Which numbers are normal?

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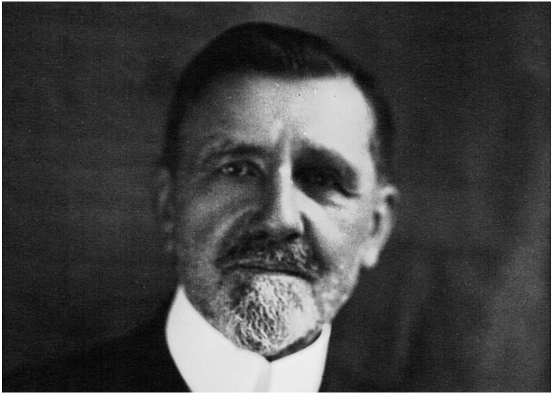
[1] Includes joint work with Simon Baker https://arxiv.org/abs/2401.01241

Émile Borel

More than 100 years ago, French mathematician Émile Borel began a quest to find certain elusive numbers.



Picture by Bibliothèque nationale de France

Normal number

Every number has an infinite decimal expansion, e.g. $\frac{1}{2} = 0.50000000....$

In 1909, Borel introduced the concept of a normal number. A number is normal if

- Each digit from 0 to 9 appears equally often in its decimal expansion (each appears roughly 10% of the time)
  For example, in the first million digits of a normal number, we expect to find roughly 100,000 of each digit.
- Each 2-digit number appears equally often
- Each 3-digit number appears equally often
  And so on

Probability of normal numbers

Borel proved the following theorem:

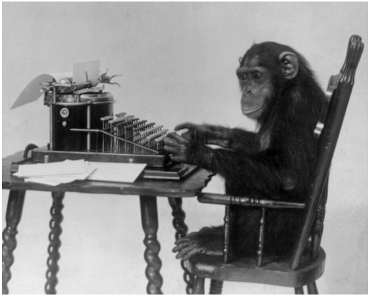
Theorem (Borel)

Normal numbers exist! In fact, if you pick a random number between 0 and 1, the probability that it will be normal is 100%.

- This means if you pick a million different numbers between 0 and 1 independently at random, almost certainly all will be normal.
- This does not mean every number is normal!

The infinite monkey thought experiment

If a monkey types digits between 0 and 9 randomly, there is a 100% chance it would eventually type a million sevens in a row (or any other string of numbers you can think of).



Picture by New York Zoological Society, public domain

- If the monkey types letters instead of numbers, there is a 100% chance that it will eventually write Shakespeare's Hamlet without a single mistake!
- But the monkey would need a ridiculously long time.

Where are the normal numbers?

- Every fraction, like $\frac{5}{7} = 0.714285\,714285...$, has a repeating decimal expansion, so it cannot be normal.
- Plenty of numbers which are not fractions are also not normal, for example if the decimal expansion is non-repeating but only contain digits 0,5,9.
- Hence began the quest to write down examples of normal numbers. It took many years to find a single one.
- Normal numbers are everywhere, but elusive!

Champernowne's number

The first person to write down the digits of a normal number was D. G. Champernowne in 1933.

Picture by no conegut, CC BY-SA 3.0

His number is

$$0.12345678910111213141516\ldots$$

(its decimal expansion is made by writing every whole number in order).

π and $\sqrt{2}$

π is the number of times a circle's diameter fits around its circumference:

$$\begin{aligned}\pi = 3.&14159265358979323846264338327950288419716939937510\\ &58209749445923078164062862089986280348253421170679\\ &82148086513282306647093844609550582231725359408128\\ &48111745028410270193852110555964462294895493038196\ldots\end{aligned}$$

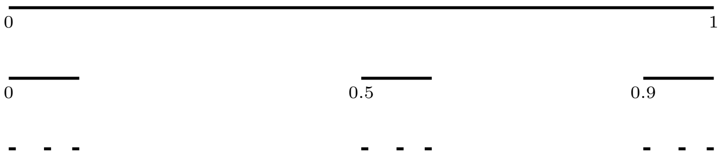
$\sqrt{2}$ is the number that gives 2 when multiplied by itself:

$$\begin{aligned}\sqrt{2} = 1.&41421356237309504880168872420969807856967187537694\\ &80731766797379907324784621070388503875343276415727\\ &35013846230912297024924836055850737212644121497099\\ &93583141322266592750559275579995050115278206057147\ldots\end{aligned}$$

These sequences look very 'random' (though of course they are not). Most mathematicians would be astonished if these numbers turn out not to be normal, but no-one has a clue how to prove it!

Fractals

The set of (non-normal) numbers between 0 and 1 whose decimal expansion consists of numbers $0, 5, 9$, for example, forms a fractal set.



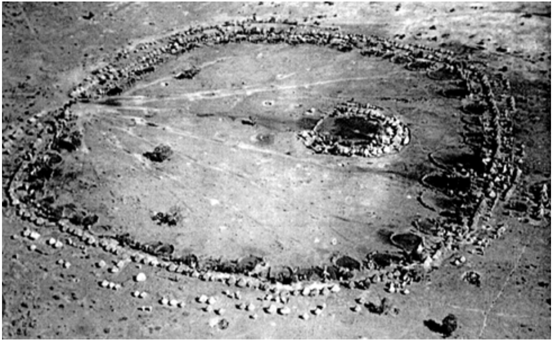
Fractals are geometric objects which display intricate and often beautiful structure at small scales. They are often formed of scaled-down images of themselves.

Fractals in nature

Romanesco broccoli. By Lauren Lester, CC0

The Barnsley Fern. By Farry, CC0

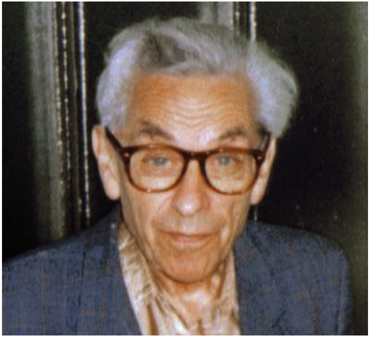
Fractals in architecture



A settlement in Zambia with a fractal design. Picture from African Fractals, CC BY-SA 4.0

Nonlinear fractals

- Question: Which fractals contain normal numbers?
- Not all of them (e.g. not missing digit fractals, which are made of strictly scaled copies of the whole fractal).
- In 1963, this problem caught the attention of Paul Erdős.



Picture by Kmhkmh, CC BY 3.0

Fourier transforms

Davenport, Erdős and LeVeque showed that if the fractal admits a weighting whose Fourier transform satisfies certain properties, then the fractal contains normal numbers.

Lots of fractals contain normal numbers

Theorem (Davenport–Erdős–LeVeque + Algom–Rodriguez Hertz–Wang / Baker–Sahlsten + Baker–B. / Algom–Chang–Wu–Wu)

If a fractal of real numbers is made of copies of itself which are scaled down in a very smooth (analytic) way, and at least one copy has some distortion (nonlinearity), then it contains normal numbers.
In fact, if we choose a 'random' point in such a fractal, then the probability of choosing a normal number is 100%.

- This is an extension of Borel's result to non-linear fractals, and shows that these fractals are 'unstructured.'
- Borel went on to study Einstein's theory of special relativity, and was later active in the French resistance to Nazi occupation.
- The quest to show that 'nice' numbers like π or $\sqrt{2}$ are normal may yet last another hundred years.

Thank you for listening!