Lower box dimension of infinitely generated self-conformal sets

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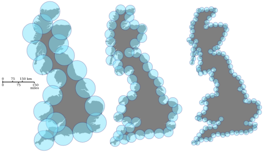


[1]Based on joint work with Alex Rutar, https://arxiv.org/abs/2406.12821

Box dimension

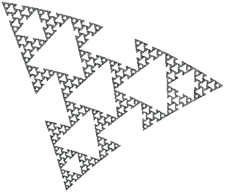
- Let $E \subset \mathbb{R}^d$ be non-empty, bounded. Let $N_r(E)$ be the smallest number of open balls of diameter r needed to cover E.
- Lower and upper box (Minkowski) dimensions:

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{r \to 0} \frac{\log N_r(E)}{\log(1/r)}, \qquad \overline{\dim}_{\mathrm{B}} E = \limsup_{r \to 0} \frac{\log N_r(E)}{\log(1/r)}.$$

- Always $\dim_{\mathrm{H}} E \leq \underline{\dim}_{\mathrm{B}} E \leq \overline{\dim}_{\mathrm{B}} E$. If the box dimension of E exists, i.e. if $\underline{\dim}_{\mathrm{B}} E = \overline{\dim}_{\mathrm{B}} E =: \dim_{\mathrm{B}} F$, then $N_r(E)$ scales like $r^{-\dim_{\mathrm{B}} F}$ at all scales.
- **Question:** for which classes of sets does the box dimension exist?

Dynamically invariant sets

If Λ is the attractor of a finite IFS of similarity/conformal maps then $\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda$ (arbitrary overlaps are allowed). (Picture by Sabrina Kombrink.)

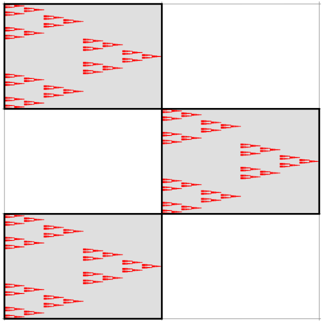


Theorem (Barreira 1996 / Gatzouras–Peres 1997)

If $f \colon M \to M$ is an expanding **conformal** C^1 map of a Riemannian manifold and a **compact** $\Lambda \subseteq M$ satisfies $f(\Lambda) = \Lambda$ and $f^{-1}(\Lambda) \cap U \subseteq \Lambda$ for a neighbourhood U of Λ, then $\dim_{\mathrm{B}} \Lambda$ exists and coincides with Hausdorff dimension.

Non-conformal dynamics

Bedford (1984) and McMullen (1984) constructed compact sets invariant under non-conformal toral endomorphisms such as $(x, y) \mapsto (2x \mod 1, 3y \mod 1)$, with distinct Hausdorff and box dimension.



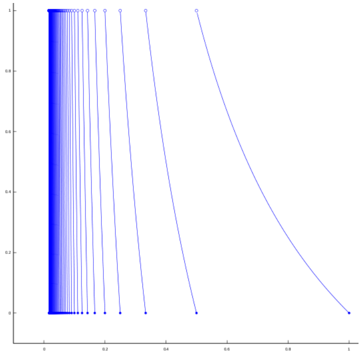
Non-conformal dynamics

- Jurga (2023) constructed a compact set Λ invariant for a non-conformal toral endomorphism with $\underline{\dim}_{\mathrm{B}} \Lambda < \overline{\dim}_{\mathrm{B}} \Lambda$.
- Jurga's example is a sub-self-affine set ($\Lambda \subset \bigcup_i S_i(\Lambda)$ for finitely many affine contractions S_i), whereas Bedford–McMullen carpets are self-affine sets ($\Lambda = \bigcup_i S_i(\Lambda)$).
- **Folklore conjecture:** the box dimension of every self-affine set should exist.

The Gauss map

The Gauss map $\mathcal{G}\colon [0,1) \to [0,1)$ is defined by

$$\mathcal{G}(x) = \begin{cases} x^{-1} - \lfloor x^{-1} \rfloor & : 0 < x < 1 \\ 0 & : x = 0. \end{cases}$$



Picture by Adam majewski, CC BY-SA 4.0

The Gauss map

Typical invariant sets are numbers whose continued fraction expansions are restricted to some $I \subset \mathbb{N}$:

$$\Lambda_I := \left\{ z \in (0,1) \setminus \mathbb{Q} : z = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots}}}, b_n \in I \text{ for all } n \in \mathbb{N} \right\}$$

satisfies $\mathcal{G}(\Lambda_I) = \Lambda_I$. If I is infinite then F_I is non-compact.

Theorem

- Mauldin & Urbański ('96, '99): there exists $I \subset \mathbb{N}$ with $\dim_{\mathrm{H}} \Lambda_I < \dim_{\mathrm{B}} \Lambda_I$.
- B.–Rutar ('24+): there exists $I \subset \mathbb{N}$ with $\dim_{\mathrm{H}} \Lambda_I < \underline{\dim}_{\mathrm{B}} \Lambda_I < \overline{\dim}_{\mathrm{B}} \Lambda_I$. In particular, the box dimension of Λ_I does not exist.

Infinite conformal IFS (Mauldin & Urbański, '96)

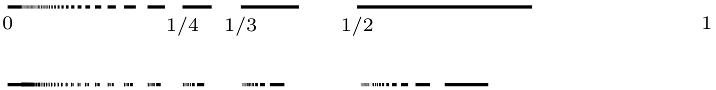
A **conformal iterated function system** is a countable family of uniformly contracting, $C^{1+\alpha}$ conformal maps $\{S_i \colon X \to X\}_{i \in I}$ on a 'nice' (e.g. non-empty convex compact) set $X \subset \mathbb{R}^d$. For continued fraction sets the maps are $\{x \mapsto (b+x)^{-1} : b \in I\}$. We always assume:

- **Open set condition:** $\mathrm{Int}(X) \neq \varnothing$ and $\bigcup_{i \in I} S_i(\mathrm{Int}(X)) \subseteq \mathrm{Int}(X)$ with the union disjoint.
- **Bounded distortion**

The **limit set** is the largest set $\Lambda \subseteq X$ satisfying

$$\Lambda = \bigcup_{i \in I} S_i(\Lambda)$$

(it is generally non-compact).



Hausdorff and box dimensions

For $w \in I^k$ let R_w be the smallest possible Lipschitz constant for $S_w := S_{w_1} \circ \cdots \circ S_{w_k}$ and define the **pressure function**

$$P(t) := \lim_{k\to\infty} \frac{1}{k} \log \sum_{w \in I^k} R_w^t,$$

Theorem (Mauldin–Urbański, '96, '99)

- $\dim_{\mathrm{H}} \Lambda = \inf\{t > 0 : P(t) < 0\}$
- $\overline{\dim}_{\mathrm{B}} \Lambda = \max\{\dim_{\mathrm{H}} \Lambda, \overline{\dim}_{\mathrm{B}} F\}$, where F is obtained by choosing exactly one point from each $S_i(X)$ (e.g. for the continued fraction sets Λ_I we can take $F = \{1/b : b \in I\}$).

Bounds for lower box dimension

Bounds for $\underline{\dim}_{\mathrm{B}} \Lambda$ that are immediate from Mauldin–Urbański:

$$\max\{\dim_{\mathrm{H}} \Lambda, \underline{\dim}_{\mathrm{B}} F\} \leq \underline{\dim}_{\mathrm{B}} \Lambda \leq \overline{\dim}_{\mathrm{B}} \Lambda = \max\{\dim_{\mathrm{H}} \Lambda, \overline{\dim}_{\mathrm{B}} F\}.$$

Theorem (B.–Rutar, '24+)

The box dimension of Λ exists if and only if these bounds coincide.

Hence the continued fraction set satisfies $\underline{\dim}_{\mathrm{B}} \Lambda_I < \overline{\dim}_{\mathrm{B}} \Lambda_I$ if I is given by removing an appropriate sequence of blocks from $\underline{\{n^2 : n \in \mathbb{N}\}}$.
In fact $\underline{\dim}_{\mathrm{B}} \Lambda$ is **not** a function of $\dim_{\mathrm{H}} \Lambda$, $\underline{\dim}_{\mathrm{B}} F$, $\overline{\dim}_{\mathrm{B}} F$:

Theorem (B.–Rutar, '24+)

The trivial lower bound for $\underline{\dim}_{\mathrm{B}} \Lambda$ is sharp, and a sharp upper bound is

$$\underline{\dim}_{\mathrm{B}} \Lambda \leq \dim_{\mathrm{H}} \Lambda + \frac{(\overline{\dim}_{\mathrm{B}} F - \dim_{\mathrm{H}} \Lambda)(d - \dim_{\mathrm{H}} \Lambda)\,\underline{\dim}_{\mathrm{B}} F}{d\,\overline{\dim}_{\mathrm{B}} F - \dim_{\mathrm{H}} \Lambda\,\underline{\dim}_{\mathrm{B}} F}.$$

An asymptotic formula

- We can derive a formula for $\underline{\dim}_{\mathrm{B}} \Lambda$ in terms of the whole function

$$r \mapsto s_F(r) := \frac{\log N_r(F)}{\log(1/r)}.$$

- Define $\psi(r) := \sup_{\theta \in (0,1]} \Psi(r, \theta)$.
  Write $f(r) \asymp g(r)$ if $f(r) - g(r) \to 0$ as $r \to 0$.

Theorem (B.–Rutar, '24+)

If Λ is the limit set of a CIFS and F is as above then

$$\frac{\log N_r(\Lambda)}{\log(1/r)} \asymp \psi(r), \qquad \text{hence} \quad \underline{\dim}_{\mathrm{B}} \Lambda = \liminf_{r \to 0} \psi(r).$$

- The formula can depend on $\dim_{\mathrm{H}} \Lambda$, even when $\dim_{\mathrm{H}} \Lambda < \underline{\dim}_{\mathrm{B}} F$.
  It only depends on the contraction ratios via $\dim_{\mathrm{H}} \Lambda$.

Alternative asymptotic formula

- We can reformulate our result using the order-reversing transformation $x = \log\log(1/r)$. This transforms the interval $[r, r^\theta]$ to $[x - \log(1/\theta), x]$.
- For $0 \le \lambda \le d$ let $\mathcal{G}(\lambda, d)$ be the set of continuous functions $g\colon \mathbb{R} \to [\lambda, d]$ such that

$$D^+ g(x) \in [\lambda - g(x), d - g(x)],$$

  where

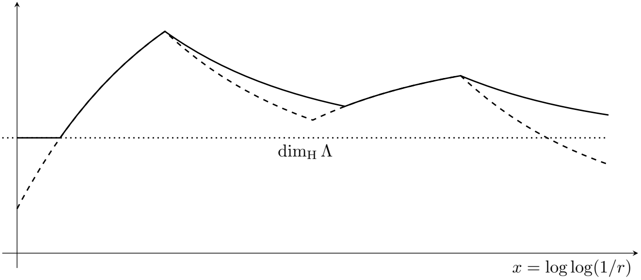
$$D^+ g(x) \coloneqq \limsup_{\varepsilon \to 0^+} \frac{g(x+\varepsilon) - g(x)}{\varepsilon}$$

  is the Dini derivative.
- In a 2022 paper we observed that if $E \subset \mathbb{R}^d$ is bounded then $s_E(\exp(\exp(x))) \asymp g(x)$ for some $g \in \mathcal{G}(0, d)$ (we say that E has covering class g), and conversely any $g \in \mathcal{G}(0, d)$ has $s_E(\exp(\exp(x))) \asymp g(x)$ for some E.

Alternative asymptotic formula

Theorem (B.–Rutar, '24+)

If F has covering class $f \in \mathcal{G}(0, d)$ and g is the pointwise minimal function $g \geq f$ satisfying $g \in \mathcal{G}(\dim_{\mathrm{H}} \Lambda, d)$ then Λ has covering class g.



Heuristically, $N_r(\Lambda) \geq N_r(F)$ is as small as possible while being at least $\dim_{\mathrm{H}} \Lambda$-dimensional between all pairs of scales.

Thank you for listening!