frac{1}{1-tf(x_1,\ldots,x_n)}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n},\qquad\text{for } t\in\mathbb{C},\ |t|\ll\infty.$$

This gives rise to a solution of a GKZ hypergeometric differential system associated to the Newton polytope of f. Expanding this integral by the formal variable t one obtains a generating series for the constant terms of exponents of f, which we call the *period sequence* and also denote by π_f:

$$\pi_f(t)=1+\sum_{k\geq1}\mathrm{coeff}_1(f^k)t^k.$$

For details see [20, Theorem 3.2] and [69, Proposition 2.3]. The period condition in Definition 3.1 tells us that we have an isomorphism between the Picard–Fuchs differential equation for a family of fibres of the map $f\colon(\mathbb{C}^\times)^n\to\mathbb{C}$, and the regularised quantum differential equation for X.

The Calabi–Yau condition is motivated by the following (see, for example, [70, Principle 32]):

Principle 3.2 (Compactification principle)**.** The relative compactification of the family of fibres of a "good" toric Landau–Ginzburg model (defined up to flops) satisfies the (B-side of the) Homological Mirror Symmetry conjecture.

From the point of view of Homological Mirror Symmetry, the fibres of the Landau–Ginzburg model for a Fano variety are Calabi–Yau varieties. The Calabi–Yau condition is designed to remove any obstructions for the compactification principle in this case. Finally, the toric condition is a generalisation of Batyrev's principle for small toric degenerations.

Claim 3.3. All toric Landau–Ginzburg models associated with the same toric degeneration of X have the same support. In other words, given a toric degeneration $X\rightsquigarrow T$, by varying a symplectic form on X one can vary the coefficients of the Laurent polynomial f_X whilst keeping the Newton polytope fixed.

Conjecture 3.4 (Strong version of Mirror Symmetry for variation of Hodge structures)**.** *Any smooth Fano variety has a toric Landau–Ginzburg model.*

Some progress has been made towards proving Conjecture 3.4 in the case of threefolds; [70, Proposition 9 and Theorem 14] and [46, Theorem 2.2 and Theorem 3.1] tell us the following:

Theorem 3.5. *Conjecture 3.4 holds for Picard rank* 1 *Fano threefolds and for complete intersections.*

Moreover, Period condition for all Fano threefolds holds by [20], and Calabi–Yau condition holds for them by [71]. Thus, Theorem 5.1 implies the following.

Corollary 3.6. *Conjecture 3.4 holds for smooth Fano threefolds.*

The compactification principle requires that the fibres of a Calabi–Yau compactification of a toric Landau–Ginzburg model for X are mirror dual to anticanonical sections of X. In the threefold case, this duality is called the *Dolgachev–Nikulin–Pinkham duality* [31, 65], and can be formulated in terms of orthogonal Picard lattices. In [32, 72] the uniqueness of compactified toric Landau–Ginzburg models satisfying these conditions is proved for rank one Fano threefolds, and so the theorem holds for all Dolgachev–Nikulin toric Landau–Ginzburg models.

4. Mutations

Mutations are a special class of birational transformations which act on Laurent polynomials and arise naturally in the context of mirror symmetry for Fano manifolds. As discussed in §3, an n-dimensional Fano manifold X is expected to correspond to a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$, with the period sequence π_f of f agreeing with the regularised quantum period of X. This correspondence is far from unique: typically there will be infinitely many Laurent polynomials corresponding to a given Fano manifold, and it is expected that these Laurent polynomials are related via mutation [2, 36, 40, 55].

We recall the definition of mutation as given in [2]. Write f in the form

$$f = \sum_{h \in \mathbb{Z}} C_h(x_1, \dots, x_{n-1}) x_n^h,$$

where each C_h is a Laurent polynomial in $n-1$ variables, and only finitely many of the C_h are non-zero. Let $F \in \mathbb{C}[x_1^{\pm1}, \dots, x_{n-1}^{\pm1}]$ be a Laurent polynomial such that C_h is divisible by $F^{|h|}$ for each $h < 0$. We call F a *factor*. Define a birational transformation $\varphi : (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$ by

$$(x_1, \dots, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-1}, F(x_1, \dots, x_{n-1})x_n).$$

The pullback of f by φ gives a Laurent polynomial

$$g \coloneqq \varphi^*(f) = \sum_{h \in \mathbb{Z}} F(x_1, \dots, x_{n-1})^h C_h(x_1, \dots, x_{n-1}) x_n^h.$$

Notice that the requirement that $F^{|h|}$ divides C_h for each $h < 0$ is essential: it is precisely this condition that ensures that g a Laurent polynomial.

Definition 4.1. A *mutation* (or *symplectomorphism of cluster type*) is the birational transformation φ, possibly pre- and post-composed with a monomial change of basis. We say that f and g are *related by the mutation* φ. If there exists a finite sequence of Laurent polynomials $f = f_0, f_1, \dots, f_k = g$, where each f_i and f_{i+1} are related by mutation, then we call f and g *mutation equivalent*.

One important property is that mutations preserve the classical period:

Lemma 4.2 ([2, Lemma 1])**.** *If the Laurent polynomials f and g are mutation equivalent then $\pi_f = \pi_g$.*

Example 4.3 (cf. [55, Example 2.6])**.** Consider the Laurent polynomial

$$f_1 \coloneqq \frac{(1 + x_1 + x_2)^3}{\prod_{i=1}^n x_i} + x_3 + \cdots + x_n \in \mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}],$$

where $n \geq 3$. This is a weak Landau–Ginzburg model for the n-dimensional cubic as described in, for example, [46, §2.1]. By, for example, [21, Corollary D.5] it has period sequence:

$$\pi(t) = \sum_{k=0}^{\infty} \frac{(3k)!((n-1)k)!}{(k!)^{n+2}} t^{(n-1)k}.$$

Set $F \coloneqq 1 + x_1 + x_2$ and define the map

$$\varphi_1 : (x_1, x_2, x_3, x_4, \dots, x_n) \mapsto (x_1, x_2, F x_3, x_4, \dots, x_n).$$

Then φ_1 defines a mutation of f_1 (in this case x_3 is playing the role of x_n in the definition above):

$$f_2 \coloneqq \varphi_1^*(f_1) = \frac{(1 + x_1 + x_2)^2}{\prod_{i=1}^n x_i} + (1 + x_1 + x_2)x_3 + x_4 + \cdots + x_n.$$

If $n \geq 4$, we can define

$$\varphi_2 : (x_1, x_2, x_3, x_4, x_5, \dots, x_n) \mapsto (x_1, x_2, x_3, F x_4, x_5, \dots, x_n),$$

giving us a second mutation:

$$f_3 := \varphi_2^*(f_2) = \frac{1+x_1+x_2}{\prod_{i=1}^n x_i} + (1+x_2+x_2)(x_3+x_4)+x_5+\cdots+x_n.$$

We could attempt to continue this process. If $n \geq 5$ we define the map

$$\varphi_3 : (x_1,\ldots,x_4,x_5,x_6,\ldots,x_n) \mapsto (x_1,\ldots,x_4,Fx_5,x_6,\ldots,x_n).$$

This gives a mutation

$$f_4 := \varphi_3^*(f_3) = \frac{1}{\prod_{i=1}^n x_i} + (1+x_1+x_2)(x_3+x_4+x_5)+x_6+\cdots+x_n.$$

However, $f_3 \cong f_4$ via the obvious monomial change of basis, and so we regard these two weak Landau–Ginzburg models as being essentially the same.

Suppose that f is mirror to an n-dimensional Fano manifold X, and consider the Newton polytope $P := \mathrm{Newt}(f) \subset N_{\mathbb{Q}}$ of f. We may assume without loss of generality that P is of maximum dimension and that it contains the origin strictly in its interior. The *spanning fan* of P – the fan whose cones in $N_{\mathbb{Q}}$ are generated by the faces of P – gives rise to a toric variety X_P, which we call a *toric model* for X. In general X_P will be singular. However, it is expected to admit a smoothing with general fibre X. Since a smooth Fano manifold can degenerate to many different singular toric varieties, we expect many different mirrors for X. This is reflected in that fact that f is mutation equivalent to many different Laurent polynomials, all of which have the same classical period by Lemma 4.2.

Example 4.4. The Laurent polynomial $f = x + y + \frac{1}{xy} \in \mathbb{C}[x^{\pm1}, y^{\pm1}]$ has Newton polytope $P = \mathrm{conv}\{(1,0),(0,1),(-1,-1)\} \subset N_{\mathbb{Q}}$ and corresponding toric variety $X_P = \mathbb{P}^2$. Via mutation, we obtain $g = y + 2x^2y^2 + x^4y^3 + \frac{1}{xy}$, giving the toric model $X_Q = \mathbb{P}(1,1,4)$. Notice that the singular point $\frac{1}{4}(1,1)$ is a T-singularity and hence admits a $\mathbb{Q}$-Gorenstein (qG) one-parameter smoothing [57, 77]. We can continue mutating, resulting in a directed graph of toric models of the form $X_{(a,b,c)} := \mathbb{P}(a^2,b^2,c^2)$, where $(a,b,c) \in \mathbb{Z}^3_{>0}$ is a solution to the Markov equation $3abc = a^2+b^2+c^2$ [4, 41]. There are infinitely many positive solutions to the Markov equation, obtainable via cluster-style mutations of the form $(a,b,c) \mapsto (b,c,3bc-a)$, and so we have infinitely many toric models $X_{(a,b,c)}$ for $\mathbb{P}^2$. In each case the weighted projective space $X_{(a,b,c)}$ has only T-singularities and so admits a qG-smoothing.

A mutation between two Laurent polynomials induces a mutation of the corresponding Newton polytopes P and Q. The corresponding toric varieties X_P and X_Q are deformation equivalent in the following precise sense:

Lemma 4.5 ([45, Theorem 1.3])**.** *Let X_P and X_Q be related by mutation. Then there exists a flat family $\mathcal{X} \to \mathbb{P}^1$ such that $\mathcal{X}_0 \cong X_P$ and $\mathcal{X}_\infty \cong X_Q$.*

Very little is known about the behaviour of mutations in general. The two-dimensional setting has been studied in [1,3,4,26,36,53,67]. In particular, the toric surface X_P is qG-smoothable to X [1, Theorem 3], and, for each del Pezzo surface X, the set of all polygons P such that X_P is qG-smoothable to X forms a single mutation-equivalence class [53, Theorem 1.2].

In dimension three a special class of Laurent polynomials called *Minkowski polynomials* were introduced in [2][2]. These are Laurent polynomials f in three variables whose Newton polytope is a reflexive polytope, with f satisfying certain additional conditions. There are 3747 Minkowski polynomials (up to monomial change of basis), and together they generate 165 periods. Furthermore, any two Minkowski polynomials have the same period sequence if and only if they are mutation equivalent. Of these periods, 98 are of so-called *manifold type*, with the remaining 67 being of *orbifold type* (these are properties of the associated Picard–Fuchs differential equations; see [20, §7] for the definitions). In [21], the period sequences of

[2]Although extremely successful at recovering mirrors for the 98 deformation families of three-dimensional Fano manifolds with very ample anticanonical bundle, there are some drawbacks to Minkowski polynomials. The two main issues are that this ansatz can only be applied to reflexive polytopes in three dimensions, and that it is not closed under mutation. Recently the definition of rigid maximally mutable Laurent polynomials (rigid MMLPs) was proposed to overcome these limitations [22]. The rigid MMLPs, and the corresponding concept of zero-mutable Laurent polynomials [26], look like they might be the correct class. It is conjectured [22, Conjecture 5.1] that rigid MMLPs are in one-to-one correspondence (up to mutation) with pairs (X, D), where X is a Fano n-fold of class TG with terminal locally toric qG-rigid singularities and $D \in |-K_X|$ is a general elephant (up to qG-deformation). Here, *class TG* is as defined in [1].

manifold type were shown to correspond under mirror symmetry to the 98 deformation families of three-dimensional Fano manifolds X with very ample anticanonical bundle $-K_X$.

Proposition 4.6. *Let $P \subset N_{\mathbb{Q}}$ be a reflexive polytope, with associated Gorenstein toric Fano variety X_P. Assume that X_P has a smooth deformation space, and is a degeneration of a generic smooth Fano variety X. Let Q be any mutation of P, with associated toric variety X_Q. Then X_Q is also a degeneration of X.*

Proof. By Lemma 4.5 there is a flat projective family over $\mathbb{P}^1$ with X_P and X_Q as special fibres. Since $[X_P] \in \mathcal{H}_X$ lies on a single irreducible component, $[X_Q]$ must also lie on this component. A general point of this component is a smooth Fano threefold deformation equivalent to X, so X degenerates to X_Q. □

By applying the above proposition, we may find many more toric degenerations of a given Fano threefold. Indeed, suppose that we have a smooth Fano threefold X which degenerates to a Gorenstein toric Fano variety X_P having a smooth deformation space, as described in Tables E, F, G, H, and I. We can use mutation to construct other Gorenstein toric Fano varieties to which X degenerates. By consulting the calculations of Tables D and K we can determine which of these have smooth deformation spaces, and then iterate using these new examples. We record the resulting degenerations in Table J.

5. Toric degenerations

Let X be a smooth Fano threefold with a very ample anticanonical divisor, and consider its anticanonical embedding $V \hookrightarrow \mathbb{P}^n$. As explained in §2.2, we are interested in finding embedded degenerations of X to Gorenstein toric Fano varieties. The main result of this section is the following:

Theorem 5.1. *Let X be a generic smooth Fano threefold with a very ample anticanonical divisor. Then X has an embedded degeneration to a Gorenstein toric Fano variety X' with smooth deformation space.*

Our proof makes use of the classification of smooth Fano threefolds [47,48,62,63] and consists of several steps:

(i) We use several techniques[3], described in §§5.1–5.5 below, to construct at least one degeneration X' for each smooth Fano X. The vast majority of these come from the models constructed in [21] of smooth Fano threefolds as complete intersections in toric varieties; see §5.4.

(ii) For any Fano variety X with very ample anticanonical divisor, let $\mathcal{H}_X$ denote the Hilbert scheme parameterising projective schemes with the same Hilbert polynomial as X in its anticanonical embedding. If X is smooth, then it corresponds to a smooth point on an irreducible component of $\mathcal{H}_X$. For each smooth Fano X, we compute the dimension of this component using [19, Proposition 3.1].

(iii) Using the comparison theorem of Kleppe [56, Theorem 3.6] we compute the tangent space dimensions for all Gorenstein toric Fano threefolds, viewed as points in relevant Hilbert schemes.

(iv) Using steps (ii) and (iii) we check that all special fibres X' of the degenerations in step (i) have tangent space dimension equal to the dimension of the Hilbert scheme component of X, and hence correspond to smooth points in the Hilbert scheme. Since the forgetful functor to the deformation space of X' is smooth [19, §2.1], each such X' has smooth deformation space.

None of our results claim to be exhaustive; for example, it is certainly possible for there to exist degenerations which do not appear in our search at step (i).

5.1. Previously known degenerations.

We make use of a number of known toric degenerations[4]. Galkin [35] classified all degenerations of smooth Fano threefolds to Gorenstein toric Fano varieties with at worst terminal singularities, which are called *small* toric degenerations. These are recorded in Table E. Notice that these include the eighteen smooth toric Fano threefolds classified by Batyrev [6] and Watanabe–Watanabe [78]. The embedded degenerations of smooth Fano threefolds with a very ample

[3] Note that after we constructed these degenerations the papers [24,32,33] appeared; they provide further tools for systematically constructing toric degenerations, however we do not make use of them here.

[4] Since writing this paper, the preprint [66] has appeared; here Prince uses the methods of [24] to construct Gorenstein toric degenerations of Fano threefolds.

anticanonical divisor to Gorenstein toric Fano varieties of degree at most twelve were classified in [19]. These are recorded in Table F.

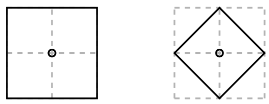
5.2. **Triangulations of the moment polytope.** We use the techniques developed in [18] to construct degenerations of rank one, index one Fano threefolds. Consider the two-dimensional simplicial complexes T_i, $i \in \{4,5,6,7,8,9,10,11\}$, defined as follows: when $i=4$ we define T_4 to be the boundary of the three-simplex; when $i=5$ we define T_5 to be the bipyramid over the boundary complex of a two-simplex; when $6 \le i \le 10$, let T_i be the unique triangulation of the sphere with i vertices having valencies four and five; when $i=11$, let T_{11} be the unique triangulation of the sphere having valencies $4,4,5,5,5,5,5,5,5,5,6$.

Theorem 5.2 ([18, §3])**.** *For $4 \le i \le 11$, let $d \coloneqq 2i-4$ and let X_d be a general rank one, index one, degree d smooth Fano threefold. Consider a three-dimensional reflexive polytope $\Delta \subset M_{\mathbb{Q}}$ whose boundary admits a regular unimodular triangulation of the form T_i, and let $X' \coloneqq X(\Delta)$ be the associated Gorenstein toric Fano variety with fan normal to Δ. If $i<11$, or if $i=11$ and $h^0(X',\mathcal{N}_{X'/\mathbb{P}^i}) = 153$, then X_d has an embedded degeneration to X'. Furthermore, the point corresponding to X' in $\mathcal{H}_{X_d}$ is smooth.*

The resulting degenerations are recorded in Table G.

5.3. **Products with del Pezzo surfaces.** A number of Fano threefolds are products of del Pezzo surfaces with $\mathbb{P}^1$: $X_{5\text{–}3}, X_{6\text{–}1}, X_{7\text{–}1}, X_{8\text{–}1}$, and $X_{9\text{–}1}$, of which the first three have very ample anticanonical divisor. Toric degenerations of these threefolds may be found by degenerating the corresponding del Pezzo surfaces. In many cases these degenerations are well understood; see for example the work of Hacking–Prokhorov [41]. The threefold degenerations constructed this way are recorded in Table H.

FIGURE 3. The spanning polytope (i) $P \subset N_{\mathbb{Q}}$ and the moment polytope (ii) $P^* \subset M_{\mathbb{Q}}$ of a toric degeneration of the del Pezzo surface of degree four.



(i) (ii)

Example 5.3. Consider the del Pezzo surface S_4 of degree four given by the blow-up of $\mathbb{P}^2$ in five points. This degenerates to the toric variety whose spanning polytope is picture in Figure 3(i). The Fano threefold $X_{7\text{–}1}$ is $S_4 \times \mathbb{P}^1$, and thus degenerates to the toric Fano threefold X_P where, up to $\mathrm{GL}_3(\mathbb{Z})$-action, $P \coloneqq \mathrm{conv}\{\pm(1,1,0), \pm(1,-1,0), \pm(0,0,1)\} \subset N_{\mathbb{Q}}$. This has Reflexive ID 510.

5.4. **Complete Intersections in toric varieties.** It appears to be common for smooth Fano varieties to arise as complete intersections in the homogeneous coordinate ring of a smooth toric Fano variety. Eight of the del Pezzo surfaces can be realised this way; so can at least 78 of the 105 smooth Fano threefolds [21], and at least 738 of the smooth Fano fourfolds [23]. This leads to the following natural construction of degenerations.

Let W be a smooth complete toric variety, with I the set of invariant prime divisors. Its homogeneous coordinate ring $R = \mathbb{C}[x_i \mid i \in I]$ is a polynomial ring graded by $\mathrm{Pic}(W)$. Then W is the quotient of $U = \mathrm{Spec}\, R \setminus Z$ by the Picard torus $T = \mathrm{Spec}\, \mathbb{C}[\mathrm{Pic}(W)]$, where Z is the so-called irrelevant, or exceptional, set. Suppose that X is a complete intersection in W of Cartier divisors $D_1, \ldots, D_k$; let $\overline{D}_i$ be the class of D_i in $\mathrm{Pic}(W)$. Each divisor D_i may be encoded by a homogeneous degree $\overline{D}_i$ polynomial f_i in R. Then X arises as the quotient

$$X = (U \cap V(f_1, \ldots, f_k)) /\!\!/ T.$$

In order to degenerate X, we may degenerate the polynomials f_i to polynomials g_i of the same multidegree. As long as the degenerate polynomials g_i still form a regular sequence in R, we get a degeneration from X

to $X' = (U \cap V(g_1, \ldots, g_k)) /\!\!/ T$. If $V(g_1, \ldots, g_k) \cap U \hookrightarrow U$ is an equivariant embedding of toric varieties, then the resulting quotient X' is toric as well.

To construct a toric X', we may choose the g_i to be binomials. If M is the character lattice of the torus of W, we have an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^I \xrightarrow{\pi} \mathrm{Pic}(W) \longrightarrow 0,$$

and each binomial g_i determines a rank-one sublattice L_i of M. Let O be the positive orthant of $\mathbb{Z}^I \otimes \mathbb{Q}$, and let O' be its image in $\left(\mathbb{Z}^I / \sum L_i\right) \otimes \mathbb{Q}$. If $M' := \mathbb{Z}^I / \sum L_i$ is torsion free and the natural map $R \to \mathbb{C}[O' \cap M']$ is surjective with kernel generated by the g_i, then $V(g_1, \ldots, g_k)$ is toric, with $U \cap V(g_1, \ldots, g_k) \hookrightarrow U$ an equivariant embedding. In order to determine the quotient in this situation, let A be any ample class in $\mathrm{Pic}(W)$. Then the quotient X' of $U \cap V(g_1, \ldots, g_k)$ is the toric variety whose moment polytope Δ is the image of $\pi^{-1}(A) \cap O$ in $M' \otimes \mathbb{Q}$.

Example 5.4. The smooth Fano threefold $X = X_{2\text{–}7}$ is a codimension two complete intersection in the toric variety $W = \mathbb{P}^1 \times \mathbb{P}^4$. The homogeneous coordinate ring $R = \mathbb{C}[x_0, x_1, y_0, \ldots, y_4]$ of W has a $\mathrm{Pic}(W) = \mathbb{Z}^2$-grading given by the columns of the weight data

x_0	x_1	y_0	y_1	y_2	y_3	y_4	
1	1	0	0	0	0	0	A
0	0	1	1	1	1	1	B

with ample cone $\overline{\mathrm{Amp}}\, W = \langle A, B \rangle$. The threefold X is the intersection of divisors $(A + 2B) \cap (2B)$. Consider the $\mathbb{Z}^2$-homogeneous polynomials

$$g_1 = x_0 y_0 y_1 - x_1 y_2 y_3$$
$$g_2 = y_3^2 - y_2 y_4.$$

Then $V(g_1, g_2)$ is toric, and the resulting quotient by the Picard torus gives the Gorenstein toric Fano variety X' with Reflexive ID 3813 given by the moment polytope

$$\Delta = \mathrm{conv}\{(1,0,0), (1,1,0), (1,0,1), (1,1,1), (0,0,1), (0,-1,1), (-1,0,-1), (-1,1,-1)\}.$$

In the above construction, the resulting toric variety X' need not be Fano. However, in the case when X' is a weak Fano with Gorenstein singularities we can construct a degeneration from X to the anticanonical model X'' of X'. Indeed, let $\pi : \mathcal{X} \to S$ be the total space of the degeneration of X to X'. Since all fibres are Cohen-Macaulay, $-K_{\mathcal{X}/S}|_F \cong -K_F$ for any fibre F [25, Theorem 3.5.1]. Furthermore, since $-K_{X'}$ is Cartier and nef (and X' is toric), all higher cohomology of $-K_F$ vanishes for every fibre F. By cohomology and base change, the restriction map $H^0(\mathcal{X}, -K_{\mathcal{X}/S}) \to H^0(F, -K_F)$ is surjective. Hence, taking Proj of the section ring of $-K_{\mathcal{X}/S}$ gives a flat family over S with X'' as the special fibre and X as the general fibre.

We apply the above discussion to find toric degenerations of many of the smooth Fano threefolds, using their descriptions in [21] as complete intersections in toric varieties. These degenerations are recorded in Table I. For each degeneration, we also record the corresponding regular sequence g_i, using the same ordering on the homogeneous coordinates as in [21]. For a similar approach to constructing degenerations, see also [43].

5.5. **The exceptional case:** $X_{2\text{–}14}$**.** The previous methods fail to construct toric degenerations for the smooth Fano threefold $X_{2\text{–}14}$. We now discuss a modification of §5.4 which does produce a toric degeneration.

The smooth Fano threefold $X = X_{2\text{–}14}$ may be realised as a divisor of bidegree $(1,1)$ in $V_5 \times \mathbb{P}^1$, where B_5 is a codimension three linear section of the Grassmannian $\mathrm{Gr}(2,5)$ in its Plücker embedding [21]. The anticanonical embedding places V_5 in $\mathbb{P}^6$ as the intersection of five quadrics. The approach of §5.2 can be applied to find degenerations of V_5. In particular, it degenerates to the Gorenstein toric Fano with Reflexive ID 68.

We can realise X as the intersection in the toric variety $\mathbb{P}^6 \times \mathbb{P}^1$ of $V_5 \times \mathbb{P}^1$ and a divisor of bidegree $(1,1)$, defined by a bihomogeneous polynomial f. Simultaneously degenerating the quadrics of V_5 and the polynomial f leads to a degeneration of X. More precisely, we may degenerate the quadrics to

$$\begin{aligned} x_1x_6 - x_2x_5, \qquad & x_1x_6 - x_3x_4, \qquad & x_0x_3 - x_1x_2, \\ & x_0x_5 - x_1x_4, \qquad & x_0x_6 - x_2x_4, \end{aligned}$$

and the polynomial f to $g = x_3x_7 - x_4x_8$. The resulting variety is the Gorenstein toric Fano threefold with Reflexive ID 2353, generated by the polytope

$$\begin{aligned} P \coloneqq \operatorname{conv}\{&(1,0,0),(0,1,0),\pm(0,0,1),\pm(1,-1,0), \\ &\pm(1,0,-1),\pm(0,1,-1),(-1,-1,0),(-1,-1,1)\} \subset N_{\mathbb{Q}}. \end{aligned}$$

6. The graph of reflexive polytopes

In [76] Sato investigated the concept of F-equivalence classes of a smooth Fano polytope. Recall that a smooth Fano polytope $P \subset N_{\mathbb{Q}}$ corresponds to a smooth toric Fano variety X_P via its spanning fan. Smooth Fano polytopes are necessarily reflexive, and a reflexive polytope $P \subset N_{\mathbb{Q}}$ is smooth if, for each facet F of P, the vertices $\operatorname{vert}(F)$ give a $\mathbb{Z}$-basis for the underlying lattice N (and hence, in particular, P needs be simplicial).

Definition 6.1 (F-equivalence of smooth Fano polytopes)**.** Two smooth Fano polytopes $P, Q \in N_{\mathbb{Q}}$ are *F-equivalent*, and we write $P \overset{F}{\sim} Q$, if there exists a finite sequence $P_0, P_1, \ldots, P_k \subset N_{\mathbb{Q}}$ of smooth Fano polytopes satisfying:

(i) P and Q are $\mathrm{GL}_n(\mathbb{Z})$-equivalent to P_0 and P_k, respectively;
(ii) for each $1 \le i \le k$ we have either that $\operatorname{vert}(P_i) = \operatorname{vert}(P_{i-1}) \cup \{w\}$, where $w \notin \operatorname{vert}(P_{i-1})$, or that $\operatorname{vert}(P_{i-1}) = \operatorname{vert}(P_i) \cup \{w\}$, where $w \notin \operatorname{vert}(P_i)$;
(iii) if $w \in \operatorname{vert}(P_i) \setminus \operatorname{vert}(P_{i-1})$ then there exists a proper face F of P_{i-1} such that
$$w = \sum_{v \in \operatorname{vert}(F)} v$$
and the set of facets of P_i containing w is equal to
$$\{\operatorname{conv}(\{w\} \cup \operatorname{vert}(F') \setminus \{v\}) \mid F' \text{ is a facet of } P_{i_1}, F \subset F', v \in \operatorname{vert}(F)\}.$$
In other words, P_i is obtained by taking a stellar subdivision of P_{i-1} with w. Similarly, if $w \in \operatorname{vert}(P_{i-1}) \setminus \operatorname{vert}(P_i)$ then P_{i-1} is given by a stellar subdivision of P_i with w.

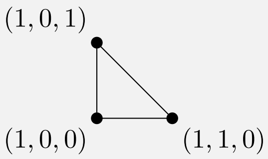
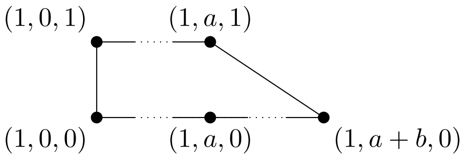
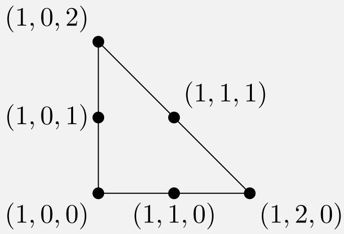
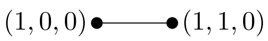
Notice that condition (iii) means that if $P \overset{F}{\sim} Q$ then the corresponding smooth toric Fano varieties X_P and X_Q are related via a sequence of equivariant blow-ups or blow-downs. Little is known about F-equivalence in general, although it is known that all smooth Fano polytopes are F-equivalent in dimensions ≤ 4, and that there exists non-F-equivalence polytopes in all dimensions ≥ 5.

We wish to generalise the notion of F-equivalence to encompass projections between reflexive polytopes. Our focus here is on three dimensions, although what follows can readily be generalised to higher dimensions. Fix a three-dimensional reflexive polytope $P \subset N_{\mathbb{Q}}$ and consider the corresponding Gorenstein toric Fano variety $X = X_P$. Toric points and curves on X correspond to, respectively, facets and edges of P. Projections from these points (or curves) correspond to blowing-up the cone generated by the relevant facet (or edge). Since we are restricting ourselves to the class of reflexive polytopes, it is clear that the original facet (or edge) only needs to be considered if it has no (relative) interior points: any interior point of that face will become an interior point of the resulting polytope, preventing it from being reflexive.

Proposition 6.2. *Let $P \subset N_{\mathbb{Q}}$ be a three-dimensional reflexive polytope, and let F be a facet or edge of P such that F contains no (relative) interior points. Up to $\mathrm{GL}_3(\mathbb{Z})$-equivalence, F is one of the four possibilities shown in Table 1.*

Proof. First consider the case when F is a facet of P. Since P is reflexive, there exists some primitive lattice point $u_F \in M$ such that $u_F(v) = 1$ for each $v \in F$. In particular, by applying any change of basis sending u_F to e_1^* we can insist that F is contained in the two-dimensional affine subspace $\Gamma \coloneqq \{(1,a,b) \mid a,b \in \mathbb{Q}\}$. Pick an edge E_1 of F such that $|E_1 \cap N|$ is as large as possible. Let $v \in \operatorname{vert}(E_1)$ be a vertex of E_1 (and hence of F), and let $E_2 \neq E_1$ be the second edge of F such that $E_1 \cap E_2 = \{v\}$. Let $v_i \in E_i \cap N$

Table 1. The possible choices of faces to be blown-up, and corresponding points to be added, when considering projections between Gorenstein toric Fano threefolds.

	Face	Points added
1		$(3, 1, 1)$
2		$(2, 1, 1)$, $(2, 2a + b - 1, 1)$
3		$(3, 2, 2)$
4		$(2, 1, 0)$

be such that $v_i - v$ is primitive. Since $F^\circ = \varnothing$ we have that $\Delta := \mathrm{conv}\{v, v_1, v_2\}$ is a lattice triangle with $|\Delta \cap N| = 3$; that is, Δ is an empty triangle. Up to $\mathrm{GL}_2(\mathbb{Z})$-equivalence, the empty triangle is unique. Hence we can apply a change of basis to the affine lattice $\Gamma \cap N$ such that $v = (1, 0, 0)$, $v_1 = (1, 1, 0)$, and $v_2 = (1, 0, 1)$. By considering the possible lengths of the edges E_1 and E_2, and remembering that $(1, 1, 1)$ cannot be an interior point, we find the first three cases in Table 1. In the case when F is an edge, it must have length one and hence (up to $\mathrm{GL}_3(\mathbb{Z})$-equivalence) $F = \mathrm{conv}\{(1, 0, 0), (1, 1, 0)\}$, the fourth case in Table 1. $\square$

Corollary 6.3. *Let $P, Q \subset N_\mathbb{Q}$ be two three-dimensional reflexive polytopes such that X_Q is obtained from X_P via a projection. Then the corresponding blow-up of the face F of P introduces new vertices as given in Table 1.*

Proof. We prove this only in case 2 in Table 1. The remaining cases are similar. We refer to Dais' survey article [28] for background, and for the combinatorial interpretation. Let

$$C = \mathrm{cone}\{(1, 0, 0), (1, a + b, 0), (1, a, 1), (1, 0, 1)\} \subset N_\mathbb{Q},$$

where $a, b \in \mathbb{Z}_{>0}$. This is defined by the intersection of four half-spaces of the form $\{v \in N_\mathbb{Q} \mid u_i(v) \geq 0\}$, where the u_i are given by $(0, 0, 1), (0, 1, 0), (1, 0, -1), (a + b, -1, -b) \in M$. Moving these half-spaces in by one, we obtain the polyhedron:

$$\bigcap_{i=1}^{4} \{v \in N_\mathbb{Q} \mid u_i(v) \geq 1\} = \mathrm{conv}\{(2, 1, 1), (2, 2a + b - 1, 1)\} + C.$$

Hence the blow-up is given by the subdivision of C into four cones generated by inserting the rays $(2, 1, 1)$ and $(2, 2a + b - 1, 1)$. $\square$

Of course case 1 in Table 1 is a specialisations of case 2. However, since the point added is what is important, we list it separately. When $a = 1$, $b = 0$, or when $a = 0$, $b = 2$ in case 2, the coordinates of the two points to be added coincide, so we add the single point $(2, 1, 1)$.

Since the choice of the basic links we are using relies on distinguishing curves of small degrees on a given toric variety, it is necessary to be able to easily calculate the anticanonical degree of a given curve. This can be done as follows:

Lemma 6.4. *Let T be a three-dimensional $\mathbb{Q}$-factorial projective toric variety with simplicial fan Δ. Let c_1 and c_2 be rays in $\Delta^{(1)}$ generating a two-dimensional cone in $\Delta^{(2)}$, with corresponding torus-invariant curve C. Here $\Delta^{(n)}$ denotes the set of n-dimensional cones in the fan Δ. Let a_1 and a_2 be rays such that $F_i = \operatorname{cone}\{c_1, c_2, a_i\} \in \Delta^{(3)}$, $F_1 \neq F_2$, and let $e_1, \ldots, e_r$ denote the remaining rays of Δ, so that $\Delta^{(1)} = \{c_1, c_2, a_1, a_2, e_1, \ldots, e_r\}$. Let $\operatorname{Vol}(F_i) = |\Gamma : N|$ be the index of the sublattice Γ_i in N generated by c_1, c_2, a_i, where by a standard abuse of notation we confuse a ray with its primitive lattice generator in N; equivalently, $\operatorname{Vol}(F_i)$ is equal to the lattice-normalised volume of the tetrahedron $\operatorname{conv}\{\mathbf{0}, c_1, c_2, a_i\}$.*

Denote the boundary divisor corresponding to c_i, a_i, or e_i by C_i, A_i, or E_i respectively. Let L_1 and L_2 be linear forms such that L_1 vanishes on c_2 but not on c_1, and L_2 vanishes on c_1 but not on c_2. Then the anticanonical degree of C is equal to

$$\deg C = \left(1 - \frac{L_1(a_1)}{L_1(c_1)} - \frac{L_2(a_1)}{L_2(c_2)}\right)\frac{1}{\operatorname{Vol}(F_1)} + \left(1 - \frac{L_1(a_2)}{L_1(c_1)} - \frac{L_2(a_2)}{L_2(c_2)}\right)\frac{1}{\operatorname{Vol}(F_2)}.$$

Proof. Recall that the anticanonical degree of C is the sum of its intersection with all boundary divisors

$$\deg C = C \cdot \left(C_1 + C - 2 + A_1 + A_2 + \sum_{i=1}^{r} E_i\right) = C \cdot C_1 + C \cdot C_2 + C \cdot A_1 + C \cdot A_2,$$

with intersection numbers given by

$$C \cdot A_1 = 1/\operatorname{Vol}(F_1), \qquad C \cdot A_2 = 1/\operatorname{Vol}(F_2).$$

Given the linear forms L_1, L_2 as above, let D_i be the principal divisor corresponding to the form L_i. Then:

$$0 \equiv D_i \equiv L_i(c_1)C_1 + L_i(c_2)C_2 + L_i(a_1)A_1 + L_i(a_2)A_2 + \sum_{j=1}^{r} L_i(e_j)E_j.$$

This gives:

$$0 = C \cdot D_i = L_i(c_i)C \cdot C_i + L_i(a_1)C \cdot A_1 + L_i(a_2)C \cdot A_2.$$

Hence:

$$C \cdot C_i = -\frac{L_i(a_1)}{L_i(c_i)}\frac{1}{\operatorname{Vol}(F_1)} - \frac{L_i(a_2)}{L_i(c_i)}\frac{1}{\operatorname{Vol}(F_2)}.$$

□

Corollary 6.5. *Let X be a toric Fano threefold and let P the corresponding three-dimensional Fano polytope. Let E be an edge of P corresponding to a curve C on X, and let F_1 and F_2 be the two facets of P meeting at E. Let c_1 and c_2 be the two vertices of P lying on E, and let a_1 and a_2 be vertices of P lying on $F_1 \setminus E$ and $F_2 \setminus E$, respectively. Then:*

$$\deg C = \frac{1}{|a_1 \cdot (c_1 \times c_2)|} + \left(1 + \frac{(c_1 - c_2) \cdot (a_1 \times a_2)}{a_1 \cdot (c_1 \times c_2)}\right)\frac{1}{|a_2 \cdot (c_1 \times c_2)|}.$$

Proof. If the point $p_1 \in X$ corresponding to the face F_1 is singular, then one can take a small resolution X' of X at this point and calculate the degree of C via that resolution. This is done by choosing a triangulation of F_1, with the result being independent of the choice. So, by picking a triangulation containing the triangle (c_1, c_2, a_1), one can assume that p_1 is a smooth point of X.

Similarly, the point $p_2 \in X$ corresponding to the face F_2 of P can also be assumed to be smooth. Therefore, X satisfies the conditions of Lemma 6.4.

Take:

$$L_1(x) = x \cdot (c_2 \times a_1)\,, \; L_2(x) = x \cdot (a_1 \times c_1)\,.$$

Since $\operatorname{Vol}(F_i) = |a_i \cdot (c_1 \times c_2)|$, have:

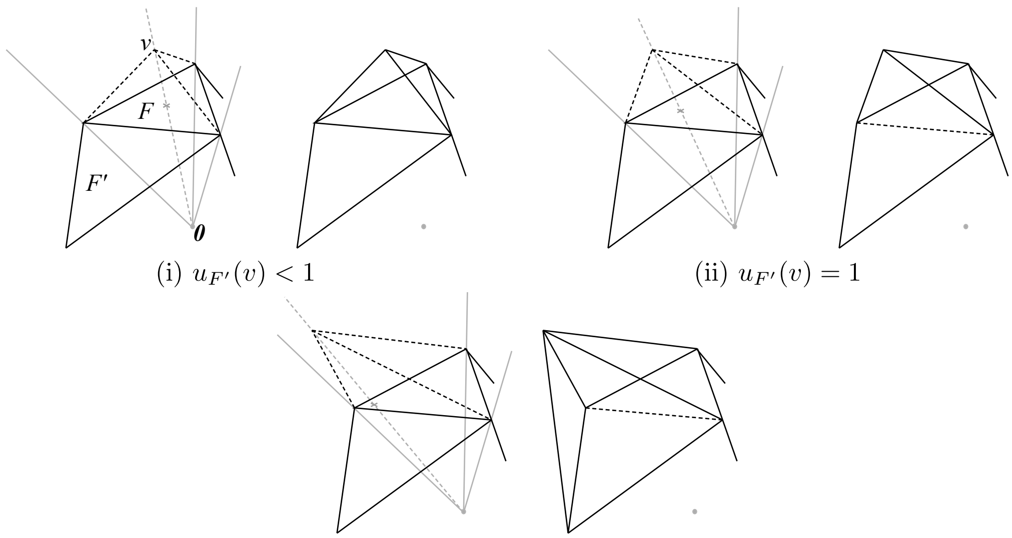
$$\deg C = \frac{1}{|a_1 \cdot (c_1 \times c_2)|} + \left(1 - \frac{a_2 \cdot (c_2 \times a_1)}{c_1 \cdot (c_2 \times a_1)} - \frac{a_2 \cdot (a_1 \times c_1)}{c_2 \cdot (a_1 \times c_1)}\right)\frac{1}{|a_2 \cdot (c_1 \times c_2)|},$$

which simplifies to the form above. □

Definition 6.6 (F-equivalence of reflexive polytopes)**.** Two three-dimensional reflexive polytopes $P, Q \in N_{\mathbb{Q}}$ are F-equivalent, and we write $P \overset{F}{\sim} Q$, if there exists a finite sequence $P_0, P_1, \ldots, P_k \subset N_{\mathbb{Q}}$ of reflexive polytopes satisfying:

(i) P and Q are $\mathrm{GL}_3(\mathbb{Z})$-equivalent to P_0 and P_k, respectively;
(ii) For each $1 \leq i \leq k$ we have that either $\operatorname{vert}(P_i) \subsetneq \operatorname{vert}(P_{i-1})$ or $\operatorname{vert}(P_{i-1}) \subsetneq \operatorname{vert}(P_i)$.

FIGURE 4. The three possible ways a facet F' adjacent to F can be modified when adding a new vertex v. See Remark 6.7 for an explanation.



(i) $u_{F'}(v) < 1$ (ii) $u_{F'}(v) = 1$

(iii) $u_{F'}(v) > 1$

(iii) If $\mathrm{vert}(P_i) \subsetneq \mathrm{vert}(P_{i-1})$ then there exists a face F of P_i and $\varphi \in \mathrm{GL}_3(\mathbb{Z})$ such that $\varphi(F)$ is one of the seven faces in Table 1. Furthermore, the points $\varphi(\mathrm{vert}(P_{i-1}) \setminus \mathrm{vert}(P_i))$ are equal to the corresponding points in Table 1, and $\partial F \subset \partial P_{i-1}$. The case when $\mathrm{vert}(P_{i-1}) \subsetneq \mathrm{vert}(P_i)$ is similar, but with the roles of P_{i-1} and P_i exchanged.

If $P \overset{F}{\sim} Q$ then the corresponding Gorenstein toric Fano threefolds X_P and X_Q are related via a sequence of projections.

Remark 6.7. The requirement that $\partial F \subset \partial P_{i-1}$ in Definition 6.6(iii) perhaps needs a little explanation. Consider the case when F is a facet. Adding the new vertices can affect a facet F' adjacent to F, with common edge E, in one of three ways. Let $u_{F'} \in M$ be the primitive dual lattice vector defining the hyperplane at height one containing F'. Let $v_1, \ldots, v_s \in N$ be the points to be added according to Table 1, so that $P_{i-1} = \mathrm{conv}(P_i \cup \{v_1, \ldots, v_s\})$.

(i) If $u_{F'}(v_i) < 1$ for $1 \leq i \leq s$ then F' is unchanged by the addition of the new vertices, and hence F' is also a facet of P_{i-1}.
(ii) Suppose that $u_{F'}(v_i) = 1$ for $1 \leq i \leq m$, and $u_{F'}(v_i) < 1$ for $m+1 \leq i \leq s$, for some $1 \leq m \leq s$. Then the facet F' is transformed to the facet $F'' := \mathrm{conv}(F' \cup \{v_i \mid 1 \leq i \leq m)$ in P_{i-1}. Notice that $F' \subset F''$, but that E is no-longer an edge of F''. This is equivalent to a blowup, followed by a contraction of the curve corresponding to E.
(iii) The final possibility is that there exists one (or more) of the v_i such that $u_{F'}(v_i) > 1$. When we pass to P_{i-1} we see that F' is no-longer contained in the boundary; in particular, $E^\circ \subset P_{i-1}^\circ$ and so $\partial F \not\subset \partial P_{i-1}$. This case is excluded since it does not correspond to a projection between the two toric varieties.

These three possibilities are illustrated in Figure 4.

Corollary 6.8 (Reflexive polytopes are F-connected)**.**

(i) *Let $P, Q \subset N_{\mathbb{Q}}$ be any two three-dimensional reflexive polytopes. Then $P \overset{F}{\sim} Q$.*
(ii) *Let $\mathcal{G}_F$ be the directed graph whose vertices are given by the three-dimensional reflexive polytopes, with an edge $P \to Q$ if and only if there exists a projection from X_P to X_Q (that is, if and only if $\mathrm{vert}(P) \subset \mathrm{vert}(Q)$, $P \overset{F}{\sim} Q$, and $k = 1$ in Definition 6.6). Then $\mathcal{G}_F$ has sixteen roots and sixteen sinks, and these are related via duality. The Reflexive IDs of the roots are*

1, 2, 3, 4, 5, 8, 9, 10, 11, 16, 18, 31, 45, 89, 102, *and* 105.

The corresponding sinks have Reflexive IDs

$$4312, 4282, 4318, 4284, 4287, 4310, 3314, 4313, 4315, 4299, 4238, 4251, 4303, 4319, 4309, \text{ and } 4317.$$

Proof. This is a simple computer calculation using the classification [58] and Table 1. □

Corollary 6.8 is somewhat surprising. Although we know of no reason to expect the reflexive polytopes to be F-connected, nor would we have expected the roots and sinks to be related via duality, we can make a small observation. Let $P, Q \subset N_{\mathbb{Q}}$ be two reflexive polytopes such that there exists a finite sequence $P_0, P_1, \ldots, P_k \subset N_{\mathbb{Q}}$ of reflexive polytopes satisfying conditions (i) and (ii) in Definition 6.1 (that is, P_{i-1} and P_i are obtained via the addition or subtraction of a vertex). Then we say that P and Q are I-*equivalent*. It is well-known that the three-dimensional reflexive polytopes are I-connected (although, once again, this is an experimental rather than theoretical fact). Furthermore, if one constructs a directed graph $\mathcal{G}_I$ in an analogous way to $\mathcal{G}_F$ in Corollary 6.8(ii), one finds the exact same list of sixteen roots and sixteen sinks (although the two graphs are different, clearly $\mathcal{G}_F$ can be included in $\mathcal{G}_I$ after possibly factoring edges). Here this duality is less mysterious: if P is minimal with respect to the removal of vertices (that is, if $\mathrm{conv}(\mathrm{vert}(P) \setminus \{v\})$ is not a three-dimensional reflexive polytope for any $v \in \mathrm{vert}(P)$) then $Q = P^*$ is maximal with respect to the addition of vertices (that is, $\mathrm{conv}(\mathrm{vert}(Q) \cup \{v\})$ is not a reflexive polytope for any $v \notin Q$).

7. Computing projections

In order to perform the associated computations, it is in most cases better to represent the toric degenerations of a smooth Fano variety by reflexive lattice polytopes. Given a toric degeneration, the associated reflexive Newton polytope is computed via the standard methods. However, it is worth noting that given a smooth Fano threefold F, it is sometimes possible to produce several different toric degenerations for F, resulting is different Newton polytopes, which give rise to different basic links. However, it is usually possible to connect these degenerations by a sequence of mutations (see [2]). One can choose to consider the set of degenerations of F either as a whole (in order to concentrate on F itself, using mutations to move between different degenerations) or as a collection of individual degenerations (to concentrate on the projections, avoiding the use of mutations).

Given a reflexive polytope, it is usually harder to determine which smooth Fano threefold it originated from. However, it is now possible due to [2, 21], where all the three-dimensional reflexive polytopes have been listed and the possibilities for the corresponding smooth Fano threefolds have been given. It is worth noting that some of the polytopes do not correspond to any smooth Fano threefold even on the level of period sequences. In this paper such polytopes are disregarded, and the projections with them as intermediate steps are avoided. In this representation, it is also possible to compute the basic links (projections) by manipulating the Newton polytopes directly.

We can take $X = \mathbb{P}^3$ and start applying the allowed toric basic links to it. This will give us birational maps between 2868 toric Fano threefolds with canonical Gorenstein singularities (containing representatives of 107 different period sequences and 74 different smooth Fano threefolds). Similarly, we can take X to be any other suitable toric Fano threefold and apply the same process, getting maps between a number of toric Fano varieties. Since there are only finitely many such Fano threefolds, there must be some minimal set of projection roots that allows us to obtain all the other ones in this way. Clearly, this minimal set of roots will contain all the varieties represented by the minimal polytopes (with respect to the removal of vertices). However, it may (depending on what projections are used) include some further toric varieties (for an example, see Remark 8.2).

We can also look at the situation in a different way: we can consider the toric Fano threefolds with canonical Gorenstein singularities primarilly as degenerations of smooth Fano threefolds. From this point of view, mutations give us a second set of basic links, connecting pairs of degenerations of the same smooth Fano threefold. We can repeat the above process, aiming to represent all the smooth Fano threefolds (via their degenerations); one can do this with or without allowing the use of mutations. Since there are only finitely many toric Fano threefolds with canonical Gorenstein singularities, it is possible to complete all the constructions described above.

Given a starting reflexive polytope (a toric Fano threefold with canonical Gorenstein singularities), the program considers all its faces (torus-invariant points) and edges (torus-invariant curves). It selects the relevant ones (according to the rules of the "allowed" projections) and projects from them, obtaining a number of new polytopes (discarding those that turn out not to be reflexive). These polytopes are then added to the processing queue, taking care to avoid duplicates to the polynomials that have been found previously (two explicitly given three-dimensional polytopes are considered to be the same if one can be mapped to the other by an action of the orthogonal group on the underlying lattice). Such a pair of duplicates is merely an indication that there are several projection paths between a pair of varieties.

For example, there is a pair of paths between polytope 232 (corresponding to variety B_3) and polytope 1969 (corresponding to variety $X_{2\text{–}8}$) – such a path can go either through polytope 428 (variety B_2) or through polytope 1599 (variety $X_{2\text{–}15}$). These maps correspond to taking a pair of points (a smooth point and a cDV point) on the degeneration of B_3 and projectiong from them, the intermediate polytope is defined by choosing the order of projections.

Aside from obtaining new polytopes via projections from previously known ones, the program can also build them by considering the possible antiprojections (i.e. inverses of projections) from the known polytopes. This serves a dual purpose: on the one hand, it makes it easier to explore ways of connecting several different polytopes (or the corresponding smooth Fano threefolds); on the other hand, since the procedures for the projections and the antiprojections have been written independently, this serves as an error-checking technique (if a projection $P_1 \rightsquigarrow P_2$ is found, not finding the corresponding antiprojection $P_2 \rightsquigarrow P_1$ would indicate an error).

8. Projection directed graphs

We are interest in constructing a three-dimensional analogue of Figure 1. As such, we wish to restrict our attention to those reflexive polytopes P whose corresponding Gorenstein Fano variety X_P is a toric degeneration of a smooth Fano threefold X. As a first approximation, it is reasonable to restrict our attention to those P such that $\mathrm{Hilb}(X_P, -K_{X_P}) = \mathrm{Hilb}(X, -K_X)$; this condition on the Hilbert series is satisfied by 4310 of the reflexive polytopes. Even if we only allow projections passing through this subset, the result is F-connected.

The calculations described above have been performed, yielding the following results. Given a toric variety T let us call a projection in an anticanonical embedding from tangent space to invariant smooth point, invariant cDV point, or an invariant smooth line (see Table 1) F-projection.

Theorem 8.1. *Given any smooth Fano threefold X, there exists a Gorenstein toric degeneration of X that can be obtained by a sequence of mutations and F-projections from a toric degeneration of one of fifteen smooth Fano threefolds (from now on referred to as the projections roots, see Table 2). The directed (sub)graph connecting all Fano varieties via the projections and mutations is presented in Table 3. Each of toric degenerations we use can be equipped with a toric Landau–Ginzburg model.*

Proof. The list of such paths that minimise the number of projections used form the graph that can be seen in Appendix A. The vertices represent the Fano varieties (one vertex can represent several different degenerations of the same variety), and the arrows represent projections between degenerations of different varieties. The existence of all the arrows is shown in Appendix A, where for each arrow an example of a relevant pair of explicit degenerations is given (according to their polytope ID from the Graded Ring Database). □

Remark 8.2. Not all the projection roots correspond to polytopes that are minimal with respect to the removal of vertices: for example, none of the polytopes corresponding to variety $X_{2\text{–}33}$ are minimal. The appearance of such roots depends on the choice of projections used – such roots can be eliminating by allowing the use of additional projection types, like projections from curves of higher degree. In fact, a degeneration of $X_{2\text{–}33}$ can be obtained by projecting $\mathbb{P}^3$ (polytope ID 1, minimal) from a curve of degree four (obtaining polytope ID 7). However, as discussed above, such projections do not represent natural geometric operations on Fano varieties, and hence are not being considered.

Note that the graph is not connected – in fact, it has three connected components. This is due to the choice of the types of projections used in the graph: using only projections from points and lines (as discussed above), the graph splits into the main component and two one-variety components (containing varieties $X_{7\text{–}1}$ and $X_{8\text{–}1}$). Variety $X_{7\text{–}1}$ (polytope ID 506) can be reached from the main component by

TABLE 2. Projection roots

Variety	Degree	Eliminated by projections from:
$\mathbb{P}^3$	64	
$X_{2\text{–}36}$	62	
Q^3	54	
$X_{3\text{–}27}$	48	
$X_{2\text{–}33}$	54	quartics
$X_{3\text{–}29}$	50	conics
$X_{3\text{–}28}$	48	conics
$X_{4\text{–}11}$	42	conics
$X_{2\text{–}28}$	40	conics
$X_{3\text{–}22}$	40	conics
$X_{5\text{–}3}$	36	conics
$X_{6\text{–}1}$	30	conics
$X_{3\text{–}9}$	26	conics
$X_{7\text{–}1}$	24	conics
$X_{8\text{–}1}$	18	conics

a projection of $X_{6\text{–}1}$ (polytope ID 357) from a conic, and $X_{8\text{–}1}$ (polytope ID 769) can be obtained by projecting $X_{7\text{–}1}$ (polytope ID 506) from a conic.

Theorem 8.3. *(i) For any smooth Fano threefold there is its toric degeneration such that all these degenerations are connected by sequences of projections with toric centres which are either cDV points or smooth lines. The directed graph of such projections containing toric degenerations of all smooth Fano threefolds with very ample anticanonical class can be chosen as a union of fifteen trees with roots shown in Table 2. The directed (sub)graph connecting all Fano varieties via the projections and mutations is presented in Figure B. Each of the toric degenerations can be equipped with a toric Landau–Ginzburg model.*

(ii) For any smooth Fano threefold with very ample anticanonical class there is its Gorenstein toric degeneration such that all these degenerations are connected by sequences of projections with toric centres which are either cDV points, smooth lines, or smooth conics. The directed (sub)graph of such projections can be chosen to have five roots which are: $\mathbb{P}^3$, $X_{2\text{–}36}$, Q^3, $X_{3\text{–}27}$, and $X_{2\text{–}33}$. The directed graph connecting all Fano varieties with very ample anticanonical class via the projections and mutations is presented in Figure C. Each of the toric degenerations can be equipped with a toric Landau–Ginzburg model.

Proof. The algorithm described in §7 was implemented to find all possible projections between degenerations of smooth Fano threefolds. After that, one can choose a toric degeneration for each of the Fano varieties as described in the statement of the theorem. An example of a choice of toric degenerations that create such graphs can be seen in Table 3. For each variety, it gives the corresponding Minkowski ID, as well as the Reflexive ID of the chosen degenerations (for the graphs in part (i) and (ii) of the theorem). If a variety has a terminal degeneration, its Minkowski ID is written in bold; if the degeneration used is terminal, its Reflexive ID is also made bold.

Note that if the choice of polytopes from part (i) is used in a graph that includes projections with centres in conics, one gets a directed graph with six roots (the additional one being $X_{3\text{–}28}$). To rectify that, a different choice of degeneration needed to be made for four varieties in part (ii). □

Remark 8.4. These graphs are not the only ones that satisfy the statement of the theorem. For many smooth Fano threefolds, a choice of degeneration needed to be made. In this case, the choice was made according to the following priorities:

(i) to minimise the number of the graphs' roots;
(ii) to minimise the number of the connected components of the graphs;
(iii) where possible, to use a terminal degeneration of the variety.

Table 3. Degeneration choice

Var.	ID	(i)	(ii)	Var.	ID	(i)	(ii)	Var.	ID	(i)	(ii)
$\mathbb{P}^3$	**1**	**1**		X_{3-27}	**45**	**31**		X_{4-2}	110	668	
X_{2-33}	**2**	**7**		B_5	**46**	**68**		X_{4-1}	**111**	**1530**	
Q^3	**3**	**4**		X_{4-11}	**48**	**85**		X_{3-8}	112	1082	
X_{2-30}	**4**	**23**		X_{3-21}	**49**	**214**		V_{22}	**113**	**1943**	
X_{2-28}	**5**	**69**		X_{4-9}	**54**	**217**		X_{3-6}	117	1501	
X_{2-36}	**6**	**8**		X_{4-8}	**57**	**425**		X_{2-12}	**118**	**2356**	
X_{2-35}	**7**	**6**		X_{2-26}	**58**	$175^{\star}$		X_{2-13}	119	1924	
X_{3-29}	**8**	**27**		X_{5-2}	**64**	**220**		X_{2-11}	120	1701	
X_{2-34}	**10**	**24**		X_{4-7}	**65**	**740**		X_{2-14}	122	2353	
X_{3-30}	**11**	**29**		X_{3-15}	**67**	**420**		V_{18}	124	2703	
X_{3-26}	**12**	**26**		X_{4-5}	**68**	**427**		X_{3-3}	135	2678	
X_{3-22}	**13**	**76**		X_{2-22}	**69**	$373^{\star}$		X_{7-1}	136	506	
X_{3-31}	**14**	**28**		X_{3-13}	**70**	**737**		X_{3-5}	138	1367	
X_{2-31}	**15**	**70**		X_{3-11}	**72**	**732**		X_{2-9}	139	3136	
X_{3-25}	**16**	**74**		X_{2-18}	74	1090		B_2	140	428	
X_{3-23}	**17**	**205**		B_4	**75**	$154^{\star}$		X_{3-4}	142	2222	
X_{3-19}	**18**	**206**		X_{5-3}	**76**	**219**		V_{16}	143	2482	
X_{2-27}	**19**	**201**		X_{2-23}	**78**	**411**		X_{2-8}	144	1969	
X_{3-14}	**21**	**203**		X_{4-6}	**81**	**426**		X_{2-10}	145	3036	
X_{3-9}	22	374		X_{4-4}	**83**	**741**		V_{14}	147	3283	
X_{2-32}	**24**	**22**		X_{2-21}	**84**	**731**		X_{2-7}	148	3239	
X_{3-28}	**28**	**30**	**81**	X_{3-12}	**85**	$723^{\star}$		X_{2-6}	149	3319	
X_{4-13}	**29**	**84**		X_{2-19}	**86**	**1109**		V_{12}	150	3966	
X_{3-24}	**31**	**78**		X_{2-20}	**87**	**1110**		X_{3-1}	154	3350	
X_{4-12}	**34**	**83**	$190^{\star}$	X_{4-3}	**88**	**735**		X_{8-1}	155	769	
X_{2-29}	**35**	**72**	**204**	X_{3-10}	**99**	**1113**		X_{3-2}	157	2791	
X_{4-10}	**37**	**215**		X_{5-1}	100	673		X_{2-5}	158	3453	
X_{3-20}	**38**	**80**		X_{2-17}	**101**	**1528**		V_{10}	160	4132	
X_{3-17}	**39**	**210**		X_{3-7}	**103**	**1529**		X_{2-4}	161	4031	
X_{3-18}	**41**	**212**	**419**	X_{2-16}	104	1485		V_8	163	4205	
X_{3-16}	**42**	**418**		B_3	106	232		V_6	164	4286	
X_{2-25}	**43**	**410**		X_{6-1}	107	357		V_4	165	4312	
X_{2-24}	**44**	**412**		X_{2-15}	109	1599					

In the choices above, the numbers of roots and connected components are indeed minimal, and a terminal degeneration was used where available except in four cases (five cases if projections with centres in smooth conics were used), marked by $\star$, where this would lead to getting additional roots or connected components. The degenerations in question are those for varieties B_4, X_{2-22}, X_{2-26}, X_{3-12} (and X_{4-12} if projections with centres at conics were used).

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Appendix A. Projection–minimizing graph

Table A. This table contains a choice of degenerations (X_1, X_2) for every arrow in the graph of projections discussed in Theorem 8.1. The "From" column gives the starting variety of the projection along with the variety's degree and Minkowski ID. The "To" column lists all the possible destination varieties, and, for each of them, an example of the corresponding pair of degenerations (X_1, X_2) (where the X_i are indicated by the Reflexive ID of the corresponding three-dimensional reflexive polytope).

	From			To		
Level	Var.	P.S.	Deg.			
0	1.1	1	64	2.35: (1 6)		
0	2.36	6	62	2.34: (8 24)		
0	1.2	3	54	3.31: (2 20)	2.30: (2 14)	
0	2.33	2	54	3.30: (7 29)	3.26: (7 26)	
0	3.29	8	50	3.26: (27 73)	3.24: (27 78)	
0	3.27	45	48	1.7: (18 43)		
0	3.28	28	48	4.12: (52 181)	2.29: (52 106)	
0	4.11	48	42	3.21: (62 184)	4.9: (191 291)	2.26: (62 163)
0	2.28	5	40	2.27: (34 305)	3.14: (34 143)	
0	3.22	13	40	3.19: (139 270)	3.17: (76 210)	3.15: (139 316)
0	5.3	76	36	4.6: (195 409)	2.21: (114 238)	
0	6.1	107	30	4.2: (284 602)		
0	3.9	22	26	2.18: (447 1999)		
0	7.1	136	24			
0	8.1	155	18			
1	2.35	7	56	2.32: (6 13)		
1	2.34	10	54	2.31: (5 21)		
1	3.31	14	52	3.25: (20 47)		
1	3.30	11	50	4.13: (29 60)		
1	2.30	4	46	3.25: (14 47)	3.23: (23 77)	
1	3.26	12	46	3.23: (26 77)	3.20: (26 44)	
1	4.12	34	44	4.10: (309 638)	3.18: (309 639)	
1	3.24	31	42	4.10: (169 390)	3.20: (169 259)	
1	1.7	46	40	1.6: (221 429)		
1	2.29	35	40	3.20: (106 259)	3.18: (19 63)	2.25: (106 251)
1	2.27	19	38	3.18: (71 212)	3.16: (71 213)	2.24: (157 322)
1	3.19	18	38	3.16: (75 213)	3.13: (35 93)	
1	3.21	49	38	4.8: (488 1350)	2.23: (488 765)	
1	4.9	54	38	4.8: (291 580)	4.7: (67 185)	2.22: (291 547)
1	3.17	39	36	3.16: (130 323)		
1	2.26	58	34	2.22: (1263 1749)	2.23: (481 765)	2.19: (957 1230)
1	3.14	21	32	2.24: (143 631)	3.11: (308 656)	3.7: (143 263)
1	3.15	67	32	2.22: (2125 2450)	3.13: (564 977)	4.4: (598 1355)
				3.11: (564 985)	3.12: (954 1435)	2.17: (564 931)
1	4.6	81	32	4.4: (1682 2179)	3.12: (1682 2110)	4.3: (1682 1976)
				2.17: (1708 2182)		
1	2.21	84	28	2.19: (123 338)	2.20: (238 1320)	3.10: (123 267)
				2.17: (1753 2252)	2.12: (2729 3077)	

	From			To		
Level	Var.	P.S.	Deg.			
1	4.2	110	26	3.8: (602 1867)	1.17: (602 930)	3.6: (1048 1437)
1	2.18	74	24	2.16: (449 1998)	2.13: (1441 1824)	
2	2.32	24	48			
2	2.31	15	46			
2	4.13	29	46			
2	3.25	16	44			
2	3.23	17	42			
2	4.10	37	40	5.2: (180 408)		
2	3.20	38	38	5.2: (44 194)		
2	3.18	41	36	4.5: (63 151)		
2	4.8	57	36			
2	3.16	42	34	4.5: (615 981)		
2	4.7	65	34			
2	1.6	75	32	1.5: (433 742)		
2	2.25	43	32			
2	2.22	69	30			
2	2.23	78	30	2.15: (1165 1599)		
2	2.24	44	30			
2	3.13	70	30	5.1: (93 285)		
2	4.4	83	30			
2	3.11	72	28			
2	3.12	85	28			
2	4.3	88	28			
2	2.19	86	26	2.15: (2066 2397)	2.11: (1392 1701)	
2	2.20	87	26	1.16: (1186 1559)		
2	3.10	99	26	4.1: (267 489)	1.16: (3571 3764)	
2	2.17	101	24	2.9: (2182 2462)		
2	3.7	103	24	2.14: (1374 1721)		
2	3.8	112	24	2.14: (2541 2958)	3.5: (984 1367)	
2	1.17	113	22	2.14: (1141 1659)	3.5: (930 1367)	1.16: (4037 4085)
				1.14: (3772 3891)		
2	2.16	104	22	1.16: (2960 3233)	2.11: (1233 1701)	
2	3.6	117	22	2.14: (3123 3921)	1.16: (2166 2494)	3.3: (3123 3445)
				3.4: (2166 2394)		
2	2.12	118	20	1.16: (1548 2500)	2.11: (1194 1701)	2.9: (2311 2606)
				2.6: (3077 3319)		
2	2.13	119	20	1.16: (3557 4085)	3.3: (1662 2070)	3.4: (2114 2544)
				1.15: (1717 2024)	2.9: (2770 3136)	
3	5.2	64	36			
3	4.5	68	32			
3	5.1	100	28			
3	1.5	106	24	1.4: (1952 2364)		
3	4.1	111	24			
3	2.15	109	22	2.8: (1599 1969)		
3	2.14	122	20	2.10: (3921 4002)		
3	3.5	138	20			
3	1.16	124	18	2.10: (3361 3617)		
3	2.11	120	18	2.8: (1701 1969)		
3	3.3	135	18	2.7: (3445 3592)		
3	3.4	142	18	2.10: (2394 2746)	3.2: (2544 2791)	

From				To		
Level	Var.	P.S.	Deg.			
3	1.15	143	16	2.7: (3373 3592)	3.2: (2482 2791)	1.13: (4050 4119)
3	2.9	139	16			
3	1.14	147	14	1.13: (3887 4119)	2.5: (3587 3736)	1.12: (4171 4200)
3	2.6	149	12	1.12: (3319 4007)		
4	1.4	140	16			
4	2.10	145	16			
4	2.7	148	14	3.1: (3102 3329)		
4	2.8	144	14			
4	3.2	157	14			
4	1.13	150	12	2.4: (3966 4031)		
4	2.5	158	12	2.4: (3736 4031)		
4	1.12	160	10	1.11: (3051 3314)		
5	3.1	154	12			
5	2.4	161	10			
5	1.11	163	8			

Figure B. Connecting varieties by projections from points and lines. The Fano threefolds are denoted by their Minkowski ID's, each arrow signify a projection between them. See Table 3(i) for the explicit choice of degenerations for each of the varieties.

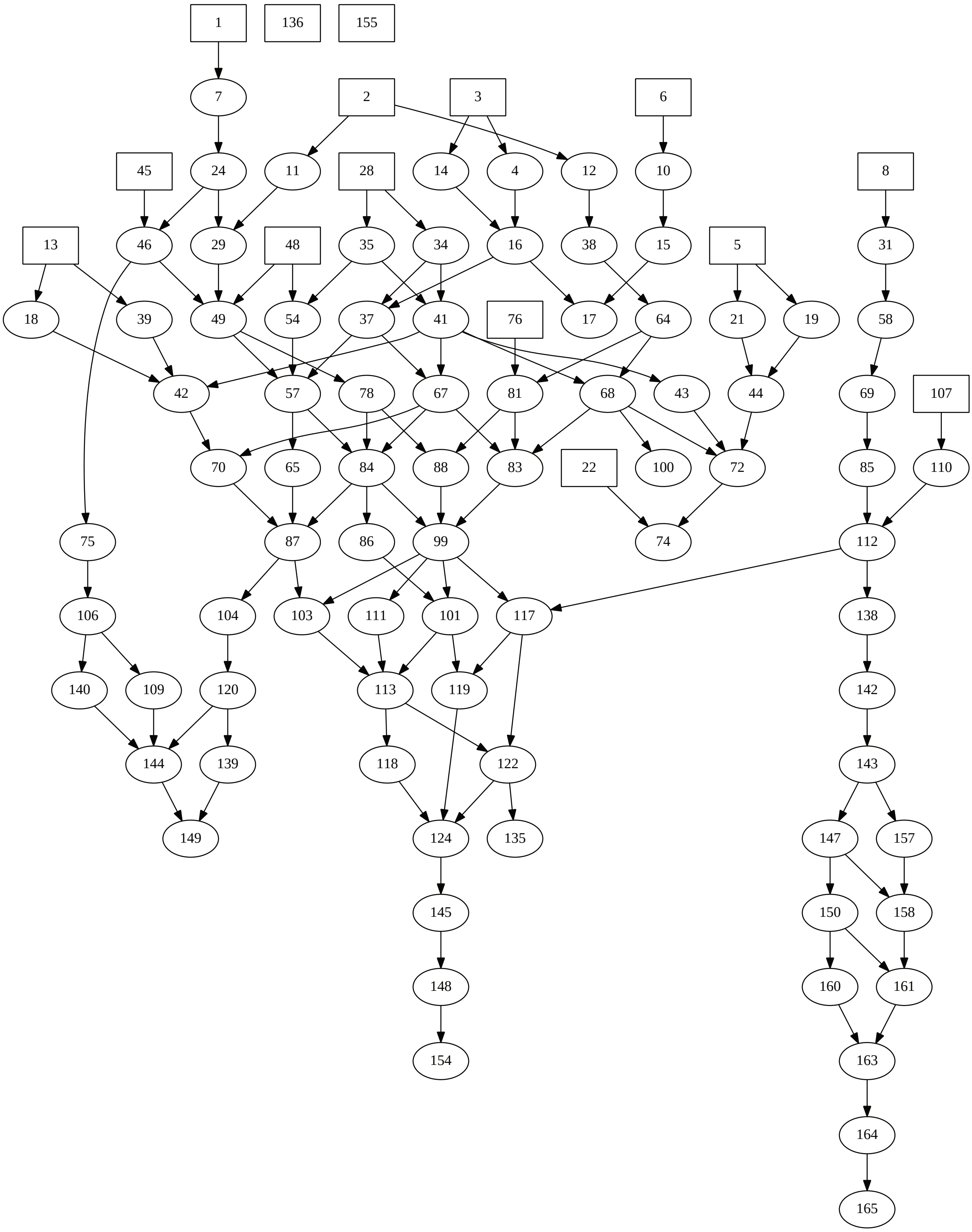
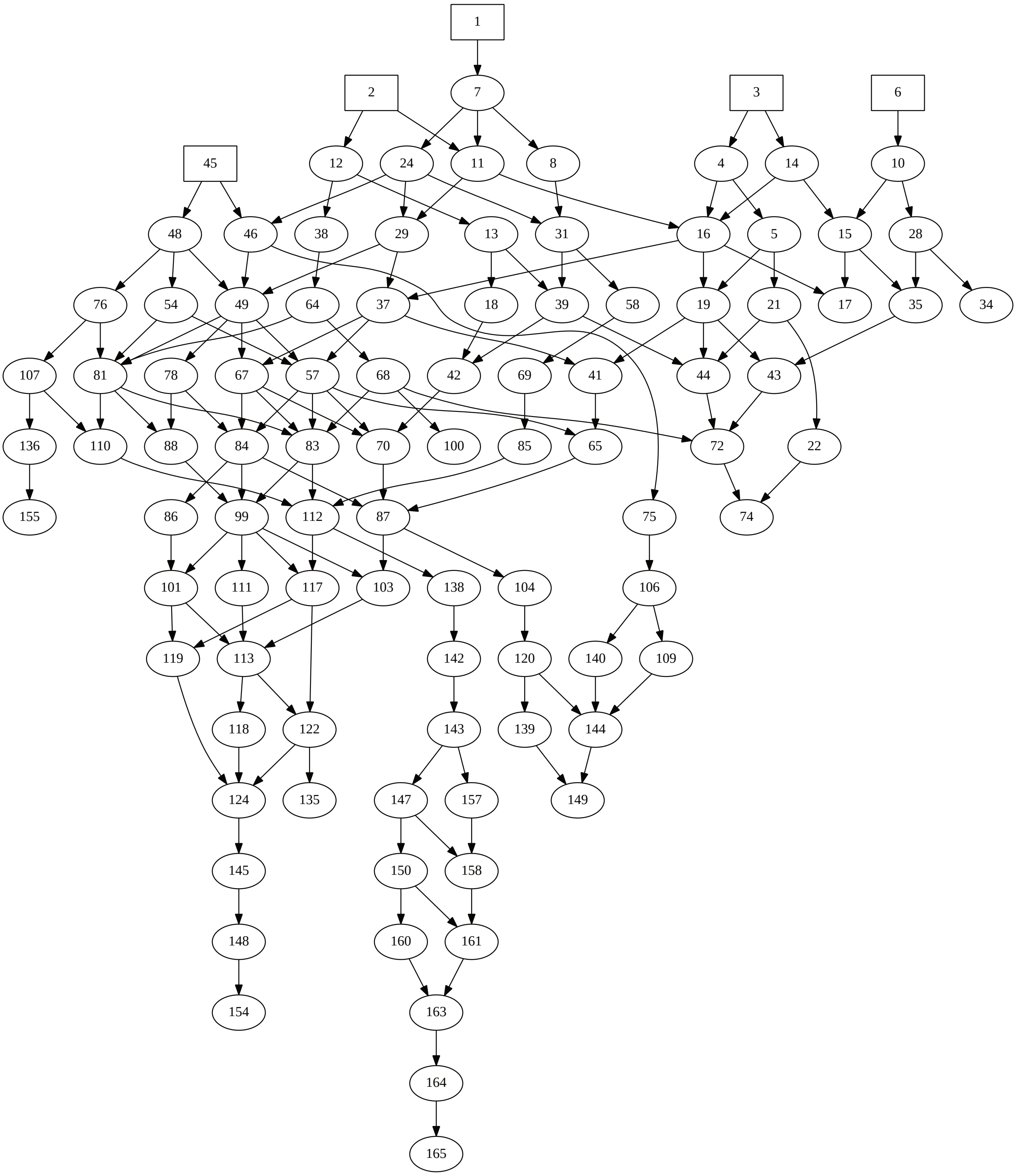


Figure C. Connecting varieties by projections from points, lines, and conics. The Fano threefolds are denoted by their Minkowski ID's, each arrow signify a projection between them. See Table 3(ii) for the explicit choice of degenerations for each of the varieties.



Appendix B. Tables for Toric Degenerations

Table D. In the following table, we compute $h^0(V, \mathcal{N}_{V/\mathbb{P}^n})$ for each smooth Fano threefold with very ample anticanonical divisor. Here $V \hookrightarrow \mathbb{P}^n$ denotes the anticanonical embedding. This value equals the dimension of the Hilbert scheme component of $Hilb_V$ corresponding to V.

Name	b_2	$b_3/2$	$(-K_V)^3$	$h^0(V, \mathcal{N}_{V/\mathbb{P}^n})$
V_4	1	30	4	69
V_6	1	20	6	69
V_8	1	14	8	75
V_{10}	1	10	10	85
V_{12}	1	7	12	98
V_{14}	1	5	14	114
V_{16}	1	3	16	132
V_{18}	1	2	18	153
V_{22}	1	0	22	201
B_2	1	10	16	139
B_3	1	5	24	234
B_4	1	2	32	363
B_5	1	0	40	525
Q^3	1	0	54	889
$\mathbb{P}^3$	1	0	64	1209
2–4	2	10	10	84
2–5	2	6	12	96
2–6	2	9	12	99
2–7	2	5	14	113
2–8	2	9	14	117
2–9	2	5	16	133
2–10	2	3	16	131
2–11	2	5	18	155
2–12	2	3	20	177
2–13	2	2	20	176
2–14	2	1	20	175
2–15	2	4	22	204
2–16	2	2	22	202
2–17	2	1	24	229
2–18	2	2	24	230
2–19	2	2	26	260
2–20	2	0	26	258
2–21	2	0	28	290
2–22	2	0	30	324
2–23	2	1	30	325
2–24	2	0	30	324
2–25	2	1	32	361
2–26	2	0	34	398
2–27	2	0	38	480
2–28	2	1	40	525
2–29	2	0	40	524
2–30	2	0	46	668
2–31	2	0	46	668
2–32	2	0	48	720
2–33	2	0	54	888
2–34	2	0	54	888
2–35	2	0	56	948
2–36	2	0	62	1140
3–1	3	8	12	97
3–2	3	3	14	110
3–3	3	3	18	152
3–4	3	2	18	151
3–5	3	0	20	173
3–6	3	1	22	200
3–7	3	1	24	228
3–8	3	0	24	227
3–9	3	3	26	260
3–10	3	0	26	257
3–11	3	1	28	290
3–12	3	0	28	289
3–13	3	0	30	323
3–14	3	1	32	360
3–15	3	0	32	359
3–16	3	0	34	397
3–17	3	0	36	437
3–18	3	0	36	437

Name	b_2	$b_3/2$	$(-K_V)^3$	$h^0(V, \mathcal{N}_{V/\mathbb{P}^n})$
3–19	3	0	38	479
3–20	3	0	38	479
3–21	3	0	38	479
3–22	3	0	40	523
3–23	3	0	42	569
3–24	3	0	42	569
3–25	3	0	44	617
3–26	3	0	46	667
3–27	3	0	48	719
3–28	3	0	48	719
3–29	3	0	50	773
3–30	3	0	50	773
3–31	3	0	52	829
4–1	4	1	24	227
4–2	4	0	26	256
4–3	4	1	28	289

Name	b_2	$b_3/2$	$(-K_V)^3$	$h^0(V, \mathcal{N}_{V/\mathbb{P}^n})$
4–4	4	0	30	322
4–5	4	0	32	358
4–6	4	0	32	358
4–7	4	0	34	396
4–8	4	0	36	436
4–9	4	0	38	478
4–10	4	0	40	522
4–11	4	0	42	568
4–12	4	0	44	616
4–13	4	0	46	666
5–1	5	0	28	287
5–2	5	0	36	435
5–3	5	0	36	435
6–1	6	0	30	320
7–1	7	0	24	223
8–1	8	0	18	144

Table E. The degenerations of smooth Fano threefolds to Gorenstein toric Fano varieties with at most terminal singularities [35]. Here "IDs" are the Reflexive IDs of the corresponding three-dimensional reflexive polytopes.

Name	IDs
V_{22}	1943
B_4	198
B_5	68
Q^3	4
$\mathbb{P}^3$	1
2–12	2356
2–17	1528
2–19	1109
2–20	1110, 1112
2–21	731, 733, 1111
2–22	414, 729
2–23	411
2–24	412
2–25	199, 410
2–26	202, 413, 734
2–27	71, 200, 201
2–28	69
2–29	72, 204
2–30	23
2–31	21, 70
2–32	22

Name	IDs
2–33	7
2–34	5, 24
2–35	6
2–36	8
3–7	1529
3–10	1113
3–11	730, 732
3–12	738, 1114
3–13	421, 737, 739
3–14	203
3–15	420, 422, 423, 736
3–16	213, 417, 418
3–17	209, 210, 415
3–18	212, 419
3–19	75, 206
3–20	80, 208, 211, 416
3–21	214
3–22	76
3–23	77, 205
3–24	78, 79, 207
3–25	25, 74

Name	IDs
3–26	26, 73
3–27	31
3–28	30, 81
3–29	27
3–30	29
3–31	28
4–1	1530
4–3	735
4–4	741
4–5	427
4–6	426
4–7	424, 740
4–8	216, 218, 425
4–9	217
4–10	82, 215
4–11	85
4–12	83
4–13	84
5–2	220
5–3	219

Table F. Degenerations of products of low degree Fano threefolds as in [19]. We only list those degenerations with smooth deformation space. Here "IDs" are the Reflexive IDs of the corresponding three-dimensional reflexive polytopes.

Name	IDs
V_4	4312
V_6	4282, 4284, 4286, 4287, 4297, 4310, 4318
V_8	3314, 4005, 4167, 4194, 4203, 4205, 4217, 4231, 4238, 4244, 4250, 4251, 4267, 4269, 4275, 4280, 4290, 4298, 4299, 4304, 4313, 4314, 4315
V_{10}	3051, 3792, 3903, 3922, 3928, 3965, 4007, 4023, 4032, 4044, 4074, 4118, 4131, 4132, 4135, 4143, 4161, 4168, 4181, 4186, 4190, 4200, 4202, 4214, 4216, 4219, 4225, 4228, 4245, 4252, 4255, 4258, 4261, 4268, 4281, 4291, 4292, 4295, 4301, 4303, 4307, 4311
V_{12}	2756, 3406, 3625, 3626, 3667, 3683, 3702, 3796, 3848, 3853, 3869, 3937, 3938, 3946, 3962, 3966, 3981, 3984, 3985, 3991, 4059, 4060, 4070, 4080, 4102, 4104, 4119, 4122, 4124, 4133, 4134, 4144, 4145, 4179, 4180, 4184, 4218, 4220, 4247, 4254, 4263, 4270, 4272, 4273, 4293, 4294
2–4	4031
2–5	3453, 3736, 3777
2–6	3319, 3349, 3416
3–1	3329, 3350

Table G. Selected degenerations of rank one index one Fano threefolds as described in §5.2. Here "IDs" are the Reflexive IDs of the corresponding three-dimensional reflexive polytopes.

Name	IDs
V_{14}	2464, 3214, 3229, 3268, 3298, 3461, 3506, 3535, 3536, 3633, 3670, 3675, 3676, 3712, 3719, 3724, 3794, 3837, 3849, 3852, 3870, 3876, 3887, 3891, 3944, 3968, 3970, 3982, 3986, 3992, 4046, 4066, 4069, 4077, 4078, 4139, 4140, 4171, 4185, 4223, 4226, 4253, 4271
V_{16}	2024, 2498, 2650, 2894, 2899, 2903, 2987, 2990, 3026, 3027, 3034, 3112, 3162, 3210, 3277, 3280, 3294, 3300, 3310, 3373, 3374, 3396, 3464, 3513, 3515, 3519, 3546, 3642, 3648, 3674, 3685, 3700, 3709, 3710, 3711, 3721, 3725, 3742, 3797, 3798, 3816, 3858, 3864, 3872, 3954, 3975, 3990, 3993, 4050, 4063, 4065, 4109, 4121, 4146, 4150, 4172, 4227
V_{18}	1559, 1827, 2116, 2187, 2223, 2289, 2302, 2337, 2480, 2494, 2500, 2520, 2539, 2605, 2646, 2648, 2658, 2683

Table H. Degenerations of products of del Pezzo surfaces with $\mathbb{P}^1$. Here "ID" is the Reflexive ID of the corresponding three-dimensional reflexive polytope.

Name	ID
5–3	219
6–1	357
7–1	510

Table I. Degenerations constructed as complete intersections in toric varieties as in §5.4. Here "ID" is the Reflexive ID of the corresponding three-dimensional reflexive polytope.

Name	ID	Regular Sequence
B_2	1978	$x_2^2x_4^2 - x_1x_3x_5$
	2364	$x_2^3x_4 - x_1x_3x_5$
	3313	$x_2^4 - x_1x_3x_5$
B_3	232	$x_1x_2x_4 - x_3x_5^2$
	742	$x_1x_2x_3 - x_5^3$
2–7	3813	$x_1x_3x_4 - x_2x_5x_6,\ x_6^2 - x_5x_7$
2–8	1969	$x_1x_2x_3x_4x_5 - x_6^2$
2–9	2606	$x_2x_5 - x_4x_7,\ x_1^2x_5 - x_2x_3x_6$
	3014	$x_2x_5 - x_4x_7,\ x_2^2x_5 - x_1x_3x_6$
	3242	$x_2x_5 - x_4x_7,\ x_2x_3x_5 - x_1^2x_6$
	3479	$x_2x_5 - x_4x_7,\ x_4^2x_5 - x_1x_3x_6$
	3480	$x_2x_5 - x_4x_7,\ x_1x_3x_5 - x_2^2x_6$
2–10	1972	$x_5x_6 - x_7^2,\ x_2x_3x_7 - x_4^2$
	2131	$x_2^2x_3^2 - x_6x_7,\ x_1^2x_3^2 - x_4x_5$
	2447	$x_2x_3x_6 - x_7^2,\ x_1x_3x_6 - x_4x_5$
	2746	$x_2x_3x_6 - x_7^2,\ x_1^2x_3^2 - x_4x_5$
	2808	$x_2x_3x_6 - x_7^2,\ x_4x_5 - x_6x_7$
	2924	$x_5x_6 - x_7^2,\ x_1x_2x_3^2 - x_4x_7$
	2928	$x_2x_3x_5 - x_6x_7,\ x_1^2x_3^2 - x_4x_5$
	2984	$x_2^2x_3^2 - x_6x_7,\ x_1x_2x_3^2 - x_4x_5$
	3019	$x_2x_3x_5 - x_6x_7,\ x_1x_2x_3^2 - x_4x_5$
	3036	$x_4x_5 - x_6x_7,\ x_1x_2x_3^2 - x_6x_7$
	3153	$x_2x_3x_6 - x_7^2,\ x_1x_3x_7 - x_4x_5$
	3362	$x_5x_6 - x_7^2,\ x_2^2x_3^2 - x_4x_7$
	3387	$x_5x_6 - x_7^2,\ x_2x_3x_4 - x_7^2$
	3481	$x_2x_3x_5 - x_6x_7,\ x_2^2x_3^2 - x_4x_5$
	3487	$x_2^2x_3^2 - x_6x_7,\ x_4x_5 - x_6x_7$
	3615	$x_2x_3x_6 - x_7^2,\ x_1x_2x_3^2 - x_4x_5$
	3617	$x_5x_6 - x_7^2,\ x_2x_3x_4 - x_6x_7$
	3789	$x_2x_3x_6 - x_7^2,\ x_2x_3x_7 - x_4x_5$
	4002	$x_5x_6 - x_7^2,\ x_2x_3x_4 - x_6^2$
	4021	$x_2x_3x_6 - x_7^2,\ x_2^2x_3^2 - x_4x_5$
2–11	1701	$x_1x_3x_4x_5 - x_2x_6^2$
	2203	$x_1x_3^2x_4^2 - x_2x_5x_6$
	2815	$x_3^3x_4^2 - x_2x_5x_6$
2–12	1497	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_8,\ x_4x_6 - x_3x_7$
	2311	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_8,\ x_4x_5 - x_3x_7$
	2350	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_6,\ x_3x_7 - x_2x_8$
	2356	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_6,\ x_2x_6 - x_3x_7$
	2996	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_8,\ x_2x_5 - x_3x_7$
	3029	$x_2x_6 - x_4x_8,\ x_1x_5 - x_2x_6,\ x_3x_6 - x_2x_7$
	1254	$x_3x_6 - x_1x_8,\ x_2x_4 - x_1x_5,\ x_7^2 - x_5x_8$
	1419	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_4x_7 - x_8^2$
	1662	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_7^2 - x_4x_8$
	2236	$x_3x_6 - x_1x_8,\ x_2x_4 - x_1x_5,\ x_5x_7 - x_8^2$

Name	ID	Regular Sequence
	2332	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_8,\ x_4x_7 - x_8^2$
	2343	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_4x_7 - x_5x_8$
	2354	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_8,\ x_4x_7 - x_6x_8$
2–13	2448	$x_3x_6 - x_1x_8,\ x_2x_4 - x_1x_5,\ x_5x_6 - x_7^2$
	2882	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_8,\ x_7^2 - x_4x_8$
	2931	$x_3x_6 - x_1x_8,\ x_2x_4 - x_1x_5,\ x_6^2 - x_5x_7$
	2940	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_5^2 - x_4x_7$
	2995	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_6^2 - x_4x_7$
	3031	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_5x_6 - x_4x_7$
	3197	$x_3x_6 - x_1x_8,\ x_2x_4 - x_1x_5,\ x_4x_7 - x_8^2$
	3616	$x_3x_6 - x_1x_8,\ x_2x_5 - x_1x_6,\ x_4x_6 - x_7^2$
	911	$x_2^2x_4x_5 - x_1x_3x_6$
2–15	1573	$x_3^3x_5 - x_1x_2x_6$
	1599	$x_1x_2x_4x_5 - x_3^2x_6$
	810	$x_3x_6 - x_2x_7,\ x_1x_4x_7 - x_5^2$
	994	$x_3^2x_4 - x_2x_7,\ x_1^2x_4^2 - x_5x_6$
	1016	$x_1x_3x_4 - x_2x_7,\ x_5x_6 - x_7^2$
	1174	$x_3^2x_4 - x_2x_7,\ x_1x_4x_5 - x_6^2$
	1233	$x_1x_3x_4 - x_2x_7,\ x_5^2 - x_6x_7$
	1464	$x_3^2x_4 - x_2x_7,\ x_1x_4x_7 - x_5x_6$
	1485	$x_3x_6 - x_2x_7,\ x_1x_4x_5 - x_6x_7$
	1520	$x_1x_3x_4 - x_2x_7,\ x_3x_4x_7 - x_5x_6$
	1669	$x_3^2x_4 - x_2x_7,\ x_5x_6 - x_7^2$
	1712	$x_3x_6 - x_2x_7,\ x_1x_4x_5 - x_7^2$
	1764	$x_3x_6 - x_2x_7,\ x_1^2x_4^2 - x_5x_7$
	1883	$x_1x_3x_4 - x_2x_7,\ x_3^2x_4^2 - x_5x_6$
	1904	$x_3^2x_4 - x_2x_7,\ x_1x_3x_4^2 - x_5x_6$
2–16	1908	$x_3x_6 - x_2x_7,\ x_1x_2x_4^2 - x_5x_7$
	1940	$x_1x_3x_4 - x_2x_7,\ x_2x_4x_7 - x_5x_6$
	1998	$x_3x_6 - x_2x_7,\ x_2x_4x_7 - x_5^2$
	2259	$x_3^2x_4 - x_2x_7,\ x_3x_4x_7 - x_5x_6$
	2264	$x_3^2x_4 - x_2x_7,\ x_1x_2x_4^2 - x_5x_6$
	2346	$x_1x_3x_4 - x_2x_7,\ x_2x_3x_4^2 - x_5x_6$
	2446	$x_1x_3x_4 - x_2x_7,\ x_3x_4x_5 - x_6^2$
	2546	$x_3x_6 - x_2x_7,\ x_2^2x_4^2 - x_5x_7$
	2629	$x_3^2x_4 - x_2x_7,\ x_2x_4x_7 - x_5x_6$
	2671	$x_1x_3x_4 - x_2x_7,\ x_2^2x_4^2 - x_5x_6$
	2809	$x_3x_6 - x_2x_7,\ x_2x_4x_5 - x_7^2$
	2960	$x_3^2x_4 - x_2x_7,\ x_2x_3x_4^2 - x_5x_6$
	3196	$x_3^2x_4 - x_2x_7,\ x_2^2x_4^2 - x_5x_6$
	3347	$x_3^2x_4 - x_2x_7,\ x_3x_4x_5 - x_6^2$
	449	$x_1x_2x_4x_5 - x_6^2$
2–18	1033	$x_1x_3x_5^2 - x_2x_4x_6$
	1250	$x_2^2x_4^2 - x_1x_5x_6$
3–2	2570	$x_1x_2x_4^3x_5^2 - x_3x_6x_7$
	2791	$xx_2^2x_3^3x_5^2 - x_4x_6x_7$
$3-3$	2678	$x_2x_4x_5^2 - x_1x_3x_6x_7$
	2005	$x_2x_3x_4x_5x_6 - x_7^2$
	2314	$x_1x_2x_3^2x_5^2 - x_4x_6x_7$

Name	ID	Regular Sequence
3–4	2544	$x_2^2x_3^2x_5^2 - x_4x_6x_7$
	2807	$x_4^2x_6^2 - x_2x_3x_5x_7$
3–5	1326	$x_2x_4^2x_6 - x_5x_8,\ x_1x_4^2x_6 - x_3x_7$
	1367	$x_2x_4^2x_6 - x_5x_8,\ x_3x_7 - x_4x_8$
	1820	$x_2x_4^2x_6 - x_5x_8,\ x_1x_4x_5x_6 - x_3x_7$
	1837	$x_2x_4^2x_6 - x_5x_8,\ x_3x_7 - x_5x_8$
	2128	$x_2x_4^2x_6 - x_5x_8,\ x_1x_5^2x_6 - x_3x_7$
	2218	$x_2x_4^2x_6 - x_5x_8,\ x_2x_4x_5x_6 - x_3x_7$
	2492	$x_2x_4^2x_6 - x_5x_8,\ x_2x_5^2x_6 - x_3x_7$
3–6	1773	$x_2x_3x_5x_7 - x_4^2x_6$
	1938	$x_1x_2x_3^2x_7 - x_4x_5x_6$
	2269	$x_2^2x_3^2x_7 - x_4x_5x_6$
3–8	1082	$x_1x_3x_6^2 - x_4x_5x_7$
	1776	$x_1x_3x_6x_7 - x_4x_5^2$
3–9	289	$x_1^4x_4^2 - x_5x_6$
	344	$x_1^3x_3x_4^2 - x_5x_6$
	354	$x_1^2x_3^2x_4^2 - x_5x_6$
	374	$x_1^2x_2x_3x_4^2 - x_5x_6$
	447	$x_1x_3x_4x_5 - x_6^2$
$4-2$	1081	$x_1x_4^2x_5x_7 - x_3x_6x_8$
$5-1$	1083	$x_3x_5x_6x_7x_8 - x_1x_2x_9$

Table J. Degenerations of smooth Fano threefolds to Gorenstein toric Fano threefolds obtained from those of Tables E, F, G, H, and I by (possibly repeated) mutations. Those toric varieties (indicated by the Reflexive ID of the corresponding three-dimensional reflexive polytope) corresponding to a smooth point of $\mathcal{H}_V$ appear in the column "Interior Points"; those corresponding to singular points of $\mathcal{H}_V$ appear in the column "Boundary Points".

Name	Interior Points	Boundary Points
V_4	4312	
V_6	4282, 4284, 4286, 4287, 4297, 4310, 4318	
V_8	3314, 4005, 4167, 4194, 4203, 4205, 4217, 4231, 4238, 4244, 4250, 4251, 4267, 4269, 4275, 4280, 4290, 4298, 4299, 4304, 4313, 4314, 4315	
V_{10}	3051, 3792, 3903, 3922, 3928, 3965, 4007, 4023, 4032, 4044, 4074, 4118, 4131, 4132, 4135, 4143, 4161, 4168, 4181, 4186, 4190, 4200, 4202, 4214, 4216, 4219, 4225, 4228, 4245, 4252, 4255, 4258, 4261, 4268, 4281, 4291, 4292, 4295, 4301, 4303, 4307, 4311	3879, 3927, 3964, 4006, 4024, 4042

Name	Interior Points	Boundary Points
V_{12}	2756, 3406, 3625, 3626, 3667, 3683, 3702, 3796, 3848, 3853, 3869, 3937, 3938, 3946, 3962, 3966, 3981, 3984, 3985, 3991, 4059, 4060, 4070, 4080, 4102, 4104, 4119, 4122, 4124, 4133, 4134, 4144, 4145, 4179, 4180, 4184, 4218, 4220, 4247, 4254, 4263, 4270, 4272, 4273, 4293, 4294	3452, 3756, 3760, 3795, 3846, 3857, 3875, 3933, 3936, 3967, 3983, 4004, 4027, 4041, 4043, 4075, 4117, 4241, 4249
V_{14}	2464, 3082, 3214, 3219, 3229, 3268, 3283, 3298, 3461, 3471, 3506, 3535, 3536, 3587, 3633, 3637, 3640, 3670, 3675, 3676, 3712, 3719, 3724, 3794, 3837, 3849, 3852, 3870, 3876, 3887, 3891, 3932, 3944, 3968, 3970, 3982, 3986, 3992, 4046, 4066, 4069, 4077, 4078, 4139, 4140, 4171, 4185, 4198, 4223, 4226, 4253, 4271	2472, 2553, 2725, 2966, 3056, 3202, 3266, 3397, 3399, 3509, 3511, 3530, 3537, 3543, 3544, 3559, 3574, 3578, 3624, 3636, 3666, 3698, 3704, 3718, 3766, 3799, 3800, 3824, 3825, 3832, 3847, 3855, 3867, 3871, 3892, 3896, 3900, 3925, 3929, 3935, 3950, 3963, 3973, 3977, 4026, 4049, 4057, 4062, 4079, 4126, 4136, 4138, 4187, 4195, 4196, 4201, 4207, 4209, 4221
V_{16}	2024, 2291, 2437, 2482, 2498, 2650, 2894, 2899, 2903, 2987, 2990, 3026, 3027, 3034, 3112, 3162, 3210, 3277, 3280, 3294, 3300, 3310, 3373, 3374, 3396, 3464, 3513, 3515, 3519, 3546, 3642, 3648, 3674, 3685, 3700, 3709, 3710, 3711, 3721, 3725, 3742, 3797, 3798, 3816, 3858, 3864, 3872, 3954, 3975, 3990, 3993, 4050, 4063, 4065, 4109, 4121, 4146, 4150, 4172, 4227	2093, 2414, 2567, 2724, 2773, 2888, 2891, 2896, 2901, 2967, 2969, 2988, 3008, 3012, 3013, 3032, 3129, 3131, 3132, 3209, 3227, 3265, 3312, 3333, 3336, 3339, 3352, 3375, 3376, 3401, 3404, 3409, 3450, 3455, 3462, 3477, 3507, 3512, 3518, 3529, 3531, 3532, 3533, 3538, 3542, 3664, 3672, 3680, 3686, 3687, 3703, 3720, 3759, 3761, 3802, 3803, 3826, 3828, 3830, 3850, 3851, 3854, 3865, 3873, 3880, 3894, 3898, 3904, 3914, 3934, 3939, 3943, 3972, 3974, 3989, 4030, 4064, 4072, 4125, 4137, 4147, 4152, 4153, 4164, 4165, 4175, 4264, 4302
V_{18}	1559, 1827, 2116, 2187, 2223, 2289, 2302, 2337, 2480, 2494, 2500, 2520, 2539, 2605, 2646, 2648, 2658, 2683, 2684, 2699, 2701, 2703, 2832, 2840, 2858, 2902, 2911, 2982, 2985, 2989, 3015, 3023, 3025, 3035, 3061, 3099, 3160, 3180, 3221, 3230, 3233, 3235, 3269, 3274, 3287, 3290, 3291, 3297, 3299, 3305, 3307, 3361, 3415, 3457, 3470, 3486, 3496, 3514, 3516, 3522, 3526, 3541, 3545, 3596, 3603, 3605, 3681, 3682, 3692, 3699, 3714, 3717, 3764, 3823, 3831, 3859, 3862, 3866, 3912, 3940, 3951, 3958, 3959, 3988, 4068, 4073, 4085, 4114, 4123, 4174, 4178, 4265	1616, 1695, 1792, 2044, 2049, 2180, 2188, 2189, 2271, 2453, 2463, 2469, 2565, 2571, 2578, 2595, 2633, 2680, 2691, 2763, 2816, 2847, 2885, 2898, 2963, 2980, 3011, 3020, 3024, 3037, 3038, 3041, 3042, 3055, 3081, 3086, 3096, 3142, 3144, 3155, 3161, 3203, 3226, 3237, 3278, 3279, 3281, 3293, 3308, 3389, 3394, 3395, 3405, 3423, 3456, 3463, 3466, 3508, 3517, 3520, 3534, 3558, 3586, 3588, 3627, 3635, 3671, 3678, 3693, 3707, 3716, 3722, 3723, 3757, 3767, 3768, 3769, 3770, 3804, 3812, 3829, 3834, 3899, 3930, 3942, 3969, 3971, 3987, 4096, 4106, 4110, 4142, 4242

Name	Interior Points	Boundary Points
V_{22}	839, 842, 843, 930, 1022, 1028, 1141, 1428, 1482, 1484, 1700, 1722, 1816, 1855, 1886, 1887, 1889, 1898, 1899, 1901, 1929, 1930, 1937, 1942, 1943, 2115, 2212, 2214, 2235, 2248, 2309, 2495, 2503, 2523, 2585, 2592, 2655, 2657, 2661, 2662, 2664, 2665, 2694, 2696, 2702, 2761, 2842, 2852, 2934, 2937, 2953, 2999, 3004, 3030, 3094, 3247, 3248, 3249, 3306, 3366, 3431, 3483, 3488, 3494, 3499, 3527, 3589, 3695, 3772, 3843, 3960, 4037, 4054	1143, 1272, 1313, 1604, 1716, 1786, 1880, 2089, 2121, 2181, 2228, 2229, 2273, 2274, 2284, 2288, 2330, 2335, 2349, 2351, 2368, 2373, 2400, 2459, 2467, 2597, 2600, 2618, 2632, 2643, 2692, 2697, 2772, 2822, 2826, 2844, 2906, 2914, 2919, 2920, 2933, 2936, 2974, 2992, 3016, 3017, 3018, 3022, 3078, 3093, 3128, 3149, 3157, 3206, 3222, 3225, 3236, 3238, 3240, 3250, 3255, 3270, 3272, 3284, 3285, 3302, 3332, 3354, 3363, 3365, 3420, 3432, 3459, 3465, 3493, 3500, 3630, 3638, 3643, 3647, 3656, 3688, 3715, 3805, 3841, 3877, 3883, 3888, 3910, 4052, 4055, 4129, 4155
B_2	428, 1978, 2078, 2364, 3313, 3316, 3317	
B_3	87, 232, 459, 742, 773, 906, 1128, 1583, 1952, 1953, 2006, 2712, 3040, 3726	
B_4	9, 92, 120, 154, 198, 234, 429, 433, 434, 437, 472, 524, 607, 825, 1138, 1558, 1596, 2709	
B_5	43, 68, 221, 245, 246, 296	444, 460, 476, 514, 1123, 1152, 1159, 1162, 2362
Q^3	2, 4, 95	746
$\mathbb{P}^3$	1, 10, 11	
2–4	4031	4024, 4056
2–5	3453, 3736, 3777	3731, 3735, 3762
2–6	3319, 3349, 3416	3756, 3790, 3846, 3857, 3875, 3933, 4004, 4043, 4229, 4236, 4241, 4249
2–7	3102, 3133, 3215, 3239, 3484, 3592, 3641, 3813, 4101	2366, 2401, 3130, 3202, 3266, 3448, 3530, 3544, 3559, 3574, 3624, 3629, 3636, 3666, 3698, 3701, 3704, 3718, 3799, 3800, 3824, 3832, 3847, 3855, 3871, 3900, 3925, 3929, 3931, 3935, 3963, 3973, 3977, 4049, 4062, 4098, 4111, 4126, 4127, 4136, 4192, 4201, 4207, 4221
2–8	1969	3137, 3263, 3351, 3996, 4009
2–9	2462, 2606, 2922, 3014, 3100, 3136, 3217, 3242, 3479, 3480, 3782	1560, 2097, 2197, 2567, 2635, 2830, 2831, 2838, 2889, 2907, 2965, 2968, 2988, 3013, 3131, 3265, 3275, 3296, 3312, 3333, 3404, 3413, 3450, 3451, 3455, 3507, 3533, 3538, 3579, 3583, 3621, 3623, 3631, 3680, 3686, 3687, 3720, 3801, 3809, 3826, 3830, 3865, 3915, 3934, 3943, 3989, 4030, 4033, 4067, 4125, 4137, 4151, 4208

Name	Interior Points	Boundary Points
2–10	1972, 2131, 2447, 2746, 2808, 2924, 2928, 2984, 3019, 3036, 3153, 3362, 3387, 3481, 3487, 3615, 3617, 3789, 4002, 4021	1954, 2093, 2367, 2410, 2414, 2773, 2897, 2967, 2969, 3032, 3129, 3209, 3275, 3339, 3357, 3375, 3376, 3409, 3473, 3477, 3532, 3542, 3601, 3614, 3645, 3703, 3783, 3802, 3803, 3854, 3873, 3914, 3934, 3939, 3972, 3998, 4016, 4072, 4152, 4164, 4165, 4175, 4264, 4302
2–11	1701, 2203, 2815	1212, 2191, 2327, 2561, 2679, 2900, 2910, 2963, 3101, 3148, 3205, 3224, 3394, 3425, 3671, 3774, 3942
2–12	1194, 1197, 1281, 1284, 1497, 1548, 1684, 1885, 2311, 2350, 2356, 2540, 2580, 2743, 2996, 3029, 3077	858, 1181, 1348, 1546, 1552, 1787, 1888, 2011, 2082, 2092, 2225, 2278, 2279, 2300, 2328, 2412, 2549, 2555, 2644, 2654, 2682, 2689, 2698, 2726, 2748, 2749, 2762, 2820, 2890, 2909, 2975, 2976, 3010, 3090, 3134, 3154, 3201, 3271, 3303, 3403, 3495, 3501, 3525, 3540, 3552, 3553, 3554, 3669, 3705, 3758, 3861, 3893, 3980, 4010, 4177, 4188
2–13	1254, 1393, 1415, 1419, 1430, 1662, 1680, 1717, 1757, 1824, 1834, 1874, 1884, 1924, 1932, 2113, 2114, 2213, 2236, 2253, 2295, 2304, 2332, 2333, 2341, 2343, 2347, 2354, 2434, 2448, 2489, 2564, 2604, 2636, 2642, 2649, 2667, 2670, 2672, 2770, 2855, 2859, 2882, 2893, 2931, 2938, 2939, 2940, 2954, 2994, 2995, 3002, 3031, 3158, 3164, 3197, 3256, 3444, 3485, 3557, 3616, 3649, 3654	1136, 1180, 1584, 1694, 1821, 2082, 2092, 2152, 2221, 2278, 2290, 2292, 2297, 2300, 2336, 2411, 2412, 2478, 2483, 2484, 2485, 2519, 2549, 2569, 2645, 2653, 2656, 2681, 2688, 2690, 2698, 2700, 2704, 2721, 2744, 2760, 2762, 2774, 2781, 2821, 2886, 2904, 2916, 2932, 2955, 2972, 2975, 2977, 2979, 2981, 2986, 3010, 3021, 3033, 3090, 3092, 3134, 3135, 3139, 3159, 3176, 3201, 3207, 3208, 3251, 3271, 3286, 3288, 3289, 3295, 3301, 3303, 3311, 3325, 3337, 3353, 3368, 3414, 3422, 3454, 3458, 3468, 3469, 3495, 3501, 3510, 3521, 3524, 3528, 3539, 3540, 3552, 3591, 3600, 3604, 3634, 3650, 3658, 3662, 3669, 3673, 3677, 3690, 3706, 3773, 3807, 3810, 3821, 3838, 3842, 3861, 3893, 3952, 3978, 4011, 4035, 4051, 4108, 4113, 4128, 4212
2–14	1193, 1322, 1416, 1659, 1721, 1835, 1878, 2151, 2233, 2234, 2263, 2312, 2342, 2353, 2524, 2598, 2626, 2693, 2930, 2958, 2978, 3028, 3124, 3252, 3261, 3273, 3388, 3417, 3503, 3663, 3817, 3921	1282, 1323, 1790, 1794, 1821, 2183, 2221, 2292, 2297, 2336, 2484, 2485, 2593, 2631, 2645, 2656, 2681, 2690, 2700, 2704, 2774, 2829, 2904, 2912, 2932, 2972, 2976, 2977, 2979, 2981, 2986, 3021, 3033, 3159, 3174, 3176, 3207, 3208, 3211, 3251, 3253, 3286, 3288, 3289, 3295, 3301, 3311, 3421, 3422, 3454, 3468, 3472, 3482, 3510, 3521, 3528, 3639, 3650, 3658, 3662, 3673, 3677, 3694, 3705, 3706, 3708, 3821, 3838, 3842, 3941, 3957, 3978, 4035, 4051, 4108, 4212
2–15	229, 837, 911, 1240, 1394, 1573, 1599, 1617, 2397	475, 1279, 1386, 1698, 1871, 2083, 2272, 2557, 2560, 2771, 2780, 2908, 3128, 3398, 3597, 4008

Name	Interior Points	Boundary Points
2–16	810, 994, 1016, 1174, 1233, 1464, 1485, 1520, 1669, 1712, 1764, 1883, 1904, 1908, 1940, 1998, 2259, 2264, 2346, 2446, 2546, 2629, 2671, 2809, 2960, 3196, 3347	782, 1135, 1272, 1420, 1427, 1603, 1785, 1818, 1832, 1907, 1926, 2219, 2273, 2274, 2275, 2284, 2313, 2331, 2349, 2373, 2374, 2375, 2419, 2426, 2459, 2486, 2590, 2647, 2686, 2853, 2892, 2914, 2918, 2926, 2974, 2992, 3017, 3204, 3225, 3234, 3245, 3270, 3334, 3363, 3365, 3426, 3594, 3630, 3907, 3910, 4155
2–17	527, 666, 836, 931, 942, 989, 1037, 1097, 1301, 1315, 1406, 1455, 1473, 1483, 1496, 1499, 1516, 1521, 1523, 1528, 1746, 1778, 1808, 1836, 1856, 1868, 1869, 1939, 2048, 2140, 2146, 2164, 2182, 2202, 2220, 2251, 2252, 2286, 2303, 2325, 2339, 2344, 2490, 2497, 2517, 2628, 2652, 2873, 2879, 2881, 2915, 3120, 3165, 3212, 3438, 3909	941, 1294, 1303, 1304, 1424, 1426, 1488, 1710, 1813, 1815, 1819, 1826, 1892, 1906, 1927, 2025, 2034, 2036, 2037, 2099, 2216, 2217, 2247, 2277, 2281, 2294, 2316, 2318, 2319, 2345, 2352, 2460, 2501, 2506, 2558, 2563, 2587, 2589, 2596, 2619, 2677, 2685, 2687, 2764, 2827, 2841, 2843, 2856, 2895, 2925, 2946, 2950, 2951, 2993, 2998, 3098, 3146, 3194, 3223, 3231, 3246, 3260, 3292, 3304, 3369, 3435, 3490, 3644, 3691, 3741, 3780, 3840, 3905, 3913, 3956
2–18	449, 451, 628, 702, 808, 1033, 1073, 1090, 1250, 1373, 1441, 1466, 1999	833, 909, 1288, 1478, 1779, 1783, 1876, 1923, 1955, 1987, 1988, 2020, 2085, 2111, 2112, 2280, 2433, 2511, 2574, 2641, 2828, 2857, 3095, 3097, 3108, 3367, 3743, 3779, 4100
2–19	338, 568, 572, 691, 882, 993, 1014, 1024, 1109, 1230, 1312, 1369, 1392, 1490, 1762, 1796, 2066, 2173, 2445	928, 934, 1306, 1345, 1414, 1476, 1590, 1740, 1782, 1817, 1875, 1877, 2104, 2109, 2141, 2195, 2475, 2515, 2562, 2651, 2867, 3145, 3169, 3358
2–20	619, 655, 701, 921, 1023, 1030, 1064, 1089, 1098, 1099, 1110, 1112, 1186, 1320, 1364, 1409, 1423, 1438, 1453, 1458, 1470, 1479, 1502, 1511, 1513, 1525, 1705, 1724, 1755, 1807, 1811, 1891, 1910, 1917, 2139, 2142, 2178, 2209, 2238, 2261, 2267, 2308, 2323, 2496, 2507, 2543, 2586, 2614, 2624, 2630, 3125, 3183, 3199	922, 940, 1018, 1298, 1310, 1487, 1495, 1634, 1703, 1795, 1800, 1804, 1831, 1872, 1881, 1895, 1896, 1897, 1905, 1928, 1936, 1941, 2122, 2133, 2215, 2227, 2276, 2306, 2421, 2435, 2518, 2581, 2582, 2584, 2588, 2612, 2616, 2651, 2668, 2673, 2674, 2676, 2695, 2846, 2850, 2851, 2854, 2869, 2913, 2943, 2945, 2957, 3005, 3103, 3106, 3145, 3173, 3185, 3188, 3259, 3377, 3491, 3492, 3502, 3590, 3653, 3655, 3661, 3776, 3822, 3885, 4040
2–21	123, 238, 295, 479, 626, 700, 703, 731, 733, 910, 918, 964, 1095, 1096, 1102, 1103, 1111, 1184, 1188, 1228, 1361, 1375, 1436, 1446, 1510, 1753, 1915, 1920, 2039, 2067, 2165, 2254, 2729	464, 513, 829, 1120, 1133, 1134, 1147, 1150, 1190, 1219, 1389, 1400, 1405, 1518, 1654, 1706, 1723, 1847, 1873, 1893, 1894, 1935, 1985, 2087, 2106, 2144, 2147, 2194, 2245, 2320, 2334, 2370, 2415, 2432, 2474, 2491, 2599, 2675, 2730, 2784, 2785, 2868, 2949, 2956, 3003, 3006, 3104, 3152, 3181, 3327, 3340, 3378, 3429, 3659, 3745, 3747, 3886

Name	Interior Points	Boundary Points
2–22	373, 389, 414, 547, 574, 662, 697, 714, 729, 870, 955, 1046, 1054, 1065, 1094, 1101, 1372, 1440, 1447, 1672, 1749, 1857, 2172, 2450	233, 785, 831, 886, 915, 926, 938, 1087, 1187, 1270, 1291, 1388, 1401, 1493, 1601, 1655, 1670, 1696, 1802, 1812, 1852, 1853, 1854, 1934, 2030, 2031, 2032, 2053, 2136, 2158, 2317, 2430, 2529, 2611, 2878, 3189, 3788
2–23	40, 266, 300, 304, 372, 411, 560, 690, 765, 806, 807, 933, 1165, 1229, 1581	521, 789, 857, 917, 927, 1036, 1341, 1346, 1413, 1475, 1621, 1625, 1781, 1849, 2057, 2123, 2162, 2510, 2572, 2782, 3070, 3372
2–24	322, 368, 412, 631, 642, 706, 973	618, 1015, 1035, 1085, 1086, 1206, 1296, 1343, 1410, 1411, 1492, 1814, 1846, 2161, 2239, 2242, 2243, 2505, 2531, 2848, 3191
2–25	108, 176, 199, 251, 384, 410, 483, 517, 549, 565, 627, 652, 685, 689, 879, 1163, 1170	461, 520, 830, 916, 1027, 1285, 1339, 1432, 1474, 1626, 1632, 1645, 2054, 2797
2–26	163, 175, 202, 387, 388, 413, 481, 554, 646, 711, 734, 957, 996, 1076, 1263, 1434, 1758	542, 616, 855, 913, 1031, 1034, 1336, 1387, 1407, 1570, 1643, 1649, 1738, 2138, 2231, 2526, 2740, 2804
2–27	71, 157, 165, 200, 201, 305, 321, 328, 375, 385, 814, 899	948, 1334, 2063
2–28	34, 55, 69	537, 1567
2–29	19, 56, 72, 106, 131, 171, 204, 253, 307, 367, 381, 503	1146, 2389
2–30	14, 23, 37, 160, 225, 273	241, 1125
2–31	21, 46, 70	535
2–32	13, 22	90, 104, 122, 156, 249
2–33	7, 50, 140	
2–34	5, 24	
2–35	6, 118	33, 42
2–36	8	
3–1	3329, 3350	3795, 3846, 3875, 3967, 4027, 4236, 4241, 4249
3–2	2570, 2791	
3–3	1307, 1726, 1805, 1833, 2070, 2120, 2299, 2594, 2602, 2638, 2678, 2941, 3001, 3166, 3178, 3445	1616, 1692, 2003, 2108, 2270, 2301, 2565, 2566, 2571, 2579, 2595, 2680, 2691, 2763, 2964, 2980, 3037, 3038, 3141, 3144, 3161, 3205, 3226, 3237, 3267, 3276, 3278, 3281, 3293, 3308, 3423, 3456, 3460, 3467, 3517, 3520, 3534, 3646, 3678, 3684, 3716, 3722, 3768, 3769, 3829, 3834, 3835, 3839, 3948, 3969, 3987, 4053, 4142
3–4	1619, 1725, 1823, 2005, 2222, 2224, 2298, 2314, 2394, 2493, 2544, 2603, 2640, 2807, 2927, 2942, 3179, 3348	1692, 1975, 2003, 2270, 2377, 2525, 2566, 2579, 2803, 2898, 2964, 3011, 3141, 3267, 3342, 3460, 3466, 3467, 3508, 3588, 3608, 3627, 3684, 3768, 3770, 3835, 3930, 3948, 4096, 4097, 4110, 4242

Name	Interior Points	Boundary Points
3–5	1326, 1367, 1820, 1837, 2128, 2218, 2492	2183, 2631, 2829, 2912, 3211, 3472, 3639
3–6	932, 997, 1318, 1357, 1429, 1437, 1501, 1720, 1773, 1829, 1861, 1900, 1911, 1931, 1938, 2166, 2167, 2269, 2296, 2321, 2326, 2479, 2542, 2620, 2622, 2625, 2627, 2929, 3123, 3195, 3241, 3424	1313, 1604, 1677, 1788, 1880, 1909, 2121, 2171, 2196, 2228, 2229, 2330, 2335, 2351, 2466, 2559, 2590, 2597, 2600, 2618, 2686, 2692, 2905, 2917, 2933, 2936, 3016, 3018, 3022, 3091, 3147, 3157, 3204, 3234, 3238, 3240, 3250, 3255, 3272, 3370, 3381, 3426, 3432, 3459, 3500, 3656, 3657, 3715, 3841, 3907, 3918, 4055, 4129
3–7	263, 630, 959, 1025, 1047, 1055, 1075, 1237, 1247, 1321, 1374, 1417, 1448, 1462, 1522, 1526, 1529, 1912, 1919, 1922, 2021, 2069, 2226, 2256, 2262, 2310, 2340, 2512, 2959	767, 1236, 1303, 1395, 1461, 1660, 1710, 1718, 1772, 1813, 1815, 1879, 1892, 1925, 1933, 1971, 2028, 2034, 2036, 2085, 2119, 2148, 2153, 2207, 2210, 2211, 2216, 2217, 2258, 2265, 2277, 2280, 2293, 2315, 2329, 2338, 2345, 2348, 2352, 2355, 2444, 2488, 2506, 2563, 2589, 2615, 2621, 2639, 2660, 2677, 2687, 2734, 2828, 2841, 2843, 2946, 2961, 2971, 2983, 2991, 2993, 2997, 2998, 3007, 3050, 3076, 3095, 3107, 3108, 3156, 3171, 3172, 3175, 3220, 3223, 3244, 3257, 3304, 3430, 3435, 3436, 3498, 3606, 3691, 3743, 3771, 3779, 3814, 3818, 4036
3–8	984, 1049, 1082, 1456, 1505, 1506, 1776, 1866, 1867, 1913, 2257, 2541	1395, 1425, 1772, 1830, 1925, 1933, 2105, 2119, 2149, 2150, 2207, 2265, 2315, 2583, 2587, 2621, 2639, 2660, 2685, 2845, 2997, 3171, 3186, 3244, 3246, 3369, 3430, 3490, 3956
3–9	289, 344, 354, 374, 447	1385, 1535, 1582, 2720, 2727, 3556, 3997
3–10	267, 336, 342, 632, 821, 890, 969, 976, 983, 1050, 1078, 1100, 1113, 1266, 1463, 1469, 1500, 1512, 1515, 1527, 1577, 1674, 1678, 1745, 1774, 1862, 1882, 1921, 2169, 2266, 2324, 2395, 2452, 2545, 2591, 2811, 2884, 3571	835, 922, 1298, 1486, 1542, 1555, 1608, 1634, 1771, 1795, 1804, 1881, 1895, 1896, 1897, 1902, 1903, 1928, 1936, 1941, 2088, 2122, 2170, 2206, 2215, 2420, 2435, 2552, 2568, 2584, 2588, 2616, 2659, 2668, 2673, 2674, 2676, 2695, 2767, 2776, 2851, 2869, 2944, 2945, 2957, 3005, 3109, 3150, 3169, 3173, 3185, 3259, 3491, 3492, 3497, 3502, 3561, 3562, 3590, 3661, 3787, 3822, 4040
3–11	401, 653, 656, 657, 724, 730, 732, 980, 985, 1005, 1009, 1079, 1088, 1452, 1465, 1489, 1754, 1761, 1769	567, 880, 925, 1069, 1295, 1297, 1391, 1402, 1404, 1459, 1477, 1480, 1517, 1519, 1602, 1624, 1675, 1733, 1752, 1803, 1848, 1850, 1865, 1918, 2143, 2168, 2193, 2230, 2285, 2491, 2513, 2536, 2547, 2623, 2877, 2947, 3170, 3380

Name	Interior Points	Boundary Points
3–12	592, 654, 665, 704, 723, 738, 1052, 1070, 1071, 1104, 1107, 1114, 1381, 1435, 1442, 1457, 1472, 1514, 1770, 1806, 1870, 1916, 2110, 2255, 2260, 2514, 2814	829, 925, 1017, 1074, 1147, 1405, 1421, 1422, 1477, 1480, 1481, 1494, 1517, 1706, 1714, 1723, 1733, 1756, 1799, 1801, 1825, 1850, 1865, 1893, 1894, 1935, 2029, 2118, 2144, 2147, 2193, 2208, 2230, 2244, 2283, 2285, 2305, 2320, 2522, 2532, 2576, 2599, 2607, 2609, 2623, 2675, 2766, 2784, 2785, 2786, 2836, 2861, 2866, 2868, 2870, 2874, 2877, 2947, 2948, 3006, 3170, 3232, 3384, 3429, 3584, 3607, 3747
3–13	93, 129, 370, 421, 556, 636, 737, 739, 977, 982, 1063, 1105, 1765	749, 757, 777, 802, 831, 937, 974, 1043, 1085, 1235, 1296, 1396, 1498, 1508, 1524, 1578, 1630, 1673, 1767, 1768, 1812, 1852, 1859, 1914, 1934, 1951, 2031, 2176, 2192, 2241, 2322, 2505, 2530, 2798, 2812, 2860, 3039, 3177, 3441
3–14	143, 186, 203, 308	522, 686, 881, 949, 1013, 1201, 1340, 1342, 1719, 2156, 2416, 2439
3–15	182, 316, 371, 402, 420, 422, 423, 561, 564, 598, 677, 698, 699, 712, 713, 717, 719, 736, 954, 1061, 1066, 1068, 1077, 1108, 1259, 1356, 1378, 1382, 1439, 1744, 1759, 1843, 2125	924, 1032, 1057, 1091, 1249, 1289, 1408, 1433, 1444, 1468, 1491, 1507, 1509, 1627, 1732, 1766, 1809, 1810, 1838, 1844, 1890, 2137, 2159, 2232, 2240, 2449, 2516, 2527, 2601, 2608, 2610, 2613, 2775, 2802, 2805, 2865, 2876, 3116, 3187, 3437, 3439, 3609, 3751
3–16	213, 323, 369, 392, 417, 418, 615, 640, 648, 708, 716, 783, 950, 1000, 1058, 1316, 1376, 1742, 2062	478, 573, 624, 643, 649, 696, 972, 1042, 1062, 1092, 1093, 1338, 1358, 1359, 1380, 1443, 1451, 1664, 1734, 1840, 1863, 2175, 2177, 2438, 2521, 2528, 2747, 2790, 2875, 3382
3–17	130, 161, 209, 210, 405, 415, 587	297, 470, 692, 707, 794, 828, 832, 853, 920, 946, 1056, 1222, 1403, 1737, 1990, 2038, 2064, 2806
3–18	63, 212, 326, 341, 383, 393, 419, 597, 639, 709, 970, 1053, 1245	552, 613, 641, 710, 1029, 1332, 1344, 1736, 1839, 2043, 2155, 2440
3–19	35, 57, 75, 164, 206, 270, 283, 339	303, 364, 471, 485, 536, 695, 967, 1041, 1258, 1532, 1566, 1702, 2391
3–20	44, 80, 174, 208, 211, 259, 404, 416, 667, 715, 888	255, 313, 548, 693, 818, 998, 1040, 1164, 1349, 1431, 1566, 1641, 2004, 2793
3–21	173, 184, 214, 382, 488, 558, 897	612, 647, 847, 947, 951, 1026, 1278, 1319, 1562, 1640, 2061, 2127, 2135, 3111
3–22	64, 76, 139	363, 611, 635, 943, 1210, 1743, 2065
3–23	77, 178, 205	127, 365, 585, 694, 850, 876
3–24	78, 79, 169, 207, 379	315, 377, 538, 584, 634, 919, 995, 1242
3–25	25, 47, 74, 136, 187	366, 462, 469, 486, 533, 589, 1563
3–26	26, 73	167, 306, 376, 621
3–27	18, 31, 133	104, 122, 156

Name	Interior Points	Boundary Points
3–28	30, 52, 81, 166	
3–29	27	132, 177
3–30	29	168, 276, 314
3–31	20, 28	240, 1118
4–1	489, 495, 578, 610, 1503, 1530, 1775, 2201, 2810	1191, 1236, 1276, 1461, 1660, 1987, 1991, 2023, 2041, 2153, 2211, 2258, 2338, 2348, 2355, 2619, 2734, 2880, 3007, 3257, 3490, 3814
4–2	602, 668, 1048, 1081, 1365, 1454, 1504, 1864	885, 936, 1486, 1574, 1663, 1668, 1771, 1800, 1902, 2206, 2306, 2537, 2581, 2659, 2944, 3106
4–3	109, 153, 603, 664, 684, 728, 735, 822, 1179, 1268, 1269, 1976	464, 513, 1120, 1133, 1134, 1149, 1172, 1213, 1389, 1400, 1460, 1518, 1654, 1658, 2087, 2132, 2396, 2415, 3340
4–4	350, 398, 622, 680, 722, 726, 741, 901, 1039, 1080, 1084, 1106, 1267, 1355, 1449, 1471, 1858, 1860, 2075, 2174, 2179	659, 802, 974, 1043, 1072, 1231, 1235, 1343, 1353, 1498, 1508, 1524, 1767, 1768, 1797, 1846, 1859, 1914, 2158, 2161, 2176, 2204, 2241, 2317, 2322, 2451, 2782, 2798, 2848, 2878, 2883, 3073, 3177, 3189, 3788
4–5	151, 196, 329, 427, 500, 591, 683, 720, 727, 820, 981, 1576	872, 924, 961, 988, 1044, 1249, 1377, 1450, 1491, 1509, 1627, 1648, 1766, 1838, 2159, 2449, 2794, 2797, 2876, 3564, 3565
4–6	406, 409, 426, 651, 661, 681, 721, 904, 1002, 1008, 1067, 1292, 1379, 1682, 1708	968, 1032, 1057, 1091, 1360, 1399, 1408, 1444, 1468, 1507, 1611, 1732, 1798, 1844, 2232, 2240, 2409, 2516, 2527, 2802, 3187
4–7	185, 352, 424, 669, 740, 1010	624, 871, 962, 972, 1060, 1092, 1264, 1358, 1359, 1443, 2175, 2231, 2521
4–8	216, 218, 391, 396, 397, 425, 580, 718, 953, 1350	675, 692, 725, 819, 828, 1056, 1059, 1243, 1445, 1467, 1629, 1667, 1841, 2064, 2154, 2399, 2538
4–9	67, 217, 291, 346, 395	255, 313, 548, 551, 693, 817, 945, 998, 1040, 1164, 1349, 1653
4–10	82, 180, 215, 320, 390, 394, 638	403, 623, 644, 958, 1001, 1331, 1362, 1607, 1731, 2422
4–11	62, 85, 191	315, 319, 377, 584, 634, 637, 995
4–12	83, 181, 190, 309	944, 1330
4–13	60, 84	269, 311, 504, 588
5–1	285, 359, 673, 903, 1007, 1083, 1354	1166, 1248, 1676, 1799, 2305, 2609, 3115
5–2	194, 220, 348, 408, 678, 898	596, 676, 1059, 1178, 1243, 1332, 1445, 1839, 2042, 2155
5–3	114, 150, 195, 219	484, 559, 675, 725, 794, 819, 853, 1966
6–1	284, 357	1231, 1353
7–1	454, 506, 510	1991, 2023, 2041
8–1	769	

Appendix C. Tangent space dimension

Table K. In the following table, we record the tangent space dimension $h^0(\mathcal{N})$ of the Hilbert scheme points corresponding to anticanonically-embedded Gorenstein toric Fano threefolds. Here "ID" is the Reflexive ID of the corresponding three-dimensional reflexive polytope.

ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$
1	1209	34	525	67	478	100	573	133	719	166	719
2	889	35	479	68	525	101	440	134	618	167	668
3	1501	36	772	69	525	102	1073	135	569	168	774
4	889	37	668	70	668	103	1074	136	617	169	569
5	888	38	668	71	480	104	721	137	480	170	618
6	948	39	1139	72	524	105	1278	138	439	171	524
7	888	40	325	73	667	106	524	139	523	172	569
8	1140	41	570	74	617	107	524	140	888	173	479
9	363	42	949	75	479	108	361	141	829	174	479
10	1209	43	525	76	523	109	289	142	773	175	398
11	1209	44	479	77	569	110	885	143	360	176	361
12	1209	45	947	78	569	111	717	144	665	177	774
13	720	46	668	79	569	112	567	145	616	178	569
14	668	47	617	80	479	113	568	146	521	179	616
15	947	48	888	81	719	114	435	147	567	180	522
16	947	49	774	82	522	115	439	148	396	181	616
17	1010	50	888	83	616	116	437	149	478	182	359
18	719	51	1010	84	666	117	324	150	435	183	522
19	524	52	719	85	568	118	948	151	358	184	479
20	829	53	667	86	442	119	442	152	435	185	396
21	668	54	668	87	234	120	363	153	289	186	360
22	720	55	525	88	442	121	618	154	363	187	617
23	668	56	524	89	1501	122	721	155	439	188	617
24	888	57	479	90	721	123	290	156	721	189	522
25	617	58	828	91	442	124	438	157	480	190	616
26	667	59	616	92	363	125	718	158	774	191	568
27	773	60	666	93	323	126	887	159	570	192	521
28	829	61	718	94	722	127	570	160	668	193	521
29	773	62	568	95	889	128	398	161	437	194	435
30	719	63	437	96	719	129	323	162	437	195	435
31	719	64	523	97	526	130	437	163	398	196	358
32	777	65	772	98	722	131	524	164	479	197	435
33	949	66	666	99	442	132	774	165	480	198	363

ID	$h^0(\mathcal{N})$
199	361
200	480
201	480
202	398
203	360
204	524
205	569
206	479
207	569
208	479
209	437
210	437
211	479
212	437
213	397
214	479
215	522
216	436
217	478
218	436
219	435
220	435
221	525
222	1209
223	669
224	401
225	668
226	400
227	667
228	293
229	204
230	256
231	668
232	234
233	325
234	363
235	401
236	260
237	481
238	290
239	829
240	830
241	669
242	526
243	483
244	571
245	525
246	525
247	294
248	1010
249	721
250	669
251	361
252	569
253	524
254	440
255	480
256	481
257	479
258	667
259	479
260	293
261	398
262	293
263	228
264	396
265	291
266	325
267	257
268	324
269	667
270	479
271	617
272	665
273	668
274	521
275	396
276	774
277	397
278	522
279	476
280	434
281	395
282	356
283	479
284	320
285	287
286	257
287	323
288	358
289	260
290	256
291	478
292	571
293	293
294	401
295	290
296	525
297	438
298	399
299	440
300	325
301	618
302	570
303	480
304	325
305	480
306	668
307	524
308	360
309	616
310	667
311	667
312	719
313	480
314	774
315	570
316	359
317	570
318	617
319	569
320	522
321	480
322	324
323	397
324	478
325	437
326	437
327	398
328	480
329	358
330	361
331	397
332	398
333	290
334	324
335	396
336	257
337	438
338	260
339	479
340	358
341	437
342	257
343	289
344	260
345	522
346	478
347	521
348	435
349	396
350	322
351	436
352	396
353	359
354	260
355	394
356	356
357	320
358	321
359	287
360	399
361	398
362	438
363	524
364	480
365	570
366	618
367	524
368	324
369	397
370	323
371	359
372	325
373	324
374	260
375	480
376	668
377	570
378	570
379	569
380	480
381	524
382	479
383	437
384	361
385	480
386	437
387	398
388	398
389	324
390	522
391	436
392	397
393	437
394	522
395	478
396	436
397	436
398	322
399	396
400	396
401	290
402	359
403	523
404	479
405	437
406	358
407	358
408	435
409	358
410	361
411	325
412	324
413	398
414	324
415	437
416	479
417	397
418	397
419	437
420	359
421	323
422	359
423	359
424	396
425	436
426	358
427	358
428	139
429	363
430	139
431	366
432	366

ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$
433	363	472	363	511	225	550	523	589	618	628	230
434	363	473	619	512	224	551	479	590	396	629	290
435	362	474	1011	513	291	552	438	591	358	630	228
436	234	475	205	514	526	553	399	592	289	631	324
437	363	476	526	515	206	554	398	593	321	632	257
438	177	477	949	516	260	555	437	594	394	633	523
439	363	478	398	517	361	556	323	595	356	634	570
440	363	479	290	518	326	557	289	596	436	635	524
441	227	480	290	519	290	558	479	597	437	636	323
442	949	481	398	520	362	559	436	598	359	637	569
443	364	482	361	521	326	560	325	599	289	638	522
444	527	483	361	522	361	561	359	600	288	639	437
445	291	484	436	523	260	562	480	601	396	640	397
446	618	485	480	524	363	563	324	602	256	641	438
447	260	486	618	525	261	564	359	603	289	642	324
448	224	487	176	526	294	565	361	604	440	643	398
449	230	488	479	527	229	566	398	605	399	644	523
450	176	489	227	528	481	567	291	606	260	645	480
451	230	490	227	529	206	568	260	607	363	646	398
452	287	491	206	530	830	569	202	608	294	647	480
453	287	492	201	531	440	570	436	609	401	648	397
454	223	493	206	532	618	571	289	610	227	649	398
455	206	494	227	533	618	572	260	611	524	650	361
456	175	495	227	534	667	573	398	612	480	651	358
457	288	496	438	535	669	574	324	613	438	652	361
458	293	497	291	536	481	575	231	614	399	653	290
459	234	498	570	537	526	576	616	615	397	654	289
460	527	499	478	538	571	577	228	616	399	655	258
461	362	500	358	539	401	578	227	617	324	656	290
462	618	501	258	540	399	579	227	618	325	657	290
463	260	502	228	541	440	580	436	619	258	658	359
464	291	503	524	542	399	581	438	620	261	659	323
465	362	504	667	543	399	582	523	621	668	660	359
466	618	505	255	544	363	583	478	622	322	661	358
467	774	506	223	545	774	584	570	623	523	662	324
468	618	507	254	546	401	585	570	624	398	663	291
469	618	508	224	547	324	586	521	625	437	664	289
470	438	509	198	548	480	587	437	626	290	665	289
471	481	510	223	549	361	588	668	627	361	666	229

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667	479	706	324	745	161	784	440	823	234	862	398
668	256	707	438	746	890	785	327	824	260	863	228
669	396	708	397	747	442	786	260	825	363	864	228
670	289	709	437	748	721	787	571	826	397	865	201
671	288	710	438	749	326	788	326	827	326	866	201
672	321	711	398	750	329	789	326	828	438	867	775
673	287	712	359	751	231	790	258	829	291	868	570
674	257	713	359	752	161	791	326	830	362	869	616
675	437	714	324	753	161	792	327	831	325	870	324
676	436	715	479	754	161	793	888	832	439	871	397
677	359	716	397	755	153	794	438	833	231	872	360
678	435	717	359	756	153	795	363	834	180	873	290
679	396	718	436	757	325	796	261	835	258	874	324
680	322	719	359	758	570	797	571	836	229	875	396
681	358	720	358	759	573	798	721	837	204	876	570
682	289	721	358	760	156	799	570	838	294	877	224
683	358	722	322	761	329	800	401	839	201	878	225
684	289	723	289	762	570	801	480	840	570	879	361
685	361	724	290	763	722	802	324	841	177	880	291
686	361	725	437	764	571	803	204	842	201	881	361
687	290	726	322	765	325	804	254	843	201	882	260
688	361	727	358	766	437	805	324	844	401	883	198
689	361	728	289	767	229	806	325	845	401	884	175
690	325	729	324	768	324	807	325	846	440	885	257
691	260	730	290	769	144	808	230	847	482	886	325
692	438	731	290	770	147	809	198	848	719	887	359
693	480	732	290	771	150	810	202	849	569	888	479
694	570	733	290	772	568	811	396	850	570	889	324
695	480	734	398	773	234	812	324	851	399	890	257
696	398	735	289	774	161	813	398	852	440	891	200
697	324	736	359	775	161	814	480	853	438	892	200
698	359	737	323	776	161	815	200	854	439	893	198
699	359	738	289	777	325	816	255	855	400	894	198
700	290	739	323	778	201	817	479	856	294	895	437
701	258	740	396	779	205	818	480	857	326	896	480
702	230	741	322	780	156	819	437	858	178	897	479
703	290	742	234	781	156	820	358	859	259	898	435
704	289	743	157	782	203	821	257	860	481	899	480
705	289	744	161	783	397	822	289	861	570	900	361

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901	322	940	259	979	361	1018	259	1057	360	1096	290
902	321	941	230	980	290	1019	291	1058	397	1097	229
903	287	942	229	981	358	1020	231	1059	437	1098	258
904	358	943	524	982	323	1021	202	1060	397	1099	258
905	228	944	618	983	257	1022	201	1061	359	1100	257
906	234	945	479	984	227	1023	258	1062	398	1101	324
907	294	946	438	985	290	1024	260	1063	323	1102	290
908	294	947	480	986	257	1025	228	1064	258	1103	290
909	231	948	481	987	290	1026	480	1065	324	1104	289
910	290	949	362	988	360	1027	362	1066	359	1105	323
911	204	950	397	989	229	1028	201	1067	358	1106	322
912	438	951	481	990	291	1029	438	1068	359	1107	289
913	399	952	261	991	231	1030	258	1069	291	1108	359
914	327	953	436	992	258	1031	399	1070	289	1109	260
915	325	954	359	993	260	1032	360	1071	289	1110	258
916	362	955	324	994	202	1033	230	1072	323	1111	290
917	326	956	202	995	570	1034	399	1073	230	1112	258
918	290	957	398	996	398	1035	325	1074	290	1113	257
919	571	958	523	997	200	1036	326	1075	228	1114	289
920	438	959	228	998	480	1037	229	1076	398	1115	527
921	258	960	437	999	437	1038	290	1077	359	1116	294
922	259	961	360	1000	397	1039	322	1078	257	1117	362
923	261	962	397	1001	524	1040	480	1079	290	1118	831
924	360	963	359	1002	358	1041	480	1080	322	1119	234
925	291	964	290	1003	228	1042	398	1081	256	1120	291
926	325	965	290	1004	257	1043	324	1082	227	1121	294
927	326	966	321	1005	290	1044	359	1083	287	1122	139
928	261	967	480	1006	321	1045	228	1084	322	1123	527
929	200	968	359	1007	287	1046	324	1085	325	1124	526
930	201	969	257	1008	358	1047	228	1086	325	1125	670
931	229	970	437	1009	290	1048	256	1087	325	1126	527
932	200	971	396	1010	396	1049	227	1088	290	1127	361
933	325	972	398	1011	228	1050	257	1089	258	1128	234
934	261	973	324	1012	261	1051	257	1090	230	1129	234
935	180	974	324	1013	361	1052	289	1091	360	1130	139
936	257	975	288	1014	260	1053	437	1092	398	1131	139
937	325	976	257	1015	325	1054	324	1093	398	1132	294
938	325	977	323	1016	202	1055	228	1094	324	1133	291
939	359	978	260	1017	290	1056	438	1095	290	1134	291

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1136	178	1175	225	1214	232	1253	175	1292	358
1137	178	1176	224	1215	201	1254	176	1293	176
1138	363	1177	176	1216	292	1255	175	1294	230
1139	399	1178	436	1217	177	1256	359	1295	291
1140	401	1179	289	1218	177	1257	398	1296	325
1141	201	1180	178	1219	291	1258	480	1297	291
1142	294	1181	178	1220	228	1259	359	1298	259
1143	204	1182	231	1221	668	1260	228	1299	261
1144	669	1183	139	1222	438	1261	291	1300	175
1145	523	1184	290	1223	571	1262	225	1301	229
1146	526	1185	291	1224	440	1263	398	1302	261
1147	291	1186	258	1225	399	1264	397	1303	230
1148	177	1187	325	1226	399	1265	257	1304	230
1149	291	1188	290	1227	570	1266	257	1305	198
1150	291	1189	260	1228	290	1267	322	1306	261
1151	228	1190	292	1229	325	1268	289	1307	152
1152	526	1191	228	1230	260	1269	289	1308	228
1153	398	1192	438	1231	323	1270	327	1309	230
1154	294	1193	175	1232	198	1271	260	1310	260
1155	397	1194	177	1233	202	1272	203	1311	180
1156	292	1195	177	1234	175	1273	177	1312	260
1157	231	1196	177	1235	324	1274	205	1313	202
1158	526	1197	177	1236	229	1275	156	1314	323
1159	526	1198	178	1237	228	1276	228	1315	229
1160	399	1199	156	1238	290	1277	201	1316	397
1161	294	1200	153	1239	324	1278	482	1317	261
1162	526	1201	361	1240	204	1279	205	1318	200
1163	361	1202	775	1241	260	1280	399	1319	481
1164	480	1203	399	1242	570	1281	177	1320	258
1165	325	1204	294	1243	437	1282	176	1321	228
1166	289	1205	294	1244	437	1283	177	1322	175
1167	522	1206	325	1245	437	1284	177	1323	176
1168	175	1207	261	1246	224	1285	362	1324	175
1169	398	1208	261	1247	228	1286	229	1325	201
1170	361	1209	229	1248	289	1287	177	1326	173
1171	396	1210	524	1249	360	1288	231	1327	175
1172	290	1211	291	1250	230	1289	360	1328	397
1173	289	1212	156	1251	176	1290	325	1329	399

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1330	618	1369	260	1408	360	1447	324	1486	258
1331	523	1370	231	1409	258	1448	228	1487	259
1332	438	1371	202	1410	325	1449	322	1488	230
1333	480	1372	324	1411	325	1450	359	1489	290
1334	481	1373	230	1412	290	1451	398	1490	260
1335	399	1374	228	1413	326	1452	290	1491	360
1336	399	1375	290	1414	261	1453	258	1492	325
1337	361	1376	397	1415	176	1454	256	1493	325
1338	398	1377	359	1416	175	1455	229	1494	290
1339	362	1378	359	1417	228	1456	227	1495	259
1340	362	1379	358	1418	203	1457	289	1496	229
1341	326	1380	398	1419	176	1458	258	1497	177
1342	361	1381	289	1420	203	1459	291	1498	324
1343	325	1382	359	1421	291	1460	290	1499	229
1344	438	1383	257	1422	290	1461	229	1500	257
1345	261	1384	180	1423	258	1462	228	1501	200
1346	326	1385	262	1424	230	1463	257	1502	258
1347	619	1386	205	1425	228	1464	202	1503	227
1348	178	1387	400	1426	230	1465	290	1504	256
1349	480	1388	325	1427	203	1466	230	1505	227
1350	436	1389	291	1428	201	1467	438	1506	227
1351	359	1390	361	1429	200	1468	360	1507	360
1352	291	1391	291	1430	176	1469	257	1508	324
1353	323	1392	260	1431	481	1470	258	1509	360
1354	287	1393	176	1432	362	1471	322	1510	290
1355	322	1394	204	1433	360	1472	289	1511	258
1356	359	1395	229	1434	398	1473	229	1512	257
1357	200	1396	325	1435	289	1474	362	1513	258
1358	398	1397	231	1436	290	1475	326	1514	289
1359	398	1398	202	1437	200	1476	261	1515	257
1360	359	1399	360	1438	258	1477	291	1516	229
1361	290	1400	291	1439	359	1478	231	1517	291
1362	524	1401	325	1440	324	1479	258	1518	291
1363	198	1402	292	1441	230	1480	291	1519	291
1364	258	1403	439	1442	289	1481	290	1520	202
1365	256	1404	291	1443	398	1482	201	1521	229
1366	204	1405	291	1444	360	1483	229	1522	228
1367	173	1406	229	1445	437	1484	201	1523	229
1368	198	1407	399	1446	290	1485	202	1524	324

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1526	228	1565	479	1604	202	1643	400	1682	358
1527	257	1566	481	1605	127	1644	399	1683	139
1528	229	1567	526	1606	291	1645	362	1684	177
1529	228	1568	363	1607	523	1646	362	1685	294
1530	227	1569	261	1608	258	1647	361	1686	156
1531	261	1570	400	1609	198	1648	360	1687	177
1532	482	1571	618	1610	202	1649	400	1688	205
1533	260	1572	481	1611	359	1650	290	1689	260
1534	363	1573	204	1612	205	1651	128	1690	156
1535	262	1574	257	1613	156	1652	482	1691	134
1536	774	1575	204	1614	156	1653	481	1692	153
1537	119	1576	358	1615	156	1654	291	1693	156
1538	263	1577	257	1616	154	1655	325	1694	178
1539	139	1578	324	1617	204	1656	175	1695	154
1540	260	1579	198	1618	204	1657	257	1696	327
1541	362	1580	325	1619	151	1658	290	1697	201
1542	261	1581	325	1620	201	1659	175	1698	205
1543	231	1582	262	1621	327	1660	229	1699	204
1544	260	1583	234	1622	668	1661	228	1700	201
1545	156	1584	178	1623	401	1662	176	1701	155
1546	178	1585	139	1624	291	1663	257	1702	480
1547	201	1586	156	1625	326	1664	398	1703	259
1548	177	1587	155	1626	362	1665	198	1704	399
1549	261	1588	203	1627	360	1666	202	1705	258
1550	260	1589	156	1628	294	1667	438	1706	291
1551	260	1590	261	1629	439	1668	257	1707	258
1552	178	1591	207	1630	326	1669	202	1708	358
1553	262	1592	154	1631	201	1670	325	1709	261
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1555	258	1594	261	1633	198	1672	324	1711	175
1556	327	1595	201	1634	259	1673	324	1712	202
1557	619	1596	363	1635	205	1674	257	1713	202
1558	363	1597	133	1636	177	1675	291	1714	292
1559	153	1598	134	1637	259	1676	289	1715	203
1560	134	1599	204	1638	156	1677	201	1716	203
1561	259	1600	261	1639	362	1678	257	1717	176
1562	482	1601	325	1640	481	1679	231	1718	230
1563	619	1602	291	1641	481	1680	176	1719	361

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1721	175	1760	202	1799	290	1838	360	1877	261
1722	201	1761	290	1800	259	1839	438	1878	175
1723	291	1762	260	1801	290	1840	398	1879	229
1724	258	1763	202	1802	325	1841	439	1880	202
1725	151	1764	202	1803	292	1842	362	1881	259
1726	152	1765	323	1804	259	1843	359	1882	257
1727	156	1766	360	1805	152	1844	360	1883	202
1728	180	1767	324	1806	289	1845	326	1884	176
1729	360	1768	324	1807	258	1846	325	1885	177
1730	571	1769	290	1808	229	1847	291	1886	201
1731	523	1770	289	1809	361	1848	291	1887	201
1732	360	1771	258	1810	361	1849	326	1888	178
1733	291	1772	229	1811	258	1850	291	1889	201
1734	398	1773	200	1812	325	1851	325	1890	361
1735	399	1774	257	1813	230	1852	325	1891	258
1736	438	1775	227	1814	325	1853	325	1892	230
1737	439	1776	227	1815	230	1854	325	1893	291
1738	400	1777	180	1816	201	1855	201	1894	291
1739	325	1778	229	1817	261	1856	229	1895	259
1740	261	1779	231	1818	203	1857	324	1896	259
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1743	524	1782	261	1821	177	1860	322	1899	201
1744	359	1783	231	1822	178	1861	200	1900	200
1745	257	1784	154	1823	151	1862	257	1901	201
1746	229	1785	203	1824	176	1863	398	1902	258
1747	290	1786	202	1825	291	1864	256	1903	259
1748	290	1787	178	1826	230	1865	291	1904	202
1749	324	1788	202	1827	153	1866	227	1905	259
1750	231	1789	156	1828	290	1867	227	1906	230
1751	231	1790	176	1829	200	1868	229	1907	203
1752	291	1791	180	1830	228	1869	229	1908	202
1753	290	1792	154	1831	259	1870	289	1909	201
1754	290	1793	326	1832	203	1871	205	1910	258
1755	258	1794	176	1833	152	1872	260	1911	200
1756	290	1795	259	1834	176	1873	292	1912	228
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2006	234
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2021	228
2022	326
2023	228
2024	132
2025	230
2026	156
2027	177
2028	229
2029	290
2030	325
2031	325

ID	$h^0(\mathcal{N})$
2032	327
2033	326
2034	230
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2036	230
2037	230
2038	439
2039	290
2040	570
2041	228
2042	437
2043	439
2044	154
2045	229
2046	177
2047	180
2048	229
2049	154
2050	177
2051	156
2052	229
2053	326
2054	362
2055	571
2056	439
2057	326
2058	361
2059	327
2060	261
2061	482
2062	397
2063	481
2064	438
2065	524
2066	260
2067	290
2068	175
2069	228
2070	152

ID	$h^0(\mathcal{N})$
2071	228
2072	228
2073	290
2074	260
2075	322
2076	175
2077	139
2078	139
2079	139
2080	139
2081	139
2082	178
2083	205
2084	156
2085	231
2086	180
2087	291
2088	261
2089	204
2090	204
2091	176
2092	178
2093	133
2094	134
2095	134
2096	156
2097	134
2098	177
2099	230
2100	230
2101	178
2102	177
2103	400
2104	261
2105	228
2106	292
2107	231
2108	154
2109	261

ID	$h^0(\mathcal{N})$
2110	289
2111	232
2112	231
2113	176
2114	176
2115	201
2116	153
2117	178
2118	291
2119	229
2120	152
2121	202
2122	259
2123	326
2124	180
2125	359
2126	258
2127	481
2128	173
2129	180
2130	178
2131	131
2132	292
2133	260
2134	360
2135	481
2136	325
2137	362
2138	400
2139	258
2140	229
2141	261
2142	258
2143	291
2144	291
2145	292
2146	229
2147	291
2148	230

ID	$h^0(\mathcal{N})$
2149	228
2150	229
2151	175
2152	178
2153	229
2154	438
2155	438
2156	361
2157	362
2158	325
2159	360
2160	326
2161	325
2162	326
2163	326
2164	229
2165	290
2166	200
2167	200
2168	291
2169	257
2170	259
2171	201
2172	324
2173	260
2174	322
2175	398
2176	324
2177	398
2178	258
2179	322
2180	155
2181	202
2182	229
2183	176
2184	134
2185	203
2186	205
2187	153

ID	$h^0(\mathcal{N})$
2188	154
2189	154
2190	132
2191	156
2192	326
2193	291
2194	292
2195	261
2196	202
2197	134
2198	291
2199	179
2200	176
2201	227
2202	229
2203	155
2204	323
2205	325
2206	258
2207	229
2208	290
2209	258
2210	230
2211	229
2212	201
2213	176
2214	201
2215	259
2216	230
2217	230
2218	173
2219	203
2220	229
2221	177
2222	151
2223	153
2224	151
2225	178
2226	228

ID	$h^0(\mathcal{N})$
2227	260
2228	202
2229	202
2230	291
2231	399
2232	360
2233	175
2234	175
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2237	439
2238	258
2239	325
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2241	324
2242	325
2243	325
2244	292
2245	292
2246	261
2247	230
2248	201
2249	259
2250	261
2251	229
2252	229
2253	176
2254	290
2255	289
2256	228
2257	227
2258	229
2259	202
2260	289
2261	258
2262	228
2263	175
2264	202
2265	229

ID	$h^0(\mathcal{N})$
2266	257
2267	258
2268	291
2269	200
2270	153
2271	154
2272	205
2273	203
2274	203
2275	203
2276	259
2277	230
2278	178
2279	178
2280	231
2281	230
2282	231
2283	292
2284	203
2285	291
2286	229
2287	156
2288	202
2289	153
2290	178
2291	132
2292	177
2293	230
2294	230
2295	176
2296	200
2297	177
2298	151
2299	152
2300	178
2301	153
2302	153
2303	229
2304	176

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2306	259	2345	230	2384	401	2423	201	2462	133
2307	230	2346	202	2385	201	2424	201	2463	155
2308	258	2347	176	2386	205	2425	156	2464	114
2309	201	2348	229	2387	203	2426	203	2465	204
2310	228	2349	203	2388	399	2427	156	2466	202
2311	177	2350	177	2389	526	2428	481	2467	202
2312	175	2351	202	2390	400	2429	524	2468	205
2313	203	2352	230	2391	480	2430	327	2469	154
2314	151	2353	175	2392	204	2431	326	2470	155
2315	229	2354	176	2393	202	2432	292	2471	134
2316	230	2355	229	2394	151	2433	232	2472	115
2317	325	2356	177	2395	257	2434	176	2473	327
2318	230	2357	204	2396	290	2435	259	2474	292
2319	230	2358	401	2397	204	2436	439	2475	261
2320	291	2359	203	2398	204	2437	132	2476	176
2321	200	2360	206	2399	438	2438	398	2477	230
2322	324	2361	670	2400	204	2439	361	2478	177
2323	258	2362	526	2401	115	2440	438	2479	200
2324	257	2363	204	2402	260	2441	362	2480	153
2325	229	2364	139	2403	156	2442	482	2481	178
2326	200	2365	119	2404	139	2443	325	2482	132
2327	156	2366	115	2405	156	2444	229	2483	178
2328	178	2367	132	2406	232	2445	260	2484	177
2329	230	2368	204	2407	204	2446	202	2485	177
2330	202	2369	294	2408	400	2447	131	2486	203
2331	203	2370	292	2409	361	2448	176	2487	325
2332	176	2371	260	2410	132	2449	360	2488	230
2333	176	2372	294	2411	178	2450	324	2489	176
2334	292	2373	203	2412	178	2451	324	2490	229
2335	202	2374	203	2413	179	2452	257	2491	292
2336	177	2375	203	2414	133	2453	157	2492	173
2337	153	2376	205	2415	291	2454	119	2493	151
2338	229	2377	153	2416	362	2455	139	2494	153
2339	229	2378	205	2417	325	2456	139	2495	201
2340	228	2379	156	2418	261	2457	119	2496	258
2341	176	2380	156	2419	203	2458	119	2497	229
2342	175	2381	203	2420	261	2459	203	2498	132
2343	176	2382	399	2421	260	2460	230	2499	156

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2501	230	2540	177	2579	153	2618	202	2657	201
2502	325	2541	227	2580	177	2619	230	2658	153
2503	201	2542	200	2581	259	2620	200	2659	258
2504	260	2543	258	2582	260	2621	229	2660	229
2505	325	2544	151	2583	228	2622	200	2661	201
2506	230	2545	257	2584	259	2623	291	2662	201
2507	258	2546	202	2585	201	2624	258	2663	203
2508	261	2547	291	2586	258	2625	200	2664	201
2509	203	2548	178	2587	230	2626	175	2665	201
2510	326	2549	178	2588	259	2627	200	2666	178
2511	231	2550	119	2589	230	2628	229	2667	176
2512	228	2551	134	2590	203	2629	202	2668	259
2513	291	2552	261	2591	257	2630	258	2669	203
2514	289	2553	115	2592	201	2631	176	2670	176
2515	261	2554	115	2593	177	2632	204	2671	202
2516	360	2555	178	2594	152	2633	154	2672	176
2517	229	2556	156	2595	154	2634	134	2673	259
2518	260	2557	205	2596	230	2635	134	2674	259
2519	177	2558	232	2597	202	2636	176	2675	291
2520	153	2559	201	2598	175	2637	260	2676	259
2521	398	2560	205	2599	291	2638	152	2677	230
2522	291	2561	156	2600	202	2639	229	2678	152
2523	201	2562	261	2601	361	2640	151	2679	156
2524	175	2563	230	2602	152	2641	231	2680	154
2525	153	2564	176	2603	151	2642	176	2681	177
2526	400	2565	154	2604	176	2643	203	2682	178
2527	360	2566	153	2605	153	2644	178	2683	153
2528	398	2567	134	2606	133	2645	177	2684	153
2529	325	2568	259	2607	291	2646	153	2685	230
2530	325	2569	178	2608	361	2647	203	2686	203
2531	325	2570	110	2609	290	2648	153	2687	230
2532	291	2571	154	2610	361	2649	176	2688	177
2533	261	2572	326	2611	325	2650	132	2689	178
2534	325	2573	134	2612	259	2651	261	2690	177
2535	261	2574	231	2613	361	2652	229	2691	154
2536	291	2575	205	2614	258	2653	177	2692	202
2537	258	2576	292	2615	230	2654	178	2693	175
2538	439	2577	231	2616	259	2655	201	2694	201

ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$	ID	$h^0(\mathcal{N})$
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2696	201	2735	178	2774	177	2813	291	2852	201
2697	203	2736	178	2775	361	2814	289	2853	203
2698	178	2737	362	2776	261	2815	155	2854	259
2699	153	2738	177	2777	177	2816	157	2855	176
2700	177	2739	362	2778	177	2817	101	2856	230
2701	153	2740	400	2779	156	2818	119	2857	231
2702	201	2741	482	2780	205	2819	179	2858	153
2703	153	2742	482	2781	178	2820	178	2859	176
2704	177	2743	177	2782	326	2821	178	2860	326
2705	178	2744	179	2783	178	2822	204	2861	291
2706	139	2745	439	2784	291	2823	203	2862	291
2707	362	2746	131	2785	291	2824	115	2863	292
2708	180	2747	398	2786	292	2825	155	2864	326
2709	363	2748	178	2787	230	2826	203	2865	361
2710	178	2749	178	2788	524	2827	230	2866	292
2711	101	2750	139	2789	399	2828	231	2867	261
2712	234	2751	139	2790	399	2829	176	2868	291
2713	139	2752	261	2791	110	2830	134	2869	259
2714	139	2753	115	2792	177	2831	134	2870	292
2715	139	2754	115	2793	481	2832	153	2871	292
2716	139	2755	139	2794	360	2833	134	2872	230
2717	119	2756	98	2795	400	2834	204	2873	229
2718	260	2757	178	2796	205	2835	205	2874	291
2719	180	2758	119	2797	362	2836	292	2875	398
2720	262	2759	134	2798	324	2837	291	2876	360
2721	178	2760	178	2799	327	2838	134	2877	291
2722	179	2761	201	2800	400	2839	178	2878	325
2723	179	2762	178	2801	439	2840	153	2879	229
2724	133	2763	154	2802	360	2841	230	2880	229
2725	115	2764	230	2803	153	2842	201	2881	229
2726	178	2765	115	2804	400	2843	230	2882	176
2727	262	2766	292	2805	361	2844	203	2883	324
2728	438	2767	261	2806	439	2845	228	2884	257
2729	290	2768	178	2807	151	2846	259	2885	155
2730	291	2769	292	2808	131	2847	154	2886	178
2731	115	2770	176	2809	202	2848	325	2887	119
2732	669	2771	205	2810	227	2849	231	2888	134
2733	619	2772	203	2811	257	2850	259	2889	134

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2891	133	2930	175	2969	133	3008	134	3047	156
2892	203	2931	176	2970	155	3009	156	3048	572
2893	176	2932	177	2971	230	3010	178	3049	157
2894	132	2933	202	2972	177	3011	154	3050	233
2895	230	2934	201	2973	156	3012	133	3051	85
2896	134	2935	203	2974	203	3013	134	3052	157
2897	133	2936	202	2975	178	3014	133	3053	101
2898	154	2937	201	2976	178	3015	153	3054	326
2899	132	2938	176	2977	177	3016	202	3055	155
2900	156	2939	176	2978	175	3017	203	3056	115
2901	134	2940	176	2979	177	3018	202	3057	156
2902	153	2941	152	2980	154	3019	131	3058	154
2903	132	2942	151	2981	177	3020	154	3059	327
2904	177	2943	260	2982	153	3021	177	3060	326
2905	202	2944	258	2983	230	3022	202	3061	153
2906	202	2945	259	2984	131	3023	153	3062	204
2907	134	2946	230	2985	153	3024	154	3063	156
2908	205	2947	291	2986	177	3025	153	3064	156
2909	178	2948	292	2987	132	3026	132	3065	325
2910	156	2949	291	2988	134	3027	132	3066	327
2911	153	2950	230	2989	153	3028	175	3067	327
2912	176	2951	230	2990	132	3029	177	3068	326
2913	259	2952	203	2991	230	3030	201	3069	325
2914	203	2953	201	2992	203	3031	176	3070	326
2915	229	2954	176	2993	230	3032	133	3071	440
2916	178	2955	178	2994	176	3033	177	3072	440
2917	202	2956	292	2995	176	3034	132	3073	327
2918	203	2957	259	2996	177	3035	153	3074	326
2919	203	2958	175	2997	229	3036	131	3075	229
2920	203	2959	228	2998	230	3037	154	3076	229
2921	154	2960	202	2999	201	3038	154	3077	177
2922	133	2961	229	3000	178	3039	327	3078	204
2923	154	2962	119	3001	152	3040	234	3079	101
2924	131	2963	156	3002	176	3041	157	3080	157
2925	230	2964	153	3003	292	3042	156	3081	157
2926	203	2965	134	3004	201	3043	101	3082	114
2927	151	2966	115	3005	259	3044	156	3083	119
2928	131	2967	133	3006	291	3045	233	3084	157

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3086	155	3125	258	3164	176	3203	155	3242	133
3087	204	3126	119	3165	229	3204	203	3243	230
3088	155	3127	119	3166	152	3205	156	3244	229
3089	400	3128	205	3167	178	3206	204	3245	203
3090	178	3129	133	3168	260	3207	177	3246	230
3091	201	3130	115	3169	261	3208	177	3247	201
3092	178	3131	134	3170	291	3209	133	3248	201
3093	204	3132	134	3171	229	3210	132	3249	201
3094	201	3133	113	3172	230	3211	176	3250	202
3095	231	3134	178	3173	259	3212	229	3251	177
3096	154	3135	178	3174	177	3213	134	3252	175
3097	232	3136	133	3175	230	3214	114	3253	177
3098	230	3137	118	3176	177	3215	113	3254	178
3099	153	3138	134	3177	324	3216	115	3255	202
3100	133	3139	180	3178	152	3217	133	3256	176
3101	156	3140	178	3179	151	3218	156	3257	229
3102	113	3141	153	3180	153	3219	114	3258	230
3103	259	3142	154	3181	292	3220	230	3259	259
3104	292	3143	204	3182	260	3221	153	3260	230
3105	231	3144	154	3183	258	3222	204	3261	175
3106	260	3145	261	3184	292	3223	230	3262	119
3107	231	3146	230	3185	259	3224	156	3263	119
3108	231	3147	202	3186	229	3225	203	3264	134
3109	261	3148	156	3187	360	3226	154	3265	134
3110	259	3149	202	3188	261	3227	134	3266	115
3111	481	3150	261	3189	325	3228	134	3267	153
3112	132	3151	154	3190	326	3229	114	3268	114
3113	439	3152	292	3191	325	3230	153	3269	153
3114	205	3153	131	3192	261	3231	230	3270	203
3115	291	3154	178	3193	292	3232	292	3271	178
3116	362	3155	154	3194	230	3233	153	3272	202
3117	291	3156	230	3195	200	3234	203	3273	175
3118	326	3157	202	3196	202	3235	153	3274	153
3119	230	3158	176	3197	176	3236	203	3275	134
3120	229	3159	177	3198	229	3237	154	3276	154
3121	326	3160	153	3199	258	3238	202	3277	132
3122	400	3161	154	3200	134	3239	113	3278	154
3123	200	3162	132	3201	178	3240	202	3279	154

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3281	154	3320	139	3359	204	3398	205	3437	361
3282	156	3321	134	3360	204	3399	115	3438	229
3283	114	3322	139	3361	153	3400	115	3439	361
3284	202	3323	139	3362	131	3401	134	3440	178
3285	203	3324	139	3363	203	3402	178	3441	325
3286	177	3325	179	3364	291	3403	178	3442	361
3287	153	3326	294	3365	203	3404	134	3443	292
3288	177	3327	292	3366	201	3405	155	3444	176
3289	177	3328	207	3367	231	3406	98	3445	152
3290	153	3329	97	3368	177	3407	118	3446	119
3291	153	3330	179	3369	231	3408	178	3447	101
3292	230	3331	101	3370	204	3409	133	3448	115
3293	154	3332	204	3371	203	3410	155	3449	155
3294	132	3333	134	3372	326	3411	155	3450	134
3295	177	3334	206	3373	132	3412	261	3451	134
3296	134	3335	115	3374	132	3413	134	3452	99
3297	153	3336	134	3375	133	3414	177	3453	96
3298	114	3337	178	3376	133	3415	153	3454	177
3299	153	3338	206	3377	260	3416	99	3455	134
3300	132	3339	133	3378	292	3417	175	3456	154
3301	177	3340	291	3379	260	3418	156	3457	153
3302	203	3341	525	3380	292	3419	178	3458	177
3303	178	3342	153	3381	202	3420	202	3459	202
3304	230	3343	400	3382	399	3421	178	3460	153
3305	153	3344	400	3383	205	3422	177	3461	114
3306	201	3345	292	3384	291	3423	154	3462	134
3307	153	3346	439	3385	261	3424	200	3463	154
3308	154	3347	202	3386	292	3425	156	3464	132
3309	134	3348	151	3387	131	3426	203	3465	204
3310	132	3349	99	3388	175	3427	260	3466	154
3311	177	3350	97	3389	157	3428	230	3467	153
3312	134	3351	118	3390	101	3429	291	3468	177
3313	139	3352	134	3391	119	3430	229	3469	177
3314	75	3353	178	3392	119	3431	201	3470	153
3315	139	3354	204	3393	101	3432	202	3471	114
3316	139	3355	178	3394	156	3433	259	3472	176
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3318	139	3357	133	3396	132	3435	230	3474	156

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3868	101	3907	203	3946	98	3985	98	4024	86
3869	98	3908	204	3947	115	3986	114	4025	119
3870	114	3909	229	3948	153	3987	154	4026	115
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3872	132	3911	156	3950	115	3989	134	4028	178
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3882	118	3921	175	3960	201	3999	180	4038	259
3883	204	3922	85	3961	230	4000	180	4039	292
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4273	98
4274	178
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4276	115
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4311	85
4312	69
4313	75
4314	75
4315	75
4316	115
4317	101
4318	69
4319	101

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