Bolzano-Weierstrass Theorem A set $A \subset R^d$ is compact if and only if it is closed and bounded in R^d.

Tychonoff theorem If A and B are compact sets, so is their Cartesian product.

Lemma A set is convex if and only if it contains all convex combinations of its points.

Convex hull Set of all convex combinations of points of a set.

Gauss-Lucas Theorem Let $f(z)$ be an n^{th} degree real polynomial. Let $z_1, z_2, \ldots, z_n$ be the roots of $f(z)$. Then $n-1$ roots of $f'(z)$ lies in $conv(\{z_1, z_2, \ldots, z_n\})$.

Caratheodory Theorem Let $S \subset R^d$. Then any $x \in conv(S)$ can be represented as a convex combination of $d+1$ points of S.

Lemma If $A \subset R^d$ is convex, then $A^o = int(A)$ is convex.

Lemma If $A \subset R^d$ is compact, then $conv(A)$ is compact.

Lemma Let $A \subset R^n$ and $u_0 \in int(A)$ be an interior point. Then for any $u_1 \in A$, $u_\alpha = (1-\alpha)u_0 + \alpha u_1 \in int(A) \quad 0 \leq \alpha < 1$.

Lemma If $\{x_1, x_2, \ldots, x_{d+1}\} \subset R^d$ are affinely independent points, then the convex hull $conv(\{x_1, x_2, \ldots, x_{d+1}\})$ has a non-zero interior.

Theorem Let $A \subset R^d$ be a convex set. If $int(A) = \phi$, then there exists an affine subspace $L \subset R^d$ such that $A \subset L$ and $dim(L) < d$.

Definition F is a face of a closed convex set if there exists a hyperplane H such that H isolates C and $F = H \cap C$. If F is a point, then F is called an exposed point. A non-empty face $F \neq C$ is called a proper face.

Lemma Let $K \subset R^d$ be a convex set with non-empty interior. Let $U \in \partial K$ be a point on the boundary. Then there exists an affine hyperplane H passing through U such that H isolates K. H is called a supporting hyperplane.

Lemma Let $K \subset R^d$ be a closed convex set. Let $U \in \partial K$. Then there exists a proper face F of K such that $U \in F$.

Definition Let $A \subset R^d$. $x \in A$ is an extreme point of A, if $x = (a+b)/2, a \in A, b \in A$ implies $x = a = b$.

Lemma A compact set has at least one extreme point.

Krein-Millman Theorem Let $K \subset R^d$ be a compact convex set. Then $K = conv(ex(K))$, where $ex(K)$ is the set of extreme points of K.

Lemma Let $K \subset R^d$ be a compact convex set. Then the linear function $f(x) = a^T x : R^d \to R$ attains a maximum or minimum on an extreme point of K.

Birkhoff-von Neumann Theorem The vertices of the Birkhoff polytope B_n are exactly the $n \times n$ permutation matrices. The polyhedron B_n of all $n \times n$ doubly stochastic matrices is called the Birkhoff polytope.

Convex set

$$\theta x + (1 - \theta) y \in C, \quad 0 \leq \theta \leq 1$$

Affine set

$$\theta x + (1 - \theta) y \in L, \quad \theta \in R$$

Cone

$$\theta_1 x + \theta_2 y \in K, \quad \theta_1, \theta_2 \geq 0$$

Proper Cone: closed, convex, solid, pointed

Dual cone

$$K^* = \{y \,|\, x^T y \geq 0 \text{ for all } x \in K\}$$

Convex function

A function $f : R^n \to R$ is convex if $domf$ is convex and for all x and $y \in domf$ with $\theta \in [0,1]$,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

Properties of convex function

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \forall x, y \in domf$$
$$\nabla^2 f(x) \succeq 0$$

Examples of convex function

- Linear and affine
- Exponential $e^{ax} \quad a \in R$
- Power $x^a \quad a \geq 1$ and $a \leq 0$
- Power of abs $|x|^p \quad p \geq 1$
- Negative entropy $x \log x$
- Norms
- Max function
- Quadratic-over-linear function
- Log-sum-exp

Examples of concave functions

- Logarithm $\log x$
- Geometric mean $\left(\prod_i x_i\right)^{1/n}$
- Log-determinant $\log\det X \quad X \in S_n^{++}$

Sublevel sets

$$S_\alpha = \{x \in domf | f(x) \leq \alpha\}$$

Epigraph

$$epif = \{(x,t) | x \in domf, t \geq f(x)\}$$

$$g(x) = \sup_{y \in C} f(x,y) \implies epig = \bigcap_{y \in C} epif(\cdot, y)$$

Operations that preserve convexity

1. Non-negative weighted sum
2. Composition with an affine mapping $g(x) = f(Ax + b)$
3. Pointwise maximum and supremum $\max\{f(x_1), \ldots, f(x_m)\}$
4. Composition $h(g(x))$

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

   f is convex when g is convex, and h is convex and non-decreasing
   f is convex when g is concave, and h is convex and non-increasing

5. Minimum over a non-empty convex set $g(x) = \inf_{y \in C} f(x,y)$
6. Perspective $g(x,t) = tf(x/t)$

Conjugate function

$$f^*(y) = \sup_{x \in dom f} (y^T x - f(x))$$

e.g. $\quad f(x) = \|x\|, \quad f^*(y) = 0, \quad \text{if } \|y\|_* \leq 1, \quad \text{else } \infty$

Conjugate of conjugate $\quad f^{**} = f$

if $epif$ is closed (or) f is convex and closed.

Fenchel's/Young's inequality

$$f(x) + f^*(y) \geq x^T y$$

Optimization Problem

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\
& h_i(x) = 0, \quad i = 1, 2, \ldots, p \\
\text{Domain} \quad & D = \bigcap_{i=0}^{m} dom f_i \cap \bigcap_{i=0}^{p} dom h_i \\
\text{Optimal value} \quad & p^* = \inf\{f_0(x) | f_i(x) \leq 0, h_i(x) = 0\} \\
\text{Optimal set} \quad & x = \{x | f_0(x) = p^*, f_i(x) \leq 0, h_i(x) = 0\} \\
\epsilon - \text{suboptimal} \quad & f_0(x) \leq p^* + \epsilon, \quad \epsilon > 0
\end{aligned}
$$

Feasibility Problem

$$
\begin{aligned}
\text{find} \quad & x \\
\text{subject to} \quad & f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\
& h_i(x) = 0, \quad i = 1, 2, \ldots, p
\end{aligned}
$$

Epigraph Problem

$$
\begin{aligned}
\text{minimize} \quad & t \\
\text{subject to} \quad & f_0(x) - t \leq 0 \\
& f_i(x) \leq 0, \quad i = 1, 2, \ldots, m \\
& h_i(x) = 0, \quad i = 1, 2, \ldots, p
\end{aligned}
$$

Convex optimization

- Objective function and inequality constraints are convex
- Equality constraints are affine $h_i(x) = a_i^T x - b_i$

Optimality criterion

If x is optimal, then,

$$
\begin{aligned}
\nabla f_0(x)^T (y - x) \geq 0 \quad & \text{for all } y \in X (\text{feasible set}) \\
\nabla f_0(x) = 0 \quad & \text{for unconstrained problem}
\end{aligned}
$$

For any COP, any local optimal point is globally optimum.

Linear Program (LP)

$$\begin{aligned} \text{minimize} \quad & c^T x + d \qquad \text{(General form)} \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & c^T x \qquad \text{(Standard form)} \\ \text{subject to} \quad & Ax = b \\ & x \succeq 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & c^T x \qquad \text{(Inequality form)} \\ \text{subject to} \quad & Ax \preceq b \end{aligned}$$

Quadratic Program (QP)

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} x^T P x + q^T x + r \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

Quadratic Constrained Quadratic Program (QCQP)

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} \quad & \frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, 2, \ldots, m \\ & Ax = b, \quad P_i \in S_+^n, \quad i = 0, 1, \ldots, m \end{aligned}$$

QCQP $\rightarrow (P_i = 0) \rightarrow$ QP $\rightarrow (P_0 = 0) \rightarrow$ LP

Second Order Cone Problem (SOCP)

$$\begin{aligned} \text{minimize} \quad & f^T x \\ \text{subject to} \quad & \|A_i x + b_i\|_2 \leq c_i^T + d, \quad i = 1, 2, \ldots, m \\ & Fx = g \end{aligned}$$

SOCP $\rightarrow (A_i = 0) \rightarrow$ LP
SOCP $\rightarrow (c_i = 0) \rightarrow$ QCQP (similar to)

Geometric Program (GP)

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 1, \quad i = 1, 2, \ldots, m \\
& h_i(x) = 1, \quad i = 1, 2, \ldots, p \\
& f_0, f_1, \ldots, f_m \text{ are posynomials} \\
& h_1, h_2, \ldots, h_p \text{ are monomials}
\end{array}
$$

Monomial :

$$f : R^n_{++} \to R, \quad f(x) = c\, x_1^{\alpha_1} x_1^{\alpha_2} \ldots x_1^{\alpha_n}, \quad c > 0, \quad \alpha_i \in R$$

Posynomial : (sum of monomials)

$$f(x) = \sum_{k=1}^{M} c_k\, x_1^{\alpha_{k1}} x_1^{\alpha_{k2}} \ldots x_1^{\alpha_{kn}}, \quad c_k \geq 0, \quad \alpha_{ki} \in R$$

Conic form Problem

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Fx + g \preceq_K 0, \quad K \text{ is a proper cone} \\
& Ax = b
\end{array}
$$

Semi-definite Program (SDP)

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\
& Ax = b, \\
& G, F_1, F_2, \ldots, F_n \in S, \quad A \in R^{p \times n}
\end{array}
$$

Lagrangian

$$\begin{aligned} min. \quad & f_0(x) \\ s.t. \quad & f_i(x) \le 0 \quad i = 1, \dots, m \\ & h_i(x) = 0 \quad i = 1, \dots, p \\ x \in R^n \quad & \text{Domain } D = \bigcap_{i=0}^{m} dom f_i \cap \bigcap_{i=0}^{p} dom h_i \end{aligned}$$

This problem need not be convex. Optimal value is p^*

$$\text{Lagrangian } L(x, \lambda, \nu) = f_0(x) + \sum_{i=0}^{m} \lambda_i f_i(x) + \sum_{i=0}^{p} \nu_i h_i(x)$$
$$L : R^n \times R^m \times R^p \to R \quad dom L = D \times R^m \times R^p$$

Dual variables (or) Lagrange multiplier vectors λ_i, ν_i

Lagrange dual function

$$g : R^m \times R^p \to R$$
$$g(\lambda.\nu) = \inf_{x \in D} L(x, \lambda, \nu) = \inf_{x \in D} \left(f_0(x) + \sum_{i=0}^{m} \lambda_i f_i(x) + \sum_{i=0}^{p} \nu_i h_i(x) \right)$$

The dual function is the pointwise infimum of a family of affine functions of (λ, ν), it is **concave**, even when the original problem is not convex

$$\text{For any } \lambda \succeq 0, \text{ and } \nu, \quad g(\lambda, \nu) \le p^*$$

Lagrange dual function and conjugate function

$$\text{Conjugate } f^*(y) = \sup_{x \in dom f} (y^T x - f(x))$$

Now consider

$$\begin{aligned} min. \quad & f_0(x) \\ s.t. \quad & Ax \preceq b \\ & Cx = d \end{aligned}$$
$$g(\lambda, \nu) = -b^T \lambda - d^T \nu - f_0^*(-A^T \lambda - C^T \nu)$$
$$dom g = \{(\lambda, \nu) | - A^T \lambda - C^T \nu \in dom f_0^*\}$$

Lagrange dual problem

$$\begin{aligned} max. \quad & g(\lambda, \nu) \\ s.t. \quad & \lambda \succeq 0 \end{aligned}$$

(λ^*, ν^*) is called dual optimal or optimal Lagrange multipliers. Dual problem is always convex, since a concave function is maximized, and the constraint is convex.

Weak and Strong duality

$$\begin{aligned} \text{Weak duality} \quad & d^* \leq p^* \\ \text{Strong duality} \quad & d^* = p^* \end{aligned}$$

Strong duality holds when both the following conditions hold

1. primal problem is convex
2. Slater's condition holds

Primal *convex* problem

$$\begin{aligned} min. \quad & f_0(x) \\ s.t. \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b \\ \text{with } & f_0, f_1, \dots, f_m \text{ convex} \end{aligned}$$

Slater's condition There exists an $x \in relint D$ such that

$$f_i(x) < 0, \quad i = 1, \dots, m, \quad Ax = b$$

Refined Slater's condition There exists an $x \in relint D$ such that

$$f_i(x) \leq 0, \quad i = 1, \dots, k, \quad f_i(x) < 0, \quad i = k+1, \dots, m, \quad Ax = b$$

Complementary slackness

$$\begin{aligned} f_0(x^*) &= g(\lambda^*, \nu^*) \quad \text{when strong duality holds} \\ &= \inf_{x \in D} \left(f_0(x) + \sum_{i=0}^{m} \lambda_i^* f_i(x) + \sum_{i=0}^{p} \nu_i^* h_i(x) \right) \\ &\leq f_0(x^*) + \sum_{i=0}^{m} \lambda_i^* f_i(x^*) + \sum_{i=0}^{p} \nu_i^* h_i(x^*) \\ &\leq f_0(x^*) \end{aligned}$$

Hence the inequality must be a strict equality. Therefore x^* minimizes $L(x, \lambda^*, \nu^*)$. Also,

$$\sum_{i=0}^{m} \lambda_i^* f_i(x^*) = 0$$

which implies

$$\text{Complementary slackness} \quad \lambda_i^* f_i(x^*) = 0 \quad i = 1, \ldots, m$$

Karush-Kuhn-Tucker (What names!) conditions (KKT)

$$\begin{aligned} f_i(x^*) \leq 0 \quad & i = 1, \ldots, m \\ h_i(x^*) = 0 \quad & i = 1, \ldots, p \\ \lambda_i^* \geq 0 \quad & i = 1, \ldots, m \\ \lambda_i^* f_i(x^*) = 0 \quad & i = 1, \ldots, m \\ \nabla f_0(x^*) + \sum_{i=0}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=0}^{p} \nu_i^* \nabla h_i(x^*) = 0 & \end{aligned}$$

Any optimization problem: Strong duality $\implies$ KKT satisfied
Convex optimization problem: KKT satisfied $\implies$ Strong duality

---

Solving primal via dual

Assume strong duality; dual optimal (λ^*, ν^*) is known; then primal optimal is

$$min \quad f_0(x^*) + \sum_{i=0}^{m} \lambda_i^* f_i(x^*) + \sum_{i=0}^{p} \nu_i^* h_i(x^*)$$

if it is primal feasible.

Perturbed problem

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq u_i, \quad i = 1, 2, \ldots, m \\ & h_i(x) = \delta_i, \quad i = 1, 2, \ldots, p \end{aligned}$$

Assume strong duality and dual optimal is attained.

$$p^*(u, \delta) \geq p^*(0, 0) - \lambda^{*T} u - \nu^{*T} \delta$$

If $p^*(u, \delta)$ is differentialble at $u = 0, \delta = 0$, then

$$\lambda_i^* = -\frac{\partial p^*(u,\delta)}{\partial u_i}\bigg|_{u=0,\delta=0}$$
$$\nu_i^* = -\frac{\partial p^*(u,\delta)}{\partial \delta_i}\bigg|_{u=0,\delta=0}$$

Weak Alternatives

At most one of the inequality systems is feasible.

$$\begin{aligned} f_i(x) < 0 \quad & i = 1, \dots, m \\ h_i(x) = 0 \quad & i = 1, \dots, p \end{aligned} \qquad\qquad \begin{aligned} \lambda \succeq 0 \quad \lambda \neq 0 \\ g(\lambda, \nu) \geq 0 \end{aligned}$$

$$\begin{aligned} f_i(x) \leq 0 \quad & i = 1, \dots, m \\ h_i(x) = 0 \quad & i = 1, \dots, p \end{aligned} \qquad\qquad \begin{aligned} \lambda \succeq 0 \\ g(\lambda, \nu) > 0 \end{aligned}$$

Strong Alternatives

These are weak alternatives with original inequality system being convex, i.e., f_i are convex and h_i are affine. Each of the inequality systems is feasible *if and only if* the other is infeasible.

$$\begin{aligned} f_i(x) < 0 \quad & i = 1, \dots, m \\ Ax = b \quad & x \in relint D \end{aligned} \qquad\qquad \begin{aligned} \lambda \succeq 0 \quad \lambda \neq 0 \\ g(\lambda, \nu) \geq 0 \end{aligned}$$

$$\begin{aligned} f_i(x) \leq 0 \quad & i = 1, \dots, m \\ Ax = b \quad & x \in relint D \end{aligned} \qquad\qquad \begin{aligned} \lambda \succeq 0 \\ g(\lambda, \nu) > 0 \end{aligned}$$

Farkas' lemma

A pair of strong alternatives

$$\begin{aligned} c^T x &< 0 \\ Ax &\preceq 0 \end{aligned} \qquad\qquad \begin{aligned} \lambda &\succeq 0 \\ c + A^T\lambda &= 0 \end{aligned}$$

Misc

$$\log detX = tr(logX)$$
$$-\log detX^{-1} = \log detX$$
$$\text{Schur complement } S = C - B^T A^{-1} B \quad \text{for } X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$
$$X \succ 0 \text{ iff } A \succ 0 \text{ and } S \succ 0$$
$$A \succ 0 \implies X \succeq 0 \text{ iff } S \succeq 0$$

$\partial/\partial X operations$	
$a^T X b$	ab^T
$a^T X^T b$	ba^T
$b^T X^T X c$	$X(bc^T + cb^T)$
$det(X)$	$det(X)(X^{-1})^T$
$det(AXB)$	$det(AXB)(X^{-1})^T$
$tr(X)$	I
$tr(XA)$	A^T
$tr(AXB)$	$A^T B^T$
$tr(AX^T B)$	BA
$tr(X^T A)$	A
$tr(A^T X)$	A
$tr(X^2)$	$2X^T$
$tr(X^T X)$	$2X$
$\ln det(X)$	X^{-1}
$\prod eig(X)$	$det(X)(X^{-1})^T$
$\sum eig(X)$	I
$tr(X^{-1}A)$	$-X^{-1}A^T X^{-1}$

$$\|z\|_a = \sup\{u^T z | \|u\|_{a*} = 1\}$$
$$\|X\|_{a,b} = \sup_{v \neq 0} \frac{\|Xv\|_a}{\|v\|_b}$$
$$= \sup\{\|Xv\|_a | \|v\|_b = 1\}$$
$$= \sup\{u^T X v | \|u\|_{a*} = 1, \|v\|_b = 1\}$$