EXTENDED ABSTRACT: RATIONAL SIMPLE CONNECTEDNESS AND FANO THREEFOLDS

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We work over the field of complex numbers.

Fano fibrations naturally appear in the birational classification of algebraic varieties. Let X be a smooth projective variety with negative Kodaira dimension, then the minimal model program (or MMP) conjecture predicts that X is birational to a Mori fibration $\pi: W \to B$, which is a "minimal" Fano fibration.

We are interested in studying the existence of rational sections for Fano fibrations. Since smooth Fano varieties are rationally connected [4, 16], the classical result by Graber, Harris and Starr [12] implies that if B is a curve, then π has a section (i.e. the generic fibre of π has a $\mathbb{C}(B)$-point).

The problem becomes much more subtle over a higher dimensional base: it is very easy to construct conic bundles over smooth surfaces having no rational sections.

Problem 1. *Can we find a sufficient condition on the fibres of π which guarantees the existence of a rational section, when B is a smooth surface?*

The first positive answer in this direction is the following classical result.

Theorem 2 (Tsen-Lang, [17])**.** *Let $K = \mathbb{C}(B)$ be the function field of an r-dimensional variety B and let $X_d \subset \mathbb{P}^n_K$ be a degree d hypersurface over K which verifies the numerical condition*

$$d^r \le n. \tag{A}$$

Then $X_d(K) \neq \varnothing$ (i.e. X_d has a K-point).

The case $r = 1$ can be rephrased as follows: for a smooth hypersurface $X_d \subset \mathbb{P}^n$ over the function field of a curve, the following conditions are equivalent:

(i) *(numerical)* $d \le n$;

(ii) *(geometric)* X_d is rationally connected.

Also, an easy computation shows that (i) is equivalent to the Fano condition.

A very intuitive way to generalise the numerical condition (A) for an arbitrary smooth Fano variety X consists in asking positivity of some Chern characters.

Definition 3. A smooth Fano variety X is 2-Fano if $\mathrm{ch}_2(X) > 0$.

This class of Fano varieties has been introduced in [8] and extensively studied in several papers [6, 2, 3, 20, 22, 1]. Although this condition is (at least in principle) easy to check on examples, it is geometrically very restrictive (see Theorem 7).

In this context, Problem 1 can be rephrased as:

Is there a geometric analogue of the numerical condition "$d^2 \leq n$" for smooth Fano varieties?

Inspired by topology, Barry Mazur proposed the following algebro-geometric analogue of simple connectedness in topology: *one asks for rational connectedness of certain moduli spaces of rational curves of sufficiently high degree* ([7, Section 1], [5, Hypothesis 6.8]). This is a *working* definition, since the geometry of moduli spaces of rational curves of high degree on Fano varieties is not well undersood (cf. [10, Section 1]).

In order to fix the ideas, we fix a definition of rational simple connectedness for Fano varieties of Picard rank one (see [7] for a more general definition for polarised varieties, [5] for a slightly different notion and [25] for a survey paper on the topic).

Let X be a smooth Fano variety verifying $\rho(X) = 1$. We denote by $\overline{\mathrm{M}}_{0,m}(X,d)$ the coarse moduli space of degree d stable rational curves with m marked points on X.
This moduli space comes with an evaluation map $\mathrm{ev}_m : \overline{\mathrm{M}}_{0,m}(X,d) \to X^m$.

Definition 4 ([7])**.** The Fano variety X is (strongly) *rationally simply connected* if for any $m \geq 2$ there exists a degree $d_0(m) > 0$ such that for all $d \geq d_0(m)$ there exists a canonical component $M_{d,m}$ of $\overline{\mathrm{M}}_{0,m}(X,d)$ for which the evaluation map

$$\mathrm{ev}_m : M_{d,m} \to X^m$$

is dominant and its general fibre is rationally connected.

What is surprising about this notion is that the results in [5] provide evidence of the following principle: *the only obstruction to a rational section for a fibration $\pi: Z \to B$ over a surface with rationally simply connected fibres lies in a Brauer class of the base B.* See also [21] for a discussion on this Brauer obstruction.

The core question is the following.

Question 5. *Which Fano varieties are rationally simply connected?*

It is surprising how very few non-trivial classes of examples (and counterexamples) of rationally simply connected Fano varieties are known. It is not too hard to see that the projective space $\mathbb{P}^n_{\mathbb{C}}$ and the smooth quadric hypersurface $Q^n \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ are both rationally simply connected (see [10]). Understanding this property for general complete intersection is much trickier and the following result has been obtained.

Theorem 6 ([7, 9])**.** *Let $X^{(n)}_{\underline{d}} \subset \mathbb{P}^N_{\mathbb{C}}$ be a general complete intersection of dimension $n \geq 3$ and degree $\underline{d} = (d_1, \dots d_c)$. Then*

$$X^{(n)}_{\underline{d}} \text{ is rationally simply connected} \iff \sum_{i=1}^{c} d_i^2 \leq N$$

The previous result shows that Tsen-Lang numerical condition (A) is equivalent to rational simple connectedness for general hypersurfaces.

Although the full classification of smooth complex Fano threefolds up to deformation is well-known ([13, 14, 15, 18, 19]) and the families have a very explicit description, we stress that rationally simple connectedness is not a birational property, namely there exist rational Fano varieties which are not rationally simply connected.

The study of rational simple connectedness for smooth Fano threefold is interesting, also because the notion of 2-Fano is too restrictive in this context, as the following result shows.

Theorem 7 ([3]). *Let X be a smooth complex Fano threefold. Then*

$$X \text{ is } 2-Fano \iff X \cong \mathbb{P}^3 \text{ or } Q^3 \subset \mathbb{P}^4.$$

In [10] and [11], the study of rational simple connectedness for smooth Fano threefolds of index two has been started, and we obtain the following result.

Theorem 8 ([10, 11]). *The smooth quintic Fano threefold $V_5 \subset \mathbb{P}^6_{\mathbb{C}}$ is rationally simply connected.*

The techniques involve an explicit description of moduli spaces of curves, looking at the birational geometry of rational Fano varieties (see [23, 24]).

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