WALL-CROSSING AND BIRATIONAL GEOMETRY

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Abstract. In this short survey (provided for Proceedings of Nottingham Algebraic Geometry Seminar), we review how Bridgeland wall-crossing is expected to appear as an operation in the Minimal Model Program and vice versa, and how this expectation fails in higher dimensions. This is based on the results by Bayer-Macrì, and also the author in [BM14a], [BM14b], [Rez20], and [Rez21].

1. Introduction

The Minimal Model Program (MMP) is a sequence of $\mathbb{Q}$-factorial surgeries (either divisorial or flipping contractions) on an algebraic variety X to turn it to either a Mori fiber space or a minimal model Y which is birational to X and whose canonical class K_Y is nef, i.e., for each curve C in Y, we require to have $C \cdot K_Y \geq 0$. On the other hand, (Bridgeland) wall-crossing is a sequence of operations on a variety (which appears as a moduli space of Bridgeland semistable objects) that replaces destabilized objects with newly stable objects after crossing a wall in the space of stability conditions. Starting from a moduli space of semistable objects M, the question is if there is a direct relashiomship between these two sequences of operations on M. For the case of surfaces, there are several positive answers to this question: For the case where M is a moduli space of (Bridgeland) stable objects on a K3 surface, this question was answered by Bayer and Macrì in [BM14a]; similar positive answers to the question are given by: [ABCH13, CH16, BMW14, LZ18] (for the Hilbert scheme of points on $\mathbb{P}^2$), [NY19] and [Bec20] (for Enriques surfaces) , [BC13] (for Hirzebruch and del Pezzo surfaces), and [Yos01, YY14, MM13] (for abelian surfaces). Also, for surfaces with rational curves of negative self-intersection, there is a description in [TX17]; note the stability conditions here lie at the boundary of the (α, β)-plane which is studied e.g., by Arcara and Bertram. In [Tod14], the MMP on any smooth projective surface is understood as a variation of the moduli spaces of Bridgeland stable objects, and in [Tod13], the first step of MMP on spaces of dimension up to three is understood as a variation of Bridgeland stable objects.

On the other hand, in dimension three, there is a description of a novel wall-crossing phenomenon by the author in [Rez20] that breaks this direct link between wall-crossing and the Minimal Model Program observed in all cases understood previously.

The rest of this brief survey outlines the results that follow the expectation (we only restrict to the results by Bayer and Macrì for K3 surfaces) and the counter-example in dimension three, after a preliminary brief section on Bridgeland stability conditions.

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2. Bridgeland stability conditions and wall-crossing

In order to have reasonable moduli spaces of (complexes of) sheaves, we need to restrict to "(semi)stable" objects among them. The notion of Bridgend stability conditions provides the right notion of stability for objects in a triangulated category ([Bri07]). Let X be an algebraic variety and $D^b(X)$ the derived category of coherent sheaves on X.

Definition 2.1. Let $\mathcal{A}$ be an abelian subcategory in $D^b(X)$. For a variety X, a *Bridgeland stability condition* on $D^b(X)$ is a pair $\sigma = (\mathcal{A}, Z)$, where

- $Z\colon K_0(\mathcal{A}) \to \mathbb{C}$ is a group homomorphism, called a *central charge*, where $K_0(\mathcal{A})$ is the Grothendieck group of $\mathcal{A}$, such that
  - For each non-zero object E in $\mathcal{A}$, we have $Im(Z(E)) \geq 0$ and if $Im(Z(E)) = 0$, then $Re(Z(E)) < 0$.
- $\mathcal{A}$ is a *heart of a bounded t-structure* on $D^b(X)$; i.e., it is a full additive subcategory of $D^b(X)$ such that
  - $\mathrm{Hom}(\mathrm{A[i]}, \mathrm{B[j]}) = 0$ for all $A, B \in \mathcal{A}$ and $i > j$.
  - Harder-Narasimhan property holds: For any non-zero object E in $\mathcal{A}$, there is a filtration
$$0 = E_0 \to E_1 \to E_2 \to \ldots \to E_n = E$$
    where E_i are objects in $\mathcal{A}$ and $A_i \colon = cone(E_{i-1} \to E_i)$ are semistable objects (as in Definition 2.2) with $\frac{-Re(Z)}{Im(Z)}(A_i) \geq \frac{-Re(Z)}{Im(Z)}(A_{i-1})$ for each i.
- For any non-zero object E in the heart, we have $Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$, where $\mathbb{H}$ is the upper half plane in $\mathbb{C}$.
- A "support property" holds.

Definition 2.2. An object $E \in \mathcal{A}$ is $\sigma-$*(semi)stable*, if $\frac{-Re(Z)}{Im(Z)}(F)(\leq) < \frac{-Re(Z)}{Im(Z)}(E)$ for any sub-object F of E.

Notice that the support property is rather technical, so we don't give a precise definition of it here. For our purpose, this property guarantees that the space of all stability conditions on X, $Stab(X)$ (which is a complex manifold by Bridgeland's deformation theorem, [Bri07]) admits a *wall-chamber decomposition*, depending on a fixed class v (e.g., Chern character), such that:

(i) *Walls* (which are locally finite codimension one submanifolds of $Stab(X)$) consist of stability conditions with strictly semistable objects of class v, and

(ii) The complement of walls in $Stab(X)$ are called *chambers*, and for a chamber C in $Stab(X)$, the moduli space of σ-semistable objects of class v, $\mathcal{M}_\sigma(v)$ is independent of the choice of $\sigma \in C$.

Notice that crossing each wall between two adjacent chambers in $Stab(X)$ changes the moduli space of semistable objects.

Definition 2.3. Crossing a wall between two adjacent chambers in $Stab(X)$ (corresponding to the moduli spaces $\mathcal{M}_{\sigma_-}(v), \mathcal{M}_{\sigma_+}(v)$) is called a *wall-crossing* $\mathcal{M}_{\sigma_-}(v) \dashrightarrow \mathcal{M}_{\sigma_+}(v)$, with respect to Bridgeland stability conditions. For the sake of simplicity, sometimes we call this a *Bridgeland wall-crossing*.

Remark 2.4. Existence of Bridgeland stability conditions and wall-chamber decomposition for K3 surfaces is given in [Bri08]. For threefolds this is conjectured in [BMT14] (and for the case of $\mathbb{P}^3$, the conjecture is proved in e.g., [Mac14], [BMS16] and [Li19]). The stability conditions we consider in what follows are based on these results.

3. Bayer-Macrì construction for surfaces

Let S be a K3 surface, and v a primitive algebraic class in the Mukai lattice with self-intersection with respect to the Mukai pairing $v^2 > 0$. Let $\mathrm{Stab}(S)$ be the space of stability conditions on S, and $\mathrm{Stab}^\dagger(S)$ the connected component of it which admits a chamber decomposition ([Bri08]). The following theorem by Bayer and Macrì leads to the implication below:

Wall-crossing $\Rightarrow$ MMP operation.

Theorem 3.1 ([BM14a, Theorem 1.1])**.** *Let σ, σ' be generic stability conditions with respect to v. Then the two moduli spaces $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_{\sigma'}(v)$ of Bridgeland-stable objects are birational to each other.*

As a consequence of Theorem 3.1, the Néron-Severi groups of $\mathcal{M}_\sigma(v)$ and $\mathcal{M}_{\sigma'}(v)$ can be canonically identified. Let C be a chamber in $\mathrm{Stab}^\dagger(S)$. The main result of [BM14b] gives a natural map $l_C\colon C \to \mathrm{NS}(\mathcal{M}_C(v))$. The global behavior of l_C is as below, which leads to the following implication:

MMP operation $\Rightarrow$ Wall-crossing.

Theorem 3.2 ([BM14a, Theorem 1.2])**.** *Fix $\sigma \in \mathrm{Stab}^\dagger(S)$. Under the identification of the Néron-Severi groups, the maps l_C glue to a piece-wise analytic continuous map $l\colon \mathrm{Stab}^\dagger(S) \to \mathrm{NS}(\mathcal{M}_\sigma(v))$. For any generic $\sigma' \in \mathrm{Stab}(S)^\dagger$, the moduli space $\mathcal{M}_{\sigma'}(v)$ is the birational model corresponding to $l(\sigma')$. In particular, every smooth K-trivial birational model of $\mathcal{M}_\sigma(v)$ appears as a moduli space $\mathcal{M}_C(v)$ of Bridgeland stable objects for some chamber $C \subset \mathrm{Stab}^\dagger(S)$.*

4. counter-example in dimension three

The main result of [Rez20] gives an example of a wall-crossing in $\mathrm{Stab}(\mathbb{P}^3)$ which induces a non-$\mathbb{Q}$-factorial model; hence it cannot induce an operation in the minimal model program:

Wall-crossing $\not\Rightarrow$ MMP operation.

Theorem 4.1 ([Rez20, Theorem 1.1])**.** *Fix $v = (1, 0, -6, 15)$. There is a wall-crossing with respect to Bridgeland stability conditions $\mathcal{M}_{\sigma_-}(v) \to \mathcal{M}_{\sigma_+}(v)$ with the following properties:*

- *$\mathcal{M}_{\sigma_-}(v)$ is a 24-dimensional smooth and irreducible variety,*
- *$\mathcal{M}_{\sigma_+}(v) = \widetilde{\mathcal{M}_{\sigma_-}(v)} \cup \mathcal{M}'$, where $\widetilde{\mathcal{M}_{\sigma_-}(v)}$ is birational to $\mathcal{M}_{\sigma_-}(v)$ and $\mathcal{M}'$ is a new irreducible component of dimension 28,*
- *There is a diagram*

$$\begin{tikzcd}
\mathcal{M}_{\sigma_-}(v) \arrow[dr, "\text{small contraction } (\phi)"'] & & \widetilde{\mathcal{M}_{\sigma_-}(v)} \arrow[dl, "\text{divisorial contraction } (\psi)"] \\
& \mathcal{M}_{\sigma_0}(v) &
\end{tikzcd}$$

where both ϕ and ψ have relative Picard rank 1. In particular, $\widetilde{\mathcal{M}_{\sigma_-}(v)}$ is not $\mathbb{Q}$-factorial.

As for the other direction, as in [Rez20], we use "flip" for the small birational map with respect to the relevant small contractions ϕ. Below is an example for non-existence of a flip as an associated wall-crossing in the stability space. This implies the following non-implication as a corollary of Theorem 4.1:

MMP operation $\not\Rightarrow$ Wall-crossing.

Corollary 4.2. *The flip of $\mathcal{M}_{\sigma_-}(v)$ (with respect to ϕ) is not induced by the associated wall inducing ϕ.*

Proof. The wall in Theorem 4.1 which induces the small contraction ϕ produces either $\widetilde{\mathcal{M}_{\sigma_-}(v)}$ which is not the desired flip (as we have the divisorial contraction ψ instead of a small contraction; therefore it induces a different birational transformation), or $\mathcal{M}'$ which is of the wrong dimension; hence this wall does not produce the desired flip. □

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