MASSEY PRODUCTS AND FUJITA DECOMPOSITION

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1. Basic Literature on Massey products and Fujita decomposition

We give an overview of the topics covered in the seminar "Fujita decomposition and Massey product for fibered varieties" held online on February 18, 2021. The presented results are mainly based on the work [RZ4].

Let $f\colon X \to B$ be a semistable fibration where X is an n-dimensional smooth complex variety and B a smooth complex curve. The study of the *relative dualizing sheaf* $\omega_{X/B} := \omega_X \otimes f^*\omega_B^\vee$ is of great interest and, by a famous result of Fujita, see [Fu1] and [Fu2], its direct image $f_*\omega_{X/B}$ has two splittings classically known as *first and second Fujita decomposition.* They are respectively

$$f_*\omega_{X/B} \cong \mathcal{O}_B^h \oplus \mathcal{E} \tag{1.1}$$

where $\mathcal{E}$ is a locally free nef sheaf on B with $h^1(B, \mathcal{E} \otimes \omega_B) = 0$ and

$$f_*\omega_{X/B} \cong \mathcal{U} \oplus \mathcal{A} \tag{1.2}$$

where $\mathcal{U}$ is a unitary flat vector bundle and $\mathcal{A}$ ample.

Fujita conjectured that $f_*\omega_{X/B}$ is always semiample, but in [CD1], [CD2], the authors prove that this in not the case and this direct image does not need to be semiample. Nevertheless it is still interesting to find conditions that ensures the semiampleness of $f_*\omega_{X/B}$. In [PT], [RZ4] this study is done by means of Massey products.

We recall that Massey products, originally called adjoint forms, have been introduced in [CP] and [PZ] and then applied in [Ra], [PR], [CNP], [G-A], [BGN], [RZ1], [RZ2], [RZ3] and recently [CRZ], [R]. They are useful for the study of infinitesimal deformations but here in this paper we show how they can be used for the study of interesting local systems of differential forms and how these local systems are related to the conjecture by Fujita.

A key reference on the higher direct images of the relative dualizing sheaf is also [K]. For other examples and applications where the unitary part of the Fujita decomposition is non trivial, see [X], [FGP], and [P] for a new perspective related to algebraic cycles.

2. Massey products and local systems

2.1. **Local systems of certain liftable holomorphic forms.** In this first section we briefly recall and discuss the main constructions of [RZ4]. Let $f\colon X \to B$ be a semistable fibration as above. We denote by B_0 the locus of singular values of f and by $B^0 = B \setminus B_0$ the open set of regular values. Consider the exact sequence

$$0 \to f^*\omega_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0 \tag{2.1}$$

defining the sheaf of relative differentials $\Omega^1_{X/B}$. It is not difficult to see that, under our hypothesis on f, the sheaf $\Omega^1_{X/B}$ is torsion free but not locally free in general.

The p-wedge of Sequence (2.1) is

$$0 \to f^*\omega_B \otimes \Omega^{p-1}_{X/B} \to \Omega^p_X \to \Omega^p_{X/B} \to 0 \tag{2.2}$$

which is also exact for f semistable. Note that for $p = n-1$, the sheaf $\Omega^{n-1}_{X/B}$ injects into the relative dualizing sheaf of the fibration $\omega_{X/B} := \omega_X \otimes f^*T_B$; furthermore their direct images via f are isomorphic when restricted to B^0.

Taking the direct image of Sequence (2.1) we obtain the long exact sequence on B

$$0 \to \omega_B \to f_*\Omega^1_X \to f_*\Omega^1_{X/B} \to R^1f_*\mathcal{O}_X \otimes \omega_B \to \dots \tag{2.3}$$

and we call K_∂ the cokernel

$$0 \to \omega_B \to f_*\Omega^1_X \to K_\partial \to 0. \tag{2.4}$$

Intuitively we can think of K_∂ as the sheaf of holomorphic one forms on the fibers of f which are locally liftable to the variety X. A key property of K_∂ is given in the following Lemma, see [PT, Lemma 3.5] or [RZ4, Lemma 2.2]

Lemma 2.1. *If $f\colon X \to B$ is a semistable fibration, the exact sequence*

$$0 \to \omega_B \to f_*\Omega^1_X \to K_\partial \to 0 \tag{2.5}$$

splits.

This means that the above intuitive idea is not only true locally around the fibers, but the liftability holds on every open subset of B. For a more complete study of K_∂ see [PT], [GST], [GT] for the case $n = 2$, [RZ4] for the general case.

If in Sequence (2.4) we consider, instead of Ω^1_X, the sheaf $\Omega^1_{X,d}$ of de Rham closed differential forms, we obtain the exact sequence

$$0 \to \omega_B \to f_*\Omega^1_{X,d} \to \mathbb{D}^1 \to 0. \tag{2.6}$$

It turns out that $\mathbb{D}^1$ is a local system on the curve B as shown in [PT] for 1-dimensional fibers and in [RZ4] for any dimension. Note that $\mathbb{D}^1$ is a subsheaf of K_∂ and we can interpret $\mathbb{D}^1$ as the local system of holomorphic one forms on the fibers of f which are liftable to *closed* holomorphic forms of the variety X. Finally note that by Lemma 2.1 also the exact Sequence (2.6) splits.

Analogously, one can define the local systems $\mathbb{D}^p$ in the following way. Take the direct image via f_* of the exact sequence (2.2) and obtain the long exact sequence

$$0 \to \omega_B \otimes f_*\Omega^{p-1}_{X/B} \to f_*\Omega^p_X \to f_*\Omega^p_{X/B} \to R^1f_*\Omega^{p-1}_{X/B} \otimes \omega_B \to \dots \tag{2.7}$$

We define $\mathbb{D}^p$ by the exact sequence

$$0 \to \omega_B \otimes f_*\Omega^{p-1}_{X/B} \to f_*\Omega^p_{X,d} \to \mathbb{D}^p \to 0 \tag{2.8}$$

where once again $\Omega^p_{X,d}$ denotes the sheaf of de Rham closed p-forms. As in the case of $p = 1$, the sheaves $\mathbb{D}^p$ are local systems of holomorphic p-forms on the fibers of f which are liftable to *closed* holomorphic p-forms on X.

The local system $\mathbb{D}^{n-1}$ is of particular interest because it turns out to be the local system of the second Fujita decomposition of $f_*\omega_{X/B}$. In fact in the second Fujita decomposition (1.2)

$$f_*\omega_{X/B} = \mathcal{A} \oplus \mathcal{U}$$

we have that $\mathcal{U} = \mathbb{D}^{n-1} \otimes \mathcal{O}_B$, see [RZ4, Theorem 3.7] for the proof.

2.2. **Massey products.** We now recall the construction of *Massey product or adjoint form*. This presentation is slightly different but equivalent to the one in [RZ4], and it will be more convenient for some future applications.

According to Lemma (2.1), Sequence (2.4) splits; from now on we choose and fix one of this splittings. The following wedge sequence also splits

$$0 \longrightarrow \bigwedge^{n-1} K_\partial \otimes \omega_B \longrightarrow \bigwedge^n f_*\Omega^1_X \longrightarrow \bigwedge^n K_\partial \longrightarrow 0 \tag{2.9}$$

and we take the composition of this splitting with the natural wedge map and obtain the morphism

$$\lambda\colon \bigwedge^n K_\partial \to \bigwedge^n f_*\Omega^1_X \to f_* \bigwedge^n \Omega^1_X = f_*\omega_X. \tag{2.10}$$

Now consider n sections $\eta_1, \dots, \eta_n \in \Gamma(A, K_\partial)$ on an open subset $A \subset B$; call $s_1, \dots, s_n \in \Gamma(A, f_*\Omega^1_{X,d})$ liftings of $\eta_1, \dots, \eta_n$ according to the above chosen splitting.

Definition 2.2. We call ω_i, $i = 1, \dots, n$, the wedge $s_1 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_n \in \Gamma(A, f_*\Omega^{n-1}_X)$ and $\mathcal{W}$ the $f_*\omega_X$-submodule generated by $\langle \omega_i \rangle \otimes \omega_B$.

Definition 2.3. The *Massey product or adjoint image* of $\eta_1, \dots, \eta_n$ is the section of $f_*\omega_X$ computed by $\lambda(\eta_1 \wedge ... \wedge \eta_n)$. We say that the sections $\eta_1, \dots, \eta_n$ are *Massey trivial* if their Massey product is contained in the submodule $\mathcal{W}$.

Remark 2.4. The Massey product is given explicitly by $s_1 \wedge \cdots \wedge s_n$ and being Massey trivial means that

$$s_1 \wedge \cdots \wedge s_n = \sum_i \omega_i \otimes \sigma_i$$

where the ω_i are as in Definition 2.2 and σ_i are sections of ω_B.

As a section of $f_*\omega_X$, the Massey product certainly depends on the choice of the splitting mentioned above. On the other hand, the condition of being Massey trivial does not; see [RZ4]. In Proposition 2.7 we will show that if the sections $\eta_1, \dots, \eta_n$ are Massey trivial, there is a very convenient choice for this splitting.

In the literature mentioned at the beginning, the construction of Massey products is done pointwise, that is for a fixed regular value $b \in B$ and working only on the fiber X_b and on an infinitesimal neighborhood of this fiber. It is not difficult to see that all the pointwise defined Massey products can be glued together and this agrees exactly with Definition 2.3 on suitable open subsets $A \subset B$.

Of course since $\mathbb{D}^1$ is a subsheaf of K_∂, it makes sense to construct Massey products starting from sections of $\mathbb{D}^1$, i.e. consider sections $\eta_i \in \Gamma(A, \mathbb{D}^1)$. One of the key points in [PT] and [RZ4] is exactly to consider this setting.

To conclude this section we recall the notion of strictness and its relation with Massey triviality. Let $A \subset B$ a contractible open subset and $W \subset \Gamma(A, K_\partial)$ a vector subspace of dimension at least n.

We give the following definition

Definition 2.5. We say that W is Massey trivial if any n-uple of sections in W is Massey trivial (according to Definition 2.3).

Definition 2.6. We say that W is strict if the map

$$\bigwedge^{n-1} W \otimes \omega_{B_{|A}} \to f_*\omega_{X_{|A}}$$

is an injection of sheaves.

The following proposition shows how Massey triviality and strictness give a preferred choice of liftings as we anticipated in Remark 2.4.

Proposition 2.7. *Let $W \subset \Gamma(B, K_\partial)$ be a strict subspace of global sections of K_∂ and let $A \subset B$ be an open contractible subset. If the sections of W are Massey trivial when restricted to A then there exist a unique lifting $\widetilde{W} \subset \Gamma(B, f_*\Omega^1_X)$ such that*

$$\bigwedge^{n} \widetilde{W} \to \Gamma(B, f_*\omega_X)$$

is zero. If furthermore $W \subset \Gamma(B, \mathbb{D}^1)$ then $\widetilde{W} \subset \Gamma(B, f_*\Omega^1_{X,d})$.

For the proof see [RZ4, Proposition 4.10]. This means that if the sections $\eta_i \in W$ are Massey trivial, for any choice of liftings s_i we have a relation of the form

$$s_1 \wedge \cdots \wedge s_n = \sum_i \omega_i \otimes \sigma_i$$

as seen in Remark 2.4. But actually there is a preferred choice of liftings $\tilde{s}_i$ such that

$$\tilde{s}_1 \wedge \cdots \wedge \tilde{s}_n = 0.$$

Remark 2.8. Given an n-dimensional variety Y, a subspace W of $H^0(Y, \Omega^1_Y)$ is usually called strict if the natural map from $\bigwedge^n W$ to $H^0(Y, \omega_Y)$ is an isomorphism on the image. See [Ca, Definition 2.1 and 2.2] and [RZ1, Definition 2.2.1].

3. Main results

In this section we recall the two main results of [RZ4].

Fist recall that a *higher irrational pencil* is a morphism with connected fibers $X \to Y$ with target a normal variety Y of maximal Albanese dimension and irregularity greater than its dimension.

Let $W \subset \Gamma(A, \mathbb{D}^1)$ be a Massey trivial subspace. Since $\mathbb{D}^1$ is a local system, it is associated to a monodromy representation. Call H its kernel, it is a normal subgroup of $\pi_1(B, b)$, and call H_W the subgroup of $\pi_1(B, b)$ which acts trivially on W. For every subgroup $K < H_W$, we denote by $B_K \to B$ the étale base change of group K and by $X_K \to B_K$ the associated fibration. We stress that in principle B_K is not necessarily compact. The following theorem is a refinement of the generalized Castelnuovo-de Franchis [Ca, Theorem 1.14]

Theorem [A]. *Let $X \to B$ be a semistable fibration with $\dim X = n$. Let $A \subset B$ be an open subset and $W \subset \Gamma(A, \mathbb{D}^1)$ a Massey trivial strict subspace. Then X_K has a higher irrational pencil $h_K \colon X_K \to Y$ over a normal $(n-1)$-dimensional variety of general type Y such that $W \subset h_K^*(H^0(Y, \Omega^1_Y))$. Furthermore if W is maximal inside $\Gamma(A, \mathbb{D}^1)$ with these properties we have the equality $W = h_K^*(H^0(Y, \Omega^1_Y))$.*

Proof. We give an idea of the proof for $K = H_W$. Since W is Massey trivial by hypothesis we can find by Proposition 2.7 liftings of the sections of W such that all the n-wedges are zero. Furthermore, up to the appropriate base change $B_K \to B$ these liftings are global one forms of X_K. Here the generalized Castelnuovo-de Franchis [Ca, Theorem 1.14] comes into play and gives the higher irrational pencil $X_K \to Y$. We point out that in this proof it is essential that $W \subset \Gamma(A, \mathbb{D}^1)$. In fact this ensures that the liftings are *closed* differential forms, hence [Ca, Theorem 1.14] can be applied even if we do not know in principle that X_K is compact.

See [RZ4, Theorem 5.8] for details. □

The second main result is the study of the monodromy of a local system generated by a Massey trivial vector space W. Consider $W < \Gamma(A, \mathbb{D}^1)$ a Massey trivial subspace. We can define the local system $\mathbb{W}$ generated by W under the monodromy action of the base B. Call $\rho_{\mathbb{W}}$ the action of the fundamental group $\pi_1(B, b)$ on the stalk of $\mathbb{W}$ and call $G_{\mathbb{W}} = \pi_1(B, b)/\ker \rho_{\mathbb{W}}$ the monodromy group. We construct an action of this group on a suitable set $\mathcal{K}$ of morphisms from the general fiber F to an $(n-1)$-dimensional variety of general type Y as in Theorem [A] and thanks to this action we prove

Theorem [B]. *Let $f\colon X \to B$ be a semistable fibration on a smooth projective curve B and let $\mathbb{W} < \mathbb{D}$ be a strict Massey trivial generated local system. Then the associated monodromy group $G_{\mathbb{W}}$ is finite and the fiber of $\mathbb{W}$ is isomorphic to*

$$\sum_{k \in \mathcal{K}} k^* H^0(Y, \Omega^1_Y).$$

Proof. We give an idea of the proof.

Let $u_{\mathbb{W}}\colon B_{\mathbb{W}} \to B$ the covering classified by the subgroup $\ker \rho_{\mathbb{W}}$ and $f_{\mathbb{W}}\colon X_{\mathbb{W}} \to B_{\mathbb{W}}$ the associated fibration.

By Theorem [A] applied to the subgroup $\ker \rho_{\mathbb{W}}$ of H_W we get a map $h\colon X_{\mathbb{W}} \to Y$ which can be composed with the action of $G_{\mathbb{W}}$ on $X_{\mathbb{W}}$ obtained from the standard action of $G_{\mathbb{W}}$ on $B_{\mathbb{W}}$. We call h_g the composition.

(3.1)
$$\begin{tikzcd} X_{\mathbb{W}} \arrow[r, "g"] \arrow[dr, "h_g"'] & X_{\mathbb{W}} \arrow[d, "h"] \\ & Y \end{tikzcd}$$

Now we take a regular point $b \in A \subset B$ and a preimage b_0 of b via $B_{\mathbb{W}} \to B$. Denote by F_0 the fibre of $f_{\mathbb{W}}$ over b_0 and consider the above diagram restricted to F_0.

(3.2)
$$\begin{tikzcd} F_0 \arrow[r, hook] \arrow[drr, "k_g"'] & X_{\mathbb{W}} \arrow[r, "g"] \arrow[dr, "h_g"] & X_{\mathbb{W}} \arrow[d, "h"] \\ & & Y \end{tikzcd}$$

It is not difficult to see that the maps k_g are surjective.

The idea of the proof is that we have an obvious action of the monodromy group on the set

$$\mathcal{K} = \{k_g\colon F_0 \to Y \mid g \in G_{\mathbb{W}}\}$$

which is a finite set because Y is of general type and the maps k_g are surjective. Here is the main difference in the proof of the analogous statement in [PT, Theorem 1.1]. This action turns out to be faithful, hence the monodromy group injects in the finite set $Aut(\mathcal{K})$ and therefore it is finite. See [RZ4, Section 5.3] for the complete proof. □

As a corollary we obtain the following result on the monodromy of $\mathbb{D}^1$ and $\mathbb{D}^{n-1}$

Corollary 3.1. *If $\mathbb{D}^1$ is Massey trivial generated by a strict subspace, then its monodromy group is finite. If furthermore the map $\bigwedge^{n-1}\mathbb{D}^1 \to \mathbb{D}^{n-1}$ is surjective, the local system $\mathbb{D}^{n-1}$ also has finite monodromy.*

Recall that the finiteness of the monodromy group of a local system is equivalent to the semi-ampleness of the unitary flat vector bundle associated, see for example [CD1, Theorem 2.5], hence since $\mathbb{D}^{n-1}$ is the local system associated to $\mathcal{U}$, $\mathcal{U} = \mathbb{D}^{n-1} \otimes \mathcal{O}_B$, this corollary is indeed a result on the semi-ampleness of $\mathcal{U}$ and hence of $f_*\omega_{X/B}$ since $f_*\omega_{X/B} = \mathcal{U} \oplus \mathcal{A}$ and $\mathcal{A}$ is ample. That is, we can rewrite it as

Corollary 3.2. *If $\mathbb{D}^1$ is Massey trivial generated by a strict subspace and furthermore the map $\bigwedge^{n-1}\mathbb{D}^1 \to \mathbb{D}^{n-1}$ is surjective, $f_*\omega_{X/B}$ is semiample.*

This holds for example when the fiber of $X \to B$ is a hyperellyptic curve or, in higher dimension, is an odd dimensional variety with an involution, see [PT] and [RZ4] for details.

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