Generalised intermediate dimensions

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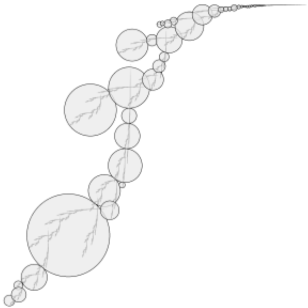
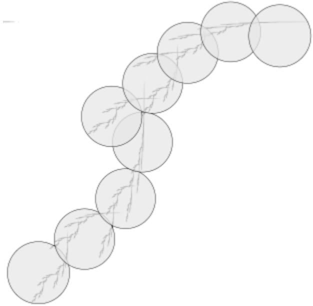
[1]Based on work in 'Generalised intermediate dimensions,' **Monatsh. Math.** (to appear), `https://arxiv.org/abs/2011.08613`.



Cover for Hausdorff dimension



Cover for box dimension

(Both pictures by Jonathan Fraser)

$$\dim_{\mathrm{H}} F \qquad \leq \qquad \overline{\dim}_{\mathrm{B}} F$$

Falconer, Fraser and Kempton ('20) defined the θ-intermediate dimensions for $\theta \in (0,1)$, satisfying

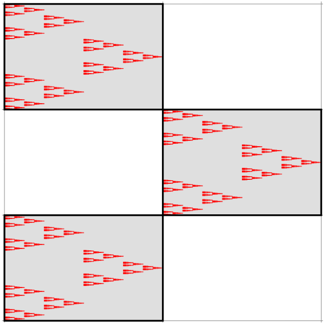
$$\dim_{\mathrm{H}} F \leq \overline{\dim}_{\theta} F \leq \overline{\dim}_{\mathrm{B}} F,$$

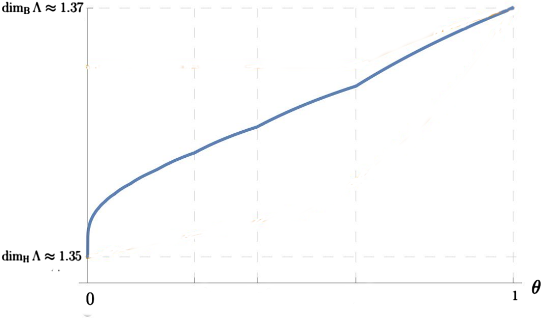
by

$$\begin{aligned}\overline{\dim}_{\theta} F := \inf\{\, s \geq 0 : &\text{ for all } \varepsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all} \\ &\delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such} \\ &\text{that } \delta^{1/\theta} \leq \mathrm{diam}(U_i) \leq \delta \text{ for all } i, \text{ and} \\ &\sum_i (\mathrm{diam}(U_i))^s \leq \varepsilon \,\}.\end{aligned}$$

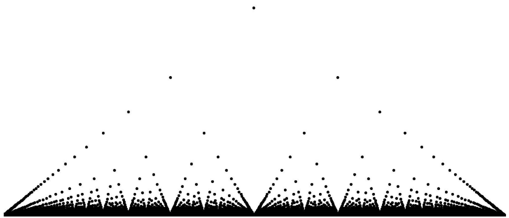
Key question: for which sets F does $\overline{\dim}_{\theta} F \to \dim_{\mathrm{H}} F$ as $\theta \to 0$ (i.e. when is $\overline{\dim}_{\theta} F$ continuous at 0)?

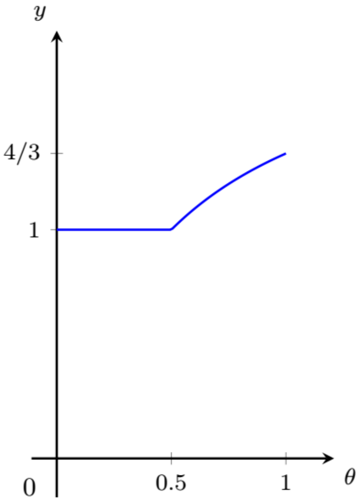
Self-affine Bedford–McMullen carpets (Falconer–Fraser–Kempton ('20), B.–Kolossváry ('21+)).





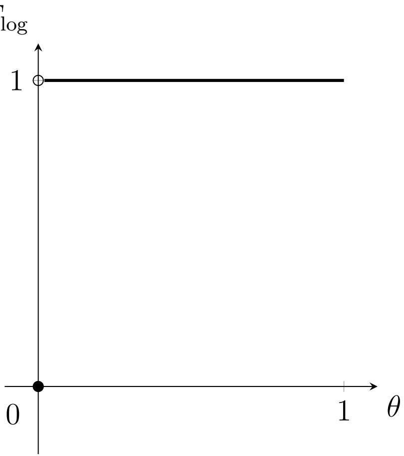
The graph of the popcorn function (B.–Chen ('22+)).





But $\dim_{\mathrm{H}} F_{\log} = 0$ while $\dim_\theta F_{\log} = 1$ for all $\theta \in (0,1]$, where

$$F_{\log} := \left\{ \frac{1}{\log n} : n \in \mathbb{N}, n \geq 3 \right\}.$$



Dimension	Allowable diameters of covering sets
Hausdorff	$[0, \delta]$
Box	$\{\delta\}$
θ-intermediate	$[\delta^{1/\theta}, \delta]$
Φ-intermediate	$[\Phi(\delta), \delta]$

I have introduced the Φ-intermediate dimensions and proved Hölder distortion estimates, bi-Lipschitz stability, mass distribution principle, Frostman-type lemma, product formulae...

Theorem (B. ('20+))

If $F \subset \mathbb{R}^d$ is non-empty and compact then for all $s \in [\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}} F]$ there exists an increasing function $\Phi_s \colon (0,1) \to (0,1)$ with $\Phi_s(\delta)/\delta \to 0$ as $\delta \to 0$, such that $\overline{\dim}^{\Phi_s} F = s$.

So using the Φ-intermediate dimensions, we can recover the interpolation between Hausdorff and box dimension.

- Example: if $\Phi_s(\delta) = \exp(-\delta^{-(1-s)})$ then $\dim^{\Phi_s} F_{\log} = s$.
- Marstrand-type projection theorems have been proved using potential theory for the θ-intermediate dimensions (Burrell–Falconer–Fraser ('21)), and Φ-intermediate dimensions (Feng, ('23+)) if for all $\varepsilon > 0$, $\delta^{\varepsilon} \log \Phi(\delta) \xrightarrow[\delta \to 0]{} 0$.

Thank you for listening!

And a *very* belated Happy Birthday, Kenneth!