Minkowski Decompositions of Lattice Polytopes

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Abstract

Given a lattice polytope P, we find all pairs of lattice polytopes Q, R such that P is the Minkowski sum of Q and R. A strongly convex cone $C(P)$ is associated to P and the indecomposable Minkowski summands of P are obtained from the Hilbert basis of this cone. The method applies to lattice polytopes of any finite dimension and has been coded as an algorithm in the computer algebra system Sage.

I would like to thank my two supervisors, Dr. Coates and Dr. Kasprzyk, for helping me get ideas onto paper and keeping me grounded in moments of apparent crisis. Without their contributions and discussions the project would not have turned out as it did.

Contents

1 Introduction

The connections between lattice polytopes and other areas of mathematics is vast: absolute irreducibility of a multivariate polynomial is implied by indecomposability of its associated Newton polytope [6]; mirror pairs of Calabi-Yao varieties, which string theorists use to model extra dimensions, are described by a special class of lattice polytopes [5]; and the several applications of counting lattice points inside lattice polytopes [3]. This list is far from exhaustive.

Following from [1], we extend the association of a strongly convex cone $C(P)$ to any lattice polytope P. The relationship between the Minkowski summands of positive multiples of P and $C(P)$ is clarified. Using this relationship, we reconstruct the maximal (lattice) decompositions of P from the Hilbert basis of $C(P)$.

Background material on graph theory and the geometry of convex sets is found in the preliminaries. In Section 3, we introduce an edge dilation measuring map ρ, which provides a route to study the Minkowski summands of P inside $C(P)$. The inverse map σ (when restricted to the correct domain) is defined in Section 4 and will enable us to recover the Minkowski summands from the image of ρ. Finally, in Section 5, we give criteria to identity which points in $C(P)$ correspond to lattice Minkowski summands of P. The computational implementation of the method described in the preceding sections, located in the appendix, is demonstrated in Section 6.

2 Preliminaries

An overview of graph theory and the geometry of convex sets is presented, with the relevant results highlighted. Those with a good grasp of the basics of these topics may wish to skim this section and move onto Section 3. Attention should be drawn to the unconventional notations introduced in Definition 2.1.19, Definition 2.1.21, Definition 2.1.22 and Definition 2.3.7.

A citation of the form [11, p. 83: Theorem 8.24], is a reference to: Theorem 8.24 on page 83 of the corresponding source (source number 11). Any additional comments, such as adaptations of the original source, are separated by a comma and written as [1, p. 446, extended to non-primitive polytopes].

2.1 Graph Theory

The definitions in this subsection are based on [4, pp. 1-9]; the intuitive notions of a graph, paths and related concepts are formalised here.

Definition 2.1.1. Let A be a set and $k \in \mathbb{Z}_{\geq 1}$. We denote by $[A]^k$ the *set of all k-element subsets* of A, that is $[A]^k = \{B \subseteq A : |B| = k\}$.

Definition 2.1.2. A *graph* is a pair of sets $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{E} \subseteq [\mathcal{V}]^2$ some subset. The elements of $\mathcal{V}$ are called the *vertices* of $\mathcal{G}$ and the elements of $\mathcal{E}$ are called the *edges* of $\mathcal{G}$.

Definition 2.1.3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. An edge $e \in \mathcal{E}$ is said to *join* two vertices $v, w \in \mathcal{V}$ if $e = \{v, w\}$.

Example 2.1.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph with vertices $\mathcal{V} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and edges $\mathcal{E} = \{\{1, 2\}, \{1, 5\}, \{2, 5\}, \{3, 6\}, \{4, 7\}, \{5, 9\}\}$. The graph $\mathcal{G}$ can be visualised as in Figure 2.1.5.*

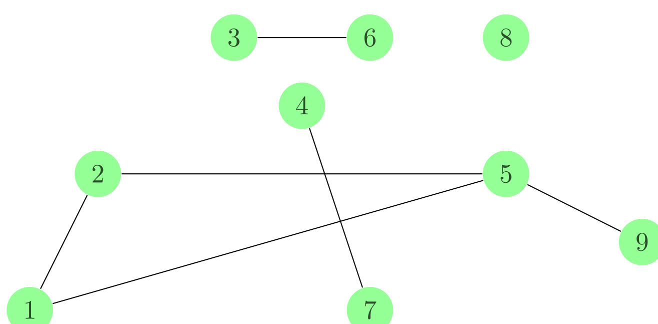

Figure 2.1.5: Diagram of the graph $\mathcal{G}$.

The vertices are distinct points on the paper and a line is drawn between two points if and only if there is an edge joining the corresponding vertices. Note that the drawn positions of the points was arbitrary, the lines do not have directions and there is at most one edge joining any two vertices.

Notation 2.1.6. For an edge $e = \{v, w\}$ of a graph $\mathcal{G}$, we will write $e = vw$, in the case that there is no ambiguity caused by the choice of symbols v, w. Since $e = \{w, v\}$, we may also write $e = wv$.

Definition 2.1.7. The *order* of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the size of the set $\mathcal{V}$. The graph $\mathcal{G}$ is called *finite* if it has finite order and *infinite* otherwise.

Definition 2.1.8. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be two graphs. If $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$ we call $\mathcal{G}'$ a *subgraph* of $\mathcal{G}$.

Definition 2.1.9. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph and $\mathcal{V}' \subseteq \mathcal{V}$ be a subset. The *induced subgraph* of $\mathcal{G}$ by $\mathcal{V}'$, is the graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, where $\mathcal{E}'$ is defined as

$$\mathcal{E}' = \{e \in \mathcal{E} : \text{e joins some } v, w \in \mathcal{V}' \subseteq \mathcal{V} \text{ in } \mathcal{G}\}.$$

Definition 2.1.10. For $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a graph and $e \in [\mathcal{V}]^2$, we define the graph $\mathcal{G} + e = (\mathcal{V}, \mathcal{E} \cup \{e\})$.

Definition 2.1.11. A *path* is a graph $\mathcal{P} = (\mathcal{V}, \mathcal{E})$ of the form $\mathcal{V} = \{v_0, ..., v_k\}$, $\mathcal{E} = \{v_0v_1, v_1v_2, ..., v_{k-1}v_k\}$, where the elements of $\mathcal{V}$ are distinct and $k \in \mathbb{Z}_{\geq 0}$. The vertices $v_0, v_k \in \mathcal{V}$ are *linked* by $\mathcal{P}$ and are called the *ends* of $\mathcal{P}$.

Definition 2.1.12. The *length* of a path $\mathcal{P} = (\mathcal{V}, \mathcal{E})$ is the number of edges $|\mathcal{E}|$.

Notation 2.1.13. If $\mathcal{P}$ is a path with ends v_0 and v_k, we may write $\mathcal{P} = v_0...v_k$ (or $\mathcal{P} = v_k...v_0$) when no ambiguity is caused. Sometimes it's useful to fix an ordering of the vertices; we would write $\mathcal{P} = v_0...v_k$ and call $\mathcal{P}$ a path *from* v_0 *to* v_k.

Definition 2.1.14. If $\mathcal{P} = v_0...v_{k-1}$ is a path and $k \in \mathbb{Z}_{\geq 3}$, then the graph $\mathcal{C} = \mathcal{P} + v_{k-1}v_0$ is called a *cycle*. The *length* of $\mathcal{C}$ is the number of edges of $\mathcal{C}$.

Notation 2.1.15. Whenever uncertainty is not caused, we may write $\mathcal{C} = v_0...v_{k-1}v_0$, and also $\mathcal{C} = v_{\tau(0)}...v_{\tau(k-1)}v_{\tau(0)}$ for any cyclic permutation τ of the k symbols $\{0, ..., k-1\}$.

Note that if $\mathcal{C} = (\mathcal{V}, \mathcal{E})$ is a cycle then the length of the cycle is the same as the order of $\mathcal{C}$.

Definition 2.1.16. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called *connected* if any two of its vertices are linked by a path in $\mathcal{G}$. Otherwise $\mathcal{G}$ is called *disconnected*.

Example 2.1.17. *Consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with vertices $\mathcal{V} = \{a, b, c, d, e, f\}$ and edges $\mathcal{E} = \{ab, bc, cd, de, ea\}$. This graph has a unique cycle given by $\mathcal{C} = abcdea$.*

The graph $\mathcal{G}$ is disconnected; there is no path linking f to any vertex in $\mathcal{V} \backslash \{f\}$. However, the graph $\mathcal{G} + af$ is connected, since we now have a path $\mathcal{P} = fab$ linking f and b.

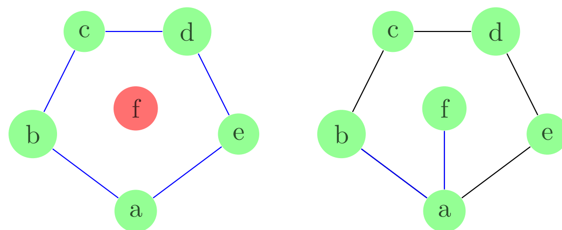

Figure 2.1.18: Diagram of the two graphs: $\mathcal{G}$ (left) and $\mathcal{G} + af$ (right). The edges of the unique cycle in $\mathcal{G}$ and a path linking b, f in $\mathcal{G} + af$ are coloured in blue.

Definition 2.1.19. [4, p. 9, adapted notation] Let $\mathcal{G} = (\mathcal{G}, \mathcal{V})$ be a graph. A *walk* $\mathcal{W}$ of *length* $k \in \mathbb{Z}_{\geq 0}$ in $\mathcal{G}$, is a sequence $(v_0, ..., v_k)$ of vertices $\{v_i\}_{i=0}^k \in \mathcal{V}$, such that $\{v_i, v_{i+1}\} \in \mathcal{E}$ for every non-negative integer $i < k$. We say $\mathcal{W}$ is a walk *from* v_0 *to* v_k. If $v_0 = v_k$, then we call the walk W *closed*.

Notation 2.1.20. As usual, in the case that no ambiguity is caused, we can write a walk as $\mathcal{W} = v_0...v_k$.

Definition 2.1.21. A walk $\mathcal{W}$ in $\mathcal{G}$ from v to v' is called *short* if it has the minimal length among all walks in $\mathcal{G}$ from v to v'.

It should be noted that every path in a graph $\mathcal{G}$ induces a pair of walks in $\mathcal{G}$. Conversely, every walk in $\mathcal{G}$ which is a sequence of *distinct* vertices of $\mathcal{G}$, induces a path in $\mathcal{G}$. In particular, if $\mathcal{G}$ is connected, then for any two vertices v, v' of $\mathcal{G}$, there is a walk from v to v'.

Definition 2.1.22. Let $C = v_0...v_{k-1}v_0$ be a cycle. The *induced orientation* of C is the closed walk W given by $W = (v_0, ..., v_{k-1}, v_0)$.

2.2 Cones and Convex Sets

We follow [11, pp. 3-4, pp. 82–83] for a description of cones and convex sets. One detour is made from [11]: we adopt the words "strongly convex" [10, p. 1], instead of "pointed". The definition of the dimension of a convex set from [7, p. 3] is also added.

For the sake of notational simplicity, we will always work over the vector space $\mathbb{R}^n$ with lattice structure $\mathbb{Z}^n$; the interested are invited to learn about the tensor products of modules [8, pp. 94-97] and general lattices [2, pp. 41-42]. Here, $n \in \mathbb{Z}_{\geq 0}$ is a non-negative integer.

Definition 2.2.1. Let $\{\mathbf{v}_i\}_{i=1}^k \subseteq \mathbb{R}^n$ and $\{\lambda_i\}_{i=1}^k \subset \mathbb{R}$ be finite subsets, some $k \in \mathbb{Z}_{\geq 1}$. Then $\sum_{i=1}^k \lambda_i \mathbf{v}_i$ is called a *linear combination* of the vectors $\{\mathbf{v}_i\}_{i=1}^k$. It is further
(i) a *conic combination*, if $\lambda_i \geq 0 \ \forall \ i \in \{1, ..., k\}$,
(ii) an *affine combination*, if $\sum_{i=1}^k \lambda_i = 1$, and
(iii) a *convex combination*, if it is both a conic combination and an affine combination.

Definition 2.2.2. Let $C \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. The *conic hull* $\operatorname{cone}(C)$ of C is the set of all conic combinations formed from all finite subsets of C. If $C = \operatorname{cone}(C)$ then we call C a *cone*.

Definition 2.2.3. Let $C \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. The *affine hull* $\operatorname{aff}(C)$ of C is the set of all affine combinations formed from all finite subsets of C.

Definition 2.2.4. Let $C \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. The *convex hull* $\operatorname{conv}(C)$ of C is the set of all convex combinations formed from all finite subsets of C. If $C = \operatorname{conv}(C)$ then we call C a *convex set*.

Example 2.2.5. *Consider the set $C = \{(-1, 2), (1, 1)\} \subset \mathbb{R}^2$. The finite subsets of C are $\{(-1, 2)\}, \{(1, 1)\}$ and $\{(-1, 2), (1, 1)\}$, hence*

$$\begin{aligned}
\operatorname{cone}(C) &= \{\lambda_1(-1, 2) + \lambda_2(1, 1) : \lambda_1, \lambda_2 \geq 0\}, \\
\operatorname{aff}(C) &= \{\lambda(-1, 2) + (1 - \lambda)(1, 1) : \lambda \in \mathbb{R}\} \\
&= \{(1, 1) + \lambda((-1, 2) - (1, 1)) : \lambda \in \mathbb{R}\}, \\
\operatorname{conv}(C) &= \{\lambda(-1, 2) + (1 - \lambda)(1, 1) : \lambda \in [0, 1]\} \\
&= \{(1, 1) + \lambda((-1, 2) - (1, 1)) : \lambda \in [0, 1]\}.
\end{aligned}$$

The affine, conic and convex hulls of C are pictured in Figure 2.2.6.

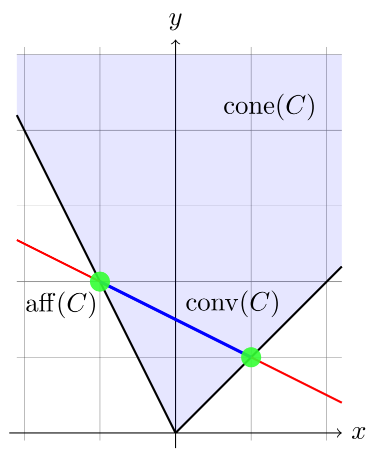

Figure 2.2.6: The affine hull (red line), convex hull (blue line segment) and conic hull (shaded region) of C.

Proposition 2.2.7. *If $C \subseteq \mathbb{R}^n$ is a non-empty subset of $\mathbb{R}^n$, then there is a unique vector subspace $V \subseteq \mathbb{R}^n$ such that $\operatorname{aff}(C) = \mathbf{v} + V$ for every $\mathbf{v} \in C$.*

Proof. See the discussion of [7, p. 3]. $\square$

Definition 2.2.8. Let C be a non-empty subset of $\mathbb{R}^n$ and suppose that $\operatorname{aff}(C) = \mathbf{v} + V$ for some $\mathbf{v} \in C$ and vector subspace V. The *dimension* of C is that of the dimension of the vector space V and is denoted $\dim C$.

Definition 2.2.9. [11, p. 4: Definition 1.3, p. 5: Theorem 1.14]. A *polyhedral cone* is a subset $C \subseteq \mathbb{R}^n$ such that there exists a finite number of vectors $\{\mathbf{b}_i\}_{i=1}^k \subseteq \mathbb{R}^n$ with $C = \mathrm{cone}(\{\mathbf{b}_i\}_{i=1}^k)$. The vectors $\{\mathbf{b}_i\}_{i=1}^k$ are called the *generators* of C.

Definition 2.2.10. A *rational polyhedral cone* is a subset $C \subseteq \mathbb{R}^n$ such that there exists a finite number of vectors $\{\mathbf{b}_i\}_{i=1}^k \subseteq \mathbb{Q}^n$ with $C = \mathrm{cone}(\{\mathbf{b}_i\}_{i=1}^k)$.

Definition 2.2.11. A point $\mathbf{v} \in \mathbb{Z}^n$ is called a *lattice point*.

Definition 2.2.12. [11, adapted and added minimality definition]. Let $\mathcal{H} = \{\mathbf{h}_i\}_{i=1}^k \subseteq \mathbb{Z}^n$ be a finite subset. We call $\mathcal{H}$ a *Hilbert basis* of the cone $C := \mathrm{cone}(\{\mathbf{h}_i\}_{i=1}^k)$ if every lattice point $\mathbf{v}$ in C can be written as $\mathbf{v} = \sum_{i=1}^k \alpha_i \mathbf{h}_i$, for some $\{\alpha_i\}_{i=1}^k \subset \mathbb{Z}_{\geq 0}$. A Hilbert basis of C is called *minimal* if it is contained in every Hilbert basis of the cone C.

It is clear that if a cone C has a minimal Hilbert basis $\mathcal{H}$ then $\mathcal{H}$ must be the unique minimal Hilbert basis of C.

Definition 2.2.13. [10, p. 1]. A convex polyhedral cone $C \subseteq \mathbb{R}^n$ is called *strongly convex* if $C \cap (-C) = \{\mathbf{0}\}$, where $-C := \{-\mathbf{v} : \mathbf{v} \in C\}$.

Theorem 2.2.14. *Let $C \subseteq \mathbb{R}^n$ be a rational polyhedral cone. Then C is generated by a Hilbert basis and if C is strongly convex then there is a unique minimal Hilbert basis of C.*

Proof. See [11, p. 83: Theorem 8.24]. □

2.3 Polytopes and the Minkowski Sum

Polytopes, in particular lattice polytopes and how they behave under the Minkowski sum operation, form the central object of study. As in the previous subsection, and throughout the entire document, we will work over the vector space $\mathbb{R}^n$ with lattice structure $\mathbb{Z}^n$, some non-negative integer n. The dual space of $\mathbb{R}^n$ is denoted $(\mathbb{R}^n)^*$.

Definition 2.3.1. [11, p. 12]. Let $A, B \subseteq \mathbb{R}^n$ be subsets. The *Minkowski sum* of A and B is denoted by $A + B$ and defined as the set

$$A + B = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A, \mathbf{b} \in B\}.$$

Notice that the Minkowski sum is associative and commutative. That is, for $A, B, C \subseteq \mathbb{R}^n$ we have $(A + B) + C = A + (B + C)$ and $A + B = B + A$. For a subset $A \subseteq \mathbb{R}^n$ and $B = \{\mathbf{b}\} \subseteq \mathbb{R}^n$ a single point set, we may denote $A + B$ by $A + \mathbf{b}$, or by $\mathbf{b} + A$. When $\mathbf{b} = \mathbf{0}$ we get $A + \mathbf{0} = A$ for any subset $A \subseteq \mathbb{R}^n$.

Let $C \subseteq \mathbb{R}^n$ and $k \in \mathbb{R}$. We write kC to denote the set

$$kC = \{k\mathbf{c} : \mathbf{c} \in C\}.$$

If $C \subseteq \mathbb{R}^n$ is a convex set and $k, h \in \mathbb{R}_{\geq 0}$ then $(k + h)C = kC + hC$ [12, p. 1: Remark 1.1.1]. However, the sum $C + (-C) := C + (-1)C$ equals $(1-1)C = \{\mathbf{0}\}$ if and only if C is a single point set.

Definition 2.3.2. [11, p. 11]. A *(convex) polytope* $P \subseteq \mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$.

Proposition 2.3.3. *Let $P \subseteq \mathbb{R}^n$ be a polytope and $k \in \mathbb{R}$. If $P = \operatorname{conv}(\{\mathbf{v}\}_{i=1}^a)$ for some finite subset $\{\mathbf{v}\}_{i=1}^a \subseteq \mathbb{R}^n$, then $kP = \operatorname{conv}(\{k\mathbf{v}\}_{i=1}^a)$. Consequently, we have kP is a polytope.*

Proof. This is immediate from the definitions of kP and the convex hull. □

Proposition 2.3.4. *If $Q = \operatorname{conv}(\{\mathbf{q}_i\}_{i=1}^a)$ and $R = \operatorname{conv}(\{\mathbf{r}_i\}_{i=1}^b)$ are two polytopes in $\mathbb{R}^n$, then*

$$Q + R = \operatorname{conv}(\{\mathbf{q}_i + \mathbf{r}_j : 1 \leq i \leq a, 1 \leq j \leq b\}).$$

Consequently, the Minkowski sum $Q + R$ is a polytope.

Proof. See [12, p. 2: Theorem 1.1.2]. □

Example 2.3.5. *Consider the polytope $P \subset \mathbb{R}^2$ given by the convex hull*

$$P = \operatorname{conv}(\{(0,0), (1,0), (2,1), (2,2), (1,2), (0,1)\}).$$

Using Proposition 2.3.4, we can write P as: the sum of two triangles $P = T_1 + T_2$ and the sum of three lines $P = L_1 + L_2 + L_3$, as pictured below in Figure 2.3.6.

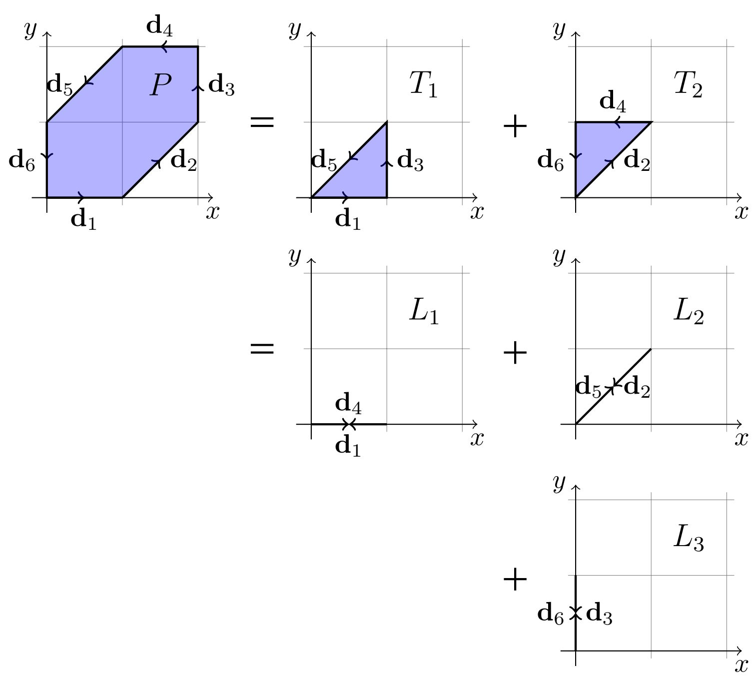

Figure 2.3.6: Expressions of P as the Minkowski sum of polytopes.

The two triangles are given by the convex hulls

$$T_1 = \operatorname{conv}(\{(0,0),(1,0),(1,1)\}),$$
$$T_2 = \operatorname{conv}(\{(0,0),(0,1),(1,1)\}),$$

and the three lines are given by

$$L_1 = \operatorname{conv}(\{(0,0),(1,0)\}),$$
$$L_2 = \operatorname{conv}(\{(0,0),(1,1))\}),$$
$$L_3 = \operatorname{conv}(\{(0,0),(0,1)\}).$$

Definition 2.3.7. Let $P \subseteq \mathbb{R}^n$ be a polytope. If $P = Q+R$, for some polytopes $Q, R \subseteq \mathbb{R}^n$, then Q, R are called *Minkowski summands* of P. We denote by $\mathrm{MS}(P)$ the set of all Minkowski summands of P.

Definition 2.3.8. Let $P \subseteq \mathbb{R}^n$ be a polytope. Suppose that P can be expressed as

$$P = \sum_{i=1}^{k} Q_i := Q_1 + \ldots + Q_k,$$

for some finite set $\{Q_i\}_{i=1}^k$ of polytopes in $\mathbb{R}^n$. The expression of P as the sum of polytopes is called a *Minkowski decomposition* of P.

Proposition 2.3.9. *Let $P \subseteq \mathbb{R}^n$ be a polytope. If $Q \in \mathrm{MS}(P)$, $Q' \in \mathrm{MS}(Q)$ then $Q' \in \mathrm{MS}(P)$.*

Proof. Let $P \subseteq \mathbb{R}^n$ and suppose $Q \in \mathrm{MS}(P), Q' \in \mathrm{MS}(Q')$. There exists polytopes $R, R' \subseteq \mathbb{R}^n$ such that $P = Q+R$ and $Q = Q'+R'$. Hence $P = Q'+(R+R')$ and so $Q' \in \mathrm{MS}(P)$. □

Definition 2.3.10. A vector $\mathbf{d} \in \mathbb{Z}^n$ is called *primitive* if it contains no interior lattice points. That is, $\operatorname{conv}(\{\mathbf{0}, \mathbf{d}\}) \cap \mathbb{Z}^n = \{\mathbf{0}, \mathbf{d}\}$.

Definition 2.3.11. [2, p. 42: Definition 1.73]. A *lattice polytope* in $\mathbb{R}^n$ is the convex hull of finitely many lattice points.

Corollary 2.3.12. *The Minkowski sum of lattice polytopes is a lattice polytope.*

Proof. This follows immediately from Proposition 2.3.4. □

Observe that for any polytope $P \subseteq \mathbb{R}^n$, the polytopes $P + \mathbf{a}$ and $\{\mathbf{a}\}$ are Minkowski summands of P for any $\mathbf{a} \in \mathbb{R}^n$.

Definition 2.3.13. [6, p. 506]. A lattice polytope $P \subseteq \mathbb{R}^n$ is called *indecomposable* if whenever we have $P = Q + R$ for some lattice polytopes Q, R then Q or R is a single point set.

Proposition 2.3.14. *Let $P \subseteq \mathbb{R}^n$ be a lattice polytope and suppose P is indecomposable. The only lattice Minkowski summands of P are the single points, $\{\mathbf{a}\}$ for any $\mathbf{a} \in \mathbb{R}^n$, and translations of P, $P + \mathbf{a}$ for any $\mathbf{a} \in \mathbb{R}^n$.*

Proof. Let $P = Q + R$ be a decomposition of P, where $Q, R \subseteq \mathbb{R}^n$ are some lattice polytopes. Since P is indecomposable, we may assume without loss of generality that $R = \{\mathbf{r}\}$ for some $\mathbf{r} \in \mathbb{R}^n$. Adding $\{-\mathbf{r}\}$ to both sides of the decomposition we find $Q = P + \{-\mathbf{r}\}$ as required. $\square$

Definition 2.3.15. A *decomposition* of a lattice polytope $P \subseteq \mathbb{R}^n$ is an expression of P as a finite Minkowski sum

$$P = \sum_{i=1}^{k} Q_i = Q_1 + ... + Q_k$$

of lattice polytopes $\{Q_i\}_{i=1}^k$ in $\mathbb{R}^n$. If $P = \sum_{i=1}^k Q_i$ for some indecomposable lattice polytopes $\{Q_i\}_{i=1}^k$ in $\mathbb{R}^n$, then $P = \sum_{i=1}^k Q_i$ is called a *maximal decomposition* of P.

Much later (Proposition 5.1.10) we will show that every lattice polytope P has a maximal decomposition. However, in general, a maximal decomposition is not unique. For example, the two decompositions in Figure 2.3.6 are different maximal decompositions of the same polytope.

Definition 2.3.16. The *ambient space* of a polytope $P \subseteq \mathbb{R}^n$ is the vector space $\mathbb{R}^n$.

We now investigate the notions of a face of a polytope and supporting hyperplanes. By studying the faces of a lattice polytope, the set of all its lattice Minkowski summands will be determined.

Proposition 2.3.17. *Let A be a subset of $\mathbb{R}^n$ and $k \in \mathbb{R} \backslash \{0\}$. Then* $\dim kA = \dim A$.

Proof. Write $\operatorname{aff} A = \mathbf{a} + V$, for some $\mathbf{a} \in A$ and vector subspace $V \subseteq \mathbb{R}^n$. By the definition of the affine hull we have $\operatorname{aff} kA = k \operatorname{aff} A$. Hence $\operatorname{aff} kA = k\mathbf{a} + kV = k\mathbf{a} + V$, and so $\dim kA = \dim V = \dim A$. $\square$

Proposition 2.3.18. *Let A, B, C be subsets of $\mathbb{R}^n$. If $A = B + C$ then* $\dim B \leq \dim A$ *and* $\dim C \leq \dim A$.

Proof. There exists vector spaces V_A, V_B, V_C and points $\mathbf{a} \in A, \mathbf{b} \in B, \mathbf{c} \in C$, such that $\operatorname{aff} A = \mathbf{a} + V_A, \operatorname{aff} B = \mathbf{b} + V_B, \operatorname{aff} C = \mathbf{c} + C$ and $\mathbf{a} = \mathbf{b} + \mathbf{c}$. It is easily checked that the inclusions $\operatorname{aff}(B + \mathbf{c}) \subseteq (\operatorname{aff} B) + \mathbf{c}$ and $\operatorname{aff}(B + \mathbf{c}) \supseteq (\operatorname{aff} B) + \mathbf{c}$ hold. We conclude $\operatorname{aff}(B + \mathbf{c}) = (\operatorname{aff} B) + \mathbf{c}$. Also, since $B + \mathbf{c} \subseteq A$ we have $\operatorname{aff} B + \mathbf{c} \subseteq \operatorname{aff} A$. Therefore $(\mathbf{b} + \mathbf{c} - \mathbf{a}) + V_B = V_B \subseteq V_A$. Finally, by linear algebra, we know $\dim V_B \leq \dim V_A$. That is, $\dim B \leq \dim A$ as required. Similarly, $\dim C \leq \dim A$ is found.

$\square$

Definition 2.3.19. [7, p. 31], [2, p. 5: Definition 1.1]. If P is a polytope with $\dim P = d$ then P is called an *d-polytope*. We also call P a *polygon* if $d = 2$.

Example 2.3.20. *The square $S \subseteq \mathbb{R}^3$ is given by the convex hull*

$$S = \mathrm{conv}(\{(0,0,0),(1,0,0),(1,1,0),(0,1,0)\}) \subseteq \mathbb{R}^3,$$

and has affine hull $\mathrm{aff}\, S = \{(a, b, 0 : a, b \in \mathbb{R}\}$*. So S is a lattice polygon, or a lattice 2-polytope, in the ambient space $\mathbb{R}^3$.*

Definition 2.3.21. [10, p. 179]. For each $u \in (\mathbb{R}^n)^*$ and $h \in \mathbb{R}$ define the *affine half-space* $H(u; h) = \{\mathbf{v} \in \mathbb{R}^n : u(\mathbf{v}) \leq h\}$.

For convenience, note that we allow $u = 0$ in the definition of an affine half-space, despite the fact $H(0, h)$ is the whole of $\mathbb{R}^n$ for any $h \geq 0$.

Theorem 2.3.22. *A subset $P \subseteq \mathbb{R}^n$ is a* polytope *if and only if P is compact and $P = \cap_{i=1}^k H(u_i; h_i)$ for some finite set of functionals $\{u_i\}_{i=1}^k \subseteq (\mathbb{R}^n)^* \backslash \{0\}$ and constants $\{h_i\}_{i=1}^k \subset \mathbb{R}$.*

Proof. See [10, p. 180: Theorem A.12]. □

Definition 2.3.23. [10, p. 182: See the remark on p. 183]. Let $P \subseteq \mathbb{R}^n$ be a polytope. The *support function for P* is the function $h_P : (\mathbb{R}^n)^* \to \mathbb{R}$ which sends a linear functional $u \in (\mathbb{R}^n)^*$ to $h_P(u) := \sup\{u(\mathbf{p}) : \mathbf{p} \in P\}$.

Definition 2.3.24. [10, p. 182]. For each $u \in (\mathbb{R}^n)^*$ we define the *affine half-space associated to P* to be $H_P(u) = \{\mathbf{v} \in \mathbb{R}^n : u(\mathbf{v}) \leq h_P(u)\}$.

By definition P is contained in $H_P(u)$ and its boundary $\partial H_P(u) = \{\mathbf{v} \in \mathbb{R}^n : u(\mathbf{v}) = h_P(u)\}$ has non-empty intersection with P, that is $P \cap \partial H_P(u) \neq \emptyset$.

Definition 2.3.25. [10, pp. 181-182, combined definitions]. Let P be a polytope. A subset $F \subseteq P$ is called a *face* and denoted $F < P$ if

$$F = P \cap \partial H_P(u), \text{ for some } u \in (\mathbb{R}^n)^*.$$

The hyperplane $H_P(u)$ is called the *supporting hyperplane* of F. The face F will be denoted by F_u^P. The set of all faces of P is denoted $\mathrm{Faces}(P)$.

Notice that if P is a polytope then $P \in \mathrm{Faces}(P)$ since $\partial H_P(0) = P$, where $0 \in (\mathbb{R}^n)^*$ is the zero functional. An element of $\mathrm{Faces}(P) \backslash \{P\}$ is called a *proper face* of P. It should also be noted that a face of P can have multiple supporting hyperplanes.

Proposition 2.3.26. *A face of a polytope is a polytope.*

Proof. Let P be a polytope, $F < P$ a face of P and $\partial H_P(u)$ a supporting hyperplane of F. By Theorem 2.3.22, we have $P = \cap_{i=1}^k H(u_i; h_i)$ for some finite set of functionals $\{u_i\}_{i=1}^k \subseteq (\mathbb{R}^n)^* \backslash \{0\}$ and constants $\{h_i\}_{i=1}^k \subset \mathbb{R}$. So we find $F = (\cap_{i=1}^k H(u_i; h_i)) \cap H(-u, -h_P(u))$. We conclude F is a polytope by Theorem 2.3.22. □

Definition 2.3.27. [7, p. 31]. Let $F < P$ be a face of a polytope $P \subseteq \mathbb{R}^n$, $\dim P = d$ and $\dim F = k$. Then F is called a k-face. The face F is also called
(i) a *vertex* of P if $k = 0$,
(ii) an *edge* of P if $k = 1$,
(iii) a *facet* of P if $k = d - 1$.
The set of all k-faces of P is denoted $k\text{-Faces}(P)$, the set of all vertices of P is denoted $\text{Verts}(P)$ and the set of all edges of P is denoted $\text{Edges}(P)$.

It is immediate from the fact $A \subseteq \text{aff}\, A$ for any subset $A \subset \mathbb{R}^n$ that if $\dim A = 0$ then $A = \{\mathbf{a}\}$ is a single point set. For a polytope $P \subseteq \mathbb{R}^n$, we will write both that $\{\mathbf{v}\} \in \text{Verts}(P)$ and $\mathbf{v} \in \text{Verts}(P)$. Usually $\text{Verts}(P)$ will be interpreted as a set of points in $\mathbb{R}^n$.

Proposition 2.3.28. *A polytope is equal to the convex hull of its vertices.*

Proof. This is a consequence of [7, p. 14: Theorem 2.3.4] and [7, p. 19: Theorem 2.4.8]. □

Proposition 2.3.29. *A polytope P has a finite number of faces. Moreover, if P is a d-polytope then there exists a k-face F of P, for any $0 \leq k \leq d$ and there are no k-faces for $k > d$.*

Proof. From [7, p. 27: Theorem 2.6.6] the number of faces of P is finite. A face F of a polytope P is a subset of P. Consequently $\text{aff}\, F \subseteq \text{aff}\, P$ and so $\dim F \leq \dim P$. The polytope P is a d-face of itself and Proposition 2.3.28 implies P has a 0-face. The existence of k-faces for all integers in the range $0 \leq k \leq d$ then follows [12, p. 98: Theorem 2.4.8]. □

Proposition 2.3.30. *Let $P \subseteq \mathbb{R}^n$ be a polytope. If F is a face of P and F' is a face of F then F' is a face of P.*

Proof. See [7, p. 33: Theorem 3.1.5].

□

Definition 2.3.31. A *triangle* is a 2-polytope with exactly three vertices.

Proposition 2.3.32. *If P is a polytope such that every 2-face of P is a triangle, then the only Minkowski summands of P are kP for $k \in [0, 1]$.*

Proof. See [12, p. 152: Corollary 3.2.13]. Note that the definition of indecomposability in [12, p. 150] is different to one defined above in Definition 2.3.13. □

Definition 2.3.33. [7, p. 212]. Let P be a polytope with vertices $\mathcal{V} := \text{Verts}(P)$ and edges $\mathcal{E} := \text{Edges}(P)$. Then the *graph* of P is the graph defined as $\text{Graph}(P) = (\mathcal{V}, \mathcal{E})$.

Theorem 2.3.34. *The graph of a polytope is a connected graph.*

Proof. See [7, p. 213: Theorem 11.3.2]. □

Definition 2.3.35. [7, p. 341, swapped "path" with "walk" for self-consistency]. Let P be a polytope. A *walk* $\mathcal{W}$ on P is a walk on the $\text{Graph}(P)$. A *closed walk* on P is a closed walk on the graph of P.

3 The Dilation of Edges of Minkowski Summands

To avoid cumbersome notation, and to reduce the required background knowledge, the ambient space of all the polytopes considered will be exclusive to the vector spaces $\mathbb{R}^n$ with lattice structures $\mathbb{Z}^n$. The theory, however, generalises to any free module $N \cong \mathbb{Z}^n$ with corresponding vector space $N_{\mathbb{R}} := N \otimes \mathbb{R}$.

3.1 Basic properties of the Minkowski sum and the edges of lattice polytopes

Given a lattice polytope $P \subseteq \mathbb{R}^n$, we want to find all lattice polytopes Q, R such that P is the Minkowski sum of Q and R. To tackle this problem, we first observe the following basic properties of the Minkowski sum.

Proposition 3.1.1. *Let P be a polytope (not necessarily a lattice polytope). If $P = Q + R$, for some polytopes Q and R, then $F_u^P = F_u^Q + F_u^R$ for all $u \in (\mathbb{R}^n)^*$.*

Proof. Let $u \in (\mathbb{R}^n)^*$ be arbitrary. The result is concluded from the below computations.

$$\begin{aligned}
\mathrm{h}_P(u) &= \sup\{u(\mathbf{v}) : \mathbf{v} \in P = Q + R\} \\
&= \sup\{u(\mathbf{q}) + u(\mathbf{r}) : \mathbf{q} \in Q, \mathbf{r} \in R\} \\
&= \sup\{u(\mathbf{q}) : \mathbf{q} \in Q\} + \sup\{u(\mathbf{r}) : \mathbf{r} \in R\} \\
&= \mathrm{h}_Q(u) + \mathrm{h}_R(u)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow F_u^P &= \partial\, \mathrm{H}_P(u) \cap P \\
&= \{\mathbf{q} + \mathbf{r} : u(\mathbf{q}) = \mathrm{h}_Q(u), u(\mathbf{r}) = \mathrm{h}_R(u), \mathbf{q} \in Q, \mathbf{r} \in R\} \\
&= \partial\, \mathrm{H}_Q(u) \cap Q + \partial\, \mathrm{H}_R(u) \cap R \\
&= F_u^Q + F_u^R.
\end{aligned}$$

The second equality of F_u^P above follows from: linearity of $u \in (\mathbb{R}^n)^*$, $u(\mathbf{q}) \leq \mathrm{h}_Q(u)$ and $u(\mathbf{r}) \leq \mathrm{h}_R(u)$. □

Thus, each face of P is the Minkowski sum of a face of Q and a face of R. Moreover, each face of P is given by a unique pair of faces from Q and R, in the following sense.

Proposition 3.1.2. *Let P be a polytope and suppose that $P = Q + R$, for some polytopes Q, R. Further suppose that $F \in \mathrm{Faces}(P)$. If $F = F_u^P = F_{u'}^P$, for some $u, u' \in (\mathbb{R}^n)^*$, then $F_u^Q = F_{u'}^Q$ and $F_u^R = F_{u'}^R$.*

Proof. Suppose that $F = F_u^P = F_{u'}^P$, for some $u, u' \in (\mathbb{R}^n)^*$. By Proposition 3.1.1, we have that $F = F_u^Q + F_u^R = F_{u'}^Q + F_{u'}^R$. So

$$\begin{aligned}
F &= \{\mathbf{q} + \mathbf{r} : u(\mathbf{q}) = \mathrm{h}_Q(u), u(\mathbf{r}) = \mathrm{h}_R(u), \mathbf{q} \in Q, \mathbf{r} \in R\} \\
&= \{\mathbf{q} + \mathbf{r} : u'(\mathbf{q}) = \mathrm{h}_Q(u'), u'(\mathbf{r}) = \mathrm{h}_R(u'), \mathbf{q} \in Q, \mathbf{r} \in R\}.
\end{aligned}$$

Pick any $\mathbf{q} \in F_u^Q$ and $\mathbf{r} \in F_u^R$, so that $\mathbf{q}+\mathbf{r} \in F$. Then $u'(\mathbf{q}+\mathbf{r}) = \mathrm{h}_P(u') = \mathrm{h}_Q(u') + \mathrm{h}_R(u')$. Since $u'(\mathbf{q}) \leq \mathrm{h}_Q(u')$ and $u'(\mathbf{r}) \leq \mathrm{h}_R(u')$, the only way we can have equality is if $u'(\mathbf{q}) = \mathrm{h}_Q(u')$ and $u'(\mathbf{r}) = \mathrm{h}_R(u')$. Hence $\mathbf{q} \in F_{u'}^Q, \mathbf{r} \in F_{u'}^R$, and so $F_u^Q \subseteq F_{u'}^Q, F_u^R \subseteq F_{u'}^R$.

Similarly, the reverse inclusions are found; thus we conclude equality.

□

Corollary 3.1.3. *Let P be a polytope and suppose that $P = Q + R$, for some polytopes Q, R. If $F_u^P \in k\text{-Faces}(P)$ then $F_u^Q \in \cup_{l=0}^k l\text{-Faces}(Q)$ and $F_u^R \in \cup_{l=0}^k l\text{-Faces}(R)$.*

Proof. We have $F_u^P = F_u^Q + F_u^R$ by Proposition 3.1.1. Then Proposition 2.3.18 tells us that $\dim F_u^P \geq \dim F_u^Q, \dim F_u^R$. □

In particular, by setting $k = 1$ in Corollary 3.1.3, and applying the uniqueness result of Proposition 3.1.2, we immediately conclude the below result.

Corollary 3.1.4. *Let P be a polytope and suppose $P = Q + R$, for some polytopes Q, R. If E is an edge of P then E is the unique sum of a vertex or an edge of Q and a vertex or an edge of R.*

Corollary 3.1.4 restricts the set of potential Minkowski summands, of a given polytope P, in a way which is easy to study. Specifically, given a Minkowski summand Q of P and an edge of P, we will measure the "*length*" or "*dilation*" of the corresponding edge or vertex of Q. For this, we specialise to lattice polytopes P.

Proposition 3.1.5. *If P is a lattice polytope with $\dim P \leq 1$ then $P = \mathrm{conv}(\{\mathbf{p}, \mathbf{p} + t\mathbf{d}\})$ for some vertex $\mathbf{p} \in \mathrm{Verts}(P)$, non-zero primitive vector $\mathbf{d} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and unique non-negative integer $t \in \mathbb{Z}_{\geq 0}$. In the case that $\dim P = 0$, the integer t is zero and $\mathbf{d} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ is arbitrary. Moreover, in the case that $\dim P = 1$, we have that $t > 0$ and $\mathbf{d}$ is uniquely defined up to a sign.*

Proof. Suppose that $\dim P = 0$, so that $\mathrm{aff}\, P = \mathbf{p} + \{\mathbf{0}\}$ for some $\mathbf{p} \in P$. Since $P \subseteq \mathrm{aff}\, P$, we conclude that $P = \{\mathbf{p}\} = \{\mathbf{p}, \mathbf{p} + 0 \cdot \mathbf{d}\}$, for any choice of $\mathbf{d} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$. Furthermore, we see that $\mathbf{p}$ is a vertex of P.

Now suppose that $\dim P = 1$. There exists a 1-dimensional vector space W, say $W = \mathrm{Span}(\mathbf{w})$ for some $\mathbf{w} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, and a point $\mathbf{q} \in P$, such that $\mathrm{aff}\, P = \mathbf{q} + W$. Since P is a lattice polytope, we can write $P = \mathrm{conv}(\{\mathbf{v}_i\}_{i=1}^m)$ for some $\mathbf{v}_i \in P \cap \mathbb{Z}^n$.

For each $i \in \{1, ..., m\}$, there exists some $t_i \in \mathbb{R}$, such that $\mathbf{v}_i = \mathbf{q} + t_i \mathbf{w}$. Without loss of generality, we may assume that $t_1 = \min\{t_i : i \in \{1, ..., m\}\}$ and $t_2 = \max\{t_i : i \in \{1, ..., m\}\}$. Since

$$\begin{aligned} \mathrm{conv}(\{\mathbf{v}_1, \mathbf{v}_2\}) &= \{(\lambda_1 + \lambda_2)\mathbf{p} + (\lambda_1 t_1 + \lambda_2 t_2)\mathbf{w} : \lambda_1 + \lambda_2 = 1\} \\ &= \{\mathbf{q} + \lambda \mathbf{w} : \lambda \in [t_1, t_2]\}, \end{aligned}$$

for each $i \in \{3, ..., m\}$ we have that $\mathbf{v}_i \in \operatorname{conv}(\{\mathbf{v}_1, \mathbf{v}_2\})$. We conclude that $P = \operatorname{conv}(\{\mathbf{v}_1, \mathbf{v}_2\})$ and $\mathbf{v}_1, \mathbf{v}_2$ are two vertices of P. Note that $\mathbf{v}_1 \neq \mathbf{v}_2$ because otherwise we would have $\dim P = 0$.

Since $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{Z}^n$, we have $\mathbf{v}_2 - \mathbf{v}_1 \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$. If $\mathbf{v}_2 - \mathbf{v}_1$ isn't primitive, that is it contains an interior lattice point, then we just rescale by the factor $1/t$, where $t - 1 \in \mathbb{Z}_{>0}$ is the number of interior lattice points. This yields a non-zero primitive vector $\mathbf{d} := \frac{1}{t}(\mathbf{v}_2 - \mathbf{v}_1) \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$. Setting $\mathbf{p} = \mathbf{v}_1$, we have by construction that $P = \operatorname{conv}(\{\mathbf{p}, \mathbf{p} + t\mathbf{d}\})$, as required.

If we can also write $P = \operatorname{conv}(\{\mathbf{p}', \mathbf{p}' + t'\mathbf{d}'\})$, for some primitive vector $\mathbf{d}' \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and positive integer $t' \in \mathbb{Z}_{>0}$, then $\operatorname{Verts}(P) = \{\mathbf{p}', \mathbf{p}' + t'\mathbf{d}'\}$. But we know $\operatorname{Verts}(P) = \{\mathbf{p}, \mathbf{p} + t\mathbf{d}\}$. Hence, either $\mathbf{p}' = \mathbf{p}$ and $\mathbf{p}' + t'\mathbf{d}' = \mathbf{p} + t\mathbf{d}$ or $\mathbf{p}' = \mathbf{p} + t\mathbf{d}$ and $\mathbf{p}' + t'\mathbf{d}' = \mathbf{p}$. The first case implies $t\mathbf{d} = t'\mathbf{d}'$ and the second case implies $t\mathbf{d} = -t'\mathbf{d}'$. Since $t, t' \in \mathbb{Z}_{>0}$ and $\mathbf{d}, \mathbf{d}'$ are both primitive vectors we conclude that $t = t'$ and $\mathbf{d} = \mathbf{d}'$, in the first case, and $t = t'$ and $\mathbf{d} = -\mathbf{d}'$, in the second case.

$\square$

Thus, given a lattice polytope P, for each edge $E \in \operatorname{Edges}(P)$ we can uniquely write $E = \operatorname{conv}(\{\mathbf{p}, \mathbf{p} + t\mathbf{d}\})$ for some vertex $\mathbf{p} \in \operatorname{Verts}(P)$, non-zero primitive vector $\mathbf{d} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and $t \in \mathbb{Z}_{>0}$, given a choice of sign of the primitive vector $\mathbf{d}$.

Definition 3.1.6. Let P be a lattice polytope and $E \in \operatorname{Edges}(P)$ an edge of P. Write E as $E = \operatorname{conv}(\{\mathbf{p}, \mathbf{p} + t\mathbf{d}\})$, for some vertex $\mathbf{p} \in \operatorname{Verts}(P)$, non-zero primitive vector $\mathbf{d} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and $t \in \mathbb{Z}_{>0}$. The *primitive edge vectors* of E are $\{-\mathbf{d}, \mathbf{d}\}$. The positive integer $t \in \mathbb{Z}_{>0}$ is the *length* of the edge E.

We use primitive vectors, instead of unit vectors, to ensure that the length of the edges of P (and its lattice Minkowski summands [Corollary 3.3.1]) are positive integers. This point will become crucial when we're ready to obtain the lattice Minkowski summands.

Now observe that, for any $k \in \mathbb{R}$, we have Q is a Minkowski summand of P if and only if kQ is a Minkowski summand of kP. That is, finding the Minkowski summands of P is equivalent to finding the Minkowski summands of non-zero multiples of P.

For reasons which will become apparent – essentially, to remove unnecessary conditions from later results – we extend our attention to the Minkowski summands of all positive multiples of P. We let $\operatorname{MS}(P)$ denote the set of all Minkowski summands of P and let $\operatorname{MS}_{k>0}(kP) := \cup_{k>0} \operatorname{MS}(kP)$ denote the set of all Minkowski summands of positive multiples of P.

Lemma 3.1.7. *Let P be a polytope. For any $u \in (\mathbb{R}^n)^*$ and $k > 0$, we have $F_u^{kP} = kF_u^P$.*

Proof. Let $u \in (\mathbb{R}^n)^*$ and $k > 0$. Notice that,

$$\begin{aligned}
\mathrm{h}_{kP}(u) &= \sup\{u(k\mathbf{p}) : \mathbf{p} \in P\} \\
&= k \sup\{u(\mathbf{p}) : \mathbf{p} \in P\} \\
&= k\, \mathrm{h}_P(u).
\end{aligned}$$

Consequently,

$$\begin{aligned} F_u^{kP} &= \{k\mathbf{p} : u(k\mathbf{p}) = \mathrm{h}_{kP}(u), \mathbf{p} \in P\} \\ &= \{k\mathbf{p} : u(\mathbf{p}) = \mathrm{h}_P(u), \mathbf{p} \in P\} \\ &= kF_u^P. \end{aligned}$$

□

Corollary 3.1.8. *Let P be a lattice polytope and $Q \in \mathrm{MS}(kP)$ a Minkowski summand of kP, for some $k > 0$. Suppose $F_u^P \in \mathrm{Edges}(P)$, with $F_u^P = \mathrm{conv}(\{\mathbf{p}, \mathbf{p} + t\mathbf{d}\})$, for some $u \in (\mathbb{R}^n)^*$, vertex $\mathbf{p} \in \mathrm{Verts}(P)$, primitive edge vector $\mathbf{d} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and $t \in \mathbb{Z}_{>0}$. Then we have $F_u^Q = \mathrm{conv}(\{\mathbf{q}, \mathbf{q} + s\mathbf{d}\})$ for some unique $\mathbf{q} \in \mathrm{Verts}(Q)$ and $s \in [0, kt]$.*

Proof. There exists some polytope R such that $kP = Q + R$, so $F_u^{kP} = F_u^Q + F_u^R$ by Proposition 3.1.1. Now $F_u^{kP} = kF_u^P = \{k\mathbf{p}, k\mathbf{p} + kt\mathbf{d}\}$ by Lemma 3.1.7. By Proposition 2.3.17, Corollary 3.1.3 and the proof of Proposition 3.1.5, we may write $F_u^Q = \mathrm{conv}(\{\mathbf{v}_1, \mathbf{v}_2\})$, for some $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{Verts}(Q)$. Let $\mathbf{r} \in R$, so that $\mathbf{v}_1 + \mathbf{r}$ and $\mathbf{v}_2 + \mathbf{r} \in kF_u^P$.

Observe that, for any two points $\mathbf{p}_1, \mathbf{p}_2 \in kF_u^P$, their difference $\mathbf{p}_2 - \mathbf{p}_1 = s\mathbf{d}$ for some $s \in [-kt, kt]$. By relabelling, if necessary, we may assume without loss of generality that $s \geq 0$. Hence $\mathbf{v}_2 - \mathbf{v}_1 = (\mathbf{v}_2 + \mathbf{r}) - (\mathbf{v}_1 + \mathbf{r}) = s\mathbf{d}$, for some $s \in [0, kt]$. Thus $\mathbf{v}_2 = \mathbf{v}_1 + s\mathbf{d}$ and so $F_u^Q = \mathrm{conv}(\{\mathbf{q}, \mathbf{q} + s\mathbf{d}\})$ as required.

The uniqueness of this expression follows from the uniqueness of writing F_u^Q as the convex hull of two points $\mathbf{v}_1, \mathbf{v}_2$ and the demand that $s \geq 0$.

□

3.2 Construction of the dilation measuring map ρ

We're now ready to formalise what was meant by "measuring the dilation of the edges of a Minkowski summand".

Definition 3.2.1. Fix a lattice polytope P. The number of edges of P is finite by Proposition 2.3.29 and so we may write the set of edges of P as $\mathrm{Edges}(P) = \{E_i\}_{i=1}^N$, for some $N \in \mathbb{Z}_{\geq 0}$. For each $i \in \{1, ..., N\}$ there exists some $u_i \in (\mathbb{R}^n)^*$ such that $E_i = F_{u_i}^P$. Arbitrarily fix a particular primitive edge vector $\mathbf{d}_i$ for each E_i, so that $E_i = \mathrm{conv}(\{\mathbf{p}_i, \mathbf{p}_i + c_i\mathbf{d}_i\})$ for some $c_i \in \mathbb{Z}_{>0}$ and $\mathbf{p}_i \in \mathrm{Verts}(P)$.

We construct a map

$$\rho : \mathrm{MS}_{k>0}(kP) \to \mathbb{R}_{\geq 0}^N, \tag{1}$$

which measures the *dilation* of the edges of P. We will also say that ρ measures the *length* of the edges of the Minkowski summands of positive multiples of P. Let $Q \in \mathrm{MS}(kP)$ for some $k > 0$. By Corollary 3.1.8 we have

$$F_{u_i}^Q = \mathrm{conv}(\{\mathbf{q}, \mathbf{q} + t_i\mathbf{d}_i\}),$$

for some unique $t_i \in [0, kc_i]$ and $\mathbf{q} \in \text{Verts}(Q)$. Set the i^{th} component of $\rho(Q) \in \mathbb{R}^N$ equal to t_i. That is, define

$$\rho(Q)_i := t_i, \text{ for each } i \in \{1, ..., N\}.$$

It should be noted that permuting the order of the listed edges of a polytope, permutes the coordinate functions of ρ. Hence an ordering of the edges must be fixed. Given an ordering of the edges, however, the choice of primitive edge vectors has no change on the values of ρ.

Example 3.2.2. *Consider the lattice polygon $P \subset \mathbb{R}^2$ given by the convex hull*

$$P = \text{conv}(\{(0,0), (1,0), (2,1), (2,2), (1,2), (0,1)\}).$$

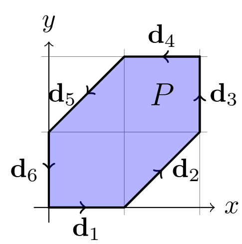

Figure 3.2.3: The polygon P.

The choice of primitive edge vectors of P depicted in Figure 3.2.3 are

$$\mathbf{d}_1 = (0,1), \mathbf{d}_2 = (1,1), \mathbf{d}_3 = (0,1), \mathbf{d}_4 = (-1,0), \mathbf{d}_5 = (-1,-1), \mathbf{d}_6 = (0,-1).$$

The edges of P are

$$E_1 = \text{conv}(\{(0,0),(1,0)\}), E_2 = \text{conv}(\{(1,0),(2,1)\}), E_3 = \text{conv}(\{(2,1),(2,2)\}),$$
$$E_4 = \text{conv}(\{(2,2),(1,2)\}), E_5 = \text{conv}(\{(1,2),(0,1)\}), E_6 = \text{conv}(\{(0,1),(0,0)\}).$$

All of these edges are primitive, that is they have no interior lattice points, or equivalently they all have unit length.

The polygon P has the two lattice Minkowski decompositions pictured below in Figure 3.2.4.

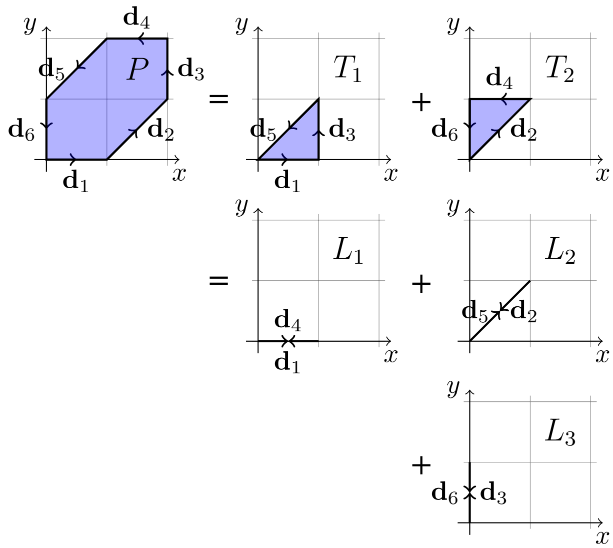

Figure 3.2.4: Two lattice Minkowski decompositions of P.

Letting e_1^*, e_2^* denote the standard dual basis of $\mathbb{R}^2$, each edge E_i of P can be written in the form $E_i = F_{u_i}^P$ for some $u_i \in (\mathbb{R}^2)^*$, as shown in the table below.

$F_{u_i}^P$	E_1	E_2	E_3	E_4	E_5	E_6
u_i	$-e_2^*$	$e_1^* - e_2^*$	e_1^*	e_2^*	$-e_1^* + e_2^*$	$-e_1^*$

Table I: Description of the edges of P in terms of linear functionals $u \in (\mathbb{R}^2)^*$.

For the Minkowski summands in Figure 3.2.4, denote the two triangles in the first decomposition by T_1, T_2, respectively, and L_1, L_2, L_3 the three lines in the second decomposition, respectively. We compute the corresponding vertices/edges for each of the Minkowski summands T_1, T_2, L_1, L_2, L_3, and present the results in two tables.

Q	T_1	T_2
$F_{u_1}^Q$	$\text{conv}(\{(0,0),(1,0)\})$	$\{(0,0)\}$
$F_{u_2}^Q$	$\{(1,0)\}$	$\text{conv}(\{(0,0),(1,1)\})$
$F_{u_3}^Q$	$\text{conv}(\{(1,0),(1,1)\})$	$\{(1,1)\}$
$F_{u_4}^Q$	$\{(1,1)\}$	$\text{conv}(\{(1,1),(0,1)\})$
$F_{u_5}^Q$	$\text{conv}(\{(1,1),(0,0)\})$	$\{(0,1)\}$
$F_{u_6}^Q$	$\{(0,0)\}$	$\text{conv}(\{(0,1),(0,0)\})$

Table II: The corresponding vertices/edges of the Minkowski summands T_1, T_2.

Q	L_1	L_2	L_3
$F_{u_1}^Q$	$\operatorname{conv}(\{(0,0),(1,0)\})$	$\{(0,0)\}$	$\{(0,0)\}$
$F_{u_2}^Q$	$\{(1,0)\}$	$\operatorname{conv}(\{(0,0),(1,1)\})$	$\{(0,0)\}$
$F_{u_3}^Q$	$\{(1,0)\}$	$\{(1,1)\}$	$\operatorname{conv}(\{(0,0),(0,1)\})$
$F_{u_4}^Q$	$\operatorname{conv}(\{(0,0),(1,0)\})$	$\{(1,1)\}$	$\{(0,1)\}$
$F_{u_5}^Q$	$\{(0,0)\}$	$\operatorname{conv}(\{(0,0),(1,1)\})$	$\{(0,1)\}$
$F_{u_6}^Q$	$\{(0,0)\}$	$\{(0,0)\}$	$\operatorname{conv}(\{(0,0),(0,1)\})$

Table III: The corresponding vertices/edges of the Minkowski summands L_1, L_2, L_3.

Using Table II and Table III, we find

$$\begin{aligned}
&\rho(P) = (1,1,1,1,1,1),\\
&\rho(T_1) = (1,0,1,0,1,0), \quad \rho(T_2) = (0,1,0,1,0,1),\\
&\rho(L_1) = (1,0,0,1,0,0), \quad \rho(L_2) = (0,1,0,0,1,0), \quad \rho(L_3) = (0,0,1,0,0,1).
\end{aligned}$$

Recall the map ρ measures the length of the corresponding edge or vertex. Notice that, the vectors involved in each of the above Minkowski decompositions, sum to $\rho(P) = (1,1,1,1,1,1)$. That is,

$$\begin{aligned}
\rho(P) &= \rho(T_1) + \rho(T_2)\\
&= \rho(L_1) + \rho(L_2) + \rho(L_3).
\end{aligned}$$

Lemma 3.2.5. *Let P be a lattice polytope. If $Q \in \mathrm{MS}(kP)$ and $Q' \in \mathrm{MS}(k'P)$ for some $k, k' > 0$, then $Q + Q' \in \mathrm{MS}((k+k')P)$ and $hQ \in \mathrm{MS}((hk)P)$ for any $h \geq 0$.*

Proof. There are two polytopes R, R' such that $kP = Q + R$ and $k'P = Q' + R'$. Hence

$$(k+k')P = (Q+Q') + (R+R'),$$

and so

$$Q + Q' \in \mathrm{MS}((k+k')P).$$

Finally, for any $h \geq 0$, we have

$$h(kP) = hQ + hR.$$

Thus $hQ \in \mathrm{MS}((hk)P)$. □

Proposition 3.2.6. *Let P be a lattice polytope and let $Q, Q' \in \mathrm{MS}_{k>0}(kP)$ be two Minkowski summands. Fix an ordering of the edges of P. Then*

$$\rho(Q + Q') = \rho(Q) + \rho(Q')$$

and

$$\rho(hQ) = h\rho(Q)$$

for any $h \geq 0$.

Proof. There exists some $k, k' > 0$ such that $Q \in \mathrm{MS}(kP)$ and $Q' \in \mathrm{MS}(k'P)$. Let $h \geq 0$. From Lemma 3.2.5, we know that $Q + Q' \in \mathrm{MS}((k + k')P)$ and $hQ \in \mathrm{MS}((hk)P)$, hence $\rho(Q + Q')$ and $\rho(hQ)$ are well-defined.

Let $i \in \{1, ..., N\}$. By Corollary 3.1.8 we have

$$F_{u_i}^{Q} = \mathrm{conv}(\{\mathbf{q}, \mathbf{q} + t_i \mathbf{d}_i\}) \text{ and } F_{u_i}^{Q'} = \mathrm{conv}(\{\mathbf{q}', \mathbf{q}' + t_i' \mathbf{d}_i\})$$

for some unique $t_i \in [0, kc_i]$, $t_i' \in [0, k'c_i]$, $\mathbf{q} \in \mathrm{Verts}(Q)$ and $\mathbf{q}' \in \mathrm{Verts}(Q')$. Then by Proposition 3.1.1 and Proposition 2.3.4 we have

$$\begin{aligned} F_{u_i}^{Q+Q'} &= F_{u_i}^{Q} + F_{u_i}^{Q'} \\ &= \mathrm{conv}(\{\mathbf{q} + \mathbf{q}', (\mathbf{q} + \mathbf{q}') + (t_i + t_i')\mathbf{d}_i\}). \end{aligned}$$

Hence

$$\rho(Q + Q')_i = t_i + t_i' = \rho(Q)_i + \rho(Q')_i.$$

We conclude that

$$\rho(Q + Q') = \rho(Q) + \rho(Q').$$

Also, by Lemma 3.1.7 and Proposition 2.3.3, we have

$$F_{u_i}^{hQ} = \mathrm{conv}(\{h\mathbf{q}, h\mathbf{q} + ht_i \mathbf{d}_i\}), \text{ for every } i \in \{1, ..., N\}.$$

Therefore

$$\rho(hQ) = h\rho(Q)$$

as required. □

Corollary 3.2.7. *Let P be a lattice polytope and fix an ordering of the edges of P. Suppose $Q, R \in \mathrm{MS}(kP)$ are two Minkowski summands of kP, for some $k > 0$, such that $kP = Q + R$. Then $k\rho(P) = \rho(Q) + \rho(R)$.*

Proof. Using both parts of Proposition 3.2.6, we find the desired result, as

$$kP = Q + R \Rightarrow \rho(kP) = \rho(Q + R) \Rightarrow k\rho(P) = \rho(Q) + \rho(R).$$

□

3.3 The closure condition

Let P be a lattice polytope with N edges. Fix an ordering of the edges of P. The strategy for finding all lattice Minkowski summands of P is:

- to determine which points in $\mathbb{R}^N_{\geq 0}$ are in the image of ρ,
- to identity which points in the image of ρ correspond to lattice Minkowski summands of P,
- to recover the Minkowski summands from these points in $\mathbb{R}^N_{\geq 0}$.

The second bullet point is partially answered by the below corollary.

Corollary 3.3.1. *Let P be a lattice polytope with N edges and fix an ordering of the edges of P. For Q a lattice Minkowski summand of P we have $\rho(Q) \in \mathbb{Z}^N_{\geq 0}$ and $\rho(Q)_i \leq \rho(P)_i$, for each $i \in \{1, ..., N\}$.*

Proof. Let $Q \in \mathrm{MS}(P)$ be a lattice Minkowski summand of P. There exists a lattice polytope R such that $P = Q + R$. By Corollary 3.2.7 we have $\rho(P) = \rho(Q) + \rho(R)$ and since $\rho(Q), \rho(R) \in \mathbb{R}^N_{\geq 0}$ we conclude $\rho(Q)_i \leq \rho(P)_i$ for each $i \in \{1, ..., N\}$.

Now suppose that $F^P_u \in \mathrm{Edges}(P)$ is an edge of P for some $u \in (\mathbb{R}^n)^*$. Let $\mathbf{d} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ be a primitive edge vector associated to F^P_u. By Corollary 3.1.8 we can write $F^Q_u = \mathrm{conv}(\{\mathbf{q}, \mathbf{q} + s\mathbf{d}\})$ for some unique $\mathbf{q} \in \mathrm{Verts}(Q)$ and $s \geq 0$. Since Q is a lattice polytope, we conclude that $s \in \mathbb{Z}_{\geq 0}$ by Proposition 3.1.5. Consequently the components of $\rho(Q)$ are integers.

□

We now concentrate our efforts on answering the question raised by the first bullet point. Of course, in general, the image of ρ is not the whole of $\mathbb{R}^N_{\geq 0}$. For example, if P is any lattice triangle then $\mathrm{MS}_{k>0}(kP) = \{kP : k \geq 0\}$ by Proposition 2.3.32. Thus, the image of ρ is the half-line $\mathrm{im}(\rho) = \{k\rho(P) : k \geq 0\}$ by Proposition 3.2.6, which is not equal to $\mathbb{R}^3_{\geq 0}$.

We first deal with the cases $\dim P \in \{0, 1\}$. If $\dim P = 0$, then $P = \mathrm{conv}(\{\mathbf{p}\})$ is the convex hull of a single point $\mathbf{p} \in P$. Hence the only lattice Minkowski summands of P are $\{\mathbf{a}\}$, for any $\mathbf{a} \in \mathbb{Z}^n$. If $\dim P = 1$ then, by Proposition 3.1.5, $P = \mathrm{conv}(\{\mathbf{p}, \mathbf{p} + t\mathbf{d}\})$, for some $\mathbf{p} \in \mathrm{Verts}(P)$, non-zero primitive vector $\mathbf{d} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and positive integer $t \in \mathbb{Z}_{>0}$. Hence, by Corollary 3.1.8, Proposition 2.3.18 and Corollary 3.3.1, the only lattice Minkowski summands of P are $\mathrm{conv}(\{\mathbf{a}, \mathbf{a} + s\mathbf{d}\})$ for any $\mathbf{a} \in \mathbb{Z}^n$ and $s \in [0, t] \cap \mathbb{Z}$.

Next consider polytopes of dimension larger than one. In this case we have existence of 2-faces by Proposition 2.3.29. Our first step will be to rule out vectors of $\mathbb{R}^N_{\geq 0}$ from $\mathrm{im}\,\rho$ using the fact Minkowski summands are polytopes. To do this, we first prove that there is a unique cycle on any 2-face of a polytope.

Lemma 3.3.2. *Let P be a lattice polytope. If $u \in (\mathbb{R}^n)^*$ then $F^P_{ru} = F^P_u$ for every $r > 0$.*

Proof. Let $u \in (\mathbb{R}^n)^*$ and $r > 0$. We compute

$$\begin{aligned} \mathrm{h}_P(ru) &= \sup\{ru(\mathbf{p}) : \mathbf{p} \in P\} \\ &= r \sup\{u(\mathbf{p}) : \mathbf{p} \in P\} \\ &= r\, \mathrm{h}_P(u). \end{aligned}$$

Therefore,

$$\begin{aligned} F^P_{ru} &= \{p : ru(\mathbf{p}) = \mathrm{h}_P(ru), \mathbf{p} \in P\} \\ &= \{p : u(\mathbf{p}) = \mathrm{h}_P(u), \mathbf{p} \in P\} \\ &= F^P_u. \end{aligned}$$

□

Lemma 3.3.3. *Let P be a lattice polytope. If there exists some $u \in (\mathbb{R}^n)^*$ such that $F_u^P = F_{-u}^P$ then $P = F_u^P$.*

Proof. Let $\mathbf{q} \in F_u^P = F_{-u}^P$. We have $u(\mathbf{q}) = \mathrm{h}_P(u)$ and $-u(\mathbf{q}) = \mathrm{h}_P(-u)$, hence $\mathrm{h}_P(u) = -\mathrm{h}_P(-u)$. Note that

$$\begin{aligned}\mathrm{h}_P(-u) &= \sup\{-u(\mathbf{p}) : \mathbf{p} \in P\} \\ &= -\inf\{u(\mathbf{p}) : \mathbf{p} \in P\}\end{aligned}$$

$$\begin{aligned}\mathrm{h}_P(-u) &= -\mathrm{h}_P(u) \\ &= -\sup\{u(\mathbf{p}) : \mathbf{p} \in P\}.\end{aligned}$$

This implies $\sup\{u(\mathbf{p}) : \mathbf{p} \in P\} = \inf\{u(\mathbf{p}) : \mathbf{p} \in P\}$ and so $u(\mathbf{p}) = \mathrm{h}_P(u)$ for all $\mathbf{p} \in P$. Therefore $P \subseteq F_u^P$ and hence $P = F_u^P$. □

Lemma 3.3.4. *Let $P \subseteq \mathbb{R}^n$ be a 1-polytope. Suppose that $\operatorname{aff} P = \mathbf{p} + W$ for some one dimensional vector space $W \subseteq \mathbb{R}^n$ and $\mathbf{p} \in \operatorname{Verts}(P)$. Further suppose there is a two dimensional vector space $U \subseteq \mathbb{R}^n$ such that $W \subset U$. If $P = F_u^P = F_{u'}^P$ for some $u, u' \in U^* \backslash \{0\}$ then $u' = ru$ for some $r \in \mathbb{R}\backslash\{0\}$.*

Proof. By the proof of Proposition 3.1.5 we can write $P = \operatorname{conv}(\{\mathbf{p}, \mathbf{p} + \mathbf{d}\})$ for some $\mathbf{d} \in \mathbb{R}^n \backslash \{0\}$. Note $\mathbf{q} \in P$ implies $\mathbf{q} = \mathbf{p} + \lambda \mathbf{d}$ for some $\lambda \in [0, 1]$. Therefore if $P = F_u^P = F_{u'}^P$ we have the vectors associated to u and u' are both perpendicular to $\mathbf{d}$ in the 2 dimensional space W. That is, $u(\mathbf{d}) = 0, u'(\mathbf{d}) = 0$. Consequently u, u' are parallel in the dual space and so $u' = ru$ for some $r \in \mathbb{R}\backslash\{0\}$. □

Proposition 3.3.5. *Let $P \subseteq \mathbb{R}^n$ be a 2-polytope. There exists a unique cycle on P.*

Proof. Let $W \subseteq \mathbb{R}^n$ be the two dimensional vector space such that $\operatorname{aff} P = \mathbf{a} + W$ for some $\mathbf{a} \in P$. Choose an orthonormal basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ of W. Extend $\{\mathbf{b}_1, \mathbf{b}_2\}$ to an orthonormal basis $\{\mathbf{b}_i\}_{i=1}^n$ of $\mathbb{R}^n$. Denote the corresponding dual basis of $\mathbb{R}^n$ by $\{b_i^*\}_{i=1}^n$. We can consider W^* as a two dimensional subspace of $(\mathbb{R}^n)^*$ by setting the value of all linear functionals in W^* on the basis elements $\{\mathbf{b}_i\}_{i=3}^n$ to be zero.

First note that for any $u \in (\mathbb{R}^n)^* \backslash W^*$ and $\mathbf{p} \in P$, we have $u(\mathbf{p}) = 0$. Similarly, in the case that $u = 0 \in (\mathbb{R}^n)^*$ is the zero functional, we have $u(\mathbf{p}) = 0$ for any $\mathbf{p} \in P$. Hence $F_u^P = P$ for all $u \in ((\mathbb{R}^n)^* \backslash W^*) \cup \{0\}$.

Next consider the case $u \in W^* \backslash \{0\}$. Parameterise the dual basis of W using polar coordinates so that each $u \in W^* \backslash \{0\}$ can be uniquely expressed as $u = r\cos(\theta) b_1^* + r \sin(\theta) b_2^*$, for some $r > 0$ and $\theta \in [0, 2\pi)$. We will denote $u_{r,\theta} := r\cos(\theta) b_1^* + r\sin(\theta) b_2^*$, for every $r > 0$ and $\theta \in [0, 2\pi)$.

By Lemma 3.3.2 we have $F_{u_{r,\theta}}^P = F_{u_{1,\theta}}^P$. Thus we only consider the elements $u_\theta := u_{1,\theta}$, for $\theta \in [0, 2\pi)$, and write $F_\theta := F_{u_\theta}^P$. As a consequence of the above discussion, every edge and vertex of P is of the form F_θ for some $\theta \in [0, 2\pi)$. The polytope P has a finite number of edges, so there exists some $\{\theta_i\}_{i=1}^N \subset [0, 2\pi)$, such that $\mathrm{Edges}(P) = \{F_{\theta_i}\}_{i=1}^N$.

Suppose $F = F_{\theta_i} = F_{\theta_i'}$ for some $i \in \{1, ..., N\}$ and $\theta_i' \in [0, 2\pi)$. Since F is an edge, this implies that $\theta_i = \pm\theta_i'$ by Lemma 3.3.4. But if $\theta_i' = -\theta_i$ then $P = F$, by Lemma 3.3.3. This is a contradiction since $\dim F = 1$ and $\dim P = 2$. Therefore $\theta_i = \theta_i'$.

Since P is two dimensional F_θ is either zero or two dimensional for any $\theta \in [0, 2\pi)\backslash\{\theta_i\}_{i=1}^N$. If F_θ were two dimensional, then there would exist non-co-linear $\mathbf{p}, \mathbf{p}', \mathbf{p}'' \in P$ such that $u_\theta(\mathbf{p}) = u_\theta(\mathbf{p}') = u_\theta(\mathbf{p}'')$. Hence $u(\mathbf{p} - \mathbf{q}') = 0$ and $u(\mathbf{p} - \mathbf{q}'') = 0$. This implies the vector in $W \subseteq \mathbb{R}^n$ corresponding to $u \in W^*\backslash\{0\}$ is perpendicular to the two non-parallel vectors $\mathbf{p} - \mathbf{q}' \in W\backslash\{0\}$ and $\mathbf{p} - \mathbf{q}'' \in W\backslash\{0\}$. But W is only two dimensional, a contradiction. We conclude that $\dim F_\theta = 0$ for all $\theta \in [0, 2\pi)\backslash\{\theta_i\}_{i=1}^N$. Moreover, we've shown every vertex of P can be written as F_θ for some $\theta \in [0, 2\pi)\backslash\{\theta_i\}_{i=1}^N$. Without loss of generality, by rotating the orthonormal basis of W^* as appropriate, we may assume that F_0 is an edge of P.

To unobscure the construction of the cycle on F: some details in this paragraph are omitted. There cannot exist three angles $\theta, \theta', \theta''$ such that $0 < \theta < \theta' < \theta'' < 2\pi$ and $\theta, \theta', \theta'' \notin \{\theta_i\}_{i=1}^N$ and $F_\theta = F_{\theta''} \neq F_{\theta'}$. Denote by $I_i := (\theta_i, \theta_{i+1})$ for each $i \in \{1, ..., N-1\}$. Then, for each i, we must have $F_\theta = F_{\theta'}$ for any $\theta, \theta' \in I_i$ and whenever $i \neq j$ we have $F_\theta \neq F_{\theta'}$ for $\theta \in I_i$ and $\theta' \in I_j$. Moreover F_θ is a vertex for any $\theta \in \cup_{i=1}^{N-1} I_i$. Label the vertices of P as follows: for each i pick any $\psi_i \in I_i$ and define $\mathbf{v}_i := F_{\psi_i}$. By construction, we have $\mathrm{Verts}(F_{\theta_1}) = \{\mathbf{v}_{N-1}, \mathbf{v}_1\}$ and $\mathrm{Verts}(F_{\theta_i}) = \{\mathbf{v}_{i-1}, \mathbf{v}_i\}$ for $i \in \{2, ..., N-1\}$. Thus $\mathcal{C} = \mathbf{v}_1...\mathbf{v}_{N-1}\mathbf{v}_1$ is a cycle on F.

Since every vertex appears in this cycle and every edge has only two vertices, there can only be one such cycle.

□

Definition 3.3.6. [1, p. 445, Extended to non-primitive polytopes]. Let P be a lattice polytope with $\dim P \geq 2$. Fix N primitive edge vectors $\{\mathbf{d}_i\}_{i=1}^N$ for each of the edges $\mathrm{Edges}(P) = \{E_i\}_{i=1}^N$. Let $c_i = \rho(P)_i \in \mathbb{Z}_{>0}$ be the length of the edge E_i. For every 2-face $F < P$, let $\mathcal{C}$ be the unique cycle on F. Further, choose an orientation of this cycle by writing $\mathcal{C} = \mathbf{v}_0...\mathbf{v}_{k-1}\mathbf{v}_0$, for some labelling of the vertices $\mathrm{Verts}(F) = \{\mathbf{v}_i\}_{i=0}^{k-1}$. We define the components of the *sign vector* $\boldsymbol{\epsilon} = (\epsilon_1, ..., \epsilon_N) \in \{0, \pm1\}^N$ associated to F as

$$\epsilon_i := \begin{cases} +1, & \text{if } E_i = \mathrm{conv}(\{\mathbf{v}_j, \mathbf{v}_{j+1}\}) \text{ and } \mathbf{v}_{j+1} = \mathbf{v}_j + c_i\mathbf{d}_i \text{ for some } j, \\ -1, & \text{if } E_i = \mathrm{conv}(\{\mathbf{v}_j, \mathbf{v}_{j+1}\}) \text{ and } \mathbf{v}_{j+1} = \mathbf{v}_j - c_i\mathbf{d}_i \text{ for some } j, \\ 0, & \text{otherwise,} \end{cases}$$

for each $i \in \{1, ..., N\}$.

For a lattice polytope P of dimension larger than one and a 2-face $F < P$, the sign vector $\boldsymbol{\epsilon}$ of F "corrects" the sign of the chosen primitive edge vectors $\{\mathbf{d}_i\}_{i=1}^N$. That is, the subset of primitive edge vectors which correspond to the edges of F form a closed cycle around the boundary of F, so that

$$\sum_{i=1}^{N} \epsilon_i \rho(P)_i \mathbf{d}_i = \mathbf{0}.$$

Given a choice of primitive edge vectors for P, the sign vector of a given 2-face is unique up to a sign; the sign depends on the orientation of the 2-face cycle.

Example 3.3.7. *Let $P \subset \mathbb{R}^3$ be the rectangular cuboid of unit height, unit width and depth two. That is, let P be the convex hull*

$$P = \operatorname{conv}(\{(1,2,1),(1,2,0),(0,2,0),(0,2,1),(0,0,1),(0,0,0),(1,0,0),(1,0,1)\}).$$

Label the vertices as

$$\begin{aligned} &\mathbf{v}_1 = (1,2,1), \quad \mathbf{v}_2 = (1,2,0), \quad \mathbf{v}_3 = (0,2,0), \quad \mathbf{v}_4 = (0,2,1), \\ &\mathbf{v}_5 = (0,0,1), \quad \mathbf{v}_6 = (0,0,0), \quad \mathbf{v}_7 = (1,0,0), \quad \mathbf{v}_8 = (1,0,1). \end{aligned}$$

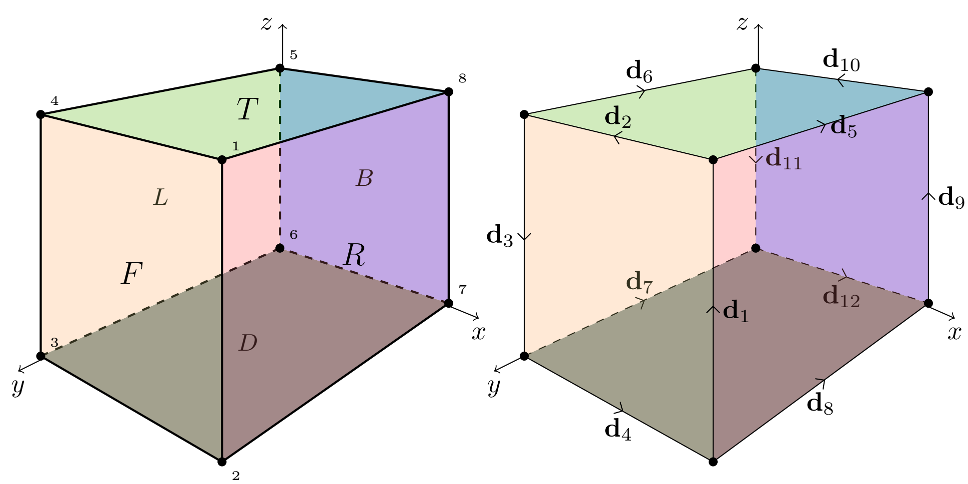

Figure 3.3.8: Annotating the vertices, primitive edge vectors and 2-faces of the rectangular cuboid P. [Adapted from: http://www.texample.net/tikz/examples/cuboid/]

The rectangular cuboid P has the six faces: Front F, Back B, Left L, Right R, Top T, Down D. The chosen primitive edge vectors depicted in Figure 3.3.8 are

$$\begin{aligned} &\mathbf{d}_1 = (0,0,1), \quad &&\mathbf{d}_2 = (-1,0,0), \quad &&\mathbf{d}_3 = (0,0,-1), \quad &&\mathbf{d}_4 = (1,0,0), \\ &\mathbf{d}_5 = (0,-1,0), \quad &&\mathbf{d}_6 = (0,-1,0), \quad &&\mathbf{d}_7 = (0,-1,0), \quad &&\mathbf{d}_8 = (0,-1,0), \\ &\mathbf{d}_9 = (0,0,1), \quad &&\mathbf{d}_{10} = (-1,0,0), \quad &&\mathbf{d}_{11} = (0,0,-1), \quad &&\mathbf{d}_{12} = (1,0,0). \end{aligned}$$

Given these primitive edge vectors, we pick the below sign vectors for the six faces of P

$$\begin{aligned}
\boldsymbol{\epsilon}^F &= (1,1,1,1,0,0,0,0,0,0,0,0), & \boldsymbol{\epsilon}^R &= (1,0,0,0,1,0,0,-1,-1,0,0,0),\\
\boldsymbol{\epsilon}^T &= (0,1,0,0,-1,1,0,0,0,-1,0,0), & \boldsymbol{\epsilon}^L &= (0,0,1,0,0,-1,1,0,0,0,-1,0),\\
\boldsymbol{\epsilon}^D &= (0,0,0,1,0,0,-1,1,0,0,0,-1), & \boldsymbol{\epsilon}^B &= (0,0,0,0,0,0,0,0,1,1,1,1).
\end{aligned}$$

The rectangular cuboid P can be decomposed into a unit square Q, in the xz-plane, and a straight line Q', of length two along the positive y-axis with a vertex at the origin. That is, if we let

$$\begin{aligned}
Q &= \operatorname{conv}(\{(1,0,1),(1,0,0),(0,0,0),(0,0,1)\}),\\
Q' &= \operatorname{conv}(\{(0,0,0),(0,2,0)\}),
\end{aligned}$$

then we have the Minkowski decomposition $P = Q + Q'$ pictured below in Figure 3.3.9.

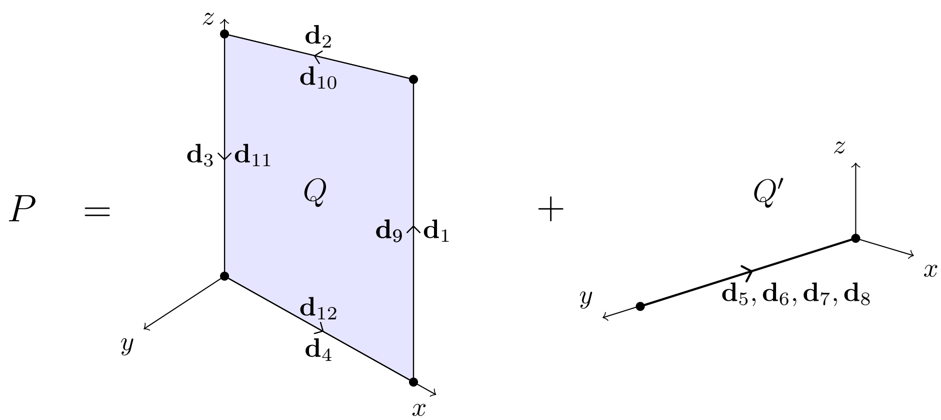

Figure 3.3.9: Minkowski decomposition of the rectangular cuboid P into the sum of a square Q and line Q'.

Here, we have

$$\begin{aligned}
\rho(P) &= (1,1,1,1,2,2,2,2,1,1,1,1),\\
\rho(Q) &= (1,1,1,1,0,0,0,0,1,1,1,1),\\
\rho(Q') &= (0,0,0,0,2,2,2,2,0,0,0,0).
\end{aligned}$$

Recall that we chose the two sign vectors

$$\begin{aligned}
\boldsymbol{\epsilon}^F &= (1,1,1,1,0,0,0,0,0,0,0,0),\\
\boldsymbol{\epsilon}^R &= (1,0,0,0,1,0,0,-1,-1,0,0,0).
\end{aligned}$$

Hence we have the following computations

$$\sum_i \epsilon_i^F \rho(Q)_i \mathbf{d}_i = \mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3 + \mathbf{d}_4 = \mathbf{0},$$
$$\sum_i \epsilon_i^F \rho(Q')_i \mathbf{d}_i = \mathbf{0},$$
$$\sum_i \epsilon_i^R \rho(Q)_i \mathbf{d}_i = \mathbf{d}_1 - \mathbf{d}_9 = \mathbf{0},$$
$$\sum_i \epsilon_i^R \rho(Q')_i \mathbf{d}_i = 2\mathbf{d}_5 - 2\mathbf{d}_9 = \mathbf{0}.$$

It can be checked that $\sum_i \epsilon_i^{F'} \rho(Q)_i \mathbf{d}_i = \mathbf{0}$ *and* $\sum_i \epsilon_i^{F'} \rho(Q')_i \mathbf{d}_i = \mathbf{0}$ *for any 2-face* F' *of the rectangular cuboid* P*.*

In fact, the observation in the Example 3.3.7 holds in general. That is, for any lattice polytope P and Minkowski summand $Q \in \mathrm{MS}_{k>0}(kP)$, we will see that (Proposition 3.3.10)

$$\sum_i \epsilon_i^F \rho(Q)_i \mathbf{d}_i = \mathbf{0}. \text{ (\textit{Closure condition})} \tag{2}$$

The closure condition simply reflects that the edges of a 2-face (of any polytope) induce a cycle and if $\{\mathbf{d}_i\}_{i\in I_F}$ are the primitive edge vectors of a 2-face $F = F_u^P$, with indexing set I_F, then a subset of $\{\mathbf{d}_i\}_{i\in I_F}$ forms the primitive edge vectors of the corresponding $0/1/2$-face F_u^Q of Q.

Therefore, given primitive edge vectors and sign vectors for a lattice polytope P, we can restrict the the range of the map $\rho : \mathrm{MS}_{k>0}(kP) \to \mathbb{R}^N_{\geq 0}$ to the set

$$C(P) := \{\mathbf{t} \in \mathbb{R}^N_{\geq 0} : \sum_i \epsilon_i^F t_i \mathbf{d}_i = \mathbf{0} \text{ for every } F \in 2\text{-Faces}(P)\}. \tag{3}$$

Proposition 3.3.10. *Let* P *be a* d*-lattice polytope with* $\dim P \geq 2$ *and edges* $\mathrm{Edges}(P) = \{E_i\}_{i=1}^n$*. Pick primitive edge vectors* $\{\mathbf{d}_i\}_{i=1}^N$ *for each edge* E_i *and pick sign vectors* $\boldsymbol{\epsilon}^F$ *for each 2-face* $F \in 2\text{-Faces}(P)$*. We have* $\mathrm{im}(\rho) \subseteq C(P)$*.*

Proof. Let F be a 2-face of P, so that $F = F_u^P$ for some $u \in (\mathbb{R}^n)^*$. Let $Q \in \mathrm{MS}(kP)$ for some $k > 0$ and let $R \in \mathrm{MS}(kP)$ be such that $kP = Q + R$. By Lemma 3.1.7 and Proposition 3.1.1, we have $kF_u^P = F_u^{kP} = F_u^Q + F_u^R$.

In the proof of Proposition 2.3.18, we showed that if $k \operatorname{aff} F_u^P = \operatorname{aff} kF_u^P = k\mathbf{p} + W$ and $\operatorname{aff} F_u^Q = \mathbf{q} + W'$ for some $\mathbf{p} \in P, \mathbf{q} \in Q$ and vector spaces $W, W' \subseteq \mathbb{R}^n$, then $W' \subseteq W$. Let $u' \in (W')^* \backslash \{0\}$ be such that $F_{u'}^P$ is an edge of F_u^P. Denote the associated vector to the functional $u' \in (W')^*$ by $\mathbf{u}' \in \mathbb{R}^n \backslash \{0\}$. Write $b_1^* := u/(|\mathbf{u}'|)$ and extend to an orthonormal basis $\{b_i^*\}_{i=1}^d$ of W^*.

Parameterise the 2-dimensional vector space $(W')^*$ by polar coordinates. Consider the sequence of vertices of F_u^P and F_u^Q found by rotating anticlockwise through the angles $[0, 2\pi)$ as in the proof of Proposition 3.3.5. This process generates cycles on F_u^P and F_u^Q such that the edges of the induced walks by the cycles have the property that the directions of the primitive edge vectors used are consistent. Hence, by the closure condition for the summand Q, we have $\rho(Q) \in \mathrm{im}(\rho)$.

$\square$

4 Recovering Minkowski Summands from $\operatorname{im}\rho$

4.1 Construction of the inverse map σ

The set $C(P)$, defined earlier in Equation 3 (page 27), was introduced by Altmann in [1, p. 446] for *primitive lattice polytopes* P (lattice polytopes with primitive edges). Altmann additionally provides a map $\sigma : C(P) \to \{\text{polytopes in } V\}$, defined for a given primitive lattice polytope P [1, p. 446].

With the notation already introduced, the map σ can be naturally extended to polytopes P which are not necessarily primitive. We will see that σ allows us to recover the Minkowski summands of P from the image of ρ (Proposition 4.2.3). Imprecisely, we'll find that $\sigma(\rho(Q)) = Q$ for every $Q \in \mathrm{MS}_{k>0}(kP)$, and in particular for lattice polytopes $Q \in \mathrm{MS}(P)$.

Let P be any polytope. To define σ we first introduce *translation classes of polytopes.* Notice that whenever $Q \in \mathrm{MS}(P)$ we have $Q + \{\mathbf{a}\} \in \mathrm{MS}(P)$ for all $\mathbf{a} \in \mathbb{R}^n$. Thus, we define an equivalence class on the set of all polytopes by saying two polytopes Q, Q' are equivalent if and only if $Q' = Q + \{\mathbf{a}\}$ for some $\mathbf{a} \in \mathbb{R}^n$. Denote by $[\mathrm{MS}(P)]$ the set of all equivalence classes of Minkowski summands of P. Similarly, let $[\mathrm{MS}_{k>0}(kP)]$ denote the set of equivalence classes of Minkowski summands of positive multiples of P.

Note that the operations $[Q] + [R] := [Q + R]$ and $k[P] := [kP]$, for $k \in \mathbb{R}$, are well-defined operations.

Proposition 4.1.1. *Let P be a lattice polytope with edges $\mathrm{Edges}(P) = \{E_i\}_{i=1}^N$. For each $i \in \{1, ..., N\}$, pick a primitive edge vector $\mathbf{d}_i \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ for the edge E_i. Then the map defined from Definition 3.2.1, page 17:*

$$\rho : \mathrm{MS}_{k>0}(kP) \to \mathbb{R}^N_{\geq 0}, \tag{4}$$

can be considered as a map

$$\rho : [\mathrm{MS}_{k>0}(kP)] \to C(P) \tag{5}$$

by defining $\rho([Q]) = \rho(Q)$ for all $[Q] \in [\mathrm{MS}_{k>0}(kP)]$, where $Q \in \mathrm{MS}_{k>0}(kP)$ is a representative element of $[Q]$.

Proof. We already saw in Proposition 3.3.10 that we can restrict the range of

$$\rho : \mathrm{MS}_{k>0}(kP) \to \mathbb{R}^N_{\geq 0},$$

to $C(P)$.

For each $i \in \{1, ..., N\}$, pick a $u_i \in (\mathbb{R}^n)^*$ such that $E_i = F^P_{u_i}$. We must check that the map

$$\rho : [\mathrm{MS}_{k>0}(kP)] \to C(P)$$

is well-defined.

Let $i \in \{1, ..., N\}$ and $[Q] \in \mathrm{MS}_{k>0}(kP)$. Suppose that $[Q] = [Q']$, for some representative Minkowski summands $Q, Q' \in \mathrm{MS}_{k>0}(kP)$. There exists some $\mathbf{a} \in \mathbb{R}^n$ such that $Q' = Q + \{\mathbf{a}\}$. By Proposition 3.1.1 we have

$$\begin{aligned} F_{u_i}^{Q'} &= F_{u_i}^{Q+\{\mathbf{a}\}} \\ &= F_{u_i}^{Q} + F_{u_i}^{\{\mathbf{a}\}} \\ &= F_{u_i}^{P} + \{\mathbf{a}\} \end{aligned}$$

By Corollary 3.3.1 we can uniquely write $F_{u_i}^{Q} = \mathrm{conv}(\{\mathbf{q}, \mathbf{q} + t_i\mathbf{d}_i\})$ and $F_{u_i}^{Q'} = \mathrm{conv}(\{\mathbf{q}', \mathbf{q}' + t_i'\mathbf{d}_i\})$ for some $t_i, t_i' \geq 0$ and $\mathbf{q} \in \mathrm{Verts}(Q), \mathbf{q}' \in \mathrm{Verts}(Q')$. Hence, by uniqueness of the above expressions and $F_{u_i}^{Q'} = F_{u_i}^{P} + \{\mathbf{a}\}$, we conclude that $t_i = t_i'$. That is $\rho(Q)_i = \rho(Q')_i$. Thus ρ, as defined in the statement of the proposition, is well-defined. □

Of course, *all* of the results proven about the map ρ with domain $\mathrm{MS}_{k>0}(kP)$ proven in the previous section can be translated into results about ρ with domain $[\mathrm{MS}_{k>0}(kP)]$. The details of which are omitted. We continue to provide terminology to define the map σ.

Definition 4.1.2. Let P be a lattice polytope and suppose $\mathrm{Edges}(P) = \{E_i\}_{i=1}^N$. Fix a choice of primitive edge vectors $\{\mathbf{d}_i\}_{i=1}^N$ for each of the edges of P. Let $\mathcal{W}$ be a walk on P and write $\mathcal{W} = \mathbf{w}_0...\mathbf{w}_m$, for some vertices $\{\mathbf{w}_i\}_{i=0}^m \subseteq \mathrm{Verts}(P)$. The *counter* of the walk $\mathcal{W}$ is the induced vector $\boldsymbol{\lambda} \in \mathbb{Z}^N$, whose components are given by

$$\lambda_i := |\Lambda_i^+| - |\Lambda_i^-|, \tag{6}$$

where

$$\begin{aligned} \Lambda_i^+ &:= \{j \in \{0, ..., m-1\} : \mathbf{w}_{j+1} = \mathbf{w}_j + \rho(P)_i\mathbf{d}_i, E_i = \mathrm{conv}(\{\mathbf{w}_j, \mathbf{w}_{j+1}\})\}, \\ \Lambda_i^- &:= \{j \in \{0, ..., m-1\} : \mathbf{w}_{j+1} = \mathbf{w}_j - \rho(P)_i\mathbf{d}_i, E_i = \mathrm{conv}(\{\mathbf{w}_j, \mathbf{w}_{j+1}\})\}, \end{aligned}$$

for each $i \in \{1, ..., N\}$.

The counter $\boldsymbol{\lambda}$ of a walk $\mathcal{W}$ on a lattice polytope P counts (taking account of directions) how many times each edge of P is crossed by $\mathcal{W}$.

Proposition 4.1.3. *Let P be a lattice polytope and suppose* $\mathrm{Edges}(P) = \{E_i\}_{i=1}^N$. *Fix a choice of primitive edge vectors $\{\mathbf{d}_i\}_{i=1}^N$ for each of the edges of P. Let $\mathcal{W}$ be a walk on P and write $\mathcal{W} = \mathbf{w}_0...\mathbf{w}_m$, for some vertices* $\{\mathbf{w}_i\}_{i=0}^m \subseteq \mathrm{Verts}(P)$. *Then*

$$\mathbf{w}_m = \mathbf{w}_0 + \sum_{i=0}^{N} \lambda_i \rho(P)_i \mathbf{d}_i.$$

Proof. We can always write

$$\mathbf{w}_m = \mathbf{w}_0 + \sum_{i=0}^{m-1} (\mathbf{w}_{i+1} - \mathbf{w}_i).$$

Hence, by definition of the counter $\boldsymbol{\lambda}$, and since $\mathcal{W}$ is a walk from $\mathbf{w}_0$ to $\mathbf{w}_m$, we have

$$\mathbf{w}_m = \mathbf{w}_0 + \sum_{i=0}^{N} \lambda_i \rho(P)_i \mathbf{d}_i.$$

The above is proven inductively on the length m of $\mathcal{W}$. $\square$

We're now ready to define the map σ at the beginning of this section.

Definition 4.1.4. [1, p. 446, Extended to non-primitive polytopes]. Let P be a lattice polytope with $\dim P \geq 2$ and suppose $\mathrm{Edges}(P) = \{E_i\}_{i=1}^{N}$. The map

$$\sigma : C(P) \to [\{\text{polytopes in } V\}] \tag{7}$$

is defined as follows. Fix a choice of primitive edge vectors $\{\mathbf{d}_i\}_{i=1}^{N}$ for each of the edges of P and fix a vertex $\mathbf{w} \in \mathrm{Verts}(P)$ of P. The vertex $\mathbf{w}$ is called the *reference vertex*. For each $\mathbf{v} \in \mathrm{Verts}(P)$ pick a walk $\mathcal{W}$ from the reference vertex $\mathbf{w}$ to $\mathbf{v}$. Write $\mathcal{W} = \mathbf{w}_0 ... \mathbf{w}_m$, with $\mathbf{w}_0 = \mathbf{w}, \mathbf{w}_m = \mathbf{v}$ and $\mathbf{w}_i \in \mathrm{Verts}(P)$ for each $i \in \{0, ..., m\}$. Let $\mathbf{t} \in C(P)$ and set

$$\sigma_{\mathbf{v}}(\mathbf{t}) := \sum_{i=1}^{N} \lambda_i t_i \mathbf{d}_i. \tag{8}$$

Define the translation class of polytopes $\sigma(\mathbf{t})$ as

$$\sigma(\mathbf{t}) := [\mathrm{conv}(\{\sigma_{\mathbf{v}}(\mathbf{t})\}_{\mathbf{v} \in \mathrm{Verts}(P)})] \in [\{\text{polytopes in } V\}]. \tag{9}$$

Recall that, given any two vertices $\mathbf{w}, \mathbf{v} \in \mathrm{Verts}(P)$ of a polytope P, there exists a path linking $\mathbf{w}$ and $\mathbf{v}$ by Theorem 2.3.25.

Example 4.1.5. *Let P be the triangle given by the convex hull*

$$P = \mathrm{conv}(\{(0,0), (2,0), (1,2)\}).$$

We pick the primitive edge vectors $\mathbf{d}_1 = (1,0), \mathbf{d}_2 = (1,2), \mathbf{d}_3 = (-1,-2)$. Label the vertices as $\mathbf{v}_1 = (0,0), \mathbf{v}_2 = (2,0), \mathbf{v}_3 = (1,2)$ and choose $\mathbf{v}_1$ as our reference vertex. Consider the anticlockwise and clockwise walks from $\mathbf{v}_1$ to $\mathbf{v}_3$, given by $\mathcal{W}_{13} = \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ and $\mathcal{W}'_{13} = \mathbf{v}_1\mathbf{v}_3$ respectively.

Suppose that we extend the domain of σ in Definition 4.1.4 from $C(P) \subseteq \mathbb{R}^3_{\geq 0}$ to the whole of $\mathbb{R}^3_{\geq 0}$. We will show this extension of σ is not well-defined.

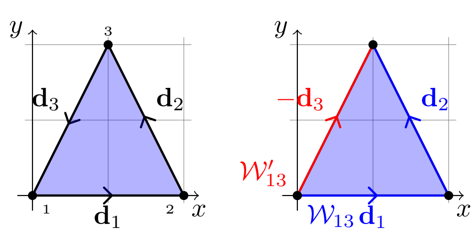

Figure 4.1.6: Labelling of the primitive edge vectors of the triangle P and two different walks $\mathcal{W}_{13}, \mathcal{W}'_{13}$ from $(0,0)$ to $(1,2)$.

The counter for the walk $\mathcal{W}_{13}$ is $\boldsymbol{\lambda}^{\mathcal{W}_{13}} = (1,1,0)$ and the counter for the walk $\mathcal{W}'_{13}$ is $\boldsymbol{\lambda}^{\mathcal{W}'_{13}} = (0,0,-1)$.

Similarly let $\mathcal{W}_{12}, \mathcal{W}'_{12}$ be the anticlockwise and clockwise walks from $\mathbf{v}_1$ to $\mathbf{v}_2$, and let $\mathcal{W}_{11} = \mathcal{W}'_{11} = \mathbf{v}_1$ be the trivial walk from $\mathbf{v}_1$ to $\mathbf{v}_1$. The corresponding counters are

$$\boldsymbol{\lambda}^{\mathcal{W}_{12}} = (1,0,0), \quad \boldsymbol{\lambda}^{\mathcal{W}'_{12}} = (0,-1,-1),$$
$$\boldsymbol{\lambda}^{\mathcal{W}_{11}} = (0,0,0), \quad \boldsymbol{\lambda}^{\mathcal{W}'_{11}} = (0,0,0).$$

Let $\mathbf{t} := \rho(P) = (2,1,1) \in C(P)$ and $\mathbf{t}' := (0,0,5) \notin C(P)$. We compute

	$\sigma_{\mathbf{v}_1}(\mathbf{t})$	$\sigma_{\mathbf{v}_2}(\mathbf{t})$	$\sigma_{\mathbf{v}_3}(\mathbf{t})$
$\mathcal{W}$	(0,0)	(2,0)	(1,2)
$\mathcal{W}'$	(0,0)	(2,0)	(1,2)

	$\sigma_{\mathbf{v}_1}(\mathbf{t}')$	$\sigma_{\mathbf{v}_2}(\mathbf{t}')$	$\sigma_{\mathbf{v}_3}(\mathbf{t}')$
$\mathcal{W}$	(0,0)	(0,0)	(0,0)
$\mathcal{W}'$	(0,0)	(-5,-10)	(-5,-10)

Table IV: Two tables showing the values of $\sigma_{\mathbf{v}}(\mathbf{t})$ and $\sigma_{\mathbf{v}}(\mathbf{t}')$ for each vertex $\mathbf{v} \in \mathrm{Verts}(P)$ of the triangle P, as computed using the anticlockwise ($\mathcal{W}$) and clockwise walks ($\mathcal{W}'$).

Denote the map σ, as determined by using the anticlockwise ($\mathcal{W}$) and clockwise ($\mathcal{W}'$) walks, by $\sigma^{\mathcal{W}}$ and $\sigma^{\mathcal{W}'}$ respectively. From Table IV, for $\mathbf{t}$ we have

$$\begin{aligned}
\sigma^{\mathcal{W}}(\mathbf{t}) &= [\mathrm{conv}(\{(0,0),(2,0),(1,2)\})] \\
&= [P], \\
\sigma^{\mathcal{W}'}(\mathbf{t}) &= [\mathrm{conv}(\{(0,0),(2,0),(1,2)\})] \\
&= [P],
\end{aligned}$$

and for $\mathbf{t}'$ we find

$$\begin{aligned}
\sigma^{\mathcal{W}}(\mathbf{t}') &= [\mathrm{conv}(\{(0,0),(0,0),(0,0)\})] \\
&= [\{(0,0)\}], \\
\sigma^{\mathcal{W}'}(\mathbf{t}') &= [\mathrm{conv}(\{(0,0),(-5,-10),(-5,-10)\})] \\
&= [\mathrm{conv}(\{(0,0),(-5,-10)\})].
\end{aligned}$$

Note that $\sigma^{\mathcal{W}}(\mathbf{t}) = \sigma^{\mathcal{W}'}(\mathbf{t})$, but $\sigma^{\mathcal{W}}(\mathbf{t}') \neq \sigma^{\mathcal{W}'}(\mathbf{t}')$. Hence in general, the domain of σ, for a polytope P with N edges, cannot be extended to the whole of $\mathbb{R}^N_{\geq 0}$.

Proposition 4.1.7. *Let P be a lattice polytope with $\dim P \geq 2$. Fix primitive edge vectors and sign vectors for P. Further let $\sigma : C(P) \to [\{\text{lattice polytopes}\}]$ be as defined in Definition 4.1.4. Then the map σ is well-defined.*

Before a proof of the well-definedness of σ is presented, the notions of winding numbers, entries, exits and branches must be introduced. As well as several lemmas proven. However, the margin is too small to contain these things and the deadline too soon. Sorry! Despite this, proving that σ is independent of the walk taken from a reference vertex isn't necessary to find all lattice Minkowski summands of a lattice polytope - as remarked in Remark 5.1.5.

We move onto our next goal of proving that σ allows us to recover the Minkowski summands of positive multiples of P.

4.2 Proving that σ is the inverse of ρ

Lemma 4.2.1. *Let P be a polytope, F_u^P a face of P and $Q \in \mathrm{MS}(kP)$ for some $k > 0$. If $F_{u'}^P \subseteq F_u^P$ then $F_{u'}^Q \subseteq F_u^Q$.*

Proof. Suppose that $F_{u'}^P \subseteq F_u^P$. That is, for all $\mathbf{p} \in P$ such that $u'(\mathbf{p}) = \mathrm{h}_P(u')$ we have $u(\mathbf{p}) = \mathrm{h}_P(u)$. There exists some polytope R such that $kF_u^P = F_u^Q + F_u^R$ and $kF_{u'}^P = F_{u'}^Q + F_{u'}^R$ by Proposition 3.1.1.

Let $\mathbf{q} \in F_{u'}^Q$ and $\mathbf{r} \in F_{u'}^R$ so that $u'(\mathbf{q}) = \mathrm{h}_Q(u')$ and $u'(\mathbf{r}) = \mathrm{h}_R(u')$. Then $\mathbf{q}+\mathbf{r} \in kF_{u'}^P$. Hence $\mathbf{q}+\mathbf{r} \in kF_u^P$ and so $u(\mathbf{q}) + u(\mathbf{r}) = \mathrm{h}_P(u) = \mathrm{h}_Q(u) + \mathrm{h}_R(u)$. Consequently $u(\mathbf{q}) = \mathrm{h}_Q(u)$. So $F_{u'}^Q \subseteq F_u^Q$. □

Lemma 4.2.2. *Let P be a polytope and $Q \in \mathrm{MS}(kP)$ for some $k > 0$. If $\mathbf{q} \in \mathrm{Verts}(Q)$ then there exists some $u \in (\mathbb{R}^n)^*$ such that $F_u^Q = \{\mathbf{q}\}$ and $F_u^P \in \mathrm{Verts}(P)$.*

Proof. There exists some $u \in (\mathbb{R}^n)^*$ such that $F_u^Q = \{\mathbf{q}\}$. Consider the face $F := F_u^P$. Let $\mathbf{p} \in \mathrm{Verts}(F)$. Then $\mathbf{p} \in \mathrm{Verts}(P)$, so there exists some $u' \in (\mathbb{R}^n)^*$ such that $F_{u'}^P = \{\mathbf{p}\}$. Now Lemma 4.2.1 states $F_{u'}^P \subseteq F_u^P$ implies $F_{u'}^Q \subseteq F_u^Q = \{\mathbf{q}\}$. Consequently $F_{u'}^Q = \{\mathbf{q}\}$ and $F_{u'}^P = \{\mathbf{p}\} \in \mathrm{Verts}(P)$ as required. □

The map σ allows us to recover the Minkowski summands of kP from the image of ρ.

Proposition 4.2.3. *Let P be a lattice polytope with $\dim P \geq 2$. Fix primitive edge vectors and sign vectors for P. If $Q \in \mathrm{MS}_{k>0}(kP)$ is a Minkowski summand then $\sigma(\rho([Q])) = [Q]$.*

Proof. Write $\mathrm{Edges}(P) = \{E_i\}_{i=1}^N$. For each $i \in \{1, ..., N\}$ there exists some $u_i \in (\mathbb{R}^n)^* \backslash \{0\}$ such that $E_i = F_{u_i}^P$ and an associated primitive edge vector $\mathbf{d}_i \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$. By the definitions we have

$$E_i = \mathrm{conv}(\{\mathbf{p}_i, \mathbf{p}_i + \rho([P])_i \mathbf{d}_i\}), \text{ for some } \mathbf{p}_i \in \mathrm{Verts}(P).$$

Let $Q \in \mathrm{MS}_{k>0}(kP)$ and compute $\rho([Q]) = (\rho([Q])_1, ..., \rho([Q])_N) \in \mathbb{R}^N_{\geq 0}$. Here we have

$$F^Q_{u_i} = \mathrm{conv}(\{\mathbf{q}_i, \mathbf{q}_i + \rho([Q])_i \mathbf{d}_i\}), \text{ for some } \mathbf{q}_i \in \mathrm{Verts}(Q).$$

Pick a reference vertex $\mathbf{w} \in \mathrm{Verts}(P)$ and let $\mathbf{q} \in \mathrm{Verts}(Q)$. By Lemma 4.2.2, there exists some $p \in (\mathbb{R}^n)^* \backslash \{0\}$ such that $F^Q_p = \{\mathbf{q}\}$ and $F^P_p \in \mathrm{Verts}(P)$. Let $\mathbf{p} \in \mathrm{Verts}(P)$ such that $F^P_p = \{\mathbf{p}\}$. Moreover, there exists some $w \in (\mathbb{R}^n)^* \backslash \{0\}$ such that $F^P_w = \{\mathbf{w}\}$. We have $F^Q_w = \{\mathbf{z}\} \in \mathrm{Verts}(Q)$ for some $\mathbf{z} \in Q$. We will show that

$$\sigma_{\mathbf{p}}(\rho([Q])) = \mathbf{q} - \mathbf{z}. \tag{10}$$

Consequently we will have shown

$$\sigma(\rho([Q])) = [Q + \{-\mathbf{z}\}] = [Q].$$

Recall that, given a walk $\mathcal{W}$ from $\mathbf{w}$ to $\mathbf{p}$, if the counter of $\mathcal{W}$ is $\boldsymbol{\lambda}$ then

$$\sigma_{\mathbf{p}}(\rho([Q])) = \sum_{i=1}^{N} \lambda_i \rho([Q])_i \mathbf{d}_i$$

We proceed by induction on the length $L(\mathbf{p})$ of the shortest walk on P from $\mathbf{w}$ to $\mathbf{p}$. If $L(\mathbf{p}) = 0$ then $\mathbf{p} = \mathbf{w}$. Let $\mathcal{W} = \mathbf{w}$ be the trivial walk from $\mathbf{w}$ to $\mathbf{w}$. The counter of $\mathcal{W}$ is $\boldsymbol{\lambda} = \mathbf{0}$. Thus, we find

$$\begin{aligned}\sigma_{\mathbf{p}}(\rho([Q])) &= \sum_{i=1}^{N} \lambda_i \rho(Q)_i \mathbf{d}_i \\ &= \mathbf{0} \\ &= \mathbf{q} - \mathbf{z}.\end{aligned}$$

Note that $\mathbf{q} = \mathbf{z}$ because $F^P_p = \{\mathbf{p}\} = \{\mathbf{w}\} = F^P_w$. Hence $F^Q_p = F^Q_w$ by Proposition 3.1.2. That is $\mathbf{q} = \mathbf{z}$.

Assume that Equation 11 is true for all $\mathbf{q} \in \mathrm{Verts}(Q)$ such that $L(\mathbf{p}) = l$. Suppose $\mathbf{q} \in \mathrm{Verts}(Q)$ is such that $L(\mathbf{p}) = l + 1$. Let $\mathcal{W} = \mathbf{w}_0 ... \mathbf{w}_{l+1}$ be a short walk from $\mathbf{w}$ to $\mathbf{p}$, where $\mathbf{w}_i \in \mathrm{Verts}(P)$ and $\mathbf{w}_0 = \mathbf{w}, \mathbf{w}_{l+1} = \mathbf{p}$.

Consider the walk of length l from $\mathbf{w}$ to $\mathbf{p}' := \mathbf{w}_l$ given by $W' = \mathbf{w}_0 ... \mathbf{w}_l$. Write $\mathbf{p}' = F^P_{p'}$ for some $p' \in (\mathbb{R}^n)^*$, then $F^Q_{p'} = \{\mathbf{q}'\}$ for some $\mathbf{q}' \in \mathrm{Verts}(Q)$. By the induction hypothesis we have $\sigma_{\mathbf{p}'}(\rho([Q])) = \mathbf{q}' - \mathbf{z}$.

Let $\boldsymbol{\lambda}$ be the counter for $\mathcal{W}$ and $\boldsymbol{\lambda}'$ the counter for $\mathcal{W}'$. For some $i \in \{1, ..., N\}$ we have $E_i = F^P_{u_i} = \mathrm{conv}(\{\mathbf{p}', \mathbf{p}\}) = \mathrm{conv}(\{\mathbf{p}', \mathbf{p}' + \mu\rho([P])_i \mathbf{d}_i\})$, where $\mu \in \{\pm 1\}$. We have $(\boldsymbol{\lambda} - \boldsymbol{\lambda}')_j = 0$ for all $j \neq i$ and $(\boldsymbol{\lambda} - \boldsymbol{\lambda}')_i = \mu$. Consider the difference

$$\mathbf{p} - \mathbf{p}' = \mu\rho([P])_i \mathbf{d}_i, \tag{11}$$

Note that

$$\sigma_{\mathbf{p}}(\rho([Q])) = \sigma_{\mathbf{p}'}(\rho([Q])) + \mu\rho([Q])_i \mathbf{d}_i \tag{12}$$

Consider the edge of Q given by $F^Q_{u_i}$. By Lemma 4.2.1 we know $F^P_p \subseteq F^P_{u_i}$ and $F^P_{p'} \subseteq F^P_{u_i}$ implies $F^Q_p \subseteq F^Q_{u_i}$ and $F^Q_{p'} \subseteq F^Q_{u_i}$. So $\mathrm{Verts}(F^Q_{u_i}) = F^Q_p \cup F^Q_{p'}$.

Hence

$$\mathbf{q} - \mathbf{q}' = \pm\mu\rho([Q])_i \mathbf{d}_i \tag{13}$$

Apply $p \in (\mathbb{R}^n)^*$ to Equation 11 and Equation 13 to get

$$0 < p(\mathbf{p} - \mathbf{p}') = \mu\rho([P])_i p(\mathbf{d}_i)$$

and

$$0 \le p(\mathbf{q} - \mathbf{q}') = \pm\mu\rho([Q])_i p(\mathbf{d}_i).$$

Note $\{\mathbf{q}\} = F^Q_p$ is the reason we get the the inequality. By a consideration of signs we can always take the plus or minus in Equation 13 to be a plus. That is,

$$\mathbf{q} - \mathbf{q}' = \mu\rho([Q])_i \mathbf{d}_i. \tag{14}$$

Also, by the inductive hypothesis

$$\begin{aligned} \mathbf{q} - \mathbf{q}' &= \mathbf{q} - (\sigma_{\mathbf{p}'}(\rho([Q])) + \mathbf{z}) && (15) \\ &= (\mathbf{q} - \mathbf{z}) - \sigma_{\mathbf{p}'}(\rho([Q])) && (16) \end{aligned}$$

Combining Equation 12, Equation 14 and Equation 16 we conclude

$$\mathbf{q} - \mathbf{z} = \sigma_{\mathbf{p}}(\rho([Q]))$$

So the result follows by induction.

□

By a small adaptation of the proof of Proposition 4.2.3 (so that not just short walks are considered), we have shown explicitly that σ is well-defined on $\operatorname{im}\rho$. That is, the values of σ are independent of the choice of walks on the image of ρ.

Corollary 4.2.4. *Let P be a lattice polytope. Fix primitive edge vectors and sign vectors for P. For any $\mathbf{t}, \mathbf{t}' \in \operatorname{im}\rho$ we have $\sigma(\mathbf{t} + \mathbf{t}') = \sigma(\mathbf{t}) + \sigma(\mathbf{t}')$.*

Proof. Let $\mathbf{t}, \mathbf{t}' \in \operatorname{im}\rho$. There exists $[Q], [Q'] \in [\mathrm{MS}_{k>0}(kP)]$ such that

$$\mathbf{t} = \rho([Q]) \text{ and } \mathbf{t}' = \rho([Q']).$$

Now

$$\rho([Q]) + \rho([Q']) = \rho([Q + Q'])$$

follows from Proposition 3.2.6, and recall

$$[Q] + [Q'] = [Q + Q']$$

by definition. Using Proposition 4.2.3 we conclude

$$\sigma(\mathbf{t} + \mathbf{t}') = \sigma(\mathbf{t}) + \sigma(\mathbf{t}').$$

□

5 Obtaining the Lattice Minkowski Summands of P from $C(P)$

5.1 Lattice Minkowski summand criteria

Let P be a lattice polytope. From the previous sections we know how to recover the Minkowski summands $Q \in \mathrm{MS}_{k>0}(kP)$ from $\operatorname{im}\rho$ (Proposition 4.2.3) and that $\operatorname{im}\rho \subseteq C(P)$ (Proposition 3.3.10). Thus identifying which points in $C(P)$ are in the image of ρ, and in particular which points correspond to lattice Minkowski summands of P, is all that remains to be solved.

Lemma 5.1.1. *Let P be a lattice polytope with $\dim P \geq 2$. Fix primitive edge vectors and sign vectors for P. Suppose that $\mathbf{t} \in \rho([MS(kP)])$ for some $k > 0$. Then:*
(i) $\sigma(\mathbf{t}) \in [\mathrm{MS}(kP)]$,
(ii) $\rho(\sigma(\mathbf{t})) = \mathbf{t}$, *and*
(iii) $\sigma(\mathbf{t}) + \sigma(k\rho([P]) - \mathbf{t}) = kP$.

Proof. Suppose $\mathbf{t} \in \rho([\mathrm{MS}(kP)])$ for some $k > 0$. Then there exists some Minkowski summand Q of kP with $\mathbf{t} = \rho([Q])$. By Proposition 4.2.3 we know

$$\sigma(\mathbf{t}) = \sigma(\rho([Q])) = [Q],$$

hence $\sigma(\mathbf{t}) \in [\mathrm{MS}(kP)]$. Moreover, by applying ρ, we find $\rho(\sigma(\mathbf{t})) = \rho([Q]) = \mathbf{t}$.

Finally we show (iii). There exists some Minkowski summand R of kP such that $kP = Q + R$. Applying ρ and rearranging, we get

$$\rho([R]) = k\rho([P]) - \rho([Q]) = k\rho([P]) - \mathbf{t}.$$

So $k\rho([P]) - \mathbf{t} \in \operatorname{im}\rho$. Hence by Corollary 4.2.4 and Proposition 3.2.6 we have

$$\sigma(\mathbf{t}) + \sigma(k\rho([P]) - \mathbf{t}) = kP.$$

□

Notation 5.1.2. Let $\mathbf{t}, \mathbf{c} \in \mathbb{R}^N$ for some $N \in \mathbb{Z}_{\geq 1}$. As in [11], we will write $\mathbf{t} \leq \mathbf{c}$ to mean that $\mathbf{t}$ is less than or equal to $\mathbf{c}$ in every component. That is, $t_i \leq c_i$ for every $i \in \{1, ..., N\}$.

Notation 5.1.3. For a lattice polytope P, the set of all lattice Minkowski summands of P is denoted $\mathrm{MS}_{\mathrm{lat}}(P)$. Correspondingly, we denote the set of translation classes of all lattice Minkowski summands of P by $[\mathrm{MS}_{\mathrm{lat}}(P)]$. Likewise, the set of all indecomposable Minkowski summands of P is denoted $\mathrm{MS}_{\mathrm{ind}}(P)$ and the set of translation classes of all indecomposable Minkowski summands of P is denoted by $[\mathrm{MS}_{\mathrm{ind}}(P)]$.

Criteria 5.1.4. *Let P be a lattice polytope with $\dim P \geq 2$ and N edges. Fix primitive edge vectors and sign vectors for P. Write $\mathbf{c} = \rho(P)$. For $\mathbf{t} \in \mathbb{R}^N_{\geq 0}$ we have $\mathbf{t} \in \rho([\mathrm{MS}_{\mathrm{lat}}(P)])$ if and only if $\mathbf{t}$ satisfies the following three ordered criteria:*
(i) $\mathbf{t} \in C(P)$, $\mathbf{t} \leq \mathbf{c}$ and $\mathbf{t} \in \mathbb{Z}^N_{\geq 0}$,
(ii) $\sigma(\mathbf{t}) + \sigma(\mathbf{c} - \mathbf{t}) = P$,
(iii) $\rho(\sigma(\mathbf{t})) = \mathbf{t}$.

By "ordered criteria" we mean: to apply the if statement of Criteria 5.1.4, the criteria should be checked in the order (i), (ii), (iii). This is because criterion (ii) makes no sense if criterion (i) isn't satisfied and criterion (iii) may make no sense if criterion (ii) isn't satisfied. It should be noted that $\mathbf{s}, \mathbf{t} \in C(P)$ and $\mathbf{s} - \mathbf{t} \geq \mathbf{0}$, implies $\mathbf{s} - \mathbf{t} \in C(P)$. This is because the equations defining $C(P)$ are linear. Also note that criterion (ii) implies $\sigma(\mathbf{t})$ is a Minkowski summand of P.

Proof. $\Rightarrow$: Suppose that $\mathbf{t} \in \rho([\mathrm{MS}_{\mathrm{lat}}(P)])$. Proposition 3.3.10 tells us that $\mathbf{t} \in C(P)$. By Lemma 5.1.1(ii) we have criterion (iii) holds. Then from criteria (iii), Lemma 5.1.1(i) and Corollary 3.3.1: criterion (i) is true. Criterion (ii) is implied by Lemma 5.1.1(iii). Therefore all of the criteria are necessary.

$\Leftarrow$: Conversely, criteria (ii) and (iii) immediately imply that $\mathbf{t} \in \rho([\mathrm{MS}(P)])$. Criterion (i), in particular $\mathbf{t} \in \mathbb{Z}^N_{\geq 0}$, ensures that $\sigma(\mathbf{t})$ is a lattice polytope by the definition of σ. So $\sigma(\mathbf{t}) \in [\mathrm{MS}_{\mathrm{lat}}(P)]$ as required. $\square$

Remark 5.1.5. Regardless of whether σ is independent of the chosen walks or not, we can still use Criteria 5.1.4; we would just have to choose walks for each vertex and not worry about the other choices. Recall that we showed that σ was well-defined on $\mathrm{im}\,\rho$.

Criteria 5.1.4, together with the results of the previous sections: Proposition 4.2.3 and Proposition 3.3.10, provide us with enough machinery to find all lattice Minkowski summands of a given lattice polytope of any dimension. Recall, the cases in which the polytope has dimension less than two are trivial and were dealt with in Section 3. The rest of this section deals with how to implement the machinery algorithmically and considers issues of computational efficiency.

Definition 5.1.6. Let P be a lattice polytope with $\dim P \geq 2$ and N ordered edges. Denote by $S(P)$ the set

$$S(P) = \{\mathbf{t} \in \mathbb{Z}^N_{\geq 0} : \mathbf{t} \leq \rho(P)\}. \tag{17}$$

First notice the below corollary.

Corollary 5.1.7. *Let P be a lattice polytope with $\dim P \geq 2$. Fix an ordering of the edges of P. Then*

$$\rho([\mathrm{MS}_{\mathrm{lat}}(P)]) \subseteq S(P).$$

Proof. This is an immediate consequence of Criteria 5.1.4(i). □

From a practical point of view, this trivial consequence is very important. It means that Criteria 5.1.4 can be applied by a computer in finite time. This is because Criteria 5.1.4(i) has eliminated all but a finite number of possible points which could be in $\rho([\mathrm{MS}_{\mathrm{lat}}(P)])$. Hence Criteria 5.1.4(ii)(iii) only have to be checked on this finite set $S(P)$. The condition that the points should also be contained in $C(P)$ further reduces the number of potential points.

Moreover, we get two more corollaries.

Corollary 5.1.8. *Let P be a lattice polytope with N edges. The number of lattice Minkowski summands of P (up to translation) is finite.*

Proof. The cases when $\dim P \leq 1$ were studied in Section 3. Now consider the case when P is lattice polytope with $\dim P \geq 2$. Here we have $\rho([\mathrm{MS}_{\mathrm{lat}}(P)]) \subseteq S(P)$ by Corollary 5.1.7. But the set $S(P)$ contains only finitely many points and, by Proposition 4.2.3, σ restricted to $\operatorname{im}\rho$ is the inverse of ρ. Hence $[\mathrm{MS}_{\mathrm{lat}}(P)]$ is finite also. □

Corollary 5.1.9. *Let P be a lattice polytope. The number of indecomposable Minkowski summands of P (up to translation) is finite.*

Proof. This follows immediately from Corollary 5.1.9, as $\mathrm{MS}_{\mathrm{ind}}(P) \subseteq \mathrm{MS}_{\mathrm{lat}}(P)$. □

Let P be a lattice polytope. Our aim is to reduce the problem of finding all lattice Minkowski summands of P, to finding all indecomposable Minkowski summands of P. We will then prove a one-to-one correspondence between a subset of $\operatorname{im}\rho$ and the set of translation classes of indecomposable Minkowski summands of P.

Proposition 5.1.10. *Every lattice polytope P has a maximal decomposition.*

Proof. Suppose P is a lattice polytope which doesn't have a maximal decomposition. So P is not a single point. A straightforward induction shows that, for every $m \in \mathbb{Z}_{\geq 1}$, there are lattice polytopes $\{Q_i\}_{i=1}^{m}$ such that

$$P = \sum_{i=1}^{m} Q_i,$$

with each Q_i not being a single point set. Hence $\dim Q_i \geq 1$ and so Q_i has an edge by Proposition 2.3.29.

Therefore, for each $i \in \{1, ..., m\}$, at least one component of $\rho(Q_i)$ is greater or equal to 1. But by Proposition 3.2.6 we have $\rho(P) = \sum_{i=1}^{m} \rho(Q_i)$. Since there are only finitely many edges of P, there must exist some $m \in \mathbb{Z}_{\geq 1}$ such that a component of $\sum_{i=1}^{m} \rho(Q_i)$ is larger than the corresponding component of $\rho(P)$. A contradiction. □

Corollary 5.1.11. *Let P be a lattice polytope with $\dim P \geq 2$. Suppose that $[\mathrm{MS}_{\mathrm{ind}}(P)] = \{[Q_i]\}_{i=1}^m$. If $P = Q + R$ for some lattice polytopes Q, R then there exists two subsets $I_Q, I_R \subseteq \{1, ..., m\}$ such that*

$$P = \textstyle\sum_{i \in I_Q \cup I_R} (\alpha_i + \beta_i) Q_i, \quad Q = \sum_{i \in I_Q} \alpha_i Q_i, \quad R = \sum_{i \in I_R} \beta_i Q_i,$$

for some non-negative integers $\{\alpha_i\}_{i \in I_Q}, \{\beta_i\}_{i \in I_R} \subset \mathbb{Z}_{\geq 0}$.

Proof. Suppose that $P = Q + R$ for some lattice polytopes Q, R, so $Q, R \in \mathrm{MS}_{\mathrm{lat}}(P)$. By Proposition 2.3.9, any Minkowski summand of Q or R is a Minkowski summand of P. Hence, using Proposition 5.1.10, we have

$$Q = \sum_{i \in I_Q} \alpha_i Q_i \text{ and } R = \sum_{i \in I_R} \beta_i Q_i$$

for some non-negative integers $\{\alpha_i\}_{i \in I_Q}, \{\beta_i\}_{i \in I_R} \subset \mathbb{Z}_{\geq 0}$ and indexing sets $I_R, I_Q \subseteq \{1, ..., m\}$.

Moreover, as $P = Q + R$, we have

$$P = \sum_{i \in I_Q \cup I_R} (\alpha_i + \beta_i) Q_i.$$

□

Corollary 5.1.12. *Let P be a lattice polytope with $\dim P \geq 2$ and fix an ordering of the edges. Suppose that $[\mathrm{MS}_{\mathrm{ind}}(P)] = \{[Q_i]\}_{i=1}^m$. If $P = Q + R$ for some lattice polytopes Q, R then there exists two subsets $I_Q, I_R \subseteq \{1, ..., m\}$ such that*

$$\rho(P) = \textstyle\sum_{i \in I_Q \cup I_R} (\alpha_i + \beta_i) \rho(Q_i), \quad \rho(Q) = \sum_{i \in I_Q} \alpha_i \rho(Q_i), \quad \rho(R) = \sum_{i \in I_R} \beta_i \rho(Q_i),$$

for some non-negative integers $\{\alpha_i\}_{i \in I_Q}, \{\beta_i\}_{i \in I_R} \subset \mathbb{Z}_{\geq 0}$.

Proof. Applying Proposition 3.2.6 to Corollary 5.1.11 yields the desired result. □

As a consequence of Corollary 5.1.12 and Criteria 5.1.4: to find all Minkowski summands of a lattice polytope P it suffices to find all indecomposable Minkowski summands of P. Before we continue further, the construction of $C(P)$ is made clear.

Suppose $P \subseteq \mathbb{R}^n$ is a lattice polytope. Pick primitive edge vectors $\{\mathbf{d}_i\}_{i=1}^N$ for each of the N edges of P. Further choose sign vectors $\{\boldsymbol{\epsilon}^i\}_{i=1}^M$ for each of the M 2-faces of P. It is useful to note that

$$C(P) = \ker(A) \cap \mathbb{R}^N_{\geq 0}, \tag{18}$$

where $A \in \mathrm{Mat}_{nM \times N}(\mathbb{Z})$ is the $n \cdot M$ by N matrix defined by

$$A^T := (A_1^T, A_2^T, ..., A_M^T), \tag{19}$$

where $A_i := (\epsilon_1^i \mathbf{d}_1, ..., \epsilon_N^i \mathbf{d}_N) \in \mathrm{Mat}_{n\times N}(\mathbb{Z})$ for each $i \in \{1, ..., M\}$ and superscript ^{T} denotes the usual matrix transposition operator. The entries of the matrices A_i are integers because $\mathbf{d}_i \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ and $\boldsymbol{\epsilon}^i \in \{0, \pm 1\}^N$.

If CPU is an issue then we can use the fact that the primitive edge vectors of a 2-face F of P lie in a two dimensional vector space V. Hence, by choosing the primitive edge vectors of two adjacent edges of F, we get a basis of V. The primitive edges of F can then be expressed as a linear combination of these two vectors and hence are described by two coefficients. This allows us to express the primitive edge vectors in terms of two numbers instead of the n components of a vector in $\mathbb{R}^n$. Therefore, the matrix A_i corresponding to this 2-face of size $n \times N$ can be replaced with a matrix of size $2 \times N$. Doing this for each 2-face of P, we can replace the $nM \times N$ matrix A with a $2M \times N$ matrix.

Lemma 5.1.13. *[1, p. 446, mentions $C(P)$ is a rational polyhedral cone]. Let P be a lattice polytope with $\dim P \geq 2$ and N edges. Fix an ordering of the edges of P. We have $C(P)$ is a strongly convex rational polyhedral cone.*

Proof. Suppose P has $N \in \mathbb{Z}_{\geq 1}$ edges, $M \in \mathbb{Z}_{\geq 1}$ 2-faces and the ambient space of P has dimension $n \in \mathbb{Z}_{\geq 2}$. Let $A \in \mathrm{Mat}_{nm\times N}(\mathbb{Z})$ be the matrix defined in Equation 19, so that $C(P) = \ker(A) \cap \mathbb{R}^N_{\geq 0}$.

Denote by I the $N \times N$ identity matrix. Notice $A\mathbf{t} = \mathbf{0} \Leftrightarrow A\mathbf{t} \geq \mathbf{0}, A\mathbf{t} \leq \mathbf{0}$ and that $\mathbf{t} \geq \mathbf{0} \Leftrightarrow (-I)\mathbf{t} \leq \mathbf{0}$, hence $C(P) = \{\mathbf{t} : (A, -A, -I)\mathbf{t} \leq \mathbf{0}\}$. Since $A, I, \mathbf{0}$ all have rational coefficients, $C(P)$ is a rational polyhedral cone by [11, p. 79: See first paragraph].

On noting $\mathbf{0} \in C(P)$, we further have

$$C(P) = \ker(A) \cap \mathbb{R}^N_{\geq 0} \Rightarrow C(P) \subseteq \mathbb{R}^n_{\geq 0}, -C(P) \subseteq \mathbb{R}^n_{\leq 0} \Rightarrow C(P) \cap (-C(P)) = \{\mathbf{0}\}.$$

Hence $C(P)$ is strongly convex. ☐

As a consequence of Lemma 5.1.13 and Theorem 2.2.14, given a lattice polytope P with $\dim P \geq 2$ and an ordering of its edges, the cone $C(P)$ has a Hilbert basis. Since every lattice point in $C(P)$ is a linear combination (with non-negative integer coefficients) of the finite number of Hilbert basis elements, we can easily find all points in the finite set $S(P) \cap C(P)$.

A simple procedure to achieve this, is as follows. Write

$$\mathrm{Hilb}(C(P)) = \{\mathbf{h}_i\}_{i=1}^m.$$

Consider the linear combinations of the form $\sum_{i=1}^m \alpha_i \mathbf{h}_i$, where $\{\alpha_i\}_{i=1}^m \subset \mathbb{Z}_{\geq 0}$. Set $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_m) \in \mathbb{Z}^m_{\geq 0}$. Now specialise to the linear combinations with $|\boldsymbol{\alpha}| = k$, for some $k \in \mathbb{Z}_{\geq 0}$. Starting from $k = 0$, we have the single possible combination yielding $\mathbf{0}$ which we know is in $S(P) \cap C(P)$. Increment k by 1, by increasing each component of $\boldsymbol{\alpha}$ independently, and check which of these are in $S(P)$. If it is in $S(P)$ then it is also in $C(P)$ as $C(P)$ is cone. If it isn't in $S(P)$ then stop incrementing in the corresponding component of $\boldsymbol{\alpha}$. Keep incrementing in any component which wasn't found to be outside $S(P)$ in the previous step, until all combinations are outside $S(P)$. At this point, all elements of $S(P) \cap C(P)$ will have been found.

Definition 5.1.14. Let P be a polytope with $\dim P \geq 2$ and choose an ordering of the edges of P. The finite set

$$S^{\times}(P) = \{\mathbf{t} \in S(P) \cap \rho([\mathrm{MS}_{\mathrm{lat}}(P)]) : \mathbf{t} \neq \mathbf{s} + \mathbf{s}' \text{ for any } \mathbf{s}, \mathbf{s}' \in \operatorname{im} \rho \backslash \{\mathbf{0}\}\}.$$

Theorem 5.1.15. *Let P be a lattice polytope with $\dim P \geq 2$. There is a 1-1 correspondence between $[\mathrm{MS}_{\mathrm{ind}}(P)] \backslash \{[\{\mathbf{0}\}]\}$ and $S^{\times}(P)$.*

Proof. From Proposition 4.2.3, we know that

$$\sigma(\rho([Q])) = [Q] \text{ for any } [Q] \in [\mathrm{MS}_{k>0}(kP)].$$

Therefore, the restricted map $\rho : [\mathrm{MS}_{\mathrm{ind}}(P)] \to \rho([\mathrm{MS}_{\mathrm{ind}}(P)])$ is a bijection onto its image. Hence it suffices to prove that

$$\rho([\mathrm{MS}_{\mathrm{ind}}(P) \backslash \{[\{\mathbf{0}\}]\} = S^{\times}(P).$$

We first show that $\rho([\mathrm{MS}_{\mathrm{ind}}(P) \backslash \{[\{\mathbf{0}\}]\} \subseteq S^{\times}(P)$. Let $\rho([Q]) \in \rho([\mathrm{MS}_{\mathrm{ind}}(P)])$ and suppose $\rho([Q]) \notin S^{\times}(P)$. Then there exists $Q_1, Q_2 \in \mathrm{MS}_{\mathrm{lat}}(P)$ such that

$$\rho([Q]) = \rho([Q_1]) + \rho([Q_2]) \tag{20}$$

with $\rho([Q_1]), \rho([Q_2])$ both non-zero. For this to be true, we must have Q_1, Q_2 are non-single point sets. Applying σ to Equation 20, and using Corollary 4.2.4 and Proposition 4.2.3 from the previous section, we find

$$[Q] = [Q_1] + [Q_2].$$

A contradiction, as Q is indecomposable. So $\rho([\mathrm{MS}_{\mathrm{ind}}(P) \backslash \{[\{\mathbf{0}\}]\} \subseteq S^{\times}(P)$.

Now let $\rho([Q]) \in S^{\times}(P)$, where $Q \in \mathrm{MS}_{\mathrm{lat}}(P)$, and suppose $\rho([Q]) \notin \rho([\mathrm{MS}_{\mathrm{ind}}(P)])$. Then there exists non-single points $Q_1, Q_2 \in \mathrm{MS}_{\mathrm{lat}}(P)$ such that

$$[Q] = [Q_1] + [Q_2].$$

Hence $\rho([Q]) = \rho([Q_1]) + \rho([Q_2])$ with $\rho([Q_1]) \neq \mathbf{0}, \rho([Q_2]) \neq \mathbf{0}$. A contradiction. Consequently $S^{\times}(P) \subseteq \rho([\mathrm{MS}_{\mathrm{ind}}(P) \backslash \{[\{\mathbf{0}\}]\}$ and thus

$$\rho([\mathrm{MS}_{\mathrm{ind}}(P) \backslash \{[\{\mathbf{0}\}]\} = S^{\times}(P).$$

□

By Corollary 5.1.11 we know that every lattice Minkowski summand Q of P is equal to the sum of the indecomposable summands of P and is part of a maximal decomposition of P. Moreover, Theorem 5.1.15, tells us which points in $C(P)$ correspond to the indecomposable summands of P. Hence to find all pairs of lattice polytopes Q, R such that $P = Q + R$ we simply identity the points in $C(P)$ which correspond to indecomposable Minkowski summands and then use Corollary 5.1.12 to find all Maximal decompositions. Then finally, bracketing the points in $C(P)$ which correspond the maximal decompositions into two pairs and then applying σ, yields all pairs Q, R of Minkowski summands of P.

Similarly to before, the points of in $S^\times(P)$ can be easily found. Write $\mathrm{Hilb}(C(P)) = \{\mathbf{h}_i\}_{i=1}^m$. Since $S^\times(P) \subseteq C(P)$ we know that every $\mathbf{t} \in S^\times(P)$ is of the form

$$\mathbf{t} = \sum_{i=1}^{m} \alpha_i \mathbf{h}_i, \tag{21}$$

where $\{\alpha_i\}_{i=1}^m \subset \mathbb{Z}_{\geq 0}$. We do not know however whether $\mathrm{Hilb}(C(P)) \subseteq \operatorname{im}\rho$ or not. To find all points in $S^\times(P)$, write $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_m)$, and start by considering all $\mathbf{t}$ of the form in Equation 21 with $|\boldsymbol{\alpha}| = 1$. Using Criteria 5.1.4, identity which of these points are in $S^\times(P)$ and which aren't. Next consider the case when $|\boldsymbol{\alpha}| = 2$ and ignore any such $\mathbf{t}$ that are a linear combination of the points found previously to be in $S^\times(P)$ with non-negative, integer coefficients. Continue incrementing until there are no points left to consider in $S(P) \cap C(P)$. This terminates with the set $S^\times(P)$ in a finite number of steps.

5.2 Summary

In summary, given a lattice polytope P of dimension larger than one, to obtain all pairs of lattice Minkowski summands of P:

- Choose primitive edge vectors for each of the edges of P and pick sign vectors for each 2-face of P.
- Compute $C(P)$ using $C(P) = \ker(A) \cap \mathbb{R}^N_{\geq 0}$.
- Calculate $\mathrm{Hilb}(C(P))$.
- Using $\mathrm{Hilb}(C(P))$ and Theorem 5.1.15, identify all elements in $\rho([\mathrm{MS}_{\mathrm{ind}}(P)])$.
- Express $\rho(P)$ as the sum of non-negative, integer multiples of the elements of $\rho([\mathrm{MS}_{\mathrm{ind}}(P)])$ in all possible ways, thus finding all maximal decompositions of P.
- Bracket each maximal decompositions in all possible ways to get all pairs of summands.
- To recover the elements of $[\mathrm{MS}_{\mathrm{lat}}(P)]$ apply the map σ.

Simple! The only last practical details are how to program the maps σ and ρ. Luckily all the tools and information to code these maps are already built into the computer algebra system Sage. Luckier still, I've written the complete program for you!

6 Sage

In this section we demonstrate the computational implementation of the method for finding the lattice Minkowski summands of a given lattice polytope. The code used can be found in the appendix. The code is written in Sage [13] and uses the standard package PALP [9].

It should be noted that the initial intentions of PALP did not involve lattice polytopes of very large dimensions. For this reason it was decided that a number of parameters should be set at compilation time. If PALP is to be used for polytopes of larger dimension than six, then the POLY_Dmax parameter must be changed appropriately. To do this through Sage use the command set_palp_dimension(d). For more information see:
http://www.sagemath.org/doc/reference/geometry/sage/geometry/lattice_polytope.html

Consider the 2-dimensional *hypercube*. That is consider the square S given by the convex hull

$$S = \text{conv}(\{(0,1),(1,1),(0,0),(1,0)\}).$$

Here we know that the only translation classes of lattice Minkowski summands of S are those of a single point, the unit horizontal line, the unit vertical line and S. We check this is what the code finds below.

```
sage: DIM = 2
sage: # S is the DIM-dimensional Hypercube.
sage: S = lattice_polytope.octahedron(DIM).polar().affine_transform(1/2,1/2)
sage: # Print the vertices of S (Output a matrix with vertices as columns)
sage: print "Vertices of S:\n", S.vertices()
sage: # Find all lattice Minkowski summands of S
sage: summands = AllSummands(S)
sage: # Print the vertices of a representation element from each translation
sage: # class of lattice Minkowski summands of S:
sage: for i in range(0,len(summands)):
...         print "\nSummand", i, "Vertices:"; summands[i].vertices()
...
...
Vertices of S:
[0 1 0 1]
[1 1 0 0]

Summand 0 Vertices:
[0]
[0]

Summand 1 Vertices:
[ 0  0]
[ 0 -1]

Summand 2 Vertices:
[0 1]
[0 0]

Summand 3 Vertices:
[ 0  1  0  1]
[ 0  0 -1 -1]
```

Now consider the pyramid P with triangular 2-faces, given by

$$P = \mathrm{conv}(\{(0,0,0),(6,0,0),(3,4,0),(3,2,1)\}).$$

Since the pyramid has unit "height" (see Figure 6.0.1), we see that kP is not a lattice polytope for any $k \in (0,1)$. Also notice that all of the 2-faces of P are triangles. Hence, by Proposition 2.3.32, we know that P is indecomposable. We therefore expect P to have exactly two translation classes of lattice Minkowski summands. We confirm this:

```
sage: # P is a lattice pyramid.
sage: P = LatticePolytope([[0,0,0],[6,0,0],[3,4,0],[3,2,1]])
sage: # Print the vertices of P (Output a matrix with vertices as columns)
sage: print "Vertices of P:\n", P.vertices()
sage: # Find all lattice Minkowski summands of P
sage: summands = AllSummands(P)
sage: # Print the vertices of a representation element from each translation
sage: # class of lattice Minkowski summands of P:
sage: for i in range(0,len(summands)):
...         print "\nSummand", i, "Vertices:"; summands[i].vertices()
...
Vertices of P:
[0 6 3 3]
[0 0 4 2]
[0 0 0 1]

Summand 0 Vertices:
[0]
[0]
[0]

Summand 1 Vertices:
[0 6 3 3]
[0 0 4 2]
[0 0 0 1]
```

To plot the pyramid P in Sage, or any other 3-dimensional lattice polytope, use the plot3d() command

```
sage: # Plot P
sage: P.plot3d()
```

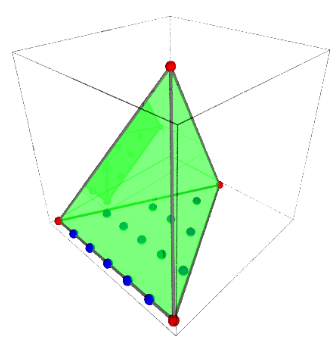

Figure 6.0.1: The indecomposable pyramid P with triangular 2-faces.

7 Further Work

There are several things I would have written about had I been given more time:

- Provide a proof that $\operatorname{im} \rho = C(P)$ for every lattice polytope P or find a counterexample.
- Extend the theory to non-compact polytopes.
- Restate the results in the more general setting of an arbitrary free module $N \cong \mathbb{Z}^n$ with corresponding vector space $N_{\mathbb{R}} := N \otimes \mathbb{R}$.
- Classification of the Minkowski summands of reflexive 3-polytopes and 4-polytopes.
- The applications of lattice polytopes to string theory.
- An explanation of the details of the proof that σ is well-defined on $C(P)$.
- In proving well-definedness of σ on $C(P)$, I have a constructive proof that every closed walk on a polytope is the "sum" of closed walks on the 2-faces of P. This should be implemented as an algorithm computationally.

8 Appendix

```
def AllSummands(P):
# Returns all lattice Minkowski summands of the lattice polytope P
    P = LatticePolytope(lattice_polytope.read_palp_matrix(P.poly_x("p", reduce_dimension=True)))
    [edges, edges_list, sign, c] = convert_user_input(P)
    return Minkowski_decomposition(edges, edges_list, sign, c, P)


def Minkowski_decomposition(edges_vector, edges_index, sign, c, P):
    N = len(edges_vector)
    M = sign.nrows()
    dimP = P.dim()
    dimAmbient = P.ambient_dim()
    ################################
    #      CASES DIM(P)=0,1        #
    ################################
    if(dimP==0):
        IrreducibleSummands_Polytope = [P]
        return IrreducibleSummands_Polytope

    if(dimP==1):
        IrreducibleSummands_Polytope = [LatticePolytope([[0 for i in range(0,dimAmbient)],
                                        edges_vector[0]])]
        return IrreducibleSummands_Polytope

    ################################
    #  CONSTRUCT VECTOR SPACE V    #
    ################################
    A = block_matrix(ZZ,[(diagonal_matrix(sign[j]))*matrix(edges_vector, sparse=True) for j in
         xrange(0,M)], ncols=M, sparse=True)
    del(sign)

    V_Basis = A.kernel(basis='LLL').basis(); del(A)
    V_Basis = V_Basis + map(lambda v: -v, V_Basis)

    V = Cone(V_Basis, check=False); del(V_Basis)

    ################################
    #      CONSTRUCT CONE C        #
    ################################
    pos_orthant = Cone(VectorSpace(QQ,N).basis(), check=False)

    C = V.intersection(pos_orthant); del(V); del(pos_orthant)


    ################################
    # FIND IRREDUCIBLE SUMMANDS    #
    ################################
    # Find Hilbert Basis
    HB = C.Hilbert_basis()
    HB_less_than_c = [vector(t.list()) for t in HB if check1(t,vector(c))]
    for t in HB_less_than_c:
        t.set_immutable()

    z = zero_vector(ZZ, len(c))
    z.set_immutable()
    HB_less_than_c.append(z)
    HB_less_than_c = Set(HB_less_than_c)

    HB_im_rho = construct_Hilb_image_rho(P,HB_less_than_c,edges_vector,edges_index,vector(c))

    HB_less_than_c == HB_im_rho
    IrreducibleSummands_Point = HB_im_rho.list()

    IrreducibleSummands_Point.remove(z)

```

```
    summands_point = FindSummands(IrreducibleSummands_Point, c, P, edges_vector, edges_index)

    num_summands = len(summands_point)

    summands_polytope = []
    for i in xrange(0,num_summands):
        Q = convert_output_to_polytope(P,summands_point[i],edges_vector, edges_index)
        summands_polytope.append(Q)

    remove_repeats_polytope([translateCentreToOrigin(Q) for Q in summands_polytope])

    return summands_polytope

def FindSummands(IrreducibleSummands_Point, c, P, edges_vector, edges_index):
    N = len(c)
    num_irreds = len(IrreducibleSummands_Point)

    R = [floor(c[i]/IrreducibleSummands_Point[0][i]) for i in xrange(0,N) if
        IrreducibleSummands_Point[0][i]!=0]
    m = min(R)

    Summands = [i*vector(IrreducibleSummands_Point[0]) for i in xrange(0,m+1)]

    for i in xrange(1,num_irreds):
        Extended_Summands = []
        for j in xrange(0,len(Summands)):
            NonZero = [k for k in xrange(0,N) if IrreducibleSummands_Point[i][k]!=0]
            R = [floor((c[k]-Summands[j][k])/IrreducibleSummands_Point[i][k]) for k in NonZero]

            if(len(R)==0):
                continue

            m = min(R)
            Extend = [Summands[j]+k*vector(IrreducibleSummands_Point[i]) for k in xrange(1,m+1)]
            Extended_Summands = Extended_Summands + Extend


        Summands = Summands+Extended_Summands

    remove_repeats(Summands)

    return Summands


def DecomposePolytope(P):
    [edges, edges_list, sign, c] = convert_user_input(P)
    return Minkowski_decomposition(edges, edges_list, sign, c, P)


def less_than_or_equal(s,t,n):
#Description:
# Determines whether or not the vector s is less than the vector t in each of their n
# components.

    for i in xrange(0,n):
        if(s[i]>t[i]):
            less_than_or_equal = False
            break

        less_than_or_equal = True

    return less_than_or_equal

def remove_repeats(list):
    list_repeats = [i for i in xrange(0,len(list)) if list[i] in list[:i]]

    num_del = 0
    for i in list_repeats:
```

```
        del(list[i-num_del])
        num_del = num_del + 1

def remove_repeats_polytope(polytope_list):
    #Note if two polytopes P,Q have same vertices but vertices in a different order then
    # P==Q returns FALSE!
    vertex_list = [Set(P.vertices().columns()) for P in polytope_list]
    list_repeats = [i for i in xrange(0,len(polytope_list)) if vertex_list[i] in
                    vertex_list[:i]]

    num_del = 0
    for i in list_repeats:
        del(polytope_list[i-num_del])
        num_del = num_del + 1

##############################
#  CONVERT USER INPUT BELOW  #
##############################
def convert_user_input(P):
    #Computes sign_vectors, rho(P), etc.
    dimP = P.dim()

    if(dimP==0):
        two_faces_list = []
        num_two_faces = 0
        edges_list = []


    elif(dimP==1):
        two_faces_list = []
        num_two_faces = 0
        edges_list = [Set([0,1])]


    else:
        #Need to have edges in same order as x.poly so made my own function.
        edges_list = get_edges(P)

    num_edges = len(edges_list)

    if(dimP==2):
        two_faces_list = [P.traverse_boundary()]
        num_two_faces = 1
    elif(dimP>2):
        two_faces_list2 = get_two_faces(P)
        two_faces_list = P.faces(dim=2)
        two_faces_list4 = [F.vertices() for F in two_faces_list]

        reorder = [two_faces_list4.index(F) for F in two_faces_list2]
        two_faces_list3 = [two_faces_list[i] for i in reorder]
        two_faces_list = map(lambda F: F.traverse_boundary(),two_faces_list3)

        num_two_faces = len(two_faces_list)

    #Create List of primitive edges and list of corresponding multiples of the primitive edges
    edges_vector = []
    c = []

    for i in xrange(0,num_edges):
        vertex_index1 = edges_list[i][0]
        vertex_index2 = edges_list[i][1]

        # Orient the edges so that their direction is from smallest vertex index to largest.
        if(vertex_index1<vertex_index2):
            edge_i = P.point(vertex_index2)-P.point(vertex_index1)
        else:
            edge_i = P.point(vertex_index1)-P.point(vertex_index2)

```

```
            c_i = lattice_polytope.integral_length(edge_i)

            edges_vector.append(edge_i/c_i)
            c.append(c_i)


    #Create Sign Matrix
    sign = []

    for i in xrange(0,num_two_faces):
        two_face = two_faces_list[i]
        num_two_face_edges = len(two_face)
        two_face_edges = [[two_face[j],two_face[j+1]] for j in xrange(0,num_two_face_edges-1)]
        two_face_edges.append([two_face[num_two_face_edges-1],two_face[0]])

        num_two_vertices = num_two_face_edges

        # Find index of each of the edges of two_face_edge
        two_face_edges_indexes = [edges_list.index(Set(e)) for e in two_face_edges]


        #Create sign vector for the 2-face
        sign_vector = [0 for j in xrange(0,num_edges)]

        for j in xrange(0,num_two_face_edges):
            if(two_face_edges[j][0]<two_face_edges[j][1]):
                pm_sign = 1
            else:
                pm_sign = -1

            sign_vector[two_face_edges_indexes[j]] = pm_sign


        sign.append(sign_vector)

    if(dimP<2):
        sign = [[1]]

    sign = matrix(sign)

    return [edges_vector, edges_list, sign, c]


def get_facet_normals(P):
    dim = P.ambient_dim()
    dimP = P.dim()

    data = lattice_polytope._palp("poly.x -e", [P], reduce_dimension=True)
    f = open(data)
    lines = f.readlines()

    binPstr = [lines[4+i].split() for i in xrange(0,P.nfacets())]

    #Reference: http://stackoverflow.com/questions/931092/reverse-a-string-in-python
    binPstr = binPstr[::-1]

    f.close()
    facet_normals = [vector([int(n[i]) for i in xrange(0,dimP)]) for n in binPstr]

    L = matrix(ZZ,facet_normals).T

    FN = LatticePolytope([[]])
    FN._points = P._embed(L)

    return facet_normals


def get_edges(P):
```

```
    dim = P.ambient_dim()
    dimP = P.dim()

    data = lattice_polytope._palp("poly.x -fi", [P], reduce_dimension=True)
    f = open(data)

    l = f.readline()
    binP = lattice_polytope._read_poly_x_incidences(f, dimP)[1]

    edge_list =[]
    for e in binP:
        edge = Set(e)
        edge_list.append(edge)

    return edge_list


def get_two_faces(P):
    dim = P.ambient_dim()
    dimP = P.dim()

    data = lattice_polytope._palp("poly.x -fi", [P], reduce_dimension=True)
    f = open(data)
    l = f.readline()
    binP = lattice_polytope._read_poly_x_incidences(f, dimP)[2]

    two_face_list =[]

    for F in binP:
        twoface = F
        two_face_list.append(twoface)

    return two_face_list


def rhoMap(Q,P,P_edge_vectors):
    dimP = P.dim()
    P_facet_normals = get_facet_normals(P)

    dataP = lattice_polytope._palp("poly.x -fi", [P], reduce_dimension=True)
    fP = open(dataP)
    linesP = fP.readlines()
    binP = linesP[dimP+4].split()

    binP = binP[1:]
    P_num_edges = len(binP) - 1

    Q_verts = Q.vertices().columns()

    rhoP = []
    for e in binP:
        e_normals = [i for i,n in enumerate(e) if n=='1']

        dual_vector = sum([P_facet_normals[i] for i in e_normals])
        dot_products = [dual_vector.dot_product(v) for v in Q_verts]
        m = min([dual_vector.dot_product(v) for v in Q_verts])

        min_verts = [v for i,v in enumerate(Q_verts) if dot_products[i]==m]

        if len(min_verts)==1:
            rhoP.append(0)
        elif len(min_verts)==2:
            rhoP.append(lattice_polytope.integral_length(min_verts[0]-min_verts[1]))
        else:
            print "Length of min_verts is bigger than 2."
            print "ERROR: Q not a Minkowski summand of P."

    return vector(rhoP)
```

```
########################################################
# Convert Program Output, i.e. create the map sigma  #
########################################################
# Note skeleton adds non-vertex points to graph of polytope so must create graph myself.

def convert_output_to_polytope(P,t,edges_vector,edges_list):
    num_vertices = P.nvertices()
    num_edges = len(edges_vector)

    skeleton = Graph(num_vertices)
    skeleton.add_edges(edges_list)

    sigma_t = [vector([0 for i in range(0,P.ambient_dim())])]

    #Choice of i is a choice of vertex of P
    for i in xrange(1, num_vertices):
        Path = skeleton.shortest_path(0,i)
        Path_Len = len(Path)-1

        Path_edges = [[Integer(Path[j+1]-Path[j]).sign(),[Path[j],Path[j+1]]] for j
                        in xrange(0,Path_Len)]

        L = [0 for j in xrange(0,num_edges)]
        used_edge_indexes = []

        for j in xrange(0,Path_Len):
            pm = Path_edges[j][0]
            current_path_edge = Set(Path_edges[j][1])
            edge_index = edges_list.index(current_path_edge)
            used_edge_indexes.append(edge_index)
            L[edge_index] = pm

        sigma_v_t = sum([edges_vector[l]*L[l]*t[l] for l in used_edge_indexes])

        sigma_t.append(sigma_v_t)

    Q = LatticePolytope(matrix(ZZ, sigma_t).T, compute_vertices=True)

    return Q


def construct_Hilb_image_rho(P,HilbCP,edges_vector,edges_list,c):
    num_edges = len(c)
    z = zero_vector(ZZ, num_edges)
    z.set_immutable()
    current_list = Set([z])

    size = 1
    l=1
    HilbCP_size = len(HilbCP)

    while l!=0:
        test_list = IntegerVectors(size,HilbCP_size)
        test_list_vector = [lin_comb_of_spanning_set(HilbCP,alpha) for alpha in test_list]
        [l,current_list] = all_checks(P,HilbCP,edges_vector,edges_list,c,current_list,
                                test_list_vector,size)
        size = size + 1

    return current_list

def all_checks(P,HilbCP,edges_vector,edges_list,c,current_list,test_list_vector,size):
    l = len(test_list_vector)
    add_vector = []
    for t in test_list_vector:

        if(check1(t,c)):
```

```
            if(check2(t,current_list)):
                if(check3(P,t,edges_vector,edges_list,vector(c))):
                    if(check4(P,t,edges_vector,edges_list)):
                        t.set_immutable()
                        add_vector.append(t)
                        current_list = current_list.union(Set([t]))
            else:
                l=l-1
        else:
            l=l-1

    current_list = current_list.union(Set(add_vector))
    return [l,current_list]

def check1(t,c):
    d = c-t
    m = min(d)
    return m>=0

def check2(t,current_list):
    V = span(current_list,ZZ,check=False)
    try:
        coordinates = V.coordinates(t)

        return any(v-floor(v)!=0 for v in coordinates)
    except ArithmeticError:
        return True

def check3(P,t,edges_vector,edges_list,c):
    sigma_t = convert_output_to_polytope(P,t,edges_vector,edges_list)
    sigma_s = convert_output_to_polytope(P,c-t,edges_vector,edges_list)
    sigma_t_plus_sigma_s = lattice_polytope.minkowski_sum(sigma_t.vertices().T,
                                sigma_s.vertices().T)
    sigma_t_plus_sigma_s = LatticePolytope(sigma_t_plus_sigma_s)

    return is_translate(P, sigma_t_plus_sigma_s)

def check4(P,t,edges_vector,edges_list):
    sigma_t = convert_output_to_polytope(P,t,edges_vector,edges_list)
    rho_sigma_t = rhoMap(sigma_t,P,edges_vector)

    return rho_sigma_t == t

def check5(P,t,edges_vector,edges_list):
    sigma_t = convert_output_to_polytope(P,t,edges_vector,edges_list)
    rho_sigma_t = rhoMap(sigma_t,P,edges_vector)
    sigma_rho_sigma_t = convert_output_to_polytope(P,rho_sigma_t,edges_vector,edges_list)
    rho_sigma_rho_sigma_t = rhoMap(sigma_rho_sigma_t,P,edges_vector)

    return rho_sigma_t == rho_sigma_rho_sigma_t


def lin_comb_of_spanning_set(S,alpha):
    return sum([S[i]*alpha[i] for i in xrange(0,len(S))])

def is_translate(P,Q):
    P = translateCentreToOrigin(P)
    Q = translateCentreToOrigin(Q)

    P_verts = Set(P.vertices().columns())
    Q_verts = Set(Q.vertices().columns())

    return P_verts==Q_verts

def translateCentreToOrigin(P):
    vertices = P.vertices().columns()
    num_vertices = len(vertices)
    centre = sum([vertices[i] for i in xrange(0,num_vertices)])/num_vertices
```

```
    centre = vector([floor(c) for c in centre])

    return P.affine_transform(b=-centre)
```

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