RESIDUAL CATEGORIES OF GRASSMANNIANS

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Abstract. This short note is an extended abstract for my talk at the Nottingham Online Algebraic Geometry Seminar on October 1, 2020. It is based on the joint works with Alexander Kuznetsov [15, 16].

We work over a fixed algebraically closed field $\Bbbk$ of characteristic zero.

1.1. **Exceptional collections.** Let X be a smooth projective variety over the field $\Bbbk$ and let $\mathbf{D}^{\mathrm{b}}(X)$ be the bounded derived category of coherent sheaves on X. An object E of $\mathbf{D}^{\mathrm{b}}(X)$ is called exceptional, if we have

$$\mathrm{Hom}(E, E) = \Bbbk id_E \quad \text{and} \quad \mathrm{Ext}^i(E, E) = 0 \quad \forall i \neq 0.$$

A sequence of exceptional objects $E_1, \dots, E_n$ is called an exceptional collection, if we have

$$\mathrm{Ext}^k(E_i, E_j) = 0 \quad \text{for } i > j \text{ and } \forall k.$$

We denote by $\langle E_1, \dots, E_n\rangle$ the smallest full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(X)$ containing the objects $E_1, \dots, E_n$. If we have

$$\mathbf{D}^{\mathrm{b}}(X) = \langle E_1, \dots, E_n\rangle,$$

then the collection $E_1, \dots, E_n$ is called full.

Example 1.1.

(1) On the projective space $\mathbb{P}^n$ we have the famous Beilinson collection [1]

$$\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n)\rangle. \tag{1.1}$$

(2) For Grassmannians $\mathrm{Gr}(k, n)$ and quadrics Q_n full exceptional collections were constructed by Kapranov [11]. In the case of $\mathrm{Gr}(2, 4)$ Kapranov's collection takes the form

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Gr}(2, 4)) = \langle \mathcal{O}, \mathcal{U}^\vee, S^2\mathcal{U}^\vee, \mathcal{O}(1), \mathcal{U}^\vee(1), \mathcal{O}(2)\rangle, \tag{1.2}$$

where $\mathcal{U}$ is the tautological subbundle on $\mathrm{Gr}(2, 4)$.

1.2. **Lefschetz exceptional collections.** In this talk we are interested in a particular class of exceptional collections, called Lefschetz collections introduced by Alexander Kuznetsov in [13] in the context of homological projective duality. The main goal of the talk is to explain that Lefschetz collections also have close connections with quantum cohomology and mirror symmetry.

Definition 1.2. Let X be a smooth projective variety over $\Bbbk$ and let $\mathcal{O}(1)$ be a line bundle on X. For an object $F \in \mathbf{D}^{\mathrm{b}}(X)$ we denote $F(1) \coloneqq F \otimes \mathcal{O}(1)$.

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(i) A **Lefschetz collection** with respect to $\mathcal{O}(1)$ is an exceptional collection, which has a block structure

$$\underbrace{E_1, E_2, \dots, E_{\sigma_0}}_{\text{block } 0}; \underbrace{E_1(1), E_2(1), \dots, E_{\sigma_1}(1)}_{\text{block } 1}; \dots; \underbrace{E_1(m-1), E_2(m-1), \dots, E_{\sigma_{m-1}}(m-1)}_{\text{block } m-1}$$

where $\sigma = (\sigma_0 \geq \sigma_1 \geq \dots \geq \sigma_{m-1} \geq 0)$ is a non-increasing sequence of non-negative integers called the **support partition** of the collection. In the above notation we use semicolons to separate the blocks. The block $(E_1, E_2, \dots, E_{\sigma_0})$ is called the **starting block**. We use notation $(E_\bullet, \sigma)$ for a Lefschetz collection with support partition σ.

(ii) If $\sigma_0 = \sigma_1 = \dots = \sigma_{m-1}$, then the Lefschetz collection is called **rectangular**. Otherwise, its **rectangular part** is defined to be the subcollection

$$E_1, E_2, \dots, E_{\sigma_{m-1}}; E_1(1), E_2(1), \dots, E_{\sigma_{m-1}}(1); \dots \\ \dots; E_1(m-1), E_2(m-1), \dots, E_{\sigma_{m-1}}(m-1). \quad (1.3)$$

(iii) The **residual category** of a Lefschetz collection is defined as the orthogonal of its rectangular part

$$\mathcal{R} = \Big\langle E_1, E_2, \dots, E_{\sigma_{m-1}}; \dots; E_1(m-1), E_2(m-1), \dots, E_{\sigma_{m-1}}(m-1) \Big\rangle^{\perp}.$$

Remark 1.3.

(1) The residual category $\mathcal{R}$ of a Lefschetz collection vanishes if and only if the Lefschetz collection is full and rectangular.

(2) The definitions imply that we have a **semiorthogonal decomposition**

$$\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{R}; E_1, E_2, \dots, E_{\sigma_{m-1}}; \dots; E_1(m-1), E_2(m-1), \dots, E_{\sigma_{m-1}}(m-1) \rangle.$$

(3) An important feature of the residual category is the existence of a natural autoequivalence $\tau_{\mathcal{R}} \colon \mathcal{R} \to \mathcal{R}$ called the **induced polarization** such that $\tau_{\mathcal{R}}^m \cong \mathbb{S}_{\mathcal{R}}^{-1}[\dim X]$, where $\mathbb{S}_{\mathcal{R}}$ is the Serre functor of $\mathcal{R}$. It is useful to think of $\tau_{\mathcal{R}}$ as the analogue of the twist by $\mathcal{O}(1)$ in $\mathbf{D}^{\mathrm{b}}(X)$. For more details on $\tau_{\mathcal{R}}$ we refer to [15, 16].

Example 1.4.

(1) Collection (1.1) is a Lefschetz collection with

$$E_\bullet = (\mathcal{O}) \quad \text{and} \quad \sigma = (1^n) \coloneqq (\underbrace{1, \dots, 1}_{n \text{ times}}).$$

(2) Collection (1.2) is a Lefschetz collection with

$$E_\bullet = (\mathcal{O}, \mathcal{U}^\vee, S^2\mathcal{U}^\vee) \quad \text{and} \quad \sigma = (3, 2, 1).$$

(3) On $\mathrm{Gr}(2,4)$ there exist a Lefschetz collection with a smaller starting block than (1.2). Namely, we have

$$\mathbf{D}^{\mathrm{b}}(\mathrm{Gr}(2,4)) = \langle \mathcal{O}, \mathcal{U}^\vee; \mathcal{O}(1), \mathcal{U}^\vee(1); \mathcal{O}(2); \mathcal{O}(3) \rangle, \quad (1.4)$$

For this collection we have

$$E_\bullet = (\mathcal{O}, \mathcal{U}^\vee) \quad \text{and} \quad \sigma = (2, 2, 1, 1).$$

1.3. **Lefschetz collections and quantum cohomology.** Let X now be a Fano variety over $\Bbbk$. Roughly speaking the main conjectures of [15, 16] say that there is a deep relation between the small quantum cohomology of X and the structure of Lefschetz collections and their residual categories. Let us now be more precise.

Let X be a Fano variety with vanishing odd cohomology. We denote by $\mathrm{QH}_{\mathsf{can}}(X)$ its small quantum cohomology specialized at the canonical class. If the Picard rank of X is one, and consequently there is only one deformation parameter q in the small quantum cohomology, $\mathrm{QH}_{\mathsf{can}}(X)$ is the small quantum cohomology at $q = 1$; this is the case for almost all examples appearing below. Thus, $\mathrm{QH}_{\mathsf{can}}(X)$ is a finite dimensional commutative $\mathbb{C}$-algebra, whose underlying vector space is canonically isomorphic to $H^*(X, \mathbb{C})$.

Now we define the quantum spectrum of X as

$$\mathrm{QS}_X \coloneqq \mathrm{Spec}(\mathrm{QH}_{\mathsf{can}}(X)),$$

which is a finite scheme endowed with an action of the group μ_m, where m is the Fano index of X. The anticanonical class $-K_X$ defines a morphism

$$\kappa\colon \mathrm{QS}_X \to \mathbb{A}^1,$$

which is equivariant with respect to the standard action of μ_m on $\mathbb{A}^1$. Finally, we define

$$\mathrm{QS}_X^\times \coloneqq \kappa^{-1}(\mathbb{A}^1 \setminus \{0\}) \quad \text{and} \quad \mathrm{QS}_X^\circ \coloneqq \mathrm{QS}_X \setminus \mathrm{QS}_X^\times .$$

The action of μ_m on $\mathrm{QS}_X^\times$ is free, as it is free on $\mathbb{A}^1 \setminus \{0\}$. We refer to [16, Introduction] for more details on the setup.

Conjecture 1.5 ([16, Conjecture 1.3])**.** *Let X be a Fano variety of index m over an algebraically closed field $\Bbbk$ of characteristic zero and assume that the big quantum cohomology $\mathrm{BQH}(X)$ is generically semisimple.*

(1) *There is an $\mathrm{Aut}(X)$-invariant exceptional collection $E_1, \dots, E_k$ in $\mathbf{D}^{\mathrm{b}}(X)$, where k is the length of $\mathrm{QS}_X^\times$ divided by m. This collection extends to a rectangular Lefschetz collection*

$$E_1, E_2, \dots, E_k; E_1(1), E_2(1), \dots, E_k(1); \dots \\ \dots; E_1(m-1), E_2(m-1), \dots, E_k(m-1). \quad (1.5)$$

in $\mathbf{D}^{\mathrm{b}}(X)$.

(2) *The residual category $\mathcal{R}$ of (1.5) has a completely orthogonal $\mathrm{Aut}(X)$-invariant decomposition*

$$\mathcal{R} = \bigoplus_{\xi \in \mathrm{QS}_X^\circ} \mathcal{R}_\xi$$

with components indexed by closed points $\xi \in \mathrm{QS}_X^\circ$. Moreover, the component $\mathcal{R}_\xi$ of $\mathcal{R}$ is generated by an exceptional collection of length equal to the length of the localization $(\mathrm{QS}_X^\circ)_\xi$ at ξ.

(3) *The induced polarization $\tau_{\mathcal{R}}$ permutes the components $\mathcal{R}_\xi$. More precisely, for each point $\xi \in \mathrm{QS}_X^\circ$ it induces an equivalence*

$$\tau_{\mathcal{R}}\colon \mathcal{R}_\xi \xrightarrow{\sim} \mathcal{R}_{g(\xi)},$$

where g is a generator of μ_m.

Thus, intuitively the points of $\mathrm{QS}_X^\times$ correspond to the rectangular part (1.5) and the twist by $\mathcal{O}(1)$ corresponds to the action of μ_m on $\mathrm{QS}_X^\times$; points of QS_X° give rise to an exceptional collection in $\mathcal{R}$.

Below we discuss two particular instances of the above conjecture. In the first case we assume that the small quantum cohomology, or rather $\mathrm{QH_{can}}(X)$, is semisimple, and in the second case we consider a particular class of homogeneous varieties called coadjoint varieties, whose $\mathrm{QH_{can}}(X)$ is almost never semisimple.

1.4. **Cases with semisimple** $\mathrm{QH_{can}}(X)$**.** The simplest example, where Conjecture 1.5 holds, is provided by $\mathbb{P}^n$. Indeed, it is well-known that we have

$$\mathrm{QH_{can}}(\mathbb{P}^n) = \mathbb{C}[h]/(h^{n+1} - 1),$$

and, therefore, the quantum spectrum $\mathrm{QS}_{\mathbb{P}^n}$ is a reduced subscheme of $\mathbb{A}^1$ supported at the points ζ^i with $i \in [0, n]$, where ζ is a primitive $(n+1)$-st root of unity. The action of μ_{n+1} on $\mathrm{QS}_{\mathbb{P}^n} = \mathrm{QS}_{\mathbb{P}^n}^\times$ is the usual action of μ_{n+1} on $(n+1)$-st roots of unity. Therefore, this action has only one orbit and, according to Conjecture 1.5, in $\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)$ we should expect to have a Lefschetz collection, whose starting block $E_\bullet$ consists of one object and whose support partition is of the form $\sigma = (1^{n+1})$. Since $\mathrm{QS}_{\mathbb{P}^n}^\circ = \emptyset$, the residual category vanishes. The Beilinson collection (1.1) satisfies all these requirements.

More generally, if $\mathrm{QH_{can}}(X)$ is semisimple, then Conjecture 1.5 gives a full description of the residual category. Indeed, since each component $\mathcal{R}_\xi$ is generated by one exceptional object Conjecture 1.5(ii) says that the residual category $\mathcal{R}$ is generated by a completely orthogonal exceptional collection (cf. [15, Conjecture 1.12]).

There is a number of cases with semisimple $\mathrm{QH_{can}}(X)$, where our conjecture is known to hold:

(1) for $\mathrm{Gr}(k, n)$ with for $k = p$ a prime number (see [15]);
(2) for quadrics Q_n this follows from Kapranov's work [11] (see [15, Example 1.6]);
(3) for $\mathrm{OG}(2, 2n + 1)$ this follows from Kuznetsov's work [14] (see [15, Example 1.9]);
(4) for $\mathrm{IG}(3, 8)$ and $\mathrm{IG}(3, 10)$ this holds by [10, 18];
(5) for $\mathrm{IG}(4, 8)$ and $\mathrm{IG}(5, 10)$ this should follow from [6, 21].
(6) for the Cayley plane $\mathrm{E}_6/\mathrm{P}_1$ this holds by [2, 5, 17];
(7) for the Cayley Grassmannian this holds by [3, 9];
(8) for the G_2-Grassmannian $\mathrm{G}_2/\mathrm{P}_2$ this holds by [12];
(9) for some horospherical varieties of Picard rank one by [7, 8, 14, 19];

In the above list $\mathrm{IG}(k, 2n)$ is the variety parametrizing k-dimensional isotropic subspaces in a $2n$-dimensional vector space with a symplectic form; this is a homogeneous space for the symplectic group Sp_{2n}. The variety $\mathrm{OG}(2, 2n + 1)$ parametrizes 2-dimensional isotropic subspaces in a $(2n+1)$-dimensional vector space with a symmetric non-degenerate form; this is a homogeneous space for the special orthogonal group SO_{2n+1}. We refrain from recalling the definitions of all the other varieties in the above list.

1.5. **Cases with non-semisimple** $\mathrm{QH_{can}}(X)$**.** If the algebra $\mathrm{QH_{can}}(X)$ is not semisimple, then Conjecture 1.5 does not give a full description of the orthogonal components $\mathcal{R}_\xi$ of the residual category. However, under mirror symmetry, the locus QS_X° corresponds to the critical points of the mirror Landau–Ginzburg model f of the Fano variety X in the fiber over zero $f^{-1}(0)$. We expect that for each $\xi \in \mathrm{QS}_X^\circ$ the component $\mathcal{R}_\xi$ is equivalent to the

Fukaya–Seidel category of the corresponding critical point in $f^{-1}(0)$. Below we illustrate this phenomenon.

Symplectic isotropic Grassmannians $\mathrm{IG}(2,2n)$**.** The variety $\mathrm{IG}(2,2n)$ parametrizes 2-dimensional isotropic subspaces in a $2n$-dimensional vector space with a symplectic form. It is embedded into the usual Grassmannian $\mathrm{IG}(2,2n) \subset \mathrm{Gr}(2,2n)$ and is, in fact, a hyperplane section of $\mathrm{Gr}(2,2n)$.

The Fano index of $\mathrm{IG}(2,2n)$ is equal to $2n-1$ and $\dim_{\mathbb{C}}(H^*(\mathrm{IG}(2,2n),\mathbb{C})) = 2n(n-1)$. We know by [4, Proposition 4.3] that $\mathrm{QS}^{\times}_{\mathrm{IG}(2,2n)}$ is a disjoint union of $n-1$ orbits of μ_{2n-1}, each of which consists of $2n-1$ reduced points, and $\mathrm{QS}^{\circ}_{\mathrm{IG}(2,2n)}$ consists of one non-reduced point ξ_0 with local algebra $\mathbb{C}[t]/t^{n-1}$.

Since $\mathbb{C}[t]/t^{n-1}$ is isomorphic to the Jacobi algebra of an isolated hypersurface singularity of type A_{n-1}, the above description of $\mathrm{QS}^{\circ}_{\mathrm{IG}(2,2n)}$ suggests that in Conjecture 1.5 we should expect the residual category $\mathcal{R} = \mathcal{R}_{\xi_0}$ to be equivalent to the Fukaya–Seidel category of an isolated hypersurface singularity of type A_{n-1}, which by [22] is equivalent to the derived category of representations of a quiver of type A_{n-1}.

A Lefschetz collection on $\mathrm{IG}(2,2n)$ satisfying these conjectures was constructed in [14]. Indeed, let us define

$$E_i = S^{i-1}\mathcal{U}^{\vee} \quad \text{for} \quad 1 \le i \le n,$$
$$\sigma = (n^{n-1}, (n-1)^n).$$

Then $(E_\bullet, \sigma)$ is a full Lefschetz collection by [14, Theorem 5.1]. Moreover, by [4, Theorem 9.6], its residual category is equivalent to the derived category of representations of A_{n-1} quiver.

Coadjoint varieties. The variety $\mathrm{IG}(2,2n)$ considered above fits naturally into a series of examples. Namely, $\mathrm{IG}(2,2n)$ is the coadjoint variety in Dynkin type C_n. In general, the coadjoint variety of a simple algebraic group G is the highest weight vector orbit in the projectivization of the irreducible G-representation, whose highest weight is the highest short root. Therefore, a coadjoint variety is uniquely determined by the Dynkin type of G, which can be $\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$. Coadjoint varieties are of Picard rank one except for type A_n, where the Picard rank is two.

The quantum spectrum of a coadjoint variety is described in [20]. To state the result we denote by T(G) the Dynkin diagram of G and by $\mathrm{T}_{\mathrm{short}}(\mathrm{G})$ the subdiagram of T(G) consisting of vertices corresponding to short roots. Thus, we have the table

T	A_n	B_n	C_n	D_n	E_n	F_4	G_2
$\mathrm{T}_{\mathrm{short}}$	A_n	A_1	A_{n-1}	D_n	E_n	A_2	A_1

Now we are ready to describe the quantum spectrum of coadjoint varieties.

Lemma 1.6 ([20])**.** *Let X be the coadjoint variety of a simple algebraic group* G*. Then*

(1) $\mathrm{QS}^{\times}(X)$ *consists of reduced points;*
(2) *if* $\mathrm{T}(\mathrm{G}) = \mathrm{A}_n$ *and n is even, then* $\mathrm{QS}^{\circ}(X) = \emptyset$*;*
(3) *otherwise,* $\mathrm{QS}^{\circ}(X)$ *has a unique point and the local algebra at this point is isomorphic to the Jacobi ring of an isolated hypersurface singularity of Dynkin type* $\mathrm{T}_{\mathrm{short}}(\mathrm{G})$*.*

Similarly to the case of $\mathrm{IG}(2,2n)$ the above description suggests the following.

Conjecture 1.7 ([16, Conjecture 1.8]). *Let X be the coadjoint variety of a simple algebraic group* G *over an algebraically closed field of characteristic zero. Then* $\mathbf{D}^{\mathrm{b}}(X)$ *has an* $\mathrm{Aut}(X)$*-invariant rectangular Lefschetz exceptional collection with residual category* $\mathcal{R}$ *and*

(1) *if* $\mathrm{T}(\mathrm{G}) = \mathrm{A}_n$ *and n is even, then* $\mathcal{R} = 0$*;*
(2) *otherwise,* $\mathcal{R}$ *is equivalent to the derived category of representations of a quiver of Dynkin type* $\mathrm{T}_{\mathrm{short}}(\mathrm{G})$.

This conjecture is by now known in all Dynkin types except for $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$. Indeed, this is [4, Theorem 9.6] for type C_n, [15, Example 1.6] for types B_n and G_2, [16, Theorem 1.9] for types A_n and D_n, [2, Theorem 1.4] for type F_4.

Remark 1.8. Above we have discussed only coadjoint varieties, but one can also define adjoint varieties. The adjoint variety of a simple algebraic group G is the highest weight vector orbit in the projectivization of the irreducible G-representation, whose highest weight is the highest root. If $\mathrm{T}(\mathrm{G})$ is simply laced, then all roots have the same length and, therefore, adjoint and coadjoint varieties coincide.

Let X be the adjoint variety of a simple algebraic group G whose Dynkin type is not simply laced, i.e. $\mathrm{B}_n, \mathrm{C}_n, \mathrm{F}_4, \mathrm{G}_2$. Then by [20, Theorem 9.1] we know that $\mathrm{QS}^{\circ}(X) = \emptyset$ and by Conjecture 1.5 we should expect a rectangular Lefschetz collection in $\mathbf{D}^{\mathrm{b}}(X)$. This is by now known in all cases: [14, Theorem 7.1] for type B_n, [15, Example 1.4] for type C_n, [23, Theorem 1.1] for type F_4, and [12, §6.4] for type G_2.

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References

[1] A. A. Beilinson, *Coherent sheaves on* $\mathbf{P}^n$ *and problems in linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 68–69.

[2] P. Belmans, A. Kuznetsov, and M. Smirnov, *Derived categories of the Cayley plane and the coadjoint Grassmannian of type* F, Transformation Groups (2021), available at `https://doi.org/10.1007/s00031-021-09657-w`. See also arXiv:2005.01989.

[3] V. Benedetti and L. Manivel, *The small quantum cohomology of the Cayley Grassmannian*, Internat. J. Math. **31** (2020), no. 3, 2050019, 23.

[4] J. A. Cruz Morales, A. Kuznetsov, A. Mellit, N. Perrin, and M. Smirnov, *On quantum cohomology of Grassmannians of isotropic lines, unfoldings of* A_n*-singularities, and Lefschetz exceptional collections*, Annales de l'Institut Fourier **69** (2019), no. 3, 955–991. See also arXiv:1705.01819.

[5] D. Faenzi and L. Manivel, *On the derived category of the Cayley plane II*, Proc. Amer. Math. Soc. **143** (2015), no. 3, 1057–1074. See also arXiv:1201.6327.

[6] A. Fonarev, *Full exceptional collections on Lagrangian Grassmannians*, Int. Math. Res. Not. (2020). Available online ahead of print as doi:10.1093/imrn/rnaa098, see also arXiv:1911.08968.

[7] ———, *On the bounded derived category of IGr(3, 7)*, Transform. Groups **27** (2022), no. 1, 89–112.

[8] R. Gonzales, C. Pech, N. Perrin, and A. Samokhin, *Geometry of horospherical varieties of Picard rank one*. Available at arXiv:1803.05063.

[9] L. Guseva, *On the derived category of the Cayley Grassmannian*. to appear.

[10] ———, *On the derived category of* $\mathrm{IGr}(3;8)$, Mat. Sb. **211** (2020), no. 7, 24–59. See also arXiv:1810.07777.

[11] M. M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Inventiones Mathematicae **92** (1988), no. 3, 479–508.

[12] A. Kuznetsov, *Hyperplane sections and derived categories*, Izv. Ross. Akad. Nauk Ser. Mat. **70** (2006), no. 3, 23–128. See also arXiv:math/0503700.

[13] ______, *Homological projective duality*, Publ. Math. Inst. Hautes Études Sci. **105** (2007), 157–220. See also arXiv:math/0507282.

[14] ______, *Exceptional collections for Grassmannians of isotropic lines*, Proceedings of the London Mathematical Society. Third Series **97** (2008), no. 1, 155–182. See also arXiv:math/0512013.

[15] A. Kuznetsov and M. Smirnov, *On residual categories for Grassmannians*, Proceedings of the London Mathematical Society **120** (2020), no. 5, 617–641.

[16] ______, *Residual categories for (co)adjoint Grassmannians in classical types*, Compositio Mathematica **157** (2021), no. 6, 1172–1206. See also arXiv:2001.04148.

[17] L. Manivel, *On the derived category of the Cayley plane*, J. Algebra **330** (2011), 177–187. See also arXiv:0907.2784.

[18] A. Novikov, *Lefcshetz Exceptional Collections on Isotropic Grassmannians* (2020). Master thesis at the Higher School of Economics, Moscow. Available at `https://www.hse.ru/en/edu/vkr/369850047`.

[19] C. Pech, *Quantum cohomology of the odd symplectic Grassmannian of lines*, J. Algebra **375** (2013), 188–215.

[20] N. Perrin and M. Smirnov, *On the big quantum cohomology of coadjoint varieties*. Available at arXiv:2112.12436.

[21] A. Polishchuk and A. Samokhin, *Full exceptional collections on the Lagrangian Grassmannians $LG(4,8)$ and $LG(5,10)$*, Journal of Geometry and Physics **61** (2011), no. 10, 1996–2014.

[22] P. Seidel, *More about vanishing cycles and mutation*, Symplectic geometry and mirror symmetry (Seoul, 2000), 2001, pp. 429–465.

[23] M. Smirnov, *On the derived category of the adjoint Grassmannian of type F*. Available at arXiv:2107.07814.

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