QUASI-PERIOD COLLAPSE FOR DUALS TO FANO POLYGONS: AN EXPLANATION ARISING FROM ALGEBRAIC GEOMETRY

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Abstract. The Ehrhart quasi-polynomial of a rational polytope P is a fundamental invariant counting lattice points in integer dilates of P. The quasi-period of this quasi-polynomial divides the denominator of P but is not always equal to it: this is called quasi-period collapse. Polytopes experiencing quasi-period collapse appear widely across algebra and geometry and yet the phenomenon remains largely mysterious. By using techniques from algebraic geometry – $\mathbb{Q}$-Gorenstein deformations of orbifold del Pezzo surfaces – we provide a sufficient condition for quasi-period collapse to occur for rational polygons dual to Fano polygons, and conjecture an explicit description of the discrepancy between the quasi-period and the denominator.

1. Introduction

Let $P \subset \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathbb{Q}$ be a convex lattice polytope of dimension d. Let $L_P(k) := \left|kP \cap \mathbb{Z}^d\right|$ count the number of lattice points in dilations kP of P, $k \in \mathbb{Z}_{\geq 0}$. Ehrhart [9] showed that L_P can be written as a degree d polynomial

$$L_P(k) = c_d k^d + \ldots + c_1 k + c_0$$

which we call the *Ehrhart polynomial* of P. The leading coefficient c_d is given by $\mathrm{Vol}(P)/d!$, c_{d-1} is equal to $\mathrm{Vol}(\partial P)/2(d-1)!$, and $c_0 = 1$. Here $\mathrm{Vol}(\,\cdot\,)$ denotes the normalised volume, and ∂P denotes the boundary of P. For example, if P is two-dimensional (that is, P is a lattice *polygon*) we obtain

$$L_P(k) = \frac{\mathrm{Vol}(P)}{2}k^2 + \frac{\left|\partial P \cap \mathbb{Z}^2\right|}{2}k + 1.$$

Setting $k = 1$ in this expression recovers Pick's Theorem [16]. The values of the Ehrhart polynomial of P form a generating function $\mathrm{Ehr}_P(t) := \sum_{k \geq 0} L_P(k) t^k$ called the *Ehrhart series* of P.

When the vertices of P are rational points the situation is more interesting. Recall that a *quasi-polynomial* with *period* $s \in \mathbb{Z}_{>0}$ is a function $q : \mathbb{Z} \to \mathbb{Q}$ defined by polynomials $q_0, q_1, \ldots, q_{s-1}$ such that

$$q(k) = q_i(k) \qquad \text{when } k \equiv i \pmod{s}.$$

The *degree* of q is the largest degree of the q_i. The minimum period of q is called the *quasi-period*, and necessarily divides any other period s. Ehrhart showed that L_P is given by a quasi-polynomial of degree d, which we call the *Ehrhart quasi-polynomial* of P. Let π_P denote the quasi-period of P. The smallest positive integer $r_P \in \mathbb{Z}_{>0}$ such that $r_P P$ is a lattice polytope is called the *denominator* of P. It is certainly the case that L_P is r_P-periodic, however it is perhaps surprising that the quasi-period of L_P does not always equal r_P; this phenomenon is called *quasi-period collapse*.

Example 1.1 (Quasi-period collapse)**.** Consider the triangle $P := \mathrm{conv}\{(5,-1),(-1,-1),(-1,1/2)\}$ with denominator $r_P = 2$. This has $L_P(k) = 9/2k^2 + 9/2k + 1$, hence $\pi_P = 1$.

Quasi-period collapse is poorly understood, although it occurs in many contexts. For example, de Loera–McAllister [7,8] consider polytopes arising naturally in the study of Lie algebras (the Gel'fand–Tsetlin polytopes and the polytopes determined by the Clebsch–Gordan coefficients) that exhibit quasi-period collapse. In dimension two McAllister–Woods [15] show that there exist rational polygons with r_P arbitrarily large but with $\pi_P = 1$ (see also Example 3.7). Haase–McAllister [10] give a constructive view of this phenomena in terms of $\mathrm{GL}_d(\mathbb{Z})$-*scissor congruence*; here a polytope is partitioned into pieces that are individually modified via $\mathrm{GL}_d(\mathbb{Z})$ transformation and lattice translation, then reassembled to give a new polytope which (by construction) has equal Ehrhart quasi-polynomial but different r_P.

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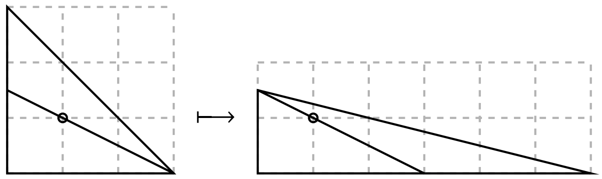
Example 1.2 ($\mathrm{GL}_2(\mathbb{Z})$-scissor congruence)**.** The lattice triangle $Q := \mathrm{conv}\{(2,-1),(-1,-1),(-1,2)\}$ with Ehrhart polynomial $L_Q(k) = 9/2k^2+9/2k+1$ can be partitioned into two rational triangles as depicted on the left below. Fix the bottom-most triangle, and transform the top-most triangle via the lattice automorphism $e_1 \mapsto (3,-1)$, $e_2 \mapsto (4,-1)$. This gives the rational triangle P (depicted on the right) from Example 1.1.



We give an explanation for quasi-period collapse in two dimensions for a certain class of polygons in terms of recent results in algebraic geometry arising from Mirror Symmetry. In §2 we explain how *mutation* – a combinatorial operation arising from the theory of cluster algebras – gives an explanation of this phenomenon, and explain how this is related to $\mathbb{Q}$*-Gorenstein (qG-) deformations* of del Pezzo surfaces as studied by Wahl [17], Kollár–Shepherd-Barron [14], Hacking–Prokhorov [11], and others. Finally, in Theorem 3.4 we provide a sufficient condition for quasi-period collapse to occur, and conjecture an explicit formula for the discrepancy between the denominator and the quasi-period for this class of polygons.

2. Mutation

In [10] Haase–McAllister propose the open problem of finding a systematic and useful technique that implements $\mathrm{GL}_d(\mathbb{Z})$-scissor congruence for rational polytopes. In the case when the dual polyhedron is a lattice polytope it was observed in [2] that one such technique is given by *mutation*.

2.1. **The combinatorics of mutation.** Let $N \cong \mathbb{Z}^d$ be a rank d lattice and set $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$. Let $P \subset N_\mathbb{Q}$ be a lattice polytope. We require – and will assume from here onwards – that P satisfies the following two conditions:

(a) P is of maximum dimension in N, $\dim(P) = d$;
(b) the origin is contained in the strict interior of P, $\mathbf{0} \in P^\circ$.

Condition (b) is not especially stringent, and can be satisfied by any polytope with $P^\circ \cap N \neq \emptyset$ by lattice translation. It is, however, an essential requirement in what follows.

Let $M := \mathrm{Hom}(N,\mathbb{Z}) \cong \mathbb{Z}^d$ denote the dual lattice. Given a polytope $P \subset N_\mathbb{Q}$, the dual polyhedron is defined by

$$P^* := \{u \in M_\mathbb{Q} \mid u(v) \geq -1 \text{ for all } v \in P\} \subset M_\mathbb{Q}.$$

Condition (b) gives that P^* is a (typically rational) polytope. It is on rational polytopes dual to lattice polytopes that we focus. In this section we will explain how mutation corresponds to a piecewise-$\mathrm{GL}_d(\mathbb{Z})$ transformation of P^*, and hence is an instance of $\mathrm{GL}_d(\mathbb{Z})$-scissor congruence for P^*.

Following [2, §3], let $w \in M$ be a primitive lattice vector. Then $w : N \to \mathbb{Z}$ determines a height function (or grading) which naturally extends to $N_\mathbb{Q} \to \mathbb{Q}$. We call $w(v)$ the *height* of $v \in N_\mathbb{Q}$. We denote the set of all points of height h by $H_{w,h}$, and write

$$w_h(P) := \mathrm{conv}\big(H_{w,h} \cap P \cap N\big) \subset N_\mathbb{Q}$$

for the (possibly empty) convex hull of lattice points in P at height h.

Definition 2.1. A *factor* of $P \subset N_\mathbb{Q}$ with respect to $w \in M$ is a lattice polytope $F \subset w^\perp$ such that for every negative integer $h \in \mathbb{Z}_{<0}$ there exists a (possibly empty) lattice polytope $R_h \subset N_\mathbb{Q}$ such that

$$H_{w,h} \cap \mathrm{vert}(P) \subseteq R_h + |h|\,F \subseteq w_h(P).$$

Here '+' denotes Minkowski sum, and we define $\emptyset + Q = \emptyset$ for every lattice polytope Q.

Definition 2.2. Let $P \subset N_\mathbb{Q}$ be a lattice polytope with $w \in M$ and $F \subset N_\mathbb{Q}$ as above. The *mutation* of P with respect to the data (w,F) is the lattice polytope

$$\mu_{(w,F)}(P) := \mathrm{conv}\left(\bigcup_{h\in\mathbb{Z}_{<0}} R_h \cup \bigcup_{h\in\mathbb{Z}_{\geq 0}} (w_h(P) + hF)\right) \subset N_\mathbb{Q}.$$

It is shown in [2, Proposition 1] that, for fixed data (w, F), any choice of $\{R_h\}$ satisfying Definition 2.1 gives $\mathrm{GL}_d(\mathbb{Z})$-equivalent mutations. Since we regard lattice polytopes as being defined only up to $\mathrm{GL}_d(\mathbb{Z})$-equivalence, this means that mutation is well-defined. One can readily see that translating the factor F by some lattice point $v \in w^\perp \cap N$ gives isomorphic mutations: $\mu_{(w,F+v)}(P) \cong \mu_{(w,F)}(P)$. In particular if $\dim(F) = 0$ then $\mu_{(w,F)}(P) \cong P$. Finally, we note that mutation is always invertible [2, Lemma 2]: if $Q := \mu_{(w,F)}(P)$ then $P = \mu_{(-w,F)}(Q)$.

Remark 2.3. Informally, mutation corresponds to the following operation on slices $w_h(P)$ of P: at height h one Minkowski adds or "subtracts" $|h|$ copies of F, depending on the sign of h. Definition 2.1 ensures that the concept of Minkowski subtraction makes sense.

Mutation has a natural description in terms of the dual polytope P^* [2, Proposition 4 and pg. 12].

Definition 2.4. The *inner-normal fan* in $M_\mathbb{Q}$ of a polytope $F \subset N_\mathbb{Q}$ is generated by the cones

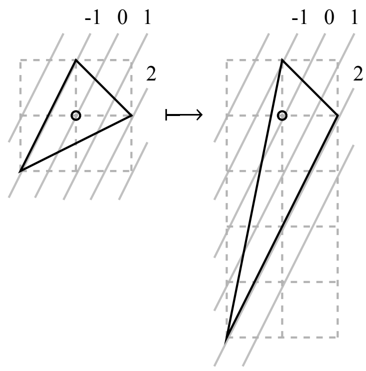
$$\sigma_{v_F} := \{u \in M_\mathbb{Q} \mid u(v_F) = \min\{u(v) \mid v \in F\}\}, \qquad \text{for each } v_F \in \mathrm{vert}(F).$$

A mutation $\mu_{(w,F)}$ induces a piecewise-$\mathrm{GL}_d(\mathbb{Z})$ transformation $\varphi_{(w,F)}$ on $M_\mathbb{Q}$ given by

$$\varphi_{(w,F)} : u \mapsto u - u_{\min} w, \qquad \text{where } u_{\min} := \min\{u(v_F) \mid v_F \in \mathrm{vert}(F)\}.$$

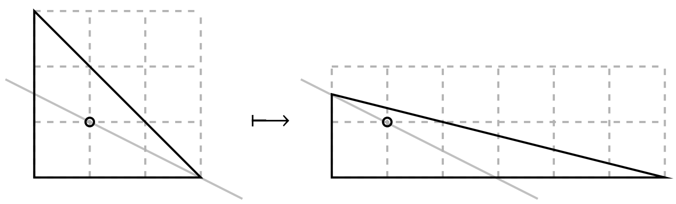
The inner-normal fan of F determines a chamber decomposition of $M_\mathbb{Q}$, and $\varphi_{(w,F)}$ acts linearly within each chamber. Let $Q := \mu_{(w,F)}(P)$. Then $\varphi_{(w,F)}(P^*) = Q^*$. It is clear that the Ehrhart quasi-polynomials L_{P^*} and L_{Q^*} for the dual polytopes are equal, since the map $\varphi_{(w,F)}$ is piecewise-linear. Hence mutation gives a systematic way to produce examples of $\mathrm{GL}_d(\mathbb{Z})$-scissor congruence.

Example 2.5 (Mutation)**.** Let $P = \mathrm{conv}\{(1,0),(0,1),(-1,-1)\} \subset N_\mathbb{Q}$ and $w = (2,-1) \in M$. Then $F = \mathrm{conv}\{(0,0),(-1,-2)\} \subset w^\perp$ is a factor. We see that $Q := \mu_{(w,F)}(P) = \mathrm{conv}\{(1,0),(0,1),(-1,-4)\}$.

$N_\mathbb{Q}$:


On the dual side we have that $M_\mathbb{Q}$ is divided into two chambers whose boundary is given by $\mathbb{Q} \cdot w$, and

$$\varphi_{(w,F)} : (u_1, u_2) \mapsto \begin{cases} (u_1, u_2), & \text{if } u_1 + 2u_2 \leq 0; \\ (3u_1 + 4u_2, -u_1 - u_2), & \text{otherwise.} \end{cases}$$

$M_\mathbb{Q}$:


Thus we recover Example 1.2 from the view-point of mutation.

From here onwards we assume that $P \subset N_\mathbb{Q}$ is *Fano*. That is, in addition to conditions (a) and (b) above, P satisfies:

(c) the vertices $\mathrm{vert}(P)$ of P are primitive lattice points.

The property of being Fano is preserved under mutation [2, Proposition 2]. We refer to [13] for a survey of Fano polytopes.

2.2. **Toric geometry.** We review some toric algebraic geometry and discuss its extensive connections with Ehrhart theory. A toric variety is a partial compactification of an algebraic torus $(\mathbb{C}^\times)^n$. They are described combinatorially by cones, fans, and polytopes. This and much more is detailed in [6].

2.2.1. *Affine toric varieties arise from cones.* Let $N \cong \mathbb{Z}^n$ be a lattice and let $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$ be the associated $\mathbb{Q}$-vector space. A *cone* σ in $N_\mathbb{Q}$ is a subset of the form

$$\mathrm{cone}(S) := \{\sum_{v\in S} \lambda_v v : \lambda_v \geq 0, \text{all but finitely many } \lambda_v \text{ are zero}\}$$

Let $M = N^\vee := \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$ be the dual lattice to N, and $M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q}$ the dual vector space to $N_\mathbb{Q}$. Define the *dual cone* to a cone $\sigma \subset N_\mathbb{Q}$ to be

$$\sigma^\vee := \{v \in M_\mathbb{Q} : \langle u, v\rangle \geq 0 \text{ for all } u \in \sigma\}$$

where $\langle\cdot,\cdot\rangle$ is the dual pairing $N_\mathbb{Q} \times M_\mathbb{Q} \to \mathbb{R}$. Suppose now that σ is a *rational polyhedral* cone: that there is a finite set of lattice points $S \subset N$ such that $\sigma = \mathrm{cone}(S)$. Such a cone σ gives an affine toric variety U_σ as follows.

- **Input:** σ, a rational polyhedral cone
- Dualise to $\sigma^\vee$
- Take lattice points $\sigma^\vee \cap M$ to obtain a semigroup
- Take the semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$; this is a finitely generated $\mathbb{C}$-algebra
- **Output:** $U_\sigma := \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$.

The cone σ (or rather $\sigma^\vee$) is describing which functions on the torus extend to global functions on U_σ, which is equivalent to describing the variety. One can describe the torus inside U_σ intrinsically as

$$T_N := N \otimes_\mathbb{Z} \mathbb{C}^\times$$

In this presentation, a vector $m \in M$ gives a function $\chi^m : T_N \to \mathbb{C}$ via

$$\chi^m(n \otimes t) = t^{\langle m,n\rangle}$$

Example 2.6. Take $N = \mathbb{Z}^2$ and let $\sigma = \mathrm{cone}(e_1, e_2)$. The dual cone is $\sigma^\vee = \mathrm{cone}(e^1, e^2)$ giving

$$\sigma^\vee \cap M = \mathbb{Z}^2_{\geq 0} \text{ and } \mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{C}[x, y]$$

Hence $U_\sigma \cong \mathbb{C}^2$. In this case, $\sigma^\vee$ prescribes that the only Laurent polynomials extending to all of U_σ are the polynomials.

2.2.2. *Toric varieties arise from fans.* To construct more complicated toric varieties we glue together affine toric varieties in an equivariant way. The combinatorial avatar of this process is collecting cones together in a *fan*. To start with, a *face* of a cone σ is a subset of σ of the form $\sigma \cap (\langle m, \cdot\rangle = 0)$ for some $m \in \sigma^\vee$. The cones forming the boundary of σ are examples of faces, as is the vertex of the cone (the origin). A *fan* in $N_\mathbb{Q}$ is a collection of cones $\Sigma = \{\sigma\}$ such that

- if $\tau \subset \sigma$ is a face, then $\tau \in \Sigma$
- for any two cones $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a face of each

A fan Σ produces a toric variety X_Σ via gluing two affine pieces $U_{\sigma_1}, U_{\sigma_2}$ according to the (potentially zero-dimensional) face they have in common.

Example 2.7. Take $N = \mathbb{Z}^2$ and Σ to be the fan containing the cones $\sigma_1 = \mathrm{cone}(e_1, e_2)$, $\sigma_2 = \mathrm{cone}(e_1, -e_1 - e_2)$, $\sigma_3 = \mathrm{cone}(e_2, -e_1 - e_2)$ and their faces. The two-dimensional cones give three copies of $\mathbb{C}^2$ and the gluing prescribed by the faces makes this into $\mathbb{P}^2$. For example, σ_1 and σ_3 share the face $\mathrm{cone}(e_2)$ that corresponds to the toric variety $\mathbb{C}^\times \times \mathbb{C}$. Gluing $\mathbb{C}^2$ to $\mathbb{C}^2$ along $\mathbb{C}^\times \times \mathbb{C}$ is familiar from the gluing construction of projective space.

2.2.3. *Compact toric varieties arise from polytopes.* Suppose $P \subset N_\mathbb{Q}$ is a lattice polytope. One can produce a fan Σ_P from P via

$$\Sigma_P := \{\mathrm{cone}(S) : S \subset \mathrm{Vert}(P) \text{ such that all } u \in S \text{ share a face}\}$$

This is called the *spanning fan* of P and defines a toric variety $X_P := X_{\Sigma_P}$ that turns out to be compact.

A polytope $Q \subset M_{\mathbb{Q}}$ also defines a toric variety V_Q. Let $L_Q = \#Q \cap M$ and define a map $\phi_Q : T_N \to \mathbb{P}^{L_Q - 1}$ by $x \mapsto (\chi^m(x))_{m \in Q \cap M}$. The toric variety V_Q is defined to be the closure of the image of ϕ_Q in $\mathbb{P}^{L_Q-1}$ in the Zariski topology (where the closed sets are the algebraic sets). If we define the dual polytope

$$P^\vee := \{v \in M_{\mathbb{Q}} : \langle u, v \rangle \geq -1\}$$

then the toric variety X_P is also described abstractly as the variety $V_{kP^\vee}$ for large enough k.

Example 2.8. A polytope for $\mathbb{P}^2$ is the triangle with vertices $e_1, e_2, -e_1 - e_2$. The dual polytope is the triangle with vertices $2e^1 - e_2, -e_1 + 2e_2, -e^1 - e^2$. This has 10 lattice points and describes the third Veronese (or anticanonical) embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$.

2.2.4. *Polytopes arise from divisors.* A (Weil) divisor on a variety is a formal $\mathbb{Z}$-linear combination of codimension one subvarieties. The irreducible codimension one subvarieties of a toric variety X_Σ that are preserved by the torus action are indexed by the rays of Σ. The set of rays is commonly denoted $\Sigma(1)$. Thus, we can express torus-invariant divisors in the forn

$$\sum_{\rho \in \Sigma(1)} a_\rho D_\rho$$

where D_ρ is the divisor corresponding to the ray $\rho \in \Sigma(1)$. One can associate a polytope $P(D)$ to a divisor of this form as follows. Let u_ρ be the primitive lattice point lying on the ray ρ. Then set

$$P(D) := \{v \in M_{\mathbb{Q}} : \langle u_\rho, v \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}$$

The hyperplanes defining the facets of $P(D)$ are given by $\langle u_\rho, \cdot \rangle = -a_\rho$ and so this construction of $P(D)$ taking in the data $(u_\rho, a_\rho)_{\rho \in \Sigma(1)}$ is often referred to as a 'facet presentation' for $P(D)$. Denote by $\mathcal{O}(D)$ the line bundle (or sheaf) associated to a divisor D.

Lemma 2.9 ([6], Proposition 4.3.3)**.** *Let $D = \sum_\rho a_\rho D_\rho$. A basis of $H^0(\mathcal{O}(D))$ is in bijection with lattice points of $P(D)$. That is,*

$$\#P(D) \cap M = L_{P(D)} = h^0(\mathcal{O}(D))$$

2.2.5. *Toric Fano varieties.* When P is a Fano polytope the toric variety X_P is a *toric Fano variety*; namely, the anticanonical divisor $-K_{X_P}$ is ample. When P is a Fano polygon X_P is a toric del Pezzo surface with at worst log terminal singularities.

The geometry, specifically the deformation theory, of toric del Pezzo surfaces will be our main tool in studying quasi-period collapse. The main deformation-theoretic invariant we will use is the *singularity content* of P introduced in [3], which we recall in Definition 2.14 below. From a combinatorial point of view the singularity content is invariant under mutations of P. In §2.5 we remark on the connection between singularity content and the qG-deformation theory of X_P, and how this gives a geometric explanation for the quasi-period collapse of P^*.

2.3. **Quotient singularities.** In order to state the definition of singularity content we first recall some of the theory of quotient or orbifold surface singularities from algebraic geometry. A point of an algebraic variety X is called singular at a point p if

$$\dim T_p X > \inf_{q \in X} \dim T_q X$$

where $T_p X$ is the tangent space to X at p. We also say that p is a singularity of X. An interesting and well-studied class of singularities is quotient singularities that are constructed as follows. Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite subgroup. G acts on $\mathbb{C}^n$ and the set of orbits $\mathbb{C}^n/G$ can be endowed with the structure of an algebraic variety. The origin in $\mathbb{C}^n$ is fixed by G and so produces a singular point of the orbit space $\mathbb{C}^n/G$. We call a singularity with a neighbourhood that is isomorphic as varieties to $\mathbb{C}^n/G$ for some G a *quotient singularity*. When G is a cyclic group, we say that such a singularity is a *cyclic quotient singularity*. Up to conjugation we can generate such cyclic G by a matrix of the form

$$\begin{pmatrix} \varepsilon & \\ & \varepsilon^a \end{pmatrix}$$

where ε is a root of unity and $a \in \mathbb{Z}$. If ε is a primitive rth root of unity we denote the corresponding quotient singularity by $\frac{1}{r}(1,a)$. Clearly the value of a only matters mod r. Since G is abelian the singularity $\frac{1}{r}(1,a)$ is a toric variety whose fan is given by the single two-dimensional cone

$$\sigma = \operatorname{cone}\{e_2, re_1 - ae_2\} \subset N_{\mathbb{Q}}.$$

The lattice height of such a cone – that is, the lattice distance between the origin and the line segment joining the two primitive ray generators of the cone (the *edge* of the cone) – is called the *local index*, and can be calculated to be

$$\ell_\sigma = \frac{r}{\gcd\{r, a+1\}}.$$

Another way of defining this is to say that the local index is the absolute value of a primitive normal vector to the cone edge evaluated on any element of the edge. The *width* of the cone is the number of unit-length lattice line segments along the edge of the cone or, equivalently, one less than the number of lattice points along the edge. The width is equal to $\gcd\{r, a+1\}$.

A cyclic quotient singularity is called a *T-singularity* if it is smoothable by a qG-deformation. This is a highly geometric definition but for our purposes we can use the following numerical characterisation of T-singularities.

Lemma 2.10 ([14, Proposition 3.11])**.** *An isolated cyclic quotient singularity is a T-singularity if and only if it takes the form*

$$\frac{1}{dn^2}(1, dnc-1)$$

for some c with $\gcd\{n, c\} = 1$*.*

The cone $\sigma \subset N_{\mathbb{Q}}$ associated to a T-singularity $\frac{1}{dn^2}(1, dnc-1)$ has local index $\ell = n$ and width dn; it is easily seen that T-singularities are characterised by having width divisible by local index. Suppose that $P \subset N_{\mathbb{Q}}$ is a Fano polygon with edge E spanning σ. Let $w \in M$ be the primitive inner-normal such that $w(E) = -\ell$, and choose $F \subset w^\perp$ of lattice length d. The mutation $\mu_{(w,F)}(P)$ collapses the edge E to a vertex, removing the cone σ. This is equivalent in geometry to a local qG-smoothing of the T-singularity.

Example 2.11. Consider the polytope $Q := \operatorname{conv}\{(1,0),(0,1),(-1,-4)\}$ appearing in Example 2.5. The corresponding spanning fan has three two-dimensional cones, two of which are smooth and one of which, $\operatorname{cone}\{(1,0),(-1,-4)\}$, corresponds to a $\frac{1}{4}(1,1)$ T-singularity.

The other relevant class of quotient singularities are the *R-singularities* introduced in [3].

Definition 2.12. A cyclic quotient singularity of local index ℓ and width k is an *R-singularity* if $k < \ell$.

Let $\sigma \subset N_{\mathbb{Q}}$ be a cone of local index ℓ and width k. Write $k = d\ell + r$, where $d, r \in \mathbb{Z}_{\geq 0}$, $0 \leq r < \ell$. If $r = 0$ then σ is a T-singularity. Assume that $r \neq 0$ and, as before, suppose that $P \subset N_{\mathbb{Q}}$ is a Fano polygon with edge E spanning σ. Let $w \in M$ be the corresponding inner-normal, and pick $F \subset w^\perp$ of lattice length d. The mutation $\mu_{(w,F)}(P)$ transforms σ to a cone τ of width r corresponding to a $\frac{1}{r\ell}(1, rc/k - 1)$ singularity. Crucially, τ has width strictly less than the local index, and so cannot be simplified via further mutation. This is equivalent to a partial qG-smoothing of the original singularity σ, resulting in a singularity τ that is rigid under qG-deformation. The R-singularity τ is independent of the choices made [3, Proposition 2.4].

Definition 2.13. Let $\sigma \subset N_{\mathbb{Q}}$ be a cone corresponding to a $\frac{1}{r}(1, c-1)$ singularity. Let ℓ be the local index and let k be the width of the cone. Write $k = d\ell + r$, where $d, r \in \mathbb{Z}_{\geq 0}$, $0 \leq r < \ell$. The *residue* of σ is

$$\operatorname{res}(\sigma) = \begin{cases} \frac{1}{r\ell}(1, rc/k - 1), & \text{if } r \neq 0; \\ \emptyset, & \text{otherwise.} \end{cases}$$

The *singularity content* of σ is the pair $(d, \operatorname{res}(\sigma))$. From a geometric perspective the singularity content contains all local qG-deformation-theoretic data about σ.

We observe that any toric variety of dimension two will only have quotient singularities (see [6]).

Definition 2.14. Let $P \subset N_{\mathbb{Q}}$ be a Fano polygon with cones $\sigma_1, \ldots, \sigma_n$. The *basket* of P is the multiset

$$\mathcal{B} := \{\operatorname{res}(\sigma_i) \mid 1 \leq i \leq n\},$$

where the empty residues are omitted[1]. The *singularity content* of P is the pair

$$(d_1 + \cdots + d_n, \mathcal{B}),$$

where the d_i are the integers appearing in the singularity content of the σ_i. Singularity content is a qG-deformation-invariant of X_P.

2.4. **Hilbert series.** Any toric variety X_P arising from a polytope P comes with a natural ample divisor D given by its 'toric boundary'

$$D = \sum_{\rho\in\Sigma(1)} D_\rho$$

When P is Fano we have $D = -K$, the anti-canonical divisor on X_P. Define the *Hilbert function* of (X_P, D) to be

$$h_{(X_P,D)}(n) := \dim H^0(\mathcal{O}(nD))$$

In this case, due to Lemma 2.9, one has that the Hilbert function of $(X_P, -K)$ equals the Ehrhart quasi-polynomial $L_{P^*}(k)$ of the rational polytope P^*. It follows that the generating function $\mathrm{Hilb}_{(X_P,-K)}(t)$ for the Hilbert function of $(X_P, -K)$, the *Hilbert series* of $(X_P, -K)$, is equal to the Ehrhart series of P^*. From here onwards we suppress $-K$ from the notation.

The Hilbert series of an orbifold del Pezzo surface X with basket $\mathcal{B}$ can be written in the form [3, Corollary 3.5]:

$$\mathrm{Hilb}_X(t) = \frac{1 + (K^2 - 2)t + t^2}{(1-t)^3} + \sum_{\sigma\in\mathcal{B}} Q_\sigma,$$

where Q_σ are *orbifold correction terms* given by certain rational functions with denominators $1 - t^{\ell_\sigma}$.

Example 2.15. The orbifold correction term for the R-singularity $\frac{1}{3}(1,1)$ is

$$Q_{\frac{1}{3}(1,1)} = \frac{-t}{3(1-t^3)} = -\frac{1}{3}(t + t^4 + t^7 + \dots)$$

which contributes $-1/3$ to the coefficient of t^d when $d \equiv 1 \pmod 3$.

It is apparent from this example how the Q_σ terms are the root of the quasi-polynomial behaviour of $h_{(X_P,D)}(d)$ and hence for the quasi-polynomial behaviour of the Ehrhart function $L_P(d)$. In general Q_σ contributes to the Hilbert function as a quasi-polynomial of period ℓ_σ.

If the sheaf $\mathcal{O}(D)$ is a line bundle, we say that D is Cartier. The smallest integer d such that $-dK$ is Cartier is called the *Gorenstein index* of X_P and denoted by ℓ_{X_P}. This is also computed by the following:

$$(*) \qquad \ell_{X_P} = \mathrm{lcm}\{\ell_\sigma : \sigma \in X_P^{\mathrm{sing}}\}$$

where X_P^{sing} is the set of singularities on X_P. We prove the following easy lemma to relate the Gorenstein index with the denominator of the dual polytope.

Lemma 2.16. *Let P be a Fano polygon. The Gorenstein index ℓ_{X_P} is equal to the denominator r_{P^*} of P^*.*

Proof. In the toric setting $-dK$ is Cartier if and only if dP^* is a lattice polytope from [6]. This gives the result. □

2.5. **Algebraic geometry and the quasi-period.** Mutations were introduced in [2] as part of an ongoing program investigating Mirror Symmetry for Fano manifolds [5]. In two dimensions the picture is very well understood: see [1] for the details. In summary, if two Fano polygons P and $Q \subset N_{\mathbb{Q}}$ are related by a sequence of mutations then there exists a qG-deformation between the corresponding toric del Pezzo surfaces X_P and X_Q. Such a qG-deformation preserves the anti-canonical Hilbert series, hence $L_{P^*} = L_{Q^*}$ and so the quasi-periods of P^* and Q^* agree. However it does not in general preserve the Gorenstein index, and hence the denominators r_{P^*} and r_{Q^*} need not be equal. The cones over the edges of P correspond to the singularities of X_P, and these admit partial qG-smoothings to the qG-rigid singularities given by the basket $\mathcal{B}$ of residues.

Suppose that the singularity content of P is $(d, \mathcal{B})$. Then X_P is qG-deformation-equivalent to a (*not necessarily toric*) del Pezzo surface X with singularities $\mathcal{B}$ and whose non-singular locus has topological

[1]In [3] the basket is cyclically ordered. Although important from the viewpoint of classification, it is not required here.

Euler number d. Since $\mathrm{Hilb}_{X_P}(t) = \mathrm{Hilb}_X(t)$, we have an explanation for quasi-period collapse of the dual polytope P^*. Specifically, the Gorenstein index of X is equal to the quasi-period of P^*.

3. Studying quasi-period collapse

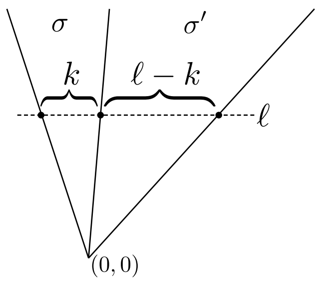
The Hilbert series of toric del Pezzo surfaces (or more generally, del Pezzo orbifolds) were studied in [18] with the aim of describing the structure of the set of possible baskets $\mathcal{B}$ of R-singularities on orbifold del Pezzo surfaces with a fixed Hilbert series. This is achieved by partitioning $\mathcal{B}$ into two pieces: a *reduced basket* and an *invisible basket*. The latter, along with the T-singularities, are not detected by the Hilbert series, and from our viewpoint it is these 'invisible' singularities that cause quasi-period collapse to occur.

Definition 3.1. A collection $\sigma_1, \ldots, \sigma_n$ of R-singularities is a *cancelling tuple* if

$$Q_{\sigma_1} + \cdots + Q_{\sigma_n} = 0.$$

A collection of R-singularities is called *invisible* if it is a disjoint union of cancelling tuples.

Example 3.2. Let σ be an R-singularity of local index ℓ and width k. Then there exists an R-singularity σ' of local index ℓ and width $\ell - k$ such that $Q_\sigma + Q_{\sigma'} = 0$. Combinatorially, this is understood by the observation that the union of the two cones gives a T-singularity.



Definition 3.3. Let X be an toric del Pezzo surface. A maximal invisible subcollection of the basket $\mathcal{B}$ of X is called an *invisible basket* for X. Notice that such a maximal subcollection is not unique, since singularities can appear in many different cancelling tuples. Given a choice of invisible basket $\mathcal{IB} \subset \mathcal{B}$, the complement $\mathcal{RB} = \mathcal{B} \setminus \mathcal{IB}$ is called the *reduced basket* for X corresponding to the choice of $\mathcal{IB}$.

Observe that

$$\sum_{\sigma \in \mathcal{IB}} Q_\sigma = 0 \tag{†}$$

however the singularities in $\mathcal{IB}$ still can contribute to the expression for the Gorenstein index (or the denominator) in (∗).

Denote the collection of T-singularities on X_P by $\mathcal{T}_P$.

Theorem 3.4. *Let $P \subset N_{\mathbb{Q}}$ be a Fano polygon with singularity content $(d, \mathcal{B})$. Let $\mathcal{IB}$ be an invisible basket for X_P with corresponding reduced basket $\mathcal{RB}$. The quasi-period of P^* is bounded above by*[2]

$$\pi_{P^*} \le \mathrm{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{RB}\}.$$

Furthermore, P^* exhibits quasi-period collapse if there exists some $\tau \in \mathcal{IB} \cup \mathcal{T}_P$ of local index not dividing $\mathrm{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{RB}\}$.

Proof. From the preceding discussion we have shown that

$$\mathrm{Ehr}_{P^*}(t) = \mathrm{Hilb}_{X_P}(t) = \frac{1 + (K^2 - 2)t + t^2}{(1-t)^3} + \sum_{\sigma \in \mathcal{B}} Q_\sigma$$

We can rewrite this as

$$\mathrm{Ehr}_{P^*}(t) = \frac{1 + (K^2 - 2)t + t^2}{(1-t)^3} + \sum_{\sigma \in \mathcal{RB}} Q_\sigma$$

using (†).

[2]We adopt the convention that $\mathrm{lcm}\{\emptyset\} = 1$.

Notice that the first term
$$\frac{1+(K^2-2)t+t^2}{(1-t)^3}$$
is the generating function for a polynomial that we denote by $i(d)$. As previously discussed, each of the orbifold correction terms Q_σ is the generating function for a quasi-polynomial $q_\sigma(d)$ with quasi-period ℓ_σ. When $\sigma \in \mathcal{RB}$ these terms are not cancelled out and so make non-zero contributions to the Ehrhart function. It follows that we can express
$$L_{P^*}(d) = i(d) + \sum_{\sigma\in\mathcal{RB}} q_\sigma(d)$$
The right hand side is a sum of a polynomial with a collection of quasi-polynomials and hence its quasi-period is bounded above by the lowest common multiple of the periods of the $q_\sigma(d)$, or
$$\operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{RB}\}$$
The period of the left hand side is by definition the quasi-period π_{P^*} of P^* and so we obtain the desired inequality.

To provide the sufficent condition for quasi-period collapse to occur, recall that the denominator r_{P^*} of P^* is equal to
$$r_{P^*} = \operatorname{lcm}\left\{\ell_\sigma \mid \sigma \in X_P^{\text{sing}}\right\} = \operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{IB}\cup\mathcal{RB}\cup\mathcal{T}_P\}$$
This is strictly greater than the upper bound for π_{P^*} if and only if there is some $\tau \in \mathcal{IB}\cup\mathcal{T}_P$ whose local index does not divide $\operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{RB}\}$, completing the proof. □

Remark 3.5. It follows from [18, §4] that the choice of $\mathcal{IB}$ is irrelevant in the statement of Theorem 3.4.

Example 3.6 (Detecting quasi-period collapse)**.** Consider the polytope $Q := \operatorname{conv}\{(1,0),(0,1),(-1,-4)\}$ appearing in Example 2.5. This has singularity content $(3,\emptyset)$, and $\mathcal{T}_Q = \{2\times\text{smooth}, \frac{1}{4}(1,1)\}$. We see that $r_{Q^*} = 2$ but $\pi_{Q^*} \le 1$, hence $\pi_{Q^*} = 1$.

Computational evidence suggests that actually this upper bound is an equality; namely that the quasi-period of P^* is given by
$$\pi_{P^*} = \operatorname{lcm}\{\ell_\sigma \mid \sigma \in \mathcal{RB}\}$$
but proving this would involve a thorough understanding of any relations between the quasi-polynomials $q_\sigma(d)$ for all σ, which is currently fairly opaque. This would give a complete characterisation of quasi-period collapse for duals to Fano polygons, and an explicit formula for the discrepancy between quasi-period and denominator.

3.1. **Application: a family of triangles with maximal quasi-period collapse.** We now give an example of an infinite family of Fano triangles, obtained via mutation, where the denominator r_{P^*} can become arbitrarily large but where $\pi_{P^*} = 1$. Let $P \subset N_\mathbb{Q}$ be a Fano triangle. Recall that the corresponding toric variety X_P is a *fake weighted projective plane* [12]: a quotient of a weighted projective plane by a finite group N/N' acting free in codimension one, where N' is the sublattice generated by the vertices of P.

Example 3.7 (Mutations of $\mathbb{P}^2$)**.** In [4, 11] the graph of mutations of $\mathbb{P}^2$ is constructed. The vertices of this graph are given by $\mathbb{P}(a^2,b^2,c^2)$, where $(a,b,c)\in\mathbb{Z}^3_{>0}$ is a *Markov triple* satisfying
$$a^2+b^2+c^2 = 3abc. \tag{3.1}$$
Let $X_P = \mathbb{P}(a^2,b^2,c^2)$ be such a weighted projective plane, with $P\subset N_\mathbb{Q}$ the corresponding Fano triangle. Since X_P is qG-deformation-equivalent to $\mathbb{P}^2$, so X_P is smoothable and its anti-canonical Hilbert function has quasi-period one. Hence $\pi_{P^*} = 1$. However, the denominator r_{P^*} of P^* can be arbitrarily large. To see this, note first that a,b,c must be pairwise coprime: if $p \mid a$ and $p \mid b$ then $p^2 \mid 3abc = a^2+b^2+c^2$, and hence $p \mid c$; but then p appears as a square on the left-hand side and as a cube on the right-hand side of (3.1). Let $\bar{b}$ be an inverse of $b \pmod{a^2}$. Note that $c^2\bar{b}^2+1 \equiv (3abc-b^2)\bar{b}^2+1 \equiv 3a\bar{b}c \pmod{a^2}$, and so the singularity $\frac{1}{a^2}(b^2,c^2)$ on X_P has local index
$$\frac{a^2}{\gcd\left\{a^2, c^2\bar{b}^2+1\right\}} = \begin{cases} a, & \text{if } a \not\equiv 0 \pmod 3;\\ a/3, & \text{if } a\equiv 0 \pmod 3.\end{cases}$$

Considering equation (3.1) (mod 3) shows that no Markov numbers are divisible by three. Hence the three local indices on X_P are a, b, and c, and so $r_{P^*} = abc$. The two triangles P and Q in Example 2.5 are the simplest examples, arising from the Markov triples $(1,1,1)$ and $(1,1,2)$ respectively, and corresponding to $\mathbb{P}^2$ and $\mathbb{P}(1,1,4)$.

We end by observing that there exist Fano triangles of quasi-period one not arising from the construction in Example 3.7.

Example 3.8. Consider

$$P = \mathrm{conv}\{(3,2), (-1,2), (-1,-2)\} \subset N_{\mathbb{Q}}.$$

The corresponding fake weighted projective plane $X_P = \mathbb{P}(1,1,2)/(\mathbb{Z}/4)$ has $2 \times \frac{1}{4}(1,3)$ and $\frac{1}{8}(1,3)$ T-singularities. We see that P^* has $r_{P^*} = 2$ and $\pi_{P^*} = 1$. In fact X_P is qG-smoothable to the nonsingular del Pezzo surface of degree two, and hence $L_{P^*}(k) = k^2 + k + 1$.

We expect that the algebro-geometric methods of this paper can be used to classify all triangles dual to Fano triangles with quasi-period 1, and hope to extend these methods to treat more general rational polygons, as well as higher dimensional polytopes.

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References

[1] Mohammad Akhtar, Tom Coates, Alessio Corti, Liana Heuberger, Alexander M. Kasprzyk, Alessandro Oneto, Andrea Petracci, Thomas Prince, and Ketil Tveiten. Mirror symmetry and the classification of orbifold del Pezzo surfaces. *Proc. Amer. Math. Soc.*, 144(2):513–527, 2016.

[2] Mohammad Akhtar, Tom Coates, Sergey Galkin, and Alexander M. Kasprzyk. Minkowski polynomials and mutations. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 8:Paper 094, 17, 2012.

[3] Mohammad Akhtar and Alexander M. Kasprzyk. Singularity content. arXiv:1401.5458 [math.AG], 2014.

[4] Mohammad Akhtar and Alexander M. Kasprzyk. Mutations of fake weighted projective planes. *Proc. Edinb. Math. Soc. (2)*, 59(2):271–285, 2016.

[5] Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, and Alexander M. Kasprzyk. Mirror symmetry and Fano manifolds. In *European Congress of Mathematics*, pages 285–300. Eur. Math. Soc., Zürich, 2013.

[6] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[7] Jesús A. De Loera and Tyrrell B. McAllister. Vertices of Gelfand-Tsetlin polytopes. *Discrete Comput. Geom.*, 32(4):459–470, 2004.

[8] Jesús A. De Loera and Tyrrell B. McAllister. On the computation of Clebsch-Gordan coefficients and the dilation effect. *Experiment. Math.*, 15(1):7–19, 2006.

[9] E. Ehrhart. Sur un problème de géométrie diophantienne linéaire. II. Systèmes diophantiens linéaires. *J. Reine Angew. Math.*, 227:25–49, 1967.

[10] Christian Haase and Tyrrell B. McAllister. Quasi-period collapse and $\mathrm{GL}_n(\mathbb{Z})$-scissors congruence in rational polytopes. In *Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics*, volume 452 of *Contemp. Math.*, pages 115–122. Amer. Math. Soc., Providence, RI, 2008.

[11] Paul Hacking and Yuri Prokhorov. Smoothable del Pezzo surfaces with quotient singularities. *Compos. Math.*, 146(1):169–192, 2010.

[12] Alexander M. Kasprzyk. Bounds on fake weighted projective space. *Kodai Math. J.*, 32(2):197–208, 2009.

[13] Alexander M. Kasprzyk and Benjamin Nill. Fano polytopes. In *Strings, gauge fields, and the geometry behind*, pages 349–364. World Sci. Publ., Hackensack, NJ, 2013.

[14] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. *Invent. Math.*, 91(2):299–338, 1988.

[15] Tyrrell B. McAllister and Kevin M. Woods. The minimum period of the Ehrhart quasi-polynomial of a rational polytope. *J. Combin. Theory Ser. A*, 109(2):345–352, 2005.

[16] G. Pick. Geometrisches zur zahlenlehre. *Sonderabdr. Naturw.-medizin. Verein f. Böhmen "Lotos"*, (8, 9):S. 8°, 1899.

[17] Jonathan M. Wahl. Elliptic deformations of minimally elliptic singularities. *Math. Ann.*, 253(3):241–262, 1980.

[18] Ben Wormleighton. Reconstruction of singularities on orbifold del Pezzo surfaces from their Hilbert series. *Communications in Algebra*, 48(1):119–140, 2020.

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