Introduction to mirror descent

Alexandr Katrutsa

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Problem statement

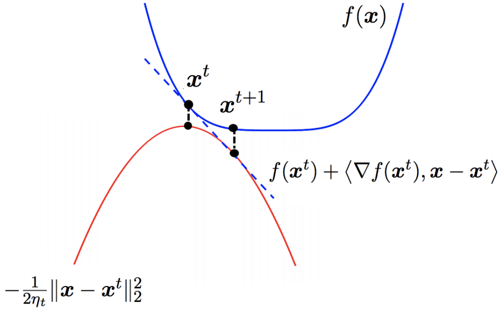
$$\min_{x \in G} f(x)$$

- f is smooth and convex
- G is convex and closed
- Condition number $\kappa = \frac{L}{\mu}$, where L is Lipschitz constant of gradient and μ is strong convexity constant

Projected gradient descent

$$x_{k+1} = \pi_G(x_k - \alpha_k f'(x_k)) = \arg\min_{x \in G} \frac{1}{2}\|(x - x_k) + \alpha_k f'(x_k)\|_2^2$$

$$= \arg\min_{x \in G} \left\{ \underbrace{f(x_k) + \langle f'(x_k), x - x_k \rangle}_{\text{linear approximation}} + \underbrace{\frac{1}{2\alpha_k}\|x - x_k\|_2^2}_{\text{proximity term}} \right\}$$



Use euclidean distance to measure discrepancy between f and FO approximation

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- **Main issue**: local geometry might sometimes be highly inhomogeneous or even non-euclidean
- Can you give some examples?

Mirror descent: main idea

Nemirovsky & Yudin, 1983

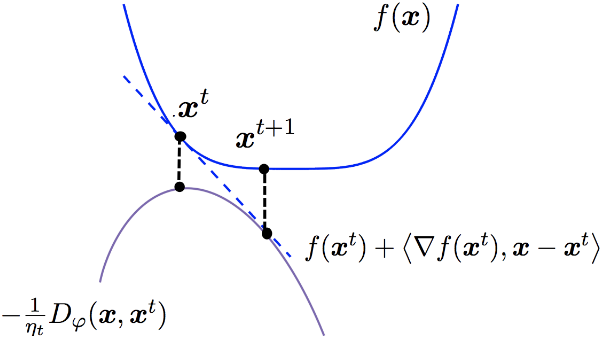
Adjust gradient updates to fit problem geometry

Mirror descent: formalism

Replace euclidean distance with distance-like function D_φ

$$x_{k+1} = \arg\min_{x \in G} \left\{ f(x_k) + \langle f'(x_k), x - x_k \rangle + \underbrace{\frac{1}{\alpha_k} D_\varphi(x, x_k)}_{\text{Bregman divergence}} \right\},$$

where $D_\varphi(x, z) = \varphi(x) - \varphi(z) - \langle \varphi'(z), x - z \rangle$ for convex and differentiable φ



Bregman divergence

Definition

Let φ be strictly convex and differentiable on G then for any $x, z \in G$

$$D_\varphi(x, z) = \varphi(x) - \varphi(z) - \langle \varphi'(z), x - z \rangle$$

- Similar to squared euclidean distance
- Locally quadratic measure:

$$D_\varphi(x, z) = (x - z)^\top \varphi''(\xi)(x - z)$$

for some ξ

How to choose Bregman divergence?

- Fit local curvature of f
- Use geometry of feasible set G
- Inexpensive computation of Bregman projection

Examples

- Squared Mahalanobis distance, $A \in \mathbb{S}^n_{++}$

$$\varphi(x) = \frac{1}{2} x^\top A x, \quad D_\varphi(x,z) = \frac{1}{2}(x-z)^\top A(x-z)$$

$$\text{MD: } x_{k+1} = x_k - \alpha_k A^{-1} f'(x_k)$$

- KL divergence for $G = \Delta$

$$\varphi(x) = \sum_i x_i \log x_i, \quad D_\varphi(x,z) = \sum_i x_i \log \frac{x_i}{z_i}$$

$$\text{MD: } x^i_{k+1} = \frac{x^i_k \exp(-\alpha_k [f'(x_k)]_i)}{\sum_{j=1}^n x^j_k \exp(-\alpha_k [f'(x_k)]_j)}$$

Also known as exponential gradient method

Some more cases

Table is from this paper

Function Name	$\varphi(x)$	$\operatorname{dom}\varphi$	$D_\varphi(x;y)$
Squared norm	$\frac{1}{2}x^2$	$(-\infty,+\infty)$	$\frac{1}{2}(x-y)^2$
Shannon entropy	$x\log x - x$	$[0,+\infty)$	$x\log\frac{x}{y} - x + y$
Bit entropy	$x\log x + (1-x)\log(1-x)$	$[0,1]$	$x\log\frac{x}{y} + (1-x)\log\frac{1-x}{1-y}$
Burg entropy	$-\log x$	$(0,+\infty)$	$\frac{x}{y} - \log\frac{x}{y} - 1$
Hellinger	$-\sqrt{1-x^2}$	$[-1,1]$	$(1-xy)(1-y^2)^{-1/2} - (1-x^2)^{1/2}$
ℓ_p quasi-norm	$-x^p \quad (0<p<1)$	$[0,+\infty)$	$-x^p + p\,x y^{p-1} - (p-1)\,y^p$
ℓ_p norm	$\lvert x\rvert^p \quad (1<p<\infty)$	$(-\infty,+\infty)$	$\lvert x\rvert^p - p\,x\operatorname{sgn} y\,\lvert y\rvert^{p-1} + (p-1)\,\lvert y\rvert^p$
Exponential	$\exp x$	$(-\infty,+\infty)$	$\exp x - (x-y+1)\exp y$
Inverse	$1/x$	$(0,+\infty)$	$1/x + x/y^2 - 2/y$

Mirror between dual and primal spaces

Assume $G = \mathbb{R}^n$, then

$$x_{k+1} = y_{k+1} = (\varphi')^{-1}(\varphi'(x_k) - \alpha_k f'(x_k))$$

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  2. Perform gradient step in dual space
  3. Project new point in primal space w.r.t. Bregman divergence proximity

Conjugacy and inversion

Lemma

$$(\varphi')^{-1} = (\varphi^*)'$$

Proof

- Assume $y = \varphi'(x)$
- By definition $\langle x, y \rangle = \varphi(x) + \varphi^*(y)$
- From convexity of φ: $\langle x, y \rangle = \varphi^{**}(x) + \varphi^*(y)$
- From definition follows $x = (\varphi^*)'(y)$
- Finally $x = (\varphi^*)'(y) = (\varphi^*)'(\varphi'(x))$

Then unconstrained MD can be written as

$$x_{k+1} = (\varphi^*)'(\varphi'(x_k) - \alpha_k f'(x_k))$$

Optimization over probability simplex with ℓ_2

Assume $G = \Delta$ and $x_0 = n^{-1}\mathbf{1}$

(1) Use euclidean proximity term: $\varphi(x) = \frac{1}{2}\|x\|_2^2$ – 1-strongly convex in $\|\cdot\|_2$. Then

$$\sup_{x \in G} D_\varphi(x, x_0) = \sup_{x \in \Delta} \frac{1}{2}\|x - n^{-1}\mathbf{1}\|_2^2 = \sup_{x \in \Delta} \frac{1}{2}\left(\|x\|_2^2 - \frac{1}{n}\right) \le \frac{1}{2}$$

and

$$f_K^{best} - f^* \le \mathcal{O}\left(L_{f,2}\frac{\log k}{\sqrt{k}}\right),$$

i.e. for all subgradients g: $\|g\|_2 \le L_{f,2}$

Optimization over probability simplex with ℓ_1

Assume $G = \Delta$ and $x_0 = n^{-1}\mathbf{1}$

(2) Use ℓ_1 proximity term: $\psi(x) = -\sum_{i=1}^n x_i \log x_i$ – 1-strongly convex in $\|\cdot\|_1$. Then

$$\sup_{x\in G} D_\psi(x, x_0) = \sup_{x\in\Delta} D_{KL}(x||x_0) = \sup_{x\in\Delta} \sum_{i=1}^n x_i \log x_i - \sum_{i=1}^n x_i \log \frac{1}{n}$$

$$= \log n + \sum_{i=1}^n x_i \log x_i \le \log n$$

and

$$f_K^{best} - f^* \le \mathcal{O}\left(L_{f,\infty}\sqrt{\log n}\frac{\log k}{\sqrt{k}}\right)$$

i.e. for all subgradients g: $\|g\|_\infty \le L_{f,\infty}$

Optimization over probability simplex: comparison

Ignore log-terms and compare

- Euclidean: $\mathcal{O}\left(\frac{L_{f,2}}{\sqrt{k}}\right)$
- D_{KL}: $\mathcal{O}\left(\frac{L_{f,\infty}}{\sqrt{k}}\right)$
- Equivalence norm

$$\|g\|_\infty \leq \|g\|_2 \leq \sqrt{n}\|g\|_\infty$$

- Why D_{KL} is better:

$$\frac{1}{\sqrt{n}} \leq \frac{L_{f,\infty}}{L_{f,2}} \leq 1$$

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- Bregman divergence is effective tool to tune distance function for your problem
- Idea from functional analysis is helpful here
- Mirror descent separates steps in primal and dual spaces