ON K-MODULI OF QUARTIC THREEFOLDS

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ABSTRACT. The family of smooth Fano 3-folds with Picard rank 1 and anticanonical volume 4 consists of quartic 3-folds and of double covers of the 3-dimensional quadric branched along an octic surface; all of them are K-stable. With the aim of investigating the compactification of the moduli space of quartic 3-folds given by K-stability, we exhibit K-polystable Fano 3-folds which deform to quartic 3-folds, but which are neither quartic 3-folds nor double covers of quadric 3-folds. The examples we find are complete intersections of two quadrics and a quartic in the weighted projective space $\mathbb{P}(1,1,1,1,1,2,2)$.

1. INTRODUCTION

A *Fano variety* is a normal projective variety over $\mathbb{C}$ for which the anticanonical divisor is $\mathbb{Q}$-Cartier and ample. There has been spectacular recent progress on constructing moduli spaces of Fano varieties using K-stability [5, 12–14, 32, 35, 39, 46, 48]. For each positive integer n and for every positive rational number v there exists an Artin stack $\mathcal{M}^{\mathrm{Kss}}_{n,v}$, called the *K-moduli stack*, which is of finite type over $\mathbb{C}$ and which parametrises K-semistable n-dimensional Fano varieties with anticanonical volume v. Moreover, this stack admits a good moduli space $M^{\mathrm{Kps}}_{n,v}$, called the *K-moduli space*, which is projective over $\mathbb{C}$. The closed points of $M^{\mathrm{Kps}}_{n,v}$ are in one-to-one correspondence with K-polystable n-dimensional Fano varieties with anticanonical volume v. We refer to [47] for a survey on these topics.

The next natural step is to investigate the geometry of K-moduli stacks, providing explicit descriptions when possible. K-moduli of smooth(able) 2-dimensional Fano varieties have been studied by Mabuchi–Mukai [40] and by Odaka–Spotti–Sun [42]. More generally, one of the most well-known classes of Fano varieties are hypersurfaces of degree d in n-dimensional projective space $\mathbb{P}^n$, with $d \leq n$; all smooth Fano hypersurfaces are conjectured to be K-polystable [47, Part 3]. Substantial progress has made towards clarifying this conjecture [1, Theorem B], however very little is currently known about their K-moduli. One readily available compact moduli for Fano hypersurfaces is the GIT moduli, and Liu–Xu [38] showed that, for cubic 3-folds, the K-moduli space coincides with the GIT moduli space. An analogous result for cubic 4-folds was recently proved by Liu [36]. There are additional results in 2 dimensions, and in other settings [7–9, 37].

Perhaps the simplest case in higher dimensions where the expected agreement between GIT moduli and K-moduli fails is the moduli of quartic 3-folds. Here it is known that their K-moduli contains elements that are not in the GIT moduli, and it is expected that all K-polystable degenerations of quartic 3-folds are embedded in the 5-dimensional weighted projective space $\mathbb{P}(1,1,1,1,1,2) =: \mathbb{P}(1^5,2)$ as complete intersections of a quartic and a quadric. We prove below that this expectation is false: we show there are many K-polystable degenerations of quartic 3-folds which do not embed in $\mathbb{P}(1^5,2)$.

1.1. **K-moduli of quartic 3-folds.** Let $\mathscr{V}_4$ denote the family of *smooth* Fano 3-folds with Picard rank 1 and anticanonical volume 4. Members of $\mathscr{V}_4$ are either:

(a) smooth quartic hypersurfaces in $\mathbb{P}^4$, simply called *smooth quartic 3-folds*; or
(b) double covers of the smooth quadric 3-fold with branch divisor of degree 8, often called *hyperelliptic*.

By [16, 23, 26] each smooth Fano 3-fold in the family $\mathscr{V}_4$ is K-stable. It is not known how to characterise singular quartic 3-folds which are K-(poly/semi)stable. Since not every member of $\mathscr{V}_4$ is a quartic 3-fold, the connected component of the K-moduli space $M^{\mathrm{Kps}}_{3,4}$ containing the members of $\mathscr{V}_4$ does not coincide with the GIT moduli space of quartic 3-folds. However, every member of $\mathscr{V}_4$ is a complete intersections of type $(4,2)$ in $\mathbb{P}(1^5,2)$; this can be seen as follows (see also [41, Example 4.3], the introduction of [43], and [6,

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Example 3.5.2]). Let $x_0, \ldots, x_4, y$ be the homogeneous coordinates of $\mathbb{P}(1^5, 2)$ with degrees $1, \ldots, 1, 2$ respectively. A $(4, 2)$-complete intersection X is given by the vanishing of

$$f_4(x_0, \ldots, x_4) + f_2(x_0, \ldots, x_4)y + ay^2 \qquad \text{and} \qquad g_2(x_0, \ldots, x_4) + by,$$

where f_2 and g_2 are quadrics, f_4 is a quartic, and $a, b \in \mathbb{C}$. If X is smooth then $a \neq 0$ or $b \neq 0$. If $b \neq 0$, from the second equation we can express y in terms of the x_i, therefore y is a redundant variable. This implies that X is a quartic hypersurface in $\mathbb{P}^4$. If $b = 0$ then $a \neq 0$ so, up to scaling, $a = 1$. By completing the square (i.e. applying the automorphism of $\mathbb{P}(1^5, 2)$ given by $y \mapsto y + \frac{1}{2}f_2(x)$) we can assume that $f_2 = 0$; in this case X is the double cover of the quadric 3-fold $\{g_2(x) = 0\} \subset \mathbb{P}^4$ branched along the surface $\{f_4(x) = g_2(x) = 0\}$.

It is natural to ask whether the connected component of the K-moduli space $M^{\mathrm{Kps}}_{3,4}$ containing the members of $\mathscr{V}_4$ parametrises $(4, 2)$-complete intersections in $\mathbb{P}(1^5, 2)$. If this were true, it might suggest that this component of K-moduli can be constructed via some form of non-reductive GIT by taking the moduli space of such complete intersections. We prove that this is not the case.

Theorem 1.1 (cf. Proposition 2.1 and Theorem 3.1)**.** *There exist K-polystable non-smooth Fano 3-folds that are smoothable to a quartic 3-fold and that are not $(4, 2)$-complete intersections in $\mathbb{P}(1^5, 2)$.*

An immediate consequence is:

Corollary 1.2. *Let M be the connected component of the K-moduli space $M^{\mathrm{Kps}}_{3,4}$ containing the members of $\mathscr{V}_4$. Then M contains K-polystable Fano 3-folds that are not $(4, 2)$-complete intersections in $\mathbb{P}(1^5, 2)$.*

We provide two types of examples of Fano 3-folds which satisfy Thorem 1.1: three toric examples that we study in §2, and an infinite family that we study in §3. These examples are all $(4, 2, 2)$-complete intersections in $\mathbb{P}(1, 1, 1, 1, 1, 2, 2) =: \mathbb{P}(1^5, 2^2)$. It is natural to ask whether the connected component of the K-moduli space $M^{\mathrm{Kps}}_{3,4}$ containing the members of $\mathscr{V}_4$ parametrises $(4, 2, 2)$-complete intersections in $\mathbb{P}(1^5, 2^2)$, however we do not know the answer to this question.

1.2. **Degenerations of quartic 3-folds inside $\mathbb{P}(1^5, 2^2)$.** Denote the variables of the weighted projective space $\mathbb{P}(1^5, 2^2)$ by $x_0, \ldots, x_4, y_0, y_1$, where y_0 and y_1 are the two variables of weight 2. Let $X \subset \mathbb{P}(1^5, 2^2)$ be a complete intersection of two quadrics and a quartic. If X is not a cone then, after a suitable change of coordinates, it is defined by the equations

$$\begin{aligned} y_0y_1 &= f(x_0, x_1, x_2, x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) &= ay_0 + by_1 \\ h(x_0, x_1, x_2, x_3, x_4) &= cy_0 + dy_1 \end{aligned}$$

where f has degree 4, g and h have degree 2, and $a, b, c, d \in \mathbb{C}$. There are three possibilities depending on the rank of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $\operatorname{rk} A = 2$ then X is a quartic hypersurface in $\mathbb{P}^4$. If $\operatorname{rk} A = 1$ then X is hyperelliptic; i.e. a $(4, 2)$-complete intersection in $\mathbb{P}(1^5, 2)$. If A is the null matrix then X has two singular points at $p_0 = [0 : 0 : 0 : 0 : 0 : 1 : 0]$ and $p_1 = [0 : 0 : 0 : 0 : 0 : 0 : 1]$. Let us now assume that we are in the latter case, so that $X \subset \mathbb{P}(1^5, 2^2)$ is defined by

$$\begin{aligned} y_0y_1 &= f(x_0, x_1, x_2, x_3, x_4) \\ g(x_0, x_1, x_2, x_3, x_4) &= 0 \\ h(x_0, x_1, x_2, x_3, x_4) &= 0 \end{aligned}$$

where f is a quartic and g and h are quadrics. In §2 we consider special binomials for f, g, h and we study certain toric varieties X. In §3 we study the infinite family of varieties X obtained by picking general f, g, h.

1.3. **Overview of the proof.** There are several methods available when verifying K-(poly)stability of a given Fano variety. For the toric examples, we use the most natural tool: a toric variety is K-polystable if and only if the barycentre of its anticanonical polytope is the origin. This translates the algebro-geometric condition into a combinatorial one amenable to computer-assisted investigation. The toric examples which satisfy Theorem 1.1 were found via a computer search amongst Fano polytopes using techniques from Mirror Symmetry [3, 18, 19]. This is explained in §1.4; the examples are studied in §2.

For the infinite family of examples satisfying Theorem 1.1, we use estimates on stability thresholds. Two of the most useful tools in K-stability are the Fujita–Li valuative criterion [27, 34] and the stability threshold, also called the δ-invariant, introduced in [28]. It follows that a Fano variety X is K-stable if and only if $\delta(X) > 1$. Although $\delta(X)$ is extremely difficult to compute in general, a method to find lower bounds for $\delta(X)$ is described in [2]. Roughly speaking, one chooses a flag on/over X and computes the refinement of the anticanonical linear system with respect to this flag [2, §2]; the refinement provides a lower bound for $\delta(X)$. With a little care, one can choose the flag so that the refinement provides a number greater than 1, implying K-stability. We do this for the infinite family in §3.

1.4. **Connection to Mirror Symmetry.** The three toric 3-folds presented in Proposition 2.1 and in Remark 2.2 were found using a computer-assisted search guided by expectations arising from Mirror Symmetry for Fano varieties [3, 18, 19]. It is expected that deformation families of smooth (or maybe mildly singular, i.e. with orbifold terminal singularities) Fano varieties of dimension n are in one-to-one correspondence with mutation-equivalence classes of certain 'special' Laurent polynomials in n variables. We need to specify: (i) the meaning of the word *mutation*; (ii) the meaning of the adjective 'special'; and (iii) how the correspondence works.

(i) By *mutation* we mean a generalisation, introduced in [4], of the notion of mutation in cluster algebra theory [25]. We do not give the definition here: see [4, Definition 2] for details. It is sufficient to know that, under certain circumstances, a Laurent polynomial $f \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ can be mutated to another Laurent polynomial $g \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Two Laurent polynomials f and g are said to be mutation-equivalent if there exists a finite sequence of mutations transforming f to g.

(ii) In place of 'special' we should have written *rigid maximally mutable*. The class of rigid maximally mutable Laurent polynomials (or *rigid MMLPs* for short) was introduced in [3] for dimension 2, and in [21] for any dimension. We remark that if $f \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a rigid MMLP in n variables, then its Newton polytope $P := \mathrm{Newt}(f) \subset \mathbb{R}^n$ is an n-dimensional lattice polytope such that the origin lies in the interior of P, and each vertex of P is a primitive lattice vector: P is an example of a *Fano polytope*. One can consider the *spanning fan* (or *face fan*) whose cones are generated by the faces of P, and the (possibly singular) Fano toric variety X_P associated to this spanning fan.

(iii) The correspondence between deformation families of Fano varieties and mutation-equivalence classes of rigid MMLPs is described in [3, 20, 21]. Briefly, we expect it to work as follows: given a rigid MMLP f with Newton polytope P, we consider the Fano toric variety X_P and associate a (partial) smoothing of X_P. The reason why this assignment should be well-defined is the following result due to Ilten [31]: if f and g are Laurent polynomials related via a mutation, then the Fano toric varieties X_P and X_Q are deformation equivalent (here Q is the Newton polytope of g); i.e. there exists a flat proper family $\mathcal{X} \to \mathbb{P}^1$ such that the fibre over $0 \in \mathbb{P}^1$ is X_P and the fibre over $\infty \in \mathbb{P}^1$ is X_Q. Notice that a singular Fano toric variety can have many different smoothings (e.g. [33, Theorem 3.1] or [44]); one needs to select the smoothing of X_P which is compatible with the mutations of f.

Now we explain how we found the toric examples. We start from the polytope P' whose vertices are the vectors in (2.2). The toric variety $X_{P'}$ associated to the spanning fan of P' is the singular quartic hypersurface $\{x_1x_2x_3x_4 = x_0^4\} \subset \mathbb{P}^4$, which clearly deforms to members of $\mathscr{V}_4$. There exists a unique rigid MMLP f such that $P' = \mathrm{Newt}(f)$, namely

$$f = \frac{(1 + x + y + z)^4}{xyz} - 24.$$

We used the computer algebra system Magma [15] to construct many Laurent polynomials g which are mutation-equivalent to f. Each of these g's gives a Fano toric variety X_Q, where $Q := \mathrm{Newt}(g)$, which, according to the Mirror Symmetry expectations described above, should be a degeneration of quartic 3-folds. We then filter for those g such that the polar of Q has barycentre at the origin; this is equivalent to requiring that the Fano toric variety X_Q is K-polystable. In this way we found three Laurent polynomials, one of which is given by:

$$\begin{aligned} g = {} & x^3y^2z^4 + 2x^2y^3z^2 + 4x^2y^2z^2 + 8x^2yz^3 + 2x^2yz^2 + xy^4 + 4xy^3 + 8xy^2z + 6xy^2 \\ & + 16xyz + 4xy + 28xz^2 + 8xz + x + 12y + 56z/y + 12/y + 8/(xz) + 16/(xyz) + 70/(xy^2) + 8/(xy^2z) \\ & + 2/(x^2yz^2) + 4/(x^2y^2z^2) + 56/(x^2y^3z) + 2/(x^2y^3z^2) + 28/(x^3y^4z^2) + 8/(x^4y^5z^3) + 1/(x^5y^6z^4). \end{aligned}$$

The Newton polytope $P := \mathrm{Newt}(g)$ is the Fano polytope given in Proposition 2.1. The Newton polytopes of the remaining two Laurent polynomials found using this method are described in Remark 2.2.

Notation and conventions. We work over an algebraically closed field of characteristic zero, denoted by $\mathbb{C}$. Every toric variety or toric singularity is assumed to be normal.

2. The toric examples

We begin by analysing a Fano toric 3-fold satisfying Theorem 1.1.

Proposition 2.1. *Let P be the polytope with vertices*

$$\begin{pmatrix}3\\2\\4\end{pmatrix}, \begin{pmatrix}1\\4\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-5\\-6\\-4\end{pmatrix} \tag{2.1}$$

in the lattice $N = \mathbb{Z}^3$ and let X be the toric variety associated to the spanning fan of P. Then:

1. *X is a $\mathbb{Q}$-factorial K-polystable Fano 3-fold;*
2. *$\mathrm{Pic}(X) \simeq \mathbb{Z}$ and $\mathrm{Cl}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$;*
3. *X is the quotient $\mathbb{P}^3/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_8)$, where $\boldsymbol{\mu}_2$ acts linearly with weights $(0,1,0,1)$ and $\boldsymbol{\mu}_8$ acts linearly with weights $(0,5,1,6)$;*
4. *the singular locus of X consists of six rational curves generically along which X has transverse A_1, A_3 or A_7 singularities;*
5. *there are exactly two non-Gorenstein points on X and both of them have Gorenstein index 2 and are not canonical;*
6. *X is not a $(4,2)$-complete intersection in $\mathbb{P}(1^5,2)$;*
7. *X is the $(4,2,2)$-complete intersection in $\mathbb{P}(1^5,2^2)$ given by the equations*

   $$\begin{aligned} x_0^4 - y_0y_1 &= 0 \\ x_2^2 - x_0x_3 &= 0 \\ x_3^2 - x_1x_4 &= 0 \end{aligned}$$

   where $x_0,\dots,x_4,y_0,y_1$ are the homogeneous coordinates of $\mathbb{P}(1^5,2^2)$ with degrees $1,\dots,1,2,2$ respectively;
8. *X deforms to a quartic 3-fold.*

Proof of (1). Let Σ be the spanning fan of P. It is clear that X is a Fano 3-fold. Since P is a tetrahedron, each cone of Σ is simplicial, therefore X is $\mathbb{Q}$-factorial. Let M be the lattice dual to N and let $\langle\cdot,\cdot\rangle\colon M \times N \to \mathbb{Z}$ be the dual pairing. We denote by the same symbol its extension to the associated real vector spaces, i.e. $M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$. Consider the polar P° of P:

$$P^\circ := \{u \in M_\mathbb{R} \mid \langle u, v\rangle \geq -1 \text{ for all } v \in P\}.$$

This is the polytope associated to the toric boundary of X, which is an anticanonical divisor. One can show that P° is the rational polytope in $M_\mathbb{R}$ with vertices $(-1,0,\frac{3}{2})$, $(-1,1,0)$, $(3,-1,-2)$, $(-1,0,\frac{1}{2})$. Since the barycentre of P° is the origin, X is K-polystable by [10]. □

Proof of (2) *and of* (3). Consider the linear map $\rho\colon \mathbb{Z}^4 \to N = \mathbb{Z}^3$ which maps the ith standard basis vector to the ith vertex of P in (2.1). Consider the transpose ${}^t\rho\colon M = \mathbb{Z}^3 \to \mathbb{Z}^4$. This is injective and its cokernel is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ via the homomorphism $\mathbb{Z}^4 \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ given by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \bar{0}^2 & \bar{1}^2 & \bar{0}^2 & \bar{1}^2 \\ \bar{0}^8 & \bar{5}^8 & \bar{1}^8 & \bar{6}^8 \end{pmatrix},$$

where $\bar{\cdot}^2$ (respectively, $\bar{\cdot}^8$) denotes the reduction modulo 2 (respectively, 8). The short exact sequence

$$0 \longrightarrow M \xrightarrow{{}^t\rho} \mathbb{Z}^4 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \longrightarrow 0$$

is the divisor sequence [22, Theorem 4.1.3], hence the divisor class group of X is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. By [22, Proposition 4.2.5] the Picard group of X is free; moreover, since X is $\mathbb{Q}$-factorial, the Picard rank of X coincides with the rank of $\mathrm{Cl}(X)$, which is 1. The presentation of X as a finite abelian quotient of $\mathbb{P}^3$ follows from [22, Exercise 5.1.13]. □

Proof of (4). Let $\rho_1, \rho_2, \rho_3, \rho_4$ be the elements in N which appear in (2.1). For $1 \le i < j \le 4$, let $\sigma_{ij} \in \Sigma$ the 2-dimensional cone with rays $\mathbb{R}_{\ge 0}\rho_i$ and $\mathbb{R}_{\ge 0}\rho_j$. Let C_{ij} be the closure of the torus orbit on X associated to the cone σ_{ij}: it is a smooth rational curve. The shape of the cone σ_{ij} determines the singularities of X generically along C_{ij}. We write down the analysis for C_{12} and we omit the other cases.

We consider the 2-dimensional lattice $N_{12} = N \cap (\mathbb{R}\rho_1 + \mathbb{R}\rho_2)$. This is the orthogonal of $(8, -2, -5) \in M$. The vectors ρ_1 and ρ_2 form an $\mathbb{R}$-basis of $\mathbb{R}\rho_1 + \mathbb{R}\rho_2 = N_{12} \otimes_{\mathbb{Z}} \mathbb{R}$, but not a $\mathbb{Z}$-basis of N_{12}. The finite abelian group $N_{12}/(\mathbb{Z}\rho_1 + \mathbb{Z}\rho_2)$ has order 2 and its generator is the class of

$$\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \in N_{12}.$$

This implies that the toric variety associated to the cone σ_{12} in the lattice N_{12} is the A_1 surface singularity $\mathrm{Spec}\, \mathbb{C}[x, y, z]/(xy - z^2)$. This implies that X has transverse A_1 singularities generically along C_{12}. □

Proof of (5). By (4) the non-Gorenstein locus of X is contained in the set of the 4 torus-fixed points: $p_{123}, p_{124}, p_{134}$, and p_{234}. Here p_{ijk} is the torus-fixed point on X corresponding to the 3-dimensional cone σ_{ijk} with rays $\mathbb{R}_{\ge 0}\rho_i$, $\mathbb{R}_{\ge 0}\rho_j$, and $\mathbb{R}_{\ge 0}\rho_k$. We need to analyse the singularities of X at these points.

Let us start from p_{123}. The three vectors ρ_1, ρ_2, and ρ_3 lie on the affine plane

$$H_{(2,0,-1),2} := \{v \in N_{\mathbb{R}} \mid \langle (2, 0, -1), v \rangle = 2\}.$$

Since the lattice vector $(2, 0, -1) \in M$ is primitive, we get that the Gorenstein index of X at p_{123} is 2. Moreover, the singularity $p_{123} \in X$ is not canonical because the lattice vector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4}\rho_1 + \frac{1}{8}\rho_2 + \frac{1}{8}\rho_3$$

lies in the interior of the polytope with vertices $0, \rho_1, \rho_2, \rho_3$ (see [22, Proposition 11.4.12b]).

Now consider p_{124}. The three vectors ρ_1, ρ_2, and ρ_4 lie on the affine plane

$$H_{(-3,1,2),1} := \{v \in N_{\mathbb{R}} \mid \langle (-3, 1, 2), v \rangle = 1\}.$$

Since the lattice vector $(-3, 1, 2) \in M$ is primitive, we get that the Gorenstein index of X at p_{123} is 1. By [22, Proposition 11.4.11] the singularity $p_{124} \in X$ is canonical.

In an analogous way we can prove that p_{134} is a Gorenstein canonical singularity, whereas p_{234} is a non-canonical singularity with Gorenstein index 2. □

Proof of (6). For brevity, set $\mathbb{P} := \mathbb{P}(1^5, 2)$. The singular locus of $\mathbb{P}$ consists of a single point $p = [0 : 0 : 0 : 0 : 0 : 1]$. For a contradiction assume that X is a $(4, 2)$-complete intersection in $\mathbb{P}$ and consider the corresponding closed embedding $X \hookrightarrow \mathbb{P}$. We have that $X \smallsetminus \{p\} \hookrightarrow \mathbb{P} \smallsetminus \{p\}$ is a regular closed embedding, because it is locally defined by the vanishing of the dehomogeneisations of the quartic and the quadric that define X inside $\mathbb{P}$. This implies that $X \smallsetminus \{p\}$ is lci, and in particular Gorenstein. Therefore the non-Gorenstein locus of X is contained in $\{p\}$, so it is either empty or consists of one point. This contradicts (5). □

Proof of (7). In the lattice $M \oplus \mathbb{Z}$ we consider the cone τ whose apex is at the origin and which is spanned by $P^\circ \times \{1\}$. In other words, τ is the cone over P° placed at height 1. The primitive generators of τ are

$$\begin{aligned} y_1 &= (-2,0,3,2), \\ x_1 &= (-1,1,0,1), \\ x_4 &= (3,-1,-2,1), \\ y_0 &= (-2,0,1,2). \end{aligned}$$

The Hilbert basis (i.e. the minimal set of generators) of the monoid $\tau \cap (M \oplus \mathbb{Z})$ is made up of y_1, x_1, x_4, y_0, $x_0 = (-1,0,1,1)$, $x_2 = (0,0,0,1)$, $x_3 = (1,0,-1,1)$.

Since P° is the moment polytope of the toric boundary of X, which is anticanonical and hence ample, we have that $X = \operatorname{Proj} \mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$, where the $\mathbb{N}$-grading is given by the projection $M \oplus \mathbb{Z} \twoheadrightarrow \mathbb{Z}$. This shows that X is a closed subvariety of the weighted projective space $\mathbb{P}(1^5, 2^2)$, equipped with homogeneous coordinates $x_0, \dots, x_4, y_0, y_1$. It is easy to see that the equations of X in $\mathbb{P}(1^5, 2^2)$ are

$$\begin{aligned} x_0^4 - y_0 y_1 &= 0, \\ x_2^2 - x_0 x_3 &= 0, \\ x_3^2 - x_1 x_4 &= 0. \end{aligned}$$

Thus X is a $(4,2,2)$-complete intersection in $\mathbb{P}(1^5, 2^2)$. □

Proof of (8). By using an argument similar to the one that appears in the introduction, one can see that the general $(4,2,2)$-complete intersection in $\mathbb{P}(1^5, 2^2)$ is a quartic 3-fold. □

Remark 2.2. In addition to the example presented in Proposition 2.1, we found two additional Fano toric 3-folds which satisfy Theorem 1.1.

(1) Let X be the toric 3-fold associated to the spanning fan of the polytope in $\mathbb{Z}^3$ with vertices $(1,3,2)$, $(1,3,0)$, $(1,0,2)$, $(1,0,0)$, $(-1,-1,2)$, $(-1,-1,-4)$, $(-1,-2,2)$, $(-1,-2,-4)$. There are isomorphisms $\mathrm{Cl}(X) \simeq \mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Pic}(X) \simeq \mathbb{Z}$. One can prove that X is the closed subvariety of $\mathbb{P}(1^5, 2^2)$ defined by the equations

$$\begin{aligned} x_0^3 x_3 - y_0 y_1 &= 0, \\ x_2^2 - x_0 x_3 &= 0, \\ x_2 x_3 - x_1 x_4 &= 0. \end{aligned}$$

(2) Let X be the toric 3-fold associated to the spanning fan of the polytope in $\mathbb{Z}^3$ with vertices $(3,4,4)$, $(3,2,4)$, $(1,2,0)$, $(1,0,0)$, $(-1,0,0)$, $(-1,-2,0)$, $(-3,-2,-4)$, $(-3,-4,-4)$. There are isomorphisms $\mathrm{Cl}(X) \simeq \mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $\mathrm{Pic}(X) \simeq \mathbb{Z}$. One can prove that X is the closed subvariety of $\mathbb{P}(1^5, 2^2)$ defined by the equations

$$\begin{aligned} x_0^2 x_4^2 - y_0 y_1 &= 0, \\ x_2^2 - x_0 x_4 &= 0, \\ x_1 x_3 - x_0 x_4 &= 0. \end{aligned}$$

Each of these examples is a $(4,2,2)$-complete intersection in $\mathbb{P}(1^5, 2^2)$ and has two singular points with Gorenstein index 2.

Remark 2.3. In addition to the three examples presented in Proposition 2.1 and in Remark 2.2 we know a further K-polystable Fano toric 3-fold which deforms to quartic 3-folds: this is the toric variety associated to the spanning fan of the polytope with vertices

$$\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \tag{2.2}$$

in $N = \mathbb{Z}^3$, and it is the $\mathbb{Q}$-factorial hypersurface $\{x_1 x_2 x_3 x_4 - x_0^4 = 0\}$ in $\mathbb{P}^4$. We conjecture that these four toric varieties are the only K-polystable Fano toric 3-folds that deform to quartic 3-folds.

Remark 2.4. A Fano polytope P is called *symmetric* if the only point which is fixed by every automorphism of P is the origin. According to [30, Conjecture 1.6] it is expected that if P is a 2-dimensional Fano polytope such that its polar P° has barycentre at the origin then either P is symmetric or P is a triangle (i.e. simplex). The examples in Remark 2.2 show that this is not true in higher dimension

Remark 2.5. According to [29] it is expected that in every mutation-equivalence class of 2-dimensional Fano polytopes there is at most one polytope P such that its polar P° has barycentre at the origin. This is not true in higher dimension; for instance, the four polytopes presented in Proposition 2.1, Remark 2.2, and Remark 2.3 are mutation-equivalent and the polar of each of them has barycentre at the origin.

3. The infinite family

3.1. **Statement.** The infinite family of examples satisfying Theorem 1.1 is given by the following:

Theorem 3.1. *Let $x_0, \dots, x_4, y_0, y_1$ be the homogeneous coordinates of the weighted projective space $\mathbb{P}(1^5, 2^2)$ with degrees $1, \dots, 1, 2, 2$ respectively. Consider $x_0, \dots, x_4$ also as the homogeneous coordinates of $\mathbb{P}^4$. Let f be a quartic in $x_0, \dots, x_4$ and let g and h be two quadrics in $x_0, \dots, x_4$. Consider the zero-loci*

$$\begin{aligned} \Delta &= \{f = 0,\ g = h = 0\} \subset \mathbb{P}^4, \\ S &= \{g = h = 0\} \subset \mathbb{P}^4, \\ X &= \{y_0 y_1 - f = 0,\ g = h = 0\} \subset \mathbb{P}(1^5, 2^2). \end{aligned} \tag{3.1}$$

Assume that Δ is a smooth curve and that S is a smooth surface. Then the following statements hold.

1. *The surface S is a smooth del Pezzo of degree 4 and contains exactly 16 lines; their intersection points form a finite subset $\Sigma \subset S$ consisting of 40 points.*
2. *X is a klt Fano 3-fold and the group $G = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ acts faithfully on X.*
3. *If $\Sigma \cap \Delta = \varnothing$, then X is K-polystable.*

Remark 3.2. The last condition $\Sigma \cap \Delta = \varnothing$ is satisfied for general choices of f, g, h. Therefore a general X in Theorem 3.1 is K-polystable. We believe that the condition $\Sigma \cap \Delta = \varnothing$ can be removed from Theorem 3.1(3).

Remark 3.3. Note that if $X \subset \mathbb{P}(1^5, 2)$ is a hyperelliptic cone, i.e. the term y^2 does not appear in the degree 4 equation, or if $X \subset \mathbb{P}(1^5, 2^2)$ is a cone, i.e. the quadratic part in the variables y_0, y_1 of the degree 4 equation f does not have rank 2, then X is unstable. This can be checked rather easily using [49, Theorem 1.4] by setting $r = 1/2$, $c = 0$, and $n = 3$ to obtain $\delta(X) \leq 5/12$.

The remaining of the paper is devoted to giving a proof of Theorem 3.1, hence we always work in the setting of Theorem 3.1. We begin with an explanation of the construction of the models, their symmetries, and the generality condition on them. We then present the proof of the main claim about their K-stability; the proof uses the theory of refinements introduced in [2], which provides a lower bound for stability thresholds. Indeed, some formulae are readily available in [6, §1.7] which compute the refinement for a flag, if the flag is geometrically realised on a Mori Dream Space birational model of the Fano variety. We will use them in the proof of Theorem 3.1(3).

3.2. **The models and their symmetries.**

Proof of Theorem 3.1(1). All of this is very classical. □

Conversely, if S is a smooth del Pezzo surface of degree 4, then we can view S as the complete intersection $\{g = h = 0\} \subset \mathbb{P}^4$ where g and h are quadrics in $x_0, \dots, x_4$. We denote by Σ the set consisting of the 40 intersection points of the lines on S.

Remark 3.4. Let p be a point in S.

1. If p is not contained in any line in S then there are exactly ten smooth conics in S that contain p; moreover, five of these ten conics can be chosen to intersect pairwise transversally at p.
2. If p is contained in a line in S and $p \notin \Sigma$, then there are exactly five smooth conics in S that contain p, and any two of them intersect transversally at p.
3. If $p \in \Sigma$, then it is contained in a unique smooth conic in S.

Proof of Theorem 3.1(2). We study the singularities of X by looking at the affine charts of $\mathbb{P}(1^5, 2^2)$.

Let us consider the chart $x_0 \neq 0$; this is isomorphic to $\mathbb{A}^6$ with affine coordinates $x_1, \ldots, x_4$, y_0, y_1. Inside this $\mathbb{A}^6$ the variety X is given by the equations

$$\begin{aligned} y_0 y_1 - \bar{f} &= 0 \\ \bar{g} &= 0 \\ \bar{h} &= 0 \end{aligned}$$

where $\bar{f} = f|_{x_0=1}$ and similarly for $\bar{g}$ and $\bar{h}$. The jacobian matrix of the equations of X in $\mathbb{A}^6$ is

$$J = \begin{pmatrix} -\partial_{x_1}\bar{f} & -\partial_{x_2}\bar{f} & -\partial_{x_3}\bar{f} & -\partial_{x_4}\bar{f} & y_1 & y_0 \\ \partial_{x_1}\bar{g} & \partial_{x_2}\bar{g} & \partial_{x_3}\bar{g} & \partial_{x_4}\bar{g} & 0 & 0 \\ \partial_{x_1}\bar{h} & \partial_{x_2}\bar{h} & \partial_{x_3}\bar{h} & \partial_{x_4}\bar{h} & 0 & 0 \end{pmatrix}.$$

Since $S \subset \mathbb{P}^4$ is a smooth surface, the bottom-left 2×4 submatrix of J has rank 2 in all points of X. Therefore it is clear that J has rank 3 at all points of X which satisfy $y_0 \neq 0$ or $y_1 \neq 0$. The points of X which satisfy $y_0 = y_1 = 0$ also satisfy $f = 0$, and from the smoothness of Δ it follows that the rank of J is 3 at these points. Therefore the intersection of X with the chart $x_0 \neq 0$ is smooth. Similarly, this holds also for the other charts $x_i \neq 0$ for $i \in \{0, \ldots, 4\}$.

In other words, we have proved that the singular points of X must satisfy $x_0 = \cdots = x_4 = 0$. There are exactly two such points:

$$p_0 = [0:0:0:0:0:1:0] \quad \text{and} \quad p_1 = [0:0:0:0:0:0:1].$$

We need to study the singularity type of these two points on X.

The chart $y_0 \neq 0$ on $\mathbb{P}(1^5, 2^2)$ gives the quotient singularity $\frac{1}{2}(1,1,1,1,1,0)$ with orbifold coordinates $x_0, \ldots, x_4, y_1$. It is easy to see that $X \cap \{y_0 \neq 0\}$ is isomorphic to $\{g = h = 0\} \subset \frac{1}{2}(1,1,1,1,1)$. Since $\frac{1}{2}(1,1,1,1,1)$ is the affine cone over $(\mathbb{P}^4, \mathcal{O}(2))$, we have that $\{g = h = 0\} \subset \frac{1}{2}(1,1,1,1,1)$ is the affine cone over $(S, -2K_S)$, which is a klt singularity. This shows that p_0 and p_1 are klt singularities of X.

By adjunction $-K_X = \mathcal{O}_{\mathbb{P}(1^5,2^2)}(1)|_X$, so X is a klt Fano 3-fold. Its degree is $(-K_X)^3 = \frac{1^3 \cdot 4 \cdot 2 \cdot 2}{2^2} = 4$.

Now we need to construct an effective action of $G = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ on X. Consider the $\mathbb{C}^*$-action given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto \left[x_0 : x_1 : x_2 : x_3 : x_4 : \lambda y_0 : \lambda^{-1} y_1\right]$$

for $\lambda \in \mathbb{C}^*$. Furthermore, the group $\mathrm{Aut}(X)$ also contains an involution σ that is given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4 : y_1 : y_0].$$

Together, they generate a subgroup $G \subset \mathrm{Aut}(X)$ that is isomorphic to $\mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$. Note that the two singular points p_0 and p_1 on X are swapped by the action of σ and fixed by the action of $\mathbb{C}^*$. This concludes the proof of Theorem 3.1(2). □

Let $\rho\colon X \dashrightarrow S$ be the rational map given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto [x_0 : x_1 : x_2 : x_3 : x_4].$$

Then ρ is undefined precisely at the points p_0 and p_1, resulting in the following G-equivariant commutative diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\eta"] & \\ X \arrow[rr, dashed, "\rho"] & & S \end{tikzcd} \tag{3.2}$$

where $\widetilde{X}$ is a smooth projective 3-fold, π is a birational morphism that contracts two irreducible smooth surfaces E_0 and E_1 to the points p_0 and p_1, respectively, and η is a G-equivariant conic bundle. Furthermore, the surfaces E_0 and E_1 are sections of the conic bundle η, so that η induces isomorphisms $E_0 \cong S$ and $E_1 \cong S$. We have $E_0|_{E_0} \sim -2K_{E_0}$ and $E_1|_{E_1} \sim -2K_{E_1}$, which gives

$$-K_{\widetilde{X}} \sim_{\mathbb{Q}} \pi^*(-K_X) + \frac{1}{2}(E_0 + E_1) \sim_{\mathbb{Q}} \eta^*(-K_S) + E_0 + E_1.$$

Corollary 3.5. *The 3-fold $\widetilde{X}$ is a Mori Dream Space.*

Proof. Let ε be a sufficiently small positive rational number. Then $\left(\widetilde{X}, \frac{1+\varepsilon}{2}(E_0 + E_1)\right)$ has Kawamata log terminal singularities, and $-(K_{\widetilde{X}} + \frac{1+\varepsilon}{2}(E_0 + E_1))$ is ample. Applying [11, Corollary 1.3.1] verifies the claim. □

Let R_0 and R_1 be the surfaces in X that are cut out by $y_0 = f = 0$ and $y_1 = f = 0$, respectively. Denote by $\widetilde{R}_0$ and $\widetilde{R}_1$ the strict transforms of R_0 and R_1 on $\widetilde{X}$, respectively. Then $\eta^*(\Delta) = \widetilde{R}_0 + \widetilde{R}_1$, and Δ is the discriminant curve of the conic bundle η. We have the following non-G-equivariant commutative diagram

$$\begin{tikzcd}
& & \widetilde{X} \arrow[dl, "\eta_0"'] \arrow[dd, "\eta"] \arrow[dr, "\eta_1"] & & \\
& U \arrow[dl, "\psi"'] \arrow[dd, "\phi"] & & U \arrow[dr, "\psi"] \arrow[dd, "\phi"'] & \\
V \arrow[dr, dashed, "\tau"'] & & S \arrow[dl, equal] \arrow[dr, equal] & & V \arrow[dl, dashed, "\tau"] \\
& S & & S &
\end{tikzcd} \tag{3.3}$$

where $U = \mathbb{P}(\mathcal{O}_S(2K_S) \oplus \mathcal{O}_S)$, ϕ is the $\mathbb{P}^1$-bundle given by the projection $\mathbb{P}(\mathcal{O}_S(2K_S) \oplus \mathcal{O}_S) \to S$, the morphism ψ is a contraction of the negative section of the $\mathbb{P}^1$-bundle ϕ, V is a cone over S, the map τ is the projection from the vertex of the cone, and morphisms η_0 and η_1 are birational contractions of the surfaces $\widetilde{R}_0$ and $\widetilde{R}_1$, respectively.

Remark 3.6. In the left hand side of (3.3), we have $\phi^*(\Delta) = \eta_0(R_1)$, the morphism ψ contracts $\eta_0(E_0)$, and $\psi \circ \eta_0(E_1)$ is a smooth hyperplane section of the cone $V \subset \mathbb{P}^{13}$. In the right hand side of (3.3), we have $\phi^*(\Delta) = \eta_1(R_0)$, ψ contracts $\eta_1(E_1)$, and $\psi \circ \eta_1(E_0)$ is a hyperplane section of the cone V. The involution σ swaps the left and the right hand sides of the diagram (3.3).

Set $Z = R_0 \cap R_1$ and $\widetilde{Z} = \widetilde{R}_0 \cap \widetilde{R}_1$. Then $Z = \pi(\widetilde{Z})$, $Z \cong \widetilde{Z} \cong \Delta$, and $Z = \{y_0 = y_1 = f = 0\} \subset X$. Observe that $p_0 \notin Z$, $p_1 \notin Z$, $\widetilde{Z} \cap E_0 = \widetilde{Z} \cap E_1 = \varnothing$, and every point in Z and $\widetilde{Z}$ is G-invariant.

Lemma 3.7. *Let q be a G-invariant point in $\widetilde{X}$, let C be a G-invariant irreducible curve in $\widetilde{X}$, and let B be a G-invariant irreducible surface in $\widetilde{X}$. Then the following assertions hold:*

(1) $q \in \widetilde{Z}$;
(2) either $C = \widetilde{Z}$ or C is a smooth fibre of the conic bundle η;
(3) $B = \eta^*(\mathscr{C})$ for some irreducible curve $\mathscr{C} \subset S$.

In particular, we have that $B \cap \widetilde{Z} \neq \varnothing$.

Proof. Left to the reader. □

3.3. **K-stability of the general models.** The proof of Theorem 3.1(3) is quite involved. However, it is less difficult to produce specific examples of X with large symmetries and prove they are K-polystable by taking advantage of the group of symmetries. Here, for illutration, we present one such case. Then we proceed by proving that a general X is K-polystable (see Theorem 3.1(3)).

Special case of Theorem 3.1(3): diagonal models. Suppose that

$$\begin{aligned}
f &= \alpha_0 x_0^4 + \alpha_1 x_1^4 + \alpha_2 x_2^4 + \alpha_3 x_3^4 + \alpha_4 x_0^4, \\
g &= x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2, \\
h &= \epsilon_0 x_0^2 + \epsilon_1 x_1^2 + \epsilon_2 x_2^2 + \epsilon_3 x_3^2 + \epsilon_4 x_4^2,
\end{aligned}$$

where $\alpha_0, \ldots, \alpha_4$ are sufficiently general complex numbers and $\epsilon_0, \ldots, \epsilon_4$ are pairwise distinct complex numbers. Then the group $\mathrm{Aut}(X)$ is larger than the group G in Theorem 3.1(2), because $\mathrm{Aut}(X)$ also contains 15 additional involutions given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : y_0 : y_1] \mapsto \left[x_0 : (-1)^a x_1 : (-1)^b x_2 : (-1)^c x_3 : (-1)^d x_4 : y_0 : y_1\right]$$

for $a, b, c, d \in \{0, 1\}$. Let Γ be the subgroup in $\mathrm{Aut}(X)$ generated by G and these 15 involutions. Then $\Gamma \cong \mathbb{C}^* \rtimes (\mathbb{Z}/2\mathbb{Z})^5$, and (3.2) is Γ-equivariant. Note that X does not contain Γ-invariant points. Furthermore, using the diagram (3.2) and [24, Theorem 6.9], we conclude that $\mathrm{Cl}^G(X) = \mathbb{Z}[-K_X]$. Now, arguing as in the proof of [6, Theorem 1.4.11], we conclude that

$$\alpha_\Gamma(X) \geq 1,$$

where $\alpha_\Gamma(X)$ is the Γ-invariant α-invariant of Tian for the 3-fold X, see [45] for the analytic definition and [17] for an algebraic definition of α. It follows that X is K-polystable by [45] as

$$\alpha_\Gamma(X) > \frac{3}{4} = \frac{\dim X}{1 + \dim X}.$$

This concludes the claim that X is K-polystable. □

Proof of Theorem 3.1(3). Suppose that X is not K-polystable. By [50, Corollary 4.14], there exists a G-invariant prime divisor $\mathbf{F}$ over X for which $\beta(\mathbf{F}) \leqslant 0$. For the definition of $\beta(\mathbf{F})$, see [27, 34]. Let $\mathfrak{C}$ be the center of the divisor $\mathbf{F}$ on X. Then, for every point $P \in \mathfrak{C}$, we must have $\delta_P(X) \leqslant 1$, where $\delta_P(X)$ is the local stability threshold of X at p. For the precise definition of $\delta_P(X)$, see [2, Definition 2.5] and [6, § 1.5]. On the other hand, by Lemma 3.7, we have the following four possibilities:

(1) $\mathfrak{C}$ is a G-invariant point in Z;
(2) $\mathfrak{C}$ is the curve Z;
(3) $\mathfrak{C}$ is an irreducible fibre of the rational map $\rho\colon X \dashrightarrow S$;
(4) $\mathfrak{C}$ is a G-invariant surface and $\mathfrak{C} \cap Z \neq \varnothing$.

Hence, we see that either $\mathfrak{C}$ is an irreducible curve that is a fibre of the rational map $\rho\colon X \dashrightarrow S$, or the curve Z contains a point x such that $\delta_x(X) \leqslant 1$. We will show that both cases are impossible, by first lifting the problem to the level of $\widetilde{X}$, then using the results obtained in [2] and [6, § 1.7] to derive a contradiction.

Since $\widetilde{X}$ is smooth and it is a Mori Dream Space, it is more convenient to work on $\widetilde{X}$ than on X. Namely, we let $L = \pi^*(-K_X)$, and we let $\widetilde{\mathfrak{C}}$ be the center of the divisor $\mathbf{F}$ on the 3-fold $\widetilde{X}$. Then $\widetilde{\mathfrak{C}}$ is the strict transform of the center $\mathfrak{C}$, and we have the following four possibilities:

(1) $\widetilde{\mathfrak{C}}$ is a G-invariant point in $\widetilde{Z}$;
(2) $\widetilde{\mathfrak{C}}$ is the curve $\widetilde{Z}$;
(3) $\widetilde{\mathfrak{C}}$ is a smooth fibre of the conic bundle $\eta\colon \widetilde{X} \to S$;
(4) $\widetilde{\mathfrak{C}}$ is a G-invariant surface and $\widetilde{\mathfrak{C}} \cap \widetilde{Z} \neq \varnothing$.

Note that $\delta_x(\widetilde{X}, L) = \delta_{\pi(x)}(X)$ for every point $x \in \widetilde{Z}$. For the definition of $\delta_x(\widetilde{X}, L)$, see [2, Definition 2.5] and [6, § 1.5]. Therefore, we conclude that one of the following two cases hold:

(◇) $\widetilde{\mathfrak{C}}$ a smooth fibre of the conic bundle $\eta\colon \widetilde{X} \to S$,
(♡) there exists a point $x \in \widetilde{Z}$ such that $\delta_x(\widetilde{X}, L) \leqslant 1$.

In both cases, let us introduce a new curve $\mathcal{C}$ in the 3-fold $\widetilde{X}$ as follows:

- in case (◇), we let $\mathcal{C} = \widetilde{\mathfrak{C}}$,
- in case (♡), we let $\mathcal{C}$ be the (singular) fibre of the conic bundle η that contains x.

Let $p = \eta(\mathcal{C})$. It follows from Remark 3.4 that S contains a smooth conic $\mathscr{C}$ such that $p \in \mathscr{C}$. Moreover, if $p \in \Delta$, it follows from Remark 3.4 that we can also choose $\mathscr{C}$ such that it intersects the curve Δ transversally at the point p, because $\Sigma \cap \Delta = \varnothing$ by assumption.

Let $B = \eta^*(\mathscr{C})$, and let $B' = \eta^*(Z)$ for a general conic $Z \in |-K_S - \mathscr{C}|$. Then B is normal, it has at most Du Val singularities of type $\mathbb{A}$, it is smooth along the curve $\mathcal{C}$, and B' is smooth.

Let us compute $\beta(B)$. We have $\beta(B) = A_X(B) - S_L(B) = 1 - S_L(B)$, where

$$S_L(B) = \frac{1}{L^3} \int_0^\infty \mathrm{vol}\big(L - uB\big) du.$$

For $u \in \mathbb{R}_{\geqslant 0}$, the divisor $L - uB$ is pseudo-effective $\iff$ $u \leqslant 1$, because

$$L - uB \sim_{\mathbb{Q}} (1-u)B + B' + \frac{1}{2}\big(E_0 + E_1\big).$$

For $u \in [0,1]$, let $P(u)$ be the positive part of the Zariski decomposition of the divisor $L - uB$, and let $N(u)$ be the negative part of the Zariski decomposition of the divisor $L - uB$. Then

$$P(u) = L - uB - \frac{u}{2}\big(E_0 + E_1\big)$$

and $N(u) = \frac{u}{2}(E_0 + E_1)$ for every $u \in [0,1]$. This gives

$$S_L(B) = \frac{1}{L^3}\int_0^\infty \mathrm{vol}\big(L - uB\big)du = \frac{1}{4}\int_0^1 \big(P(u)\big)^3 du = \frac{1}{4}\int_0^1 (2u^3 - 6u + 4)du = \frac{3}{8},$$

which implies that $\beta(B) = \frac{5}{8}$. Here, we used the following intersections on the 3-fold $\widetilde{X}$:

$$B^3 = 0, (B')^3 = 0, E_0^3 = 16, E_1^3 = 16, E_0^2 \cdot E_1 = 0, E_0 \cdot E_1^2 = 0, E_0 \cdot B \cdot E_1 = 0,$$
$$E_0 \cdot B' \cdot E_1 = 0, E_0 \cdot B \cdot B' = 2, B \cdot B' \cdot E_1 = 2, E_0 \cdot B^2 = 0, B^2 \cdot B' = 0, E_0 \cdot (B')^2 = 0, B \cdot (B')^2 = 0,$$
$$B^2 \cdot E_1 = 0, E_0^2 \cdot B = -4, B \cdot E_1^2 = -4, E_0^2 \cdot B' = -4, B' \cdot E_1^2 = -4, (B')^2 \cdot E_1 = 0.$$

We set $\mathbf{e}_0 = E_0|_B$ and $\mathbf{e}_1 = E_1|_B$. Then $\mathbf{e}_0$ and $\mathbf{e}_1$ are smooth irreducible rational disjoint curves, and B is smooth along $\mathbf{e}_0$ and $\mathbf{e}_1$. On B, we have $\mathbf{e}_0^2 = \mathbf{e}_1^2 = -4$, $\mathbf{e}_0 \cdot \mathbf{e}_1 = 0$, $\mathbf{e}_0 \cdot \mathcal{C} = \mathbf{e}_1 \cdot \mathcal{C} = 1$.

Suppose that case ($\diamond$) holds. Then $\widetilde{\mathfrak{C}}$ is a smooth fibre of the conic bundle η, and $\mathcal{C} = \widetilde{\mathfrak{C}}$. Set

$$S\big(W^B_{\bullet,\bullet};\mathcal{C}\big) = \frac{3}{L^3}\int_0^1\int_0^\infty \mathrm{vol}\big(P(u)\big|_B - v\mathcal{C}\big)dvdu.$$

Using [6, Corollary 1.7.26], we get $S(W^B_{\bullet,\bullet};\mathcal{C}) \geqslant 1$, since $\beta(\mathbf{F}) \leqslant 0$, $\beta(B) > 0$ and $\mathcal{C} \not\subset \mathrm{Supp}(N(u))$. On the other hand, it is easy to compute $S(W^B_{\bullet,\bullet};\mathcal{C})$. Indeed, take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$P(u)\big|_B - v\mathcal{C} \sim_{\mathbb{R}} (2-v)\mathcal{C} + \frac{1-u}{2}\big(\mathbf{e}_0 + \mathbf{e}_1\big).$$

Therefore, the divisor $P(u)|_B - v\mathcal{C}$ is nef for $v \leqslant 2u$, and it is not pseudo-effective for $v > 2$. Moreover, if $v \in [2u, 2]$, the positive part of its Zariski decomposition is $\frac{2-v}{4}(4\mathcal{C} + \mathbf{e}_0 + \mathbf{e}_1)$. Then

$$\mathrm{vol}\big(P(u)|_B - v\mathcal{C}\big) = \begin{cases} 2 - 2u^2 + 2uv - 2v \text{ if } 0 \leqslant v \leqslant 2u, \\ \dfrac{(v-2)^2}{2} \text{ if } 2u \leqslant v \leqslant 2, \\ 0 \text{ if } v \geqslant 2. \end{cases}$$

Integrating, we get $S(W^B_{\bullet,\bullet};\mathcal{C}) = \frac{3}{4}$, which is a contradiction. Thus, case ($\diamond$) does not hold.

Now, we assume that the case ($\heartsuit$) holds, and derive a contradiction. Recall that this means that $\delta_x(\widetilde{X}, L) \leqslant 1$ for some point $x \in \widetilde{Z}$. In this case, the curve $\mathcal{C}$ is singular. Namely, we have $\mathcal{C} = C_0 + C_1$, where C_0 and C_1 are smooth irreducible rational curves that intersect each other transversally at x. Without loss of generality, we may assume that $C_0 \cap E_0 \neq \varnothing$. Moreover, since the surface B is smooth along $\mathcal{C}$, the numerical intersections of the curves C_0, C_1, $\mathbf{e}_0$, $\mathbf{e}_1$ on the surface B are given in the following table:

	C_0	C_1	$\mathbf{e}_0$	$\mathbf{e}_1$
C_0	-1	1	1	0
C_1	1	-1	0	1
$\mathbf{e}_0$	1	0	-4	0
$\mathbf{e}_1$	0	1	0	-4

Let us estimate $\delta_x(\widetilde{X}, L)$ using [6, Theorem 1.7.30]. We have $N(u)|_B = \frac{u}{2}(\mathbf{e}_0 + \mathbf{e}_1)$ and $x \notin \mathbf{e}_0 \cup \mathbf{e}_1$. For every $u \in [0,1]$, we let

$$t(u) = \inf\Big\{v \in \mathbb{R}_{\geqslant 0} \;\Big|\; \text{the divisor } P(u)\big|_B - vC_0 \text{ is pseudo-effective}\Big\}.$$

For $v \in [0, t(u)]$, we let $P(u,v)$ be the positive part of the Zariski decomposition of $P(u)|_B - vC_0$, and we let $N(u,v)$ be its negative part. Then we set

$$S\big(W^B_{\bullet,\bullet}; C_0\big) = \frac{3}{L^3}\int_0^1 \int_0^{t(u)} \mathrm{vol}\big(P(u)\big|_B - vC_0\big) dv du.$$

Note that $C_0 \not\subset \mathrm{Supp}(N(u,v))$ for every $u \in [0,1)$ and $v \in (0, t(u))$. Thus, we can let

$$F_x\big(W^{B,C_0}_{\bullet,\bullet,\bullet}\big) = \frac{6}{L^3}\int_0^1 \int_0^{t(u)} \big(P(u,v)\cdot C_0\big)\cdot \mathrm{ord}_x\big(N(u,v)\big|_{C_0}\big) dv du.$$

Finally, we let

$$S\big(W^{B,C_0}_{\bullet,\bullet,\bullet}; x\big) = \frac{3}{L^3}\int_0^1 \int_0^{t(u)} \big(P(u,v)\cdot C_0\big)^2 dv du + F_x\big(W^{B,C_0}_{\bullet,\bullet,\bullet}\big).$$

Then, since $C_0 \not\subset \mathrm{Supp}(N(u))$, it follows from [6, Theorem 1.7.30] that

$$1 \geqslant \delta_x(\widetilde{X}, L) \geqslant \min\left\{\frac{1}{S(W^{B,C_0}_{\bullet,\bullet,\bullet}; x)}, \frac{1}{S(W^B_{\bullet,\bullet}; C_0)}, \frac{1}{S_L(B)}\right\}.$$

Recall that $S_L(B) = \frac{3}{8}$. Thus, either $S(W^{B,C_0}_{\bullet,\bullet,\bullet}; x) \geqslant 1$ or $S(W^B_{\bullet,\bullet}; C_0) \geqslant 1$ (or both).

Let us compute $S(W^{B,C_0}_{\bullet,\bullet,\bullet}; x)$ and $S(W^B_{\bullet,\bullet}; C_0)$. As above, take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$P(u)\big|_B - vC_0 \sim_{\mathbb{R}} (2-v)C_0 + 2C_1 + \frac{1-u}{2}\big(\mathbf{e}_0 + \mathbf{e}_1\big).$$

Therefore, since the intersection form of the curves C_1, $\mathbf{e}_0$, $\mathbf{e}_1$ on the surface B is negative definite, we see that $P(u)|_B - vC_0$ is pseudoeffective $\iff v \leqslant 2$, so $t(u) = 2$. Moreover, if $0 \leqslant u \leqslant \frac{1}{5}$, then

$$P(u,v) = \begin{cases} (2-v)C_0 + 2C_1 + \frac{1}{2}(1-u)\big(\mathbf{e}_0 + \mathbf{e}_1\big) & \text{if } 0 \leqslant v \leqslant 2u, \\ \frac{1}{4}(2-v)\big(4C_0 + \mathbf{e}_0\big) + 2C_1 + \frac{1}{2}(1-u)\mathbf{e}_1 & \text{if } 2u \leqslant v \leqslant \frac{1}{2}(1-u), \\ \frac{1}{4}(2-v)\big(4C_0 + \mathbf{e}_0\big) + \frac{1}{2}(5-2v-u)C_1 + \frac{1}{2}(1-u)\mathbf{e}_1 & \text{if } 1-u \leqslant 2v \leqslant 1+3u, \\ \frac{1}{12}(2-v)(12C_0 + 16C_1 + 3\mathbf{e}_0 + 4\mathbf{e}_1) & \text{if } \frac{1}{2}(1+3u) \leqslant v \leqslant 2, \end{cases}$$

and,

$$N(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant 2u, \\ \frac{1}{4}(v-2u)\mathbf{e}_0 & \text{if } 2u \leqslant v \leqslant \frac{1}{2}(1-u), \\ \frac{1}{4}(v-2u)\mathbf{e}_0 + \frac{1}{2}(2v+u-1)C_1 & \text{if } 1-u \leqslant 2v \leqslant 1+3u, \\ \frac{1}{4}(v-2u)\mathbf{e}_0 + \frac{1}{3}(4v-2)C_1 + \frac{1}{6}(2v-3u-1)\mathbf{e}_1 & \text{if } \frac{1}{2}(1+3u) \leqslant v \leqslant 2, \end{cases}$$

which compute

$$P(u,v)\cdot C_0 = \begin{cases} \frac{1}{2}(1-u+2v) & \text{if } 0 \leqslant v \leqslant 2u, \\ \frac{1}{4}(2+3v) & \text{if } 2u \leqslant v \leqslant \frac{1}{2}(1-u), \\ \frac{1}{4}(4-2u-v) & \text{if } 1-u \leqslant 2v \leqslant 1+3u, \\ \frac{1}{12}(14-7v) & \text{if } \frac{1+3u}{2} \leqslant v \leqslant 2, \end{cases}$$

$$\mathrm{vol}\big(P(u)|_B - vC_0\big) = \begin{cases} 2 - 2u^2 + uv - v^2 - v & \text{if } 0 \leqslant v \leqslant 2u, \\ 2 - v - u^2 - \frac{3}{4}v^2 & \text{if } 2u \leqslant v \leqslant \frac{1}{2}(1-u), \\ \frac{1}{4}(9 - 3u^2 + 4uv + v^2 - 2u - 8v) & \text{if } 1-u \leqslant 2v \leqslant 1+3u, \\ \frac{7}{12}(2-v)^2 & \text{if } \frac{1}{2}(1+3u) \leqslant v \leqslant 2. \end{cases}$$

Similarly, if $\frac{1}{5} \leqslant u \leqslant 1$, then

$$P(u,v) = \begin{cases} (2-v)C_0 + 2C_1 + \frac{1}{2}(1-u)\big(\mathbf{e}_0 + \mathbf{e}_1\big) & \text{if } 0 \leqslant v \leqslant \frac{1}{2}(1-u), \\ (2-v)C_0 + \frac{1}{2}(5-2v-u)C_1 + \frac{1}{2}(1-u)\big(\mathbf{e}_0 + \mathbf{e}_1\big) & \text{if } \frac{1}{2}(1-u) \leqslant v \leqslant 2u, \\ \frac{1}{4}(2-v)\big(4C_0 + \mathbf{e}_0\big) + \frac{1}{2}(5-2v-u)C_1 + \frac{1}{2}(1-u)\mathbf{e}_1 & \text{if } 1-u \leqslant 2v \leqslant 1+3u, \\ \frac{1}{12}(2-v)(12C_0 + 16C_1 + 3\mathbf{e}_0 + 4\mathbf{e}_1) & \text{if } \frac{1}{2}(1+3u) \leqslant v \leqslant 2, \end{cases}$$

$$N(u,v) = \begin{cases} 0 & \text{if } 0 \leqslant v \leqslant \frac{1}{2}(1-u), \\ \frac{1}{2}(2v+u-1)C_1 & \text{if } \frac{1}{2}(1-u) \leqslant v \leqslant 2u, \\ \frac{1}{4}(v-2u)\mathbf{e}_0 + \frac{1}{2}(2v+u-1)C_1 & \text{if } 1-u \leqslant 2v \leqslant 1+3u, \\ \frac{1}{4}(v-2u)\mathbf{e}_0 + \frac{1}{3}(4v-2)C_1 + \frac{1}{6}(2v-3u-1)\mathbf{e}_1 & \text{if } \frac{1}{2}(1+3u) \leqslant v \leqslant 2, \end{cases}$$

$$P(u,v)\cdot C_0 = \begin{cases} \frac{1}{2}(1-u+2v) & \text{if } 0 \leqslant v \leqslant \frac{1}{2}(1-u), \\ 1-u & \text{if } \frac{1}{2}(1-u) \leqslant v \leqslant 2u, \\ \frac{1}{4}(4-2u-v) & \text{if } 1-u \leqslant 2v \leqslant 1+3u, \\ \frac{1}{12}(14-7v) & \text{if } \frac{1}{2}(1+3u) \leqslant v \leqslant 2, \end{cases}$$

$$\mathrm{vol}\big(P(u)|_B - vC_0\big) = \begin{cases} 2-2u^2+uv-v^2-v & \text{if } 0 \leqslant v \leqslant \frac{1}{2}(1-u), \\ \frac{1}{4}(1-u)(7u-8v+9) & \text{if } \frac{1}{2}(1-u) \leqslant v \leqslant 2u, \\ \frac{1}{4}(9-3u^2+4uv+v^2-2u-8v) & \text{if } 1-u \leqslant 2v \leqslant 1+3u, \\ \frac{7}{12}(2-v)^2 & \text{if } \frac{1}{2}(1+3u) \leqslant v \leqslant 2. \end{cases}$$

Now, integrating $\mathrm{vol}(P(u)|_B - vC_0)$ and $(P(u,v)\cdot C_0)^2$, we obtain $S(W^B_{\bullet,\bullet};C_0) = \frac{13}{16}$ and

$$S\big(W^{B,C_0}_{\bullet,\bullet,\bullet};x\big) = \frac{77}{320} + F_x\big(W^{B,C_0}_{\bullet,\bullet,\bullet}\big).$$

Furthermore, since $x \notin \mathbf{e}_0 \cup \mathbf{e}_1$ and the curves C_0 and C_1 intersect transversally at x, we have

$$\begin{aligned} F_x\big(W^{B,C_0}_{\bullet,\bullet,\bullet}\big) &= \frac{3}{2}\int_0^1\int_0^2 \mathrm{ord}_x\big(N(u,v)\big|_{C_0}\big) \times \big(P(u,v)\cdot C_0\big)dvdu \\ &= \frac{3}{2}\int_0^{\frac{1}{5}}\int_{\frac{1-u}{2}}^{\frac{1+3u}{2}} \frac{2v+u-1}{2}\times\big(P(u,v)\cdot C_0\big)dvdu + \frac{3}{2}\int_0^{\frac{1}{5}}\int_{\frac{1+3u}{2}}^{2}\frac{4v-2}{3}\times\big(P(u,v)\cdot C_0\big)dvdu \\ &+ \frac{3}{2}\int_{\frac{1}{5}}^1\int_{\frac{1-u}{2}}^{\frac{1+3u}{2}} \frac{2v+u-1}{2}\times\big(P(u,v)\cdot C_0\big)dvdu + \frac{3}{2}\int_{\frac{1}{5}}^1\int_{\frac{1+3u}{2}}^{2}\frac{4v-2}{3}\times\big(P(u,v)\cdot C_0\big)dvdu \\ &= \frac{3}{2}\int_0^{\frac{1}{5}}\int_{\frac{1-u}{2}}^{\frac{1+3u}{2}} \frac{2v+u-1}{2}\times\frac{4-2u-v}{4}dvdu + \frac{3}{2}\int_0^{\frac{1}{5}}\int_{\frac{1+3u}{2}}^{2}\frac{4v-2}{3}\times\frac{14-7v}{12}dvdu \\ &+ \frac{3}{2}\int_{\frac{1}{5}}^1\int_{\frac{1-u}{2}}^{2u} \frac{2v+u-1}{2}\times(1-u)dvdu + \frac{3}{2}\int_{\frac{1}{5}}^1\int_{2u}^{\frac{1+3u}{2}}\frac{2v+u-1}{2}\times\frac{4-2u-v}{4}dvdu \\ &+ \frac{3}{2}\int_{\frac{1}{5}}^1\int_{\frac{1+3u}{2}}^{2}\frac{4v-2}{3}\times\frac{14-7v}{12}dvdu = \frac{183}{320}, \end{aligned}$$

which gives $S(W^{B,C_0}_{\bullet,\bullet,\bullet};x) = \frac{13}{16}$. Thus, we see that $S(W^{B,C_0}_{\bullet,\bullet,\bullet};x) = S(W^B_{\bullet,\bullet};C_0) = \frac{13}{16} < 1$, which is a contradiction. This completes the proof of Theorem 3.1. ◻

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