Homological projective duality for the Segre cubic

Thorsten Beckmann, Pieter Belmans

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Abstract

The Segre cubic and Castelnuovo–Richmond quartic are two projectively dual hypersurfaces in $\mathbb{P}^4$, with a long and rich history starting in the 19th century. We will explain how Kuznetsov's theory of homological projective duality lifts this projective duality to a relationship between the derived category of a small resolution of the Segre cubic and a small resolution of the Coble fourfold, the double cover of $\mathbb{P}^4$ ramified along the Castelnuovo–Richmond quartic.

Homological projective duality then provides a description of the derived categories of linear sections, which we will describe to illustrate the theory. The case of the Segre cubic and Coble fourfold is non-trivial enough to exhibit interesting behavior, whilst being easy enough to explain the general machinery in this special and very classical case.

Contents

1 Introduction

Projective duality has been a cornerstone of (algebraic) geometry since a long time, going back to the 19th century and before. It provides a way to recover a projective variety from the set of its tangent hyperplanes in the dual projective space. For a modern treatment one is referred to [**MR2113135**, **MR1264417**, **MR1234494**].

An interesting example of the theory is provided by

1. the *Segre cubic* S_3,

the up to projective equivalence unique singular cubic threefold with the maximal number of 10 nodes, studied by Segre in 1887 [**segre-cubic**]. An explicit description is given in (26). Its projective dual is

    2. the *Castelnuovo–Richmond quartic* CR_4,

an explicit singular quartic threefold, described in (28). It was studied by Castelnuovo in 1891 [**castelnuovo-quartic**] and independently by Richmond in 1902 [**richmond-quartic**]. This threefold is also referred to as the *Igusa quartic*[1], because its modular properties were studied by Igusa in 1962 [**MR0141643**], The third main character is

    3. the *Coble fourfold* Cob,

the double cover $\pi\colon \text{Cob} \to \mathbb{P}^4$ ramified along CR_4, thoroughly studied by Coble in his 1929 book [**coble-book**]. We will recall some of the many interesting geometric and modular properties of S_3, CR_4 and Cob in Section 3.

Recently Kuznetsov has introduced a *homological* version of projective duality [**MR2354207**]. The original motivation is to study derived categories of linear sections, and it has blossomed into a rich theory of "homological projective geometry" [**MR3948688**, **1902.09824v2**, **MR4280866**, **MR3948688**] where classical constructions in projective geometry (such as cones and joins) have an analogous construction on the level of derived categories. For an introduction to the theory one is referred to [**MR3728631**, **MR3821163**, **2111.00527v1**].

The theory roughly states that, for a morphism $f\colon X \to \mathbb{P}(V)$ where X is a smooth projective variety the derived category of a linear section $X \times_{\mathbb{P}(V)} \mathbb{P}(L)$ for some $L \subseteq V$ is described in terms of a *homological projective dual* variety $f^\natural\colon X^\natural \to \mathbb{P}(V^\vee)$ (possibly noncommutative) and its dual linear sections. To make this possible we consider a semiorthogonal decomposition of $\mathbf{D}^b(X)$ which is compatible with $\mathcal{O}_X(1) = f^*(\mathcal{O}_{\mathbb{P}(V)}(1))$ in a suitable way, a so-called *Lefschetz decomposition*. The general theory is developed in the geometric setting in [**MR2354207**], and in complete generality without geometricity conditions in [**MR3948688**]. We will briefly introduce the framework in Section 2.1.

The main result of the theory is the existence of a homological projective dual together with a recipe to understand derived categories of hyperplane sections. The name homological projective duality refers to the fact that, if X is in fact a closed subvariety of $\mathbb{P}(V)$, then the classical projective dual arises as the critical locus of the dual morphism $f^\natural$. However, whilst the theory provides an abstract existence result it does not give an explicit geometric description of the dual in concrete situations (much like an explicit description of the classical projective dual is usually hard to come by), which is necessary to interpret the output of the machinery.

Instances in which the homological projective dual has a nice geometric description are listed in [**MR3728631**], and include cases like quadrics, Grassmannians of lines (the Grassmannian–Pfaffian duality, which is still incomplete), or Veronese embeddings (the Veronese–Clifford duality).

The goal of this (mostly expository) paper is to give a case study of homological projective duality for the classically relevant case of the Segre cubic S_3. It has the benefit that the linear sections appearing are all well-known varieties, such as (smooth and singular) cubic surfaces, Kummer quartics, elliptic curves as plane cubics and double covers. Because all the objects are commutative (in particular, no need for noncommutative or categorical resolutions), reasonably small, completely classical, and their derived categories are easy to describe, the workings and the output of the abstract machinery becomes tractable.

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[1]This might be the more common name nowadays, but Dolgachev writes the following in [**MR2964027**]: "The quartic hypersurface isomorphic to CR_4 is often referred to in modern literature as an *Igusa quartic* (apparently, reference [**MR1007155**] is responsible for this unfortunate terminology)." Therefore we will write Castelnuovo–Richmond quartic.

The main result   The version of homological projective duality from [**MR2354207**] works for a morphism $X \to \mathbb{P}(V)$ where X is smooth projective. Because the Segre cubic is singular, we will recall in Section 3.1 a certain small resolution $\varpi\colon X \to S_3$ giving rise to the natural map $f\colon X \to \mathbb{P}(V)$ which is the composition of the resolution and the closed immersion.

For its homological projective dual we will consider a small resolution of singularities $\rho\colon Y \to \mathrm{Cob}$, which induces a natural map $g\colon Y \to \mathbb{P}(V^\vee)$. The short version of the main result is then:

Main theorem. *Let $f\colon X \to \mathbb{P}(V)$ be the composition of the small resolution ϖ and the inclusion of the Segre cubic* S_3*. Consider the Lefschetz structure from* (40)*.*

The homological projective dual is $g\colon Y \to \mathbb{P}(V^\vee)$, where g is the composition of the small resolution ρ and the double cover ramified along the Castelnuovo–Richmond quartic.

In Theorem 11 we will give the precise version. The proof of this theorem is given in Section 3.2. It is an application of homological projective duality for projective bundles [**MR2354207**] recalled in Section 2.2.

More interesting than the theorem itself are the applications of the machinery of homological projective duality. Linear sections of X and Y are easy to describe and often have a very classical interpretation. The description of their derived categories is performed in Section 4, and we will observe how homological projective duality gives a new point of view on their usual descriptions.

By [**2206.01549v1**] the relation between the Segre cubic and the Coble fourfold is actually just one instance for which homological projective duality involving very classical objects can be understood very explicitly. The interested reader is warmly recommended to have a look in Appendix A of op. cit. for inspiration.

Comparison with homological projective duality for quadrics   Arguably one of the easiest non-trivial case of homological projective duality is that of (smooth) quadrics [**MR4283549**]. By Theorem 1.1 of op. cit. the homological projective dual of a quadric hypersurface (with respect to the Lefschetz decomposition from Kapranov's exceptional collection) is the classical projective dual hypersurface (if the dimension is even), resp. the double cover in the classical projective dual hypersurface (if the dimension is odd). Going up one degree, already the homological projective dual of a smooth cubic threefold is very complicated, as explained in Section 3.3, and the classical projective dual is highly singular hypersurface of degree 24 (by (27)).

However, the many singularities of the Segre cubic make its dual hypersurface manageable. This, together with the existence of a natural rectangular Lefschetz decomposition for a small resolution, and an understanding of the geometry in very classical terms, make it possible to prove the main theorem, with a statement analogous to that of odd-dimensional quadrics, except that it involves singular hypersurfaces of degrees 3 and 4, and double covers thereof. We will recall homological projective duality for quadrics in Example 5 and the reader is invited to compare this description to the statement of Theorem 11.

A second Lefschetz decomposition   The input for homological projective duality is not just the morphism $f\colon X \to \mathbb{P}(V)$, one also needs to specify a Lefschetz structure on $\mathbf{D}^{\mathrm{b}}(X)$. In the context of homogeneous varieties this leads to the search for *minimal* Lefschetz centers [**MR3137200**, **MR4017162**], which are in some sense optimal: their homological projective duals will be as small as possible. Many examples of homological projective duality are considered for minimal Lefschetz centers.

The Lefschetz decomposition in the main theorem is rectangular, so the Lefschetz center is in particular minimal, so optimal from the point-of-view of homological projective duality.

But we can construct a *second* rectangular Lefschetz decomposition. We will first do this by using a different small resolution of S_3 in Proposition 10. Then in Proposition 30 we will construct another rectangular Lefschetz decomposition using a modular interpretation of S_3 and its resolutions in terms of quiver representations. We will compare the Lefschetz centers in Proposition 32 and leave a description of the homological projective dual for future work.

Conventions   Throughout k will be an algebraically closed field of characteristic 0.

2   Homological projective duality

We will now give a brief introduction to homological projective duality and discuss the case of projective bundles. For more information the reader is referred to [**MR3728631**, **MR3821163**] or the original [**MR2354207**]. In our presentation we will include the noncommutative picture from [**MR3948688**].

2.1   Lefschetz categories and homological projective duality

Let X be a smooth and proper variety over k, let V be a finite-dimensional vector space, and let $f\colon X \to \mathbb{P}(V)$ be a morphism. We set $\mathcal{O}_X(1) = f^*(\mathcal{O}_{\mathbb{P}(V)}(1))$.

A *Lefschetz center* of $\mathbf{D}^{\mathrm{b}}(X)$ is an admissible subcategory $\mathcal{A}_0$ of $\mathbf{D}^{\mathrm{b}}(X)$, for which

- there exist admissible subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_{m-1}$ of $\mathbf{D}^{\mathrm{b}}(X)$,
- which fit into a chain of admissible subcategories

$$(1) \qquad 0 \subseteq \mathcal{A}_{m-1} \subseteq \ldots \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_0$$

such that there exists a semiorthogonal decomposition

$$(2) \qquad \mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{m-2}(m-2), \mathcal{A}_{m-1}(m-1) \rangle.$$

Here we write $\mathcal{A}_i(i)$ for the image of $\mathcal{A}_i$ under the tensor product with $\mathcal{O}_X(i)$. The $\mathcal{A}_i$ are the *Lefschetz components* and the semiorthogonal decomposition in (2) is a *Lefschetz decomposition* of $\mathbf{D}^{\mathrm{b}}(X)$. One can, moreover, show that the choice of $\mathcal{A}_0$ determines all the components $\mathcal{A}_i \subseteq \mathcal{A}_0$.

The main motivation for this definition is that, if one takes a hyperplane $H \subseteq \mathbb{P}(V)$ such that the "hyperplane section" $X \times_{\mathbb{P}(V)} H$ has dimension[2] $\dim X - 1$, one gets an induced semiorthogonal decomposition

$$(3) \qquad \mathbf{D}^{\mathrm{b}}(X \times_{\mathbb{P}(V)} H) = \langle \mathcal{C}_H, \mathcal{A}_1(1), \mathcal{A}_2(2), \ldots, \mathcal{A}_{m-2}(m-2), \mathcal{A}_{m-1}(m-1) \rangle$$

where one can consider the categories $\mathcal{A}_i$ as admissible subcategories for the hyperplane section, and preserve their semiorthogonality as long as one discards $\mathcal{A}_0$. The category $\mathcal{C}_H$ is by definition the right orthogonal to the restricted decomposition.

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[2]Without this condition one needs to take the derived fiber product. We will discuss derived fiber products in Section 4.

The homological projective dual   The main construction of homological projective duality is a family of categories over $\mathbb{P}(V^\vee)$, the dual projective space whose points correspond to the hyperplanes H, whose fibers are the categories $\mathcal{C}_H$. To make this construction, we consider the universal hyperplane section $\mathbf{H} \subset \mathbb{P}(V) \times \mathbb{P}(V^\vee)$ and set $\mathbf{H}(X) := X \times_{\mathbb{P}(V)} \mathbf{H}$. One can show that there exists a $\mathbb{P}(V^\vee)$-linear[3] semiorthogonal decomposition

$$(4) \qquad \mathbf{D}^b(\mathbf{H}(X)) = \langle \mathbf{D}^b(X)^\natural, \mathcal{A}_1(1) \boxtimes \mathbf{D}^b(\mathbb{P}(V^\vee)), \ldots, \mathcal{A}_{m-1}(m-1) \boxtimes \mathbf{D}^b(\mathbb{P}(V^\vee)) \rangle.$$

Here $\mathbf{D}^b(X)^\natural$ is defined as the right orthogonal to the "standard" components, and we have surpressed some of the notation for the embedding functors. The category $\mathbf{D}^b(X)^\natural$ is $\mathbb{P}(V^\vee)$-linear, and one can construct a canonical Lefschetz structure on it, whose Lefschetz center $\mathcal{B}_0$ is equivalent to the Lefschetz center $\mathcal{A}_0$.

The category $\mathbf{D}^b(X)^\natural$ is not necessarily the derived category of a variety. But if it is we make the following definition.

Definition 1. We say that the *homological projective dual* to $f: X \to \mathbb{P}(V)$ equipped with a Lefschetz decomposition is a morphism $f^\natural: X^\natural \to \mathbb{P}(V)$ such that

- $\mathbf{D}^b(X^\natural)$ is equipped with a Lefschetz decomposition,
- there exists a $\mathbb{P}(V^\vee)$-linear Fourier–Mukai functor from $\mathbf{D}^b(X^\natural)$ to $\mathbf{D}^b(\mathbf{H}(X))$, i.e. the kernel needs to live in the derived category of the fiber product $(X \times Y) \times_{\mathbb{P}(V) \times \mathbb{P}(V^\vee)} Q$ where Q is the incidence variety,

which induces a Lefschetz equivalence $\mathbf{D}^b(X^\natural) \cong \mathbf{D}^b(X)^\natural$, i.e. an equivalence which also gives an equivalence of Lefschetz centers, between the canonical Lefschetz center from the construction and the Lefschetz center from the first point.

For the theory to work it is not actually necessary that $f^\natural: X^\natural \to \mathbb{P}(V)$ is really a morphism from a (smooth proper) variety $X^\natural$ to the dual projective space. This will rarely be the case in fact! Rather it suffices to treat $\mathbf{D}^b(X)^\natural$ as an enhanced triangulated category, which is linear over $\mathbb{P}(V^\vee)$. One can then just formally consider this as the homological projective dual. The general machinery of [**MR3948688**] makes this possible and proves the existence of a homological projective dual without any geometricity conditions.

We can now discuss the two main parts of the main theorem of homological projective duality [**MR2354207**].

First part of the main theorem: Dual Lefschetz decompositions   Homological projective duality is indeed a duality, so its output is again a derived category with a Lefschetz structure to which the machinery can be applied again.

The Lefschetz structure on $\mathbf{D}^b(X)^\natural$ is the *dual Lefschetz decomposition*[4]

$$(5) \qquad \mathbf{D}^b(X)^\natural = \langle \mathcal{B}_{n-1}(1-n), \ldots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle$$

for some Lefschetz center $\mathcal{B}_0$ which is equivalent to $\mathcal{A}_0$. The length n is given in [**MR2354207**] and reads

$$(6) \qquad n := \dim V - 1 - \max\{i \mid \mathcal{A}_i = \mathcal{A}_0\}.$$

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[3]To be taken in the sense of [**MR2238172**], i.e. it commutes with tensor products of pullbacks along $\mathbf{H}(X) \to \mathbb{P}(V^\vee)$ of objects in $\mathbf{D}^b(\mathbb{P}(V^\vee))$.

[4]We will not go into the details of left and right homological projective duality: by [**MR3948688**] these notions agree because we are working with smooth and proper varieties.

The relationship between the $\mathcal{A}_i$ and $\mathcal{B}_j$ can be described as follows. The chain of inclusions (1) allows one to obtain semiorthogonal decompositions

$$(7) \qquad \mathcal{A}_i = \langle \mathfrak{a}_i, \mathcal{A}_{i+1} \rangle = \ldots = \langle \mathfrak{a}_i, \ldots, \mathfrak{a}_{m-1} \rangle$$

and the categories $\mathfrak{a}_i$ are the *primitive categories* of the Lefschetz decomposition. The categories $\mathcal{B}_j$ are built using the same primitive categories, namely there exist semiorthogonal decompositions

$$(8) \qquad \mathcal{B}_j = \langle \mathfrak{a}_0, \ldots, \mathfrak{a}_{\dim V - j - 2} \rangle.$$

In all examples mentioned in this paper, except Example 5, the Lefschetz decomposition is in fact *rectangular*: $\mathcal{A}_0 = \ldots = \mathcal{A}_m$. This means that there is but one non-zero primitive subcategory, $\mathfrak{a}_0 = \ldots = \mathfrak{a}_{m-2} = 0$ and $\mathfrak{a}_{m-1} = \mathcal{A}_0$. We have that $n = \dim V - n$, and the dual Lefschetz decomposition is also rectangular.

Second part of the main theorem: Linear sections The main application of homological projective duality is to provide a description of the derived category of a linear section of X, in terms of the Lefschetz decomposition *and* the derived category of the dual linear section of $X^\natural$. We will from now on write $Y = X^\natural$.

For this we consider a linear subspace $L \subseteq V$, so that we can consider $L^\perp \subseteq V^\vee$, and define

$$(9) \qquad X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L), \qquad Y_{L^\perp} := Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L^\perp).$$

For now we will assume that $L \subseteq V$ is *admissible*, i.e. that X_L and $Y_{L^\perp}$ have their expected dimension, so

$$(10) \qquad \dim X_L = \dim X - \dim_k L^\perp, \qquad \dim Y_{L^\perp} = \dim Y - \dim_k L.$$

We comment in Remark 2 on what to do in general.

With this setup, the second part of the main theorem of homological projective duality states that the derived categories of X_L and $Y_{L^\perp}$ have induced semiorthogonal decompositions

$$(11) \quad \begin{aligned} \mathbf{D}^b(X_L) &= \langle \mathcal{C}_L, \mathcal{A}_{\dim_k L^\perp}(1), \ldots, \mathcal{A}_{m-1}(m - \dim_k L^\perp) \rangle, \\ \mathbf{D}^b(Y_{L^\perp}) &= \langle \mathcal{B}_{n-1}(\dim_k L - n), \ldots, \mathcal{B}_{\dim_k L}(-1), \mathcal{C}_L \rangle. \end{aligned}$$

The important observation is that they have the component $\mathcal{C}_L$ in common. So understanding a semiorthogonal decomposition for one variety gives information on the semiorthogonal decomposition of a possibly very different variety, of "complementary dimension". This of course requires a decent enough understanding of the homological projective dual variety, and a way to understand $\mathcal{C}_L$.

Remark 2. The admissibility condition that X_L and $Y_{L^\perp}$ from (9) are of expected dimension can be removed using methods from derived algebraic geometry, provided that one takes the *derived fiber products*

$$(12) \quad X_L := X \times^{\mathbf{L}}_{\mathbb{P}(V)} \mathbb{P}(L), \qquad Y_{L^\perp} := Y \times^{\mathbf{L}}_{\mathbb{P}(V^\vee)} \mathbb{P}(L^\perp).$$

This is worked out in [**MR3948688**]. From now on X_L and $Y_{L^\perp}$ refer to these derived fiber products.

To explicitly compute the derived fiber product X_L we take a locally free resolution of $\mathcal{O}_{\mathbb{P}(V)}$-modules of either $\mathcal{O}_X$ or $\mathcal{O}_{\mathbb{P}(L)}$ which has the structure of a dg algebra, and take the tensor product of dg algebras instead of the tensor product of algebras in the formation of the fiber product. In this way we obtain a derived scheme structure on the ordinary fiber product. We will discuss how to do this in practice in Section 4.

Let us now illustrate these results in two examples, which also explain how to build intuition for the structure of the homological projective dual.

Example 3. The easiest example of a Lefschetz decomposition is provided by Beilinson's exceptional collection [**MR0509388**]

(13) $\quad \mathbf{D}^b(\mathbb{P}(V)) = \langle \mathcal{O}_{\mathbb{P}(V)}, \mathcal{O}_{\mathbb{P}(V)}(1), \ldots, \mathcal{O}_{\mathbb{P}(V)}(\dim V - 1) \rangle$

where one takes $f\colon \mathbb{P}(V) \to \mathbb{P}(V)$ the identity morphism, and takes as Lefschetz center

(14) $\quad \mathcal{A}_0 := \langle \mathcal{O}_{\mathbb{P}(V)} \rangle.$

Because a hyperplane section (and linear section) is always just a lower-dimensional projective space, for which the restricted decomposition is again a full exceptional collection, all the categories $\mathcal{C}_H$ (and $\mathcal{C}_L$) are zero, because the part in the semiorthogonal decomposition (3) (and (11)) determined by the $\mathcal{A}_i$ already generates the derived category of the section. In fact, the homological projective dual of projective space (with respect to this choice of Lefschetz center!) is just the zero category.

More interesting to consider is an inclusion of vector spaces $W \subset V$ and the resulting morphism $f\colon \mathbb{P}(W) \hookrightarrow \mathbb{P}(V)$, using Beilinson's collection for $\mathbb{P}(W)$ and Lefschetz center (14). Reasoning as in the case of $W = V$ considered before, one can see that the homological projective dual is supported on $\mathbb{P}(W^\perp) \subset \mathbb{P}(V^\vee)$: for a point h outside $\mathbb{P}(W^\perp)$ corresponding to a hyperplane $H \subset \mathbb{P}(V)$ the intersection $H \cap \mathbb{P}(W)$ has the expected dimension and the restricted collection is again a full exceptional collection. One can in fact show that the homological projective dual *is* the inclusion $\mathbb{P}(W^\perp) \subset \mathbb{P}(V^\vee)$ [**MR3728631**].

Whilst especially the first example looks quite pathological and the generalisation isn't too complicated either, it turns out that their relative version (i.e. for projective bundles) is exactly the ingredient we need for homological projective duality for the Segre cubic, and also a convenient way to prove homological projective duality for $\mathbb{P}(W) \subset \mathbb{P}(V)$. We will discuss this relative version in detail in Section 2.2.

Remark 4. We also emphasise that in Example 3 we consider a Lefschetz decomposition with respect to $\mathcal{O}_{\mathbb{P}(V)}(1)$. If one instead considers the second Veronese embedding $\mathbb{P}(V) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 V)$, it is possible to consider Lefschetz structures with respect to $\mathcal{O}_{\mathbb{P}(V)}(2)$. This is done in [**MR2419925**], and gives rise to a description of the derived categories of intersections of quadrics, as these correspond to linear sections with respect to this choice of line bundle.

Instead of describing homological projective duality for the Veronese embedding in more detail we describe homological projective duality for quadrics, as there are some interesting parallels between this situation and that of the Segre cubic.

Example 5. Let $Q \subseteq \mathbb{P}(V)$ be a smooth quadric hypersurface. On an odd-dimensional quadric Q there exists a unique *spinor bundle* $\mathcal{S}$, on an even-dimensional quadric Q there exist two *spinor bundles* $\mathcal{S}_+, \mathcal{S}_-$ which are related by the isomorphism $\mathcal{S}_\pm \otimes \mathcal{O}_Q(1) \cong \mathcal{S}_\pm^\vee$ or $\mathcal{S}_\mp^\vee$ depending on the parity of $\dim Q/2$. By mutating Kapranov's decomposition from [**MR0939472**] these bundles give rise to the Lefschetz decompositions

(15) $\quad \mathbf{D}^b(Q) = \begin{cases} \langle \mathcal{S}, \mathcal{O}_Q, \mathcal{O}_Q(1), \ldots, \mathcal{O}_Q(\dim Q - 1) \rangle & \dim Q \text{ odd}, \\ \langle \mathcal{S}_+, \mathcal{O}_Q, \mathcal{S}_+(1), \mathcal{O}_Q(1), \ldots, \mathcal{O}_Q(\dim Q - 1) \rangle & \dim Q \text{ even}. \end{cases}$

In both cases the Lefschetz center $\mathcal{A}_0$ is generated by two objects, namely $\mathcal{A}_0 = \langle \mathcal{S}, \mathcal{O}_Q \rangle$ resp. $\langle \mathcal{S}_+, \mathcal{O}_Q \rangle$. In the odd-dimensional case we have that $\mathcal{A}_1 = \ldots = \mathcal{A}_{\dim Q - 1} = \langle \mathcal{O}_Q \rangle$, whilst in the even-dimensional case we have that $\mathcal{A}_0 = \mathcal{A}_1$, and $\mathcal{A}_2 = \ldots = \mathcal{A}_{\dim Q - 1} = \langle \mathcal{O}_Q \rangle$.

To heuristically understand the homological projective dual, observe that a general hyperplane section $Q \cap H$ is again a smooth quadric hypersurface. Here $H = \mathbb{P}(L)$ for subspace $L \subseteq V$ of codimension 1, so that $\mathbb{P}(L^\perp) = \mathrm{pt}$. The answer depends on the parity of $\dim Q \cap H$.

If the hyperplane section is odd-dimensional, then the restriction from the Lefschetz collection on Q provides the full exceptional collection in (15) because the restriction of a spinor bundle is again a spinor bundle. This suggests that the homological projective dual is supported on the classical projective dual, because only for singular hyperplane sections a non-trivial contribution can exist.

If the hyperplane section is even-dimensional, then the restriction from the Lefschetz collection on Q provides all the twists of $\mathcal{O}_{Q\cap H}$ in (15), but not the two spinor bundles. These are orthogonal exceptional objects, so that we may interpret $\mathcal{C}_H$ in (3) as corresponding to a double cover with discriminant the classical projective dual.

This heuristic picture can be made fully rigorous, and this is indeed the main result of [**MR4283549**]. It is quite rare that the homological projective dual is supported on the classical projective dual, as this requires the full exceptional collection to restrict to a full exceptional collection for a smooth hyperplane section. Whilst it will turn out not to be the case for the Segre cubic, it is conjecturally true for the Cartan cubic in $\mathbb{P}^{26}$, see [**cayley-plane**].

2.2 Homological projective duality for projective bundles

We will now introduce the version of homological projective duality we will need to study the Segre cubic. To do so we will first recall the setup of [**MR2354207**], which sets up a relative homological projective duality, and then we will explain how this leads to the absolute version.

Let S be a smooth variety, and $\mathcal{E}$ a vector bundle of rank r on S, such that the dual $\mathcal{E}^\vee$ is globally generated. Let $X = \mathbb{P}_S(\mathcal{E})$ be the projective bundle over S parameterising line subbundles of $\mathcal{E}$, with structure morphism

(16) $p \colon X \to S$,

and let $\mathcal{O}_p(-1)$ denote the tautological line subbundle of $p^*\mathcal{E}$ such that $p_*\mathcal{O}_p(1) = \mathcal{E}^\vee$.

We will also need a map to some projective space. For this we take

(17) $V := \mathrm{H}^0(X, \mathcal{O}_p(1))^\vee = \mathrm{H}^0(S, \mathcal{E}^\vee)^\vee$,

so that we can consider the morphism

(18) $f \colon X \to \mathbb{P}(V)$.

Orlov's projective bundle formula gives a $\mathbf{D}^{\mathrm{b}}(S)$-linear Lefschetz decomposition

(19) $\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{\mathrm{rk}\,\mathcal{E}-1}(\mathrm{rk}\,\mathcal{E} - 1)\rangle$

with respect to $\mathcal{O}_X(1) := \mathcal{O}_p(1)$, such that the Lefschetz center is $\mathcal{A}_0 = p^*(\mathbf{D}^{\mathrm{b}}(S))$. We have

(20) $\mathcal{A}_0 = \ldots = \mathcal{A}_{\mathrm{rk}\,\mathcal{E}-1} = p^*(\mathbf{D}^{\mathrm{b}}(S))$,

so this is a *rectangular* Lefschetz decomposition. Observe that $\mathcal{O}_p(1) \cong f^*(\mathcal{O}_{\mathbb{P}(V)}(1))$, so that we also have a Lefschetz decomposition with respect to (18).

The (relative) homological projective dual to $X = \mathbb{P}_S(\mathcal{E})$ is constructed using the orthogonal bundle, i.e. we set

(21) $\mathcal{E}^\perp := \ker(V^\vee \otimes \mathcal{O}_S \to \mathcal{E}^\vee)$

and consider $Y := \mathbb{P}_S(\mathcal{E}^\perp)$, with

(22) $\quad q \colon Y \to S.$

Equip the derived category with the (again rectangular) $\mathbf{D}^b(S)$-linear Lefschetz structure

(23) $\quad \mathbf{D}^b(Y) = \langle \mathcal{B}_{\mathrm{rk}\, \mathcal{E}^\perp - 1}(-\dim V + \mathrm{rk}\, \mathcal{E} + 1), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle$

with respect to $\mathcal{O}_Y(1) := \mathcal{O}_q(1)$, where $\mathcal{B}_0 = q^*(\mathbf{D}^b(S))$.

Again we will need a map to some projective space. In the definition of Y we have $q_* \mathcal{O}_q(1) = \mathcal{E}^{\perp, \vee}$ where

(24) $\quad 0 \to \mathcal{E} \to V \otimes \mathcal{O}_S \to \mathcal{E}^{\perp, \vee} \to 0$

is the dual of the defining sequence (21) so that $\mathrm{H}^0(Y, \mathcal{O}_q(1)) = \mathrm{H}^0(S, \mathcal{E}^{\perp, \vee}) = V$, and we get the morphism

(25) $\quad g \colon Y \to \mathbb{P}(V^\vee).$

As before, we have that $\mathcal{O}_q(1) \cong g^*(\mathcal{O}_{\mathbb{P}(V^\vee)}(1))$ so (23) is also a Lefschetz decomposition with respect to (25) In [**MR2354207**] homological projective duality for projective bundles is then stated[5] as follows.

Theorem 6 (Relative version). *Let S be a smooth variety, and $\mathcal{E}$ a vector bundle of rank r on S such that $\mathcal{E}^\vee$ is globally generated. The* relative *homological projective dual of $p \colon X \to S$ (where $X := \mathbb{P}_S(\mathcal{E})$) with respect to the Lefschetz structure from (19), is $q \colon Y \to S$ (where $Y := \mathbb{P}_S(\mathcal{E}^\perp)$) with respect to the Lefschetz structure from (23).*

This relative version of homological projective duality is explained in [**MR2354207**]: instead of the morphism $X \to \mathbb{P}(V)$ the morphism $X \to \mathbb{P}(V) \times S$ is used. One can apply this relative version of homological projective duality using arbitrary morphisms of vector bundles $\phi \colon \mathcal{F} \to \mathcal{E}^\vee$, as explained in [**MR2354207**]. This e.g. leads to a derived equivalence for certain flops, see Corollary 8.9 in op. cit.

Rather we want to consider homological projective duality for the morphism (18), so that we will only consider linear sections given by $L \subseteq V$.

Corollary 7 (Absolute version). *With the setup of Theorem 6, define $V := \mathrm{H}^0(S, \mathcal{E}^\vee)^\vee$. The homological projective dual of $X \to \mathbb{P}(V)$ with respect to the Lefschetz structure from (19) is $Y \to \mathbb{P}(V^\vee)$, with respect to the Lefschetz structure from (23).*

In the setting of Theorem 6 and Corollary 7 the homological projective dual is again a very explicit variety (which a priori does not need to be true at all). In order to apply this absolute version of homological projective duality, one needs to determine $\mathcal{E}^\perp$ and understand the induced morphism $Y \to \mathbb{P}(V^\vee)$.

The interested reader is referred to [**MR3490767**] for other applications of Corollary 7, where this setup is the main ingredient in the proof of homological projective duality for determinantal varieties. The geometric input in op. cit. is provided by the Springer resolution of the space of matrices of bounded rank, so that $\mathcal{E}^\perp$ has an interpretation in terms of representation theory. In the next section we will describe the geometric input for the Segre cubic.

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[5]There is a typo in the statement: in the notation of op. cit. one has to require that E^* is generated by global sections.

3 The Segre cubic, the Castelnuovo–Richmond quartic and the Coble fourfold

The Segre cubic S_3 The *Segre cubic* S_3 lives in $\mathbb{P}^4$, but we can define it more symmetrically in $\mathbb{P}^5$ as the subvariety defined by

$$(26) \quad \begin{cases} 0 = \displaystyle\sum_{i=0}^{5} x_i, \\[2mm] 0 = \displaystyle\sum_{i=0}^{5} x_i^3. \end{cases}$$

We will consider S_3 as embedded in the invariant hyperplane given by the first equation. This is a more invariant description than a description directly within $\mathbb{P}^4$, as this exhibits S_3 as the zero locus of the first and third symmetric polynomial, and it is the starting point to prove that $\mathrm{Aut}(S_3) \cong \mathfrak{S}_6$.

The Segre cubic is characterised as the unique cubic threefold with ten nodes, up to projective equivalence. For more about its projective geometry we refer to Section 4.1.

It appears often in algebraic geometry. For instance, by Coble [**MR1501008**] (see also [**MR2964027**]) S_3 is isomorphic to the GIT quotient $(\mathbb{P}^1)^6 /\!\!/ \mathrm{SL}_2$. If one blows up the 10 nodes on S_3 the resulting variety is $\overline{\mathrm{M}}_{0,6}$, the Deligne–Mumford compactification of the moduli space of stable rational curves with 6 marked points [**MR4076812**].

It is also the Igusa compactification of the moduli space of principally polarised abelian surfaces with a level-2 structure, as explained in [**MR1929793**]. One can moreover show that S_3 is the Satake compactification of $Y^*(\sqrt{-3})$ [**MR1274130**], the arithmetic quotient $(\Gamma(\sqrt{-3}) \setminus \mathbb{B}_3)^*$, where $\Gamma(\sqrt{-3})$ is the corresponding lattice in $\mathrm{U}(3, 1, \mathcal{O}_{\mathbb{Q}(\sqrt{-3})})$ and $\mathbb{B}_3$ is the complex 3-ball.

Another modular interpretation of S_3 and some of its resolutions in terms of quiver representations, closely related to the GIT description, will be discussed in Section 5.

What is relevant to us about the Segre cubic is that its projective dual is of degree 4, which is remarkably low. Indeed, the *Plücker–Teissier formula* [**MR2964027**] for a hypersurface X of degree d in $\mathbb{P}^n$ with m nodal singularities (and no other singularities) gives

$$(27) \quad \deg X^\vee = d(d-1)^{n-1} - 2m$$

for the projectively dual hypersurface $X^\vee$. We will describe the projective dual of the Segre cubic explicitly next. On the other hand, the projective dual of a smooth cubic threefold is a highly singular hypersurface of degree 24, and we explain in Section 3.3 why homological projective duality in this case is not very enlightening.

The Castelnuovo–Richmond quartic CR_4 The *Castelnuovo–Richmond quartic* CR_4 (or *Igusa quartic*) in $\mathbb{P}^5$ can be defined as the complete intersection by the two equations

$$(28) \quad \begin{cases} 0 = \displaystyle\sum_{i=0}^{5} x_i, \\[3mm] 0 = \left(\displaystyle\sum_{i=0}^{5} x_i^2 \right)^2 - 4 \displaystyle\sum_{i=0}^{5} x_i^4. \end{cases}$$

We will consider CR_4 as embedded in the invariant hyperplane given by the first equation. By [**MR2964027**] CR_4 is projectively dual to S_3. It is singular along 15 lines, which intersect in 15 points,

with 3 lines through each point, and 3 points on each line, giving rise to the *Cremona–Richmond configuration*. We will further discuss its singularities and projective geometry in Section 4.1.

As for S_3, the Castelnuovo–Richmond quartic has a rich modular theory. From the projective duality S_3 and CR_4 are birational. This manifests itself for instance as an interpretation of CR_4 as the compactification of the moduli of 6 ordered points on a conic, and hence also has a quotient description as (another) GIT quotient $(\mathbb{P}^1)^6 /\!\!/ SL_2$. It is also the Satake compactification of the moduli space of principally polarised abelian surfaces with a level-2 structure, and it is the Satake compactification $X^*(2)$ [**MR1274130**], the arithmetic quotient $(\Gamma(2) \setminus \mathbb{S}_2)^*$, where $\Gamma(2)$ is the corresponding lattice in $Sp(4, \mathbb{Z})$, and $\mathbb{S}_2$ is the Siegel space of degree 2.

The Coble fourfold Cob   The last remaining player to be introduced in this section is the Coble fourfold Cob, defined as the double cover

(29) $\pi\colon \mathrm{Cob} \to \mathbb{P}^4$

branched along the Castelnuovo–Richmond quartic.

The Coble fourfold also appears often in algebraic geometry. For instance, by Coble [**MR1501008**] (see [**MR1007155**] for a modern account) Cob is isomorphic to the GIT quotient $(\mathbb{P}^2)^6 /\!\!/ SL_3$, and thus related to the moduli space of marked cubic surfaces (with the marking corresponding to the ordering of the 6 points being blown up). It also has a modular interpretation as the Baily–Borel compactification of an arithmetic quotient, parametrising K3 surfaces which are double covers of $\mathbb{P}^2$ branched along an (ordered) set of 6 lines [**MR1438547**].

The geometry of its singularities is the same as that of the Castelnuovo–Richmond quartic.

3.1   Resolutions of singularities

In order to apply the theory of homological projective duality as described in Section 2 we need to suitably replace the Segre cubic (and the Coble fourfold), because the input (at least in the version from [**MR2354207**]) is required to be smooth and proper. We comment on a variation without resolution in Section 3.3.

The small resolutions of the Segre cubic were classified by Finkelnberg in [**MR0914085**]. There are two resolutions in the overview table in §5 of op. cit. whose automorphism group is $\mathfrak{S}_5$, corresponding to type IV and VI. The geometry of one of these resolutions is linked to a resolution of the Coble fourfold Cob, and we will now recall this from [**MR4076812**]. An alternative discussion of the same geometry is given in [**1906.12295v2**]. We comment on the other resolution in Remark 9.

The quintic del Pezzo surface   The key in the description of the resolutions of S_3 and Cob is the del Pezzo surface S of degree 5. There is (up to isomorphism) a unique such surface, isomorphic to the blowup of $\mathbb{P}^2$ in 4 points, no three of which are collinear. Its automorphism group is $\mathfrak{S}_5$, whose irreducible representations are described in Table 1. The most important representation for us will be the irreducible 5-dimensional representation W with trivial determinant. It arises, together with $W \otimes -1$, as the summands of $\mathrm{Ind}_{\mathfrak{S}_4}^{\mathfrak{S}_5} U$ where U is the unique 2-dimensional irreducible representation of $\mathfrak{S}_4$.

As recalled in [**MR4076812**], the canonical embedding

(30) $S \hookrightarrow \mathbb{P}(\mathrm{H}^0(S, \omega_S^\vee)) = \mathbb{P}(\bigwedge^2 V) = \mathbb{P}^5$

involves the unique irreducible $\mathfrak{S}_5$-representation $\bigwedge^2 V$ of dimension 6. From this description of S as the (non-complete) intersection of 5 quadrics, we obtain the irreducible 5-dimensional $\mathfrak{S}_5$-representation

(31) $\mathrm{H}^0(\mathbb{P}^5, \mathcal{I}_S(2))^\vee = W$

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description	notation	dimension	partition	determinant
trivial representation	1	1	(5)	1
sign representation	-1	1	(1^5)	-1
standard representation	V	4	$(4,1)$	-1
twisted standard representation	$V \otimes -1$	4	$(2,1,1,1)$	-1
half of induced from $\mathfrak{S}_4$	W	5	$(3,2)$	1
other half of induced from $\mathfrak{S}_4$	$W \otimes -1$	5	$(2,2,1)$	-1
exterior square of standard	$\bigwedge^2 V$	6	$(3,1,1)$	-1

Table 1: Irreducible representations of $\mathfrak{S}_5$

as per [**MR4076812**].

Two manifestations of the Grassmannians will play a role in what follows: $\mathrm{Gr}(2, W^\vee)$ and $\mathrm{Gr}(3, W)$. These come equipped with universal subbundles of ranks 2 respectively 3, which will be denoted $\mathcal{U}_2$ resp. $\mathcal{U}_3$. We have the universal short exact sequences

(32) $\quad 0 \to \mathcal{U}_2 \to W^\vee \otimes \mathcal{O}_{\mathrm{Gr}(2,W^\vee)} \to \mathcal{Q}_3 \to 0$

and

(33) $\quad 0 \to \mathcal{U}_3 \to W \otimes \mathcal{O}_{\mathrm{Gr}(3,W)} \to \mathcal{Q}_2 \to 0.$

Using the embedding $\mathbb{P}(\bigwedge^2 V) \hookrightarrow \mathbb{P}(\bigwedge^3 W)$ from [**MR4076812**] we can describe S as the intersection of $\mathrm{Gr}(3, W)$ in its Plücker embedding with $\mathbb{P}(\bigwedge^2 V)$. We remark that the linear $\mathfrak{S}_5$-invariant embedding $\mathbb{P}(\bigwedge^2 V) \hookrightarrow \mathbb{P}(\bigwedge^3 W)$ corresponds to the irreducible representation $V \wedge V$ appearing as the only six-dimensional irreducible representation in $\bigwedge^3 W$, see also [**MR3851123**].

This allows us to restrict $\mathcal{U}_2$ and $\mathcal{U}_3$ to $S \subseteq \mathrm{Gr}(2, W^\vee) \cong \mathrm{Gr}(3, W)$, and we will use the same notation for these rank 2 and 3 bundles on S.

Small resolutions of the Segre cubic and Coble fourfold   We now define the two main varieties, whose homological projective duality we will prove in Theorem 11. Namely we set

(34) $\quad X := \mathbb{P}_S(\mathcal{U}_2), \qquad Y := \mathbb{P}_S(\mathcal{U}_3).$

Because $\mathcal{U}_2$ respectively $\mathcal{U}_3$ are naturally subbundles of $W^\vee \otimes \mathcal{O}_S$ respectively $W \otimes \mathcal{O}_S$ we obtain two ($\mathfrak{S}_5$-equivariant) morphisms:

(35) $\quad f \colon X \to \mathbb{P}(W^\vee), \qquad g \colon Y \to \mathbb{P}(W).$

The following proposition explains our interest in these projective bundles. It is a combination of [**MR4076812**].

Proposition 8 (Cheltsov–Kuznetsov–Shramov). *The natural map $f \colon X \to \mathbb{P}(W^\vee)$ has the Segre cubic S_3 as its image, and exhibits X as a small resolution $\varpi \colon X \to S_3$.*

The Stein factorisation of the natural map $g \colon Y \to \mathbb{P}(W)$ has the Coble fourfold Cob as intermediate variety, and it is the composition of a small resolution $\rho \colon Y \to \mathrm{Cob}$ and the Coble fourfold's defining double cover ramified in the Castelnuovo–Richmond quartic CR_4.

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We can summarise the situation in the following diagrams: for the Segre cubic S_3 we have

(36)

$$\begin{array}{ccc}
 & X = \mathbb{P}_S(\mathcal{U}_2) \xrightarrow{\quad f \quad} \mathbb{P}(W^\vee) \\
\mathbb{P}^1\text{-bundle} \swarrow p \quad \searrow \omega \quad \nearrow i \\
S \quad\quad \begin{array}{c}\text{small}\\ \text{resolution}\end{array} \quad S_3
\end{array}$$

and the Coble fourfold Cob (and Castelnuovo–Richmond quartic CR_4) we have

(37)

$$\begin{array}{c}
CR_4 \\
\downarrow \\
Y = \mathbb{P}_S(\mathcal{U}_3) \xrightarrow{\quad g \quad} \mathbb{P}(W) \\
\mathbb{P}^2\text{-bundle} \swarrow q \quad \searrow \rho \quad \nearrow \pi \\
S \quad \begin{array}{c}\text{small}\\ \text{resolution}\end{array} \quad \text{Cob} \quad {\scriptstyle 2:1}
\end{array}$$

We will discuss some properties of these resolutions in Section 4, as we need to understand the properties of the fibers in order to describe how (derived categories of) linear sections of X and Y are related.

Remark 9. All small resolutions of the Segre cubic are all obtained by blowing up the 10 nodes, and contracting a factor of every exceptional $\mathbb{P}^1 \times \mathbb{P}^1$. This gives 2^{10} resolutions, which can be grouped into 13 isomorphism classes [**MR0914085**]. Of these, 6 correspond to smooth projective varieties, the other 7 correspond to smooth proper non-projective algebraic spaces. They are all related via flops, hence by [**alg-geom/9506012**] (or the more general [**MR1893007**]) they are all derived equivalent.

The preferred resolution used here, and taken from [**MR4076812**], has the benefit of giving an easy-to-describe Lefschetz structure for which we can immediately apply an existing result to determine the homological projective dual.

One could also use another resolution. By [**MR2605172**] the blowup $X' := \mathrm{Bl}_5 \mathbb{P}^3$ of 5 points in general position realises the second isomorphism type of small resolutions of S_3 with automorphism group $\mathfrak{S}_5$. This is again a weak Fano threefold of index 2, whose half-anticanonical divisor $\mathcal{O}_{X'}(-\frac{1}{2}K_{X'})$ in this description is $\mathcal{O}_{X'}(2H - E)$, where the E_i are the exceptional divisors, we set $E := \sum_{i=1}^5 E_i$, and H is the pullback of $\mathcal{O}_{\mathbb{P}^3}(1)$. It is basepoint-free, and defines a morphism to $\mathbb{P}^4$ with image the Segre cubic, which realises it as the resolution of S_3.

Proposition 10. *Let $X' \to S_3$ be the small resolution of the Segre cubic $S_3 \hookrightarrow \mathbb{P}(W^\vee)$ given by [MR2605172], so that $X' = \mathrm{Bl}_5 \mathbb{P}^3$ of 5 points in general position. Let $\mathcal{O}_{X'}(H)$ be the pullback of $\mathcal{O}_{\mathbb{P}^3}(1)$, and $E_1, \ldots, E_5$ the exceptional divisors. Then*

(38)
$$\begin{aligned}
\mathbf{D}^b(X') = \langle &\mathcal{O}_{X'}, \mathcal{O}_{X'}(H), \mathcal{O}_{E_i} \mid i = 1, \ldots, 5; \\
&\mathcal{O}_{X'}(2H - E), \mathcal{O}_{X'}(3H - E); \mathcal{O}_{E_i}(E_i) \mid i = 1, \ldots, 5 \rangle,
\end{aligned}$$

is a rectangular Lefschetz decomposition of $\mathbf{D}^b(X') \cong \mathbf{D}^b(X)$.

Proof. From Orlov's blowup formula we have the full exceptional collection

(39) $\quad \mathbf{D}^b(X') = \langle \mathcal{O}_{X'}, \mathcal{O}_{X'}(H), \mathcal{O}_{X'}(2H), \mathcal{O}_{X'}(3H); \mathcal{O}_{E_i}, \mathcal{O}_{E_i}(-E_i) \mid i = 1, \ldots, 5 \rangle.$

Now right mutate $\langle \mathcal{O}_{X'}(2H), \mathcal{O}_{X'}(3H) \rangle$ with respect to $\langle \mathcal{O}_{E_i} \mid i = 1, \ldots, 5 \rangle$ to obtain (38). Here we have used that $\mathcal{O}_{E_i}(H) = \mathcal{O}_{E_i}$ and $\mathcal{O}_{E_i}(E_j) = \mathcal{O}_E$ if $i \neq j$. $\quad\square$

We will come back to this Lefschetz structure in Proposition 31.

3.2 Homological projective duality for the Segre cubic

We can now state and prove the main theorem, which was stated somewhat imprecisely in the introduction. Because the small resolutions discussed in Section 3.1 are both described by projective bundles over the quintic del Pezzo surface S we only need to check the relation between the defining bundles to establish homological projective duality.

Theorem 11. *Let $X \to S_3$ be the small resolution of the Segre cubic $S_3 \hookrightarrow \mathbb{P}(W^\vee)$ given in Proposition 8, so that $p\colon X = \mathbb{P}_S(\mathcal{U}_2) \to S$ is a $\mathbb{P}^1$-bundle over the del Pezzo surface S of degree 5. Consider the morphism $f\colon X \to \mathbb{P}(W^\vee)$, and the rectangular Lefschetz decomposition*

$$(40) \quad \mathbf{D}^b(X) = \langle p^*(\mathbf{D}^b(S)), p^*(\mathbf{D}^b(S)) \otimes \mathcal{O}_X(1) \rangle$$

where $\mathcal{O}_X(1) = f^*(\mathcal{O}_{\mathbb{P}(W^\vee)}(1))$.

Its homological projective dual is $g\colon Y \to \mathbb{P}(W)$, where $Y \to \mathrm{Cob}$ is the small resolution of the Coble fourfold given in Proposition 8, so that $q\colon Y = \mathbb{P}_S(\mathcal{U}_3) \to S$ is a $\mathbb{P}^2$-bundle over S. The dual Lefschetz decomposition is given by

$$(41) \quad \mathbf{D}^b(Y) = \langle q^*(\mathbf{D}^b(S)) \otimes \mathcal{O}_Y(-2), q^*(\mathbf{D}^b(S)) \otimes \mathcal{O}_Y(-1), q^*(\mathbf{D}^b(S)) \otimes \mathcal{O}_Y \rangle$$

where $\mathcal{O}_Y(1) = g^*(\mathcal{O}_{\mathbb{P}(W)}(1))$.

Proof. By the setup in Section 2.2 we need to check that $\mathcal{U}_2$ and $\mathcal{U}_3$ are related by taking the orthogonal bundle (21). But they are restrictions of the universal subbundle on the Grassmannians $\mathrm{Gr}(2, W^\vee) \cong \mathrm{Gr}(3, W)$. Before the restriction we have that $\mathcal{U}_2^\vee$ on $\mathrm{Gr}(2, W^\vee)$ in (32) is identified with $\mathcal{Q}_2$ on $\mathrm{Gr}(3, W)$ in (33) by this isomorphism, so that $\mathcal{U}_2^\perp$ is identified with $\mathcal{U}_3$ by this isomorphism. It now suffices to restrict this identification to S, and apply Corollary 7. □

Classical projective duality The motivation for the terminology *homological projective duality* is explained on [**MR2354207**], at least when the homological projective dual is an honest variety. Namely if $X \to \mathbb{P}(V)$ is a closed immersion (so that it has a projective dual $X^\vee \subseteq \mathbb{P}(V^\vee)$ in the usual sense), then we have an equality of subsets of $\mathbb{P}(V^\vee)$ between

- the projective dual $X^\vee$;
- the critical locus of the homological projective dual variety $Y \to \mathbb{P}(V^\vee)$.

Here critical locus refers to the complement of the dense open locus where $Y \to \mathbb{P}(V^\vee)$ is smooth (by generic smoothness). This allows homological projective duality to be interpreted as a categorification, or homological version, of classical projective duality.

More generally, in [**MR2354207**] a definition of $X^\vee$ is given when $X \to \mathbb{P}(V)$ is not necessarily a closed immersion. This is called the *classical projective dual*, a terminology explained by [**MR2354207**]. This is further generalised to a noncommutative setting in [**MR3948688**].

We can conclude with the following corollary, which shows that replacing the Segre cubic with a small resolution in order to apply the theory of homological projective duality did not change the link with classical projective duality. Observe that the critical locus of $Y \to \mathbb{P}(W)$ is the critical locus of $\mathrm{Cob} \to \mathbb{P}(W)$, which is CR_4.

Corollary 12. *The classical projective dual of $f\colon X \to \mathbb{P}(W^\vee)$ is the Castelnuovo–Richmond quartic $\mathrm{CR}_4 \subseteq \mathbb{P}(W)$. The classical projective dual of $g\colon Y \to \mathbb{P}(W)$ is the Segre cubic $S_3 \subseteq \mathbb{P}(W^\vee)$. In particular, for all $L \subseteq W^\vee$ the following are equivalent:*

- *X_L is singular;*
- *$Y_{L^\perp}$ is singular.*

Here we take the *derived fiber product* in the sense of derived algebraic geometry, so that smooth and singular are to be taken in the sense of [**MR3948688**].

Remark 13. We observe that for the equivalence in Corollary 12 it is important that we use resolutions and derived fiber products. As we will explain in Lemma 18, for a point $H \in \mathbb{P}(W)$ we have that the corresponding hyperplane section of S_3 is smooth if and only if it lies outside the union of CR_4 and the 10 hyperplanes P_i corresponding to the 10 nodes, making the classical projective dual too big. But for a point H of $P_i \setminus CR_4$ we have that $H \cap Cob = Y_{L^\perp} = \{2 \text{ points}\}$ is smooth (and the fiber product agrees with the derived fiber product). By considering the resolution $X \to \mathbb{P}(W^\vee)$ instead of $S_3 \subseteq \mathbb{P}(W^\vee)$ the hyperplane section X_H is smooth as we will explain in Section 4.2.

Secondly, for most points in the singular locus of CR_4 the ordinary fiber product is a smooth rational curve by [**MR4076812**]. But the relative dimension of the morphism $Y \to \mathbb{P}(W)$ is 0, not 1.

Remark 14. If L is admissible in the sense of [**MR2354207**], i.e. *both* derived fiber products are in fact underived because they are of the expected dimension, [**MR2354207**] gives an identification of the singularity categories of $X \times_{\mathbb{P}(W^\vee)} \mathbb{P}(L)$ and $Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L^\perp)$. If L is not admissible (because at least one fiber product does not have the expected dimension) then for the derived fiber product(s) one has to use a notion of singularity category from derived algebraic geometry. We will not go into this.

3.3 What about other cubic threefolds?

We now discuss some possible variations on the theme of homological projective duality for our chosen resolution of the Segre cubic.

Smooth cubics If one would like to study homological projective duality for a *smooth* cubic threefold, one runs into the following problem. Let X denote a smooth cubic threefold for now, then we have the semiorthogonal decomposition

(42) $\quad \mathbf{D}^b(X) = \langle \mathcal{R}_X, \mathcal{O}_X, \mathcal{O}_X(1) \rangle$

where $\mathcal{R}_X$ is a $\frac{5}{3}$-Calabi–Yau category [**MR3987870**].

Whilst it is not clear whether $\mathcal{R}_X$ is indecomposable, the decomposition (42) cannot in any way be refined to an interesting Lefschetz decomposition. By considering the Hodge numbers of X together with the additivity of Hochschild homology we see that the longest length of an exceptional collection is 4, so $\mathcal{A}_1$ can at most consist of 2 exceptional objects, hence the interesting component (if such a Lefschetz decomposition exists at all) consists of 7 exceptional objects with a complicated structure.

If on the other hand we were to take $\langle \mathcal{R}_X, \mathcal{O}_X \rangle$ as the initial block, so that $\mathcal{A}_1 = \langle \mathcal{O}_X \rangle$, then the only contribution to the derived category of a hyperplane section is a single exceptional object. Hence the homological projective dual is a very complicated object, and we refer to [**MR3626552**] for a description of derived categories of hypersurfaces in terms of gauged Landau–Ginzburg models, which can be used as a possible starting point for a description of the interesting component (which in this case is a $\frac{4}{3}$-Calabi–Yau category consisting of 8 exceptional objects).

One could argue that homological projective duality is a balancing act, where one tries to make the input data and the resulting homological projective dual have roughly equal complexity, so that one can leverage information about one to understand the other. The resolution of the Segre cubic seems to provide a particularly good solution to this balancing act.

The unresolved Segre cubic The smoothness assumption for the initial input which is present in [**MR2354207**] has been removed in [**MR3948688**]. But currently lacking an interesting Lefschetz

decomposition for $\mathbf{D}^b(S_3)$ to start the machinery with we have focused in this article on the smooth case. It would be interesting to find a similar Lefschetz decomposition for the Segre cubic, or rule out the existence of one.

Resolutions of singular cubic threefolds with fewer nodes   If instead of the Segre cubic or a smooth cubic threefold one wishes to analyse homological projective duality for (a resolution of) a cubic threefold with 1 to 9 nodes, one can consider the description and classification obtained in [**MR1005050**]. A starting point for the description of the derived categories of the singular cubics is provided by [**MR4277855**].

Starting with the 1-nodal case, by [**MR3050698**] the derived category of a small resolution has a semiorthogonal decomposition in terms of the 2 exceptional line bundles $\mathcal{O}_X, \mathcal{O}_X(1)$ and the derived category of a genus 4 curve. This brings us in a situation similar to that of the smooth cubic and the homological projective dual is seemingly a complicated object.

If the nodal cubic threefold has a *determinantal* presentation, we can bootstrap from homological projective duality for determinantal varieties [**MR3490767**].

Consider first the 6-nodal case, which is the general determinantal cubic threefold and corresponds to the case J_9 in [**MR1005050**]. The derived category of a small resolution has a semiorthogonal decomposition in terms of the 2 exceptional line bundles $\mathcal{O}_X, \mathcal{O}_X(1)$ and 4 additional exceptional objects [**MR3490767**]. Assume for now that we can turn these 6 exceptional objects into a rectangular Lefschetz decomposition, then the theory provides 3 exceptional objects for the derived category of a hyperplane section. A general section is a smooth cubic surface, for which a full exceptional collection contains 9 exceptional objects. Therefore the general fiber of the homological projective dual $X^\natural \to \mathbb{P}(V^\vee)$ consists of 6 exceptional objects.

We can make this prediction precise by the following proposition, which also covers cases with more nodes. We recall the setup for homological projective duality for determinantal hypersurfaces from [**MR3490767**] for this specific instance.

Setup 15. Let A and B be 3-dimensional vector spaces, and set $V := A \otimes_k B$. Let $X \subset \mathbb{P}(V^\vee) = \mathbb{P}^8$ be the cubic determinantal 7-fold corresponding to the locus of 3×3-matrices of rank at most 2. Let $\widetilde{X} \to X$ be its Springer resolution, given as the projectivisation of $\Omega^1_{\mathbb{P}(A)}(1) \otimes_k B$. We will consider the composition

$$(43) \quad f \colon \widetilde{X} \to \mathbb{P}(V^\vee).$$

On the dual side we let

$$(44) \quad g \colon Y = \mathbb{P}^2 \times \mathbb{P}^2 = \mathbb{P}(A) \times \mathbb{P}(B) \hookrightarrow \mathbb{P}(V)$$

be the Segre embedding. These have rectangular Lefschetz decompositions

$$
(45) \quad
\begin{aligned}
\mathbf{D}^b(\widetilde{X}) &= \langle \mathcal{A}_0, \mathcal{A}_1(1), \mathcal{A}_2(2), \mathcal{A}_3(3), \mathcal{A}_4(4), \mathcal{A}_5(5) \rangle \\
\mathbf{D}^b(Y) &= \langle \mathcal{B}_2(-2), \mathcal{B}_1(-1), \mathcal{B}_0 \rangle
\end{aligned}
$$

with Lefschetz center $\mathcal{A}_0 = \mathcal{B}_0 = \mathbf{D}^b(\mathbb{P}(A))$, and by [**MR3490767**] we have homological projective duality for $\widetilde{X}$ and Y with respect to these choices. This result is another instance of homological projective duality for projective bundles given in Section 2.2.

We need to reduce the case of cubic in $\mathbb{P}^8$ to that of a cubic in $\mathbb{P}^4$, and deduce homological projective duality for this linear section. The following proposition describes the result of this procedure.

Proposition 16. *Let $L \subset V^\vee$ be a 5-dimensional linear subspace, such that $Y \cap \mathbb{P}(L^\perp) = \emptyset$, and hence $X \cap \mathbb{P}(L)$ is a 6-nodal determinantal cubic threefold.*

The linear projection $g_L \colon Y \to \mathbb{P}(L^\vee)$ is homological projective dual to the restriction $f_L \colon \widetilde{X}_L \to \mathbb{P}(L)$, where $\mathbf{D}^{\mathrm{b}}(Y)$ and $\mathbf{D}^{\mathrm{b}}(\widetilde{X}_L)$ are equipped with the rectangular Lefschetz decompositions

(46)
$$\mathbf{D}^{\mathrm{b}}(\widetilde{X}_L) = \langle \mathcal{A}_0, \mathcal{A}_1(1) \rangle$$
$$\mathbf{D}^{\mathrm{b}}(Y) = \langle \mathcal{B}_2(-2), \mathcal{B}_1(-1), \mathcal{B}_0 \rangle$$

with $\mathcal{A}_0 = \mathcal{B}_0 = \mathbf{D}^{\mathrm{b}}(\mathbb{P}^2)$.

Proof. This follows from applying homological projective duality for linear systems with a base locus [**MR4176839**] (see also [**MR4280866**] for an abstract version with empty base locus), with the roles of $\widetilde{X}$ and Y reversed in the notation of op. cit. The linear section $\widetilde{X}_L$ is the crepant resolution of a determinantal cubic threefold, and $Y_{L^\perp} = \emptyset$ by our assumption. These are smooth, and satisfy the expected dimension condition, hence we obtain the stated homological projective duality for $\widetilde{X}_L$ and $Y = \mathrm{Bl}_{Y_{L^\perp}} Y$. $\qquad\square$

We have that the linear projection $Y = \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}(L) = \mathbb{P}^4$ is a $6 : 1$-cover, ramified along the classical projective dual of the cubic. By the Plücker–Teissier formula (27) this is a hypersurface of degree 12. Because this is a $6 : 1$-cover, we have that a smooth hyperplane section of $\widetilde{X}$ has 3 exceptional objects coming from the Lefschetz center $\mathcal{A}_0 = \mathbf{D}^{\mathrm{b}}(\mathbb{P}^2)$, and 6 orthogonal exceptional objects from the cover, which is an incarnation of Orlov's blowup formula for the smooth cubic surface $\widetilde{X}_L \cong \mathrm{Bl}_6 \mathbb{P}^2$.

We can also consider a general plane section of $\widetilde{X}_L$, which is a smooth cubic curve. On the dual side we have a line section of $Y_{L^\perp}$, which is a $6 : 1$-cover of $\mathbb{P}^1$ ramified along 12 points, and by the Riemann–Hurwitz formula this is a curve of genus 1. Homological projective duality thus provides an equivalence of categories for these curves by (11).

4   Applying homological projective duality

In this section we will describe linear sections of the resolved Segre cubic and its homological projective dual. This illustrates how the abstract machinery describes derived categories of linear sections, which in this setting have an explicit and classical description. We will describe two methods for this:

- a bottom-up approach, starting from a linear section of S_3 resp. Cob, and then understanding how the resolution affects the description;

- a top-down approach, starting from a linear section of X resp. Y, and then understanding how the restriction of the resolution can be interpreted.

We will be exhaustive in our discussion of hyperplane sections of the (resolved) Segre cubic in Section 4.2, but restrict ourselves to only discussing some interesting examples in the other settings in Section 4.3, as the amount of cases to be covered in a complete case-by-case analysis is large. For the analysis we need to understand both the projective geometry of S_3, CR_4 and Cob (which we recall in Section 4.1) and the geometry of the resolutions defined in Section 3.1.

Before we do this we will explain the recipe of describing the derived category in the ideal situation, when the resolutions do not play a role in the description.

Definition 17. We will say that $L \subseteq W^\vee$ of dimension $2, 3, 4$ is *generic* if

- $\mathbb{P}(L) \cap S_3$ is smooth of dimension $\dim_k L - 2$, and it avoids the singular locus of S_3;

- $\mathbb{P}(L^\perp) \cap CR_4$ is smooth of dimension $3 - \dim_k L$, and it avoids the singular locus of CR_4.

$\dim_k L$	$\mathbb{P}(L) \cap S_3$ $\mathbf{D}^b(\mathbb{P}(L) \cap S_3)$	$\mathbb{P}(L^\perp) \cap CR_4$	$\mathbb{P}(L^\perp) \times_{\mathbb{P}(W)} \mathrm{Cob}$ $\mathbf{D}^b(\mathbb{P}(L^\perp) \times_{\mathbb{P}(W)} \mathrm{Cob})$
2	3 points 3 orthogonal objects	smooth quartic curve	double cover of $\mathbb{P}^2$ ramified in quartic $10 = 7 + 3$ exceptional objects
3	smooth cubic curve C $\mathbf{D}^b(C)$	4 points	double cover C of $\mathbb{P}^1$ ramified in 4 points $\mathbf{D}^b(C)$
4	smooth cubic surface $9 = 7 + 2$ exceptional objects	$\emptyset$	2 points 2 orthogonal objects

Table 2: Description of the generic situation

What happens in this case is that $\mathbb{P}(L) \cap S_3 \cong X_L$ and $\mathbb{P}(L^\perp) \times_{\mathbb{P}(W)} \mathrm{Cob} \cong Y_{L^\perp}$. This makes describing the output of homological projective duality using the bottom-up approach straightforward. If $\dim_k L = 1$ then $\mathbb{P}(L^\perp) \cap CR_4$ always hits the singular locus of the Castelnuovo–Richmond quartic as we will explain in Section 4.1, so there is no generic $L \subseteq W^\vee$ in this case.

We have summarised the description in Table 2. Let us explain what is written there. If $\dim_k L = 2$, then on the Segre side we see that X_L consists of 3 points, whilst on the Coble side we obtain a del Pezzo surface $Y_{L^\perp}$ of degree 2 as the double cover of $\mathbb{P}(L^\perp) \cong \mathbb{P}^2$ ramified in the smooth quartic curve $\mathbb{P}(L^\perp) \cap CR_4$ which is isomorphic to the blowup $\mathrm{Bl}_7 \mathbb{P}^2$ of 7 points in general position. By (11) we get

$$
(47) \quad
\begin{aligned}
\mathbf{D}^b(X_L) &= \mathcal{C}_L = \langle E_1, E_2, E_3 \rangle \\
\mathbf{D}^b(Y_{L^\perp}) &= \langle \mathcal{B}_1(-1), \mathcal{C}_L \rangle = \langle \mathbf{D}^b(S), E_1, E_2, E_3 \rangle.
\end{aligned}
$$

Here E_1, E_2, E_3 are completely orthogonal exceptional objects. In fact, the composition $Y_{L^\perp} \hookrightarrow Y \twoheadrightarrow S$ obtained similarly to (66) exhibits $Y_{L^\perp}$ as $\mathrm{Bl}_3 S$, and thus the semiorthogonal decomposition induced by homological projective duality can be interpreted as an instance of Orlov's blowup formula.

If $\dim_k L = 3$, then on the Segre side we obtain a smooth cubic curve X_L, whilst on the Coble side we obtain a double cover $Y_{L^\perp}$ of $\mathbb{P}(L^\perp) \cong \mathbb{P}^1$ ramified in the 4 points $\mathbb{P}(L^\perp) \cap CR_4$, which is *also* a curve of genus 1. By (11) we get

$$
(48) \quad \mathbf{D}^b(X_L) \cong \mathbf{D}^b(Y_{L^\perp})
$$

as there are no contributions by the Lefschetz components on either side. By the reconstruction of curves from their derived category (see e.g. [**MR2244106**]) we even obtain $X_L \cong Y_{L^\perp}$.

If $\dim_k L = 4$, then on the Segre side we obtain a smooth cubic surface, whilst on the Coble side we see that $Y_{L^\perp}$ consists of 2 points, as $\mathbb{P}(L^\perp) \cap CR_4 = \emptyset$. By (11) we get

$$
(49) \quad
\begin{aligned}
\mathbf{D}^b(X_L) &= \langle \mathcal{C}_L, \mathcal{A}_1(1) \rangle = \langle E_1, E_2, \mathbf{D}^b(S) \rangle \\
\mathbf{D}^b(Y_{L^\perp}) &= \mathcal{C}_L = \langle E_1, E_2 \rangle.
\end{aligned}
$$

Here E_1, E_2 are completely orthogonal exceptional objects. As for the case of $\dim_k L = 2$ we can consider the composition $X_L \hookrightarrow X \twoheadrightarrow S$ obtained similarly to (66) and this exhibits X_L as $\mathrm{Bl}_2 S$, so the same comment as for $Y_{L^\perp}$ with $\dim_k L = 2$ applies.

In what follows we will discuss what happens in the *non-generic* situation, and explain how the resolutions $\varpi \colon X \to S_3$ and $\rho \colon Y \to \mathrm{Cob}$ change the resulting descriptions of (some) linear sections and their derived categories.

4.1 The projective geometry of S_3, CR_4 and Cob

We will now discuss the geometry of the Segre cubic, Castelnuovo–Richmond quartic and Coble fourfold in more detail, to prepare for the description and (partial) classification of linear sections of X and Y.

Segre cubic The Segre cubic contains ten nodes $p_1, \ldots, p_{10}$, which are the $\mathfrak{S}_6$-orbit of the point $(1 : 1 : 1 : -1 : -1 : -1)$. Moreover, S_3 contains exactly 15 planes, called *Segre planes*, defined as

$$(50) \quad P_\sigma := \{x_{\sigma(0)} + x_{\sigma(3)} = x_{\sigma(1)} + x_{\sigma(4)} = x_{\sigma(2)} + x_{\sigma(5)} = 0\}$$

for $\sigma \in \mathfrak{S}_6$. No three of the ten nodes are collinear, and each Segre plane contains exactly four of the ten nodes. Each node is contained in exactly six Segre planes. This gives a $(15_4, 10_6)$-configuration.

Let us also mention two sets of hyperplanes in $\mathbb{P}^5$ (which define hyperplanes in $\mathbb{P}(W^\vee)$). The first set is given by the 15 hyperplanes

$$(51) \quad T_{i,j} := \{x_i - x_j = 0\} \qquad 0 \le i < j \le 5,$$

whose intersection with S_3 is the Cayley cubic surface, the unique cubic surface with 4 nodes. The other set of 15 hyperplanes is given by

$$(52) \quad H_{i,j} := \{x_i + x_j = 0\} \qquad 0 \le i < j \le 5,$$

which have the property that they each contain exactly three of the fifteen Segre planes so that their intersection with S_3 is the union of three projective planes.

Castelnuovo–Richmond quartic The Castelnuovo–Richmond quartic CR_4 is singular along fifteen lines $\ell_1, \ldots, \ell_{15}$. The singular set of CR_4^{sing} consists of fifteen points $q_1, \ldots, q_{15}$ dual to the $H_{i,j}$. and each q_i is contained in 3 lines, so that each ℓ_i intersects 6 other lines. This makes up the *Cremona–Richmond configuration*, which has type $(15_3, 15_3)$.

The Segre cubic and the Castelnuovo–Richmond quartic are projectively dual in the classical sense and we will use the duality throughout the next section. We will therefore elaborate here a bit about it.

There are ten hyperplanes $P_i \subseteq \mathbb{P}(W)$ which correspond to hyperplanes containing the nodes $p_i \in S_3$. The hyperplanes P_i intersect the Castelnuovo–Richmond quartic in a smooth quadric Q_i with non-reduced structure.

The fifteen singular lines of CR_4 are exactly the projective duals of the fifteen Segre planes. Their fifteen intersection points are dual to the fifteen hyperplanes $H_{i,j} \subseteq \mathbb{P}(W)$.

We can also describe the (rational) duality map

$$(53) \quad d \colon S_3 \dashrightarrow CR_4.$$

The map d restricted to the complement of the fifteen Segre planes is an isomorphism onto the complement of the ten hyperplanes P_i, i.e.

$$(54) \quad d \colon S_3 \setminus \bigcup_{\sigma \in \mathfrak{S}_6} P_\sigma \xrightarrow{\sim} CR_4 \setminus \bigcup_{i=1,\ldots,10} P_i,$$

see [**MR1438547**]. There are several implications of this, from which we mention two.

Firstly, for two points $x \neq y \in S_3$ the tangent hyperplanes $T_x S_3, T_y S_3$ agree if and only if both points are contained in a common $P_\sigma \subset S_3$ and the line joining x and y passes through one of the four nodes contained in P_σ. Secondly, any tangent hyperplane $T_x S_3$ at a point $x \in S_3 \setminus \bigcup_{\sigma \in \mathfrak{S}_6} P_\sigma$ does not contain a node.

Coble fourfold   Recall that the Coble fourfold Cob was defined as the double cover $\pi\colon \text{Cob} \to \mathbb{P}^4$ branched along the Castelnuovo–Richmond quartic CR_4. It follows from the definition that the singular locus of Cob is isomorphic to the singular locus of CR_4.

4.2   Hyperplane sections of the (resolved) Segre cubic

We can now describe hyperplane sections of X and their derived categories using the analysis of the projective geometry of S_3 and CR_4 and their resolutions.

Bottom-up approach   For this approach we consider hyperplane sections of S_3, we have to explain which type of singularities a hyperplane section can obtain, and then how the restriction of the resolution interacts with the hyperplane section. Depending on the origin of the singularity, the resolution will interact differently with the hyperplane section. We have summarised the conclusions in Table 3.

We write $H = \mathbb{P}(L) \subset \mathbb{P}(W^\vee)$, where $L \subset W^\vee$ is a codimension-one subspace, whilst we denote the projective dual by $h = \text{pt} = \mathbb{P}(L^\perp) \in \mathbb{P}(W)$, and we will write

$$(55) \quad X_H := X \times_{\mathbb{P}(W^\vee)} \mathbb{P}(L), \qquad Y_h := Y \times_{\mathbb{P}(W^\vee)} \mathbb{P}(L^\perp).$$

These are denoted X_L and $Y_{L^\perp}$ in Section 2.

Lemma 18. *A hyperplane section $H \cap S_3 \subset \mathbb{P}(W^\vee)$ is smooth if and only if*

$$(56) \quad h \in \mathbb{P}(W) \setminus \left(\bigcup_{i=1,\ldots,10} P_i \cup \text{CR}_4 \right).$$

Proof. This follows from classical projective duality for varieties with isolated nodal singularities. Any hyperplane h which lies in P_i for some i will inherit the node p_i and any hyperplane $h \in \text{CR}_4$ will be tangent to S_3 at some point. $\square$

The restriction of the resolution ρ in this case is an isomorphism and one obtains a smooth cubic surface. This is precisely the generic situation already discussed in Section 4.1.

The following two lemmas describe the cases in which there is a single isolated singularity.

Lemma 19. *A hyperplane section $H \cap S_3 \subset \mathbb{P}(W^\vee)$ is a one-nodal cubic surface if and only if $h \in P_i \setminus (\text{CR}_4 \cup \bigcup_{j \neq i} P_j)$ or $h \in \text{CR}_4 \setminus \bigcup_{i=1,\ldots,10} P_i$.*

Proof. Let us first show that these two cases yield one-nodal cubic surfaces.

In the first case, the point h will lie in some P_i and therefore the intersection $S_3 \cap H$ contains the node p_i. Since we assume that the hyperplane H contains no other node, we also deduce that it cannot be tangent at a point contained in a Segre plane, since such a hyperplane would contain the Segre plane and therefore four nodes. Moreover, the duality map restricts to an isomorphism between $S_3 \setminus \bigcup_{\sigma \in \mathfrak{S}_6} P_\sigma$ and $\text{CR}_4 \setminus \bigcup_{i=1,\ldots,10} P_i$ and therefore the hyperplane $h \in P_i$ cannot be tangent at a point not contained in a Segre plane. To conclude, since the hyperplane h does not lie on CR_4, the hyperplane H does not acquire a singularity worse than a node at the point p_i.

This also shows that a tangent hyperplane $T_p S_3$ at a point $p \in S_3 \setminus \bigcup_{\sigma \in \mathfrak{S}_6} P_\sigma$ can never contain a node p_i and is only tangent at the point p. Indeed, the duality map shows that $T_p S_3$ at a point $p \in S_3 \setminus \bigcup_\sigma P_\sigma$ does not contain any of the ten nodes of the Segre cubic. Since the duality map is an isomorphism when restricted to the complement of the Segre planes, we infer that it will only be tangent at the point p.

Conversely, if a hyperplane section is a one-nodal cubic surface, then it must intersect the Segre cubic non-transversely or contain a node.

To finish the proof, just note that a hyperplane H intersecting the Segre cubic at a general point $p \in P_\sigma$ not transversely will contain the whole Segre plane $P_\sigma \subset H$. □

The two cases in Lemma 19 behave differently with respect to the resolution ρ. In the latter case, the resolution is an isomorphism and we have a 1-nodal cubic surface. In the first case, we blow up the node of the cubic surface and end up with a weak del Pezzo surface.

On the dual side, the picture is also twofold. In the latter case, the h is contained in the smooth locus of the Castelnuovo–Richmond quartic and

(57) $\quad Y_h = \mathrm{Cob} \times_{\mathbb{P}(W)} h = \mathrm{Spec}\, k[\epsilon]/(\epsilon^2).$

In the first case $h \in P_i$ does not lie on CR_4. Therefore

(58) $\quad Y_h = \{2 \text{ points}\}.$

The last possible case for a hyperplane section $H \cap \mathrm{CR}_4$ to have exactly one isolated singularity is the following.

Lemma 20. *The hyperplane section $H \cap S_3 \subset \mathbb{P}(W^\vee)$ is a cubic surface with an isolated A_2-singularity if and only if $h \in (\mathrm{CR}_4 \cap P_i) \setminus \bigcup_{j \neq i} P_j$.*

Proof. A point $h \in \mathrm{CR}_4 \cap P_i$ induces a hyperplane section $H \cap S_3$ such that $p_i \in H \cap S_3$ is a singularity worse than an A_1-singularity.

More precisely, the hyperplane H will be at the node p_i tangent to the smooth tangent cone of the node. That means that the tangent cone of the point p_i considered in the hyperplane section $H \cap S_3$ is the cone over a smooth conic. We choose coordinates so that the node p_i is $[1:0:0:0] \in P_i \cong \mathbb{P}^3_{x_0:x_1:x_2:x_3}$. Writing

(59) $\quad f = \displaystyle\sum_{i=0}^{3} x_0^i f_i(x_1, x_2, x_3)$

with f_i homogenous of degree i, we have that $f_0 = 0$ as $p_i \in S_3$ and the point p_i being singular translates into $f_1 = 0$. The above discussion shows that f_2 is a singular conic and we can rewrite f after projective transformation to have the form

(60) $\quad f = x_0(x_1^2 + x_2^2) + f_3(x_1, x_2, x_3)$

for f_3 a homogenous polynomial of degree 3.

The cubic surface $H \cap S_3$ will only be singular at the point p_i. Indeed, no tangent hyperplane at a point outside the Segre planes contains a node and tangent hyperplanes at smooth points on Segre planes lie on the singular locus of the Castelnuovo–Richmond quartic. Since $\mathrm{CR}_4^{\mathrm{sing}} \subset \bigcup_{j \neq i} P_j$ we know that $H \cap S_3$ will neither be reducible nor contain any other node. This genericity implies that $f_3(x_1, x_2, x_3)$ defines a smooth elliptic curve inside the projective plane $x_0 = 0$ and that the completion of the local ring of the surface $H \cap S_3$ at p_i is isomorphic to $\mathbb{C}[[x_1, x_2, x_3]]/(x_1^2 + x_2^2 + x_3^3)$ showing that the isolated singularity is an A_2-singularity.

For the converse, by Lemma 19 no tangent hyperplane at a point not lying on a Segre plane acquires a worse singularity than a node. Moreover, tangent hyperplanes at smooth points lying on a Segre plane produce reducible hyperplane sections.

Thus, a hyperplane H such that $H \cap S_3$ has an A_2-singularity must contain a node p_i and therefore h must lie on some P_i. Moreover, since the singularity is worse than nodal, h is contained in the intersection $P_i \cap CR_4$.

To conclude, we need to show that $h \notin P_j$ for $j \neq i$. This follows as the cubic surface is assumed to have a unique isolated singularity. $\qquad\square$

Remark 21. In fact any hyperplane H which is tangent at a node p_i is either nowhere else tangent and contains no other node or must contain a Segre plane. This follows from the above proof together with the fact that $CR_4 \cap P_i \cap P_j$ is the union of two non-reduced lines, and therefore is contained in the singular locus of the Castelnuovo–Richmond quartic. For the latter claim, see [**MR1438547**].

The small resolution of the Segre cubic produces a *partial* resolution for the A_2 singularity, and we end up with a nodal singularity.

Since we are again considering a point h in the smooth locus of the Castelnuovo–Richmond quartic, we have

(61) $\quad Y_h = \mathrm{Cob} \times_{\mathbb{P}(W)} h = \mathrm{Spec}\, k[\epsilon]/(\epsilon^2).$

Next, we treat the case of several isolated singularities.

Lemma 22. *Let $r \in \{2, 3, 4\}$. A hyperplane section $H \cap S_3 \subset \mathbb{P}(W^\vee)$ is an r-nodal cubic surface if and only if $h \in \mathbb{P}(W)$ lies on exactly r different hyperplanes P_i's, but not on the Castelnuovo–Richmond quartic. Such points $h \in \mathbb{P}(W)$ exist for all $r \in \{2, 3, 4\}$.*

Proof. Let us first prove the if direction. For a point $h \in \mathbb{P}(W)$ which is contained in exactly r of the hyperplanes P_i the intersection $H \cap S_3$ will contain exactly r nodes of the Segre cubic. Moreover, the intersection is transversal at all other points of the Segre cubic, since we assume that h does not lie on the singular locus of the Castelnuovo–Richmond quartic and therefore does not contain a Segre plane (transversality outside the Segre planes again follows from (54)).

To prove the converse note that for a tangent hyperplane at a point of the Segre cubic we have two possibilities. Either the point lies on a Segre plane and the hyperplane section will therefore contain some P_σ. The second option is that the point does not lie on any of the Segre planes which by Lemma 19 is always one-nodal. Remark 21 shows that such a hyperplane h cannot lie on CR_4, since otherwise the hyperplane section would be a cone over an elliptic curve.

It remains to show that such hyperplanes exist. For this we can consider the hyperplanes $T_{i,j}$ which contain exactly four of the nodes and the intersection $T_{i,j} \cap S_3$ is isomorphic to the Cayley cubic surface, the (unique up to isomorphism) cubic surface with four nodes. By what we have already proven, we know that for each hyperplane $T_{i,j}$ there are exactly four distinct integers $a, b, c, d \in \{1, \dots, 10\}$ such that

(62) $\quad t_{i,j} = P_a \cap P_b \cap P_c \cap P_d \in \mathbb{P}(W) \setminus CR_4.$

Thus a general point of the plane $P_a \cap P_b$ corresponds to a hyperplane containing exactly two nodes and a general point of $P_a \cap P_b \cap P_c$ yields a hyperplane containing exactly three of the nodes such that the hyperplanes do not contain any Segre plane. $\qquad\square$

Since all nodes of the hyperplane section were already nodes on the Segre cubic, they are resolved by the restriction of ρ and we again obtain a smooth weak del Pezzo surface.

On the dual side we have that $h \notin CR_4$, so that we always obtain that Y_h is 2 points. Hence either by Orlov's blowup formula for X_H as an (iterated) blowup or homological projective duality we obtain 9 exceptional objects in the hyperplane section.

Remark 23. The above cases of smooth, nodal and A_2 singular cubic surfaces correspond exactly to hyperplanes $h \in \mathbb{P}(W)$ such that h does not lie on the singular locus of the Castelnuovo–Richmond quartic.

Lemma 24. *All other hyperplane sections* $H \cap S_3$ *yield reducible cubic surfaces which are either the union of a plane and a quadric or the union of three Segre planes.*

Proof. It is immediate from the above discussion that we have already exhausted all possibilities for hyperplanes $H \subset \mathbb{P}(W^\vee)$ such that the corresponding point $h \in \mathbb{P}(W)$ does not lie on the singular locus of the Castelnuovo–Richmond quartic. This yields the assertion. □

We just remark that in the first case the residual quadric is the hyperplane section of the tangent cone of one of the nodes.

We will not provide an explicit description of X_H, as this is a non-normal surface. On the dual side something interesting happens with Y_h: because h lies in the singular locus of CR_4 we have that $h \cap \mathrm{Cob}$ is a double point, *but* Y_h is not of the expected dimension. By [**MR4076812**] the fiber of $Y \to \mathbb{P}(W)$ over h is either a line or a conic. This means we have to compute Y_h as a derived fiber product. We will illustrate such a computation for a hyperplane section of Y instead, see Lemma 28 and the ensuing discussion.

Top-down approach It is also possible to directly consider hyperplane sections of X and fibers of $\varpi \colon Y \to \mathbb{P}(W)$, using the description as projective bundles. Only *a posteriori* do we make the link to the more classical picture of the singular varieties S_3 and Cob. We will only briefly explain this method, to avoid too much redundancy with the earlier discussion.

The hyperplane section of $\mathbb{P}(W^\vee)$ corresponding to $L \subset W^\vee$ gives a surjective morphism $W^\vee \to k$, so that the fiber product X_L corresponding to the hyperplane section

$$(63) \quad \begin{array}{ccc} X_L & \hookrightarrow & X \\ \downarrow & & \downarrow f \\ \mathbb{P}(L) & \hookrightarrow & \mathbb{P}(W^\vee) \end{array}$$

can be written as $\mathbb{P}_S(\mathcal{R})$, where

$$(64) \quad \mathcal{R} := \mathrm{im}(\mathcal{U}_2 \to \mathcal{O}_S)$$

using the composition

$$(65) \quad \mathcal{U}_2 \hookrightarrow W^\vee \otimes_k \mathcal{O}_S \twoheadrightarrow \mathcal{O}_S.$$

We need to understand what the degeneracy locus of the morphism $\mathcal{U}_2 \to \mathcal{O}_S$ is. The analysis is similar to that of [**MR4076812**]: the zero locus of a non-zero section $\sigma \in \mathrm{H}^0(\mathrm{Gr}(2, W^\vee), \mathcal{U}_2^\vee)$ is $\mathrm{Gr}(2, 4)$, and from the description $S = \mathrm{Gr}(2, W^\vee) \cap \mathbb{P}^5$ we see that there are 3 cases:

1. a 0-dimensional scheme Z of length 2;
2. a line $L \subset S$;
3. a conic $C \subset S$.

In the first case we get that $\mathcal{R} \cong \mathcal{I}_Z$ and thus the composition

$$(66) \quad X_H \cong \mathbb{P}_S(\mathcal{I}_Z) \hookrightarrow X = \mathbb{P}_S(\mathcal{U}_2) \twoheadrightarrow S$$

$H \cap S_3$	X_H	$\mathcal{C}_H$	Y_h	$h \cap \text{Cob}$
smooth cubic surface	smooth cubic surface	2 orthogonal objects	2 points	2 points
1-nodal cubic surface	singular cubic surface	$\mathbf{D}^b(k[\epsilon]/(\epsilon^2))$	double point	double point
	weak del Pezzo surface	2 orthogonal objects	2 points	2 points
2-nodal cubic surface	weak del Pezzo surface	2 orthogonal objects	2 points	2 points
3-nodal cubic surface	weak del Pezzo surface	2 orthogonal objects	2 points	2 points
4-nodal Cayley cubic surface	weak del Pezzo surface	2 orthogonal objects	2 points	2 points
cubic surface with A_2 singularity	nodal weak del Pezzo surface	$\mathbf{D}^b(k[\epsilon]/(\epsilon^2))$	double point	double point
union of Segre plane and quadric				double point
union of 3 Segre planes				double point

Table 3: Description of all possible hyperplane sections of $X \to \mathbb{P}(W^\vee)$ and the homological projective dual sections

$H \cap \text{Cob}$	Y_H	$\mathcal{C}_H$	X_h	$h \cap S_3$
15-nodal quartic double solid	$\mathrm{Bl}_4\mathbb{P}(\mathrm{T}_{\mathbb{P}^2})$	zero	$\emptyset$	$\emptyset$
non-reduced quadric				point
16-nodal quartic double solid	nodal weak Fano threefold	$\mathbf{D}^b(k[\epsilon]/(\epsilon^2))$	$k[\epsilon]/(\epsilon^2),\ |\epsilon| = -1$	point

Table 4: Description of some interesting possible hyperplane sections of $Y \to \mathbb{P}(W)$ and the homological projective dual sections

is equal to $\mathrm{Bl}_Z S \to S$. This can now be compared to the results in Table 3: when Z is reduced the position of the two points with respect to the 10 lines on S determines which case we are in, when Z is non-reduced we are in the case that X_H is a nodal weak del Pezzo surface.

In the second case we have that $\mathcal{R} \cong \mathcal{I}_L$, and the restriction of $\mathcal{U}_2$ to L is $\mathcal{O}_L \oplus \mathcal{O}_L(-1)$, as in the proof of [**MR4076812**]. We get that

(67) $\quad X_H \cong \mathbb{P}_L(\mathcal{U}_2|_L) \cup_L S \hookrightarrow X.$

The composition $\mathbb{P}_L(\mathcal{U}_2|_L) \cong \mathbb{F}_1 \hookrightarrow X \to \mathrm{S}_3$ has as image a Segre plane (with the restriction to this $\mathbb{P}^2$ being the blowup), and S gets blown down to the residual quadric of the hyperplane section through the Segre plane.

In the third case the restriction of $\mathcal{U}_2$ to C is now $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$, and we get that

(68) $\quad X_H \cong \mathbb{P}_C(\mathcal{U}_2|_C) \cup_L S.$

If C is a smooth conic we get that $\mathbb{P}^1 \times \mathbb{P}^1$ is mapped isomorphically onto a quadric in S_3, and S gets blown down to a Segre plane.

4.3 Other linear sections

We will now discuss hyperplane sections of the Coble fourfold, and codimension 2 linear sections of both the Segre cubic and the Coble fourfold. Doing so we can describe the derived categories of the linear sections of the resolutions.

Hyperplane sections of the Coble fourfold Dual to the discussion in Section 4.2 we are considering hyperplane sections of the resolution Y of $\mathrm{Cob} \subset \mathbb{P}(W)$. We will now write $H \subset \mathbb{P}(W)$ for a hyperplane corresponding to $L \subset W$ of codimension 1, and $h \in \mathbb{P}(W^\vee)$ for the dual point $\mathbb{P}(L^\perp)$. Likewise X_h and Y_H denote the (derived) fiber products. Observe that we are now applying Section 2 to Y and consider X as its homological projective dual.

Because the singular locus of the Castelnuovo–Richmond quartic consists of 15 lines, every hyperplane section $H \subset \mathbb{P}(W)$ will necessarily intersect the singular locus. An overview of the cases we will discuss is given in Table 4.

The generic case is when this happens in exactly 15 nodes, so that we obtain a singular quartic surface with 15 nodes. On the Segre side this corresponds to the hyperplane $h \in \mathbb{P}(W^\vee)$ not lying on the Segre cubic and not being contained in one of the hyperplanes $H_{i,j}$ the hyperplanes dual to the singular locus of the singular locus of the Castelnuovo–Richmond quartic.

The corresponding hyperplane section of Cob is a quartic double solid with 15 nodes, branched along the singular quartic surface. The following result shows how the Castelnuovo–Richmond quartic and Coble fourfold are universal for such varieties [**MR3954304**],

Proposition 25 (Avilov). *Let T be a quartic surface, singular in precisely 15 nodes. Then T is a hyperplane section of the Castelnuovo–Richmond quartic. Let Z be a quartic double solid, singular in precisely 15 nodes. Then Z is a hyperplane section of the Coble fourfold.*

Incorporating the resolution ϖ into the picture, the results of Avilov imply the following.

Lemma 26. *Let $H \subseteq \mathbb{P}(W)$ be a hyperplane as in Proposition 25. Then Y_H is a smooth projective weak Fano threefold, obtained as a small resolution of a quartic double solid singular in 15 nodes. Its derived category has semiorthogonal decompositions*

(69)
$$\begin{aligned}
\mathbf{D}^{\mathrm{b}}(Y_H) &= \langle \mathbf{D}^{\mathrm{b}}(S), \mathbf{D}^{\mathrm{b}}(S) \rangle \\
&= \langle \mathbf{D}^{\mathrm{b}}(\mathbb{P}(\mathrm{T}_{\mathbb{P}^2})), E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}, E_{3,1}, E_{3,2}, E_{4,1}, E_{4,2} \rangle
\end{aligned}$$

where the $E_{i,j}$ are exceptional objects.

Proof. The hyperplane section Y_H is a smooth projective weak Fano threefold because it is a small resolution of a quartic double solid. The first semiorthogonal decomposition is induced by homological projective duality and consists of $14 = 2 \times 7$ exceptional objects, as X_h is empty in this case. The second semiorthogonal decomposition follows from an explicit description of the hyperplane section given in [**MR3954304**], as the blowup of the Fano threefold $\mathbb{P}(T_{\mathbb{P}^2})$, isomorphic to a $(1,1)$-section of $\mathbb{P}^2 \times \mathbb{P}^2$, in 4 points in general position, and applying Orlov's blowup formula. We again count $14 = 6 + 4 \times 2$ exceptional objects. □

Remark 27. Observe that the description of Y_H in the proof of Lemma 26 parallels that of the *second* small resolution of singularities of the Coble fourfold discussed in [**MR4076812**]. This small resolution is obtained as the blowup of $\mathbb{P}^2 \times \mathbb{P}^2$ in 4 points, and the small resolution in Lemma 26 is a hyperplane section of it.

The next case we consider is that of a tangent hyperplane.

Lemma 28. *For any point $p \in \mathrm{CR}_4 \cap P_i \setminus \mathrm{CR}_4^{\mathrm{sing}}$ the tangent hyperplane is the non-reduced quadric Q_i. For any other point $p \in \mathrm{CR}_4 \setminus (\bigcup_i P_i)$ the tangent hyperplane section $\mathrm{T}_p\mathrm{CR}_4 \cap \mathrm{CR}_4$ is a singular Kummer quartic surface.*

Proof. The first statement is clear. For the second, use that the inverse of the duality map restricted to the complement of the hyperplanes P_i is an isomorphism. This shows that the hyperplane will only be tangent at $p \in \mathrm{CR}_4$. For more details, see [**MR1438547**]. □

There are 16 singularities on the Kummer quartic surface $H \cap \mathrm{CR}_4$, and $H \cap \mathrm{Cob}$ is a double cover ramified in the Kummer quartic, a 16-nodal double quartic solid. The restriction of the resolution $Y \to \mathrm{Cob}$ resolves all nodes except the one coming from the tangency point, so that Y_H is a singular weak Fano fold.

To compute X_h we need to take the derived fiber product, as $h \in \mathrm{S}_3 \setminus \mathrm{S}_3^{\mathrm{sing}}$ implies that the usual fiber product $X \times_{\mathbb{P}(W^\vee)} h$ is a single point, which is not of the expected dimension -1. To compute the derived fiber product X_h we can take a Koszul resolution for $\mathrm{S}_3 \subset \mathbb{P}(W^\vee)$:

$$(70) \quad 0 \to \mathcal{O}_{\mathbb{P}(W^\vee)}(-3) \to \mathcal{O}_{\mathbb{P}(W^\vee)} \to \mathcal{O}_{\mathrm{S}_3} \to 0$$

and consider the restriction of the sheaf of dg algebras $[\mathcal{O}_{\mathbb{P}(W^\vee)}(-3) \to \mathcal{O}_{\mathbb{P}(W^\vee)}]$ living in degrees -1 and 0 to h. This gives a dg algebra A given by $[k \to k]$ living in degrees -1 and 0. Because the morphism in the Koszul resolution is multiplication with the defining equation, and $h \in \mathrm{S}_3$ we obtain that the restricted differential vanishes. Hence $A = k[\epsilon]/(\epsilon^2)$ is the formal dg algebra where $|\epsilon| = -1$. The dg algebra A is an ingredient in the theory of absorption of nodal singularities as introduced in [**2207.06477v1**] (see also [**2111.00527v1**]).

What is interesting in this case is that X_h has a derived structure, but Y_H is a singular variety. This frequently happens when X is (similar to) a closed subvariety. In the next example *both* fiber products will have a derived structure.

Plane sections of the Segre cubic   Let $L \subset W^\vee$ be a subspace of codimension 2. We will write $P = \mathbb{P}(L) \subset \mathbb{P}(W^\vee)$, and $p \subset \mathbb{P}(W)$ for the dual projective line. We wish to describe X_P and Y_p.

If $P \subseteq \mathrm{S}_3$ it is one of the fifteen Segre planes, and $X \times_{\mathbb{P}(W^\vee)} P$ is either $\mathbb{F}_1 \cong \mathrm{Bl}_1 \mathbb{P}^2$ or $S \cong \mathrm{Bl}_4 \mathbb{P}^2$. This can be deduced from the arguments at the end of Section 4.2 for hyperplane sections of S_3 containing a Segre plane. In any case it is not of the expected dimension. On the dual side p is one of the 15 lines of the singular locus of CR_4, and $Y \times_{\mathbb{P}(W)} p$ is described in [**MR4076812**]. It is again

not of the expected dimension. In (11) there are no contributions from the Lefschetz center, and we obtain an equivalence of categories

$$(71) \quad \mathbf{D}^b(X_P) \cong \mathbf{D}^b(Y_p)$$

where both X_P and Y_p are surfaces equipped with a derived structure.

If P is not strictly contained within the Segre cubic, we obtain a plane cubic curve. Let us first consider a plane cubic $P \cap S_3$ contained in a smooth hyperplane section of S_3 (as discussed in Lemma 18) such that $p \cap CR_4^{\text{sing}} = \emptyset$. Then $P \cap S_3$ is a reduced (but possibly reducible) plane cubic. All cases except three lines meeting in one point arise for a smooth cubic surface. For this last case to occur the cubic surface needs to contain an Eckardt point.

Again we obtain an equivalence

$$(72) \quad \mathbf{D}^b(X_P) \cong \mathbf{D}^b(Y_p),$$

now for a (possibly singular) plane cubic and a (possibly singular) double cover of $\mathbb{P}^1$ ramified in a subscheme of length 4. The reconstruction result [**MR3217412**] shows that we in fact have an isomorphism $X_P \cong Y_p$. We leave the non-generic situation to the interested reader.

Plane sections of the Coble fourfold   Finally, let $L \subset W$ be a subspace of codimension 2. We will write $P = \mathbb{P}(L) \subset \mathbb{P}(W)$, and $p \subset \mathbb{P}(W^\vee)$ for the dual projective line.

There are 2 possible scenarios for $p \cap S_3$:

- $p \subset S_3$: there is a Fano surface of lines on S_3 described in [**MR3776662**], such that X_p acquires a derived structure and its geometry moreover depends on the position of p with respect to S_3^{sing};
- $\dim(p \cap S_3) = 0$: the intersection is a scheme of length 3.

Let us consider the generic case of a zero-dimensional intersection. Then X_p consists of 3 points which avoid the singular locus of S_3. On the dual side we obtain a double cover of $P = \mathbb{P}^2$, ramified in a quartic $P \cap CR_4$ which avoids CR_4^{sing}, so that Y_P is a (smooth) del Pezzo double plane, a del Pezzo surface of degree 2.

By (11) we obtain semiorthogonal decompositions

$$(73) \quad \begin{aligned} \mathbf{D}^b(X_p) &= \langle E_1, E_2, E_3 \rangle \\ \mathbf{D}^b(Y_p) &= \langle \mathbf{D}^b(X_p), \mathbf{D}^b(S) \rangle \end{aligned}$$

where E_1, E_2, E_3 are completely orthogonal. Similar to (66) we get that the composition $Y_p \to S$ is the blowup in 3 points. Generically it will give a del Pezzo surface as in Table 2, but we will not perform the case-by-case analysis of when the 3 points are not in general position with respect to the 10 lines on S.

If $p \cap S_3^{\text{sing}} = \emptyset$ and X_p is still reduced but $P \cap CR_4^{\text{sing}} \neq \emptyset$ the fiber product Y_P is a smooth weak del Pezzo surface of degree 2, and (73) still holds. If on the other hand $p \cap S_3^{\text{sing}} = \emptyset$ and X_p non-reduced, we obtain that Y_P is a singular weak del Pezzo surface of degree 2, such that $\mathbf{D}^b(X_p)$ in (73) becomes $\mathbf{D}^b(\text{Spec}\, k[\epsilon]/(\epsilon^2) \times k)$ or $\mathbf{D}^b(\text{Spec}\, k[\epsilon]/(\epsilon^3))$. We leave it to the interested reader to match this up with the classification of [**MR2964027**]

If $p \cap S_3^{\text{sing}} \neq \emptyset$ the fiber product X_p acquires a derived structure and we will not discuss it further.

5 The Segre cubic vs. moduli of quiver representations

As explained in Section 3 the Segre cubic has a modular interpretation as the Satake compactification of a Picard modular variety. This interpretation a priori does not yield anything interesting for the resolution X from the point of view of homological projective duality. But there is a different modular interpretation of S_3 and the resolution X, which gives rise to a *second* rectangular Lefschetz decomposition.

By [**MR1007155**] and [**MR4352662**] we have that S_3 is the moduli space of semistable quiver representations for the 6-subspace quiver

(74) S_6 :

where we use the dimension vector $(1, 1, 1, 1, 1, 1; 2)$, and the canonical stability condition as discussed in [**MR4352662**]. This stability condition lies on a wall, and by considering a small perturbation of the stability condition we can obtain small desingularisations [**MR3644807**]. Two particular choices were studied in [**MR3778138**], and by the description of their automorphisms we know that they correspond to types IV and VI in [**MR0914085**], hence by Remark 9 we know that they are isomorphic to $\mathbb{P}_S(\mathcal{U}_2)$ and $\mathrm{Bl}_5\,\mathbb{P}^3$.

A fully faithful functor in modular settings   On the other hand, in full generality there is an expected relationship between $\mathbf{D}^b(kQ)$ and $\mathbf{D}^b(M)$, where Q is an acyclic quiver and M is a suitable moduli space of semistable representations. Namely we expect that, for the right choice of dimension vector and stability condition, there is a fully faithful embedding of $\mathbf{D}^b(kQ)$ into $\mathbf{D}^b(M)$ given by the universal representation.

There exists a rich literature on similar admissible embeddings into derived categories of moduli spaces:

- for curves of $g \geq 2$ and moduli spaces of vector bundles [**MR3764066**, **MR3713871**, **MR3954042**, **2106.04857v1**];

- for Hilbert schemes of points [**MR3397451**, **MR3950704**]

In the case of noncommutative algebra, we refer to

- for two noncommutative surfaces and Hilbert schemes of points [**MR4165471**, **MR3488782**];

- for quivers, provided the dimension vector is thin (i.e. the moduli space is toric) [**MR1688469**].

Mutating the original Lefschetz structure   Starting from the Lefschetz structure in Theorem 11 we can perform a sequence of mutations to find a new Lefschetz structure, whose existence is related to the expectation outlined above.

Proposition 29. *Let S be the del Pezzo surface of degree 5. There exists a 3-block exceptional collection*

(75) $\mathbf{D}^b(S) = \langle \mathcal{O}_S; \mathcal{O}_S(h - e_1), \mathcal{O}_S(h - e_2), \mathcal{O}_S(h - e_3), \mathcal{O}_S(h - e_4), \mathcal{O}_S(2h - e); \mathcal{U}_2^\vee \rangle$

where h is the pullback of the hyperplane class of $\mathbb{P}^2$, $e_1, \ldots, e_4$ are the classes of the exceptional divisors, we set $e := e_1 + e_2 + e_3 + e_4$, and $\mathcal{U}_2$ is as in Section 3.1. The 5 line bundles in the middle block correspond to the 5 conic bundle structures on S.

Moreover, we have isomorphisms

(76) $\mathrm{Ext}_S^\bullet(\mathcal{O}_S(h - e_i), \mathcal{U}_2^\vee) \cong k[0], \qquad \mathrm{Ext}_S^\bullet(\mathcal{O}_S(2h - e), \mathcal{U}_2^\vee) \cong k[0].$

Proof. We obtain this collection by mutating the 3-block exceptional collection from [**MR1642152**]. Their collection, written using the notation of the statement of the proposition, is

$$(77) \quad \mathbf{D}^b(S) = \langle \mathcal{O}_S; \mathcal{F}; \mathcal{O}_S(h), \mathcal{O}_S(e_1 - \omega_S - h), \mathcal{O}_S(e_2 - \omega_S - h), \mathcal{O}_S(e_3 - \omega_S - h), \mathcal{O}_S(e_4 - \omega_S - h) \rangle$$

where $\mathcal{F}$ is the vector bundle obtained as the universal extension

$$(78) \quad 0 \to \mathcal{O}_S(-\omega_S - h) \to \mathcal{F} \to \mathcal{O}_S(h) \to 0.$$

We have that $\mathcal{F} \cong \mathcal{U}_2^\vee$. Indeed, the Chern classes of $\mathcal{F}$ and $\mathcal{U}_2^\vee$ satisfy

$$(79) \quad c_1(\mathcal{F}) = -\omega_S = c_1(\mathcal{U}_2^\vee), \quad c_2(\mathcal{F}) = -\omega_S h - h^2 = 2 = c_2(\mathcal{U}_2^\vee).$$

The last equality follows from classical Schubert calculus saying that $\sigma_1^4 \sigma_{1,1} = 2$ on $\mathrm{Gr}(2,5)$. Since both bundles are of rank two and exceptional (for $\mathcal{U}_2^\vee$ this is a Koszul computation) they must be isomorphic by [**MR1642152**].

Mutating the third block of the last 5 line bundles in (77) to the very left corresponds to tensoring the objects with ω_S. Now perform a right mutation on the first two blocks. The resulting mutated exceptional sheaves are computed using *division*, i.e. they are the cokernels in the short exact sequences

$$(80) \quad 0 \to \mathcal{O}_S(\omega_S + h) \to \mathrm{Hom}_S(\mathcal{O}_S(\omega_S + h), \mathcal{O}_S)^\vee \otimes \mathcal{O}_S \to \mathcal{O}_S(2h - e) \to 0$$

and

$$(81) \quad 0 \to \mathcal{O}_S(e_i - h) \to \mathrm{Hom}_S(\mathcal{O}_S(e_i - h), \mathcal{O}_S)^\vee \otimes \mathcal{O}_S \to \mathcal{O}_S(h - e_i) \to 0.$$

The Hom-spaces in the middle term are 2-dimensional, as they are identified with the global sections of the 5 conic bundle structures on S, and the identification of the cokernels is a Chern class computation. The result is the 3-block exceptional collection in (75).

The identifications in (76) follow from (78), so that we obtain

$$(82) \quad \begin{aligned} \mathrm{Ext}_S^\bullet(\mathcal{O}_S(h - e_i), \mathcal{U}_2^\vee) &\cong \mathrm{H}^\bullet(S, \mathcal{O}_S(e_i)) \\ \mathrm{Ext}_S^\bullet(\mathcal{O}_S(2h - e), \mathcal{U}_2^\vee) &\cong \mathrm{H}^\bullet(S, \mathcal{O}_S). \end{aligned}$$

by the vanishing of $\mathrm{H}^\bullet(S, \mathcal{O}_S(h - e + e_i))$ and $\mathrm{H}^\bullet(S, \mathcal{O}_S(e - h))$ thanks to the projection formula. $\quad\square$

From Orlov's projective bundle formula applied to $p \colon X \to S$ we therefore obtain the semiorthogonal decomposition

$$(83) \quad \begin{aligned} \mathbf{D}^b(X) = \langle &\mathcal{O}_X; \mathcal{O}_X(h - e_i) \mid i = 1, 2, 3, 4; \mathcal{O}_X(2h - e), p^*\mathcal{U}_2^\vee; \\ &\mathcal{O}_X(s); \mathcal{O}_X(s + h - e_i) \mid i = 1, 2, 3, 4; \mathcal{O}_X(s + 2h - e), p^*\mathcal{U}_2^\vee(s) \rangle \end{aligned}$$

where we denote the relative hyperplane class by s. This is of course nothing but the decomposition (40) written as a full exceptional collection using (75). We can now modify this collection into a *new* Lefschetz decomposition, with a Lefschetz center that is not equivalent to the Lefschetz center in (40).

Proposition 30. *There exists a rectangular Lefschetz decomposition*

$$(84) \quad \mathbf{D}^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1) \rangle$$

with respect to the line bundle $\mathcal{O}_X(1) = \mathcal{O}_X(s) = f^*(\mathcal{O}_{\mathbb{P}(W^\vee)}(1))$, where the Lefschetz center is

$$(85) \quad \mathcal{A}_0 = \langle \mathcal{O}_X(h - e_i) \mid i = 1, \ldots, 4; \mathcal{O}_X(2h - e), \mathcal{O}_X(3h - e - s), p^*\mathcal{U}_2^\vee \rangle$$

such that $\mathcal{A}_0 = \mathcal{A}_1 \cong \mathbf{D}^b(kS_6)$ is the derived category of the 6-subspace quiver from (74).

Proof. The right mutation of $\mathcal{O}_X$ with respect to its orthogonal complement gives the anticanonical line bundle $\mathcal{O}_X(2s)$ as the final object. We obtain a rectangular Lefschetz decomposition with Lefschetz center

(86) $\langle \mathcal{O}_X(h - e_i) \mid i = 1, \ldots, 4; \mathcal{O}_X(2h - e), p^*\mathcal{U}_2^\vee, \mathcal{O}_X(s)\rangle$

To identify this with (85) and show that $\mathcal{A}_0 \cong \mathbf{D}^{\mathrm{b}}(k S_6)$ we do the left mutation on the objects $p^*\mathcal{U}_2^\vee$ and $\mathcal{O}_X(s)$ which does not change the Lefschetz center. The left mutation is defined by the kernel in the short exact sequence

(87) $0 \to \mathrm{L}_{p^*\mathcal{U}_2^\vee}\mathcal{O}_X(s) \to \mathrm{Hom}_X(p^*\mathcal{U}_2^\vee, \mathcal{O}_X(s)) \otimes p^*\mathcal{U}_2^\vee \to \mathcal{O}_X(s) \to 0,$

using the identification $\mathcal{U}_2^\vee \cong p_*\mathcal{O}_X(s)$ and the exceptionality of $\mathcal{U}_2^\vee$, which gives the surjectivity and the kernel is a line bundle by a rank computation. The identification with $\mathcal{O}_X(3h - e - s)$ follows from a Chern class computation using (78).

We can now check that $\mathcal{A}_0 \cong \mathbf{D}^{\mathrm{b}}(k S_6)$. We have that

(88) $\mathrm{Ext}_X^\bullet(\mathcal{O}_X(h - e_i), p^*\mathcal{U}_2^\vee) \cong \mathrm{Ext}_X^\bullet(\mathcal{O}_X(2h - e), p^*\mathcal{U}_2^\vee) \cong k[0]$

by the last part of Proposition 29. The first 6 objects are completely orthogonal: the first 5 because they originate from a block in the 3-block collection, the orthogonality with the 6th object follows from the vanishing of $\mathbf{R}p_*\mathcal{O}_X(-s)$ in one direction and from the exceptional sequence in the other.   □

Comparison of Lefschetz structures   We have now obtained 3 (rectangular) Lefschetz structures on $\mathbf{D}^{\mathrm{b}}(X)$:

1. the projective bundle Lefschetz structure used in Theorem 11, where the Lefschetz center is $\mathbf{D}^{\mathrm{b}}(S)$;

2. the blowup Lefschetz structure from (38) in Proposition 10, where the Lefschetz center is $\mathbf{D}^{\mathrm{b}}(A)$ for the finite-dimensional algebra $A = kQ/I$ where Q is the quiver

(89)
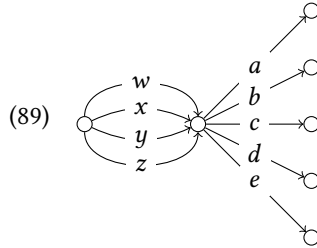
and I is the ideal of relations

(90) $I = (xa, ya, za, wb, yb, zb, wc, xc, zc, wd, xd, yd, we - xe, xe - ye, ye - ze)$

which encodes 5 points in general position on $\mathbb{P}^3_{w:x:y:z}$ as up to the action of PGL_4 we can take these to be

(91) $(1 : 0 : 0 : 0), (0 : 1 : 0 : 0), (0 : 0 : 1 : 0), (0 : 0 : 0 : 1), (1 : 1 : 1 : 1),$

so that the structure follows from the composition law in (38);

3. the quiver Lefschetz structure from Proposition 30, where the Lefschetz center is $\mathbf{D}^{\mathrm{b}}(k S_6)$.

The following propositions gives a comparison between these Lefschetz structures, in the sense of [**MR3948688**]. We have that

- the blowup and quiver Lefschetz structures agree;

- the projective bundle Lefschetz structure is different from the other two.

Proposition 31. *The projective bundle Lefschetz structure is not Lefschetz equivalent to the quiver Lefschetz structure.*

Proof. We observe that the Lefschetz centers are already inequivalent. Indeed, $\mathbf{D}(S)$ is not equivalent to $\mathbf{D}^{\mathrm{b}}(k\mathrm{S}_6)$, as $\mathrm{HH}^2(S) \cong \mathrm{H}^0(S, \omega_S^\vee) = k^6$ by the Hochschild–Kostant–Rosenberg theorem, whilst $\mathrm{HH}^2(k\mathrm{S}_6) = 0$ as $k\mathrm{S}_6$ is hereditary (and as S_6 is a tree we moreover have $\mathrm{HH}^1(k\mathrm{S}_6) = 0$). □

Hence we have found a genuinely new Lefschetz decomposition for $\mathbf{D}^{\mathrm{b}}(X)$. On the other hand we have the following identification.

Proposition 32. *The blowup Lefschetz center is equivalent to the quiver Lefschetz center.*

Proof. We want to prove that $\mathbf{D}^{\mathrm{b}}(A) \cong \mathbf{D}^{\mathrm{b}}(k\mathrm{S}_6)$. One can compute (e.g. using [**qpa**] and the explicit presentation given above) that $\mathrm{HH}^1(A) = \mathrm{HH}^2(A) = 0$, and $\mathrm{HH}^{\geq 3}(A) = 0$ by global dimension reasons. Now consider A as a one-point extension of the 5-quotient quiver by the representation M. By the long exact Hochschild cohomology sequence [**MR1035222**] we see that M is an exceptional representation.

On the other hand $k\mathrm{S}_6$ can also be seen as a one-point extension, of the 5-subspace quiver, using the indecomposable projective (thus exceptional) representation P concentrated at the sink.

Let M' be the image of P under the composition of the Nakayama functor (sending it to the indecomposable injective at the sink) and the reflection functor at the sink. This is also an exceptional representation for the 5-quotient quiver, so $M \cong M'$ as we have $\mathbf{dim}\, M = \mathbf{dim}\, M' = (1, 1, 1, 1, 1; 4)$ if we put the source as the last vertex.

By considering the derived versions of the Nakayama functor and the reflection functor we get that their composition is an equivalence of derived categories for the 5-subspace and 5-quotient quiver, which sends P to M. Hence by [**MR1980681**] we obtain an induced derived equivalence between one-point extensions. This gives an identification of the Lefschetz centers. □

It would be interesting to further understand (noncommutative) homological projective duality for this second Lefschetz center.

Pieter Belmans, `pieter.belmans@uni.lu`
Université du Luxembourg, Avenue de la Fonte 6, L-4364 Esch-sur-Alzette, Luxembourg

Thorsten Beckmann, `beckmann@math.uni-bonn.de`
Max–Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany