ON THE NON-EXISTENCE OF SYMPATHETIC LIE ALGEBRAS OF LOW DIMENSION

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Joint work w/ Gil Salgado.

I. OVERVIEW OF SYMPATHETIC LIE ALGEBRAS.

Let $\mathfrak{g}$ be a Lie algebra. finite dim / $\mathbb{C}$

Fact: $\mathfrak{g}$ semi-simple
then

1) $Z(\mathfrak{g}) = \{0\}$
2) $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$
3) $\underbrace{Der(\mathfrak{g})}_{\text{space of derivations}} = \underbrace{Inn(\mathfrak{g})}_{\text{space of inner derivations}}$

<u>Question</u>. Angelopolous (mid 80's)
Do properties 1,2 & 3 completely characterise semi-simple Lie algebras?

<u>Answer</u>: No! In 1988 he gives an example of $\mathfrak{g}$ non-semisimple satisfying 1,2,3.

Defn. $\mathfrak{g}$ is sympathetic if 1,2,3 hold.

Non trivial examples:

Angelopolous	1988	dim 35
Benayadi	1993	dim 48
Benayadi	1996	dim 25

What is the smallest possible dimension of a (non semi-simple) sympathetic Lie algebra? ---- (*)

- smallest known dim = 25 (Benayadi)
- $\mathfrak{g}$ sympathetic $\Rightarrow \dim \mathfrak{g} \geq 10$ (J. Simon)

Theory was developed to describe some structural properties.
(Angelopoulos, Benayadi, etc.)

For example, Benayadi shows that $\mathfrak{g}$ splits into sympathetic direct factors.

At that time,

- knowledge of classification of Lie algebras was very limited.
(symp. were meant to characterise semi-simple ones!)
- computational tools were almost inexistent.

II. NON-EXISTENCE OF SYMPATHETIC LIE ALGEBRAS

From here onwards this is
new work joint w/Gil Salgado

<u>Levi decomposition</u>.

$\mathfrak{g}$ Lie algebra $\dim \mathfrak{g} < \infty$

$$\mathfrak{g} = \underbrace{\mathfrak{g}_L}_{\substack{\text{semi-simple}\\ \text{subalgebra}}} \ltimes \underbrace{\mathcal{H}}_{\substack{\text{maximal solvable}\\ \text{ideal } \mathrm{Rad}(\mathfrak{g})}}$$

We consider $\mathfrak{g}_L = \mathfrak{sl}_2(\mathbb{C})$ since

- $\mathfrak{sl}_2\mathbb{C}$ is the simple Lie alg. w/smallest dimension
- $\mathfrak{g}$ is indecomposable into direct symp. factors ☆

- The $sl_2\mathbb{C}$-action $[\ ,\ ] : sl_2\mathbb{C} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ gives a decomposition of $\mathfrak{h}$ into irreps

  $\mathfrak{h} = V_{n_1} \oplus \cdots \oplus V_{n_N}$ $\quad\quad \dim V_{n_i} = n_i + 1$

  $[sl_2\mathbb{C}, V_{n_i}] \subseteq V_{n_i}$

  In fact, $[sl_2\mathbb{C}, V_{n_i}] = \begin{cases} V_{n_i} \\ 0 \end{cases}$ (Schur's lemma)

- $Z(\mathfrak{g}) = \{0\} \Rightarrow \dim V_{n_i} > 1$
  (scalars commute with everything)

  $\parallel$

- The only irrep where $sl_2\mathbb{C}$ acts trivially is $\dim V_{n_i} = 1$.

  $\Rightarrow \quad [sl_2\mathbb{C}, V_{n_i}] = V_{n_i}$

- Classical result: $\mathfrak{g}$ perfect $\Rightarrow \mathfrak{h}$ nilpotent

  $\Rightarrow \quad Z(\mathfrak{h}) \neq \{0\}$

Now

$$\dim \mathfrak{g} = \dim \mathfrak{sl}_2\mathbb{C} + \sum_{k=1}^{N} \dim V_{n_k} \qquad n_k \geq 1$$

$$= 3 + N + \sum_{k=1}^{N} n_k$$

Can we rule out small N?

<u>Theorem</u> (G.P., Salgado)

$N = 1,2,3$ then $\mathfrak{g}$ is not sympathetic.

Theorem (G.P., Salgado)

$\mathfrak{g}$ sympathetic with $N=4$.

Then $[\mathfrak{n},\mathfrak{n}] = V_{n_3} \oplus V_{n_4}$ with $V_{n_4} = Z(\mathfrak{n})$

and the Lie bracket is unique, given by

	$\mathfrak{sl}_2\mathbb{C}$	V_{n_1}	V_{n_2}	V_{n_3}	V_{n_4}
$\mathfrak{sl}_2\mathbb{C}$	$\mathfrak{sl}_2\mathbb{C}$	V_{n_1}	V_{n_2}	V_{n_3}	V_{n_4}
V_{n_1}	V_{n_1}	V_{n_3}	V_{n_4}	V_{n_4}	0
V_{n_2}	V_{n_2}	V_{n_4}	V_{n_4}	0	0
V_{n_3}	V_{n_3}	V_{n_4}	0	0	0
V_{n_4}	V_{n_4}	0	0	0	0

Moreover, $n_i \equiv 0 \bmod 2$

$$\dim \mathfrak{g} = 7 + \sum_{k=1}^{4} 2m_k \qquad m_k \geq 1$$

III. COMPUTATIONAL METHODS

Input: For every partition (m_1, m_2, m_3, m_4) with $\dim \mathfrak{g} \leq 25$ we

1) Construct the candidate $\mathfrak{g}$
2) Is $\mathfrak{g}$ Lie?
3) If 2) is true: construct $\mathrm{Der}(\mathfrak{g})$
4) Is $\dim \mathrm{Der}(\mathfrak{g}) = \dim \mathrm{Inn}(\mathfrak{g})$?
5) If 4 is true: $\mathfrak{g}$ is symp
   else $\mathfrak{g}$ non-symp.

Theorem (G.P., Salgado)
$\mathfrak{g} = \mathfrak{sl}_2\mathbb{C} \ltimes V_{n_1} \oplus \cdots \oplus V_{n_4}$ symp. $\Rightarrow \dim \mathfrak{g} \geq 25$

Corollary.
$\mathfrak{g} = \mathfrak{sl}_2\mathbb{C} \ltimes \mathfrak{n}$ symp $\Rightarrow \dim \mathfrak{g} \geq 15$

III. FINAL REMARKS.

- We've are currently proving that this is true for any $\mathfrak{g}_L$ simple with 4 irreps.

- We have results concerning a family of nilpotent Lie alg's:
This gave us a systematic approach to dealing with nilpotent Lie algebras

Thank you!