CONSTRUCTION OF NON-KÄHLER CALABI-YAU MANIFOLDS BY LOG DEFORMATIONS

TARO SANO

ABSTRACT. We survey recent construction of non-Kähler Calabi-Yau manifolds by smoothing SNC varieties obtained by non-trivial isomorphisms of strict Calabi-Yau manifolds. We also give a new example by smoothing an SNC 3-fold which are constructed from automorphisms of an abelian surface.

1. INTRODUCTION

In this note, we survey the recent construction of non-Kähler Calabi-Yau manifolds by using log deformation theory of simple normal crossing (SNC) varieties (cf. [HS19], [San21]). A compact complex manifold X of dimension n is called a *(strict) Calabi-Yau manifold* (*CY n-fold*) if $\omega_X \simeq \mathcal{O}_X$ and $H^i(X, \mathcal{O}_X) = 0 = H^0(X, \Omega^i_X)$ for $0 < i < \dim X$.

Projective Calabi-Yau manifolds are one of building blocks in the classification of algebraic varieties. Note that 1-dimensional Calabi-Yau manifolds are elliptic curves and 2-dimensional Calabi-Yau manifolds are K3 surfaces. It is an open problem whether there are only finitely many topological types of projective Calabi-Yau manifolds in a fixed dimension $N \geq 3$.

On the other hand, it had been known that there are infinitely many topological types of non-Kähler Calabi-Yau 3-folds (cf. [Fri91], [HS19] and references therein).

Our examples in [San21] are the following.

Theorem 1.1. *Let $N \geq 3$ and $m \in \mathbb{Z}_{>0}$.*

There exists a non-Kähler Calabi–Yau N-fold $X(m)$ with the following properties

(i) *The 2nd Betti number is* $b_2(X(m)) = \begin{cases} m+1 & (N=3) \\ m+10 & (N=4) \\ m+2 & (N \geq 5) \end{cases}$.

(ii) *The algebraic dimension of $X(m)$ is $N-2$ and $X(m)$ admits a K3 fibration $X(m) \to T$ to a smooth rational variety T.*

(iii) *The Hodge to de Rham spectral sequence on $X(m)$ degenerates at E_1.*

An essential ingredient of the construction is the log deformation theory of normal crossing varieties developed by Kawamata–Namikawa (See also recent results by [CLM19] and [FFR21]).

Remark 1.2. Lee [Lee21] constructed an example of a non-Kähler Calabi-Yau 4-fold by smoothing SNC varieties.

Remark 1.3. Note that, if X and Y are Calabi–Yau manifolds such that $\dim X, \dim Y > 0$, then $X \times Y$ also has a trivial canonical bundle, but it does not satisfy $h^i(\mathcal{O}_{X \times Y}) = 0$ for $0 < i < \dim(X \times Y)$.

Let us explain the contents of this note. In section 2, we explain the smoothing result of an SNC CY variety and give a brief review on previous examples of non-Kähler CY 3-folds. In section 3, we give a brief explanation on the examples in [San21]. In section 4, we give further examples of non-Kähler CY 3-folds from an SNC CY 3-folds which are union of two smooth varieties glued along an abelian surface. We also give some comments on problems.

2. Log deformation theory of SNC Calabi–Yau varieties

In this section, we shall explain log deformation theory. We say that a proper SNC $\mathbb{C}$-scheme X is an *SNC Calabi–Yau variety* if $\omega_X \simeq \mathcal{O}_X$. The following is an essential ingredient.

Theorem 2.1. *(*[KN94], [CLM19]*) Let X be an SNC Calabi–Yau variety. Assume that X is "d-semistable".*

Then X admits a deformation $\phi \colon \mathcal{X} \to \Delta^1$ over a unit disk Δ^1 such that $\mathcal{X}$ is smooth and $\mathcal{X}_t := \phi^{-1}(t)$ is smooth for $t \neq 0$.

Remark 2.2. Let us recall the definition of d-semistability [Fri83, Definition 1.13] in a simple case. Let $X = X_1 \cup X_2$ be an SNC variety which is a union of two smooth varieties X_1 and X_2. Let $D := X_1 \cap X_2$ be the singular locus of X. Then X is d-semistable if $\mathcal{N}_{D/X_1} \otimes \mathcal{N}_{D/X_2} \simeq \mathcal{O}_D$. It is also equivalent to the existence of a log smooth structure on X (cf. [KN94], [Kat96]).

In the above theorem, if $\dim X \geq 3$, $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$ and X is projective, then the general fiber $\mathcal{X}_t$ is a projective CY manifold by $H^2(X, \mathcal{O}_X) = 0$ and Grothendieck's existence theorem. Note that Theorem 2.1 holds even when X is not projective. Then the smoothing $\mathcal{X}_t$ can be non-projective in general.

Example 2.3. We can construct an SNC variety $X_0 = X_1 \cup X_2$ by gluing smooth varieties X_1 and X_2 with smooth divisors $D_i \subset X_i$ for $i = 1, 2$ with an isomorphism $\phi \colon D_1 \xrightarrow{\simeq} D_2$. If $D_i \in |-K_{X_i}|$ for $i = 1, 2$ and they are connected, then we see that X_0 satisfies that $\omega_{X_0} \simeq \mathcal{O}_{X_0}$, thus X_0 is an SNC CY. We write $X_0 =: X_1 \cup^{\phi} X_2$.

Note that we really need the connectedness assumption. Indeed, if E is an elliptic curve and $X_1 = X_2 = \mathbb{P}^1 \times E$, then we can glue X_1 and X_2 along $\{0, \infty\} \times E$ via the identity $\mathrm{id} \in \mathrm{Aut}(0 \times E)$ and the negation $\phi_{-1} \in \mathrm{Aut}(\infty \times E)$. Then we see that $X_0 := X_1 \cup X_2$ satisfies that $\omega_{X_0} \not\simeq \mathcal{O}_{X_0}$ since we compute $H^0(\omega_{X_0}) = 0$ by the exact sequence

$$0 \to \omega_{X_0} \to \omega_{X_1}(D_1) \oplus \omega_{X_2}(D_2) \xrightarrow{(r_1, -r_2)} \omega_{D_1} \to 0,$$

where r_1 is the usual residue map and r_2 is a composition of the residue map and ϕ^* for $\phi\colon D_1 \xrightarrow{\sim} D_2$.

We give a review on non-Kähler Calabi-Yau 3-folds constructed in 90's.

Example 2.4. We can construct infinitely many topological types of non-Kähler CY 3-folds as analytic flops of smooth rational curves of large degrees on a projective CY 3-fold. Let us briefly recall the construction in the following (cf. [Cle83], [Ogu94]).

Let $S = (q = 0) \subset \mathbb{P}^3$ be a smooth quartic surface defined by $q \in |\mathcal{O}_{\mathbb{P}^3}(4)|$ with a smooth rational curve $C_d \subset S$ of degree d which exists by [Mor84]. Let $l \in |\mathcal{O}_{\mathbb{P}^3}(1)|$ be a linear form and embed $\mathbb{P}^3$ as a hyperplane $(z_4 = 0) \subset \mathbb{P}^4$. Let

$$V_0 := (q \cdot l + z_4 \cdot f = 0) \subset \mathbb{P}^4$$

be a quintic hypersurface, where $f \in |\mathcal{O}_{\mathbb{P}^4}(4)|$ is a general quartic form. Then, for a general choice of l, f, we see that

$$\operatorname{Sing} V_0 = (q = l = z_4 = f = 0) \subset \mathbb{P}^4$$

and they are 16 nodes. Note that the family of hyperplanes $(az_4 - bl = 0) \subset \mathbb{P}^4$ for $[a : b] \in \mathbb{P}^1$ induces a family of quartic surfaces $\{S_{[a:b]} \mid [a : b] \in \mathbb{P}^1\}$ such that $S_{[1:0]} = S$ and $C_d \subset S$ does not lift to the deformation since f is general (cf. [Ogu94, Lemma 2.6]). Hence we see that the normal bundle $\mathcal{N}_{C_d/V_0} \simeq \mathcal{O}_{C_d}(-1)^{\oplus 2}$, that is, C_d is a $(-1,-1)$-curve on V. This implies that a general quintic 3-fold $V \subset \mathbb{P}^4$ contains a $(-1,-1)$-curve C of degree d. By taking an analytic flop $V \dashrightarrow V_d$ of C, we obtain a Moishezon CY 3-fold V_d such that $\operatorname{Pic} V_d = \mathbb{Z}H_d$, where $H_d \subset V_d$ is the strict transform of the hyperplane $H = \mathcal{O}_V(1)$. We check that $H_d^3 = 5 - d^3$, thus we see that V_d is non-Kähler when $d \geq 2$. This implies that $\{V_d \mid d \geq 2\}$ provides infinitely many topological types of non-Kähler CY 3-folds since the cubic forms on $H^2(V_d, \mathbb{Z})$ are different.

Clemens, Friedman and Reid ([Cle83], [Fri91], [Rei87]) constructed infinitely many topological types of non-Kähler CY 3-folds as conifold transitions of a general smooth quintic 3-fold V with infinitely many pairwise disjoint $(-1,-1)$-curves $\Gamma_1, \Gamma_2, \cdots$. We give a review on the construction in the following.

Let $V := (ql + z_4 f = 0) \subset \mathbb{P}^4$ be a quintic 3-fold as above. Let $D_i \subset |\mathcal{O}_{\mathbb{P}^3}(4)|$ be the divisor which contains the locus of smooth quartic surfaces with (-2)-curves of degrees i (For example, the image of the corresponding flag Hilbert schme). The family of quartic surfaces $\{S_{[a:b]} \mid [a : b] \in \mathbb{P}^1\}$ induces a curve

$$\gamma = \gamma_{q,l,f}\colon \mathbb{P}^1 \to |\mathcal{O}_{\mathbb{P}^3}(4)|$$

of degree 5. For a general choice of q, l, f, we see that γ and D_i intersects (transversely) at (distinct) points which are not on any $D_j (j \neq i)$ and correspond to smooth quartic surfaces. We can pick distinct $[a_i\colon b_i] \in \mathbb{P}^1$ so that the corresponding quartic surface $S_{[a_i:b_i]}$ contains a (-2)-curve Γ_i of degree i. By choosing general l and f, we may take Γ_i which avoids the nodes of V. Hence $\{\Gamma_i \mid i = 1, 2, \ldots\}$ are pairwise disjoint. By smoothing the nodes, we see that a general quintic 3-fold V contains pairwise disjoint smooth rational curves Γ_i of degrees i for $i = 1, 2, \ldots$.

Let $m \in \mathbb{Z}_{>0}$ be any positive integer and choose $\Gamma_1, \ldots, \Gamma_m \subset V$. Then we have a birational contraction $\mu\colon V \to \bar{V}_m$ of $\Gamma_1, \ldots, \Gamma_m$ to a Moishezon space $\bar{V}_m$. Then $\bar{V}_m$ has m ordinary double points $p_i := \mu(\Gamma_i)$ and there is a smoothing of $\bar{V}_m$ to a Calabi-Yau 3-fold Y_m by [Fri86, Corollary 4.7]. Then we see that $b_2(Y_m) = 0$ and $e(Y_m) = -200 - 2m$, where $b_2(Y_m)$ is the 2nd Betti number and $e(Y_m)$ is the topological Euler number.

We are not sure whether these constructions can be generalized in higher dimensions.

3. Construction of non-Kähler Calabi-Yau manifolds

In the construction of our non-Kähler Calabi-Yau manifolds, the choice of an isomorphism $\phi\colon D_1 \to D_2$ to construct an SNC variety is crucial. We give a brief review of the examples in Theorem 1.1.

Example 3.1. Let $S := (sF_1 + tF_2 = 0) \subset \mathbb{P}^2 \times \mathbb{P}^1$ be a general hypersurface of bi-degree $(3, 1)$, where $F_1, F_2 \in |\mathcal{O}_{\mathbb{P}^2}(3)|$ are general cubics and $[s : t] \in \mathbb{P}^1$ is the homogeneous coordinate. Then we see that the 1st projection $\pi_1\colon S \to \mathbb{P}^2$ is a blow-up at 9 points $\{p_1, \ldots, p_9\} = (F_1 = F_2 = 0) \subset \mathbb{P}^2$. We also see that $\pi_2\colon S \to \mathbb{P}^1$ is an elliptic fibration, thus S is a rational elliptic surface.

A quadratic transformation $\psi_{123}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ at p_1, p_2, p_3 induces an isomorphism $\phi_{123}\colon S \xrightarrow{\simeq} S_{123}$ to another hypersurface $S_{123} \subset \mathbb{P}^2 \times \mathbb{P}^1$ of bi-degree $(3, 1)$. We can perform the same operations to construct isomorphisms

$$S \to S_{123} \to S_{456} \to S_{789} = S' \to S'_{123} \to S'_{456} \to S'_{789} =: S_1.$$

Now let $\psi_1 : S \to S_1$ be the composition of these 6 isomorphisms. Finally, for any $m \in \mathbb{Z}_{>0}$, we repeat this process m-times to obtain

$$\phi_m\colon S \xrightarrow{\psi_1} S_1 \to \cdots \to S_{m-1} \xrightarrow{\psi_m} S_m.$$

An important feature of ϕ_m is the following:

Proposition 3.2. [San21, Proposition 2.6(iii)] *Let $S, S_m \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the $(3,1)$-hypersurfaces as above. Let $H_S := \mu_S^* \mathcal{O}_{\mathbb{P}^2}(1)$ and $H_{S_m} := \mu_{S_m}^* \mathcal{O}_{\mathbb{P}^2}(1)$, where $\mu_S\colon S \to \mathbb{P}^2$ and $\mu_{S_m}\colon S_m \to \mathbb{P}^2$ are the projections. Let $\phi_m\colon S \to S_m$ be the isomorphism as above.*

Then the linear system $|H_S + \phi_m^* H_{S_m} + mK_S|$ is ample and free.

It is not clear that the isomorphism ϕ_m can be realized as an automorphism of $S \subset \mathbb{P}^2 \times \mathbb{P}^1$ or not, but this is enough for our purpose. We also use the hypersurface T as follows.

Proposition 3.3. [San21, Proposition 2.9] *Let $n \geq 2$ and let*

$$T := (sG_1 + tG_2 = 0) \subset \mathbb{P}^1 \times \mathbb{P}^n$$

be a general hypersurface of bi-degree $(1, n+1)$, where $[s\colon t] \in \mathbb{P}^1$ is the coordinate and $G_1, G_2 \in |\mathcal{O}_{\mathbb{P}^n}(n+1)|$ are general.

(i) *Then the projection $T \to \mathbb{P}^1$ is a Calabi-Yau fibration and the projection $T \to \mathbb{P}^n$ is the blow-up along the subvariety $(G_1 = G_2 = 0) \subset \mathbb{P}^n$.*

(ii) *Let*

$$D_S := S \times \mathbb{P}^n, D_T := \mathbb{P}^2 \times T \subset \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^n$$

be divisors and $D_{ST} := D_S \cap D_T$. Then D_{ST} is a projective Calabi-Yau manifold and we have an isomorphism $D_{ST} \simeq S \times_{\mathbb{P}^1} T$.

(**Construction of examples**) Let $S, S_m \subset \mathbb{P}^2 \times \mathbb{P}^1$ and $T \subset \mathbb{P}^1 \times \mathbb{P}^n$ be the hypersurfaces as above. Let $Y_1 := \mathbb{P}^2 \times T =: Y_2$ and $D_1 := D_{ST}$ and $D_2 := D_{S_m T}$. Then the isomorphism $\phi_m\colon S \to S_m$ induces an isomorphism $\Phi_m\colon D_1 \to D_2$ via natural isomorphisms $D_1 \simeq S \times_{\mathbb{P}^1} T$ and $D_2 \simeq S_m \times_{\mathbb{P}^1} T$. Hence we have a diagram

$$\begin{tikzcd} Y_1 & Y_2 \\ D_1 \arrow[u, hook] \arrow[r, "\simeq", "\Phi_m"'] & D_2 \arrow[u, hook] \end{tikzcd}$$

and define $Y_0 := Y_1 \cup^{\Phi_m} Y_2$. Then we see that Y_0 is an SNC CY variety since $D_i \in |-K_{Y_i}|$ and it is connected for $i = 1, 2$. Then Y_0 is not d-semistable by

$$\begin{aligned} &\mathcal{N}_{D_1/Y_1} \otimes \Phi_m^* \mathcal{N}_{D_2/Y_2} \\ &\quad \simeq \mathcal{O}_{D_1}(p_S^*(3(H_S + \phi_m^* H_{S_m}) + 2(-K_S))) \simeq \mathcal{O}_{D_1}(F_1 + \cdots + F_m + \Gamma_m), \end{aligned}$$

where $p_S\colon D_1 \to S$ is the projection and $F_i := p_S^{-1}(f_i)$ for smooth $f_i \in |-K_S|$ for $i = 1, \dots m$ and $\Gamma_m := p_S^{-1}(C_m)$ for smooth $C_m \in |3(H_S + \phi_m^* H_{S_m}) + (2-m)(-K_S)|$. (One may see that the tensor product is "quite positive" since it decomposes to many divisors $F_1, \dots, F_m, \Gamma_m$. This is why the 2nd Betti number of our examples $X(m)$ in the below can be large.)

Now let $\mu_1\colon X_1 \to Y_1$ be the blow-up of $F_1, \dots, F_m$ and the strict transform of Γ_m and $X_2 := Y_2$. Let $\tilde{D}_1 \subset X_1$ be the strict transform of $D_1 \subset Y_1$ with an isomorphism $\nu_1 := \mu_1|_{\tilde{D}_1}$. Then we have an isomorphism

$$\tilde{\Phi}_m := \Phi_m \circ \nu_1\colon \tilde{D}_1 \to D_2$$

and let

$$X_0(m) := X_1 \cup^{\tilde{\Phi}_m} X_2.$$

Then we see that $X_0(m)$ is an SNC CY variety by $\tilde{D}_1 \in |-K_{X_1}|$. We also see that $X_0(m)$ is d-semistable by the choice of the centers of the blow-up. By Theorem 2.1, there exists a smoothing $\mathcal{X}(m) \to \Delta^1$ of $X_0(m)$ and let $X(a)$ be its general fiber. Then we see that $X(m)$ is a Calabi-Yau manifold. This gives $X(m)$ as in Theorem 1.1. The properties can be checked as in [San21].

4. Further examples and problems

4.1. **Further examples.** In [HS19], we constructed non-Kähler Calabi-Yau 3-folds by smoothing SNC CY 3-folds which are unions of two smooth "quasi-Fano 3-folds" glued along K3 surfaces. (More examples can be found in the earlier version of [HS19] on arXiv.) We can also construct examples from SNC CY 3-folds whose intersection locus are abelian surfaces as follows.

Example 4.1. Let $X = X_{(3,3)} \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a general hypersurface of bi-degree $(3,3)$. Then it degenerates to an SNC hypersurface $Y_0 := E \times \mathbb{P}^2 \cup \mathbb{P}^2 \times E$, where $E \subset \mathbb{P}^2$ is a cubic curve. Its intersection locus is the abelian surface $E \times E$. Y_0 is not d-semistable, but we may construct a d-semistable SNC CY 3-fold by blowing up suitable curves.

Let $X := X_{(3,0,1))} \cap X_{(0,3,1)} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ be the complete intersection of two hypersurfaces of bi-degrees $(3,0,1)$ and $(0,3,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$. This X also has a fibration to $\mathbb{P}^1$ whose general fiber is an abelian surface of product type.

We can also construct infinitely many topological types of non-Kähler CY 3-folds from SNC CY 3-folds whose singular locus are abelian surfaces as follows. Note that we need the result of [CLM19] since $H^1(\mathcal{O}_{X_i}) \neq 0$ for the irreducible component X_i in the following example.

Example 4.2. Let $E \subset \mathbb{P}^2$ be a general cubic curve without complex multiplications. Let $f_i \subset E \times E$ be the fibers of the i-th projection $E \times E \to E$ for $i = 1, 2$ and $\Delta \subset E \times E$ be the diagonal. It is well known that the Néron–Severi group $\mathrm{NS}(E \times E)$ is freely generated by the divisors f_1, f_2, Δ.

Let $a \in \mathbb{Z}_{>1}$. Let $Y_1 := \mathbb{P}^2 \times E$ and $Y_2 := E \times \mathbb{P}^2$. Let $D_1 := E \times E \subset Y_1$ and $D_2 := E \times E \subset Y_2$. Let

$$\phi_1 \colon E \times E \to E \times E; (x, y) \mapsto (x - ay, y)$$

be the isomorphism induced by the matrix $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$ (use column vectors and $x - ay \in E$ is defined by the addition of E.). Similarly, let $\phi_2 \colon E \times E \to E \times E$ be the isomorphism induced by the matrix $\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$. Then we have the following diagram:

$$\begin{tikzcd}
Y_1 = \mathbb{P}^2 \times E & & & E \times \mathbb{P}^2 = Y_2 . \\
D_1 = E \times E \arrow[u, hook] & E \times E \arrow[l, "\phi_1"'] \arrow[r, "\phi_2"] & & E \times E = D_2 \arrow[u, hook]
\end{tikzcd}$$

Let $C_a := \phi_1^* f_1$ and $D_a := \phi_2^* f_2$. Then C_a is numerically equivalent to the curve $\{(ay, y) \mid y \in E\} \subset E \times E$. Then we compute that

$$C_a \cdot f_1 = a^2, \ C_a \cdot f_2 = 1, \ C_a \cdot \Delta = (a-1)^2.$$

From this, we obtain $C_a \equiv (1-a)f_1+(a^2-a)f_2+a\Delta$. Similarly, we compute $D_a \equiv (a^2-a)f_1+(1-a)f_2+a\Delta$. By these and $\deg \mathcal{O}_{\mathbb{P}^2}(3)|_E = 9$, we obtain

$$\phi_1^*\mathcal{N}_{D_1/Y_1} \otimes \phi_2^*\mathcal{N}_{D_2/Y_2} \equiv 9(\phi_1^*(f_1) + \phi_2^*(f_2)) \equiv 9((a-1)^2(f_1+f_2)+2a\Delta).$$

By this, we see that the linear system

$$|\phi_1^*\mathcal{N}_{D_1/Y_1} \otimes \phi_2^*\mathcal{N}_{D_2/Y_2} \otimes \mathcal{O}_{E\times E}(-\Delta_1 - \cdots - \Delta_a)|$$

is ample and free, thus it contains a smooth member Γ_a.

Let $\mu_1 \colon X_1 \to Y_1$ be the blow-up of smooth curves $\phi_1(\Delta_1), \ldots, \phi_1(\Delta_a)$ such that $\Delta_i \equiv \Delta$ for $i = 1, \ldots, a$ and the strict transform of $\phi_1(\Gamma_a)$. Then μ_1 induces an isomorphism

$$\nu_1 := \mu_1|_{\tilde{D}_1} \colon \tilde{D}_1 \to D_1$$

from the strict transform $\tilde{D}_1 \subset X_1$ of D_1. Now let

$$X_0 := X_0(a) := X_1 \cup^{\psi_a} Y_2$$

be an SNC variety which is a union of X_1 and Y_2 glued along the isomorphism

$$\psi_a := \phi_2 \circ \phi_1^{-1} \circ \nu_1 \colon \tilde{D}_1 \to D_1 \to D_2.$$

Since $\tilde{D}_1 \in |-K_{X_1}|$ and $D_2 \in |-K_{Y_2}|$, we see that $\omega_{X_0} \simeq \mathcal{O}_{X_0}$. By the choice of the blow-up centers of μ_1, we see that X_0 is d-semistable. Hence we can apply Theorem 2.1 to obtain a smoothing $\mathcal{X}(a) \to \Delta^1$ and let $X(a)$ be its general fiber.

Proposition 4.3. *Let $X = X(a)$ be the 3-fold constructed as above. Then we have the following.*
(i) *X and X_0 are simply connected.*
(ii) *$H^i(X, \mathcal{O}_X) = 0 = H^0(X, \Omega^i_X)$ for $i = 1, 2$. Thus $X(a)$ is a Calabi-Yau 3-fold.*
(iii) *The 2nd Betti number of X is $b_2(X) = a + 1$.*
(iv) *The algebraic dimension $a(X)$ is 0.*

Proof. Recall that $X_1 \to \mathbb{P}^2 \times E$ is the blow-up of the $a+1$ smooth curves as above, $X_2 = E \times \mathbb{P}^2$ and $X_{12} \simeq E \times E$.
(i) This can be shown by a similar argument as in [HS19, Proposition 3.10] using the Clemens map $c \colon X \to X_0$. Recall that c is a diffeomorphism over $X_0 \setminus X_{12}$, where $X_{12} := X_1 \cap X_2 \simeq E \times E$. Indeed, by van Kampen's theorem, we see that

$$\pi_1(X_0) \simeq \pi_1(X_1) *_{\pi_1(X_{12})} \pi_1(X_2) \simeq \{1\}.$$

Moreover, we see that

$$\pi_1(X) \simeq \pi_1(X_1') *_{\pi_1(\tilde{X}_{12})} \pi_1(X_2'),$$

where $X_i' := X_i \setminus X_{12}$ for $i = 1, 2$ and $\tilde{X}_{12} := c^{-1}(X_{12})$ is an S^1-bundle over X_{12}. Then we can show that $\pi_1(X) \simeq \{1\}$ as in [HS19, Claim 3.12].

(ii) Since we obtain $H^1(X, \mathbb{Z}) = 0$ by $\pi_1(X) = \{1\}$, we see that $H^1(X, \mathcal{O}_X) = 0 = H^0(X, \Omega^1_X)$ by the E_1-degeneration of the Hodge to de Rham spectral sequence on X (cf. [HS19, Remark 3.8]). By the Serre duality, we see that $H^2(X, \mathcal{O}_X) = 0$. We see that $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$ by the exponential sequence. This implies that $H^0(X, \Omega^2_X) = 0$.

(iii) We have an exact sequence

$$0 \to H^2(X_0, \mathbb{Z}) \to H^2(X_1, \mathbb{Z}) \oplus H^2(X_2, \mathbb{Z}) \xrightarrow{\alpha} H^2(X_{12}, \mathbb{Z}).$$

Note that the injectivity on the left follows from the surjectivity of the homomorphism $H^1(X_1, \mathbb{Z}) \oplus H^1(X_2, \mathbb{Z}) \to H^1(X_{12}, \mathbb{Z})$ by explicit calculation. Since $\mathrm{Im}\, \alpha \simeq \mathbb{Z}^3$, $H^2(X_1, \mathbb{Z}) \simeq \mathbb{Z}^{2+a+1}$ and $H^2(X_2, \mathbb{Z}) \simeq \mathbb{Z}^2$, we see that $H^2(X_0, \mathbb{Z}) \simeq \mathbb{Z}^{a+2}$. By the standard argument as in [Fri19, 2.6], we obtain an exact sequence

$$H^1(X, \mathbb{C}) \to H^0(X_{12}, \mathbb{C}) \to H^2(X_0, \mathbb{C}) \to H^2(X, \mathbb{C}) \to 0 \tag{1}$$

and see that $H^2(X, \mathbb{C}) \simeq \mathbb{C}^{a+1}$ by this sequence. Hence we obtain $b_2(X) = a+1$.

(iv) Note that $H^2(X_0, \mathbb{Z}) \simeq H^2(\mathcal{X}, \mathbb{Z}) \simeq \mathrm{Pic}\, \mathcal{X}$ and $H^2(X, \mathbb{Z}) \simeq \mathrm{Pic}\, X$ by $H^i(X, \mathcal{O}_X) = 0 = H^i(X_0, \mathcal{O}_{X_0})$ for $i = 1, 2$. Hence we have a surjection $\mathrm{Pic}\, \mathcal{X} \to \mathrm{Pic}\, X$ by the sequence (1).

Suppose that there exists a line bundle $\mathcal{L}_t \in \mathrm{Pic}\, X$ with $\kappa(\mathcal{L}_t) \geq 1$. Then there exists $\mathcal{L}_0 \in \mathrm{Pic}\, X_0$ such that $\kappa(\mathcal{L}_0) \geq 1$ and $\mathcal{L}_0|_{X_i}$ is effective for $i = 1, 2$ by the same argument as in [HS19, Proposition 3.19 (iii)]. This contradicts the following claim.

Claim 4.4. Let $\mathcal{L}_0 \in \mathrm{Pic}\, X_0$ be a line bundle such that $\mathcal{L}_i := \mathcal{L}_0|_{X_i}$ is effective for $i = 1, 2$. Let $F_1, \ldots, F_{a+1} \subset X_1$ be the exceptional divisors over $\phi_1(\Delta_1), \ldots, \phi_1(\Delta_a)$ and $\phi_1(\Gamma_a)$. Write

$$\mathcal{L}_1 = \mu_1^*(\mathcal{O}_{\mathbb{P}^2}(\alpha) \boxtimes \mathcal{O}_E(D_1)) \otimes \mathcal{O}_{X_1}\left(\sum_{j=1}^{a+1} b_j F_j\right), \quad \mathcal{L}_2 = \mathcal{O}_E(D_2) \boxtimes \mathcal{O}_{\mathbb{P}^2}(\alpha')$$

for some integers $\alpha, \alpha', b_1, \ldots, b_{a+1}$ and let $d_i := \deg D_i$ for $i = 1, 2$.

Then we see that $\alpha = \alpha' = 0$ and $d_1 = d_2 = 0$. In particular, $\kappa(\mathcal{L}_0) = 0$.

Proof of Claim. Since $\mathcal{L}_i$ is effective, we see that $\alpha, \alpha', d_1, d_2 \geq 0$. Suppose that $(\alpha, d_1) \neq (0, 0)$. Then we have $(\alpha', d_2) \neq (0, 0)$. We shall see a contradiction in the following. We use the identification

$$\mathrm{NS}(E \times E) \simeq \mathbb{Z}^3; \quad a_1 f_1 + a_2 f_2 + a_3 \Delta \mapsto (a_1, a_2, a_3).$$

By the isomorphism $\mathcal{L}_1|_{X_{12}} \simeq \mathcal{L}_2|_{X_{12}}$, we see that

$$3\alpha \cdot C_a + p_2^*(D_1) + \sum_{j=1}^{a} b_j \Delta + b_{a+1} \Gamma_a \equiv p_1^*(D_2) + 3\alpha' \cdot D_a.$$

By the above identification, this corresponds to the equality

$$3\alpha(1-a, a-a^2, a) + (0, d_1, 0) + \sum_{j=1}^{a} b_j(0,0,1) + 9b_{a+1}\left((a-1)^2, (a-1)^2, a\right) \\ = (d_2, 0, 0) + 3\alpha'(a^2-a, 1-a, a) \in \mathbb{Z}^3.$$

The equality on the 1st and 2nd coordinates implies the equality

$$\left(3\alpha(1-a), 3\alpha(a^2-a)+d_1\right) + 9b_{a+1}\left((a-1)^2, (a-1)^2\right) \\ = \left(d_2 + 3\alpha'(a^2-a), 3\alpha'(1-a)\right)$$

and we see that

$$3\alpha(1-a) - (d_2 + 3\alpha'(a^2-a)) = (3\alpha(a^2-a)+d_1) - 3\alpha'(1-a).$$

This is a contradiction since the L.H.S. is negative and the R.H.S. is positive. This finishes the proof of Claim 4.4 □

By Claim 4.4, we see that there does not exist $\mathcal{L}_t \in \operatorname{Pic} X$ such that $\kappa(\mathcal{L}_t) \geq 1$. Hence we see that $a(X) = 0$. □

4.2. **Problems.** Let $X_0 = X_1 \cup X_2$ be an SNC CY variety with two smooth irreducible components X_1 and X_2. Degenerations of a smooth CY manifold to such an SNC CY variety are called *Tyurin degenerations* although it is often assumed that X_1, X_2 are "quasi-Fano" varieties and $X_1 \cap X_2$ is a (strict) CY manifold (cf. [Lee06, 5.4], [DHT17]). In Example 4.2, we treated CY 3-folds which admit degenerations to SNC CY 3-folds of the form $X_1 \cup X_2$ such that $X_1 \cap X_2$ are abelian surfaces. One can ask which kind of varieties can appear as the intersection locus of such SNC Calabi-Yau varieties.

Example 4.5. Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be a smooth hypersurface of bi-degree $(n+1, m+1)$. Then it degenerates to an SNC CY variety $X_0 = \mathbb{P}^n \times X_{m+1} \cup X_{n+1} \times \mathbb{P}^m$. The intersection locus of X_0 is $X_{n+1} \times X_{m+1}$, thus a product of two Calabi-Yau manifolds can appear as the intersection locus.

Tyurin [Tyu04] called a CY 3-fold with Tyurin degenerations a *constructive* CY 3-fold. He speculated that every CY 3-fold is constructive. This is widely open to the author's knowledge. It is also written that topological types of constructive CY 3-folds should be finite. Although there are infinitely many topological types of non-Kähler Calabi-Yau manifolds with Tyurin degenerations as in examples in Section 3, it is still not clear in the projective case.

It would also be interesting to see bimeromorphic relations between our examples of non-Kähler CY manifolds (cf. [Rei87]).

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Department of Mathematics, Faculty of Science, Kobe University, Kobe, 657-8501, Japan

E-mail address: `tarosano@math.kobe-u.ac.jp`