Geometry and arithmetic of schemes

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Abstract

This dissertation gives a general introduction to the theory and applications of schemes. Chapter 2 defines a scheme and gives some basic constructions. Chapter 3 studies the geometric properties. Chapter 4 gives an application to number theory.

Contents

1 Introduction

In the 1960s, Grothendieck built the theory of schemes, which shortly became the central object in modern algebraic geometry and arithmetic geometry. This dissertation is a quick introduction to the theory and application of schemes.

The second chapter of this dissertation is the basic theory of schemes. As a generalisation of algebraic varieties, we use a topological way to define a scheme and functions on it. Like varieties, maps of schemes will also be discussed. Then various constructions on schemes will be introduced. This chapter is about defining the basic terms of schemes for later chapters (In this chapter, the main reference I follow is [1]).

The third chapter of this dissertation is about how schemes can be viewed as geometric objects rather than abstract sets. The notation of dimension, regularity and smoothness will be studied, which are the basic notions of a geometric object. Various results about them will also be proven. Then the focus will move to abstract geometric constructions like vector bundles and divisors. Some algebraic and homological method will be used to study them, in particular Čech cohomology Picard groups will be introduced to help the proofs (In this chapter, the main reference I follow is [1]).

The fourth chapter of this dissertation focuses on how to use the theory developed from the previous chapters to study number theory. A very interesting class of schemes called arithmetic surface will be discussed. There is also a section introducing a very modern tool to study number theory, which is called p-divisible groups (In this chapter, the main references I follow are [9] and [8]).

The final chapter will be a quick recap, and discusses what follows from each section, and what iterature to read.

2 Sheaves and schemes

We have seen in MATH4002 the notion of algebraic varieties, which is the common zero set of some polynomials. In this chapter, the construction of schemes will be introduced, which can be considered as a generalisation of algebraic varieties. In classical algebraic geometry, the study of affine varieties plays an important role. And we can proof there is an equivalence between the category of affine varieties and the category of finitely generated reduced algebras. This essentially means that study affine varieties is the same as study finitely generated reduced algebras, which are coordinate rings of some affine varieties. However, here is a problem :

What about a general commutative rings?

The problem was solved by Grothendieck, who introduced the theory of schemes. Roughly speaking he considered every commutative unital ring A as the 'coordinate ring' of some geometric object, more precisely $\mathrm{Spec}(A)$.

In this chapter, A is a commutative unital ring, unless otherwise stated.

2.1 Sheaves

Defining an affine scheme is not as easy as defining it as $\mathrm{Spec}(A)$, we need more information. In particular, we need to understand the functions on it, and the notion of sheaves helps us to do so.

But defining a sheaf directly is rather complicated, so we first introduce the notion of a presheaf.

Definition 2.1.1 (Presheaf). A **presheaf** $\mathscr{F}$ on a topological space X is the following data:

(1) For each open subset $U \subseteq X$, we have the set $\mathscr{F}(U) = \Gamma(U, \mathscr{F})$, which is called a **section** of $\mathscr{F}$ over U. When $U = X$, we have $\mathscr{F}(X) = \Gamma(X, \mathscr{F})$, we call it the **global section** of $\mathscr{F}$.

(2) For each inclusion of open sets $U \subset V$, we have the restriction map $\mathrm{res}_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ such that $\mathrm{res}_{U,U} = \mathrm{id}_{\mathscr{F}(U)}$. For the inclusion $U \subset V \subset W$ of open sets, the following diagram commutes:

$$\begin{tikzcd} & \mathscr{F}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathscr{F}(W) \arrow[ur, "\mathrm{res}_{W,V}"] \arrow[r, "\mathrm{res}_{W,U}"] & \mathscr{F}(U) \end{tikzcd} .$$

Definition 2.1.2 (Germ and stalk). Let $\mathscr{F}$ be a presheaf on X, let $p \in X$. A **germ** at p is the sections over some open subsets of X containing p. More precisely, germs are in the form:

$$\{(f,U) : p \in U, f \in \mathscr{F}(U), U \subseteq X \text{ open}\}.$$

Consider the equivalence relation: $(f,U) \sim (g,V)$ if there is an open subset $W \subset U,V$ where $p \in W$ and $\text{res}_{U,W} f = \text{res}_{V,W} g$. The **stalk** at p (denoted by $\mathscr{F}_p$) is defined to be the equivalence class:

$$\{(f,U) : p \in U, f \in \mathscr{F}(U), U \subseteq X \text{ open}\}/\sim.$$

Now we have constructed everything we need to define a sheaf.

Definition 2.1.3 (Sheaf)**.** A presheaf $\mathscr{F}$ on X is called a **sheaf** if it satisfies the following two conditions:

(Identity) If $\{U_i\}_{i\in I}$ is an open cover of $U \subseteq X$, and $f_1, f_2 \in \mathscr{F}(U)$, and f_1 and f_2 agree on U_i under the restriction map for all $i \in I$, then $f_1 = f_2$.

(Gluability) If $\{U_i\}_{i\in I}$ is an open cover of U, then given $f_i \in \mathscr{F}(U_i)\ \forall i,j \in I$, we have $\text{res}_{U_i,U_i\cap U_j} f_i = \text{res}_{U_j,U_i\cap U_j} f_j\ \forall i,j \in I$, then there exists $f \in \mathscr{F}(U)$ such that $\text{res}_{U,U_i} f = f_i\ \forall i \in I$.

Although defining a sheaf is complicated, we can see it captures information about nicely-behaved functions on certain topological spaces.

Example 2.1.4 (Regular functions on quasi-projective varieties)*.* Consider a quasi-projective variety V over some algebraically closed field k. Denote the set of all regular functions on V to be $\mathscr{O}_V$. It is esay to check $\mathscr{O}_V$ is a presheaf on V. We are going to check $\mathscr{O}_V$ is a sheaf.

To check the identity axiom, consider $\{U_i\}_{i\in I}$ an open cover of U an open subset of V. Let $f_1, f_2 \in \mathscr{O}_V(U)$ and $f_1 = f_2$ on each sections $\mathscr{O}_V(U_i)$. So by definition of regular functions, $f_1 = f_2$ on $\mathscr{O}_V(U)$.

To check the gluability axiom, let $f_i, f_j \in \mathscr{O}_V(U_i)$ and $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. Then consider the function:

$$f(x) := \begin{cases} f_i(x), & \text{if } x \in U_i \\ f_j(x), & \text{if } x \in U_j \end{cases}$$

We can see the gluability axiom is satisfied. Hence $\mathscr{O}_V$ is a sheaf on V and it is called the structure sheaf on V.

Example 2.1.5 (Smooth functions on differential manifolds)*.* Let M be a differential manifold in $\mathbb{R}^n$. The set of all smooth functions on M denoted by $\mathscr{O}_M$ forms a sheaf, and for each open subset $U \subseteq M$ we have the section $\mathscr{O}_M(U)$. Checking $\mathscr{O}_M$ is a sheaf is analogue to Example 2.1.4.

We have now seen two illustrative examples to show that sheaf works well in the context of geometry.

Remark 2.1.6*.* Consider the structure sheaf $\mathscr{O}_V$ as in Example 2.1.4. Consider the stalk $\mathscr{O}_p$ for some $p \in U$. Then immediately from the definition of a sheaf, $\mathscr{O}_p$ is a ring. Consider the functions vanishing at p, denoted by $\mathfrak{m}_p$. This is clearly an ideal of $\mathscr{O}_p$ as if $f, g \in \mathfrak{m}_p$, so is $f + g$, and if $f \in \mathscr{O}_p$, $g \in \mathfrak{m}_p$ then $fg \in \mathfrak{m}_p$.

In fact $\mathfrak{m}_p$ is the only maximal ideal of $\mathscr{O}_p$. Note that a regular function at p is invertible if and only if it is non-zero. Take the quotient ring $\mathscr{O}_p/\mathfrak{m}_p$, take $f + g \in \mathscr{O}_p/\mathfrak{m}_p$ where $f \in \mathscr{O}_p \setminus \mathfrak{m}_p,\ g \in \mathfrak{m}_p$. Notice that $f + g \neq 0$ as $\mathfrak{m}_p$ contains all functions vanishing at p, so $f \neq 0$ so is $f + g$, hence $f + g$ is invertible in this function ring, also all elements in $\mathscr{O}_p \setminus \mathfrak{m}_p$ are invertible, so $\mathscr{O}_p/\mathfrak{m}_p$ is a field, so $\mathfrak{m}_p$ is maximal.

Take $\mathfrak{n}_p$ be a maximal ideal of $\mathscr{O}_p$. So $\mathscr{O}_p/\mathfrak{n}_p$ is a field. So $f + g$ is invertible for $f \in \mathscr{O}_p \setminus \mathfrak{n}_p$, $g \in \mathfrak{n}_p$. But all elements in $\mathscr{O}_p \setminus \mathfrak{n}_p$ are invertible, so we may conclude $\mathfrak{n}_p \subseteq \mathfrak{m}_p$ or $\mathfrak{m}_p \subseteq \mathfrak{n}_p$. But $\mathfrak{n}_p$ and $\mathfrak{m}_p$ are maximal, so $\mathfrak{n}_p = \mathfrak{m}_p$. Hence $\mathfrak{m}_p$ is the only maximal ideal of $\mathscr{O}_p$. Thus $\mathscr{O}_p$ is a local ring with the residue field $\mathscr{O}_p/\mathfrak{m}_p$.

In addition, we can show that $\mathscr{O}_p/\mathfrak{m}_p \cong k$ by $f \mapsto f(p)$. And $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a k-vector space, and we can hence to define the tangent space to V at p by $\mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k)$. The scheme-theoretic version of tangent spaces will be discussed in detail in Chapter 3.

Now we are going to see an important example of sheaves, a structure sheaf on $\mathrm{Spec}(A)$ for some commutative unital ring A. At this point, we need to assume that $\mathrm{Spec}(A)$ is a topological space, this will be proved in the next section. And the structure sheaf is one of the reasons why the geometric theory behind $\mathrm{Spec}(A)$ is very rich. This also answers the question (at least partially), it is possible to study geometry over a general commutative unital ring.

Example 2.1.7 (The structure sheaf on $\mathrm{Spec}(A)$)*.* Let us take the topological space $\mathrm{Spec}(A)$, points in $\mathrm{Spec}(A)$ are prime ideals $\mathfrak{p}$, but for the sake of unambiguity, let us denote points in Spec(A) by $[\mathfrak{p}]$. So we have the structure sheaf $\mathscr{O}_{\text{Spec}(A)}$. We use the letter $\mathscr{O}$ because we want to think of functions on $[\mathfrak{p}]$. This might seem strange: because the stalks will be rings, and germs are elements of the rings, this means we consider elements in A as functions on prime ideals, but the reason will become clear later. To compute the value of the function $a \in A$ at the point $[\mathfrak{p}]$, we just take $a([\mathfrak{p}]) = a \pmod{\mathfrak{p}}$. For example, take $A = \mathbb{Z}$, take $6 \in \mathbb{Z}$ as a function on (2), then $6([(2)]) = 6 \pmod{(2)} = 0$.

$\mathscr{O}_{\mathfrak{p}}$ is a local ring, which is isomorphic to the localisation of A at $[\mathfrak{p}]$, i.e $A_{\mathfrak{p}}$. The proof is very similar to the previous remark 2.1.6.

Definition 2.1.8 (Pushforward)**.** Let $\pi : X \to Y$ be a continuous map, and $\mathscr{F}$ is a sheaf on X. Let $V \subseteq Y$ be an open subset, then define $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$ to be the **pushforward of $\mathscr{F}$ by** π.

Definition 2.1.9 (Restriction of sheaves)**.** Let $\mathscr{F}$ is a sheaf on X, and U is an open subset of X. Define the **restriction of $\mathscr{F}$ to** U, denoted by $\mathscr{F}|_U$ to be the collection $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for all open subsets $V \subseteq U$.

Definition 2.1.10 (Morphism of sheaves)**.** Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. A **morphism** between $\mathscr{F}$ and $\mathscr{G}$ consists of maps $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ such that for $U \subset V$:

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "\text{res}_{V,U}"] & \mathscr{G}(V) \arrow[d, "\text{res}_{V,U}"] \\
\mathscr{F}(U) \arrow[r, "\phi(U)"] & \mathscr{G}(U)
\end{tikzcd}$$

commutes.

This is basically everything we need in the theory of sheaves to formally define an affine scheme. However, sheaf theory is also a very elegant subject on its own right, for example the theory of quasi-coherent sheaves has nice interactions with line bundles, which will be discussed later.

2.2 Properties of Spec(A)

We have seen in Example 2.1.7, Spec(A) forms a topological space. Here we are going to prove this fact, before that we need some preparation.

Definition 2.2.1 (Vanishing set)**.** Let S be a subset of a ring A. Define the **vanishing set** of S by: $\mathbb{V}(S) := \{[\mathfrak{p}] \in \mathrm{Spec}(A) : S \subset \mathfrak{p}\}$.

Lemma 2.2.2 (Zariski topology on Spec(A))**.** *Let A be a ring, then there is a unique topology on Spec(A) by setting the topology to be $\mathfrak{T} := \{Spec(A) \setminus \mathbb{V}(S)\}$, for all subset $S \subset A$. The topology is called the Zariski topology.*

Proof. Firstly, consider $S = \emptyset$ and $S = \mathrm{Spec}(A)$. Then we have $\mathbb{V}(\emptyset) = \mathrm{Spec}(A)$ and $\mathbb{V}(\mathrm{Spec}(A)) = \emptyset$, which are in $\mathfrak{T}$.

Now consider $\bigcap_{i \in I} \mathbb{V}(S_i)$. From the definition, we can easily check that $\bigcap_{i \in I} \mathbb{V}(S_i) = \mathbb{V}(\bigcup_{i \in I} S_i)$. So taking intersection is preserved in $\mathfrak{T}$.

Finally we need to check union is also preserved in $\mathfrak{T}$. Let S_1 and S_2 be two subsets of A. Consider $\mathbb{V}(S_1) \cup \mathbb{V}(S_2)$, now define $S_1 S_2 := \{fg : f \in S_1, g \in S_2\}$. It is clear that $\mathbb{V}(S_1) \cup \mathbb{V}(S_2) \subseteq \mathbb{V}(S_1 S_2)$. Conversely, consider $\mathbb{V}(S_1 S_2) = \{[\mathfrak{p}] \in \mathrm{Spec}(A) : S_1 S_2 \subset \mathfrak{p}\}$. We have the condition $S_1 S_2 \subset \mathfrak{p}$, which gives that $fg \in \mathfrak{p}$ for $f \in S_1, g \in S_2$. Since $\mathfrak{p}$ is a prime ideal, so $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Which implies that $\mathbb{V}(S_1 S_2) \subseteq \mathbb{V}(S_1)$ or $\mathbb{V}(S_2)$. Hence $\mathbb{V}(S_1) \cup \mathbb{V}(S_2) = \mathbb{V}(S_1 S_2)$. And this can be generalised to any arbitrary number of sets of unions. Hence taking union is preserved in $\mathfrak{T}$. Thus $\mathfrak{T}$ is a topology on Spec(A). Hence Spec(A) is a topological space. $\square$

Definition 2.2.3 (Distinguished open set)**.** Let $f \in A$, and define the **distinguished open set**: $D(f) := \{[\mathfrak{p}] \in \mathrm{Spec}(A) : f \notin \mathfrak{p}\} = \{[\mathfrak{p}] \in \mathrm{Spec}(A) : f([\mathfrak{p}]) \neq 0\}$.

We can easily observe that $D(f)$ is indeed an open subset of Spec(A). We want to show that the collection of all $D(f)$ forms a basis for the Zariski topology.

Lemma 2.2.4 (Basis for the Zariski topology)**.** *The collection of all $D(f)$ forms a basis for the Zariski topology.*

Proof. Let us take U be an open subset of Spec(A). Then it can be written as the complement of closed sets. In particular, $U^c = \mathbb{V}(S) = \bigcap_{f \in S} \mathbb{V}((f))$. So by set theory, we have $U = \bigcup_{f \in S} D(f)$. Hence the collection of $D(f)$ forms a basis for the Zariski topology. $\square$

Before we really see the definition of affine schemes, it is good to see some examples where geometry matters on Spec(A).

Example 2.2.5 (The affine n-space). For an algebraically closed field k, we have the polynomial ring $k[x_1, \ldots, x_n]$. By Nullstellensatz, maximal ideals of $k[x_1, \ldots, x_n]$ are in the form $((x - a_1, \ldots, x - a_n))$, i.e points in k^n. Also $k[x_1, \ldots, x_n]$ is a UFD, so any irreducible polynomial generate a prime ideal. So in particular, (0) is a prime ideal. So we may now define the affine n-space over k, denoted by $\mathbb{A}^n_k$, to be $\mathrm{Spec}(k[x_1, \ldots, x_n])$. We can see that $\mathrm{Spec}(k[x_1, \ldots, x_n])$ captures all points in k^n (we do not consider irreducible polynomials like $x_1^2 + x_2^2 + x_3^2$ here, they will not provide us new information other than k, because k is algebraically closed, more precisely the only finite extension of k is itself), and in addition, it captures $[(0)]$, which we call a generic point. One way to think of it is to, consider all lines passing through all points in k^n. The generic point lies on the intersection of all these lines. We will make this clear and rigorous later.

On the other hand, we use Spec because it is functorial, while maxSpec is not. Being functorial is very important, we will see the reason shortly.

Example 2.2.6 (Affine line over $\mathbb{Q}$). For the sake of number-theoretic interests, we want to consider the affine line $\mathbb{A}^1_{\mathbb{Q}}$. Which we can copy the definition above, $\mathbb{A}^1_{\mathbb{Q}} := \mathrm{Spec}(\mathbb{Q}[x])$. We know that $\mathbb{Q}$ is not algebraically closed. So there are non-linear irreducible polynomials. So $\mathrm{Spec}(\mathbb{Q}[x])$ contains information not only about rationals $\mathbb{Q}$, but also about all finite extension of rationals, i.e number fields. For example, for $(x^2 - 2) \in \mathrm{Spec}(\mathbb{Q}[x])$, we have $\mathbb{Q}[x]/(x^2 - 2) \cong \mathbb{Q}(\sqrt{2})$.

2.3 Affine schemes

We are now ready to introduce an affine scheme, which is one of the central object of this dissertation. Recall from Example 2.1.7 we have the notion of structure sheaf on Spec(A).

Theorem 2.3.1. *$\mathscr{O}_{Spec(A)}$ is a sheaf on Spec(A).*

Proof. See Theorem 5.1 in [2]. □

Proposition 2.3.2 (Sections on distinguished open sets)**.** *The section $\mathscr{O}_{Spec(A)}(D(f))$ is isomorphic to the localisation of A at f, i.e. $A_f := \{\frac{a}{f^n} : a \in A, n \in \mathbb{N}_0\}$.*

Proof. Notice that $\mathscr{O}_{\text{Spec}(A)}(D(f))$ is the collection of all functions on the points of Spec(A) who does not vanish at f. So we can consider the ring homomorphism:

$$\mathscr{O}_{\text{Spec}(A)}(D(f)) \to A_f$$

$$g \mapsto gf$$

with the inverse

$$A_f \to \mathscr{O}_{\text{Spec}(A)}(D(f))$$
$$h \mapsto \frac{h}{f}.$$

So we have an isomorphism of rings: $\mathscr{O}_{\text{Spec}(A)}(D(f)) \cong A_f$. □

Definition 2.3.3 (Ringed space). A **ringed space** is defined by the pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is the sheaf of functions on X.

Definition 2.3.4 (Isomorphism of ringed spaces). An **isomorphism** between ringed spaces $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ is a homeomorphism of topological spaces $\pi : X \to Y$, and an isomorphism of sheaves $\phi : \mathscr{O}_X \to \mathscr{O}_Y$.

Definition 2.3.5 (Affine schemes). An **affine scheme** is a ringed space $(X, \mathscr{O}_X)$ isomorphic to $(\text{Spec}(A), \mathscr{O}_{\text{Spec}(A)})$ for some ring A.

Definition 2.3.6 (Schemes). A **scheme** is a ringed space $(X, \mathscr{O}_X)$ such that for each point of X there is an open neighborhood U such that $(U, \mathscr{O}_X|_U)$ is an affine scheme.

Remark 2.3.7. From Defintion 2.3.5, we can see that taking global section on Spec(A) recovers the ring, i.e. $\Gamma(\text{Spec}(A), \mathscr{O}_{\text{Spec}(A)}) = A$. However, we will see later, this is not generally true, this only works with affine schemes.

From the definition, we can already see some motivations of why schemes are generalisation of varieties. Affine schemes are "building blocks" of a general scheme. Conversely, a general scheme can be locally viewed as an affine scheme.

Definition 2.3.8 (Open subschemes). Let $(X, \mathscr{O}_X)$ be a scheme, let $U \subseteq X$ be an open subset. We define $(U, \mathscr{O}_X|_U)$ to be an **open subscheme** of X. If $(U, \mathscr{O}_X|_U)$ is an affine scheme, then we call U an **affine open subset**.

Definition 2.3.9 (Locally ringed space)**.** A ringed space is called a **locally ringed space** if all of its stalks are local rings. From Example 2.1.7 affine schemes are locally ringed spaces. The local ring $\mathscr{O}_{\mathrm{Spec}(A),\mathfrak{p}}$ has the only maximal ideal $\mathfrak{m}_{\mathfrak{p}}$, and we denote the residue field $\mathscr{O}_{\mathrm{Spec}(A),\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ by $k(\mathfrak{p})$.

Now we can move to some examples of schemes.

Example 2.3.10 (Affine varieties as schemes). This is more or less a trivial example, but it is worth thinking about it.

Let V be an affine variety over some field k. Then by the equivalence between the category of affine varieties and the category of finitely generated reduced k-algebras, studying V is the same thing as studying its coordinate ring $k[V]$. So we can consider $(\mathrm{Spec}(k[V]), \mathscr{O}_{\mathrm{Spec}(k[V])})$ as an affine scheme, which corresponds to the affine variety V. In particular, we have $V \cong \mathrm{Spec}(k[V])$.

Example 2.3.11 (Affine line without 0). Let $A = k[x]$ for some algebraically closed field k. Then $\mathbb{A}^1 = \mathrm{Spec}(A)$. Now let us find the functions on the open set: $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{A}^1 \setminus \{[(x)]\}$. Obviously, we can see $U = D(x)$. So by Proposition 2.3.2, $\Gamma(U, \mathscr{O}_{\mathbb{A}^1}) = \Gamma(D(x), \mathscr{O}_{\mathrm{Spec}(A)}) \cong A_x \cong k[x, \frac{1}{x}]$.

Now consider the ringed space $(U, \mathscr{O}_{\mathbb{A}^1}|_U)$. Taking the global section $\Gamma(U, \mathscr{O}_{\mathbb{A}^1}|_U)$ we will get A_x, i.e all functions whose denominators are powers of x. Therefore, $(U, \mathscr{O}_{\mathbb{A}^1}|_U)$ is affine as a scheme which is isomorphic to $(\mathrm{Spec}(A_x), \mathscr{O}_{\mathrm{Spec}(A_x)})$.

Example 2.3.12 (Affine 3-space without 0). Unlike Example 2.3.11, $\mathbb{A}^3 \setminus \{0\} = \mathbb{A}^3 \setminus \{[(x, y, z)]\}$ fails to be affine as a scheme.

Let us take $A = k[x, y, z]$, $\mathrm{Spec}(A) = \mathbb{A}^3$, $U = \mathbb{A}^3 \setminus \{[(x, y, z)]\}$. Here is the problem: U can not be identified as a single distinguished open set, but as a union of distinguished open sets. In particular, we find that $U = D(x) \cup D(y) \cup D(z)$. So if we take the section $\Gamma(U, \mathscr{O}_{\mathbb{A}^3})$, we should have the ring of functions whose denoinators are powers of x and y and z. In particular, that ring must be the ring A, as the only possible denominators are $x^0 = y^0 = z^0 = 1$. So $\Gamma(U, \mathscr{O}_{\mathbb{A}^3}) \cong A$. Let us assume $(U, \mathscr{O}_{\mathbb{A}^3}|_U)$ is affine. So we take the global section $\Gamma(U, \mathscr{O}_{\mathbb{A}^3}|_U) \cong A$. But $\Gamma(U, \mathscr{O}_{\mathbb{A}^3}) \cong A$, so we have $U \cong \mathbb{A}^3$. But this is clearly a contradiction, hence $(U, \mathscr{O}_{\mathbb{A}^3}|_U)$ is not affine as a scheme.

Definition 2.3.13 (Generic point)**.** Let X be a scheme, let $p \in X$. We say p is a **generic point** if $\{p\}$ is dense. In other words, $\overline{\{p\}} = X$.

Definition 2.3.14 (Associated point)**.** Let $(X, \mathscr{O}_X)$ be a scheme, let $p \in X$. We say p is an **associated point** of X if for all $f \in \mathfrak{m}_X \subseteq \mathscr{O}_{X,p}$, we have $\mathscr{O}_{X,p} \xrightarrow{\times f} \mathscr{O}_{X,p}$ is not injective.

Definition 2.3.15 (Rational functions)**.** Let X be a scheme. The **rational functions** on X are defined to be the equivalence class $\{(U, f)\}/\sim$, where $U \subseteq X$ contains all associated points and $f \in \Gamma(U, \mathscr{O}_X)$ and $(U, f) \sim (U', f')$ if f and f' agrees on $U \cap U'$. When X is an integral scheme(see Definition 2.7.6), then all rational functions forms a field, called the **function field** of X, denoted by $K(X)$.

2.4 Glueing schemes

Once we have the notion of affine schemes, it is natural to ask if there is a type of schemes called projective schemes. The answer is yes, but we are not yet there.

In the classical language, we learn projective varieties as glueing affine patches together, these affine patches are affine varieties. And we could do the same thing to study projective schemes, as gluing affine schemes together. This is a natural thing to do because glueing spectrums is basically glueing topological spaces.

Theorem 2.4.1 (Glueing schemes)**.** *Let X_i be schemes, where $i \in I$. Suppose for all $i, j \in I$, we have $X_{i,j} \subseteq X_i = X_{i,i}$ is an open subscheme and the isomorphism $\phi_{i,j} : X_{i,j} \to X_{j,i}$ such that $\phi_{i,k} = \phi_{j,k}(\phi_{i,j})$ for all $i, j, k \in I$ on $X_{i,j,k} = X_{i,j} \cap X_{j,k}$. Then we have a new scheme covered by X_i, i.e:*

$$X = \bigcup_{i \in I} X_i.$$

Proof. See Section 26.14 in [4] for a proof. □

Theorem 2.4.1 tells us that we can really glue schemes together to get a new scheme, and this process is unique up to isomorphism.

Example 2.4.2 (The projective line, Example 4.4.6 in [1]). Let us take two affine schemes we had in Exmaple 2.3.11, say, $U = \mathrm{Spec}(k[x, \frac{1}{x}])$ and $V = \mathrm{Spec}(k[y, \frac{1}{y}])$. Consider the isomorphism via $x \mapsto \frac{1}{y}$ We have glued two affine lines to get a new scheme, called the projective line over k, denoted by $\mathbb{P}^1_k$.

As the name suggests, $\mathbb{P}^1$ is not affine as a scheme. To see this, we just compute the global section $\Gamma(\mathrm{Spec}(\mathbb{P}^1), \mathscr{O}_{\mathbb{P}^1})$. By construction, on $\mathbb{P}^1$. we need functions on $\mathrm{Spec}(k[x])$ to agree with the functions on $\mathrm{Spec}(k[y])$. That is, $f(x) = g(\frac{1}{x})$. But we need both $f(x)$ and $g(\frac{1}{x})$ be polynomials, which cannot happen at same time unless $\deg(f) = \deg(g) = 0$. Hence the global section is the set of all degree zero polynomials over k, i.e. the field k. But this is a contradiction, as $\mathrm{Spec}(k)$ has only one point, where $\mathbb{P}^1$ clearly contains more points than one.

Now we can generalise this glueing construction to dimension n to get the projective n-space, denoted by $\mathbb{P}^n$. And we can also see $\mathbb{P}^n$ is not affine: the argument is similar to the example above.

Definition 2.4.3 (Projective space over A)**.** We may define the **projective n-space** over any ring A, denoted by $\mathbb{P}^n_A$, in a similar way to above.

It seems strange at first to define the projective n-space over any ring. But in the context of schemes this is natural to do and particularly useful when we want to study projective schemes. Because one way to define a projective scheme is to say a scheme is projective over A if it is isomorphic to a closed subscheme of $\mathbb{P}^n_A$.

2.5 Proj and projective schemes

Definition 2.5.1 (Graded ring)**.** A **$\mathbb{Z}$-graded ring** is a ring $S = \bigoplus_{n\in\mathbb{Z}} S_n$ where each S_n is an S_0-module and S is an S_0-algebra. The elements of S_n are called **homogeneous elements** of S. An ideal $I \subset S$ is called a **homogeneous ideal** if it is generated by homogeneous elements. If $s \in S_n$ is a homogeneous element in S, then we say s is of degree n.

In this section, all graded rings are in the form $\bigoplus_{n\in\mathbb{N}_0} S_n$.

We saw in Example 2.4.2, spectrum does not work well with projective spaces as a functor, so we need the projective version of spectrum in order to define a projective scheme. In particular, if we take $S = A[x_0, \dots, x_n]$ to be a graded ring over A (which means $S_0 = A$), we want the "projective" spectrum recovers $\mathbb{P}^n_A$. Moreover, we do not want to include a bad point. For example, $[0 : 0 : \cdots : 0]$ is not even defined in $\mathbb{P}^n_A$. So we want to exclude some ideal, called the **irrevelant ideal** $S_+ := \bigoplus_{n>0} S_n$.

Definition 2.5.2 (Proj construction)**.** Define the **homogeneous spectrum** by the set of all homogeneous prime ideals of S not containing the irrevelant ideal S_+, we denote it by $\mathrm{Proj}(S)$.

Like we did for $\mathrm{Spec}(A)$, we need the vanishing sets and the distinguished open sets, hence the Zariski topology on $\mathrm{Proj}(S)$ in order to have a scheme structure.

Definition 2.5.3 (Vanishing set and distinguished open set)**.** Let $T \subset S$ be a subset consists of homogeneous elements of positive degree. Then the **vanishing set** of T is defined by $\mathbb{V}(T) := \{I \in \mathrm{Proj}(S) : T \subset I\}$.

Let $f \in S$, we define the **projective distinguished open set** $D(f) := \{I \in \mathrm{Proj}(S) : f \notin I\}$.

We can also prove there is a topology on $\mathrm{Proj}(S)$ called the Zariski topology like we did in Lemma 2.2.2. The proof is the same idea, which will be omitted here.

Definition 2.5.4 (Projective n-space)**.** We may now define the **projective n-space** by:

$$\mathbb{P}^n_A := \mathrm{Proj}(A[x_0, \ldots, x_n])$$

and the variables $x_0, \ldots, x_n$ are called projective coordinates on $\mathbb{P}^n_A$.

Remark 2.5.5. This is a better definition of projective space than the previous one in 2.4.3, as this definition is more intrinsic, i.e. does not require a particular glueing process or choices of affine patches.

Definition 2.5.6 (Projective A-scheme)**.** Let S be a finitely generated graded ring over A, a ringed space that isomorphic to $(\mathrm{Proj}(S), \mathscr{O}_{\mathrm{Proj}(S)})$ is called a **projective A-scheme**.

Example 2.5.7 (Projective varieties as schemes)*.* Like what we did in Example 2.3.10, we have similar situation here. However, we need our projective variety V to be generated by a homogeneous prime ideal I. Then we have the homogeneous coordinate ring $k[V]$, which is a k-graded ring. So we have the projective k-scheme $\mathrm{Proj}(k[V]) \cong V$.

2.6 Morphisms

We can define morphisms between schemes as a map between ring spectrums together with a map between sheaves. However, it makes more sense to define a morphism of ringed spaces first.

Definition 2.6.1 (Morphism of ringed spaces)**.** A **morphism of ringed space** is a continuous map $\pi : X \to Y$ together with a map of sheaves $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$.

If $U, V \subseteq Y$ are an open subsets, then define the open embedding to be the isomorphism $(U, \mathscr{O}_U) \cong (V, \mathscr{O}_Y|_V)$ via $U \cong V$.

Lemma 2.6.2. *Let $\pi : X \to Y$ be a morphism of ringed space such that $\pi(p) = q$. Then there is a map of stalks $\mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$.*

Proof. Let $U \subseteq \mathscr{O}_Y$ be an open subset. By definition we have a map of sections:

$$\mathscr{O}_Y(U) \to \pi_* \mathscr{O}_X(\pi^{-1}(U)).$$

Without loss of generality, let us say $q \in U$, so by continuity of π, we have $p \in \pi^{-1}(U)$. In particular, we have the map of stalks induced by the map of sections above: $\mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$. □

Definition 2.6.3 (Morphism of locally ringed spaces)**.** A **morphism of locally ringed spaces** $\pi : X \to Y$ is a morphism of ringed spaces together with the map of stalks in Lemma 2.6.2 which send the maximal ideal in $\mathscr{O}_{Y,q}$ to the maximal in $\mathscr{O}_{X,p}$, i.e.:

$$\pi^{\#} : \mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}.$$

Lemma 2.6.4. *Let $\pi : Spec(A) \to Spec(B)$ be a morphism of locally ringed space. Then it is the morphism of locally ringed space induced by $\pi^{\#} : B \to A$.*

Proof. This is a quite technical proof, see Proposition 6.3.2 in [1]. □

Definition 2.6.5 (Morphism of schemes)**.** Let X and Y be schemes. A morphism $\pi : X \to Y$ as locally ringed spaces is called a **morphism of schemes**.

Remark 2.6.6. It seems that defining a morphism of schemes does need the technical Lemma 2.6.4. However, Lemma 2.6.4 tells us that a morphism between ring spectrums as locally ringed spaces is actually well-defined, hence we can define the morphism of schemes in the obvious way.

Example 2.6.7. Let $X = \mathrm{Spec}(k[x, y])$, $Y = \mathrm{Spec}(k[t])$ for some field k. Then we have an obvious morphism of locally ringed spaces:

$$\pi : \mathrm{Spec}(k[x, y]) \to \mathrm{Spec}(k[t])$$

$$\mathbb{A}^2 \to \mathbb{A}^1$$

$$(x, y) \mapsto x,$$

together with the map of stalks:

$$\pi^{\#} : \mathscr{O}_{Y,t} \to \mathscr{O}_{X,(x,y)}$$

$$\mathfrak{m}_t \mapsto \mathfrak{m}_{(x,y)}.$$

Hence this is a morphism of affine schemes.

Proposition 2.6.8. *Let X, Y be schemes and $\pi : X \to Y$ is a morphism of schemes, Spec$(A) \subseteq X$, Spec$(B) \subseteq Y$ be affine open subsets. If $\pi($Spec$(A)) \subseteq$ Spec(B), then the induced morphism of ringed spaces is a morphism of affine schemes.*

Proof. Note that $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are affine schemes, and open subschemes of X and Y respectively. So we have the morphisms of schemes:

$$f : \mathrm{Spec}(A) \to X$$

and

$$g : Y \to \mathrm{Spec}(B).$$

As f, π, g are morphisms of schemes, so they are continuous. Hence we have a map of locally ringed space by composition:

$$g \circ \pi \circ f : \mathrm{Spec}(A) \to \mathrm{Spec}(B).$$

Also we have $\pi(\mathrm{Spec}(A)) \subseteq \mathrm{Spec}(B)$. In particular, let $[\mathfrak{p}] \in \mathrm{Spec}(A)$, then $\pi([\mathfrak{p}]) = [\mathfrak{q}] \in \mathrm{Spec}(B)$. So by Lemma 2.6.2, we have an morphism of stalks:

$$\pi^{\#} : \mathscr{O}_{\mathrm{Spec}(B),\mathfrak{q}} \to \mathscr{O}_{\mathrm{Spec}(A),\mathfrak{p}}.$$

So we have a morphism of locally ringed space: $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$, hence a morphism of affine schemes. □

Definition 2.6.9 (Closed embedding and closed subscheme)**.** A morphism of schemes $\pi : X \to Y$ is a **closed embedding** if every point $q \in Y$ has an affine open neighbourhood U such that $\pi^{-1}(U) \subseteq X$ is an affine subscheme and the the morphism $\mathscr{O}_Y(U) \to \pi_*\mathscr{O}_X(\pi^{-1}(U))$ is surjective. In this case we say X is a **closed subscheme** of Y.

Proposition 2.6.10. *Let $X \subseteq \mathbb{P}^n_A$ be a closed subscheme, then X is a projective A-scheme.*

Proof. Immediately from Theorem 2.4.1 and Definition 2.6.9. □

Definition 2.6.11 (Finite morphism)**.** Let $\pi : X \to Y$ be a morphism, we say π is **finite** if for every affine open subset $\mathrm{Spec}(B)$ of Y, $\pi^{-1}(\mathrm{Spec}(B))$ is the spectrum of a B-algebra that is finitely generated B-module.

2.7 Varieties

We have seen a bit of the theory of schemes. Now we can try to define varieties using the scheme-theoretic language to review the classical theory of algebraic geometry.

Definition 2.7.1 (Reduced schemes)**.** A scheme X is said to be **reduced** if $\mathscr{O}_X(U)$ is a reduced ring for all open subset U of X.

This is a satisfying definition, because we have seen in the classical language: affine varieties correspond to finitely generated reduced algebras.

Lemma 2.7.2. *A scheme is reduced if and only if all of its stalks are reduced as rings.*

Proof. Let X be a scheme, $p \in X$.

If X is reduced, then $\mathscr{O}_X(U)$ is reduced $\forall U \subseteq X$. So without loss of generality, we may assume $p \in U$. So by the definition of stalks, we immediately get $\mathscr{O}_{X,p}$ is reduced.

Conversely, if $\mathscr{O}_{X,p}$ is reduced. It is obvious that $\mathscr{O}_X(U)$ is reduced $\forall p \in U$, but p is arbitrary, so X is reduced. □

Corollary 2.7.3. *Let A be a reduced ring, then Spec(A) is a reduced affine scheme.*

Proof. Let A be reduced, then A_f is reduced $\forall f \in A$. That means $\mathscr{O}_{\mathrm{Spec}(A),f}$ is reduced. So by the lemma above, $\mathrm{Spec}(A)$ is reduced. □

The results above are quite useful to determine whether a scheme is reduced or not. Let us see an easy example.

Example 2.7.4. $\mathrm{Spec}(k[x]/(x^3))$ is not reduced. Let us assume $\mathrm{Spec}(k[x]/(x^3))$ is reduced, so in particular, $\mathscr{O}_{\mathrm{Spec}(k[x]/(x^3)),x}$ is a reduced local ring, however $\frac{x^6}{x^2} = (\frac{x^3}{x})^2 = 0 \implies \frac{x^3}{x} = 0$. But $x \neq 0$, so $x^3 = 0$ hence x is nilpotent, which is a contradiction, so $\mathrm{Spec}(k[x]/(x^3))$ is not reduced.

Definition 2.7.5 (Noetherian schemes)**.** Let X be a scheme. If X can be coverd by affine open sets $\mathrm{Spec}(A_i)$ for $i \in I$, and A_i is Noetherian $\forall i \in I$, then X is said to be a **locally Noetherian scheme**, if $|I| < \infty$, then X is said to be a **Noetherian scheme**.

Definition 2.7.6 (Normal scheme)**.** Let X be a scheme. We say X is **normal** if all of its stalks $\mathscr{O}_{X,p}$ are integral domains and integrally closed in their fractional fields. If $\mathscr{O}_{X,p}$ happens not to be integrally closed, we say X is **integral**.

Definition 2.7.7 (Scheme of finite type over A)**.** Let A be a ring. We say X is an A-scheme if $\mathscr{O}_X(U)$ is an A-algebra for all open subset U of X and all restriction maps of $\mathscr{O}_X$ are maps of A-algebras.

Now let X be an A-scheme. If X can be covered by affine open sets $\mathrm{Spec}(B_i)$, $i \in I$, where B_i is a finitely generated A-algebra $\forall i \in I$, then we say X is a **locally finite type A-scheme**, if $|I| < \infty$, we say X is a **finite type A-scheme**.

Definition 2.7.8 (Degree of a point)**.** Let X be a locally finite type k-scheme, and p a closed point, then the **degree** of p is defined to be $[k(p) : k]$.

Now we can come to the scheme-theoretic definition of varieties.

Definition 2.7.9 (Varieties)**.** Let k be a field. An **affine variety** over k is a reduced affine k-scheme of finite type. A **projective variety** over k is a reduced projective k-scheme.

Example 2.7.10. Let $I = (f_1, \ldots, f_s) \subset k[x_1, \ldots, x_n]$ be an ideal. In the classical language, we define the affine variety generated by I to be $\mathbb{V}(I) := \{a \in k^n : f_1(a) = \cdots = f_s(a) = 0\}$. And this affine variety corresponds to its coordinate ring $k[x_1, \ldots, x_n]/I$. Let us now consider its scheme-theoretic version $\mathrm{Spec}(k[x_1, \ldots, x_n]/I)$. Notice that coordinate rings are finitely generated reduced k-algebras so in particualr $k[x_1, \ldots, x_n]/I$ is reduced. Hence $\mathrm{Spec}(k[x_1, \ldots, x_n]/I)$ is reduced as a k-scheme by Corollary 2.7.3. The fact that $k[x_1, \ldots, x_n]/I$ is a finitely generted implies that $\mathrm{Spec}(k[x_1, \ldots, x_n]/I)$ is a of finite type over k. So we have checked this is indeed coincident with the definition above.

Now we have seen that scheme theory indeed enlarges the classical theory of varieties, and we have more tools to work with. On the other hand, our theory is self-consistent, which means we can recover the notions of classical language using scheme-theoretic language.

2.8 Fibred product of schemes

This section is more or less categorical. We have been working with Cartesian product in the language of algebraic objects like groups, rings, etc. However, there are shortages of Cartesian products when things get more complicated (and this is the reason why we need the notion of direct sums rather than Cartesian products in most cases). For example, the product of schemes. We can see that Cartesian product can not possibly work for schemes, in particular how could the Cartesian product of two structure sheaves make sense?

We need a new version of product. More precisely we need a universal version of product. As mentioned above, we need to introduce the categorial product, in particular we want the product to have the universal property.

Let $\mathcal{C}$ be a category. $X, Y \in \mathrm{Obj}(\mathcal{C})$. We define the product $X \times Y \cong Z$, where the following diagram commutes:

$$\begin{tikzcd}
 & Z \arrow[dl, "\phi_1"'] \arrow[d, "\cong"] \arrow[dr, "\phi_2"] & \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y,
\end{tikzcd}$$

where π_1 and π_2 are obvious projection maps, and Z is uniquely determined upto isomorphism. This categorical definition of product is intrinsic, it does not depend on the choice of categories, objects and morphisms. We say that it has the universal property. The fact that product is universal is non-trivial, but we will not prove it here. A proof can be found in any textbooks of category theory.

Now we should have a taste of how the product of schemes would be, but before we really define it, we need a preliminary definition.

Definition 2.8.1 (The category of S-schemes)**.** Let S be a scheme, the **category of S-schemes** denoted by $\mathcal{S}$ is the collection of:

(1)All schemes over S. A scheme over S is a scheme X where there exists a morphism: $\phi : X \to S$. Denoted by $\mathrm{Obj}(\mathcal{S})$.

(2)All morphism over S. That means for any $X, Y \in \mathrm{Obj}(\mathcal{S})$, we have a morphism between X and Y via S. Symbolically $\phi : X \to S$, $\varphi : Y \to S$, then the morphism $f : X \to Y$ over S satisfies $\varphi \circ f = \phi$. Denoted by $\mathrm{Mor}(\mathcal{S})$.

Definition 2.8.2 (Fibred product of S-schemes)**.** Let $X, Y \in \mathrm{Obj}(\mathcal{S})$, and $\phi_1 : X \to S; \phi_2 : Y \to S$ where $\phi_1, \phi_2 \in \mathrm{Mor}(\mathcal{S})$. Then we define the **fibred product** of X and Y to be $X \times_S Y \cong Z$, where $Z \in \mathrm{Obj}(\mathcal{S})$, where the following diagram commutes:

$$\begin{tikzcd}
X \times_S Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X \arrow[d, "\phi_1"] \\
Y \arrow[r, "\phi_2"] & S.
\end{tikzcd}$$

Theorem 2.8.3 (Universal property of fibred product)**.** *Z defined in Definition 2.8.2 is uniquely determined upto isomorphism.*

Proof. This proof is quite long, and kind of unimportant. See Lemma 26.16.1 in [4]. □

Example 2.8.4. Let k be a field, A, B be finitely generated k-algebras. Consider the tensor product $A \otimes_k B$, we have the commutative diagram:

$$\begin{array}{ccc} A \otimes_k B & \xrightarrow{\varphi_1} & A \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow \\ B & \longrightarrow & k. \end{array}$$

By the functoriality of Spec, the following diagram is also commutative:

$$\begin{array}{ccc} \mathrm{Spec}(A \otimes_k B) & \xrightarrow{\phi_1} & \mathrm{Spec}(A) \\ \downarrow{\scriptstyle \phi_2} & & \downarrow \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(k). \end{array}$$

And by the universal property of tensor product, $A \otimes_k B$ is uniquely determined upto isomorphism, hence $\mathrm{Spec}(A \otimes_k B)$ is also unique upto isomorphism. Thus we may define the fibred product of affine k-schemes $\mathrm{Spec}(A) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(B)$ to be $\mathrm{Spec}(A \otimes_k B)$. And this construction can be extended to any ring R rather than the field k.

Definition 2.8.5 (Universally closed morphism)**.** Let $\pi : X \to Y$ be a morphism of schemes. We say π is **universally closed** if for every morphism $Z \to Y$, the morphism $Z \times_Y X$ closed.

3 Geometric properties of schemes

We have studied varieties as a geometric object via several notions like smoothness, dimension, tangent spaces, etc. Now we want to study the same geometric properties of schemes, and hence prove some interesting result using the language of schemes. We also want to study the constructions of vector bundles and line bundles, divisors and Picard groups. The reason is some results can be proven in the classical geometric argument, but they are difficult to proceed. For example, proving an elliptic curve forms an abelian group is quite complicated in classical projective geometry. However, when we introduce the Picard groups, we will be able to get a much simpler proof.

3.1 Dimension

The language of scheme is abstract, but we can still do geometry on it by defining geometric object like curves and surfaces. The precise way of doing this is by finding its dimension. We all know curves are 1-dimensional object and surfaces are 2-dimensional object from a topological point of view. We will make this intuition rigorous in a scheme-theoretic version.

Definition 3.1.1 (Krull dimension of a topological space)**.** Let X be a topological space. Then the **Krull dimension** of X is defined to be the length of the longest chain of closed irreducible subsets of X. That is, if $X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_d$ is the longest possible chain of closed irreducible subsets of X, then $\dim(X) := d$.

Note that irreducible subsets of spectrum of a ring are exactly those prime ideals, so we can hence to define the Krull dimension of a ring A.

Definition 3.1.2 (Krull dimension of a ring)**.** Let A be a ring. Then the **Krull dimension** of A is the Krull dimension of $\mathrm{Spec}(A)$ as a topological space. That is:

$$\dim(A) := \dim(\mathrm{Spec}(A)).$$

From now on, we will be using dimension rather than the Krull dimension for convenience.

Remark 3.1.3. We will not consider the dimension of an empty set, but strictly speaking we may assign any negative value to the dimension of an empty set. Also we need to be aware that dimension could be infinite and it makes sense.

Definition 3.1.4 (Equidimensional space)**.** Let X be a topological space. Then X is said to be **equidimensional** if each of its irreducible components has the same dimension.

Definition 3.1.5 (Codimension)**.** Let X be a topological space. Let $Y \subset X$ be an irredcuible subset. Then we define the **codimension** of Y in X to be the length of the longest increasing chain of irreducible closed set starting with $\overline{Y}$. We denote the codimension of Y in X to be $\mathrm{codim}_X(Y)$.

Let $\mathfrak{p} \in \mathrm{Spec}(A)$, then $\mathfrak{p}$ has codimension equal to the length of the longest chain of decreasing prime ideals starting at $\mathfrak{p}$.

Definition 3.1.6 (Transcendence degree)**.** Let A be an k-algebra. Then the **transcendence degree** of A is defined to be:

$$\mathrm{trdeg}(A) := \sup\{|T| : T \subseteq A, T \text{ is finite and algebraically independent}\}.$$

This means that the transcendence degree of A is the largest size of finite subset of A whose elements cannot be a solution of any non-trivial polynomials over k.

Remark 3.1.7. The transcendence degree of a non-trivial k-algebra can be any natural number(including 0) or infinity, again we do not consider the trivial ring or the empty set.

The next result is non-trivial.

Theorem 3.1.8. *Let k be an infinite field, A be a finite generated k-algebra which is also an integral domain. Then:*

$$dim(A) \leq trdeg_k(A).$$

Proof. Note that $\mathrm{trdeg}_k(A) \in \mathbb{N}_0$ which means we can use induction on it. Let us denote $n = \mathrm{trdeg}_k(A)$. Consider the base case, for $n = 0$. There really is nothing to prove as if the transcendence degree of A is 0, then A will be an algebraic extension of k which means A is a field, so the dimension of A will be 0. Now let us assume the inequality holds for $n - 1$. Let us consider the case n. Take the chain of prime ideals in $\mathrm{Spec}(A)$:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_s$$

for some $s > 0$. Now let us assume $\dim(A) = s$, and factor the chain by $\mathfrak{p}_i$ for some $0 \leq i \leq s$. Then we get the quotien algebra $A/\mathfrak{p}_i$ whose dimension is $s - 1$. Now assume for a contradication $\mathrm{trdeg}(A/\mathfrak{p}_i) = n$. Let $T := \{x_1 + \mathfrak{p}, \ldots, x_n + \mathfrak{p} : x_i \in A\}$ be a transcendental basis of $A/\mathfrak{p}_i$, i.e $|T| = \mathrm{trdeg}(A/\mathfrak{p}_i) = n$. Let $F := \mathrm{Frac}(k[x_1, \ldots, x_n])$, so in particular x_i is algebraic over F hence $\mathrm{Frac}(A)$ is an algebraic extension of F. Therefore there exists $a \in \mathfrak{p}_i$ is algebraic over F, in other words, there is a non-trivial polynomial $g \in F[x]$ such that $g(a) = 0$. So we have:

$$g(a) = g_0 + g_1 a^1 + \cdots + g_m a^m = 0.$$

Note that if $a = 0$, then the case will be trivial, so we would assume $a \neq 0$. So we must also have $g_0 \neq 0$. So we can arrange the equality to get:

$$g_0 = -\sum_{j=1}^{m} g_j a^j.$$

But for note that $g_i \in F$ for all i, so can be written as a quotient $g_i = \frac{h_i}{f_i}$, in particular, we can mutiply both sides by some $f_i, h_i \in k[x_1, \ldots, x_n]$ to clear the denominator:

$$g'_0 = -\sum_{j=1}^{m} g'_j a^j \in \mathfrak{p}_i.$$

However, this means that g'_0 is algebraic over k as we would have $g'_0(x_1+\mathfrak{p}_i, \ldots, x_n+\mathfrak{p}_i) = 0$, but this is a contradiction as we are assuming $T = \{x_1 + \mathfrak{p}_i, \ldots, x_n + \mathfrak{p}_i : x_i \in A\}$ is a transcendental basis. Then we can conclude that $\mathrm{trdeg}(A/\mathfrak{p}_i) \neq n$, and moreover, $\mathrm{trdeg}(A/\mathfrak{p}_i) < n$ as it does not make sense to exceed n. However, by induction hypothesis, we have $\dim(A/\mathfrak{p}_i) = s - 1 < n$ so $s \leq n$ as s and n are both natural numbers. So we have proved the inquality for the case n. □

This is a technical result, but it yields a very interesting corollary.

Corollary 3.1.9. *$dim(\mathbb{A}^n_k) = n$.*

Proof. Note that $\mathbb{A}^n_k = \mathrm{Spec}(k[x_1, \ldots, x_n])$. So $\dim(\mathbb{A}^n_k) = \dim(k[x_1, \ldots, x_n])$. Now by theorem 3.1.8, we have $\dim(k[x_1, \ldots, x_n]) \leq \mathrm{trdeg}(k[x_1, \ldots, x_n]) = n$. Now we only need to prove $\dim(k[x_1, \ldots, x_n]) \geq n$. We can do this by considering the chain of prime ideals:

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \ldots, x_n).$$

So we have construct a chain of prime ideals of length n, and the length is bounded above by n, so we have achieve the equality. Hence $\dim(\mathbb{A}^n_k) = n$. □

Remark 3.1.10. It is really a non-trivial result that $\dim(\mathbb{A}^n_k) = n$, and there is another proof using Noether's normalisation, which can be found as 11.2.7 in [1]. We can also prove $\dim(\mathbb{P}^n_k) = n$ using the same idea.

Theorem 3.1.11 (Krull's principal ideal theorem)**.** *Let X be a locally Noetherian scheme and $f \in \mathscr{O}_X$, then the irreducible components of $\mathbb{V}(f)$ are of codimension 0 or 1. If X is reduced, then the irreducible components of $\mathbb{V}(f)$ are of codimension exactly 1.*

Proof. The proof is omitted. See 11.5 in [1]. □

Remark 3.1.12. Krull's principal ideal theorem is a handy tool to determine the dimension of a given scheme. For example, let $f = x^3 + y^2 + z \in k[x, y, z]$ for some algebraically closed field k. Then the scheme $\mathrm{Spec}(k[x, y, z]/(f))$ is of dimension 2. We know that $\mathbb{A}_k^3$ is a locally Noetherian reduced scheme. So by Krull's principal ideal theorem, the irreducible components of $\mathbb{V}(f)$ are of codimension 1, which essentially tells us that $\mathrm{Spec}(k[x,y,z]/(f))$ is a scheme of dimension 2 and we would call it a surface.

The next result is fundamental to our theory, and yields a third possible proof of corollary 3.1.9.

Theorem 3.1.13. *Let X, Y be schemes of finite type over the field k, then:*

$$dim(X \times_{Spec(k)} Y) = dim(X) + dim(Y).$$

Proof. Let $X = \bigcup_{i=1}^{n} \mathrm{Spec}(A_i)$, $Y = \bigcup_{j=1}^{m} \mathrm{Spec}(B_j)$ be covered by affine schemes, where A_i, B_j are finitely generated k-algebras. Now take A_s, B_t for some $1 \leq s \leq n$, $1 \leq t \leq m$. We want to firstly proof:

$$\dim(\mathrm{Spec}(A_s) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(B_t)) = \dim(\mathrm{Spec}(A_s)) + \dim(\mathrm{Spec}(B_t)).$$

By example 2.8.4, we have $\mathrm{Spec}(A_s) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(B_t) = \mathrm{Spec}(A_s \otimes_k B_t)$. Now let T_1, T_2, T_3 be the generating sets of A_s, B_t, $A_s \otimes_k B_t$ respectively. We know that $\dim(A_s) = |T_1|$, $\dim(B_t) = |T_2|$ and $\dim(A_s \otimes_k B_t) = |T_3|$. Let $\alpha \in T_3$, then by the universal property of tensor product, α can be expressed uniquely by $e_1 \otimes e_2$ for $e_1 \in T_1$, $e_2 \in T_2$. Which means that $|T_3| = |T_1 \times T_2| = |T_1| + |T_2|$. So we have:

$$\dim(A_s \otimes_k B_t) = \dim(A_s) + \dim(B_t).$$

And so:

$$\dim(\mathrm{Spec}(A_s) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(B_t)) = \dim(\mathrm{Spec}(A_s)) + \dim(\mathrm{Spec}(B_t)).$$

Hence we have proved that for any affine cover of X and Y and their fibred product, the equality holds. Now without the loss of generality, consider X, we would have: $\dim(X) = \max_{1 \leq i \leq n}\{\dim(\mathrm{Spec}(A_i))\}$, and the same result holds for Y. Let $\dim(X) = \dim(\mathrm{Spec}(A_l))$, $\dim(Y) = \dim(\mathrm{Spec}(B_h))$. And we have:

$$\dim(X \times_{\mathrm{Spec}(k)} Y) = \dim(\mathrm{Spec}(A_l \otimes_k B_h)).$$

But we have proved that for affine schemes the equality holds, hence result follows. □

Remark 3.1.14. As mentioned above, Theorem 3.1.13 gives a third possible way to prove $\dim(\mathbb{A}^n) = n$, which is by induction on n. It can be easily proven that $\dim(\mathbb{A}^1) = 1$, and assume that $\dim(\mathbb{A}^{n-1}) = n - 1$. We just need to show that $\mathbb{A}^n = \mathbb{A}^{n-1} \times_{\mathrm{Spec}(k)} \mathbb{A}^1$, then the result follows. $\mathbb{A}^n = \mathbb{A}^{n-1} \times_{\mathrm{Spec}(k)} \mathbb{A}^1$ is not very hard to prove, we will not do it here.

Definition 3.1.15 (Dimension at a point)**.** Let X be a locally finite type k-scheme, $p \in X$ be a closed point (which means $\{p\}$ is a closed subset). We define the **dimension of** X **at the point** p to be $\dim(\mathscr{O}_{X,p})$.

Definition 3.1.16. A **curve** is an equidimensional scheme of dimension 1. A **surface** is an equidimensional scheme of dimension 2. We call an equidimensional scheme of dimension n an n**-fold**.

3.2 Regularity and Smoothness I

Smoothness is a measure to determine whether a geometric object has a bad point or not. In general, we want to work with objects without a bad point, we call that object smooth. To determine smoothness at a point, we usually compute the tangent space at that point. So in our context of algebraic geometry, we use the notion of the Zariski tangent space.

Definition 3.2.1 (The Zariski tangent space)**.** Let X be a scheme over k, $p \in X$. We define the **Zariski tangent space** of X at p to be the k-vector space $T_{X,p} := \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k)$. And $\mathfrak{m}_p/\mathfrak{m}_p^2$ is called the **Zariski cotangent space** at p of X.

Remark 3.2.2. It might be strange that the elements in the Zariski cotangent space are called differentials, but it has nice reasons. Because elements in the Zariski cotangent space are more naturally arose as functions on the scheme, and we call it differentials because we want to think of derivations (although we do not define a derivation here).

Example 3.2.3. Let us consider the surface given by $f(x, y, z) = x^3 + x + y + x^2z + y^2z \in k[x, y, z]$ for some algebraically closed field k. From calculus, we know that the differential of f at 0 is defined to be the linear terms of f, i.e. $df|_0 = x + y$. Now consider the Zariski cotangent space $\mathfrak{m}_{[(x,y,z)]}/\mathfrak{m}^2_{[(x,y,z)]}$. We can see that $df|_0 \in \mathfrak{m}_{[(x,y,z)]}/\mathfrak{m}^2_{[(x,y,z)]}$, as we moduli higher order terms out, and remains only the linear terms.

So this example more or less gives an explaination why we call elements in $\mathfrak{m}_p/\mathfrak{m}_p^2$ differentials. And this definition of differentials is very intrinsic, and do not rely on differentiability. Hence we do not require the notions from differential geometry to compute the tangent space. Furthermore, if we consider the ringed space on a differential manifold $(M, \mathscr{O}_M)$, the definition of the Zariski tangent space coincident with the classical definition of tangent space in differential geometry.

Lemma 3.2.4. *Let A be a Noetherian local ring with residue field $k = A/\mathfrak{m}$. Then $dim(A) \leq dim_k(\mathfrak{m}/\mathfrak{m}^2)$.*

Proof. Follows from Nakayama's lemma and Krull's height theorem, details omitted. See theorem 12.2.1 in [1]. □

Definition 3.2.5 (Regular local ring)**.** Let A is a Noetherian local ring with residue field k. If $\dim(A) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$, then we say A is a **regular local ring**.

Definition 3.2.6 (Regularity of schemes)**.** Let X be a locally Noetherian scheme. We say X is **regular** at a point p if the local ring $\mathscr{O}_{X,p}$ is regular. X is said to be **regular** if it is regular at all points.

Definition 3.2.7 (Singularity)**.** A point $p \in X$ is called a **singularity** if it is not regular. And such scheme X is called a **singular** scheme.

Now we are going to introduce a particular useful result to determine regularity of affine n-folds.

Theorem 3.2.8 (The Jacobian criterion)**.** *Let $X = Spec(k[x_1, \ldots, x_n]/(f_1, \ldots, f_s))$ be equidimensional of dimension d for some algebraically closed field k. Consider the Jacobian matrix at a point $p \in X$:*

$$J_p = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_s}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_s}{\partial x_n}(p) \end{pmatrix}.$$

Then p is regular if and only if $corank(J_p) = d$. Where $corank(J_p) := dim(coker(J_p))$.

Sketch of the proof. First we want to show that $\mathfrak{m}_p/\mathfrak{m}_p^2 = \mathrm{coker}(J_p)$, which is the same to show that $\ker(J_p) = \mathrm{Hom}_k(k, \mathfrak{m}_p/\mathfrak{m}_p^2)$. This is a standard result in differential geometry, requires quite a few calculus and linear algebra to work out, we will assume it is true here. Hence we have $\mathfrak{m}_p/\mathfrak{m}_p^2 = \mathrm{coker}(J_p)$.

Now assume $\mathrm{corank}(J_p) = d$, which means $\dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2) = d$. We know that X is equidimensional, in particular, the dimension of each point is d, which means $\dim(\mathscr{O}_{X,p}) = d = \dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2)$, which means $\mathscr{O}_{X,p}$ is regular, hence p is regular.

Conversely, if p is regular, then $\dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2) = \dim(\mathscr{O}_{X,p})$, but X is equidimensional of dimension d, so $\dim(\mathfrak{m}_p/\mathfrak{m}_p^2) = \mathrm{corank}(J_p) = d$. □

Remark 3.2.9. In fact the condition of k being algebraically closed is not very necessary, we have the condition here is to simplify the proof, if we drop the condition the proof need a bit more argument, in particular, if k is not algebraically closed, then there are some points missing.

Definition 3.2.10 (Smoothness)**.** Let X be a scheme over k where k is a field. We say X is **smooth of dimension** d over k if there exists an affine cover of the form: $\mathrm{Spec}(k[x_1, \dots, x_n]/(f_1, \dots, f_s))$ which satisfies the Jacobian criterion at all points. We say X is smooth over k if X is smooth of some dimesion.

Example 3.2.11. We can show that $\mathbb{A}_k^n$ is smooth for any n and any field k. Because we can regard $\mathbb{A}_k^n$ as $\mathrm{Spec}(k[x_1, \dots, x_n]/(0))$, in any cases, the Jacobian criterion is automatically satisfied. So we may conclude that $\mathbb{A}_k^n$ is smooth, and this implies that $\mathbb{P}_k^n$ is also smooth for any n and field k, as we may cover $\mathbb{P}_k^n$ by $n+1$ copies of $\mathbb{A}_k^n$.

We have seen how smoothness interact with regularity, but we have not really observed that the connection between them rather than the Jacobian criterion, and we will see it in the next result.

Theorem 3.2.12 (Smoothness-regularity theorem)**.**

(1) If k is a perfect field(i.e. either a characteristic zero field or a finite field). Then every regular finite type k-scheme is smooth over k.

(2)Every smooth k-scheme is regular.

We are not ready to proof both parts of this theorem, the proof will be left later.

Remark 3.2.13. As stated in smoothness-regularity theorem, smoothness is not equivalent to regularity for fields that is not perfect, see 12.2.11 in [1] for an example.

Example 3.2.14. Let k be a field, then by smoothness-regularity theorem, $\mathbb{A}_k^n$ is regular as it is smooth(see 3.2.11). So $\mathbb{P}_k^n$ is regular. Furthermore, without using smoothness-regularity theorem, we can still prove that $\mathbb{A}_k^n$ is regular, but it is more complicated, and need a technical result, see proposition 12.3.6 in [1].

Although smoothness-regularity theorem is a powerful tool to determine whether a scheme is regular or not(which is quite hard in general without the theorem), there are some examples that cannot be determined using smoothness-regularity theorem.

Example 3.2.15. Consider $\mathrm{Spec}(\mathbb{Z})$. This is a regular scheme. However, the regularity of $\mathrm{Spec}(\mathbb{Z})$ cannot be determined by smoothness-regularity theorem, as it is a scheme over a ring.

To show $\mathrm{Spec}(\mathbb{Z})$ is regular, we want to show it straightly from the definition. Take $[(p)] \in \mathrm{Spec}(\mathbb{Z})$, think of the localisation at $[(p)]$, i.e $\mathbb{Z}_{(p)} := \{\frac{a}{s} : a \in \mathbb{Z}, s \notin (p)\}$. Notice that $\mathbb{Z}_{(p)}$ is a Noetherian local ring, so if we can show that $\dim(\mathbb{Z}_{(p)}) = \dim_{\mathbb{F}_p}(\mathfrak{m}/\mathfrak{m}^2)$, where $\mathfrak{m} = (p)\mathbb{Z}_{(p)} = \{\frac{a}{b} : a \in (p), b \notin (p)\}$, then we are done.

Consider $\dim(\mathbb{Z}_{(p)}) = \dim(\mathrm{Spec}(\mathbb{Z}_{(p)}))$. Notice that $\mathrm{Spec}(\mathbb{Z}_{(p)}) = \{(q)\mathbb{Z}_{(p)} : (q) \in \mathrm{Spec}(\mathbb{Z}), (q) \subseteq (p)\}$. But $\mathbb{Z}$ is a PID, hence every prime ideal is maximal, so there is only one element in $\mathrm{Spec}(\mathbb{Z}_{(p)})$, thus $\dim(\mathbb{Z}_{(p)}) = 1$. Now consider the maximal ideal $\mathfrak{m}$, by lemma 3.2.4, we have $\dim_{\mathbb{F}_p}(\mathfrak{m}/\mathfrak{m}^2) \geq 1$. We can count the number of elements in $\mathfrak{m}/\mathfrak{m}^2$, in particular, it has p elements, so $\mathfrak{m}/\mathfrak{m}^2$ is of dimension 1 over $\mathbb{F}_p$. Thus $\mathbb{Z}_{(p)}$ is regular, hence $\mathrm{Spec}(\mathbb{Z})$ is regular as a scheme.

And this result is true for general ring of integers, but proving it is rather difficult, requires a lot of homological algebra.

Theorem 3.2.16. *$\mathrm{Spec}(\mathcal{O}_K)$ is regular as a scheme, where $\mathcal{O}_K$ is the ring of integers of some number field K.*

Outline of the proof. As mentioned above, this result requires many notions in homological algebra. In particular, J.P Serre has shown that for a Noetherian local ring A, A is regular if and only if A has a finite homological dimension.

In our case, we know that $\mathcal{O}_K$ is a Dedekind domain, so is Noetherian, hence has Noetherian localisations. So we may compute the homological dimension for a localisation of $\mathcal{O}_K$ which happens to be finite, so $\mathcal{O}_K$ is regular at each of its localisations. Hence we may conclude that $\operatorname{Spec}(\mathcal{O}_K)$ is a regular scheme. $\square$

3.3 Vector bunldes and Čech cohomology

This section is slightly off the theme of schemes, it is very sheaf-theoretic. However, it has nice interactions with schemes. In particular, algebraic K-theory is all about studying vector bundles generated by schemes (although we will not go that far, we will not even define a K-group).

Definition 3.3.1 ($\mathscr{O}_X$-modules)**.** Let X be a scheme with structure sheaf $\mathscr{O}_X$. Then an **$\mathscr{O}_X$-module $\mathscr{M}$** is a sheaf of abelian groups such that for each open subset $U \subseteq X$, we have $\mathscr{M}(U)$ is an $\mathscr{O}_X(U)$-module. Also sections should respect restriction maps.

Definition 3.3.2 ($\mathscr{O}_X$-algebras)**.** An **$\mathscr{O}_X$-algebra $\mathscr{A}$** is a sheaf of rings that is also an $\mathscr{O}_X$ module.

Definition 3.3.3 (Affine space over a scheme)**.** Let S be a scheme. Then the affine n-space over S is defined to be:

$$\mathbb{A}^n_S := \operatorname{Spec}(\mathscr{O}_S[x_1, \ldots, x_n]).$$

Definition 3.3.4 (Vector bundles)**.** A **rank n vector bundle** on a S-scheme X is an S-scheme E with a projection map: $\pi : E \to X$ such that there exists an affine cover $\bigcup_i \operatorname{Spec}(A_i)$ of X where:

$$\begin{tikzcd}
\pi^{-1}(\operatorname{Spec}(A_i)) \arrow[r, "\phi_i"] \arrow[d, "\pi"] & \operatorname{Spec}(A_i) \times_S \mathbb{A}^n_S \arrow[dl, "p"] \\
\operatorname{Spec}(A_i) &
\end{tikzcd}$$

commutes. Where ϕ_i is an isomorhism called a **trivialisation** over $\operatorname{Spec}(A_i)$. A rank 1 vector bundle is called a **line bundle**.

Definition 3.3.5 (Transition functions)**.** Let $\{U_i\}$ covers X, and for any U_i and U_j, consider the trivialisations over U_i and U_j over their intersection, then there is a function associate to them, denoted by T_{ij}. And $\{T_{ij}\}$ must satisfy the following:

$$T_{ij}|_{U_i\cap U_j\cap U_k} \circ T_{jk}|_{U_i\cap U_j\cap U_k} = T_{ik}|_{U_i\cap U_j\cap U_k}.$$

Elements of $\{T_{ij}\}$ are called **transition functions** for the corresponding trivialisations.

Definition 3.3.6 (Sheaf of sections)**.** Take a rank n vector bundle $E \to X$. The **sheaf of sections** $\mathscr{F}$ of E is an $\mathscr{O}_X$-module, where for any open subset $U \subseteq X$, take $f \in \mathscr{O}_X(U)$, then $f\mathscr{F}(U)$ defines another section.

And for each open subset $U_i \subseteq X$, we have an isomorphism via trivialisation:$\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^{\oplus n}$. We call $\mathscr{F}$ a **locally free sheaf of rank** n.

Definition 3.3.7 (Invertible sheaf)**.** A locally free sheaf of rank 1 is called an **invertible sheaf**.

Remark 3.3.8. An invertible sheaf is essentially a line bundle. The term invertible refers the invertible elements in the monoid of $\mathscr{O}_X$-module.

Proposition 3.3.9.

(1)Let $\mathscr{F}$ and $\mathscr{G}$ are locally free sheaves over a scheme X of rank m and n respectively. Define $Hom_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}) := \{Hom_{\mathscr{O}_X|_U}(\mathscr{F}|_U,\mathscr{G}|_U) : U \subseteq X$ open subset$\}$. Then $Hom_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is a locally free sheaf of rank mn.

(2)Let $\mathscr{H}$ be a locally free sheaf of rank n on X, then $\mathscr{H}^* := Hom_{\mathscr{O}_X}(\mathscr{H},\mathscr{O}_X)$ is also a locally free sheaf of rank n. We call $\mathscr{H}^*$ the dual of $\mathscr{H}$.

(3)Let $\mathscr{F}$ and $\mathscr{G}$ be the same as part (1), define $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} := \{\mathscr{F}|_U \otimes_{\mathscr{O}_U} \mathscr{G}|_U : U \subseteq X$ open subset$\}$. Then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is a locally free sheaf of rank mn.

Proof.

(1)By definition, take $U \subseteq X$ be an open subset, we have: $\mathscr{F}|_U \cong \mathscr{O}_U^{\oplus m}$ and $\mathscr{G}|_U \cong \mathscr{O}_U^{\oplus n}$. So we have $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\mathscr{G}|_U) \cong \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U^{\oplus m},\mathscr{O}_U^{\oplus n}) \cong \mathscr{O}_U^{\oplus mn}$. Hence $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is a locally free sheaf of rank n on X.

(2)Straightly from the definition.

(3)Similar to part (1), standard result from commutative algebra. $\square$

Example 3.3.10. Let X be a scheme, consider the set of all invertible sheaves on X: $\mathrm{Pic}(X)$ (this notation will become clear in a moment). Clearly $\mathscr{O}_X \in \mathrm{Pic}(X)$. And take

$A \in \mathrm{Pic}(X)$, then by part (2) of Proposition 3.3.9, we have $A^* \in \mathrm{Pic}(X)$, and we have $A \otimes_{\mathscr{O}_X} A^* \cong \mathscr{O}_X$. For $A, B, C \in \mathrm{Pic}(X)$, we have $A \otimes_{\mathscr{O}_X} (B \otimes_{\mathscr{O}_X} C) = (A \otimes_{\mathscr{O}_X} B) \otimes_{\mathscr{O}_X} C$. Now consider $A \otimes_{\mathscr{O}_X} B$, for any open subset $U \subseteq X$, we have $A|_U \otimes_{\mathscr{O}_U} B|_U = B|_U \otimes_{\mathscr{O}_U} A|_U$. Hence $A \otimes_{\mathscr{O}_X} B = B \otimes_{\mathscr{O}_X} A$. So we have verified that $(\mathrm{Pic}(X), \otimes_{\mathscr{O}_X})$ forms an abelian group, and we call it the Picard group of X.

Example 3.3.11 (Line bundle on projective space). Let us consider a line bundle on $\mathbb{P}^n_k$. First let us take two affine covers, for $i \neq j$, define $U_i = \mathrm{Spec}(k[\frac{x_0}{x_i}, \ldots, \frac{x_i}{x_i}, \ldots, \frac{x_n}{x_i}])$ and $U_j = \mathrm{Spec}(k[\frac{x_0}{x_j}, \ldots, \frac{x_j}{x_j}, \ldots, \frac{x_n}{x_j}])$. For simplicity, let us denote $U_i = \mathrm{Spec}(A_i)$ and $U_j = \mathrm{Spec}(A_j)$. Notice that we have the transition function defined by:

$$A_i \xrightarrow{\times \frac{x_i}{x_j}} A_j$$

$$A_i \xleftarrow[\times \frac{x_j}{x_i}]{} A_j.$$

Now think of the functions $f \in A_i, g \in A_j$, recall from example 2.4.2, we use the same method to compute functions agree on $A_i \cap A_j$. That is, for $f(X_1, \ldots, X_t)$ and $g(Y_1, \ldots, Y_n)$ for $X_m := \frac{x_m}{x_i}, Y_m := \frac{x_m}{x_j}$, we require $f(Y_1, \ldots, Y_n)\frac{Y_m}{X_m} = g(Y_1, \ldots, Y_n)$. So we will have f and g are linear polynomials in n variables. All such f, g on all affine covers give an invertible sheaf on $\mathbb{P}^n_k$, denoted by $\mathfrak{L}(1)$. Note that $\dim(\Gamma(\mathbb{P}^n_k, \mathfrak{L}(1))) = n+1$. If we let our transition functions $A_i \to A_j$ by multplying $(\frac{x_j}{x_i})^m$ and vice versa, then we have the line bundle $\mathfrak{L}(m)$. Notice that: $\dim(\Gamma(\mathbb{P}^n_k, \mathfrak{L}(n))) = (\,{}^{n+m}_{\ m}\,)$.

Now we are going to introduce an elegant construction: Čech cohomology. We first introduce some preliminary definitions.

Definition 3.3.12 (Quasicoherent sheaf)**.** Let X be a scheme, $\mathscr{F}$ be an $\mathscr{O}_X$-module. We say $\mathscr{F}$ is **quasicoherent** if for every affine open subset $\mathrm{Spec}(A) \subseteq X$, we have $\mathscr{F}|_{\mathrm{Spec}(A)} \cong \widetilde{M}$. Where M is an A-module and $\widetilde{M}$ is an $\mathscr{O}_{\mathrm{Spec}(A)}$-module such that $\widetilde{M}(D(f)) = M_f$.

Definition 3.3.13 (Separated schemes)**.** A morphism of schemes: $\pi : X \to Y$ is called **separated** if there is a morphism: $\delta_\pi : X \to X \times_Y X$ is a closed embedding. An A-scheme X is separated over A if the morphism $X \to \mathrm{Spec}(A)$ is separated.

Definition 3.3.14 (Quasicompact schemes)**.** A scheme X is **quasicompact** if for any open cover $\bigcup_i U_i = X$, there is a finite subcover $\bigcup_i^n U_i = X$.

Remark 3.3.15. Some people use compact instead of our quasicompact, but compact in our sense means that quasicompact and Hausdorff. We will not use the confusing notion though.

Definition 3.3.16 (Čech complex)**.** Let $\mathscr{F}$ be a quasicoherent sheaf on X, where X is a quasicompact separated scheme. Let $\mathfrak{U} := \{U_i\}_{i=1}^n$ be a finite collection of affine covers of X. For $I \subseteq \{1, 2, \ldots, n\}$, define $U_I = \bigcap_{i \in I} U_i$. Then the **Čech complex** is defined to be the sequence:

$$0 \longrightarrow \prod_{|I|=1} \mathscr{F}(U_I) \xrightarrow{f_0} \ldots \xrightarrow{f_{i-2}} \prod_{|I|=i} \mathscr{F}(U_I) \xrightarrow{f_{i-1}} \prod_{|I|=i+1} \mathscr{F}(U_I) \xrightarrow{f_i} \ldots.$$

Where the map $\mathscr{F}(U_I) \to \mathscr{F}(U_J)$ is 0 unless $I \cup \{j\} = J$. If j is the k-th element of J, then the map is defined to be $(-1)^{k-1}\text{res}_{U_I, U_J}$.

Remark 3.3.17. Čech complex is so called a complex is because $f_k \circ f_{k+1}$ is the zero morphism, which satisfies the definition of a chain complex.

Definition 3.3.18 (Čech cohomology)**.** Given the Čech complex, define $H^i_{\mathfrak{U}}(X, \mathscr{F}) := \ker(f_i)/\text{im}(f_{i-1})$. We call $H^i_{\mathfrak{U}}(X, \mathscr{F})$ the ***i*-th cohomology group** of the Čech complex, for simplicity, we call it the **Čech cohomology group**.

Remark 3.3.19. We will not study homology groups here, so we use f_i in stead of f^i for the sake of clarity. Another reason is that we do not use the notion of differentials, so it has not point to use f^i (as in usual sense of cochain complex, we use differentials, so the maps denoted as d^i, called i-th differental).

Remark 3.3.20. Notice that if X is a k-scheme, then $H^i_{\mathfrak{U}}(X, \mathscr{F})$ is a k-vector space, if X is an A-scheme, then $H^i_{\mathfrak{U}}(X, \mathscr{F})$ is an A-module.

Example 3.3.21. $H^0_{\mathfrak{U}}(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

Proof. By definition, $H^0_{\mathfrak{U}}(X, \mathscr{F}) = \ker(f_0)/0 = \ker(f_0)$. Fix $|I| = 1$, $|J| = 2$ let $U_I = \bigcap_{i \in I} U_i$ and $U_J = \bigcap_{j \in J} U_j$. Now by definition we have $f_0 = \sum(\text{res}_{U_{I_a}, U_{J_a}} - \text{res}_{U_{I_b}, U_{J_b}})$ where $I_a \cup \{j_a\} = J_a$, j_a is the first element of J_a, and $I_b \cup \{j_b\} = J_b$, j_b is the second element of J_b. To make f_0 vanish, we need elements such that $\text{res}_{U_{I_a}, U_{J_a}}$ and $\text{res}_{U_{I_b}, U_{J_b}}$ agrees, which are exactly elements from the global section $\Gamma(X, \mathscr{F})$. Hence $H^0_{\mathfrak{U}}(X, \mathscr{F}) = \ker(f_0) = \Gamma(X, \mathscr{F})$. □

We now want to make a first use of Čech cohomology, which is quite helpful to determine whether a given scheme is affine or not.

Theorem 3.3.22 (Serre's Cohomological criterion)**.** *Let X be an quasicompact separated scheme, and for any quasicoherent sheaf $\mathscr{F}$ on X, $H^i_{\mathfrak{U}}(X,\mathscr{F})=0$ for all $i>0$, then X is an affine scheme.*

Proof. Omitted, see Lemma 30.3.1 and 30.3.2 in [4]. □

The converse is also true, i.e for affine scheme $\mathrm{Spec}(A)$, $H^i_{\mathfrak{U}}(\mathrm{Spec}(A),\mathscr{F})=0$ for all $i>0$ and $\mathscr{F}$ is a quasicoherent sheaf on $\mathrm{Spec}(A)$. See Theorem 18.2.4 in [1] for a proof.

We can observe that Čech cohomology groups are in general very difficult to compute, but for certain classes of sheaves they are computable.

Theorem 3.3.23 (Cohomology of projective bundles)**.** *Let k be a field, then:*

$$H^i_{\mathfrak{U}}(\mathbb{P}^n_k,\mathfrak{L}(d))\cong\begin{cases}\Gamma(\mathbb{P}^n,\mathfrak{L}(d)), & \text{if } i=0\\ 0, & \text{if } 0<i<n\\ \left(\frac{1}{x_0x_1\dots x_n}k[x_0^{-1},\dots,x_n^{-1}]\right)_{deg\leq d}, & \text{if } i=n\end{cases}.$$

Proof. Omitted, see Lemma 30.8.1 in [4] □

Definition 3.3.24 (Euler's characteristic)**.** Let X be a projective k-scheme. Define $h^i(X,\mathscr{F}):=\dim_k H^i_{\mathfrak{U}}(X,\mathscr{F})$, then the **Euler's characteristic** of $\mathscr{F}$ is:

$$\chi(X,\mathscr{F}):=\sum_{i=0}^{\dim(X)}(-1)^i h^i(X,\mathscr{F}).$$

Example 3.3.25. Consider $\mathbb{P}^2_{\mathbb{R}}$. The Euler's characteristic of $\mathbb{P}^2_{\mathbb{R}}$ by definition is $\chi(\mathbb{P}^2_{\mathbb{R}},\mathscr{O}_{\mathbb{P}^2_{\mathbb{R}}})=\sum_{i=0}^{2}(-1)^i h^i(\mathbb{P}_{\mathbb{R}^2},\mathscr{O}_{\mathbb{P}^2_{\mathbb{R}}})$. So we first want to compute the Čech cohomology group $H^i_{\mathfrak{U}}(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2})$. Consider the Čech complex on $\mathscr{O}_{\mathbb{P}^2}$:

$$0\longrightarrow\mathscr{O}(U_1)\oplus\mathscr{O}(U_2)\oplus\mathscr{O}(U_3)\xrightarrow{f_0}\mathscr{O}(U_1\cap U_2)\oplus\mathscr{O}(U_1\cap U_3)\oplus\mathscr{O}(U_2\cap U_3)\xrightarrow{f_1}\mathscr{O}(U_1\cap U_2\cap U_3)\xrightarrow{f_2}0$$

where U_i is the i-th affine patch covers $\mathbb{P}^2$, i.e $U_i=\mathrm{Spec}(\mathbb{R}[\frac{x_j}{x_i},\frac{x_k}{x_i}])$ for $j,k\neq i$. We may think of $\mathscr{O}_{\mathbb{P}^2}$ to be the trivial line bundle on $\mathbb{P}^2$, i.e $\mathfrak{L}(0)$. So apply theorem 3.3.23, we have $H^0_{\mathfrak{U}}(\mathbb{P}^2,\mathfrak{L}(0))=\mathbb{R}$, $H^1_{\mathfrak{U}}(\mathbb{P}^2,\mathfrak{L}(0))=0$ and $H^2_{\mathfrak{U}}(\mathbb{P}^2,\mathfrak{L}(0))=\frac{1}{x_0x_1x_2}\mathbb{R}[x_0^{-1},x_1^{-1},x_2^{-1}]$. Hence we have $h^0(\mathbb{P}^2,\mathfrak{L}(0))=1$, $h^1(\mathbb{P}^2,\mathfrak{L}(0))=h^2(\mathbb{P}^2,\mathfrak{L}(0))=0$. Thus $\chi(\mathbb{P}^2_{\mathbb{R}},\mathscr{O}_{\mathbb{P}^2_{\mathbb{R}}})=1$, which coincident with the topological version of Euler's characteristic of a real projective plane.

3.4 Divisors and Picard groups

This section has a particular goal, which is to use the notion of divisors and the Picard group to show an elliptic curve forms an abelian group. This is a nice example of how abstract theory helps to solve a classical geometric problem. It is possible to prove an elliptic curve forms an abelian group using only projective geometry, but the proof is quite long, and the part of proving the associativity is especially hard. In comparison, the abstract proof is neat and explain things well.

Definition 3.4.1 (Weil divisor). Let X be a Noetherian scheme, then a **Weil divisor** is a formal sum of codimension 1 irreducible closed subsets of X:

$$D := \sum_{\mathrm{codim}_X(Y)=1} n_Y[Y].$$

Where $n_Y \in \mathbb{Z}$, and finitely many of them are zero. And we say $[Y]$ is an irreducible divisor. A Weil divisor D is effective if all of its coefficients are non-negative, we use the notion $D \geq 0$.

Definition 3.4.2. Let D be a Weil divisor, a **support** of D is defined to be: $\mathrm{Supp}(D) := \bigcup_{n_Y \neq 0} Y$

Definition 3.4.3. Notice that the set of all Weil divisors forms an abelian group, denoted by $\mathcal{W}(X)$. Let $U \subseteq X$ be an open subset, then the restriction map is defined to be:

$$\rho : \mathcal{W}(X) \to \mathcal{W}(U)$$

$$\sum n_Y[Y] \mapsto \sum_{Y \cap U \neq \emptyset} n_Y[Y \cap U].$$

Before we move on, we need a bit of algebraic preparations.

Definition 3.4.4 (Valuation ring). Let K be a field, a **discrete vaulation** on K is a surjective homomorphism: $v : K^\times \to \mathbb{Z}$ such that $v(x+y) \geq \min(v(x), v(y))$. Then $\{x \in K^\times : v(x) \geq 0\} \cup \{0\}$ is a ring and we call it the **valuation ring** of v, denoted by O_v.

Definition 3.4.5 (Discrete valuation ring). A **discrete valuation ring** (DVR) is an integral domain A where there is a discrete valuation on $\mathrm{Frac}(A)$ where $O_v = A$.

Theorem 3.4.6. *Let A be a Noetherian regular local ring of dimension 1, then A is a DVR with a unique valutaion v.*

Proof. We will use this theorem without proving it here, although the proof is not difficult. See Section 12.5 in [1] for a proof (it is split into parts though). $\square$

Definition 3.4.7. Let A be a Noetherian regular local ring of dimension 1. Let $s \in \mathrm{Frac}(A)$, if $v(s) = n > 0$, we say s has a **zero of order** n. If $v(s) = n < 0$, we say s has a **pole of order** n.

Definition 3.4.8 (Rational section)**.** A **rational section** of a line bundle $\mathscr{L}$ is the equivalence class:

$$\{(s, U)\}/\sim$$

where $U \subseteq X$ is an open subset contains all the associated points of X, and s is a section of $\mathscr{L}$ over U. $(s, U) \sim (s', U')$ if s and s' agree on $U \cap U'$.

Now we can move on to further topics in divisors.

Definition 3.4.9. Let X be a reduced regular Noetherian scheme where each of its stalk is a reduced regular Noetherian local ring of dimension 1. Let $\mathscr{L}$ be a line bundle on X and s be a rational section not vanishing everywhere on any irreducible components of X. Then s gives a Weil divisor:

$$\mathrm{div}(s) := \sum_Y v_Y(s)[Y]$$

where Y is an irreducible divisor. We call $\mathrm{div}(s)$ the **divisor of zeros and poles** of the rational section s.

Remark 3.4.10. This definition might be strange at first, because we have not really use the line bundle $\mathscr{L}$ on X. However, we have to indirectly use $\mathscr{L}$. In particular, if we want to determine the valuation v_Y, we take any open subset U that contains the generic point of Y where we can find a trivialisation of $\mathscr{L}$ over U. And here s will be a nonzero rational function on U, so has an valuation.

Theorem 3.4.11. *Let X be a scheme as in Definition 3.4.9. Consider the set of all pairs of line bundles with nonzero rational sections $\{(\mathscr{L}, s)\}$, with the binary operation $\otimes := \otimes_{\mathscr{O}_X}$, forms an abelian group.*

Proof. Denote $\{\mathscr{L}, s\} = G$, notice that $(\mathscr{O}_X, 1) \in G$, and we set this element to be the identity. Let $(\mathscr{L}_1, s_1)$ and $(\mathscr{L}_2, s_2)$ lie in G, consider:

$$(\mathscr{L}_1, s_1) \otimes (\mathscr{L}_2, s_2) = (\mathscr{L}_1 \otimes \mathscr{L}_2, s_1 s_2).$$

By proposition 3.3.9, $\mathscr{L}_1 \otimes \mathscr{L}_2$ is also a line bundle on X, also $s_1 s_2$ is also a nonzero rational section. So $(\mathscr{L}_1 \otimes \mathscr{L}_2, s_1 s_2) \in G$.

Now take $(\mathscr{L}_i, s_i) \in G$ for $i = 1, 2, 3$. We have:

$$((\mathscr{L}_1, s_1) \otimes (\mathscr{L}_2, s_2)) \otimes (\mathscr{L}_3, s_3) = (\mathscr{L}_1 \otimes \mathscr{L}_2, s_1 s_2) \otimes (\mathscr{L}_3, s_3) = (\bigotimes_{i=1}^{3} \mathscr{L}_i, s_1 s_2 s_3)$$

and

$$(\mathscr{L}_1, s_1) \otimes ((\mathscr{L}_2, s_2) \otimes (\mathscr{L}_3, s_3)) = (\mathscr{L}_1, s_1) \otimes (\mathscr{L}_2 \otimes \mathscr{L}_3, s_2 s_3) = (\bigotimes_{i=1}^{3} \mathscr{L}_i, s_1 s_2 s_3).$$

so we have verified associativity. And the commutativity is obvious.

Now we need to find an inverse element. Take $(\mathscr{L}, s) \in G$, so we need an element $(\mathscr{L}', s')$ such that $(\mathscr{L} \otimes, \mathscr{L}', s s') = (\mathscr{O}_X, 1)$. By proposition 3.3.9, $\mathscr{L}^*$ is also a line bundle on X, so we may set $\mathscr{L}' = \mathscr{L}^*$ and $s' = s^{-1}$, and we have the inverse element $(\mathscr{L}^*, s^{-1})$. ◻

Corollary 3.4.12. *There is a group homomorphism:*

$$div : G \to \mathcal{W}(X).$$

Proof. Just consider:

$$\mathrm{div} : G \to \mathcal{W}(X)$$

$$s \to \mathrm{div}(s)$$

where $\mathrm{div}(1) = \sum [Y]$. And also $\mathrm{div}(s_1 s_2) = \sum_Y v_Y(s_1 s_2)[Y] = \sum_Y v_Y(s_1) v_Y(s_2)[Y] = \sum_Y v_Y(s_1)[Y] \sum_Y v_Y(s_2)[Y]$. ◻

Definition 3.4.13. Let X be an irreducible normal scheme, $U \subseteq X$ be an open subset. Let $D \in \mathcal{W}$, define the sheaf $\mathscr{O}_X(D)$ by:

$$\Gamma(U, \mathscr{O}_X(D)) := \{x \in K(X)^\times : \mathrm{div}|_U x + D|_U \geq 0\} \cup \{0\}.$$

Here $\mathrm{div}|_U x$ means to take the divisor of the form $\mathrm{div}(x)$ considered as a rational function on U.

Definition 3.4.14 (Principal divisor)**.** Let X be a Noetherian normal irreducible scheme, and $D \in \mathcal{W}(X)$. We say D is **principal** if for some rational function f, we have $D = \mathrm{div}(f)$. We denote the set of all principal Weil divisors to be $\mathcal{P}(X)$. If X can be covered by $\{U_i\}$, and on each U_i D is principal, then we say D is **locally principal**. The set of all locally principal Weil divisors is denoted by $\mathcal{L}(X)$.

Remark 3.4.15. Both $\mathcal{P}(X)$ and $\mathcal{L}(X)$ are subgroups of $\mathcal{W}(X)$.

Proposition 3.4.16. *Let X be a Noetherian normal irreducible scheme and $D \in \mathcal{W}(X)$, then $\mathscr{O}_X(D)$ is an invertible sheaf if and only if D is locally principal.*

Sketch of the proof. Suppose $\mathscr{O}_X(D)$ is an invertible sheaf, then we consider the isomorphism: $\phi_U : \mathscr{O}_U(D) \to \mathscr{L}(U)$ where $\mathscr{L}$ is an invertible sheaf and $U \subseteq X$ is an open set such that $\mathscr{O}_X(D)$ is locally principal on U. But $\mathscr{O}_X(D)$ is an invertible sheaf, in other words all such U must form a basis for the Zariski topology on X, so we may conclude that D is locally principal on X.

Coversely, let D be locally principal, that is for some open subset $U \subseteq X$, we have $\mathscr{O}_U(D) = \mathscr{O}_U(\mathrm{div}(s))$ for some nonzero rational function on U. Now consider the morphism:

$$\phi_U : \mathscr{O}_U(D) \to \mathscr{L}(U)$$

$$s \mapsto st$$

we can also find an inverse morphism to ϕ_U:

$$\phi_U^{-1} : \mathscr{L}(U) \to \mathscr{O}_U(D)$$

$$l \mapsto ls^{-1}$$

so $\mathscr{O}_U(D)$ is locally isomorhic to some invertible sheaf on U. Now we want to show that all such U form a basis for the Zariski topology. But from Definition 3.4.13 we know that D is associated with a locally principal divisor, so all such U must form a basis for the Zariski topology. Hence $\mathscr{O}_X(D)$ is an invertible sheaf. $\square$

Definition 3.4.17 (Divisor class group)**.** Let X be a Noetherian normal irreducible scheme, then the **divisor class group** of X is defined to be: $\mathcal{C}(X) := \mathcal{W}(X)/\mathcal{P}(X)$.

The divisor class group is closely related to the Picard group and hence a use tool to compute the Picard group. In particular we have the following theorem.

Theorem 3.4.18. *Let X be a Noetherian normal irreducible scheme, then Pic$(X) \cong \mathcal{L}(X)/\mathcal{P}(X)$. Hence Pic$(X)$ is a subgroup of $\mathcal{C}(X)$.*

Proof. Take $\mathscr{L} \in \mathrm{Pic}(X)$, then by Proposition 3.4.16, we have the group map:

$$g : \mathrm{Pic}(X) \to \mathcal{L}(X)/\mathcal{P}(X)$$

$$\mathscr{O}_X(D) \mapsto D$$

where there is also an inverse:

$$g^{-1} : \mathcal{L}(X)/\mathcal{P}(X) \to \mathrm{Pic}(X)$$

$$D \mapsto \mathscr{O}_X(D)$$

hence we have the isomorphism: $\mathrm{Pic}(X) \cong \mathcal{L}(X)/\mathcal{P}(X) \hookrightarrow \mathcal{C}(X)$. $\square$

Example 3.4.19. Let R be a UFD, then we can show that all codimension 1 prime ideals in Spec(R) are principal. Let $\mathfrak{p} \in \mathrm{Spec}(R)$ is of codimension 1. Let $0 \neq x \in \mathfrak{p}$. If x is irreducible, then we are done as $\mathfrak{p} = (x)$. Assume x is not irreducible, then $x = yz$ where at least one of y and z is irreducible, say y is irreducible. So $(0) \subseteq (y) \subseteq \mathfrak{p}$, but $\mathfrak{p}$ is of codimension 1, so $\mathfrak{p} = (y)$. Notice that each codimension 1 prime ideal in Spec(R) corresponds to a Weil divisor of Spec(R). And we just proved that all prime ideals in Spec(R) of codimension 1 are principal, hence all Weil divisors are principal. Thus $\mathcal{C}(\mathrm{Spec}(R)) = 0$, but Pic(Spec$(R)$) is a subgroup of $\mathcal{C}(\mathrm{Spec}(R))$ by Theorem 3.4.18, so Pic(Spec(R)) $= 0$. So in particular, $\mathrm{Pic}(\mathbb{A}^n_R) = 0$ as $R[x_1, \ldots, x_n]$ is a UFD. Which means all line bundles on affine spaces are trivial (it is possible to prove this result directly from the definition of vector bundles). More generally, any vector bundles on affine spaces are also trivial. But this is a far more difficult fact to prove, which is known as the Quillen-Suslin theorem.

Definition 3.4.20 (Degree of a divisor)**.** Let $D := \sum_{p \in C} a_p[p]$ be a divisor on a regular projective curve C (a dimension 1 regular projective k-scheme) over some field k. Then the **degree** of D is defined to be: $\deg(D) := \sum_{p \in C} a_p \deg(p)$.

Theorem 3.4.21 (Riemann-Roch theorem for line bundles on projective curves)**.** *Let C be a regular projective curve, and D be a divisor. Then:*

$$\chi(C, \mathscr{O}_C(D)) = deg(D) + \chi(C, \mathscr{O}_C).$$

Proof. Omitted, see Theorem 7.3.17 in [6]. □

Definition 3.4.22 (Degree of an invertible sheaf)**.** Let $\mathscr{L}$ be an invertible sheaf of C, where C is a regular projective curve over k a field. Then the **degree** of $\mathscr{L}$ is defined to be : $\deg(\mathscr{L}) := \chi(C, \mathscr{L}) - \chi(C, \mathscr{O}_C)$.

Remark 3.4.23. By Proposition 3.4.16, we have $\mathscr{L} = \mathscr{O}_C(D)$ for some D, then by Theorem 3.4.21 we have $\deg(\mathscr{L}) = \deg(\mathscr{O}_C(D))$.

Definition 3.4.24 (Arithmetic genus)**.** Let X be a scheme, then the **arithmetic genus** of X is defined to be: $g_a(X) := 1 - \chi(X, \mathscr{O}_X)$.

Theorem 3.4.25. *Let C be a regular projective curve over a field k, if $deg(\mathscr{L}) > 2g_a(C) - 2$ where $\mathscr{L}$ is an invertible sheaf on C, then we have:*

$$h^0(C, \mathscr{L}) = deg(\mathscr{L}) - g_a(C) + 1.$$

Proof. This fact is very difficult to prove, need to use Serre's duality. See 19.2.5 in [1] for the idea of the proof. □

Definition 3.4.26 (Elliptic curve)**.** Let E be a cubic regular projective curve over some field k, where the characteristic of k is not 2 or 3. Then E is an **elliptic curve** if $g_a(E) = 1$.

Lemma 3.4.27. *Let C be a regular projective curve of the form: $Proj(k[x, y, z]/(f))$ for some homogeneous polynomial f with $deg(f) \geq 1$. Then: $g_a(C) = \frac{(d-1)(d-2)}{2}$.*

Proof. Uses Hilbert's polynomial and Hilbert's function, see Section 18.6.6 in [1] for the idea of the proof. □

Remark 3.4.28. There is an alternative way to define an elliptic curve is to via Weierstrass form, let $f = x^3 - axz^2 - y^2z + bz^3 \in k[x, y, z]$, then an elliptic curve may be defined as: $E := \text{Proj}(k[x, y, z]/(f))$. We can see here E is regular by using differentials. And E is also of genus 1 by the lemma above. So the two definitions are really equivalent.

Now we are ready to prove the final and central result of this section, the group law (Although for simplicity, we only prove the case for algebraically closed field, but the general proof uses the same idea).

Theorem 3.4.29 (The group law)**.** *Let E be an elliptic curve defined over an algebraically closed field k. Then all points in E forms an abelian group.*

Proof. Notice that E is defined over an algebraically closed field k, hence all points in E are of degree 1. On the other hand, we for any principal divisor D, we have $\deg(D) = 0$. Where $D = \sum_{p \in E} a_p[p]$. Which means for each D, there are some points $p \in E$ associated with it. Hence we may define a bijection:

$$\phi : \mathscr{O}_E(D) \mapsto \operatorname{div}(\mathscr{O}_E(D)(p))$$

and the inverse:

$$\phi^{-1} : q \mapsto \mathscr{O}_E(q - p).$$

But notice that all $\mathscr{O}_E(D)$ are invertible sheaves by 3.4.16, hence we build a bijection: $E \leftrightarrow \operatorname{Pic}(E)$, and $\operatorname{Pic}(E)$ is an abelian group, so E must also be an abelian group. $\square$

Remark 3.4.30. We have only proved the case for algebraically closed field, but for non-algebraically closed field, the idea will be similar, but we need to classify points of degree more than 1. Moreover, ellipitc curves are naturally identified as a group variety hence an abelian variety.

3.5 Regularity and Smoothness II

In this section, we are going to develop various tools to proof smoothness-regularity theorem in Section 3.2. In particular, we need we need to build a sheaf-theoretic version of differentials to define smoothness.

Definition 3.5.1 (Module of Kähler differentials)**.** Let A be an B-algebra, with the morphism of rings: $\phi : B \to A$ and the morphism of schemes: $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$. Then the **module of Kähler differentials** is defined to be the A-module $\Omega_{A/B}$ together with a map: $d : A \to \Omega_{A/B}$, such that the following conditions are satisfied:

(1)$d(a_1 + a_2) = da_1 + da_2$

(2)$d(a_1 a_2) = a_1 da_2 + a_2 da_1$

(3)$db = 0$ if $b \in \phi(B)$.

Remark 3.5.2. We can see that this is really an algebraic version of differentials, which is B-linear, and satisfies product rule, and vanishes at constants. And the chain rule is also satisfied by the linearity of differentials.

Example 3.5.3. Let $B = \mathbb{C}$ and $A = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_s)$, with $\phi :=$ id. Then $\Omega_{A/B}$ is generated by dx_i, with the relation $df_1 = df_2 = \cdots = df_s = 0$. Which means we can describe $\Omega_{A/B}$ by the cokernel of the Jacobian matrix:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_s}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_s}{\partial x_n} \end{pmatrix}.$$

So we have proved a special case the follwing theorem. And the general proof uses exactly the same idea.

Theorem 3.5.4 (The Jacobian criterion of $\Omega_{A/B}$)**.** *Let $A = B[x_1, \dots, x_n]/(f_1, \dots, f_s)$. Then $\Omega_{A/B}$ can be described as coker(J), where J is the Jacobian matrix:*

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_s}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_s}{\partial x_n} \end{pmatrix}.$$

Definition 3.5.5 (Conormal sheaf)**.** Let $\pi : X \hookrightarrow Y$ be a closed embedding of cut out by the ideal sheaf $\mathscr{I}$, that is kernel of the map of sheaves $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$. Then the **conormal sheaf** is defined by $\mathscr{I}/\mathscr{I}^2$, denoted by $N^*_{X/Y}$. Hence we may define the normal sheaf $N_{X/Y} = \mathrm{Hom}(N^*_{X/Y}, \mathscr{O}_X)$. If the normal sheaf is locally free, then we call it a **normal bundle**.

Definition 3.5.6. Let $\pi : X \to Y$ be a morphism of schemes, then the **relative cotangent sheaf** is defined to be: $\Omega_{X/Y} := N^*_{X/X \times_Y X}$. If X is a k-scheme for some field k, then $\Omega_{X/k}$ is called the **cotangent sheaf**. We can define the relative tangent sheaf and tangent sheaf in the obvious way.

Definition 3.5.7 (Smoothness over k)**.** Let k be a field, X be a k-scheme, we say X is **smooth of dimension** n **over** k if X is locally of finite type and equidimensional of dimension n, and $\Omega_{X/k}$ is locally free of rank n.

Remark 3.5.8. It is not hard to check that this definition is equivalent to Definition 3.2.10. Immediate from the definition of $\Omega_{X/k}$.

Now we are ready to proof part (1) of the smoothness-regularity theorem. We will assume that k is algebraically closed, which means all points are either generic or closed. The method of the proof will be the same when k is an arbitrary perfect field, but need a bit more justification.

Proof of smoothness-regularity theorem (1). Let X be a regular k-scheme of finite type. So for each $p \in X$, we have $\dim(\mathscr{O}_{X,p}) = \dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2)$. Consider the affine cover $\{U_i\}$ of X, where $U_i = \mathrm{Spec}(A_i)$ are regular over k. Assume that $d = \dim_k(\mathfrak{m}_p/\mathfrak{m}_p^2)$ for $p \in U_i$ for some i. Notice that $\mathrm{Spec}(A_i)$ is a Noetherian k-scheme of finite type, so we may conclude that $A_i = k[x_1, \ldots, x_n]/(f_1, \ldots, f_s)$, and $\mathrm{corank}(J_p) = d$. Which means $\Omega_{U_i/k}$ is of rank d by 3.5.4. But $\Omega_{U_i/k}$ is of rank d means $\Omega_{X/k}(U_i)$ is of rank d. So $\Omega_{X/k}$ is of rank d as an $\mathscr{O}_X$-module. Now consider the map:

$$\Omega_{U_i/k} \to \mathscr{O}_{U_i}^{\oplus d}$$

$$(g_1, \ldots, g_d) \mapsto (h_1, \ldots, h_d)$$

where g_i and h_j are linear terms of the polynomials in the ideal $(f_1, \ldots, f_s)$, so there is an inverse map. So we have proved that $\Omega_{U_i/k} \cong \mathscr{O}_{U_i}$, which means $\Omega_{X/k}$ is a locally free sheaf of rank d. Hence X is smooth over k of dimension d. □

Now we want to define étale morphisms in order to prove the second part of smoothness-regularity theorem.

Definition 3.5.9 (Étale morphism)**.** Let $\pi : X \to Y$ be a morphism of schemes. We say it is **smooth of relative dimension** n if there exist open covers $\{U_i\} of X$ and $\{V_i\}$ of Y such that $\pi(U_i) \subset V_i$, where for every i we have the following commutative diagram:

$$\begin{array}{ccc} U_i & \xrightarrow{\sim} & W \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ V_i & \xrightarrow{\sim} & \mathrm{Spec}(B). \end{array}$$

Where there is a morphism $\rho : \mathrm{Spec}(B[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r)) \to \mathrm{Spec}(B)$ and W is an open subscheme of $\mathrm{Spec}(B[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r))$ where the determinant of the Jacobian matrix:

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_r} \end{vmatrix}$$

is a non-zero function on W. We say π is **étale** if it is smooth of relative dimension 0.

Lemma 3.5.10. *Let $\pi : X \to Y$ be a smooth morphism of relative dimension n, then for every $p \in X$, there is an open neighborhood U of p, where $\pi|_U = \beta \circ \alpha$, where $\alpha : U \to \mathbb{A}^n_Y$ is étale and $\beta : \mathbb{A}^n_Y \to Y$ is the projection map.*

Proof. Let $\pi : X \to Y$ be a smooth morphism of relative dimension n. Then we have the following diagram relation:

$$\begin{tikzcd}
X \arrow[dd, "\pi"] & U_i \arrow[l, hook'] \arrow[r, "\sim"] \arrow[dd, "\pi"] & W \arrow[dd, "\rho"] \arrow[dr, hook] & & \\
& & & \mathrm{Spec}(B[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r)) \arrow[r, hook] \arrow[dl, "\rho"] & \mathbb{A}^{n+r}_B. \\
Y & V_i \arrow[l, hook'] \arrow[r, "\sim"] & \mathrm{Spec}(B) \arrow[urr, hook, bend right] & &
\end{tikzcd}$$

On ther other hand, we have an étale morphism : $\mathbb{A}^{n+r}_B \to \mathbb{A}^n_Y$. Hence we have an étale morphism: $U_i \to \mathbb{A}^n_Y$ for each U_i, so the result is proven. $\square$

Lemma 3.5.11. *Let $\pi : X \to Y$ be étale, then for any $\pi^{-1}(q) \in X$ for some regular point $q \in Y$ of dimension n, $\mathscr{O}_{X,\pi^{-1}(q)}$ is regular of dimension n.*

Sketch of the proof. Let $\pi : X \to Y$ be an étale morphism. Let $\dim(\mathscr{O}_{Y,q})$ is of dimension n. By the definition of étale morphisms, $q \in V$ for some affine open subset $V \subset Y$. And notice that $\pi^{-1}(V) = \rho^{-1}(\mathrm{Spec}(B))$, so $\pi^{-1}(q) = \rho^{-1}(q) \in \mathrm{Spec}(B[x_1, \dots, x_r]/(f_1, \dots, f_r))$ is a regular point (this fact is because of the slicing criterion for regularity, see 12.2.C in [1]). So $\pi^{-1}(q) \in U \subset X$ is a regular point of dimension n. $\square$

Proof of smoothness-regularity theorem (2). Let X be a smooth k-scheme of dimension n. By Lemma 3.5.10, we know that for every open subset $U \subseteq X$, there is an étale morphism to $\mathbb{A}_k^n$, which means X locally covers $\mathbb{A}_k^n$. But we know that $\mathbb{A}^n$ is regular, so by Lemma 3.5.11, X is regular. $\square$

4 Arithmetic applications

It is clear that scheme theory gives a good interaction with number theory, as we can study schemes over $\mathrm{Spec}(\mathcal{O}_K)$ for some number field K. Where we can find rational points of curves and surfaces over $\mathrm{Spec}(\mathcal{O}_K)$, essentially gives certain solutions to some diophantine equations. There is a famous slogan of this: geometry determines arithmetic.

4.1 Arithmetic schemes

In this section, we are going to mainly study $\mathrm{Spec}(\mathcal{O}_K)$ and schemes over it. However, there will not be a full theory on it, we will proceed using results and construction in the previous sections.

Example 4.1.1. Let us consider $\mathrm{Spec}(\mathbb{Z})$. And let I be a fractional ideal of $\mathbb{Z}$, that means I is a finitely generated $\mathbb{Z}$-module in $\mathbb{Q}$ where there is $0 \neq a \in \mathbb{Z}$ such that $aI \subset \mathbb{Z}$. The claim is that I is an invertible sheaf on $\mathrm{Spec}(\mathbb{Z})$.

Notice that the function field $K(\mathrm{Spec}(\mathbb{Z})) = \mathbb{Q}$, so I consists of some rational functions on $\mathrm{Spec}(\mathbb{Z})$. Hence an $\mathscr{O}_{\mathrm{Spec}(\mathbb{Z})}$-module, thus a locally free sheaf on $\mathrm{Spec}(\mathbb{Z})$. Now we need to show I is of rank 1. Now we take an open cover of $\mathrm{Spec}(\mathbb{Z})$ of the form $\{D(f_i)\}_{i>0}$ for some $f \in \mathbb{Z}$. Note that for each f_i we have $\mathscr{O}_{D(f_i)} \cong \mathbb{Z}_{(f_i)}$, also $I|_{D(f_i)} \cong \mathbb{Z}_{(f_i)}$ by the localisation of $\mathbb{Z}$-module. Hence I is an invertible sheaf on $\mathrm{Spec}(\mathbb{Z})$. And therefore $I \in \mathrm{Pic}(\mathrm{Spec}(\mathbb{Z}))$. Thus all fractional ideals of $\mathbb{Z}$ are invertible sheaves.

Recall that we say two fractional ideals I, J are equivalent if there are non-zero $a, b \in \mathbb{Z}$ such that $(a)I = (b)J$, and the set of all fractional ideals modulo this equivalent relation is called the ideal class group of $\mathbb{Z}$ denoted $\mathscr{C}(\mathbb{Z})$, with binary operation: $I \otimes_{\mathbb{Z}} J$. And hence we have an isomorphism of groups:

$$\mathscr{C}(\mathbb{Z}) \to \mathrm{Pic}(\mathrm{Spec}(\mathbb{Z})).$$

Notice that $\mathrm{Pic}(\mathrm{Spec}(\mathbb{Z}))$ is clearly trivial as $\mathbb{Z}$ is a UFD and by Example 3.4.19. So $\mathbb{Z}$ is trivial. Hence all fractional ideals of $\mathbb{Z}$ are equivalent to $\mathbb{Z}$. Geometrically, it means all invertible sheaves on Spec($\mathbb{Z}$) are isomorphic to the structure sheaf $\mathscr{O}_{\text{Spec}(\mathbb{Z})}$. Moreover, by Proposition 3.4.16, we know that all invertible sheaves are associated with certain principal divisors. Let $a \in \mathbb{Z}$, we have the principal ideal given by $(a) = (p_1)^{n_1} \dots (p_r)^{n_r}$, where p_i is prime and $i > 0$. Then we have the principal divisor of Spec($\mathbb{Z}$) on a given by $\text{div}(a) = \sum_{i=1}^{r} n_i[(p_i)]$. So we have the divisor class group $\mathcal{C}(\text{Spec}(\mathbb{Z}))$, which again is trivial because all divisors are principal, which is because $\mathbb{Z}$ is a UFD. So in fact the true group isomorphism should be:

$$\mathscr{C}(\mathbb{Z}) \to \mathcal{C}(\text{Spec}(\mathbb{Z})).$$

All discussion above are not very interesting but quite important. Because it hints us of a arithmetic-geometric correpondence, and we can generalise the argument to arbitrary ring of integers of number fields.

Theorem 4.1.2. *Let K be a number field, let $\mathcal{O}_K$ be its ring of integers. Then there is an isomorphism of groups:*

$$\mathscr{C}(\mathcal{O}_K) \cong \mathcal{C}(\mathit{Spec}(\mathcal{O}_K)).$$

Proof. Notice that $\mathcal{O}_K$ is a Dedekind domain, so Spec($\mathcal{O}_K$) is normal, so we may conclude that all divisors are locally principal. In particular, we have $\mathcal{C}(\text{Spec}(\mathcal{O}_X)) \cong \text{Pic}(\text{Spec}(\mathcal{O}_X))$. So now we want to prove:

$$\mathscr{C}(\mathcal{O}_K) \cong \text{Pic}(\text{Spec}(\mathcal{O}_K))$$

in other words, we need to identify the fractional ideals of $\mathcal{O}_K$ with the invertible sheaves of Spec($\mathcal{O}_K$). Note that a fractional ideal I of $\mathcal{O}_K$ is an finitely generated $\mathcal{O}_K$-module in K, and $\mathcal{O}_K = \Gamma(\text{Spec}(\mathcal{O}_K, \mathscr{O}_{\text{Spec}(\mathcal{O}_K)}))$. So I is an $\mathscr{O}_{\text{Spec}(\mathcal{O}_K)}$-module, hence a locally free sheaf. By the arguement of Example 4.1.1, we have I is of rank 1, hence an invertible sheaf. So there is an induced isomorphism:

$$\mathscr{C}(\mathcal{O}_K) \to \text{Pic}(\text{Spec}(\mathcal{O}_K)).$$

$$I \mapsto \mathscr{L}$$

$$\mathcal{O}_K \mapsto \mathscr{O}_{\text{Spec}(\mathcal{O}_K)}.$$

□

By the theorem above, we have build a correspondence between fractional ideals of $\mathcal{O}_K$ and invertible sheaves of $\mathrm{Spec}(\mathcal{O}_K)$:

- Principal ideals in $\mathcal{O}_K \leftrightarrow$ principal divisors in $\mathrm{Spec}(\mathcal{O}_K)$;
- Fractional ideals of $\mathcal{O}_K \leftrightarrow$ invertible sheaves of $\mathrm{Spec}(\mathcal{O}_K)$;
- $\mathscr{C}(\mathcal{O}_K) \leftrightarrow \mathrm{Pic}(\mathrm{Spec}(\mathcal{O}_K))$.

We have see a typical example of arithmetic schemes, now we are going to see the general definition of an arithmetic scheme.

Definition 4.1.3 (Flatness)**.** Let M be an A-module. We say M is **flat** if $M \otimes_A$ - is an right exact functor. That means for any short exact sequence of A-modules:

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

then

$$M \otimes_A F \longrightarrow M \otimes_A G \longrightarrow M \otimes_A H \longrightarrow 0$$

is also exact. We say a ring homomorphism: $A \to B$ is **flat** if B is a flat A-module.

Definition 4.1.4 (Proper morphism)**.** Let $\pi : X \to Y$ be a morphism of schemes. We say π is **proper** if it is universally closed, finite and separated.

Definition 4.1.5 (Flat morphism)**.** Let $\pi : X \to Y$ be a morphism of schemes. We say π is **flat** at $p \in X$ if the pullback of the morphism of stalks is flat, i.e $\pi_p^\# : \mathscr{O}_{Y,\pi(p)} \to \mathscr{O}_{X,p}$ is a flat morphism of rings. We say π is flat if it is flat for every point in X.

Definition 4.1.6 (Functor of points)**.** Let X be an S-scheme. Then define the S**-valued points** of X to be $X(S)$, which consists all morphisms: $S \to X$. If S is a ring, then $X(S)$ consists all morphisms: $\mathrm{Spec}(S) \to X$.

Definition 4.1.7 (Fibre)**.** Let $\pi : X \to Y$ be a morphism of schemes, let $p \in Y$. A **fibre of** π **over** p is defined to be: $X \times_Y \mathrm{Spec}(k(p))$, where $k(p)$ is the residue field at p. If p is a closed point, we say the fibre over p is the **special fibre**. If p is a generic point, then we say the fibre over p is the **generic fibre**.

Definition 4.1.8 (Arithmetic scheme)**.** Write $Z = \mathrm{Spec}(\mathbb{Z})$. Let X be a Z-scheme, with a structure morphism: $\pi : X \to Z$. We say X is an **arithmetic scheme** if X is a normal scheme of finite type over Z, also π is a flat morphism. (Note here we dropped one technical stuff, which is to require X is excellent, but we will not use the excellent assumption, and an excellent scheme is hard to define, so we just ignore that)

Examples 4.1.9.

(1)Clearly $\mathrm{Spec}(\mathcal{O}_K)$ is an arithmetic scheme, where the structure morphism is induced by the ring morphism: $\mathbb{Z} \to \mathcal{O}_K$.

(2)Affine spaces and projective spaces over $\mathcal{O}_K$ are arithmetic schemes. For the case of affine space, just consider the ring morphism: $\mathbb{Z} \to \mathcal{O}_K[x_1, \ldots, x_n]$.

(3)A closed subscheme of $\mathbb{P}^n_{\mathcal{O}_K}$ is an arithmetic scheme.

The theory behind arithmetic schemes is very rich, but in general we are particularly interested in a special class of arithmetic schemes, called the arithmetic surface.

Definition 4.1.10 (Arithmetic surface)**.** Let $\mathfrak{S}$ be an arithmetic scheme. we say $\mathfrak{S}$ is an **arithmetic surface** if its generic fibre is a regular connected projective curve over $\mathbb{Q}$ and the special fibres are unions of curves over their residue fields.

Note that the term surface is a bit of confusing, as we will se in the next example.

Example 4.1.11. $\mathbb{P}^1_{\mathbb{Z}}$ is an arithmetic surface.

Proof. Take the structure morphism: $\pi : \mathbb{P}^1 \to \mathrm{Spec}(\mathbb{Z})$, we have the unique generic point of $\mathrm{Spec}(\mathbb{Z})$, namely $[(0)]$. Whose residue field is $\mathbb{Q}$, hence the generic fibre is $\mathbb{P}^1_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Q}) \cong \mathbb{P}^1_{\mathbb{Q}}$. And for those special fibres, we have $\mathbb{P}^1_{\mathbb{F}_p}$, as non-generic points in $\mathrm{Spec}(\mathbb{Z})$ is closed. Hence $\mathbb{P}^1_{\mathbb{Z}}$ is an arithmetic surface. □

Now we see why the term surface is confusing, as $\mathbb{P}^1_{\mathbb{Z}}$ looks like a curve.

Now we are going to introduce a very interesting example of arithmetic surface, which is the arithmetic surface given by the Weierstrass equation.

Example 4.1.12. Let $\mathfrak{S} \subset \mathbb{P}^2_{\mathbb{Z}}$ be an closed subscheme, which is an arithmetic surface, given by the equation:

$$y^2 = x^3 + ax + b, \ a, b \in \mathbb{Z}.$$

Then the generic fibre E is an elliptic curve defined over $\mathbb{Q}$. Recall that the discriminant of E is defined to be: $\Delta := -16(4a^3 + 27b^2)$. So take the closed point (p) where $p \nmid \Delta$, we have the special fibres to be the good reductions of E modulo p, for those $p|\Delta$, we have the bad reductions of E modulo p, where the special fibres with bad reductions are singular.

Theorem 4.1.13. *Let $\mathfrak{S}$ be an arithmetic surface proper over $\mathbb{Z}$, with generic fibre C, then $\mathfrak{S}(\mathbb{Z}) = C(\mathbb{Q})$.*

Remark 4.1.14. This is a very strong result, for example, let $\mathfrak{S}$ be an arithmetic surface given by the Weierstrass equation as in 4.1.12, then the theorem tells us that the integral solutions to the equation are really just the points of the generic fibre over $\mathbb{Q}$. And by Mordell-Weil theorem, elliptic curves over $\mathbb{Q}$ forms a finitely generated abelian group, which means the integral solutions of the equation have a groups structure.

Before we can prove this result, we need a lemma.

Lemma 4.1.15. *Let $\pi : X \to Y$ be a finite morphism, where X Noetherian. Then π is proper if and only if for every DVR A, and for every morphism of $U = Spec(Frac(A)) \to X$ and $T = Spec(A) \to Y$, we have the following diagram commutative:*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ T \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Proof. See Theorem 4.7 in chapter 2 in [5] □

Proof of theorem 4.1.13. Notice that we require the generic fibre C to be regular, so $\mathfrak{S}(\mathbb{Z}) \hookrightarrow C(\mathbb{Q})$ is clear.

Conversely, by lemma 4.1.15, we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{Q}) \arrow[r] \arrow[d] & \mathfrak{S} \arrow[d, "\pi"] \\ \mathrm{Spec}(\mathbb{Z}) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(\mathbb{Z}). \end{tikzcd}$$

Which tells us that there is a unique inverse morphism to the structure morphism: π :

$\mathfrak{S} \to \mathrm{Spec}(\mathbb{Z})$. Hence we have the commutative diagram:

$$\begin{tikzcd}
C = \mathfrak{S} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Q}) \arrow[d, "\text{projection}"] & \mathrm{Spec}(\mathbb{Q}) \arrow[l] \arrow[d, "\text{id}"] \\
\mathfrak{S} & \mathrm{Spec}(\mathbb{Z}). \arrow[l, "\pi^{-1}"]
\end{tikzcd}$$

Now let us take $p \in C(\mathbb{Q})$. Note that p gives a map :Spec($\mathbb{Q}$) $\to C$, and by the commutative diagram above, we know that π^{-1} is uniquely associated with p, so we can conclude that $C(\mathbb{Q}) \hookrightarrow \mathfrak{S}(\mathbb{Z})$. Hence we have $C(\mathbb{Q}) = \mathfrak{S}(\mathbb{Z})$. $\square$

For rational points of curves there is a deep result called Faltings' theorem, which was conjectured by Mordell and proven by Gerd Faltings.

Theorem 4.1.16 (Faltins' theorem)**.** *Let C be a projective curve of genus more than 1 defined over $\mathbb{Q}$, then $|C(\mathbb{Q})| < \infty$.*

We can see this result largely interacts with the theory of arithmetic surfaces, as to determine $|C(\mathbb{Q})|$ is the same to determine $|\mathfrak{S}(\mathbb{Z})|$ for some arithmetic surfaces $\mathfrak{S}$ whose generic fibre is C. The proof of this theorem is very hard, and Faltings won the Fields medal for his proof (And it would be impossible for Andrew Wiles to prove the Fermat's Last Theorem without Faltings' theorem). The proof requires a special class of arithmetic surface with good properties, which is called the Néron model.

4.2 Introduction to p-divisible groups

In this section we will give a brief introduction to the theory of p-divisible groups(or Barsotti-Tate groups), which plays an important role in modern arithmetic geometry include p-adic Hodge theory.

Definition 4.2.1 (Group scheme)**.** Let $\mathcal{S}$ be the category of S-schemes. A **group object** is an element$X \in \mathrm{Obj}(\mathcal{S})$ with three morphisms:

(1)$m : X \times_S X \to X$

(2)$i : X \to X$

(3)$e : S \to X$ where the following properties must be satisfied:

(1)Associativity:

$$\begin{array}{ccc} X \times_S X \times_S X & \xrightarrow{(m,\mathrm{id})} & X \times_S X \\ \downarrow{\scriptstyle (\mathrm{id},m)} & & \downarrow{\scriptstyle m} \\ X \times_S X & \xrightarrow{m} & X \end{array}$$

is commutative, where id is the identity map.

(2)Identity:

$$X \xrightarrow{\cong} S \times_S X \xrightarrow{e\times\mathrm{id}} X \times_S X \xrightarrow{m} X$$

and

$$X \xrightarrow{\cong} S \times_S X \xrightarrow{\mathrm{id}\times e} X \times_S X \xrightarrow{m} X$$

are both the identity map $X \to X$.

(3)Inverse:

$$X \xrightarrow{i,\mathrm{id}} X \times_S X \xrightarrow{m} X$$

and

$$X \xrightarrow{\mathrm{id},i} X \times_S X \xrightarrow{m} X$$

are both the map can be identified with the composition: $X \longrightarrow Z \xrightarrow{e} X$. Such X is called a **group scheme**.

Remark 4.2.2. This is a rather complicated and categorical definition, but it is really just a fancy way to define a group structure.

Example 4.2.3. Take $S = \mathrm{Spec}(\mathbb{Z})$. Consider $\mathrm{Spec}(\mathbb{Z}[X]) = \mathbb{A}^1_{\mathbb{Z}} \in \mathcal{S}$. Let us take:

$$m : \mathbb{A}^1 \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{A}^1 \to \mathbb{A}^1$$

$$(p, q) \mapsto p + q$$

and

$$i : \mathbb{A}^1 \to \mathbb{A}^1$$

$$p \mapsto -p$$

and

$$e : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{A}^1$$

$$(0) \mapsto 0.$$

It is elementary to check all three properties are satisfied, so $\mathbb{A}^1$ is a group scheme.

Example 4.2.4. Elliptic curves over $\mathbb{Q}$ are group schemes, with the group structure given by the geometric group law, see section 2 of chapter 3 in [7].

Example 4.2.5. In this example, we will illustrate a few classes of important groups schemes.

(1) Additive group schemes, denoted by $G_a(X)$ for X a Spec($\mathbb{Z}$)-scheme. Which is defined to be the additive subgroup of $\Gamma(X, \mathscr{O}_X)$. If X is affine, then X itself is an additive group scheme.

(2)General linear group schemes, denoted by $\mathbf{GL}_n(X)$. Which is defined to be the invertible $n \times n$ with whose entries are elements in $\Gamma(X, \mathscr{O}_X)$, with group operation being multiplication of matrix. This is a non-commutative group scheme, as in general matrix multiplication is non-commutative. Let us consider an example, let $X = \mathbb{A}^1_{\mathbb{Z}}$, then $\mathbf{GL}_n(\mathbb{A}^1_{\mathbb{Z}})$ is just the general linear group over $\mathbb{Z}$. If we fix $n = 1$, then we have the multiplicative group scheme $G_m(X) := \Gamma(X, \mathscr{O}_X)^{\times}$. This group scheme is commutative. And it is natural to replace **GL** by **SL**, then we have the special linear group scheme $\mathbf{SL}_n(X)$, with obvious definition.

(3)Group of unity, denoted by $\zeta_n(X) := \{x \in G_m(X) : x^n = 1\}$. For example, let X be an arithmetic surface proper over Spec($\mathbb{Z}$), C be the generic fibre of X, then by theorem 4.1.13, $\zeta_n(X) = \zeta_n(C)$.

Definition 4.2.6. Let $S = \mathrm{Spec}(\mathbb{Z}_p)$ (here $\mathbb{Z}_p$ means the p-adic completion of $\mathbb{Z}$, where $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$), and $\mathcal{S}$ is the category of S-schemes. Let X be an S-scheme, such that:

(1)X is of finite type over S.

(2)$\Gamma(X, \mathscr{O}_X)$ is a finite length $\mathbb{Z}_p$-module(length means the maximal length of the chain of submodules).

Let $\mathcal{F}$ be a functor from $\mathcal{S}$ to the category of sets. We say $\mathcal{F}$ is a **formal group functor** if $\mathcal{F}(X)$ is a group scheme for any X satisfies (1) and (2). If $\mathcal{F}$ is a group functor (which means it sends schemes to group schemes), then we can restrict it to the schemes satisfies (1) and (2) to obtain a formal group functor, we call this process the **formal completion** of $\mathcal{F}$, denoted by $\hat{\mathcal{F}}$.

Remark 4.2.7. For convenience, we call (1) and (2) the finite property. This is a non-standard notation.

Definition 4.2.8 (Profinite algebra)**.** Let A be a commutative $\mathbb{Z}_p$-algebra and $(I_n) \subseteq A$ is a sequence of ideals. We say A is **profinite** if $A = \varprojlim A/I_n$ where each A/I_n is a finitely generated $\mathbb{Z}_p$-module of finite length.

Definition 4.2.9 (Formal spectrum)**.** Let A be a profinite $\mathbb{Z}_p$-algebra, the **formal spectrum** of A is defined to be:

$$\mathbf{Spf}_A(X) := \{\pi : A \to \Gamma(X, \mathscr{O}_X)\}$$

where π is a morphism of $\mathbb{Z}_p$-algebras but also a continuous between topological spaces, and X has the finite property.

Proposition 4.2.10. *Let $\mathcal{S}$ be the category of Spec$(\mathbb{Z}_p)$-schemes. Then $\boldsymbol{Spf}_A$ defines a formal group functor.*

Proof. Let $X \in \mathrm{Obj}(\mathcal{S})$ with the finite property. We just need to determine the group structure of $\mathbf{Spf}_A(X)$. Notice that $\mathbf{Spf}_A(X) \subseteq \mathrm{Hom}_{\mathbb{Z}_p}(A, \Gamma(X, \mathscr{O}_X))$, which means $\mathbf{Spf}_A(X)$ respect the group structure of $\mathrm{Hom}_{\mathbb{Z}_p}(A, \Gamma(X, \mathscr{O}_X))$, moreover, continuous functions respect the group structure. Thus $\mathbf{Spf}_A(X)$ forms an abelian group, hence $\mathbf{Spf}_A$ defines a formal group functor. □

Hence we have defined a formal group scheme, which is a scheme given by $\mathbf{Spf}_A$.

Example 4.2.11. Here we will see an example where A is not necessarily profinite $\mathbb{Z}_p$-algebra. Suppose $A = \mathbb{Z}_p[X]$ which is clearly not profinite. So $\mathrm{Spec}(A)$ is not a formal group scheme. But we may convert it into a formal group scheme. We just compute $\hat{A}$ by using the profinite completion of rings. Which gives $\hat{A} = \mathbb{Z}_p[[X]]$, i.e the formal power series over $\mathbb{Z}_p$. So the profinite completion of $\mathrm{Spec}(A)$ arose naturally from the proposition above. Just set $\widehat{\mathrm{Spec}(A)} = \mathbf{Spf}_{\hat{A}}(\mathrm{Spec}(A))$. Notice that we have $\widehat{\mathrm{Spec}(A)} = \mathrm{Spec}(\hat{A})$.

Definition 4.2.12 (p-divisible group)**.** Let p be a prime, a p**-divisible group** over $\mathbb{Z}_p$ of height h is a sequence (G_n, i_n) where:

(1)G_n is a finite commutative group scheme over $\mathbb{Z}_p$ of order p^{nh}.

(2)For each n, we have the following exact sequence:

$$0 \longrightarrow G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{m} G_{n+1}$$

where m is given by multiplying by p^n.

Example 4.2.13. From definition we can immediately think of an example. Let us take $G_n = \mathrm{Spec}(A_n)$, where $A_n = \mathbb{Z}/p^n\mathbb{Z}$, let $A = \varprojlim A_i = \mathbb{Z}_p$, which is the p-adic integer. Then $G = \mathbf{Spf}_A(A)$ is the height 1 p-divisible group over $\mathbb{Z}_p$.

However, this is more or less a trivial example, we now consider $G_n = \mathrm{Spec}(B_n)$, where $B_n = \mathbb{F}_{p^n}[X]$. So $B = \mathbb{Z}_p[X]$, so $G = \mathbf{Spf}_B(B) = \mathbb{A}^n_{\mathbb{Z}_p}$. In general if G_n is affine, i.e $G_n = \mathrm{Spec}(R_n)$, then the p-divisible group is $\mathbf{Spf}_R(R)$, where $R = \varprojlim R_n$.

The main reason why p-divisible groups are interesting is not because the group itself, but because it leads to the definition of Tate modules, which is the starting point of the theory of Galois representations.

Definition 4.2.14 (Tate module)**.** Let p be a prime, X be a formal group scheme over $\mathbb{Z}_p$, then the **Tate module** of X, denoted by $T_p(X)$, is defined to be:

$$T_p(X) := \varprojlim_n \ker(f)$$

where $f : X(\bar{\mathbb{Q}}) \to X(\bar{\mathbb{Q}})$ is given by multiplying p^n. If X is a p-divisible group, given by G_n, then $T_p(X) = \varprojlim G_n(\bar{\mathbb{Q}})$.

The study of Tate modules is very hard, as it relates to the group $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, it is still unknow that whether there exists a direct description of this group(something like its presentation or Dynkin diagram, etc). On the other hand, the study of Tate modules lead to the study of Galois representations, which is a key tool of Wiles' proof of Fermat's last theorem.

5 Summaries and further stuff

In this dissertation, we have investigated the basic definitions and examples of schemes, and have seen various geometric facts about schemes. On the other hand, we have also discovered some usages of schemes in number theory.

In the second chapter, we have seen various definitions and constructions about schemes. Scheme is a generalisation of algebraic variety, and there is even a generalisation of scheme, which is called **moduli space**. Moduli space is used to classifying geometric object, and

David Mumford's *Geometric invariant theory* and [4] provide a general introduction to the theory of moduli space.

In the third chapter, we studied some geometric properties and constructions on schemes. For the two sections on smoothness and regularity, we have mostly used algebraic methods. However they can be studied in a very analytic way, our introduction of Kähler differentials is a good starting point. To investigate more interesting theory in an analytic way, the best book to read is J.P Serre's *GAGA*, which connects algebraic geometry and complex differential geometry. We have also seen some basic fact about vector bundles. Then general theory of vector bundles is very rich, so called K-theory, which is still an active area of research. The algebraic aspect of K-theory, so called algebriac K-theory has the most interactions with algebraic geometry and algebraic number theory, a proof of quadratic reciprocity is given by the context of algebraic K-theory, Daniel Quillen's *Algebraic K-theory* series are good references to have a taste of algebraic K-theory. Čech cohomology is the first cohomology theory in algebraic geometry. There are many other cohomology theories in algebriac geometry, and the Grothendieck contributed many of them, and his famous *SGA* series discuss quite a lot cohomology theories.

In the fourth chapter, we firstly looked at a correspondence between line bundles on $\mathrm{Spec}(\mathcal{O}_K)$ and fractional ideals of $\mathcal{O}_K$. The general correspondence between the algebraic geometry and algebraic number theory is very deep. We have only discussed the correspondence over $\mathrm{Spec}(\mathcal{O}_K)$, but the it can be enlarged to any Dedekind domains. Other than ring of integers, another important class of Dedekind domains, is the coordinate ring of regular integral affine curve over a field. So there is a large amount of analogies in both contexts, the study of such correspondence is called the geometic Langlands correspondence, which is a big theme in Langlands program. Also in Langlands program, the study of Zeta functions plays an important role. The theory of Zeta functions over arithmetic schemes is one of the major interests in arithmetic geometry, which also uses large amount of tools in complex analysis, especially, modular forms and automorphic forms. Arithmetic surface is highly connected with abelian varieties and elliptic curves and [9] provides a general introduction to these objects.

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