Bunches

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What follows is a summary of a recent paper by Berchtold and Hausen [BH03] in which the language of bunches applied to toric varieties is presented and investigated. I would like to thank Jarosław Wiśniewski for drawing this paper to my attention and helping me appreciate the methods described.

In order to motivate the definitions, we first present an example in which the weighted projected space $\mathbb{P}(w_1, \ldots, w_n)$ is constructed.

1 Motivating Example

Take $K = \mathbb{Z}, n \in \mathbb{N}$, and $E = \mathbb{Z}^n$ with standard basis $\{e_1, \ldots, e_n\}$. Let $\gamma = \text{Cone}\{e_1, \ldots, e_n\}$ be the first quadrant in $E_{\mathbb{Q}}$. Choose $w_1, \ldots, w_n \in \mathbb{N}$ and define $Q : e_i \mapsto w_i$. We have (see Definitions 4 and 5) that $\Theta = \{\mathbb{Q}_{\geq 0}\} \subset K_{\mathbb{Q}}$ and $\text{cov}(\Theta) = \{\rho_1, \ldots, \rho_n\}$, where $\rho_i = \text{Cone}\, e_i$.

Consider the short exact sequence

$$0 \longrightarrow M := \ker Q \overset{\iota}{\hookrightarrow} E \xrightarrow{Q} K \longrightarrow 0. \tag{1}$$

Let $x = \sum_{i=1}^{n} a_i e_i \in E$, then

$$\begin{aligned} x \in M = \ker Q &\iff Q(\sum_{i=1}^{n} a_i e_i) = 0 \\ &\iff \sum_{i=1}^{n} a_i w_i = 0 \\ &\iff x \in \text{span}\left\{\sum_{i=1}^{n} w_i e_i\right\}^{\perp} \end{aligned}$$

and so we see that $M = \text{span}\{\sum_{i=1}^{n} w_i e_i\}^{\perp}$.

We apply $\text{Hom}(\cdot, \mathbb{Z})$ to the short exact sequence (1) to obtain the dual sequence (see Definition 7)

$$0 \longrightarrow L \cong \ker P \xrightarrow{Q^*} F \xrightarrow{P} N \longrightarrow 0.$$

We need to understand $P : \text{Hom}(E, \mathbb{Z}) \to \text{Hom}(M, \mathbb{Z})$. Let $g \in F = \text{Hom}(E, \mathbb{Z})$, so $g = \sum_{i=1}^{n} b_i e_i^*$, and let $x = \sum_{i=1}^{n} a_i e_i \in M$. Then

$$\begin{aligned} P(g)(x) &= g(\iota(x)) \in \mathbb{Z} \\ &= g(x) \\ &= \sum_{i=1}^{n} a_i b_i. \end{aligned}$$

Thus we see that

$$g \in L = \ker P \Longleftrightarrow \sum_{i=1}^{n} a_i b_i = 0 \text{ for all } x \in M = \operatorname{span}\left\{\sum_{i=1}^{n} w_i e_i\right\}^{\perp}$$
$$\Longleftrightarrow g \in \operatorname{span}\left\{\sum_{i=1}^{n} w_i e_i^*\right\}$$

and so $L = \operatorname{span}\{\sum_{i=1}^{n} w_i e_i^*\}$. Hence P is the projection of F along the line L onto N.

Define $\delta := \gamma^\vee$ and $\gamma_0^* := \gamma_0^\perp \cap \delta$ for $\gamma_0 \preceq \gamma$, and (see Lemma 2)

$$\begin{aligned}\Sigma &:= \{\sigma \preceq \delta \mid \sigma \preceq \gamma_0^* \text{ for some } \gamma_0 \in \operatorname{cov}(\Theta)\} \ \text{(a fan in } F_{\mathbb{Q}}) \\ &= \left\{sigma \preceq \delta \mid \sigma \preceq \rho_i^\perp \cap \delta \text{ for some } i\right\} \\ &= \ \text{fan generated by Cone}\left\{e_1^*, \ldots, e_{i-1}^*, e_{i+1}^*, \ldots, e_n^*\right\}_{i=1}^{n}.\end{aligned}$$

Thus we see that the fan Δ obtained from the cones (see Definition 10)

$$\{P(\sigma) \mid \sigma \text{ is a maximal cone in } \Sigma\}$$

is precisely that corresponding to the weighted projective space $\mathbb{P}(w_1, \ldots, w_n)$.

2 The Construction and Basic Equivalences

Definition 1. A *projected cone* is a pair $(E \xrightarrow{Q} K, \gamma)$ where $Q : E \to K$ is an epimorphism of lattices and $\gamma \subset E_{\mathbb{Q}}$ is a strictly convex simplicial cone of full dimension.

Definition 2. A *morphism of projected cones*$(E_i \xrightarrow{Q_i} K_i, \gamma_i)$ is a homomorphism $\Phi : E_1 \to E_2$ such that $\Phi(\gamma_1) \subset \gamma_2$ and the diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \downarrow Q_1 & & \downarrow Q_2 \\ K_1 & \xrightarrow{\bar{\Phi}} & K_2 \end{array}$$

Definition 3. A *projected face* in K is the image $Q(\gamma_0)$ of a face $\gamma_0 \preceq \gamma$.

Definition 4. A *bunch* in $(E \xrightarrow{Q} K, \gamma)$ is a collection Θ of projected faces in K such that a projected face $\tau_0 \subset K_{\mathbb{Q}}$ belongs to Θ if and only if

$$\emptyset \neq \tau_0^\circ \cap \tau^\circ \neq \tau^\circ \text{ for all } \tau \in \Theta \text{ such that } \tau \neq \tau_0 \tag{2}$$

Thus for no pair τ_0, τ with $\tau_0^\circ \subsetneq \tau^\circ$ do τ_0 and τ lie in Θ. If $\tau_0 = Q(\gamma_0)$ overlaps every cone of Θ then at least some $Q(\gamma_1)$ with $\gamma_1 \preceq \gamma_0$ belongs to Θ.

Definition 5. Let Θ be a bunch in a projected cone $(E \xrightarrow{Q} K, \gamma)$. The *covering collection* of Θ is

$$\operatorname{cov}(\Theta) := \{\gamma_0 \preceq \gamma \mid \gamma_0 \text{ minimal with } \tau \subset Q(\gamma_0) \text{ for some } \tau \in \Theta\}.$$

Definition 6. Let Θ_i be bunches in $(E_i \xrightarrow{Q_i} K_i, \gamma_i)$. A *morphism* from Θ_1 to Θ_2 is a morphism $\Phi : E_1 \to E_2$ of the projected cones such that for every $\alpha_2 \in \operatorname{cov}(\Theta_2)$ there exists $\alpha_1 \in \operatorname{cov}(\Theta_1)$ with $\Phi(\alpha_1) \subset \alpha_2$. It is an *isomorphism* if Φ is an isomorphism of the projected cones and $\bar{\Phi}$ gives a bijection $\Theta_1 \to \Theta_2$.

Definition 7. Given a projected cone $(E \xrightarrow{Q} K, \gamma)$ we have two exact sequences of lattice homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{Q} & K & \longrightarrow & 0 \\ 0 & \longrightarrow & L & \longrightarrow & F & \xrightarrow{P} & N & \longrightarrow & 0 \end{array}$$

where $M \cong \ker Q$ and the second sequence is obtained from the first by applying $\mathrm{Hom}(\cdot, \mathbb{Z})$. Let $\delta := \gamma^\vee$ denote the dual cone. The δ is a strictly convex simplicial cone of full dimension. We call $(F \xrightarrow{P} N, \delta)$ the *dual projected cone* of $(E \xrightarrow{Q} K, \gamma)$. The sets of faces of the cone $\gamma \subset E_{\mathbb{Q}}$ and $\delta \subset F_{\mathbb{Q}}$ are in order reversing correspondence via

$$\begin{array}{rcl} \mathrm{faces}(\gamma) & \to & \mathrm{faces}(\delta) \\ \gamma_0 & \mapsto & \gamma_0^* := \gamma_0^\perp \cap \delta. \end{array}$$

Lemma 1 (Invariant Separation Lemma). *Let $\gamma_1, \gamma_2 \preceq \gamma$ and $\delta_i := \gamma_i^*$. Let $L := \ker P$. There is an L-invariant separating linear form δ_1, δ_2 if and only if $Q(\gamma_1)^\circ \cap Q(\gamma_2)^\circ \neq \emptyset$.*

Definition 8. Let $(F \xrightarrow{P} N, \delta)$ be a projected cone, and let $L := \ker P$. A *projectable fan* in $(F \xrightarrow{P} N, \delta)$ is a fan Σ consisting of faces of δ such that any two maximal cones of Σ can be separated by an L-invariant linear form. We call a projectable fan Σ *maximal* if any $\delta_0 \preceq \delta$ which can be separated by L-invariant linear forms from the maximal cones of Σ belongs to Σ.

Definition 9. A *morphism of projectable fans* Σ_i in projected cones $(F_i \xrightarrow{P_i} N_i, \delta_i)$ is a morphism $\Psi : F_1 \to F_2$ of projected cones which is in addition a map of the fans Σ_i.

Lemma 2. *Let Θ be a bunch in the projected cone $(E \xrightarrow{Q} K, \gamma)$ and consider the dual projected cone $(F \xrightarrow{P} N, \delta)$. Then*

$$\Sigma := \{\sigma \preceq \delta \mid \sigma \preceq \gamma_0^* \text{ for some } \gamma_0 \in cov(\Theta)\}$$

is a maximal projectable fan in $(F \xrightarrow{P} N, \delta)$.

Lemma 3. *For every morphism $\Phi : E_1 \to E_2$ of the bunches Θ_1 and Θ_2 the dual map $\Psi : F_2 \to F_1$ is a morphism of the maximal projectable fans Σ_2 and Σ_1.*

Thus we have that the assignments $\Theta \mapsto \Sigma$ and $\Phi \mapsto \Psi$ give a contravariant functor $\mathcal{F}$ from the category of bunches to the category of maximal projectable fans.

Lemma 4. *Let Σ be a maximal projectable fan in a projected cone $(F \xrightarrow{P} N, \delta)$ and let $(E \xrightarrow{Q} K, \gamma)$ denote the associated dual projected cone. Define*

$$\Theta := \{\tau_0 \mid \tau_0 \text{ is minimal with } \tau_0 = Q(\delta_0^*) \text{ for some } \delta_0 \in \Sigma^{max}\}$$

where Σ^{max} is the set of maximal cones of Σ. Then Θ is a bunch in $(E \xrightarrow{Q} K, \gamma)$.

Thus we can see that the assignments $\Sigma \mapsto \Theta$ and $\Psi \mapsto \Phi$ define a contravariant functor $\mathcal{B}$ from the category of bunches. The functors $\mathcal{F}$ and $\mathcal{B}$ are inverse to each other. In particular, the categories of bunches and maximal projectable fans are dual to each other.

Definition 10. Let Σ be a maximal projectable fan in $(F \xrightarrow{P} N, \delta)$. The images $P(\sigma), \sigma \in \Sigma^{\max}$ are the maximal cones of a quasifan Σ' in $N_{\mathbb{Q}}$. We reduce Σ' to a fan as follows: Let $L' \subset N$ be the primitive sublattice generating the minimal cone of Σ', let $N' := \frac{N}{L'}$, and let $P' : N \to N'$ denote the projection. Then the *quotient fan* of Σ is $\Delta := \{P'(\sigma') \mid \sigma' \in \Sigma'\}$.

Note that $R := P' \circ P : F \to N'$ is a map of the fans Σ and *Delta*. Everything is compatible with morphisms, and we obtain

Proposition 1. *The assignment $\Sigma \mapsto \Delta$ defines a covariant functor $\mathcal{D}$ from the category of maximal projectable fans to the category of fans.*

Consider the composition $\mathcal{D} \circ \mathcal{F}$. Let Θ be a bunch in $(E \xrightarrow{Q} K, \gamma)$ with associated maximal projectable fan Σ in $(F \xrightarrow{P} N, \delta)$. Let Δ be the quotient fan of Σ, and $R : F \to N'$ be the projection. Then there is a canonical order reversing bijection

$$\begin{array}{rcl} \{\gamma_0 \preceq \gamma \mid \tau^\circ \subset Q(\gamma_0)^\circ \text{ for some } \tau \in \Theta\} & \to & \Delta \\ \gamma_0 & \mapsto & R(\gamma_0^*) \end{array}$$

whose inverse map is given by

$$\sigma \mapsto (R^{-1}(\sigma) \cap |\Sigma|)^*.$$

2.1 The Main Results

Let X_Δ be the toric variety associated with the fan Δ in $N_\mathbb{Q}$. In Table 1 we recall three definitions we will make use of, along with the corresponding conditions on the fan Δ.

$X := X_\Delta$	Fan Δ
X is *nondegenerate* if X admits no toric decomposition $X \cong X' \times \mathbb{K}^*$.	The support $\|\Delta\|$ generates $\mathbb{N}_\mathbb{Q}$ as a vector space.
X is 2-*complete* if X does not admit a toric open embedding $X \subset X'$ with $X' \backslash X$ nonempty of codimension at least two.	Δ cannot be enlarged without adding new rays.
X is *full* if X is 2-complete and every T_X-orbit has a fixed point in its closure.	Δ is as above, and every maximal cone of Δ is of full dimension.

Table 1: Equivalences between the toric variety X_Δ and the fan Δ.

Definition 11. A bunch Θ in a projected cone $(E \xrightarrow{Q} K, \gamma)$ is said to be *free* if γ is a regular cone in $E_\mathbb{Q}$.

Fix a projected cone $(E \xrightarrow{Q} K, \gamma)$ and let $\gamma_1, \dots, \gamma_n$ denote the facets of γ. Set

$$\begin{aligned} \nu &:= Q(\gamma) \\ \nu_i &:= Q(\gamma_i) \\ \mathbb{K}_i &:= Q(\operatorname{lin}\{\gamma_i\} \cap E) \end{aligned}$$

Definition 12. We say that a bunch Θ in $(E \xrightarrow{Q} K, \gamma)$ is a *standard bunch* if

(i) for all i, $K_i = K$

(ii) for all $\tau \in \Theta$, $\tau^\circ \cap \nu^\circ \neq \emptyset$

(iii) every nu_i contains an element $\tau \in \Theta$

(iv) for all i, j, $\nu_i^\circ \cap \nu_j^\circ \neq \emptyset$.

Definition 13. Let Θ be a standard bunch in $(E \xrightarrow{Q} K, \gamma)$ and let $\Delta := \mathcal{D}(\mathcal{F}(\Theta))$ be the quotient fan of the maximal projectable fan corresponding to Θ. The *toric variety associated to* Θ is $X_\Theta := X_\Delta$.

Theorem 1. *The assignment $\mathcal{I} : \Theta \mapsto X_\Theta$ defines a contravariant functor from the category of standard bunches to the category of nondegenerate 2-complete toric varieties. Moreover*

(i) every nondegenerate 2-complete toric variety is isomorphic to a toric variety X_Θ with standard bunch Θ

(ii) $\mathcal{I}$ induces a bijection from the isomorphism classes of free standard bunches to the isomorphism classes of nondegenerate 2-complete toric varieties with free class group.

In fact the equivalences in Table 2 can be established between the toric variety X_Θ and the standard bunch Θ.

As above let Θ be a standard bunch in a projected cone $(E \xrightarrow{Q} K, \gamma)$, and let $X := X_\Theta$ be the associated toric variety with its torus $T := T_X$. Let $(F \xrightarrow{P} N, \delta)$ be the dual projected cone, Σ the maximal projectable fan corresponding to Θ, and Δ the quotient fan of Σ. Let $v_1, \dots, v_r$ be the primitive generators of the one-dimensional cones of Δ, and let $\widehat{v_1}, \dots, \widehat{v_r}$ be the primitive generators of the rays of δ, such that $P(\widehat{v_i}) = v_i$.

Note that every ray $\widehat{\rho_i} := \mathbb{Q}_{\geq 0}\widehat{v_i}$ of δ has a unique opposite ray $\widehat{\eta_i} \in \gamma(1)$, ie. $\widehat{\eta_i}$ is the ray not contained in $\widehat{\rho_i}^\perp$. We denote the primitive generator of $\widehat{\eta_i}$ by $\widehat{w_i}$, and its unique image in K by $w_i := Q(\widehat{w_i})$.

$X := X_\Theta$	Standard Bunch Θ
$\mathbb{Q}$-Factorial	Θ consists of cones of full dimension in K
Smooth	For all $\gamma_0 \in \mathrm{cov}(\Theta)$, γ_0^* is a regular cone in F, and Q maps $\mathrm{lin}\,\{\gamma_0\} \cap E$ onto K.
$\mathcal{O}(X) = \mathbb{K}$	Q contracts no ray of γ to a point, and the image $Q(\gamma)$ is strictly convex.
Full	All the cones $Q(\gamma_0)$, where $\gamma_0 \in \mathrm{cov}(\Theta)$, are simplicial.
Complete	Θ consists of simplicial cones, and any face $\gamma_0 \preceq \gamma$ satisfying $\tau^\circ \subsetneq Q(\gamma_0)^\circ$ for some $\tau \in \Theta$ and $\gamma_1 \preceq \gamma_0$ for precisely one $\gamma_1 \in \mathrm{cov}(\Theta)$ belongs to $\mathrm{cov}(\Theta)$.

Table 2: Equivalences between the toric variety X_Θ and the standard bunch Θ.

Each ray ρ_i determines an invariant prime divisor D_i in X. There is a canonical injective mapping E into the lattice $\mathrm{WDiv}^T(X)$ of invariant Weil divisors on X given by

$$\begin{array}{rcl} \mathcal{W} : E & \to & \mathrm{WDiv}^T(X) \\ \widehat{w} & \mapsto & \mathcal{W}(\widehat{w}) := \sum_{i=1}^r \widehat{w}(\widehat{v}_i) D_i \end{array}$$

By construction, an element $u \in M$ is mapped to the principal divisor $\mathrm{div}(\chi^u)$ of X, and we obtain the commutative diagram with exact rows and injective upwards arrows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{PDiv}^T(X) & \longrightarrow & \mathrm{WDiv}^T(X) & \longrightarrow & \mathrm{ClDiv}(X) & \longrightarrow & 0 \\ & & \uparrow\cong & & \uparrow\mathcal{W} & & \uparrow\bar{\mathcal{W}} & & \\ 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

By tensoring this diagram with $\mathbb{Q}$ we obtain isomorphisms of rational vector spaces

$$\begin{array}{rcl} \mathcal{W} : \; E_\mathbb{Q} & \longrightarrow & \mathrm{WDiv}^T_\mathbb{Q}(X) \\ \bar{\mathcal{W}} : \; K_\mathbb{Q} & \longrightarrow & \mathrm{ClDiv}_\mathbb{Q}(X) \end{array}$$

Consider the group $\mathrm{Pic}_\mathbb{Q}(X) \subset \mathrm{ClDiv}_\mathbb{Q}(X)$ of rational Cartier divisor classes. Let $C^{\mathrm{sa}}(X) \subset \mathrm{Pic}_\mathbb{Q}(X)$ be the cone of semiample classes, and let $C^{\mathrm{a}}(X) \subset \mathrm{Pic}_\mathbb{Q}(X)$ be the cone of ample classes.

Theorem 2. *The map $\bar{\mathcal{W}} : K_\mathbb{Q} \to ClDiv_\mathbb{Q}(X)$ defines canonical isomorphisms*

$$Pic_\mathbb{Q}(X) \cong \bigcap_{\tau\in\Theta} lin\{\tau\}, \qquad C^{sa}(X) \cong \bigcap_{\tau\in\Theta} \tau, \qquad C^{a}(X) \cong \bigcap_{\tau\in\Theta} \tau^\circ.$$

In the case where Θ is a free standard bunch we obtain

$$K \cong \mathrm{ClDiv}(X), \qquad \mathrm{Pic}(X) \cong \bigcap_{\gamma_0\in\mathrm{cov}(\Theta)} Q(\mathrm{lin}\,\{\gamma_0\} \cap E)$$

along with the nice equivalences shown in Table 3.

$X := X_\Theta$	Free Standard Bunch Θ
$\mathbb{Q}$-Gorenstein	$\sum_{i=1}^r w_i \in \bigcap_{\tau\in\Theta} \mathrm{lin}\,\{\tau\}$
Fano	$\sum_{i=1}^r w_i \in \bigcap_{\gamma_0\in\mathrm{cov}(\Theta)} Q(\gamma_0 \cap E)^\circ$

$X := X_\Theta$ is Smooth	Free Standard Bunch Θ
Fano	$\sum_{i=1}^r w_i \in \bigcap_{\tau\in\Theta} \tau^\circ$

Table 3: Equivalences between the toric variety X_Θ and the free standard bunch Θ.

References

[BH03] F. Berchtold and J. Hausen, *Bunches of Cones in the Divisor Class Group - A New Combinatorial Language for Toric Varieties.*