A TOUR OF FLOPS AND INDUCED AUTOEQUIVALENCES

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Algebraic Geometry is a vast area of Mathematics and nowadays picturing its state-of-the-art is extremely difficult because of the various branches it is made of. Despite the many ideas which have come into play during the years, Algebraic Geometry is still based on the following belief: geometric questions can be studied using algebraic techniques.

Let us consider X a smooth, projective variety over a field k of characteristic zero. Then, naturally attached to X there are many different algebraic objects, each of which builds a connection between algebra and geometry. For example, one can consider the Hochschild homology $\mathrm{HH}_\bullet(X)$, the Chow ring $\mathrm{CH}(X)$, the Grothendieck group $\mathrm{K}_0(X)$, etcetera. Among these gadgets, there is one that has played a prominent role since its introduction and which will be the main topic of interest in this abstract: the bounded derived category of coherent sheaves $\mathrm{D^b}(X) = \mathrm{D^b}(\mathrm{Coh}(X))$.

When passing from a geometric object to an algebraic one, it is natural to wonder whether we gave up some information. In our context, this question takes the form: does $\mathrm{D^b}(X)$ recover X? It is known that the answer is yes if ω_X, the canonical bundle of X, is (anti)ample, see [BO01]. However, this not need to be true if, for example, X is Calabi–Yau: $\omega_X \simeq \mathcal{O}_X$. As a consequence, one might wonder whether, being less rigid than X, there are more symmetries of $\mathrm{D^b}(X)$ than X.

It is clear that any automorphism $f\colon X \to X$ induces an autoequivalence, *i.e.*, endofunctor which is an equivalence, of $\mathrm{D^b}(X)$ by considering the functor[1] $f_*\colon \mathrm{D^b}(X) \to \mathrm{D^b}(X)$. Moreover, the resulting homomorphism of groups $\mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{D^b}(X))$ is injective but not surjective, and therefore there are more symmetries in the algebraic world than in the geometric one. However, we should distinguish between autoequivalences that arise because of the new technical setup in which we put ourselves, and *genuinely* new autoequivalences.

The derived category is a triangulated category, therefore among its autoequivalences we find the shift [1], which generates a subgroup $\mathbb{Z} \simeq \mathbb{Z}\cdot[1] \subset \mathrm{Aut}(\mathrm{D^b}(X))$. Furthermore, any line bundle induces an autoequivalence by tensor product. Hence, we have an inclusion

$$\mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)) \subset \mathrm{Aut}(\mathrm{D^b}(X))$$

where the left hand side is called the *subgroup of standard autoequivalences*. The above inclusion is known to be an equality when ω_X is (anti)ample, see [BO01], which tells us that in this case nothing new is happening.

In this abstract, we will focus on the case in which X is Calabi–Yau, and we will see how a geometric relation (K-equivalence) conjecturally produces non-standard autoequivalences.

1. Autoequivalences of (enhanced) triangulated categories

Even though $\mathrm{D^b}(X)$ is defined as a triangulated category, and the question we are interested in is to describe its group of autoequivalences as a triangulated category, we need to introduce a stronger formalism.

Technically speaking, the problem we have to face is that if $\mathcal{T}$ and $\mathcal{T}'$ are triangulated categories, then there is no canonical triangulated structure on $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{T}, \mathcal{T}')$, the category of exact functors between $\mathcal{T}$ and $\mathcal{T}'$. The pitfall resides in one of the defining axioms of a

[1] As we will work only with derived categories, we will adopt the convention that our functors are implicitly derived.↑

triangulated category: we require the existence of a certain map, but not uniqueness. This results in the lack of the functoriality needed to endow $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{T}, \mathcal{T}')$ with a triangulated structure.

To *enhance* triangulated categories, we consider k-linear differential graded (dg-) categories, *i.e.* categories $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(-,-)$ is a dg-k-vector space for every pair of objects and the composition maps are maps of dg-k-vector spaces. To every such $\mathcal{D}$ we can associate another category, which we denote by $\mathrm{H}^0(\mathcal{D})$, called its *homotopy category* and that has the same objects of $\mathcal{D}$ but $\mathrm{Hom}_{\mathrm{H}^0(\mathcal{D})}(-,-) = \mathrm{H}^0(\mathrm{Hom}_{\mathcal{D}}(-,-))$. In this way, we get a functor $\mathrm{H}^0 \colon \mathrm{dgcat}_k \to \mathrm{Cat}_k$ between the category of k-linear dg-categories and the category of k-linear categories.

Among all dg-categories we can select those for which a functorial construction of cones is possible; we call them *pre-triangulated* dg-categories and we denote them as $\mathrm{dgcat}_k^{\mathrm{tr}} \subset \mathrm{dgcat}_k$.[2] Then, H^0 sends pre-triangulated dg-categories to triangulated categories, and we have set up the enhanced framework we need. We will call the (functors of) triangulated categories in the essential image of H^0 *enhanced* (functors).

Example 1.0.1. Given a smooth projective variety X, the bounded derived category $\mathrm{D}^{\mathrm{b}}(X)$ is an enhanced triangulated category. There exist many possible different enhancements. However, they are all equivalent to each other, where the notion of *equivalence of enhancements* is suitably defined. We refer the interested reader to [CS17] for more details.

What we gained in passing to pre-triangulated dg-categories is that $\mathrm{dgcat}_k^{\mathrm{tr}}$ has an internal hom functor (induced by an internal hom in dgcat_k) which can be derived to induce an internal hom for $\mathrm{H}^0(\mathrm{dgcat}_k^{\mathrm{tr}})$. In particular, functors between pre-triangulated dg-categories have a natural pre-triangulated structure, and therefore enhanced functors of enhanced triangulated categories have the same structure.

Using $\mathrm{dgcat}_k^{\mathrm{tr}}$ we can introduce the concept of a *spherical functor*, see [AL17]. Given two enhanced triangulated categories $\mathcal{T}$, $\mathcal{T}'$ together with an enhanced functor $f \colon \mathcal{T} \to \mathcal{T}'$, we say that f is spherical if it has enhanced adjoints f^R, f^L, and the functors sitting in the following distinguished triangles are equivalences

$$ff^R \to \mathrm{id} \to t_f \qquad c_f \to \mathrm{id} \to f^R f.$$

The functor t_f is called the *twist* around f and c_f is called the *cotwist*.

Example 1.0.2. The first examples of spherical functors were given by spherical twists around spherical objects, see [ST01]. A spherical object $E \in \mathrm{D}^{\mathrm{b}}(X)$ is an object such that $E \otimes_{\mathcal{O}_X} \omega_X \simeq E$ and

$$\mathrm{Hom}^\bullet_{\mathrm{D}^{\mathrm{b}}(X)}(E,E) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(E, E[n])[-n] \simeq \mathrm{H}^\bullet(S^{\dim X}, k)$$

as graded algebras (notice that the first condition is always satisfied if X is Calabi–Yau). The spherical functor associated to E is $f = - \otimes_k E : \mathrm{D}^{\mathrm{b}}(k) \to \mathrm{D}^{\mathrm{b}}(X)$, its cotwist is given by $[-\dim X - 1]$, and its twist sits in the distinguished triangle

$$\mathrm{RHom}_X(E,-) \otimes_k E \to \mathrm{id} \to t_E.$$

Let us list a few examples of spherical objects:

(1) If X is strict Calabi–Yau, *i.e.* $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $1 \le i \le \dim X - 1$, then any line bundle is a spherical object.
(2) If C is a smooth projective curve, then $\mathcal{O}_x$ is a spherical object for any closed point $x \in C$ and $t_{\mathcal{O}_x}(-) \simeq - \otimes_{\mathcal{O}_C} \mathcal{O}_C(x)$ (this is due to $\dim C = 1$ as we will see in the next section).

[2]For a survey of these notions, see [Toë11] or [AL17].↑

(3) If $X = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1})$, then $\mathcal{O}_{\mathbb{P}}$ (the structure sheaf of the zero section) is a spherical object.

Apart from the theoretical beauty of spherical functors, the reason why they are interesting is that not only their twists give examples of autoequivalences, but every autoequivalence is the spherical twist around a spherical functor, see [Seg18]. Hence, for each autoequivalence we could ask ourselves the question: how can it be represented as a spherical twist?

This question is not well posed because for a single autoequivalence there exist infinitely many presentations as a spherical twist. However, as most of them depend on the fact that, of the source category $\mathcal{T}$, the twist sees only the essential image of f^R, not all the presentations are interesting. Regardless, the existence of more than one presentation requires us to change the initial question: given an autoequivalence, does it have a good presentation as a spherical twist? Clearly, we now have to decide what "good presentation" means, but that will depend on the situation.

2. From K-equivalence to autoequivalences

For this section, let us assume $k = \mathbb{C}$. At the beginning of this abstract, we said that, when studying $\mathrm{Aut}(\mathrm{D^b}(X))$, we should distinguish between standard autoequivalences, namely those lying in the subgroup generated by shifts, pushforward along automorphisms, and tensor product with line bundles, and everything else. The reason is that we expect non-standard autoequivalences to tell us something new about $\mathrm{D^b}(X)$ that we cannot reach from X.

From now on, we will consider X a smooth, Calabi–Yau, projective variety over $\mathbb{C}$ of dimension $\dim X \geq 2$, and we will look for non-standard autoequivalences of $\mathrm{D^b}(X)$.

The first example of a non-standard autoequivalence we propose is given by the spherical twist t_E around a spherical object $E \in \mathrm{D^b}(X)$ such that

$$E^{\perp} := \left\{ F \in \mathrm{D^b}(X) : \mathrm{Hom}^{\bullet}_{\mathrm{D^b}(X)}(E, F) = 0 \right\} \neq 0.$$

Indeed, assume we had an isomorphism $t_E \simeq f_* \left(L \otimes_{\mathcal{O}_X} -\right)[n]$. Then, given any $0 \neq F \in E^{\perp}$, the isomorphism $t_E(F) \simeq F$ would force $n = 0$, but that would contradict $t_E(E) \simeq E[1 - \dim X]$ as $\dim X \geq 2$.

The geometric *rasion d'être* behind spherical objects is given by mirror symmetry: a spherical object E should be thought of as a mirror to a Lagrangian sphere L, and the spherical twist t_E is the mirror to the Dehn twist around L, see [ST01] for more details.

In order to present other examples of non-standard autoequivalences, let us give the following

Definition 2.0.1. Two smooth, projective varieties X_- and X_+ are K-equivalent if there exists a smooth, projective variety $\widehat{X}$ and a roof $X_- \xleftarrow{p_-} \widehat{X} \xrightarrow{p_+} X_+$ of birational maps such that $p_-^*\omega_{X_-} \simeq p_+^*\omega_{X_+}$.

A long-standing conjecture due to Bondal–Orlov, [BO02], and Kawamata, [Kaw02], predicts that K-equivalent varieties have equivalent derived categories: they are D-equivalent. As a particular case, the conjecture predicts that birational Calabi–Yau varieties are derived equivalent.

Let us now walk through two examples of K-equivalent varieties for which the above conjecture is true, and let us see how using the induced equivalence we can produce examples of non-standard autoequivalences.

2.1. **The Atiyah flop.** Let us consider[3] $X_\pm = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$ and the 3-fold ordinary double point $Y = \mathrm{Spec}(\mathrm{H}^0(X_\pm, \mathcal{O}_{X_\pm})) = V(ad-bc) \subset \mathbb{A}^4_{\mathbb{C}}$, which is the affinization of $X_\pm$ obtained contracting the zero section $\mathbb{P}^1 \subset X_\pm$.

The varieties $X_\pm$ are clearly K-equivalent (and D-equivalent) being isomorphic, but they are not isomorphic relative to Y. We will now introduce a K-equivalence between X_- and X_+ that makes the picture linear over Y.

The fibre product $X_- \times_Y X_+$ is identified with $X_- \xleftarrow{p_-} \widehat{X} = \mathrm{Bl}_{\mathbb{P}^1} X_\pm \xrightarrow{p_+} X_+$, and it realises a K-equivalence between X_- and X_+. It was proven by Bondal and Orlov, [BO95], that the Fourier–Mukai transform associated to $\mathcal{O}_{\widehat{X}}$ gives an equivalence $\mathrm{D^b}(X_\pm) \to \mathrm{D^b}(X_\mp)$. Hence, we get equivalences which are linear over Y, and we get an autoequivalence

$$\mathrm{FF} = (p_+)_* p_-^* (p_-)_* p_+^* \in \mathrm{Aut}(\mathrm{D^b}(X_+)).$$

We claim that FF is a non-standard autoequivalence. Indeed, it is the identity on the open, dense subset $X_+ \setminus \mathbb{P}^1$, and therefore, as $\mathbb{P}^1$ has codimension 2, if it were a standard autoequivalence, it would have to be the identity. However, one can easily see that $\mathrm{FF}(\mathcal{O}_{X_+}(-1)) \simeq \mathcal{I}_{\mathbb{P}^1}$, and therefore FF is not the identity.

It turns out that FF admits a description as the *inverse* of a spherical twist around a spherical object. Indeed, as we saw in Example 1.0.2, the structure sheaf of the zero section $\mathbb{P}^1 \subset X_+$ and all its twists are spherical objects, and we have an isomorphism of functors $\mathrm{FF} \simeq t^{-1}_{\mathcal{O}_{\mathbb{P}^1}(-1)}$, see e.g. [Seg11].

2.2. **Calabi–Yau threefolds.** It is a theorem that two birational Calabi–Yau threefolds X_- and X_+ are derived equivalent, see [Bri02]. It turns out that, when the birational transformation is given by flopping a single curve, the equivalence can be implemented using the fibre product: the functors $(p_+)_* p_-^*$ and $(p_-)_* p_+^*$ (where p_- and p_+ are the projections from the fibre product) give equivalences between $\mathrm{D^b}(X_\mp)$ and $\mathrm{D^b}(X_\pm)$ [Che02]; therefore, as for the Atiyah flop example, we get a non-standard autoequivalence $\mathrm{FF} = (p_+)_* p_-^* (p_-)_* p_+^*$.

A very explicit description of FF was given in [DW13]. There, the authors attach to the geometric picture $X_- \to Y \leftarrow X_+$ (where Y is the common singularity resolved by $X_\pm$) an algebra A_{con}, the *contraction algebra*, and they construct an object $E \in \mathrm{D^b}(X_+)$ that carries an action of A_{con} and such that the inverse of FF sits in the following distinguished triangle of functors

$$\mathrm{RHom}_{X_+}(E, -) \overset{L}{\otimes}_{A_{\mathrm{con}}} E \to \mathrm{id} \to \mathrm{FF}^{-1}.$$

2.3. **The general strategy.** Now that we have described a couple of examples, let us outline the general strategy we would like to apply.

One way to approach the Bondal–Orlov–Kawamata conjecture is to consider, when possible, the common singularity Y resolved by X_- and X_+ and the fibre product $X_- \times_Y X_+$. Then, a candidate functor for the equivalence is given by the Fourier–Mukai transform associated to $\mathcal{O}_{X_- \times_Y X_+} \in \mathrm{D^b}(X_- \times X_+)$:

$$\mathrm{FM}^{X_- \to X_+}_{\mathcal{O}_{X_- \times_Y X_+}}(-) := (p_+)_* (\mathcal{O}_{X_- \times_Y X_+} \otimes_{\mathcal{O}_{X_- \times X_+}} (p_-)^*(-))$$

where $p_\pm : X_- \times X_+ \to X_\pm$ denotes the projection. Even though this does not work in general, see [Nam04], it does in many cases, and therefore we get the equivalences

$$\Phi = \mathrm{FM}^{X_- \to X_+}_{\mathcal{O}_{X_- \times_Y X_+}}, \qquad \Psi = \mathrm{FM}^{X_+ \to X_-}_{\mathcal{O}_{X_- \times_Y X_+}} \quad \text{and} \quad \mathrm{FF} = \mathrm{FM}^{X_- \to X_+}_{\mathcal{O}_{X_- \times_Y X_+}} \circ \mathrm{FM}^{X_+ \to X_-}_{\mathcal{O}_{X_- \times_Y X_+}}.$$

[3]Notice that the varieties of this example are projective-over-affine rather than projective.↑

As in the previous cases, FF is a non-standard autoequivalence[4] because it is the identity on the open subset where X_- and X_+ are isomorphic, but it is not the identity. Thus, we would like to understand FF better.

Remark 2.3.1. As we remarked above, in general the structure sheaf of the fibre product does not induce an equivalence. In [BDF17] the authors use VGIT (Variation of Geometric Invariant Theory) methods to construct an alternative functor to the Fourier–Mukai transform induced by the fibre product. It is not known under which conditions this functor gives rise to an equivalence.

3. An intrinsic description of the autoequivalence as a spherical twist

In the examples of the Atiyah flop and of Calabi–Yau threefolds, we encountered a phenomenon which is generally described as "Flop-Flop = (inverse) twist". Namely, in both cases the Flop-Flop autoequivalence was be described as the inverse of a spherical twist. What we aim to show now is that this is in fact a general behaviour.

From now on, we consider X_-, X_+, and $\widehat{X}$ three separated, finite type schemes over a field k of characteristic zero, together with a roof $X_- \xleftarrow{p_-} \widehat{X} \xrightarrow{p_+} X_+$ of maps of finite type satisfying the following assumptions:

(1) $(p_+)_* p_-^* \colon \mathrm{D}_{\mathrm{qc}}(X_-) \to \mathrm{D}_{\mathrm{qc}}(X_+)$ and $(p_+)_* p_-^* \colon \mathrm{D}_{\mathrm{qc}}(X_+) \to \mathrm{D}_{\mathrm{qc}}(X_-)$ are equivalences; here $\mathrm{D}_{\mathrm{qc}}(-)$ is the unbounded derived category of complexes with quasi-coherent cohomologies.[5]
(2) $(p_-)_* \mathcal{O}_{\widehat{X}} \simeq \mathcal{O}_{X_-}$ and $(p_+)_* \mathcal{O}_{\widehat{X}} \simeq \mathcal{O}_{X_+}$.

We will denote $\mathrm{FF} := (p_+)_* p_-^* (p_-)_* p_+^*$ and we will call it the Flop-Flop autoequivalence. We will show that in the above general setup, which comprises the case of two K-equivalent varieties (when the assumptions (1) and (2) are satisfied), the functor FF can be described as the inverse of a spherical twist.

The whole point of presenting FF as the inverse of a spherical twist is, of course, to do so in a way that arises from the geometry. Let us explain how we intend to achieve this goal with the following example.

Example 3.0.1. Let us consider the setting of the Atiyah flop. Then, one sees $X_- \times_Y X_+ \simeq \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1))$. The SOD (semiorthogonal decomposition) for a blow-up together with Beilinson's exceptional sequence tell us

$$\mathrm{D}^{\mathrm{b}}(X_- \times_Y X_+) = \langle \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1), p_-^* \mathrm{D}^{\mathrm{b}}(X_-) \rangle.$$

Hence, by taking the Verdier quotient, we have

$$\mathrm{D}^{\mathrm{b}}(X_- \times_Y X_+)/\langle \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1) \rangle = \langle \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1), p_-^* \mathrm{D}^{\mathrm{b}}(X_-) \rangle,$$

and one can show that $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1)$ is exceptional in the quotient category, see [Bar20b]. Hence, the functor

$$\langle \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1) \rangle \hookrightarrow \mathrm{D}^{\mathrm{b}}(X_- \times_Y X_+)/\langle \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1) \rangle \xrightarrow{(p_+)_*} \mathrm{D}^{\mathrm{b}}(X_+)$$

is identified with the spherical functor attached to $\mathcal{O}_{\mathbb{P}^1}(-1)$, and the inverse of the twist around is it FF.

To generalise the above example let us consider the functors $(p_-)_*$ and $(p_+)_*$, and denote the common kernel $\mathcal{K} = \ker(p_-)_* \cap \ker(p_+)_*$. The objects of $\mathcal{K}$ are invisible to FF, and furthermore p_-^* and p_+^* don't land in $\mathcal{K}$ because of assumption (2) above. Hence, it makes

[4]Here we are implicitly assuming that the locus where $X_\pm$ are not isomorphic has codimension at least 2.↑

[5]The passage from bounded complexes with coherent cohomologies to unbounded complexes with quasi-coherent cohomologies is of technical nature: we need to work with *cocomplete* categories.↑

sense to consider the quotient category $Q\colon \mathrm{D}_{qc}(\widehat{X}) \to \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$ and the induced functors $(\overline{p}_+)_*\colon \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K} \to \mathrm{D}_{\mathrm{qc}}(X_+)$, $\overline{p}_+^* = Q \circ p_+^*$, etcetera.

Remark 3.0.2. One can prove that in Example 3.0.1 we have $\mathcal{K} = \langle \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)\rangle$.

It turns out that this technical passage is the only thing we need to reach our goal. More precisely, we have[6]

Theorem 3.0.3 ([Bar20b])**.** *The functor* $\ker(\overline{p}_-)_* \hookrightarrow \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K} \xrightarrow{(\overline{p}_+)_*} \mathrm{D}_{\mathrm{qc}}(X_+)$ *is conservative and spherical, and the inverse of the twist around it is the Flop-Flop autoequivalence* FF*.*

Remark 3.0.4. Example 3.0.1 unravels the techincalities behind the above theorem: in that case, $\ker(\overline{p}_-)_* = \langle \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1)\rangle$.

Remark 3.0.5. Albeit the above theorem is stated for $\mathrm{D}_{\mathrm{qc}}(-)$, we would like to be able to get back to $\mathrm{D}^{\mathrm{b}}(-)$. When X_- and X_+ are smooth varieties, a compactness argument shows that the functor $\big(\ker(\overline{p}_-)_*\big)^{\mathrm{c}} \xrightarrow{(\overline{p}_+)_*} \mathrm{D}_{\mathrm{qc}}(X_+)^{\mathrm{c}} \simeq \mathrm{D}^{\mathrm{b}}(X_+)$ is spherical, and the inverse of the twist around it is $\mathrm{FF}|_{\mathrm{D}^{\mathrm{b}}(X_+)}$. Here, given a triangulated category $\mathcal{T}$ with arbitrary direct sums, $\mathcal{T}^{\mathrm{c}}$ denotes the subcategory of compact objects.

The theorem we just stated achieves the aim of producing a source category which is built in the geometry and that can supply us information about FF. However, in general it is quite hard to describe $\ker(\overline{p}_-)_*$, and saying something meaningful about it is an interesting challenge.

In the rest of this section we will focus on some explicit examples, and we will see how much can be said about $\ker(\overline{p}_-)_*$.

3.1. **Relative dimension** 1. Let us assume $k = \overline{k}$. The first paper where the quotient category $\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$ was considered was [BB15]. There, Bodzenta and Bondal focused on flopping contractions of relative dimension 1, *i.e.*, birational maps $f_-\colon X_- \to Y$ whose fibres have dimension at most 1 (satisfying certain further assumptions), together with a flop $f_+\colon X_+ \to Y$.

The relative dimension assumption implies that the category $\mathcal{A}_{f_+} := \{E \in \mathrm{Coh}(X_+) : (f_+)_*E \simeq 0\}$ is an abelian subcategory of $\mathrm{Coh}(X_+)$. Then, the inclusion functor $\mathcal{A}_{f_+} \hookrightarrow \mathrm{Coh}(X_+)$ can be derived, and Bodzenta and Bondal show that $\iota : \mathrm{D}^{\mathrm{b}}(\mathcal{A}_{f_+}) \to \mathrm{D}^{\mathrm{b}}(X_+)$ is spherical and the inverse of the twist around it is the Flop-Flop functor FF.

The relation between what we just said and Theorem 3.0.3 is the following: using results in [BB15] and [Bar20b], one sees that there is an equivalence of categories $\mathrm{D}^{\mathrm{b}}(\mathcal{A}_{f_+}) \simeq (\ker(\overline{p}_-)_*)^{\mathrm{b}}$, where the latter category denotes the kernel of the functor $(\overline{p}_-)_*\colon \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}} \to \mathrm{D}^{\mathrm{b}}(X_+)$, $\mathcal{K}^{\mathrm{b}} = \mathcal{K} \cap \mathrm{D}^{\mathrm{b}}(\widehat{X})$ (that everything can be done with the bounded derived category follows from the particular setup we work in); under this equivalence, the functor $(\overline{p}_+)_*|_{(\ker(\overline{p}_-)_*)^{\mathrm{b}}}$ corresponds to ι.

3.2. **Standard flops.** We now consider examples that are a generalisation of the Atiyah flop. We set $X_\pm = \mathrm{Tot}\,(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1})$ and $Y = \mathrm{Spec}(\mathrm{H}^0(X_\pm, \mathcal{O}_{X_\pm}))$. Then, X_- and X_+ are Calabi–Yau and birational over Y. The fibre product can be identified with $X_- \xleftarrow{p_-} \widehat{X} := X_- \times_Y X_+ \simeq \mathrm{Bl}_{\mathbb{P}^n} X_\pm \xrightarrow{p_+} X_+$, and the Fourier-Mukai transform $\mathrm{FM}_{\mathcal{O}_{\widehat{X}}}$ gives an equivalence between $\mathrm{D}_{\mathrm{qc}}(X_-)$ and $\mathrm{D}_{\mathrm{qc}}(X_+)$, see [BO95].

[6] Recall that a functor is called conservative if it has no kernel.↑

The Flop-Flop autoequivalence FF is then a non-standard autoequivalence, and it was proven in [ADM19] that we have an isomorphism of functors

$$\mathrm{FF} \simeq t^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-1)} \circ t^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-2)} \circ \cdots \circ t^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-n)}.$$

This decomposition of FF is reflected (and induced) by $\ker(\overline{p}_-)_*$. More precisely, in [Bar20b] it is shown that the objects $\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j) \in \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$, $1 \leq j \leq n$, generate $\ker(\overline{p}_-)_*$, and that we have

$$\mathrm{Hom}^\bullet_{\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-i), \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)) = \begin{cases} \mathrm{Hom}^\bullet_{\mathrm{D}_{\mathrm{qc}}(X_+)}(\mathcal{O}_{\mathbb{P}^n}(-i), \mathcal{O}_{\mathbb{P}^n}(-j)) & i > j \\ k & i = j \\ 0 & \text{otherwise} \end{cases}.$$

Hence, we have a semiorthogonal decomposition

$$\ker(\overline{p}_-)_* \simeq \langle \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-n), \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-n+1), \dots, \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-1)\rangle \simeq \langle \mathrm{D}(k), \dots, \mathrm{D}(k)\rangle,$$

where n copies of $\mathrm{D}(k)$ appear. Then, one can recover the splitting of T^{-1}_Ψ from the above SOD using [HLS16, Theorem 4.14] and [Bar20a, Remark 3.2.2].

Remark 3.2.1. The picture above gives an example of the general construction the author described in [Bar20a]. There, we explain how to glue spherical functors. We refer the interested reader to *ibidem* for a more detailed explanation.

Here we only remark that the passage to the quotient category in Theorem 3.0.3 is necessary. Indeed, if we want to glue the spherical functors realising $t_{\mathcal{O}_{\mathbb{P}^n}(-i)}$, as it is explained in [Bar20a], we must have

$$\mathrm{Hom}^\bullet_{\mathrm{D}_{\mathrm{qc}}(\widehat{X})}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-i), \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)) = \mathrm{Hom}^\bullet_{\mathrm{D}_{\mathrm{qc}}(X_+)}(\mathcal{O}_{\mathbb{P}^n}(-i), \mathcal{O}_{\mathbb{P}^n}(-j))$$

for $1 \leq j \leq i \leq n$. It is easy to show that this equality does not hold in $\mathrm{D}_{\mathrm{qc}}(\widehat{X})$ unless $i = 1$, but it does hold for all $1 \leq j \leq i \leq n$ if we pass to the quotient category $\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$.

Remark 3.2.2. The previous example can be generalised to the non-local case, *i.e.*, we can take X_- to be any smooth, projective variety containing a copy of $\mathbb{P}V_-$ such that $N_{\mathbb{P}V_-/Z_-} \simeq \mathcal{O}_{\mathbb{P}V_-}(-1) \otimes \pi_Z^* V_+$; here V_- and V_+ are locally free sheaves of rank $n+1$ over a smooth projective variety Z. See [Bar20b] and references therein.

3.3. **Mukai flops.** In this section, we keep employing the notation we introduced for standard flops.

Using the Euler's sequence $\Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n}$, we see that we have an inclusion of vector bundles $Z_\pm := \mathrm{Tot}(\Omega^1_{\mathbb{P}^n}) \subset X_\pm$. Blowing up the zero section $\mathbb{P}^n \subset Z_\pm$, we get a K-equivance roof $Z_- \xleftarrow{\tilde{q}_-} \tilde{Z} := \mathrm{Bl}_{\mathbb{P}^n} Z_\pm \xrightarrow{\tilde{q}_+} Z_+$. The fibre product $Z_- \times_W Z_+$, where $W = \mathrm{Spec}(\mathrm{H}^0(Z_\pm, \mathcal{O}_{Z_\pm}))$, is identified with[7]

$$Z_- \xleftarrow{q_-} \tilde{Z} \cup_{\mathbb{P}(\Omega^1_{\mathbb{P}^n})} (\mathbb{P}^n \times \mathbb{P}^n) \xrightarrow{q_+} Z_+,$$

and it was proven in [Kaw02], [Nam03], that the Fourier-Mukai transform $\mathrm{FM}_{\mathcal{O}_{Z_-\times_W Z_+}}$ induces an equivalence between $\mathrm{D}_{\mathrm{qc}}(Z_-)$ and $\mathrm{D}_{\mathrm{qc}}(Z_+)$.

In [ADM19], a description of the autoequivalence FF is given, but to present it here, we have to take a step back.

In [HT06], Huybrechts and Thomas introduced the notion of a $\mathbb{P}^n$-object: an object $E \in \mathrm{D}^{\mathrm{b}}(X)$, for X a smooth projective variety over k of dimension $2n$, such that $E \otimes \omega_X \simeq E$ and

$$\mathrm{Hom}^\bullet_{\mathrm{D}^{\mathrm{b}}(X)}(E, E) \simeq \mathrm{H}^\bullet(\mathbb{P}^n, k)$$

[7] $\mathbb{P}(\Omega^1_{\mathbb{P}^n})$ is embedded in $\mathbb{P}^n \times \mathbb{P}^n$ via the Euler exact sequence.↑

as graded algebras. To a $\mathbb{P}^n$-object E, Huybrechts and Thomas associate an autoequivalence P_E called the $\mathbb{P}$-twist around E.

The objects $\mathcal{O}_{\mathbb{P}^n}(-j)$ are $\mathbb{P}^n$-objects in $\mathrm{D^b}(Z_+)$, and in [ADM19] it is proved that there is an isomorphism of functors

$$\mathrm{FF} \simeq P^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-1)} \circ P^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-2)} \circ \cdots \circ P^{-1}_{\mathcal{O}_{\mathbb{P}^n}(-n)}. \tag{1}$$

In [Seg18], the author explained how to present P_E as the spherical twist around a spherical functor. One should consider the dg-algebra $k[\varepsilon]/\varepsilon^2$, where $\deg \varepsilon = -1$ and $d(\varepsilon) = 0$, and the dg-module M over $k[\varepsilon]/\varepsilon^2$ associated to the generator of degree 2 of $\mathrm{Hom}^\bullet_{\mathrm{D^b}(X)}(E, E)$, *i.e.*, $M = \mathrm{cone}(E[-2] \to E)$. Then, the functor $f = - \overset{L}{\otimes}_{k[\varepsilon]/\varepsilon^2} M : \mathrm{D}(k[\varepsilon]/\varepsilon^2) \to \mathrm{D_{qc}}(X)$ is spherical, and the twist around it is given by P_E.

Once again, in [Bar20b] we proved that the isomorphism of functors (1) is reflected by the structure of $\ker(\overline{p}_-)_*$. More precisely, we can prove that $\ker(\overline{p}_-)_*$ has a semiorthogonal decomposition

$$\ker(\overline{p}_-)_* \simeq \langle \mathrm{D}(k[\varepsilon]/\varepsilon^2), \mathrm{D}(k[\varepsilon]/\varepsilon^2), \dots, \mathrm{D}(k[\varepsilon]/\varepsilon^2) \rangle$$

where n copies of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$ appear. The generators of these subcategories can be identified by realising that $\widehat{Z}$ is the zero locus of a regular function inside $\widehat{X}$. Then, if $r\colon \widehat{Z} \hookrightarrow \widehat{X}$ denotes the closed immersion, the j-th copy of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$, counting right to left, is generated by $r^*\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0, -j)$.

Hence, $\ker(\overline{p}_-)_*$ is a gluing of copies of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$, and furthermore we can describe the non-trivial homomorphisms among the generators: if for $1 \le i \le n$ we denote

$$M_{-i} = \mathrm{cone}(\mathcal{O}_{\mathbb{P}^n}(-i)[-2] \to \mathcal{O}_{\mathbb{P}^n}(-i)) \in \mathrm{D_{qc}}(Z_+)$$

the canonical extension, then we have

$$\mathrm{Hom}^\bullet_{\mathrm{D_{qc}}(\widehat{Z})/\mathcal{K}}(r^*\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0, -i), r^*\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0, -j)) \simeq \mathrm{Hom}^\bullet_{\mathrm{D_{qc}}(Z_+)}(M_{-i}, M_{-j})$$

for $1 \le j < i \le n$. Pictorially, we can think of $\ker(\overline{p}_-)_*$ as the category of dg-modules over the dg-algebra given by the quiver (we draw the case $n = 4$ for simplicity)

$$\begin{tikzcd}
4 \arrow[loop above] \arrow[r] \arrow[rr, bend right] \arrow[rrr, bend right] & 3 \arrow[loop above] \arrow[r] \arrow[rr, bend left] & 2 \arrow[loop above] \arrow[r] & 1 \arrow[loop above]
\end{tikzcd}$$

where the arrows from i to j are given by 0 if $i < j$, $k[\varepsilon]/\varepsilon^2$ if $i = j$, and by a dg $k[\varepsilon]/\varepsilon^2$-$k[\varepsilon]/\varepsilon^2$ bimodule enhancing $\mathrm{Hom}^\bullet_{\mathrm{D^b}(Z_+)}(M_{-i}, M_{-j})$ if $j < i$.

3.4. **A few more examples.** There are many examples in the literature satisfying assumptions (1) and (2). However, as we remarked, describing $\ker(\overline{p}_-)_*$ is often complicated. We will now list a couple of further examples and expectations about the description of the source category $\ker(\overline{p}_-)_*$.

The example of Grassmannian flops has been studied in various papers, e.g. [DS14], [BCF+19]. Putting together results from these papers, we know that the Flop-Flop autoequivalence splits as the composition of twists around various spherical functors. Therefore, it would be natural to expect $\ker(\overline{p}_-)_*$ to have a SOD reflecting this splitting, in line with what happens for standard and Mukai flops.

Another intesting flop is the Abuaf flop, which has the peculiar property of having non-isomorphic contracting loci. It was proven to induce a derived equivalence in [Seg16], and the picture was further studied in [Har17]. One can prove that the Flop-Flop autoequivalence for the Abuaf flop splits as the composition of the inverses of five spherical twists around spherical objects, and therefore one would expect ker $(\bar{p}_-)_*$ to have a full exceptional sequence of length 5.

Unfortunately, in both cases the picture is more complicated. The point is that we conjecture the existence of the SODs basing ourselves on the gluing construction of [Bar20a]. However, the spherical functors obtained by gluing in *ibidem* might have kernel, whereas the spherical functor in Theorem 3.0.3 is conservative.

Because of this, it is tempting to conjecture that the spherical functors of Theorem 3.0.3, when the Flop-Flop autoequivalence splits, are given by the gluing procedure of [Bar20a], up to quotienting by the kernel. Unfortunately, at the present moment we do not know how to prove or disprove this statement, and therefore we leave it as a wishful thought.

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