FOURIER TRANSFORMS AND NONLINEAR IMAGES OF SELF-SIMILAR SETS AND MEASURES

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ABSTRACT. We show that under mild and almost optimal conditions, nonlinear images of self-similar measures have polynomial Fourier decay. In certain cases, we can also explicitly bound the decay exponent from below. As applications, we prove that

- Every complex-analytic and nonlinear self-conformal measure on $\mathbb{C}$ whose support is not contained in any analytic curves has polynomial Fourier decay. This result is based on a recent breakthrough [arXiv:2407.11688].
- for any self-similar set $F \subset (0,1]$, $\dim_{\mathrm{H}} F > (\sqrt{65}-5)/4 = 0.766...$, the arithmetic product set $F \cdot F = \{xy : x, y \in F\}$ has a positive Lebesgue measure.
- for any self-similar set $F \subset (0,1]$, $\dim_{\mathrm{H}} F > (-3+\sqrt{41})/4 = 0.850...$, the set $F \cdot F \cdot F$ has non-empty interiors.

CONTENTS

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1. Introduction

sec:intro

In this paper, the main topic is about Fourier decay of nonlinear images of self-similar measures. Recall that the Fourier transform of a Borel probability measure μ supported on $\mathbb{R}^k$ is the function $\widehat{\mu}\colon \mathbb{R}^k \to \mathbb{C}$ given by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^k} e^{-2\pi i \langle \xi, x\rangle} d\mu(x).$$

The question of which measures have Fourier transform decaying to 0 as $|\xi| \to \infty$, and the speed of decay if so, has received a great deal of attention in the literature. If $|\widehat{\mu}(\xi)| \to 0$ as $|\xi| \to \infty$ then μ is called *Rajchman*, and if there exist $C, \sigma > 0$ such that $|\widehat{\mu}(\xi)| \leqslant C|\xi|^{-\sigma}$ for all $\xi \in \mathbb{R}^k \setminus \{0\}$ then μ is said to have *polynomial Fourier decay* (or *power Fourier decay*).

There are many self-similar measures, for example the Cantor–Lebesgue measure, which are not Rajchman. Determining which self-similar measures have polynomial Fourier decay is a problem that is wide open even in the line. We refer the reader to [16, 70, 44, 2, 13, 75] for work on self-similar measures in the line, and to [63] which determines which self-similar measures in higher dimensions are Rajchman without considering the rate of decay. This paper, however, is in a different line of work that seeks to verify that in many different cases that if the measures are generated by an iterated function system

(IFS) that is sufficiently nonlinear then the stationary measures will have polynomial Fourier decay. Other papers in this direction include [38, 62, 35, 11, 43, 66, 2, 3, 9, 4]. In particular, pushforwards of self-similar measures under nonlinear maps can have polynomial Fourier decay even if the original self-similar measure does not. For nonlinear maps $\mathbb{R} \to \mathbb{R}$ these questions have been studied recently in [1, 39, 53]. Baker and Banaji [7] also consider pushforwards of a wide class of measures which they call fibre product measures on $\mathbb{R}^k$, and Mosquera and Olivo [54, Theorem 3.1] consider analytic pushforwards of homogeneous self-similar measures on $\mathbb{C}$. We also refer the reader to the recent survey [64] for an overview of these topics.

We make no further delay of the introduction of our main results. Section 2 gives applications which (hopefully) provide motivation for the results we are about to state, and the terminology and notation will be explained in detail in Section 3. We use Vinogradov and Bachmann–Landau notation: given complex-valued functions f, g we write $f \ll g$ or $f = O(g)$ to mean $|f| \leqslant C|g|$ pointwise for some constant $C > 0$, and write $f \asymp g$ if $f \ll g$ and $g \ll f$.

1.1. **Polynomial Fourier decay with an explicit decay estimate.**

Our first result concerns explicit decay exponents of nonlinear images of self-similar measures.

thm: image fourier decay

Theorem 1.1. *Let μ be a non-expanding self-similar measure on $\mathbb{R}^k$. Let $f\colon \mathbb{R}^k \to \mathbb{R}$ be a smooth function so that the graph $\Gamma_f = (x, f(x))_{x\in\mathbb{R}^k}$ has non-vanishing Gaussian curvature over the support of μ. Consider the pushforward measure $f_*\mu$ under f. Then assuming $\kappa_F > k/2$, $|\widehat{\mu}_f(\xi)| \ll |\xi|^{-\sigma}$ for some $\sigma > 0$. More precisely, let*

$$\tilde{\sigma}\colon x \in \mathbb{R} \mapsto \frac{\frac{\kappa_F - x}{1+\kappa_* - x}}{2 - \frac{\kappa_F - x}{1+\kappa_* - x}}.$$

Then we can choose arbitrarily

$$0 < \sigma < \max\left\{\tilde{\sigma}(k - \kappa_1), \tilde{\sigma}((k + B - \kappa_2)/2)\right\}.$$

Note in particular that we can take the exponent σ to be independent of f.

Remark 1.2. *We briefly explain the terminologies. For more details, see Section 3. A self-similar is non-expanding if its rotation semi-group is in some sense simple. We remark that all self-similar sets in $\mathbb{R}^k, k \leqslant 2$ are non-expanding. Here κ_F is the Frostman exponent of μ, κ_1 is the l^1-dimension of μ, κ_2 is the l^2-dimension of μ, B is the* *box dimension of* $\operatorname{supp}(\mu)$[1]*, and* κ_* *is the Assouad dimension of the set* $\operatorname{supp}(\mu)$ *(also called Furstenberg's *-exponent). For general self-similar measures,* $k \geqslant \kappa_* \geqslant B \geqslant \kappa_2 \geqslant \kappa_{\mathrm{F}} \geqslant \kappa_1$*, and for* μ *being AD-regular,* $\kappa_* = B = \kappa_2 = \kappa_F \geqslant \kappa_1$*. For* μ *being a missing-digit measure, it is possible to estimate* κ_1 *and in some cases it is almost equal to* κ_2 [15].

The above result does not require any separation conditions for the self-similar measures. It is known that for each self-similar set E, there is a self-similar measure μ on E such that $\kappa_2 = B = \dim_{\mathrm{H}} E$. As long as only the underlying set is concerned, we can assume that $B = \kappa_2$. Note that Theorem 1.1 is only non-trivial if $\kappa_F > k/2$. This will require that $\kappa_2 > k/2$. In case $\kappa_F < k/2$, the theorem gives no new information.

rem: large linear space

Remark 1.3. *The need for some conditions like* $\kappa_2 > k/2$ *is not due to our incapability. Let us see why such a condition is essential to get any non-trivial polynomial decay for the Fourier transform of its image. For* $k = 2$*, consider hyperboloids in* $\mathbb{R}^3$ *with negative curvatures. Some hyperboloids contain (many) lines. For larger* k*, let* $F \subset \mathbb{R}^{k+1}$ *be a smooth hypersurface with non-vanishing Gaussian curvature. It is known that* F *cannot contain any affine subspace with dimension bigger than* $k/2$*. On the other hand, some surfaces with non-vanishing Gaussian curvature contain large affine subspaces. Consider for example the following algebraic hypersurface with an even number* $k > 0$*,*

$$F_k = \{x_1^3 + \cdots + x_{k+1}^3 = 1\}.$$

Then the linear subspace

$$L_k : \{x_1 = -x_2, x_3 = -x_4, \ldots, x_{k-1} = -x_k, x_{k+1} = 1\}$$

is contained in F_k*. This linear subspace has dimension* $k+1-(k/2)-1 = k/2$*. If* k *is odd, then* F_k *contains*

$$L_k : \{x_1 = -x_2, x_3 = -x_4, \ldots, x_{k-2} = -x_{k-1}, x_k = 2^{-1/3}, x_{k+1} = 2^{-1/3}\}$$

which has dimension $k+1-(k-1)/2-2 = (k-1)/2$*. The Gaussian curvature of* F_k *vanishes only if* $x_1 x_2 \ldots x_k = 0$*. Therefore,* L_k *contains some non-trivial open sets on which* F_k *has non-vanishing Gaussian curvature. From here we see that there is a non-trivial open set* U *in* $\mathbb{R}^k$ *so that* $F_k \cap (U \times \mathbb{R})$ *is the graph of some nonlinear function* f*. The projection* $\pi(L_k)$ *of* L_k *to the first* k*-coordinates satisfy the property that* $\pi(L_k) \cap U$ *contains an non-trivial open set of* $\pi(L_k)$ *which is a linear subspace of dimension* $[k/2]$*. Consider any self-similar measure*

[1]It is well-known that for a self-similar set, box and Hausdorff dimension coincide, see Falconer's implicit theorem described in Section 3.7. Here we choose to use the box dimension for semantic clarity.

μ supported in $\pi(L_k) \cap U$. We see that $f|\mathrm{supp}(\mu)$ is in fact linear, although f is itself a nonlinear function. Thus in general there is no hope to prove polynomial Fourier decay for $f_*\mu$, which is a linear copy of μ.

We make the following quick corollary about pushforwards of self-similar measures on $\mathbb{R}^k$. In Section 2 we will discuss more results of this type.

Corollary 1.4. *Fix any integer $k \geqslant 5$, and define $f\colon \mathbb{R}^k \to \mathbb{R}$ by $f(x_1, \dots, x_k) = x_1^2 + \cdots + x_k^2$. Let μ be a self-similar measure on $\mathbb{R}^k$ without rotations (in particular, it is non-expanding), satisfying the open set condition and with natural weights. If*

$$\kappa_2 > 2 + \frac{k}{2},$$

then $f_*\mu$ is absolutely continuous with respect to the Lebesgue measure with an L^2 Radon–Nikodym derivative.

Proof. Fix $y \in (2+k/2, k)$ and note that $2/3 < \frac{y-k/2}{1+y-k/2} < 1$. Let μ be a self-similar measure with dimension y that is generated by an IFS that has no rotations, satisfying the open set condition and with natural weights. These assumptions imply that μ is AD regular and $\kappa_F = \kappa_* = \kappa_2 = y$. Note also that f has non-vanishing Gaussian curvature on $\mathbb{R}^k$, so we can apply our quantitative theorem. Indeed, $\tilde{\sigma}(k/2) > 1/2$, so $\int_{\mathbb{R}} |\widehat{f_*\mu}(\xi)|^2 d\xi < \infty$. Therefore by Plancherel's theorem $f_*\mu$ is absolutely continuous and its Radon–Nikodym derivative is in L^2. $\square$

A class of self-similar measures can be seen as generalised Cantor construction, for example, the famous middle-third Cantor set and the Cantor–Lebesgue measure on it. For those measures, we have much to say about their l^1-dimensions and therefore we can exploit the $\tilde{\sigma}(k-\kappa_1)$ part in Theorem 1.1.

Example 1.5 (Cantor–Lebesgue measure)**.** *If μ is the Cantor–Lebesgue measure then it was shown in [15] that $\kappa_1 < 1/2$, so we use the l^2 bound instead. By Lemma 3.3, $\kappa_2 = \kappa_F = \log 2/\log 3 > 1/2$, so the pushforward decays with exponent at least* 0.061*, improving the* 0.016 *achieved in* [53].

The Cantor–Lebesgue measure is in fact a special case of a more general class of self-similar measures called missing digit measures.

Example 1.6 (missing digit measures)**.** *Fix $b \geqslant 4$, divide the interval $[0, 1]$ into pieces of size $1/b$ and choose $b-1$ of the pieces corresponding to a set $D \subset \{0, \dots, b-1\}$ (in the case $b = 4$ we additionally assume* *that $D = \{0,1,2\}$ or $D = \{1,2,3\}$). Let μ be the self-similar measure with equal weights corresponding to the IFS of maps sending $[0,1]$ to each of the $b-1$ intervals. Then $\kappa_F = \kappa_2 = \kappa_* = \log(b-1)/\log b$.*

In this case the l^2 estimate proves that the exponent of Fourier decay is at least $\frac{2\log(b-1)-\log b}{2\log(b-1)+3\log b}$. But Chow, Varjú and Yu [15, Proposition 2.4] have recently shown that $\kappa_1 > 1/2$, so the l^1 estimate in fact there exists $\varepsilon_b > 0$ such that the true exponent is at least $\frac{2\log(b-1)-\log b}{2\log(b-1)+3\log b} + \varepsilon_b$. Moreover, it is shown in [15, Theorem 2.6] that $\kappa_1 \to 1$ as $b \to \infty$, so a lower bound for the Fourier decay exponent is $1/3 - o_b(1)$. For comparison, if these are Salem measures then the Fourier decay would tend to $1/2$ as $b \to \infty$.

1.2. **Qualitative results.** If the explicit lower bound for the Fourier decay is not a concern, we have the following qualitative result. Recall that an analytic map is said to be degenerate if its graph lies in a proper affine subspace.

thm: weaker estimate

Theorem 1.7. *Let μ be a self-similar measure for a non-expanding and irreducible IFS on $\mathbb{R}^k$. Let $U \subset \mathbb{R}^k$ be an open neighbourhood of $\mathrm{supp}(\mu)$ and let $\mathbf{f}\colon U \to \mathbb{R}^d$ be a real analytic and non-degenerate map. Consider the pushforward measure $\mathbf{f}_*\mu$ under $\mathbf{f}$. Then there is some $\sigma > 0$ so that*

$$|\widehat{\mathbf{f}_*\mu}(\xi)| \ll |\xi|^{-\sigma}.$$

Remark 1.8. *The non-expanding condition trivially holds for $k \leqslant 2$. In any case, it can be dropped by requiring the strong separation condition for μ and by replacing the non-degeneracy of $\mathbf{f}$ with a slightly stronger condition, see Theorem 5.4.*

Remark 1.9. *Polynomial Fourier decay of nonlinear pushforwards $\mathbb{R}^k \to \mathbb{R}$ of a class of dynamically defined measures have been obtained under a non-trivial fibre condition by [7, Theorem 1.3]. Our result is almost optimal apart from the non-expanding condition. See Section 5.*

One of the key steps in the proof of Theorem 1.7 is showing that the self-similar measures in the statement of that theorem have polynomial Fourier decay outside a very sparse set of frequencies. More precisely, following the terminology from [8], we make the following definition.

d:afd

Definition 1.10. *Let μ be a probability measure on $\mathbb{R}^k$. We say that μ (or $\widehat{\mu}$) has* polynomial (Fourier) decay on average *if for each $\varepsilon > 0$ there exist $\delta, C > 0$ so that*

$$|\{\xi \in \mathbb{R}^k : |\widehat{\mu}(\xi)| \geqslant R^{-\delta}, |\xi| < R\}| \leqslant CR^{\varepsilon}$$

for all $R > 0$, where $|\cdot|$ denotes the Lebesgue measure.

A simplified version of what we prove is the following result, which builds on [74] and may be of interest in its own right. Note that the non-expanding condition is automatically satisfied if $k \in \{1, 2\}$, or if the IFS is rotation-free.

thm:babyafd

Theorem 1.11. *Every non-expanding and irreducible self-similar measure on $\mathbb{R}^k$ has polynomial Fourier decay on average.*

Remark 1.12. *It is in principle possible to effectively compute the decay rate σ in Theorem 1.7. However, our proof uses Theorem 1.11 which is proved using an Erdős–Kahane-type argument. Therefore the computed σ will be far from being optimal, and we do not pursue this further.*

Finally, we consider self-conformal measures. Among other results, by combining Theorem 1.7 with a recent breakthrough due to Algom, Rodriguez Hertz and Wang [5], we will prove that all self-conformal measures on $\mathbb{C}$ which are not self-similar and not supported inside an analytic curve have polynomial Fourier decay. In particular, if $\dim_{\mathrm{H}} \mathrm{supp}(\mu) > 1$ then μ cannot be supported inside any analytic curve. Therefore such μ has polynomial Fourier decay.

thm:analyticconformal

Theorem 1.13. *Let μ be a complex analytic self-conformal measure on $\mathbb{C}$. Assume that the analytic contractions do not simultaneously preserve any analytic curve, and that at least one contraction is non-affine. Then μ has polynomial Fourier decay.*

For measures supported inside analytic curves which are not straight lines, there are several results in the literature proving decay of L^p averages of Fourier transforms of such measures (which may not be dynamically defined) [18, 20, 57, 58], but pointwise Fourier decay estimates do not seem to have been considered previously. There is good reason for this: the lift of the Cantor–Lebesgue measure on $[0, 1]$ under $x \mapsto (x, x^2)$, for example, has no Fourier decay in the horizontal direction (we will describe this example in more detail in Example 7.1). However, we see fix a direction that is not in a small set of exceptional directions that we describe precisely, then there will be polynomial Fourier decay along this direction. Note, however, that the multiplicative constant and time taken to begin to see the decay can depend on the direction.

cor:graphdecay

Corollary 1.14. *In the setting of Theorem 1.7, if ν is the pushforward of μ under $\mathbf{x} \mapsto (\mathbf{x}, \mathbf{f}(\mathbf{x}))$ onto the graph of μ, if we fix $\xi \in \mathbb{R}^{k+d}$ whose last d coordinates are not all 0, then there exists $\sigma > 0$ such that for all $t > 0$ we have $|\widehat{\nu}(t\xi)| \ll t^{-\sigma}$.*

Proof. There exist $\mathbf{w} \in \mathbb{R}^k$ and $\mathbf{v} \in \mathbb{R}^d \setminus \{0\}$ such that if χ is the pushforward of μ under $g\colon \mathbf{x} \mapsto \mathbf{w} \cdot \mathbf{x} + \mathbf{v} \cdot \mathbf{f}(\mathbf{x})$, then $\widehat{\chi}(t) = \widehat{\nu}(t\xi)$. Since $\mathbf{f}$ is non-degenerate, so is g, so Theorem 1.7 applied to g gives that there exists $\sigma > 0$ such that $|\widehat{\chi}(t)| \ll t^{-\sigma}$. □

Structure of paper. In Section 2 we present applications of our Fourier decay results related to nonlinear arithmetic of self-similar sets and measures, normality in higher dimensions, Fourier restriction estimates, and fractal uncertainty principles.

In Section 3 we formally introduce the notions and terminology (especially from fractal geometry) needed for the proof of the results in Sections 1 and 2.1, and prove some preliminary results.

Section 4 concerns quantitative Fourier decay and we prove Theorem 1.1.

Section 5 relates to qualitative polynomial Fourier decay. We formulate a conjecture regarding analytic pushforwards and make significant progress towards it in Theorems 5.4 and 5.3. There are some similarities but also many differences between the proofs in Sections 4 and 5.

Section 6 concerns the fact that many self-similar measures decay polynomially outside very sparse frequencies. We formulate a conjecture and make significant progress towards it in Theorems 6.8 (which implies Theorem 1.11) and 6.3. Theorem 6.8 combines with Theorem 5.3 to prove the qualitative Fourier decay result Theorem 1.7, but the proofs in Section 6 have a different flavour to the proofs in the rest of the paper and this section can be read independently.

Finally, in Section 7 we prove consequences of our main Fourier decay results. We prove the results from Section 2.1 on nonlinear arithmetic, and then Theorem 1.13 and some other results related to self-conformal measures.

2. Applications

sec: applications

Polynomial Fourier decay of measures has many applications. In this section, we briefly discuss some of them. They are all direct consequences of Theorems 1.1 and 1.7.

subsec: arithmetic

2.1. Nonlinear arithmetic of self-similar measures/sets.

The result in this section is motivated by the following problem. Given a fractal set K, how to tell if $K \cdot K$ has positive Lebesgue measure? Here we used the following notion

$$K \cdot K = \{xy : x, y \in K\}.$$

A more general problem is to consider the image $f(K\times K)$ for a certain smooth map $f\colon \mathbb{R}^2\to\mathbb{R}$. Such a problem was studied in [25] from number theory, in [52, 55] from smooth dynamical systems and in [31, 47] from geometric measure theory. Recently, we have many new results. More precisely, we have [14, 78, 79] extending [25]; [65, 71, 72, 73, 77, 83] extending [52, 55]; [33, 32, 67, 76] extending [31]; and [37, 56, 81, 82] extending [47]. This is certainly not a complete list. For example, we have omitted great achievements regarding Falconer's distance conjecture. See [68] for a recent account.

Our starting point is the following general result due to Marstrand. We do not state the most general version (which requires the notion of the transversal family as in [60]). In fact, we will state the theorem as high-level idea and then supply two concrete examples.

Theorem 2.1 ((general) Marstrand projection theorem)**.** *Let P_a be a parametrised family of smooth maps $\mathbb{R}^2\to\mathbb{R}$. Let $A\subset\mathbb{R}^2$ be a compact set. For almost all parameter a (w.r.t. some natural measure), $P_a(A)$ has a positive Lebesgue measure as long as $\dim_{\mathrm{H}}A>1$.*

Example 2.2. *For $a\in\mathbb{S}^1$, we can define $P_a\colon (x,y)\mapsto x\cos a+y\sin a$. It is the orthogonal projection on direction a. The corresponding result is then the classical Marstrand's projection theorem for linear projections.*

Example 2.3. *For $a\in\mathbb{R}^2$, we can define $P_a\colon \mathbf{x}\mapsto(\mathbf{x}-a)/|(\mathbf{x}-a)|\in\mathbb{S}^1$. This is the radial projection centred at a. The corresponding result is then the classical Marstrand projection theorem for radial projections.*

There is no hope to improve this general result. However, suppose that K has additional structures, it is very likely to greatly improve this result. See [81, 82]. One interesting structure for us to explore is coded as the notion of self-similarity. In this direction, a question can be found on the internet [46][2]. Various results can be found on that webpage including the following. See also [33].

Theorem 2.4. *Let E,F be two self-similar sets in $\mathbb{R}$. Suppose that some contraction ratio r_E for E and some contraction ratio r_F for F satisfy $\log r_E/\log r_F\notin\mathbb{Q}$. Then $\dim_{\mathrm{H}}(E\cdot F)=\min\{\dim_{\mathrm{H}}E+\dim_{\mathrm{H}}F,1\}$.*

Unlike Marstrand's projection theorem, there is no uncertainty in the above result. The idea of proving the above result relies on relating the product set $E\cdot F$ to $\log E+\log F$ and then to the sum set $E+F$. It is a

[2]Last accessed by the authors on September 14, 2024.

certain linear projection of $E \times F$. Marstrand's projection cannot say much. However, under the assumption, $E + F$ will have the indicated dimension because in small scales, E and F look quite different and there should be no additive collision. As far as $E \cdot F$ is concerned, we note that $\log E, \log F$ will not have common "additive structure" even if they have all the same contraction ratios! This consideration leads us to believe that even stronger results can hold.

conj: multiplication

Conjecture 2.5. *Let $E_1, \ldots, E_k$ be $k \geqslant 2$ self-similar sets in $\mathbb{R}$. If $\sum_{i=1}^k \dim_{\mathrm{H}} E_i > 1$, then $E_1 \cdot E_2 \cdot \cdots \cdot E_k$ has positive Lebesgue measure.*

This conjecture is wildly open. A recent result in [59] says that for Borel probability measures $\mu_1, \ldots, \mu_k$ with $\kappa_2(\mu_1) + \cdots + \kappa_2(\mu_k) > 1$, the measure $\mu_1 \cdot \mu_2 \cdot \cdots \cdot \mu_k$ has polynomial Fourier decay. The point there is that $\mu_1, \ldots, \mu_k$ are not required to be self-similar.

We will work towards Conjecture 2.5.

thm: enough variables mul

Theorem 2.6. *Let $E_1, E_2 \subset \mathbb{R}$ be self-similar with $\dim_{\mathrm{H}} E_1, \dim_{\mathrm{H}} E_2 > (\sqrt{65} - 5)/4$ then $E_1 \cdot E_2$ has a positive Lebesgue measure. Let E_1, E_2, E_3 be self-similar with $\dim_{\mathrm{H}} E_1, \dim_{\mathrm{H}} E_2, \dim_{\mathrm{H}} E_3 > (-3 + \sqrt{41})/4$ then $E_1 \cdot E_2 \cdot E_3$ has non-empty interior.*

Remark 2.7. *Similar results were obtained in [81, 82] conditioned on l^1-dimensions of measures. The threshold there is better than the threshold here. However, we note that Hausdorff dimension is in general larger in value and much easier to compute than l^1-dimension.*

The two sets part of the above theorem follows from the following stronger results. Of course, it is also possible to explicitly work out sufficient conditions involving more than three sets. We do not continue this route.

thm: measure main

Theorem 2.8. *Let $f \colon (x, y) \mapsto xy$ be the multiplication function. Let μ be a canonical[3] self-similar measure on $[0, 1]$. Consider the image measure $f_*(\mu \times \mu)$. If μ has the SSC (therefore μ is AD-regular), then $f_*(\mu \times \mu)$ has an L^2-density whenever $\kappa_2 > (\sqrt{65} - 5)/4 \approx 0.766\ldots$. More generally, for two SSC self-similar measures μ, ν, if*

$$\kappa_2(\mu)\kappa_2(\nu) + \max\{1.5\kappa_2(\mu) + \kappa_2(\nu), 1.5\kappa_2(\nu) + \kappa_2(\mu)\} > 2.5$$

then $f_*(\mu \times \nu)$ has an L^2-density.

Remark 2.9. *The result continues to hold without the SSC although the dimensional threshold would be much worse. For example, if we only assume the ESC then the result holds for $\kappa_2 > 7/9$.*

[3] A canonical self-similar measure μ is so that $\kappa_2(\mu) = \dim_{\mathrm{H}} \mathrm{supp}(\mu)$.

Regarding sets rather than measures, we have the following seemingly stronger result which is in fact a consequence of Theorem 2.8. We emphasise that in the following result, we do not require that there are any separation conditions.[4]

thm: set main

Theorem 2.10. *Let $f\colon (x,y)\mapsto xy$ be the multiplication function. Let F be a self-similar set on $[0,1]$. Suppose that $\dim_{\mathrm{H}}F>(\sqrt{65}-5)/4$, then $f(F,F)$ has a positive Lebesgue measure. More generally, for each self-similar sets E,F on $[0,1]$, if*

$$\dim_{\mathrm{H}}E\dim_{\mathrm{H}}F+\max\{1.5\dim_{\mathrm{H}}E+\dim_{\mathrm{H}}F,1.5\dim_{\mathrm{H}}F+\dim_{\mathrm{H}}E\}>2.5$$

then $f(E\times F)$ has a positive Lebesgue measure.

We say more about Conjecture 2.5. We have the following result which is a very simple consequence of Theorems 1.1, 1.7.

thm: multi

Theorem 2.11. *Let $E_1,\dots,E_k\subset\mathbb{R}$ be $k\geqslant 3$ self-similar sets. Suppose that $\sum_{i=1}^k E_i>1+(k-1)^{-1}$, then $E_1\cdot E_2\cdot\dots\cdot E_k$ has a positive Lebesgue measure.*

Remark 2.12. *We only use Theorem 1.7 for $d=1$. Therefore this result can be deduced from the results in [7].*

Remark 2.13. *The function $f\colon (x,y)\mapsto xy$ is not special. In fact, our result holds for quite general functions. For example, a sufficient condition is that f is C^2 and the graph $\Gamma_f\subset\mathbb{R}^3$ has non-vanishing Gaussian curvature over the support of $E\times F$.*

For example, consider the radial projection $R_{(a,b)}\colon (x,y)\to(x-a,y-b)/|(x-a,y-b)|$ for $(a,b)\in\mathbb{R}^2$ being fixed. After some smooth identification, we can consider the map

$$L_{(a,b)}(x,y)=\log(x-a)-\log(y-b).$$

This map has Hessian matrix

$$\begin{bmatrix}\frac{-1}{(x-a)^2} & 0\\ 0 & \frac{1}{(y-b)^2}\end{bmatrix}.$$

Outside of the singularities $x=a$ or $y=b$, the graph of $L_{(a,b)}$ has non-vanishing Gaussian curvature. The following result follows by using the same arguments as in Theorems 2.8, 2.10.

[4]We hide a technical point here that it is extremely difficult to determine the box dimension of a self-similar set without any separation conditions, see [32, 67]. Recall however than under the ESC, which holds extremely generically, box dimension coincides with the similarity dimension.

thm: radial

Theorem 2.14. *Let F be a self-similar set on $[0,1]$. Suppose that $\dim_{\mathrm{H}} F > (\sqrt{65}-5)/4$, then $R_{(a,b)}(F,F)$ has a positive Lebesgue measure for all $(a,b) \notin F \times F$. More generally, for each self-similar sets E, F on $[0,1]$, if*

$$\dim_{\mathrm{H}} E \cdot \dim_{\mathrm{H}} F + \max\{1.5\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F, 1.5\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F\} > 2.5$$

then $R_{(a,b)}(E \times F)$ has a positive Lebesgue measure for all $(a,b) \notin E \times F$.

Remark 2.15. *According to Marstrand's theorem, we saw that the conclusion holds for Lebesgue almost all $(a,b) \in \mathbb{R}^2$. Here, we took care of the exceptions.*

2.2. **Other applications.**

We now present several other applications which are straightforward to deduce from the results in this paper together with existing results in the literature. One-dimensional versions of these have been considered in [7, Section 2].

Normality. We say that a point $x \in \mathbb{R}^k$ is *normal* if for every expanding integer matrix $A \in M_{k\times k}(\mathbb{Z})$, $(A^k x)_{n=1}^{\infty}$ is equidistributed in the torus $\mathbb{R}^k/\mathbb{Z}^k$ when reduced mod 1.

If the Fourier transform of a probability measure μ decays polynomially (in fact a weaker decay condition suffices) then μ-a.e. point is normal; see [19, 61] for a proof in one dimension and [8, Theorem A.1] (stated in a paper of Baker, Khalil and Sahlsten but attributed to Fraser and Sahlsten) for the result in higher dimensions. We can use this result to deduce, for example, that with respect to the Cantor–Lebesgue measure μ on the middle-third Cantor set, μ-.a.e x satisfies the condition that $(x^2, x^3, x^4, \ldots, x^{k+1}) \in \mathbb{R}^k$ is normal.

Fourier restriction. Fourier restriction theory relates to the interesting fact that, for $1 < p < 2$, the Fourier transforms of L^p functions can be restricted to certain sets of zero Lebesgue measure in a meaningful way, despite *a priori* just belonging to L^q where $1/p + 1/q = 1$. From work such as [51, Theorem 4.1], [50, Corollary 3.1] and [69, page 353] it is known that if a measure μ on $\mathbb{R}^k$ has polynomial Fourier decay then there is some $p_\mu > 1$ such that for all $p \in [1, p_\mu]$ the Fourier transform can be thought of as a bounded linear operator $L^p(\mathbb{R}^k, \text{Lebesgue}) \to L^2(\text{supp}(\mu), \mu)$. In particular, this is the case for the measures which we have seen have polynomial Fourier decay from Theorems 1.13, 5.3, 5.4

Fractal uncertainty principles. Fractal uncertainty principles state (roughly) that a function cannot be localised in position and frequency near a fractal set [21, 24, 12]. Proving fractal uncertainty principles for neighbourhoods of fractal sets arising from hyperbolic dynamics has applications in quantum chaos [22, 23]. Using [9, Proposition 1.5] there are several possible statements one could prove, for instance from Theorem 1.13 we have the following.

Theorem 2.16. *Let K, J be attractors of complex analytic conformal IFSs on $\mathbb{C} \simeq \mathbb{R}^2$ with non-vanishing derivative, which satisfy the strong separation condition and do not preserve any analytic curve. Assume that the IFS generating J contains a non-affine map, and that $\dim_{\mathrm{H}} K + \dim_{\mathrm{H}} J < 2$. Then for all $\kappa \in (\dim_{\mathrm{H}} K + \dim_{\mathrm{H}} J, 2)$ there exists $C > 0$ such that for all $h > 0$, if we let X and Y be the open h-neighbourhoods of K and J respectively, then for all $f \in L^2(\mathbb{R}^2)$ satisfying $\{\xi \in \mathbb{R}^2 : \widehat{f}(\xi) \neq 0\} \subset h^{-1}Y$, we have $||f||_{L^2(X)} \leqslant Ch^{(1-\kappa)/2}||f||_{L^2(\mathbb{R}^2)}$, where the L^2 spaces are with respect to Lebesgue measure.*

3. Preliminaries

sec: pre

A general account of fractal geometry can be found in textbooks such as [28, 29, 48, 49].

ss:ifs

3.1. Self-similar sets and measures.

Let $k \geqslant 1$ and $N > 1$ be integers, and let $D \subset \mathbb{R}^k$ be compact. Let $\Lambda = \{f_i \colon D \to D\}_{1 \leqslant i \leqslant N}$ be contraction maps. Let $p_1, \ldots, p_N \in (0, 1)$ be such that $\sum_i p_i = 1$. By Hutchinson's theorem [34], there is a unique compact set K and a Borel probability measure on K such that

$$K = \bigcup_i f_i(K), \qquad \mu = \sum_i p_i f_i(\mu).$$

We will always assume that the contractions do not share a common fixed point, ensuring that K is uncountable and μ is non-atomic.

If there exist $r_1, \ldots, r_N \in (0, 1)$ and $O_1, \ldots, O_N \in O_k(\mathbb{R})$ (possibly reflected) rotations and $t_1, \ldots, t_N \in \mathbb{R}^k$ such that Λ consists of the similarity maps

$$\Lambda = \{f_i(\cdot) = r_i O_i(\cdot) + t_i\}_{i \in \{1, \ldots, N\}},$$

then we say that Λ, K, μ are self-similar. In this case, we say that Λ, μ, K are homogeneous if $r_i O_i$ are all the same for $i \in \{1, \ldots, N\}$.

e:affineirred

Definition 3.1 (Irredicibility of self-similar IFS)**.** *We say that an IFS of contracting similarity maps $\{f_i \colon \mathbb{R}^k \to \mathbb{R}^k\}_i$ is* irreducible *(or sometimes* affinely irreducible*) if there does not exist a $k-1$ dimensional* *affine subspace V of $\mathbb{R}^k$ and $t \in \mathbb{R}^k$ such that $f_i(V+t) = V+t$ for all i. We say that a self-similar set in $\mathbb{R}^k$ is (affinely) irreducible if it is the attractor of an irreducible IFS, or equivalently is not contained in any proper hyperplane. Similarly, a self-similar measure is irreducible if it is the stationary measure for an irreducible IFS with positive weights, or equivalently is not supported in any proper hyperplane.*

As an example, note that an IFS of similarities on $\mathbb{R}$ is irreducible if and only if the maps do not all have a common fixed point.

3.2. **Hausdorff and box dimensions.** Let $k \geqslant 1$ be an integer. Let $E \subset \mathbb{R}^k$ be a non-empty Borel set. Let $g\colon [0,1) \to [0,\infty)$ be a continuous function such that $g(0) = 0$. Then for all $\delta > 0$ we define the quantity

$$\mathcal{H}^g_\delta(E) = \inf\left\{ \sum_{i=1}^{\infty} g(\mathrm{diam}(U_i)) : \bigcup_i U_i \supset E, \mathrm{diam}(U_i) < \delta \right\}.$$

The g-Hausdorff measure of E is

$$\mathcal{H}^g(E) = \lim_{\delta \to 0} \mathcal{H}^g_\delta(E).$$

When $g(x) = x^s$ then $\mathcal{H}^g = \mathcal{H}^s$ is the s-Hausdorff measure and Hausdorff dimension of E is

$$\dim_{\mathrm{H}} E = \inf\{s \geqslant 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geqslant 0 : \mathcal{H}^s(E) = \infty\}.$$

If we assume E is bounded then we can also define the box dimensions of E. For all $\delta > 0$ let $N_\delta(E)$ be the smallest number of open balls of diameter δ needed to cover E. Then the lower and upper box dimensions of E are respectively defined by

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{\delta \to 0^+} \frac{\log N_\delta(E)}{\log(1/\delta)}; \qquad \overline{\dim}_{\mathrm{B}} E = \limsup_{\delta \to 0^+} \frac{\log N_\delta(E)}{\log(1/\delta)}.$$

If the two values coincide, we call the common value the box dimension of E, denoted $\dim_{\mathrm{B}} E$.

3.3. **Assouad dimension and microsets.** The Assouad dimension, also known as Furstenberg's *-exponent, of a non-empty set $E \subset \mathbb{R}^k$ is defined by

$$dim_{\mathrm{A}} E \coloneqq \inf\Big\{\alpha \geqslant 0 : \exists C > 0 \text{ such that for all } 0 < r < R \text{ and } x \in E,$$
$$N_r(B_R(x) \cap E) \leqslant C \left(\frac{R}{r}\right)^{\alpha} \Big\}.$$

For more on the Assouad dimension we refer the reader to [6, 30]. The following inequalities hold for all non-empty bounded $E \subset \mathbb{R}^k$:

$$\dim_{\mathrm{H}} E \leqslant \underline{\dim}_{\mathrm{B}} E \leqslant \overline{\dim}_{\mathrm{B}} E \leqslant \dim_{\mathrm{A}} E.$$

An important result related to the Assouad dimension is the fact that the Assouad dimension of any closed set $E \subset \mathbb{R}^k$ is the maximum of Hausdorff dimensions of microsets of E, see [36, Proposition 5.7], [10, Lemma 2.4.4], [26, Theorem 5.1].

3.4. **Dimension exponents of Borel probability measures.**

3.4.1. *Frostman exponent.* Let μ be a Borel probability measure on $\mathbb{R}^k$. The (uniform) Frostman exponent of μ is the supremum of $\kappa \geqslant 0$ such that

$$\mu(B_r(x)) \ll r^\kappa$$

uniformly for all $x \in \mathbb{R}^k, r > 0$.

3.4.2. *AD-regularity.* Let μ be a Borel probability measure on $\mathbb{R}^k$. If there exists $s \geqslant 0$ such that

$$r^s \ll \mu(B_r(x)) \ll r^s$$

uniformly for all $x \in \mathrm{supp}(\mu)$, $r > 0$, then μ and $\mathrm{supp}(\mu)$ are called *s-Ahlfors–David regular*, or simply *AD regular* if the value of s is clear from the context. We call s the (uniform) AD exponent of μ.

3.5. **l^p-dimension of Borel probability measures.**

Recall that κ_p denotes the Fourier l^p dimension of a measure μ, where $1 \leqslant p$. We first record the following inequalities related to the Fourier l^p dimensions. We mostly use $p = 1, 2$ in this paper. In fact, l^2-dimension is naturally related to the notion of Hausdorff dimension. The less-known l^1-dimension was studied in detail in [15, 80].

Lemma 3.2. *If μ is a Borel probability measure with compact support in $\mathbb{R}^k$ and $1 \leqslant p \leqslant q \leqslant 2$, then*

$$\frac{p}{q} \cdot \kappa_q \leqslant \kappa_p \leqslant \kappa_q.$$

Proof. The second inequality holds since μ is a probability measure, so $|\widehat{\mu}(\xi)| \leqslant 1$ and hence $|\widehat{\mu}(\xi)|^p \geqslant |\widehat{\mu}(\xi)|^q$ for all ξ. For the first inequality, we apply Hölder's inequality with $p' := q/p$ and q' such that $1/p' + 1/q' = 1$ to the functions $|\widehat{\mu}(\xi)|^p$ and the indicator function of $B(0, R)$. This gives

$$\int_{|\xi| \leqslant R} |\widehat{\mu}(\xi)|^p d\xi \ll R^{k/q'} \left(\int_{|\xi| \leqslant R} |\widehat{\mu}(\xi)|^{pp'} \right)^{1/p'} \leqslant R^{k/q' + (k - \kappa_{pp'})/p'}.$$

In particular,

$$\kappa_p \geqslant k - \frac{k-\kappa_q}{p'} - \frac{k}{q'} = \frac{p}{q} \cdot \kappa_q,$$

as required. □

We will use the following fact.

l:l2dimHausdorff

Lemma 3.3. *If μ is an s-AD regular Borel probability measure then*

$$\kappa_2 = s.$$

Proof. This follows from [49, Section 3.8]. □

3.6. **l^2-dimension and Plancherel's theorem.** By using Plancherel it is possible to see that for any Borel probability measure on $\mathbb{R}^k$, any diffeomorphism f on $\mathbb{R}^k$, the pushforward $f_*\mu$ and μ share the same κ_2 value. In the set setting, this result reflect the fact that Hausdorff dimension is kept invariant under diffeomorphisms.

It is natural to ask the following.

Question 3.4. *What happens to the other l^p-dimensions under diffeomorphisms? In particular, what can one say about l^1-dimension under diffeomorphisms?*

ss:dims

3.7. **Dimensions of self-similar sets and measures.** The dimension of self-similar sets/measures is an important topic in Geometric Measure Theory. Here, we review some results that will be useful in this paper.

- Given a self-similar measure μ as above, $\kappa_\sim := \frac{\sum_i p_i \log p_i}{\sum_i p_i \log r_i}$ is called the similarity dimension, and it is always an upper bound for Hausdorff dimension of μ. The unique $s \geqslant 0$ satisfying $\sum_i r_i^s = 1$ is called the similarity dimension of the IFS (or of the self-similar set). It is always an upper bound for upper box dimension of the set.
- It is well known that the box and Hausdorff dimension of any self-similar set coincide, see [27, Example 2]. This follows from Falconer's so-called 'implicit theorems.'
- Given an IFS, one often assumes that certain separation conditions. We say that Λ, K, μ has the strong separation condition (SSC) if $\forall i \neq j$, $f_i(K) \cap f_j(K) = \varnothing$.

  A weaker condition is the open set condition (OSC), which means there exists a non-empty bounded open set V such that $V \subseteq \bigcup_i f_i(V)$ with the union disjoint.

  If the IFS consists of similarity maps on $\mathbb{R}$, then another condition that is even weaker than the OSC is the exponential separation condition (ESC), introduced in [32]. In this case writing $r_i = r_{i_1} \cdots r_{i_n}$ for $i \in \{1, \ldots, N\}^n$, define the distance $d(i, j)$ between cylinders $i, j \in \{1, \ldots, N\}^n$ to be ∞ if $r_i \neq r_j$, and $|f_i(0) - f_j(0)|$ if $r_i = r_j$. Then the ESC holds if there exists $c > 0$ such that $d(i, j) > c^n$ for all distinct $i, j \in \{1, \ldots, N\}^n$. The ESC holds extremely generically: if a family of self-similar IFSs is parameterised in a real analytic manner then the set of 'exceptional' parameters where it fails has zero Hausdorff dimension.

Under any of these separation conditions, the dimension of the self-similar set coincides with the similarity dimension, and the dimension of the self-similar sets and measures coincide with their similarity dimensions [28, 32].

- Given a self-similar set satisfying the ESC, there exists a unique self-similar measure, called the measure of maximal entropy, namely the one corresponding to weights r_i^s, for which $\kappa_2 = B$. Moreover, $\kappa_F = \kappa_2 = \kappa_\sim$ [32, 67]. Under the OSC, this measure is AD-regular.

3.8. Non-expanding generators.

Definition 3.5. *Let G be a group. Let $F \subset G$ be a finite set. We say that F is non-expanding if for each $\varepsilon > 0$,*

$$\#F_n \ll e^{\varepsilon n},$$

where $F_n \subset G$ is the collection of $f_1 \cdots f_n$ for $f_1, \ldots, f_n \in F$.

This notion is closely related to the amenability of G. In fact, if G is amenable, then any finite $F \subset G$ is non-expanding. This holds in particular for abelian G.

Definition 3.6 (non-expanding self-similar system)**.** *Let Λ be a self-similar IFS. We say that it is non-expanding if the collection of linear parts $\{r_i O_i\}$ is non-expanding viewed as a subset of the Euclidean group on $\mathbb{R}^k$. We also say that a self-similar set/measure is non-expanding if one of its corresponding self-similar IFS is non-expanding.*

Any self-similar system on $\mathbb{R}, \mathbb{R}^2$ is non-expanding. For a self-similar system on $\mathbb{R}^3$, it is possible that it is not non-expanding. However, if no rotations are presented, then it is non-expanding. If all linear parts commute (or equal), then the non-expanding condition is automatically satisfied.

3.9. **Gaussian curvature.** First recall the following definition.

def:analytic

Definition 3.7. *If $U \subset \mathbb{R}^k$ is open then $f: U \to \mathbb{R}$ is (real) analytic if for each $x_0 \in U$ there is a neighbourhood V of x_0 and a sequence of homogeneous polynomials P_n of degree n in k variables such that $f(x) = \sum_{n=0}^{\infty} P_n(x - x_0)$ for all $x \in V$. A function $f: U \to \mathbb{R}^d$ is analytic if each of its d components is analytic.*

Now let $M \subset \mathbb{R}^k$ be a smooth hypersurface. The notion of Gaussian curvature is defined on M as a function of $m \in M$. For each $m \in M$, we denote the unit normal vector $N_m \in \mathbb{R}^{k-1}$. We need to fix an orientation beforehand and we will assume this. Then we can define a smooth map $G: m \in M \mapsto N_m \in \mathbb{S}^{k-1}$. This map is only well-defined locally unless M is orientable in which case it is defined globally. The shape operator at $m \in M$ is defined to be dG at m. It is a real symmetric operator and therefore it has eigenvalues and eigenvectors. Its eigenvalues are principle curvatures, its trace is the mean curvature and its determinant is the Gaussian curvature.

If M is the graph of a smooth function f, then it is well known that at each point $(\mathbf{x}, f(\mathbf{x})) \in M$, the Gaussian curvature is proportional to the determinant of the Hessian matrix of f at $\mathbf{x}$. The proportion is a smooth function and it is nowhere zero.

3.10. **Non-degeneracy.** For $M \subset \mathbb{R}^k$ being a smooth submanifold, the notion of curvature is more delicate. The basic idea of curvature captures the non-flatness of M. There are many existing theories of curvatures, i.e. that of the Ricci flow. In our paper, we only need a qualitative notion of curvature. For a simpler introduction, we assume that M is a real analytic submanifold, i.e. the transition maps (for overlapping charts) are real analytic. In this case, we say that M is non-degenerate if and only if no connected component of M is contained in a proper affine subspace of $\mathbb{R}^k$. See Kleinbock and Margulis [42] for a more detailed introduction to the non-degeneracy for possibly non-analytic manifolds. Given an open subset $U \subset \mathbb{R}^k$, we say that a smooth map $f: U \to \mathbb{R}^d$ is non-degenerate if and only if its graph $(\mathbf{x}, f(\mathbf{x}))$ is a non-degenerate manifold. We say f degenerates along an affine subspace of dimension $k' < k$ if the restriction of f to the subspace is a degenerate when thought of as a map $\mathbb{R}^{k'} \to \mathbb{R}^d$. We record the following basic lemma whose proof is left to the reader.

l:degeneratecharacterisation

Lemma 3.8. *Given an open set $U \subseteq \mathbb{R}^k$, a real analytic map $f: U \to \mathbb{R}^d$ is degenerate if and only if there exists some $\mathbf{v} \in \mathbb{S}^{d-1}$ such that $\mathbf{x} \mapsto \mathbf{v} \cdot f(x)$ is an affine function on some connected component of U.*

4. Quantitative Fourier decay and proof of Theorem 1.1

sec:provequantitative

4.1. **Dyadic decomposition of $\mathbb{R}^k$.** For each integer $n \geqslant 1$, let $\mathcal{D}_n$ be the decomposition of $\mathbb{R}^k$ into disjoint union of dyadic cubes of size $1/2^n$. We align all the cubes so that $\mathbf{0}$ is the corner of at least one (thus 2^k) many such dyadic cubes.

4.2. **Dyadic decomposition of self-similar measures.** Let μ be a self-similar measure. Let Λ be the symbolic coding space for μ. For each $\omega \in \Lambda$, following $\omega_0\omega_1 \dots$, there is a least $l \geqslant 0$ such that the contraction ratio of ω_0^l is at most $1/2^n$. We write $l = l_{\omega,n}$. Notice that the finite collection of paths

$$\{\omega_0^{l_{\omega,n}}\}_{\omega\in\Lambda}$$

corresponds to a covering of $\operatorname{supp}(\mu)$. Such a covering uses similar copies of $\operatorname{supp}(\mu)$ of sizes at most $1/2^n$ and at least $\rho_m 2^{-n}$ where ρ_m is smallest contraction ratio for μ. Clearly, each such branch $\mu_{\omega,n}$ of μ intersects at least one and at most 2^k many dyadic cubes in $\mathcal{D}_n$. We choose only one such dyadic cube and associate it with $\mu_{\omega,n}$. As a result, for each dyadic cube $D_n \in \mathcal{D}_n$, there is a collection of similar copies $\mu_{\omega,n}$ that are associated with this cube. We write $\mu_{\omega,n} \sim D_n$ for this association. We will call such a collection of ω to be Λ_{D_n}. We then obtain the following decomposition of Λ

$$\{\Lambda_{D_n}\}_{D_n\in\mathcal{D}_n}$$

which induces a decomposition (disintegration) of μ. For each $\omega \in \Lambda_{D_n}$, the branch $\mu_{\omega,n}$ is supported in $2D_n$, the doubling of D_n with the same centre. Notice that $2D_n$ is not in $\mathcal{D}_{n-1}$.

4.3. **Separation of tangents.** Let $f\colon \mathbb{R}^k \to \mathbb{R}$ be a smooth function with non-vanishing Hessian over a compact set K. Then the manifold $\Gamma_f = (x, f(x))_{x\in\mathbb{R}^k}$ has non-vanishing Gaussian curvature over K. Let $n \geqslant 1000$ and consider the decomposition $\mathcal{D}_n$. For each $D_n \in \mathcal{D}_n$, let x_{D_n} be its centre. Consider the tangent plane $T_{D_n} = T_{x_{D_n}}$ of Γ_f on the point x_{D_n}. We identify T_{D_n} with its normal unit vector on the unit sphere S^k.

lma: bounded overlaps

Lemma 4.1. *Let $n \geqslant 1000$. Consider all the dyadic cubes that $D_n \cap K \neq \varnothing$. Consider the corresponding discrete set $\{T_{D_n}\}_{D_n}$ for D_n ranging over this collection. Then the union of balls*

$$\bigcup_{T_{D_n}} B_{2^{-n}}(T_{D_n})$$

has bounded multiplicity with a bound that depends on f.

Proof. First, we consider two adjacent cubes $D, D' \in \mathcal{D}_n$. Since f has a non-vanishing Hessian over K, we see that for large enough n, f has a uniformly bounded (from below) Hessian for all D such that $D \cap K \neq \varnothing$. Then the distance $T_D, T_{D'}$ on the unit sphere is $\gg 2^{-n}$. The implicit constant here depends on the second-order derivatives of f. Following this argument, we see that there is a constant $c > 0$ such that for each $x \in K$, for all $y \in K$ with $|y - x| < c$,

$$|T_x - T_y| \gg |x - y|.$$

This proves the result with K being replaced by $K \cap B_c$ where B_c is any c-ball. Since K is compact, we only need to consider finitely many such c-balls. From here the lemma is proved. □

Proof of Theorem 1.1. Consider the map $T\colon (x, 0) \mapsto (x, f(x))$ and the image measure $\mu_T = T_*\mu$. We will consider the bounded continuous function $\widehat{\mu_T}\colon \mathbb{R}^{k+1} \to \mathbb{C}$. Observe that for each $\xi = (0, \ldots, 0, \xi_{k+1})$, $\widehat{\mu_T}(\xi) = \widehat{f_*\mu}(\xi_{k+1})$. For each $n > 1000$, consider the dyadic decomposition Λ_{D_n} for $D_n \in \mathcal{D}_n$. We obtain

$$\begin{aligned}\widehat{\mu_T}(\xi) &= \int e^{-2\pi i (x,\xi)} d\mu_T(x)\\ &= \sum_{D_n \in \mathcal{D}_n} \sum_{\mu_{\omega,n} : \mu_{\omega,n} \sim D_n} \int e^{-2\pi i (x,\xi)} d\mu_{\omega,n,T}(x).\end{aligned}$$

where $\mu_{\omega,n,T} = T_*\mu_{\omega,n}$. Our aim is to estimate for each $\mu_{\omega,n}$ the integral

$$\int e^{-2\pi i (x,\xi)} d\mu_{\omega,n,T}(x).$$

Let $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^k\}$ be the graph of f. Let $T_{\omega,n}$ be the tangent plane of Γ_f at $(x_0, f(x_0))$ for x_0 being the centre of D_n. Consider the image measure $\mu'_{\omega,n,T}$ obtained by pushing $\mu_{\omega,n}$ to $T_{\omega,n}$ in the direction that is parallel to the $(k+1)$th coordinate axis. Observe that as long as $|\xi| \leqslant 2^{2n}$,

$$\begin{aligned}&\left| \int e^{-2\pi i (x,\xi)} d\mu_{\omega,n,T}(x) - \int e^{-2\pi i (x,\xi)} d\mu'_{\omega,n,T}(x) \right|\\ &= \left| \int e^{-2\pi i ((x,f(x)),\xi)} d\mu_{\omega,n}(x) - \int e^{-2\pi i ((x, f(x_0) + \nabla f(x_0)\cdot(x - x_0)),\xi)} d\mu_{\omega,n}(x) \right|\\ &= O(|\mu_{\omega,n}||\xi|/2^{2n})\end{aligned}$$

where $|\mu_{\omega,n}|$ is the total mass of $\mu_{\omega,n}$.[5] Next, consider the latter integral

$$\int e^{-2\pi i(x,\xi)}d\mu'_{\omega,n,T}(x).$$

We write $\xi = \xi_\perp + \xi_\parallel$ where $\xi_\perp$ is perpendicular to $T_{\omega,n}$ and $\xi_\parallel$ is parallel to $T_{\omega,n}$. We see that

$$\begin{aligned}\Big|\int e^{-2\pi i(x,\xi)}d\mu'_{\omega,n,T}(x)\Big| &= \Big|\int e^{-2\pi i(x-x_0,\xi)}d\mu'_{\omega,n,T}(x-x_0)\Big| \\ &= \Big|\int e^{-2\pi i(x-x_0,\xi_\perp+\xi_\parallel)}d\mu'_{\omega,n,T}(x-x_0)\Big| \\ &= \Big|\int e^{-2\pi i(x-x_0,\xi_\parallel)}d\mu'_{\omega,n,T}(x-x_0)\Big| \\ &= \Big|\widehat{\mu}_{\omega,n}(\xi_{\omega,n})\Big|\end{aligned}$$

where $\xi_{\omega,n}$ is the projection of $\xi_\parallel$ to $\mathbb{R}^k \times \{0\}$ in the direction that is orthogonal to $T_{\omega,n}$. In other words, $\xi_{\omega,n}$ is equal to the projection of ξ to $\mathbb{R}^k \times \{0\}$ in the direction that is orthogonal to $T_{\omega,n}$.

We will now assume that μ is homogeneous. Later on, we will replace this condition with the much weaker non-expanding condition. Now as μ is homogeneous, $\mu_{\omega,n}$ are all at exactly the same scale and orientation (related to the scale and orientation of μ). Now, we have

eqn: inhomogeneous measure decomposition

$$|\widehat{\mu_T}(\xi)| \leqslant \sum_{D_n}\sum_{\mu_{\omega,n}\sim D_n}|\widehat{\mu}_{\omega,n}(\xi_{\omega,n})| + O(|\xi|/2^{2n}) \tag{1}$$

eqn: measure decomposition

$$\leqslant \sum_{D_n}\sum_{\mu_{\omega,n}\sim D_n}|\mu_{\omega,n}||\widehat{\mu}(r_nO_n^{-1}(\xi_{\omega,n}))| + O(|\xi|/2^{2n}). \tag{2}$$

where $r_n \asymp 2^{-n}$ is the contraction ratio of the copy $\mu_{\omega,n}$ and O_n is its orientation, i.e. $\mu_{\omega,n} = (r_nO_n)_*(\mu)$.[6] Since O_n are all the same for each $\mu_{\omega,n}$, we ignore its appearance. We write $\xi_{D_n} = \xi_{\omega,n}$ since there is no dependence on ω as long as $\mu_{\omega,n} \sim D_n$. Thus

$$|\widehat{\mu}_T(\xi)| \ll \sum_{D_n}\frac{1}{2^{\kappa_F n}}|\widehat{\mu}(\xi_{D_n}r_n)| + O(|\xi|/2^{2n})$$

[5]This step approximates μ_T locally by a suitable linear image of μ. The error term here is crude and it is the main reason that σ cannot be chosen to be close to 1/2.

[6]Note that at this step in the proof we use the self-similarity of μ in a crucial way.

where we have used the fact that for each D_n,

$$\sum_{\mu_{\omega,n}\sim D_n} |\mu_{\omega,n}| \ll \frac{1}{2^{\kappa_F n}}.$$

This follows from the definition of the exponent κ_F. Since $(x, f(x))_{x\in\mathbb{R}^k}$ has non-vanishing Gaussian curvature over $\mathrm{supp}(\mu)$, we see that for different D_n, $\xi_{D_n} r_n$ are $\asymp |\xi| 2^{-2n}$ separated with a bounded multiplicity. For this, we mean that the discrete set

$$F = \{\xi_{D_n} r_n\}_{D_n\in\mathcal{D}_n} \subset \mathbb{R}^k$$

can be divided into disjoint cubes of size $|\xi|/2^{2n}$ with the property that each cube contains at most a bounded number of points in F. See Lemma 4.1. Moreover, for each cube D_0 of size 1, the cardinality of the set $\{\xi_{D_n} r_n\}_{D_n\in\mathcal{D}_n} \cap D_0$ counts how many D_n in a range $\asymp 2^n/|\xi|$ that intersect $\mathrm{supp}(\mu)$. To see this, note that this number is $\ll \#(\{\xi_{D_n}\}\cap C')$ for some cube C' of size $1/r^n \asymp 2^n$. We note that C' may not be itself a dyadic cube. We then scale by $1/|\xi|$ and count

$$\{\xi_{D_n}/|\xi|\} \cap C$$

where C is a cube of size $\asymp 2^n/|\xi|$. The set $\{\xi_{D_n}/|\xi|\}$ is $\{\eta_{D_n}\}$ for the unit vector $\eta = (0, \dots, 1)$. For each $x \in \mathbb{R}^k$, let T_x be the tangent space of Γ_f at $(x, f(x))$. Let $p(x)$ be the projection of η to $\mathbb{R}^k \times \{0\}$ in the direction that is orthogonal to T_x. This map is smooth and it is a local diffeomorphism near all x. Thus $\#(\{\eta_{D_n}\} \cap C)$ is at most the maximal cardinality of a 2^{-n} separated set A in $\mathrm{supp}(\mu)$ with the property that $p(A) \subset C$. This number is then

$$\ll N_{2^{-n}}(\mathrm{supp}(\mu) \cap B_{2^n/|\xi|}) \ll (2^{2n}/|\xi|)^{\kappa_*}$$

by the definition of κ_*.

Next, we denote all D_0 that intersects $\{\xi_{D_n} r_n\}_{D_n\in\mathcal{D}_n}$ to be $\mathcal{C}_n$. From here we obtain that

$$|\widehat{\mu}_T(\xi)| \ll \frac{1}{2^{\kappa_F n}} (2^{2n}/|\xi|)^{\kappa_*} \sum_{D_0\in\mathcal{C}_n} \int_{D_0} |\widehat{\mu}(\xi')| d\xi' + |\xi|/2^{2n}. \tag{3}$$

Notice that the elements in F are $\ll |\xi|/2^n$ away from the origin. The sum over D_0 can be bounded by considering the l^1-dimension of μ or with Cauchy–Schwarz and using the l^2-dimension of μ.

l^1-method: We trivially have

$$\sum_{D_0\in\mathcal{C}_n} \int_{D_0} |\widehat{\mu}(\xi')| d\xi' \ll (|\xi|/2^n)^{k-\kappa_1}.$$

Then by choosing a suitable $2^n \ll |\xi| \ll 2^{2n}$, namely $|\xi| = 2^{\gamma n}$ for some $\gamma \in (1,2)$, to achieve a minimisation of

$$\frac{1}{2^{\kappa_F n}}(2^{2n}/|\xi|)^{\kappa_*}(|\xi|/2^n)^{k-\kappa_1} + |\xi|/2^{2n}.$$

We obtain that

$$|\widehat{\mu}_T(\xi)| \ll |\xi|^{-\sigma}$$

for

$$\sigma = \tilde{\sigma}(k-\kappa_1) = \frac{\frac{\kappa_F+\kappa_1-k}{1+\kappa_*+\kappa_1-k}}{2-\frac{\kappa_F+\kappa_1-k}{1+\kappa_*+\kappa_1-k}}.$$

l^2*-method:* We can use Cauchy–Schwarz and see that

$$\sum_{D_0\in\mathcal{C}_n}\int_{D_0}|\widehat{\mu}(\xi')|d\xi' \ll (\#\mathcal{C}_n)^{1/2}(|\xi|/2^n)^{(k-\kappa_2)/2}.$$

Here $\#\mathcal{C}_n \ll (|\xi|/2^n)^B$. By the same argument we see that

$$|\widehat{\mu}_T(\xi)| \ll |\xi|^{-\sigma}$$

for

$$\sigma = \tilde{\sigma}((k+B-\kappa_2)/2).$$

The above asymptotic $|\widehat{\mu}_T(\xi)| \ll |\xi|^{-\sigma}$ holds for all $|\xi| = [2^{-2+\gamma n}, 2^{2+\gamma n}]$ with a fixed choice $\gamma \in (1,2)$. The value of γ depends on the two different approaches above. We then see that

$$|\widehat{\mu}_T(\xi)| \ll |\xi|^{-\sigma}$$

holds for $|\xi| \to 0$. This finishes the proof under the assumption that μ is homogeneous.

Now we only assume that μ is non-expanding We cannot arrive at (2) because there is no uniform r_n in this case. We step back to (1). In this case $\mu_{\omega,n}$ are not at the same scale and orientation. Let the set L_n be the set of all possible scales and orientations of $\mu_{\omega,n}$. Then we see that for each $\varepsilon > 0$,

$$\#L_n \ll 2^{\varepsilon n}.$$

For each $g \in L_n$, we can apply the argument for the collection of $\mu_{\omega,n}$ that has the scale and orientation determined by g. We write this collection as g. We obtain the following contribution towards $|\widehat{\mu}_T(\xi)|$,

$$\sum_{D_n}\sum_{\mu_{\omega,n}\in g,\mu_{\omega,n}\sim D_n}|\mu_{\omega,n}||\widehat{\mu}(r_nO_n^{-1}(\xi_{\omega,n}))| + O\left(\sum_{\mu_{\omega,n}\in g}||\mu_{\omega,n}|\xi|/2^{2n}\right).$$

Here r_n, O_n depend on the choice of g. We could have written $r_{n,g}, O_{n,g}$ to signify this dependency. However, this is not necessary because we as long as we fix one g, we treat r_n, O_n as being constants. Summing up all such contributions for different g we have

$$
\begin{aligned}
|\widehat{\mu}_T(\xi)| &\ll \sum_{g\in L_n}\sum_{D_n}\sum_{\mu_{\omega,n}\in g,\mu_{\omega,n}\sim D_n} |\mu_{\omega,n}||\widehat{\mu}(\xi_{\omega,n}r_n)| + O(|\xi|/2^{2n}) \\
&\ll \frac{1}{2^{\kappa_F n}}(2^{2n}/|\xi|)^{\kappa_*}\sum_{D_0\in\mathcal{C}_n} m_n(D_0)\int_{D_0}|\widehat{\mu}(\xi')|d\xi' + |\xi|/2^{2n}
\end{aligned}
$$

where $m_n(D_0)$ counts the number of $g\in L_n$ such that there exist $\mu_{\omega,n}\in g$ whose corresponding $\xi_{\omega,n}$ is contained in D_0. It is trivially true that uniformly for all D_0,

$$m_n(D_0)\leqslant \#L_n \ll 2^{\varepsilon n}.$$

From here, the rest of the arguments are the same as in the case when μ is homogeneous. This finishes the proof. □

In fact, we have the following more general quantitative estimate for Fourier decay. Given $1\leqslant p\leqslant 2$ and a compactly supported Borel measure μ on $\mathbb{R}^k$, the Fourier l^p dimension of μ is given by

$$\sup\left\{s\geqslant 0 : \int_{|\xi|\leqslant R}|\widehat{\mu}(\xi)|^p \ll R^{k-s}\right\}.$$

thm:allparams

Theorem 4.2. *Let μ be a non-expanding self-similar measure on $\mathbb{R}^k$. Let $f\colon\mathbb{R}^k\to\mathbb{R}$ be a C^2 function so that the graph $\Gamma_f=(x,f(x))_{x\in\mathbb{R}^k}$ has non-vanishing Gaussian curvature over the support of μ. Consider the pushforward measure $f_*\mu$ under f. Recall that κ_F and κ_p are the Frostman and Fourier l^p dimensions of the measure μ respectively, and that B and κ_* are the box and Assouad dimensions of the self-similar set supp(μ) respectively. Assume that for some $1\leqslant p\leqslant 2$, letting q be such that $1/p+1/q=1$, we have*

e:lpcondition

$$\kappa_F > \frac{B}{q}+\frac{k-\kappa_p}{p}. \tag{4}$$

Then

$$|\widehat{\mu}_f(\xi)|\ll|\xi|^{-\sigma_p},$$

where

$$\sigma_p :- \frac{\kappa_F-\frac{B}{q}-\frac{k-\kappa_p}{p}}{2\kappa_*-\frac{B}{q}-\frac{k-\kappa_p}{p}+2-\kappa_F}>0.$$

Proof. First note that (4) together with $\kappa_F\leqslant\kappa_*$ gives that

e:impliesassouadcondition

$$\kappa_*-\frac{B}{q}-\frac{k-\kappa_p}{p}>0, \tag{5}$$

and also gives that indeed $\sigma_p < 0$. Now, we begin from (3). Applying Hölder's inequality to the indicator function of $\mathcal{C}_n$ and $\xi' \mapsto \widehat{\mu}(\xi')$ gives

$$|\widehat{\mu}_T(\xi)| \ll \frac{1}{2^{\kappa_F n}}(2^{2n}/|\xi|)^{\kappa_*} \cdot (\#\mathcal{C}_n)^{1/q} \cdot (|\xi|/2^n)^{\frac{k-\kappa_p}{p}} + |\xi|/2^{2n}.$$

But $\#\mathcal{C}_n \ll (|\xi|/2^n)^B$, so if $|\xi| = 2^{\gamma n}$ for some $\gamma \in (1,2)$, then this expression simplifies to

e:firstgammaequation

$$|\widehat{\mu}_T(\xi)| \ll 2^{n(-\kappa_F+2\kappa_*-\gamma\kappa_*+(\gamma-1)B/q+(\gamma-1)(\frac{k-\kappa_p}{p}))} + 2^{(\gamma-2)n}. \tag{6}$$

Now, (5) implies that as γ increases, the first term in (6) will decrease but the second term will increase. After some algebraic manipulation, the value of γ at which both expressions are approximately equal is

$$\gamma_p := \frac{\kappa_F + \frac{B}{q} + \frac{k-\kappa_p}{p} - 2\kappa_* - 2}{\frac{B}{q} + \frac{k-\kappa_p}{p} - \kappa_* - 1}.$$

One can verify from (4) and $\kappa_F \leqslant \kappa_*$ that indeed $\gamma_p \in (1,2)$. Then $|\widehat{\mu}_T(\xi)| \ll 2^{(\gamma_p-2)n} = |\xi|^{(2-\gamma_p)/\gamma_p}$. But another algebraic manipulation gives $(2-\gamma_p)/\gamma_p = \sigma_p$. $\square$

Remark 4.3. *If the assumptions of Theorem 4.2 holds, then the value of $p \in [1,2]$ which maximises σ_p and gives the best bound for Fourier decay is the value of $p \in [1,2]$ which minimises $\frac{B}{q} + \frac{k-\kappa_p}{p}$.*

5. Qualitative Fourier decay and proof of Theorem 1.7

sec: qual

Definition 5.1. *Let μ be a self-similar measure. Let $\mathbf{f}\colon \mathbb{R}^k \to \mathbb{R}^d$ be continuous. We say that it twists μ if $\mathbf{f}_*\mu$ has polynomial Fourier decay (also called power Fourier decay), i.e. there exists $\sigma, C > 0$ such that $|\widehat{\mathbf{f}_*\mu}(\boldsymbol{\xi})| \leqslant C|\boldsymbol{\xi}|^{-\sigma}$ for all $\boldsymbol{\xi} \in \mathbb{R}^k \setminus \{0\}$ with $\|\boldsymbol{\xi}\| \geqslant 1$.*

It is now useful to look at an obvious obstruction to the twisting property. Consider the map $\mathbf{f}\colon (x_1, \dots, x_k) \mapsto (f_1(\mathbf{x}), \dots, f_d(\mathbf{x}))$. Let $\mathbf{v} \in \mathbb{R}^d$. Then for each measure μ on $\mathbb{R}^d, \xi \in \mathbb{R}$ we have

$$\widehat{\mathbf{f}_*\mu}(\xi\mathbf{v}) = \int_{\mathbb{R}^k} e^{-2\pi i (\mathbf{f}(\mathbf{x}), \xi\mathbf{v})} d\mu(\mathbf{x}).$$

To simplify the situation, we let $k = 1$. Suppose that for some non-zero vector $\mathbf{v} \in \mathbb{R}^d$, and real numbers a, b,

$$v_1 f_1(x) + \dots + v_d f_d(x) = a + bx$$

for all $x \in \mathbb{R}$. Then we have

$$\widehat{\mathbf{f}_*\mu}(\xi\mathbf{v}) = \int_{\mathbb{R}} e^{-2\pi i \xi (a+bx)} d\mu(x) = e^{-2\pi i \xi a}\widehat{\mu}_b(\xi)$$

for some μ_b which is a linearly scaled copy of μ. Then as long as μ does not have polynomial Fourier decay, nor does $\mu_{\mathbf{f}}$ along the direction $\mathbf{v}$. More generally, if for some $\mathbf{v}$, the function $f_{\mathbf{v}} = \mathbf{f} \cdot \mathbf{v}$ is an affine function, then $\mu_{\mathbf{f}}$ may not have Fourier decay along the direction $\mathbf{v}$. Notice that if $\mathbf{f}$ is analytic and $f_{\mathbf{v}}$ is affine for some $\mathbf{v}$, then the graph of $\mathbf{f}$ is contained in some proper affine subspace and $\mathbf{f}$ degenerates, recall Lemma 3.8. Thus a degenerate function may not twist some self-similar measures. In fact, the situation is more delicate. In fact, if $f_{\mathbf{v}}$ for some $\mathbf{v}$ is affine over some affine subspace $L \subset \mathbb{R}^k$, then $\mathbf{f}$ may not twist some self-similar measures which are supported in L.

We pose the following conjecture which completely characterises the twisting property.

conj:pushforward

Conjecture 5.2. *Let μ be a self-similar measure on $\mathbb{R}^k$. Let $U \subseteq \mathbb{R}^k$ be an open neighbourhood of* $\operatorname{supp}(\mu)$ *and let $\mathbf{f}\colon U \to \mathbb{R}^d$ be analytic. Then f twists μ unless μ is supported on some affine subspace $L \subseteq \mathbb{R}^k$ and $f_{\mathbf{v}}$ restricted to L is an affine function for some $\mathbf{v} \in \mathbb{S}^{d-1}$.*

In order to work towards this conjecture, it is necessary to recall the definition of polynomial Fourier decay on average from Definition 1.10. We defer the detailed study of this property to later (Section 6). Here, we note that all non-expanding irreducible self-similar measures have this property (noting that the non-expanding condition is automatically satisfied if $k \in \{1, 2\}$ or the IFS is rotation-free), as do all irreducible self-similar measures with the strong separation condition, see Section 6.

Our results towards Conjecture 5.2 concern mostly non-degenerate functions. For this reason, we will also require that μ does not give positive measure to proper subspaces which means that it is irreducible. The main results in this section are as follows.

thm: NE qual

Theorem 5.3. *Let μ be a polynomially Fourier decaying on average and non-expanding self-similar measure on $\mathbb{R}^k$. Let $U \subset \mathbb{R}^k$ be an open neighbourhood of* $\operatorname{supp}(\mu)$ *and let $\mathbf{f}\colon U \to \mathbb{R}^d$ be real analytic and non-degenerate. Then f twists μ.*

Theorem 1.7 follows immediately from Theorem 5.3 together with the fact from Section 6 that all non-expanding irreducible self-similar measures have polynomial Fourier decay on average. We therefore see that Conjecture 5.2 is true under the additional assumption that μ is non-expanding.

Our next result removes this non-expanding condition at the expense of adding an assumption on $\mathbf{f}$ that is slightly stronger than non-degeneracy.

thm: qual

Theorem 5.4. *Let μ be a polynomially Fourier decaying on average self-similar measure on $\mathbb{R}^k$. Let $U \subset \mathbb{R}^k$ be an open neighbourhood of* $\mathrm{supp}(\mu)$ *and let $\mathbf{f}\colon U \to \mathbb{R}^d$ be real analytic and such that $|P_{\mathbf{v}}|$ is not constant for any $\mathbf{v} \in \mathbb{S}^{d-1}$. Then $\mathbf{f}$ twists μ.*

Here $P_{\mathbf{v}}\colon U \to \mathbb{R}^k$ is the analytic function with $P_{\mathbf{v}}(\mathbf{x}) = (\nabla f_{\mathbf{v}}(\mathbf{x}))^T(1)$, and $|\cdot|$ denotes the Euclidean norm. The geometric significance of this function will be revealed later. Theorem 5.4 is not tight. The condition that $|P_{\mathbf{v}}|$ is not constant is needed in the proof but may not be an actual obstruction. For $k > 1$, the function $f\colon \mathbb{R}^k \to \mathbb{R}$, $f(\mathbf{x}) = |\mathbf{x}|$, whose graph is a cone, is an example of a function for which $|P_{\mathbf{v}}|$ is constant. Of course, f is not analytic, but we can restrict f to a subset not containing the origin to obtain an analytic and non-degenerate map. On the other hand, if $f_{\mathbf{v}}$ is a linear function (f being degenerate), then $P_{\mathbf{v}}$ is constant and therefore the same holds for $|P_{\mathbf{v}}|$. We see that. in general, the condition of the non-constancy of $|P_{\mathbf{v}}|$ for all $\mathbf{v} \in \mathbb{S}^{d-1}$ is stronger than that of the non-degeneracy for $\mathbf{f}$.

5.1. **Proof of Theorem 5.3.** Recall that $\mathbf{f}\colon U \to \mathbb{R}^d$ is real analytic. For each $\mathbf{v} \in \mathbb{S}^{d-1}$ we defined $f_{\mathbf{v}} = \mathbf{f} \cdot \mathbf{v}$. Next, for $\xi > 0$, we define $P_{\xi\mathbf{v}}\colon \mathbf{x} \mapsto (\nabla f_{\mathbf{v}}(\mathbf{x}))^T(\xi) \in \mathbb{R}^k$. We see that $P_{\mathbf{v}}$ is real analytic. For each $\mathbf{y} \in \mathbb{R}^k$, consider the set

$$E_{\mathbf{v},\mathbf{y}} = P_{\mathbf{v}}^{-1}(\mathbf{y}).$$

We see that $E_{\mathbf{v},\mathbf{y}}$ is a (not necessarily proper/non-singular) analytic variety of $\mathbb{R}^k$. Similarly, we use $E_{\mathbf{v}}$ to denote the points $\mathbf{x} \in \mathbb{R}^k$ such that the map $f_{\mathbf{v}}$ is degenerate, i.e. the shape operator $dG_{f_{\mathbf{v}}}$ has rank zero. Here, $G_{f_{\mathbf{v}}}(\mathbf{x})$ is the unit normal vector to the tangent space of the graph of $f_{\mathbf{v}}$ at $(\mathbf{x}, f_{\mathbf{v}}(\mathbf{x}))$. In general, $dG_{g_{\mathbf{v}}}(\mathbf{x})$ has orthogonal eigenspaces (possibly with eigenvalue zero). Thus outside of $E_{\mathbf{v}}$, $dG_{g_{\mathbf{v}}}$ has a positive rank, thus its smallest non-zero eigenvalue is well-defined.

Let $K \subset \mathbb{R}^k$ be compact. Because of the analyticity (of $f_{\mathbf{v}}, P_{\mathbf{v}}$ etc.), we see that there are an integer $l \geqslant 1$, numbers $C > 0$ so that for $0 < \delta < 1$, $\mathbf{y} \in \mathbb{R}^k$,

eqn: sep

$$\mathbf{x} \in K, d(\mathbf{x}, E^{\delta}_{\mathbf{v}} \cup E^{\delta}_{\mathbf{v},\mathbf{y}}) > \delta \implies d(P_{\mathbf{v}}(\mathbf{x}), \mathbf{y}) \geqslant C\delta^l. \tag{7}$$

As long as K is fixed, the numbers l, C can be chosen independently w.r.t $\mathbf{v}$.

We first deal with the degenerate case. Suppose that for some $\mathbf{v}$, $E_{\mathbf{v}}$ has dimension k. Then $f_{\mathbf{v}}$ is degenerate and it is a linear function. This implies that f itself is degenerate.

Similarly, if for some $\mathbf{v}, \mathbf{y}$ we have $E_{\mathbf{v},\mathbf{y}}$ to have dimension k, then $E_{\mathbf{v},\mathbf{y}}$ is the whole of $\mathbb{R}^k$. In this case we see that $P_{\mathbf{v}}$ is a constant map. This implies that $\nabla f_{\mathbf{v}}$ is a constant map. This implies that the graph of $f_{\mathbf{v}}$ is always normal to a fixed vector. Thus $f_{\mathbf{v}}$ is a linear function. This again implies that f is degenerate.

Next, we consider the case that $E_{\mathbf{v}}, E_{\mathbf{v},\mathbf{y}}$ never has dimension k. Therefore, they are always proper analytic subvarieties of $\mathbb{R}^k$. We take this collection of proper subvarieties as $\mathcal{M}_f$.

thm: gen NE twisters

Theorem 5.5. *Let μ be a polynomially Fourier decaying on average and non-expanding self-similar measure on $\mathbb{R}^k$, and let U be an open neighbourhood of* $\operatorname{supp}(\mu)$. *Let $f\colon U \to \mathbb{R}^d$ be real analytic and non-degenerate. Suppose μ uniformly decays near $\mathcal{M}_f$, then $f_*\mu$ has polynomial Fourier decay.*

Here, μ uniformly decays near $\mathcal{M}_f$ means that for some numbers $C, \eta > 0$, for all $\mathbf{v} \in \mathbb{S}^{d-1}, \mathbf{y} \in \mathbb{R}^k, \delta > 0$,

$$\mu(E^\delta_{\mathbf{v}}), \mu(E^\delta_{\mathbf{v},\mathbf{y}}) \leqslant C\delta^\eta.$$

Proof. Consider the map $T\colon \mathbf{x} \mapsto (\mathbf{x}, \mathbf{f}(\mathbf{x}))$. Consider the pushed-forwarded measure $\mu_T = T_*\mu$. Observe that for each $\boldsymbol{\xi} \in \{(0,\ldots,0) \in \mathbb{R}^k\} \times \mathbb{R}^d$,

$$\widehat{\mu}_T(\boldsymbol{\xi}) = \widehat{\mu}_f(\boldsymbol{\xi}_d)$$

where $\boldsymbol{\xi}_d$ is the last d components of $\boldsymbol{\xi}$.

For each $n > 1000$, consider the dyadic decomposition Λ_{D_n} for $D_n \in \mathcal{D}_n$. We obtain

$$\begin{aligned}\widehat{\mu}_T(\boldsymbol{\xi}) &= \int e^{-2\pi i(\mathbf{x},\boldsymbol{\xi})} d\mu_T(\mathbf{x})\\ &= \sum_{D_n\in\mathcal{D}_n} \sum_{\mu_{\omega,n}:\mu_{\omega,n}\sim D_n} \int e^{-2\pi i(\mathbf{x},\boldsymbol{\xi})} d\mu_{\omega,n,T}(\mathbf{x}).\end{aligned}$$

where $\mu_{\omega,n,T} = T_*\mu_{\omega,n}$. Our aim is to estimate for each $\mu_{\omega,n}$ the integral

$$\int e^{-2\pi i(\mathbf{x},\boldsymbol{\xi})} d\mu_{\omega,n,T}(\mathbf{x}).$$

Let $\Gamma_{\mathbf{f}} = \{(\mathbf{x}, \mathbf{f}(\mathbf{x})) : \mathbf{x} \in \mathbb{R}^k\}$ be the graph of $\mathbf{f}$. Let $T_{\omega,n}$ be the (k-dimensional) tangent subspace of $\Gamma_{\mathbf{f}}$ at $(\mathbf{x}_0, \mathbf{f}(\mathbf{x}_0))$ for $\mathbf{x}_0$ being the centre of D_n. Consider the image measure $\mu'_{\omega,n,T}$ obtained by pushing

$\mu_{\omega,n}$ to $T_{\omega,n}$. Observe that as long as $|\xi| \leqslant 2^{2n}$,

$$\begin{aligned}&\left|\int e^{-2\pi i(\mathbf{x},\boldsymbol{\xi})} d\mu_{\omega,n,T}(\mathbf{x}) - \int e^{-2\pi i(\mathbf{x},\boldsymbol{\xi})} d\mu'_{\omega,n,T}(\mathbf{x})\right|\\ =&\left|\int e^{-2\pi i((\mathbf{x},f(\mathbf{x})),\xi)} d\mu_{\omega,n}(\mathbf{x}) - \int e^{-2\pi i((\mathbf{x},f(\mathbf{x}_0)+\nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)),\xi)} d\mu_{\omega,n}(\mathbf{x})\right|\\ =&O(|\mu_{\omega,n}||\boldsymbol{\xi}|/2^{2n})\end{aligned}$$

where $|\mu_{\omega,n}|$ is the total mass of $\mu_{\omega,n}$. The $O(\cdot)$ symbol is absolute (i.e. it does not depend on ω, n, ξ). This is because $\mathbf{f}$ is analytic and μ is compactly supported. Next, consider the integral

$$\int e^{-2\pi i(\mathbf{x},\boldsymbol{\xi})} d\mu'_{\omega,n,T}(\mathbf{x}).$$

We write $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \boldsymbol{\xi}_\parallel$ where $\boldsymbol{\xi}_\perp$ is perpendicular to $T_{\omega,n}$ and $\boldsymbol{\xi}_\parallel$ is parallel to $T_{\omega,n}$. We see that

$$\begin{aligned}\left|\int e^{-2\pi i(\mathbf{x},\boldsymbol{\xi})} d\mu'_{\omega,n,T}(\mathbf{x})\right| &= \left|\int e^{-2\pi i(\mathbf{x}-\mathbf{x}_0,\boldsymbol{\xi})} d\mu'_{\omega,n,T}(\mathbf{x}-\mathbf{x}_0)\right|\\ &= \left|\int e^{-2\pi i(\mathbf{x}-\mathbf{x}_0,\boldsymbol{\xi}_\perp+\boldsymbol{\xi}_\parallel)} d\mu'_{\omega,n,T}(\mathbf{x}-\mathbf{x}_0)\right|\\ &= \left|\int e^{-2\pi i(\mathbf{x}-\mathbf{x}_0,\boldsymbol{\xi}_\parallel)} d\mu'_{\omega,n,T}(\mathbf{x}-\mathbf{x}_0)\right|\\ &= \left|\widehat{\mu}_{\omega,n}(\boldsymbol{\xi}_{\omega,n})\right|\end{aligned}$$

where $\boldsymbol{\xi}_{\omega,n}$ is the projection of $\boldsymbol{\xi}_\parallel$ to $\mathbb{R}^k \times \{(0,\dots,0)\}$ along the direction that is orthogonal to $T_{\omega,n}$. In other words, $\boldsymbol{\xi}_{\omega,n}$ is the vector in $\mathbb{R}^k \times \{(0,\dots,0)\}$ with the property that

$$(\boldsymbol{\xi}_{\omega,n} - \boldsymbol{\xi}) \perp T_{\omega,n}.$$

We can regard $\boldsymbol{\xi}_{\omega,n}$ as an element in $\mathbb{R}^k$.

Each $\mu_{\omega,n}$ is a scaled and rotated copy of the original self-similar measure μ. More precisely, we have $\widehat{\mu}_{\omega,n}(\boldsymbol{\xi}_{\omega,n}) = |\mu_{\omega,n}|\widehat{\mu}(r_{\omega,n}O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}))$ for some scaling $r_{\omega,n} \asymp 2^{-n}$ and rotation $O_{\omega,n} \in \mathbb{O}(k)$.

From here we have the following estimate,

$$\widehat{\mu}_T(\boldsymbol{\xi}) = \sum_{g\in L_n} \sum_{D_n\in\mathcal{D}_n} \sum_{\mu_{\omega,n}\in g, \mu_{\omega,n}\sim D_n} |\mu_{\omega,n}||\widehat{\mu}(r_{\omega,n}O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}))| + O(|\xi|/2^{2n}).$$

Here, L_n is the following set of probability weights and scaling ratios of $\mu_{\omega,n}$,

$$\begin{aligned}L_n = \{&(p,r,R) : p,r,R \text{ are the probability weight, scaling ratio}\\ &\text{and rotation for some } \mu_{\omega,n}\}.\end{aligned}$$

For each $g \in L_n$, we write $\mu_{\omega,n} \sim g$ if it has the indicated (by g) probability weight and scaling ratio. Notice that $\#L_n \ll 2^{\varepsilon n}$ for each $\varepsilon > 0$ because of the non-expanding property.

Therefore we see that

$$\begin{aligned}|\widehat{\mu_T}(\boldsymbol{\xi})| &\ll \sum_{D_0 \in \mathcal{C}_n} \sum_{g \in L_n} \sum_{\mu_{\omega,n} \in g, r_{\omega,n} O_{\omega,n} \boldsymbol{\xi}_{\omega,n} \in D_0} |\mu_{\omega,n}| |\widehat{\mu}(r_{\omega,n} O_{\omega,n} \boldsymbol{\xi}_{\omega,n})| + O(|\boldsymbol{\xi}|/2^{2n}) \\ &\ll \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 \in \mathcal{C}_n} \sum_{\mu_{\omega,n} \in g, r_{\omega,n} O_{\omega,n} \boldsymbol{\xi}_{\omega,n} \in D_0} |\widehat{\mu}(r_{\omega,n} O_{\omega,n} \boldsymbol{\xi}_{\omega,n})| + O(|\boldsymbol{\xi}|/2^{2n}).\end{aligned}$$

Here $2^{-\kappa_g n} = |\mu_{\omega,n}|$ for $\mu_{\omega,n} \sim g$ indicates the probability weight of those $\mu_{\omega,n}$. We recall that $\mathcal{C}_n$ is the collection of all D_0 (dyadic cubes of unit scale) that intersects $\{\xi_{D_n} r_n\}_{D_n \in \mathcal{D}_n}$ (recall ξ_{D_n}). From here we write

$$N_{D_0}(g) = \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}) r_{\omega,n} \in D_0\}.$$

Here $O_{\omega,n}$ is the rotation part of $\mu_{\omega,n}$. It is fixed according to g as well. Then we see that

$$|\widehat{\mu_T}(\boldsymbol{\xi})| \ll \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 \in \mathcal{C}_n} N_{D_0}(g) \int_{D_0} |\widehat{\mu}(\boldsymbol{\xi}')| d\boldsymbol{\xi}' + O(|\boldsymbol{\xi}|/2^{2n}).$$

For each $\varepsilon > 0$ we can find a $\delta > 0$ so that

$$\begin{aligned}&\sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 \in \mathcal{C}_n} N_{D_0}(g) \int_{D_0} |\widehat{\mu}(\boldsymbol{\xi}')| d\boldsymbol{\xi}' \\ \ll &\sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 \in \mathcal{C}_n} N_{D_0}(g) |\boldsymbol{\xi}/2^n|^{-\delta} \\ &+ \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 : |\widehat{\mu}(\boldsymbol{\xi}')| \gg |\boldsymbol{\xi}/2^n|^{-\delta}, \boldsymbol{\xi}' \in D_0} N_{D_0}(g) + O(|\boldsymbol{\xi}|/2^{2n}).\end{aligned}$$

Observe that

$$\begin{aligned}\sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 \in \mathcal{C}_n} N_{D_0}(g) &\ll \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g\} \\ &\ll 1.\end{aligned}$$

On the other hand (because of the polynomial Fourier decay on average) we have

$$\sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 : |\widehat{\mu}(\boldsymbol{\xi}')| \gg |\boldsymbol{\xi}/2^n|^{-\delta}, \boldsymbol{\xi}' \in D_0} N_{D_0}(g) \ll \sum_{g \in L_n} \frac{1}{2^{\kappa_g n}} 2^{\varepsilon n} \max_{D_0} N_{D_0}(g).$$

We claim that for some $\eta > 0$,

$$N_{D_0}(g) = \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, O_{\omega,n}(\boldsymbol{\xi}_{\omega,n})r_{\omega,n} \in D_0\} \quad \text{eqn: preimage} \tag{8}$$
$$\ll \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, \boldsymbol{\xi}_{\omega,n} \in r_g^{-1}R_g^{-1}(D_0)\} \tag{9}$$
$$\ll \left(\frac{2^n}{|\boldsymbol{\xi}|}\right)^{\eta} 2^{\kappa_g n}. \tag{10}$$

Here r_g is the scaling ratio indicated by g and R_g is the rotation indicated by g. From this claim, we obtain

$$\sum_{g\in L_n} \frac{1}{2^{\kappa_g n}} \sum_{D_0 : |\widehat{\mu}(\boldsymbol{\xi}')| \gg |\boldsymbol{\xi}/2^n|^{-\delta}, \boldsymbol{\xi}' \in D_0} N_{D_0}(g) \ll 2^{2\varepsilon n} \left(\frac{2^n}{|\boldsymbol{\xi}|}\right)^{\eta}.$$

From here, we have

$$|\widehat{\mu_T}(\boldsymbol{\xi})| \ll |\boldsymbol{\xi}/2^n|^{-\delta} + 2^{2\varepsilon n}(2^n/|\boldsymbol{\xi}|)^{\eta} + O(|\boldsymbol{\xi}|/2^{2n}).$$

This implies that

$$|\widehat{\mu_T}(\boldsymbol{\xi})| \ll |\boldsymbol{\xi}|^{-\sigma}$$

for some $\sigma > 0$.

We now prove the claim. We need the decay property for μ near $\mathcal{M}_f$. The idea is that the support of each $\mu_{\omega,n} \in g$ so that $\boldsymbol{\xi}_{\omega,n} \in r_g^{-1}R_g^{-1}(D_0)$ is contained in a thin neighbourhood of an analytic submanifold of $\mathbb{R}^k$. More precisely, recall the map $P_{\xi\mathbf{v}} \colon \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$P_{\xi\mathbf{v}} \colon \mathbf{x} \in \mathbb{R}^k \mapsto (\nabla\mathbf{f}(\mathbf{x}))^T(\xi\mathbf{v}) = (\nabla\mathbf{f}_{\mathbf{v}}(\mathbf{x}))^T(\xi)$$

where $\mathbf{v} \in \mathbb{S}^{d-1}$, and $\xi \geqslant 0$ is such that $\boldsymbol{\xi} = \xi\mathbf{v}$, and $(\nabla\mathbf{f}(\mathbf{x}))^T \colon \mathbb{R}^d \to \mathbb{R}^k$ denotes the transpose of the linear map describing by the derivative of $\mathbf{f}$ at $\mathbf{x}$. This map has the property that for $\mathbf{x}_0 \in \mathbb{R}^k$ being the center of D_n as in the beginning of this proof, and $\boldsymbol{\xi} = \xi\mathbf{v}$, the corresponding $\boldsymbol{\xi}_{\omega,n}$ is precisely $P_{\xi\mathbf{v}}(\mathbf{x}_0)$. Thus, the branches $\mu_{\omega,n}$ so that $\boldsymbol{\xi}_{\omega,n} \in R_g^{-1}(D_0)$ is located in the inverse image $P_{\xi\mathbf{v}}^{-1}(D_0')$ for some cube D_0' of size $\asymp 1/r_g \asymp 2^n$. That is to say, $P_{\mathbf{v}}^{-1}(D'')$ for some cube D'' of size $\asymp 2^n/\xi$.

Now, we can consider $E_{\mathbf{v}}, E_{\mathbf{v},\mathbf{y}}$ for a suitable $\mathbf{y} \in \mathbb{R}^k$ (the center of D''). We can use the Property (7) together with the decay property of μ on $\mathcal{M}_f$ (in particular $E_{\mathbf{v}}, E_{\mathbf{v},\mathbf{y}}$) to see that for some $\eta > 0$, the total μ mass of all above discussed branches $\mu_{\omega,n}$ is

$$\ll (2^n/\xi)^{\eta}$$

this implies that the number of such branches is

$$\ll (2^n/\xi)^{\eta} 2^{\kappa_g n}.$$

This proves the claim. The proof is finished. □

5.1.1. *The decay property near $\mathcal{M}_f$.* The decay property does not come for free. For example, consider a cylindrical surface in $\mathbb{R}^3$ (a graph of some analytic $f\colon \mathbb{R}^2 \to \mathbb{R}$ restricted to some compact subset). It is non-degenerate but contains lines. Therefore, if μ would be supported on some certain line, then $f^*\mu$ would be a linear copy of μ. This indicates that μ cannot decay at all near some certain $E_{\mathbf{v},\mathbf{y}}$.

We want to prove the following sufficient condition which will prove Theorem 5.3 together with Theorem 5.5.

lma: decay

Lemma 5.6. *Let μ be an irreducible self-similar measure on $\mathbb{R}^k$. Let $U \subseteq \mathbb{R}^k$ be an open neighbourhood of $\operatorname{supp}(\mu)$ and let $f\colon U \to \mathbb{R}^d$ be real analytic and non-degenerate. Then μ uniformly decays near $\mathcal{M}_f$.*

For this lemma, we need the following result which is a special case of this lemma and at the same time an initial step of its proof. For this reason, the proof of the following lemma in fact leads to a stronger result.

lma: red to linear

Lemma 5.7. *Let μ be a self-similar measure on $\mathbb{R}^k$ and let $M \subset \mathbb{R}^k$ be a proper analytic subvariety. Then if $\mu(M) > 0$, then M contains an affine linear subspace and μ gives a positive measure to this linear subspace.*

Proof. Since M is a proper analytic subvariety, it is at most a countable union of analytic submanifolds. We can then assume without loss of generality that M does not have singularities. Assume that μ is irreducible. Then for some small enough $r > 0$, all affine linear subspaces must miss at least one r-branch of μ. As M is analytic, we see that for some small enough $r > 0$, M must miss at least one r-branch of μ. The analyticity of M indicates that inside each r-ball, the portion of M is close to an affine linear subspace. As long as r is small enough, this closeness is uniform across all points on M inside any fixed compact set. This is not a restriction because the support of μ is compact.

We have seen that for some small enough $r > 0$, M misses at least one r-branch of μ. This implies that $\mu(M^r) < \rho$ for some $\rho < 1$. For each r-ball hosting r-branches of μ, we can zoom it to have a unit length and see that M inside this ball will miss at least one r^2-branch of μ. This this holds inside all r-branches of μ, we conclude that

$$\mu(M^{r^2}) \leqslant \rho^2.$$

We can iterate the above consideration, and as a result, we see that $\mu(M^r) \ll r^\eta$ for some $\eta > 0$ and $r \in (0,1)$. This contradicts the fact that $\mu(M) > 0$.

To get out of this contradiction, we need that μ gives a positive measure on some affine subspaces. Repeating the above arguments we see that M must contain at least one such affine subspace. In the case when M is an analytic manifold, this implies that M is itself an affine subspace. This finishes the proof. $\square$

Proof of Lemma 5.6. We claim that there is some number $r > 0$ such that all minisets of $\mathcal{M}_f$ omit at least one r-branch of μ (we will precisely formulate and prove this claim below). Given this claim, for each $\delta = r^s$, $s \geqslant 1$, consider $\mu(E^\delta)$ for some $E \in \mathcal{M}_f$. We see that E misses at least one r-branch of μ. This implies that for some $\rho < 1$,

$$\mu(E^r) \leqslant \rho.$$

Next, consider any r-branch of μ that gives a positive measure to E^r, we can zoom in this r-branch and obtain a miniset E_1 of E that misses at least one r-branch of μ again. This implies that

$$\mu(E^{r^2}) \leqslant \rho^2.$$

Iterating this procedure we obtain that $\mu(E^\delta) \leqslant C\delta^\eta$ for some $C, \eta > 0$. This concludes the proof of the lemma.

We now prove the claim. Let $r > 0, \mathbf{x} \in \mathbb{R}^k, \mathbf{v} \in \mathbb{S}^{d-1}, \mathbf{y} \in \mathbb{R}^k$. We consider the miniset $M_{\mathbf{v},\mathbf{x},\mathbf{y},r}$ be the unit scale copy of the set $B_r(\mathbf{x}) \cap E_{\mathbf{v},\mathbf{y}}$. More precisely, $M_{\mathbf{v},\mathbf{x},\mathbf{y},r}$ is $L(B_r(\mathbf{x}) \cap E_{\mathbf{v},\mathbf{y}})$ where L is a scaling and translation map that maps $B_r(\mathbf{x})$ to the unit ball and $\mathbf{x}$ to $\mathbf{0}$.

We see that for some sequence $\delta_i \to 0$, we can find $r_1, \mathbf{x}_i, \mathbf{y}_i, \mathbf{v}_i$ so that

$$M^{2\delta_i}_{\mathbf{v}_i,\mathbf{x}_i,\mathbf{y}_i,r_i} \supset \mathrm{supp}(\mu).$$

Assume that r_i's are bounded away from zero, then by taking a subsequence if necessary, we obtain $\lim_{i\to 0} M^{2\delta_i}_{\mathbf{v}_i,\mathbf{x}_i,\mathbf{y}_i,r_i} \supset \mathrm{supp}(\mu)$. The limit is taken under the Hausdorff metric. Since $\mathbf{v}_i, \mathbf{x}_i, \mathbf{y}_i, r_i$ are all ranged in compact sets, we can take the limits and obtain that

$$M_{\mathbf{v},\mathbf{x},\mathbf{y},r} \supset \mathrm{supp}(\mu)$$

for some $\mathbf{v}, \mathbf{x}, \mathbf{y}, r > 0$. Since $M_{\mathbf{v},\mathbf{x},\mathbf{y},r}$ is a proper analytic subvariety, we see that μ must be (affinely) reducible and $M_{\mathbf{v},\mathbf{x},\mathbf{y},r}$ is linear. This is Lemma 5.7.

Thus we must have $r_i \to 0$. In this case, $M_\infty = \lim_{i\to 0} M^{2\delta_i}_{\mathbf{v}_i,\mathbf{x}_i,\mathbf{y}_i,r_i}$ may not be a miniset of $\mathcal{M}_f$. However, it is still a tangent set of $\mathcal{M}_f$. Given all the analytical regularities, it is plausible that M_∞ preserves some regularities and it cannot contain $\mathrm{supp}(\mu)$.

We now study the "minisets" of $f_{\mathbf{v}}$. For each $\mathbf{v}, \mathbf{x}, r$, consider the function

$$M_{\mathbf{v},\mathbf{x},r} f(\mathbf{z}) = f_{\mathbf{v}}(\mathbf{x} + r\mathbf{z}) - f_{\mathbf{v}}(\mathbf{x}) - L_1 f(r\mathbf{z})$$

where $L_1 f(rz)$ is the linear part of $f_{\mathbf{v}}(\mathbf{x}+r\mathbf{z})-f_{\mathbf{v}}(\mathbf{x})$. By the analyticity, we can write it as a power series

$$M_{\mathbf{v},\mathbf{x},r}f(\mathbf{z}) = \sum_I r^{|I|} a_I(\mathbf{x},\mathbf{v}) z^I$$

that convergent uniformly in some open neighbourhood of the origin. The radius of this open ball can be taken to be uniform across $\mathbf{v}, \mathbf{x}, 0 < r < 1$. Here I is a multi-index to indicate monomials z^I, and $|I|$ is the total degree of z^I. Coefficients $a_I(\mathbf{x},\mathbf{v})$ are coefficients of the function $f_{\mathbf{v}}$ at $\mathbf{x}$. They are thus uniformly bounded (for each fixed I) across $\mathbf{x}, \mathbf{v}$.

We claim that there is some $c > 0$ and $D > 1$ so that $|a_I(\mathbf{x},\mathbf{v})| > c$ for some $|I| < D$ and for all $\mathbf{x}, \mathbf{v}$. If this is not the case, then for some sequence $\mathbf{x}_i, \mathbf{v}_i$ and $c_i \to 0, D_i \to \infty$ we have

$$|a_I(x_i, v_i)| < c_i$$

for all $1 < |I| < D_i$. Thus for each fixed $D > 1$, we see that after taking the limit,

$$a_I(\mathbf{x},\mathbf{v}) = 0$$

for all $|I| < D$. We can let $D \to \infty$ and retaking the $\mathbf{x}, \mathbf{v}$ if necessary,

$$a_I(\mathbf{x},\mathbf{v}) = 0$$

for all $|I| > 1$. This implies that $f_{\mathbf{v}}$ a linear power series near $\mathbf{x}$. Because of the analyticity, this happens only if $f_{\mathbf{v}}$ is a linear function which was ruled out because of the non-degeneracy.

From the above argument, we see that for some $D > 1$,

$$\lim_{i\to\infty} M_{\mathbf{v}_i,\mathbf{x}_i,r_i} f(\mathbf{z})/r_i^D = P_D(\mathbf{z}) + \lim_{r\to 0} O(r) = P_D(\mathbf{z})$$

for some non-zero homogeneous polynomial P_D of degree D.

Next, we take a closer look at $E_{\mathbf{v}}, E_{\mathbf{v},\mathbf{y}}$. Observe that over each connected component of $E_{\mathbf{v}}$ or $E_{\mathbf{v},\mathbf{y}}$ the graph of $f_{\mathbf{v}}$ must be contained in some affine hyperplane and the graph of $f_{\mathbf{v}}$ is tangent to this affine hyperplane. To see this, take $E_{\mathbf{v},\mathbf{y}}$. Since $\nabla f_{\mathbf{v}}$ is constant in $E_{\mathbf{v},\mathbf{y}}$, we see that the graph of $f_{\mathbf{v}}$ over $E_{\mathbf{v},\mathbf{y}}$ has a constant normal vector. This implies the claim. Therefore, for each $\mathbf{v}, \mathbf{x}, r, \mathbf{y}$, locally around $\mathbf{x}$, the graph of $f_{\mathbf{v}}$ over $E_{\mathbf{v},\mathbf{y}}$, if not empty, is the intersection of the graph of $f_{\mathbf{v}}$ and the tangent plane at a point on the graph of $f_{\mathbf{v}}$ whose $\mathbb{R}^k$ component $\tilde{\mathbf{x}}$ is at most r-away from $\mathbf{x}$. We can then consider the miniset $M_{\mathbf{v},\tilde{\mathbf{x}},\mathbf{y},r}$ in addition to $M_{\mathbf{v},\mathbf{x},\mathbf{y},r}$.

Recall the sequence $M_{\mathbf{v}_i,\mathbf{x}_i,\mathbf{y}_i,r_i}$. Assume that none of the terms is empty. Then for each i, we have $M_{\mathbf{v}_i,\tilde{\mathbf{x}}_i,\mathbf{y}_i,r_i}$. Denote $\tilde{M}_\infty$ as the limit of the second sequence. From the construction, it is almost a translated copy of M_∞. More precisely, the limit of $2M_{\mathbf{v}_i,\tilde{\mathbf{x}}_i,\mathbf{y}_i,2r_i}$ contains a translated copy of M_∞. Thus the geometry of $\tilde{M}_\infty$, being a subset of $2M_{\mathbf{v}_i,\tilde{\mathbf{x}}_i,\mathbf{y}_i,2r_i}$, can be regarded as a translated copy of M_∞.

Since the set of affine hyperplanes with compact translations form a compact set. We see that the limit set $\tilde{M}_\infty$ is equal to the solution set (which is an analytic variety)

$$\{z : P_D(z) = L(z)\}$$

for some affine linear function $L : \mathbb{R}^k \to \mathbb{R}$. If it is not a proper subvariety, then it must be the whole of $\mathbb{R}^k$. Then we have $P_D = L$ has functions. By the real analyticity, $P_D = L$ as power series which is a contradiction. This implies that M_∞, being almost a translated copy of $\tilde{M}_\infty$, is also a proper analytic subvariety of $\mathbb{R}^k$. Then M_∞ cannot contain $\mathrm{supp}(\mu)$ unless M_∞ is linear and $\mathrm{supp}(\mu)$ is reducible. □

5.2. **Proof of Theorem 5.4.** We show in this section that if f is such that $|P_\mathbf{v}|$ is not constant for any $\mathbf{v}$, then f twists all irreducible self-similar measures regardless of the rotation group.

If $P_\mathbf{v}$ is not constant, then for each $r \in [0, \infty)$, the set

$$E_{\mathbf{v},r} = \{|P_\mathbf{v}(\mathbf{x})| = r\}$$

is an analytic variety. Its dimension cannot be k, otherwise $|P_\mathbf{v}|^2$ (which is analytic) is a constant function. Those subvarieties form a collection $\mathcal{M}^K_f$ (K for "cone"). We have the following result.

Theorem 5.8. *Let μ be a polynomially Fourier decaying on average self-similar measure on $\mathbb{R}^k$. Let $f\colon \mathbb{R}^k \to \mathbb{R}^d$ be real analytic and such that $|P_\mathbf{v}|$ is not constant for any $\mathbf{v} \in \mathbb{S}^{d-1}$. Suppose μ uniformly decays near $\mathcal{M}^K_f$, then $f_*\mu$ has polynomial Fourier decay.*

Proof. The proof is very similar to that of Theorem 5.5. We need to rectify the difficulty that our self-similar system may now have rotations.

We can follow the proof of Theorem 5.5 until the construction of L_n. Because the rotation group is not non-expanding we must not include it in L_n. The new L_n is defined as

$$L_n = \{(p, r) : p, r \text{ is the probability weight and scaling ratio for some } \mu_{\omega,n}\}.$$

Notice that $\#L_n$ grows only polynomially w.r.t. n. We can then follow the rest of the proof of Theorem 5.5 until we hit the construction of $N_{D_0}(g)$. It is now

$$N_{D_0}(g) = \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}) r_{\omega,n} \in D_0\}.$$

We can no longer control the rotation part $O_{\omega,n}$. Nonetheless, we claim that

$$\begin{aligned} N_{D_0}(g) &= \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, O_{\omega,n}(\boldsymbol{\xi}_{\omega,n}) r_{\omega,n} \in D_0\} && (11)\\ &\ll \#\{\mu_{\omega,n} : \mu_{\omega,n} \in g, \boldsymbol{\xi}_{\omega,n} \in A_{D_0/r_g}\} && (12)\\ &\ll \left(\frac{2^n}{|\boldsymbol{\xi}|}\right)^{\eta} 2^{\kappa_g n}. && (13)\end{aligned}$$

Here r_g is the scaling ratio indicated by g and A_{D_0/r_g} is the set of points $\boldsymbol{\xi}$ in $\mathbb{R}^k$ so that $R(\boldsymbol{\xi}) \in D_0/r_g$ for some rotation R. Thus A_{D_0/r_g} is an annulus of thickness $\asymp 1/r_g \asymp 2^n$. This annulus can have an inner radius 0.

Thus, the effect of not controlling the rotation part is that instead of considering preimages of $P_{\xi\mathbf{v}}$ of cubes, we need to consider preimages of $P_{\xi\mathbf{v}}$ of annuli which is much larger than cubes.[7] The condition that $|P_{\mathbf{v}}|$ is not constant for all $\mathbf{v}$ and the fact that μ uniformly decays near $\mathcal{M}_f^K$ prove the claim. This is similar to the last part of the proof of Theorem 5.5. This finishes the proof. □

5.2.1. *The decay property near $\mathcal{M}_f^K$.* As the non-expanding case, we now consider the decay property near $\mathcal{M}_f^K$. We have the following result which will finish the proof of Theorem 5.4.

lma: cone decay

Lemma 5.9. *Let μ be an irreducible self-similar measure on $\mathbb{R}^k$. Let $f\colon \mathbb{R}^k \to \mathbb{R}^d$ be real analytic and $|P_{\mathbf{v}}|$ is not constant for all $\mathbf{v}$. Then μ uniformly decays near $\mathcal{M}_f^K$.*

Proof. The proof is very similar to that of Lemma 5.6. In fact, notice that we only need to consider $E_{\mathbf{v},r}$ for $r \in [0, L]$ for some large enough L (therefore it has a compact range). The rest of the proof of Lemma 5.6 can be put here with minor changes. We omit the full details. □

6. Polynomial Fourier decay on average

sec: AFD

6.1. **Background.** In order to deduce Theorem 1.7 from Theorem 5.3, we need to prove that non-expanding irreducible self-similar measures satisfy polynomial Fourier decay on average property from Definition 1.10. Since the Fourier transform of μ is Lipschitz, this property is clearly equivalent to saying that $\{\xi \in \mathbb{R}^k : |\widehat{\mu}(\xi)| \geqslant R^{-\delta}, |\xi| < R\}$ can be covered by $\ll R^{\varepsilon}$ balls in $\mathbb{R}^k$ of radius 1. The natural measure on the middle-third Cantor set, for example, is already known to have polynomial Fourier decay on average, despite being non-Rajchman. It

[7] Cubes are in some sense zero-dimensional objects and annuli are $k-1$ dimensional objects.

is straightforward to see that self-similar measures with polynomial Fourier decay on average are necessarily irreducible, and we strongly conjecture that the converse also holds:

conj:afd

Conjecture 6.1. *Every irreducible self-similar measure on* $\mathbb{R}^k$ *has polynomial Fourier decay on average.*

In this section we will make progress towards this conjecture. The following classes of self-similar measures are already known to have polynomial Fourier decay on average.

- Conjecture 6.1 holds when $k = 1$, i.e. every non-atomic self-similar measure in $\mathbb{R}$ has polynomial Fourier decay on average [74].
- Every irreducible homogeneous self-similar measure without rotations has polynomial Fourier decay on average, see [54, Section 5].
- Every measure with polynomial Fourier decay automatically has polynomial Fourier decay on average. In particular this is the case for every irreducible self-similar measure whose associated transfer operator has a spectral gap, see [45] and [64, Section 4].
- We say that a Borel measure μ on $\mathbb{R}^k$ is uniformly affinely non-concentrated if there exist $C \geqslant 1$ and $\alpha > 0$ such that for all $\delta \in (0,1)$, $x \in \operatorname{supp}\mu$, $0 < r \leqslant 1$, and every proper affine subspace K, we have

  e:localnonconc

  $$\mu(K^{\delta r} \cap B(x,r)) \leqslant C\delta^{\alpha}\mu(B(x,r)), \tag{14}$$

  Khalil [40, Corollary 1.8] proved that every such measure has polynomial Fourier decay on average. This was in fact a step towards proving that the geodesic flow on a geometrically finite locally symmetric space of negative curvature is exponentially mixing with respect to the Bowen–Margulis–Sullivan measure. In fact a weaker non-concentration condition suffices for polynomial Fourier decay on average (see [40, Definition 11.1 and Corollary 11.5]). For irreducible self-similar measures satisfying the OSC, the uniform affine non-concentration condition holds for the measure of maximal dimension (see [17, Theorem 1.5]), but can fail in general (for example for the $(1/3, 2/3)$ measure on $\{x \mapsto x/2, x \mapsto (x+1)/2\}$).

In this section we will provide more such sufficient conditions for μ to have polynomial Fourier decay on average, without assuming separation conditions on μ. There are many reasons why it is useful to verify that a measure has polynomial Fourier decay on average. In particular, the assumption was used in Theorems 5.3,5.4. Moreover, the polynomial Fourier decay on average property has recently been used to prove polynomial Fourier decay for certain Patterson–Sullivan measures in a recent work of Baker, Khalil and Sahlsten [8]. Furthermore, [8, Theorem 1.6] says that for every self-similar measure μ that has polynomial Fourier decay on average and comes from an IFS which has two contraction ratios r, r' such that $\log r/\log r'$ is Diophantine, there exist $C, \eta > 0$ such that $|\widehat{\mu}(\xi)| \leqslant C(\log|\xi|)^{-\eta}$ whenever $|\xi| > 2$. Combining these results with Theorems 6.8 and 6.3 below therefore proves these properties for a wide class of self-similar measures, and in particular the proofs of [8, Corollary 1.7, Theorem 1.9] use Theorem 6.8, see also [8, Remark 1.8].

6.2. **Doubling self-similar measures.** We next show that irreducible and doubling self-similar measures have polynomial Fourier decay on average using a result of Khalil [40]. First, we need the following basic lemma. Given $\gamma > 0$ and $K \subseteq \mathbb{R}^k$ we write K^γ for the open γ-neighbourhood of K.

l:tube

Lemma 6.2. *Let μ be a finite compactly supported Borel measure on $\mathbb{R}^k$ whose support does not lie in any proper affine subspace. Then there exists $\delta, \gamma \in (0, 1)$ such that for all proper affine subspaces K of $\mathbb{R}^k$,*

$$\mu(K^\gamma) \leqslant (1 - \delta)\mu(\mathbb{R}^k).$$

Proof. Assume for contradiction this is false. Then there exists a sequence $(S_n)_{n=1}^\infty$ of $k-1$ dimensional affine subspace of $\mathbb{R}^k$ such that $\mu(S_n^{1/n}) \geqslant \mu(\mathbb{R}^k) - 1/n$. Let B be a ball containing the support of μ. By compactness we can find a subsequence of S_n which converges to some $k-1$ dimensional affine subspace S, and take a further subsequence (call it $(S_{k_n})_{n=1}^\infty$) such that $S_{k_n}^{1/k_n} \cap B \supseteq S_{k_{n+1}}^{1/k_{n+1}} \cap B$ for all $n \in \mathbb{N}$. Therefore since $\mu(S_{k_n}^{1/k_n}) \geqslant \mu(\mathbb{R}^k) - 1/k_n$ for all n, we must have $\mu(S_{k_n}^{1/k_n}) = \mu(\mathbb{R}^k)$ for all n. But this means that $\mu(S) = \mu(\mathbb{R}^k)$, contradicting our assumption that μ is not supported in a proper affine subspace. □

Recall that a finite measure μ is doubling if for all $x \in \mathrm{supp}\mu$ and $r > 0$ there is a constant $D > 0$ such that $\mu(B(x, 2r)) \leqslant DB(x, r)$. Self-similar measures satisfying the strong separation condition (and with arbitrary positive weights) are known to be doubling (see [30, Theorem 7.4.1], for example), so if such a measure is also uniformly affinely non-concentrated then the following theorem will apply to it.

thm:doublingafd

Theorem 6.3. *Let μ be a doubling, irreducible, self-similar measure on $\mathbb{R}^k$. Then μ is uniformly affinely non-concentrated and has polynomial Fourier decay on average.*

Proof. Without loss of generality we may assume that the support of μ is contained in a ball of diameter 1. Let δ, γ be as in Lemma 6.2, and let $r_{\min} \in (0,1)$ be the smallest contraction ratio of the IFS defining μ. Write $\mathcal{F}$ for the set of all proper affine hyperplanes in $\mathbb{R}^k$.

Claim: For all $K \in \mathcal{F}$ and $n \geqslant 1$, if we let $r_n := (r/10) \cdot (\gamma r_{\min}/4)^{n-1}$ then $\mu(K^{r_n} \cap B(x, r + r_n)) \leqslant (1-\delta)^n \mu(B(x, 2r))$.

Proof of claim: The proof goes by induction, and holds for $n = 1$ since μ is a probability measure. Assume that the claim holds for some $n \geqslant 1$. Write $\mu = \sum_{w \in \Omega} p_w f_w \mu$ where Ω is a subset of finite words from the alphabet of the IFS such that contraction ratios of the corresponding compositions f_w all lie in the interval $[r_n r_{\min}/2, r_n/2]$. Let

$$\Omega' := \{w \in \Omega : \operatorname{supp} f_w(\mu) \cap K^{r_{n+1}} \cap B(x, r + r_{n+1}) \neq \varnothing\},$$

noting that if $w \in \Omega'$ then $\operatorname{supp} f_w \mu \subseteq K^{r_n} \cap B(x, r + r_n)$. Now by Lemma 6.2,

$$\begin{aligned} \mu(K^{r_{n+1}} \cap B(x, r + r_{n+1})) &= \sum_{w \in \Omega'} p_w f_w \mu(K^{r_{n+1}} \cap B(x, r + r_{n+1})) \\ &\leqslant \sum_{w \in \Omega'} p_w (1-\delta) f_w \mu(K^{r_n} \cap B(x, r + r_n)) \\ &\leqslant (1-\delta) \mu(K^{r_n} \cap B(x, r + r_n)) \\ &\leqslant (1-\delta)^{n+1} \mu(B(x, 2r)). \end{aligned}$$

This completes the proof of the claim.

It follows from this claim that

$$\begin{aligned} \mu(K^{r_n} \cap B(x, r)) &\leqslant \mu(K^{r_n} \cap B(x, r + r_n)) \\ &\leqslant (1-\delta)^n B(x, 2r) \\ &\leqslant D(1-\delta)^n B(x, r), \end{aligned}$$

where D is the doubling constant of μ, so (14) follows.

We have proved that μ is uniformly affinely non-concentrated, so by Khalil's result [40, Corollary 11.5], μ has polynomial Fourier decay on average. $\square$

6.3. **Non-expanding self-similar measures.** We now work towards the proof of Theorem 6.8, which implies Theorem 1.11.

l:dimslice

Lemma 6.4. *For every irreducible self-similar set $K \subset \mathbb{R}^k$ satisfying the strong separation condition there exists $\varepsilon := \varepsilon(K) > 0$ such that for all affine hyperplanes L of dimension less than k, we have $\overline{\dim}_{\mathrm{B}}(L \cap K) \leqslant \dim K - \varepsilon$.*

Proof. Without loss of generality assume that the diameter of K is 1. Since K is irreducible it is not contained in any affine hyperplane. Therefore by standard compactness arguments, there is a positive lower bound for $\inf_{x \in L} \sup_{y \in K} |x - y|$ that is uniform across all hyperplanes L. By taking a high enough power of an IFS generating K, we see that there there exists $R_0 \in (0,1)$ such that for every hyperplane L there is a similar copy of K with diameter at least R_0 which does not intersect the $2R_0$-neighbourhood of L. By the strong separation condition, we can write K as a disjoint union $\bigcup_{i \in I_n} K_i$ (for some index set I_n) of similar copies of K with diameters in the interval $[R_0^n r_{\min}^n, R_0^n r_{\min}^{n-1}]$, where $r_{\min}$ is the minimum contraction ratio of the defining IFS. Let $M := \# I_0$. Each element of I_{n-1} intersecting the $R_0^{n-1} r_{\min}^{n-2}$-neighbourhood of L contains a copy of K with diameter at least $R_0^n r_{\min}^{n-1}$ which does not intersect the $R_0^n r_{\min}^{n-1}$-neighbourhood of L. Therefore a standard induction argument shows that the ratio of the number of elements of I_n which intersect the $R_0^n r_{\min}^{n-1}$-neighbourhood of L is $\lesssim \left(\frac{M-1}{M}\right)^n \cdot \# I_n$. Therefore $\overline{\dim}_{\mathrm{B}}(L \cap K) \leqslant \dim K - \varepsilon$, where $\varepsilon := \frac{\log(M/(M-1))}{-\log(R_0 r_{\min})}$. □

l:lineareq

Lemma 6.5. *Let μ be a stationary measure (with positive weights) for an IFS of similarities such that there exists some iterate with an irreducible and non-expanding subsystem. Then some iterate of the IFS has a subsystem of maps of the form $\{r_* O_*(\cdot) + t_i\}_{i \in J}$ where $J = \{1, \ldots k+1\}$ for some $r_* \in (0,1)$ and $O_* \in O_k(\mathbb{R})$ satisfying the following conditions:*

- *The $t_i \in \mathbb{R}^k$ are such that $\mathcal{T} := \{t_1 - t_j : 1 < j \leqslant k+1\}$ is a vector space basis of $\mathbb{R}^k$. In particular this subsystem is irreducible.*
- *This subsystem satisfies the strong separation condition.*
- *The probability weights corresponding to each element of this subsystem are equal to some common value $p_* \in (0,1)$.*

Proof. We denote the non-expanding subsystem by Φ_1. Since Φ_1 is irreducible, there exist $x_1, \ldots, x_{k+1}$ in the attractor of Φ_1 which are not contained in any affine hyperplane. By standard compactness arguments, there is a positive lower bound (call it δ_1) for $\inf_{y \in L} \sup_i |x_i - y|$ that is uniform across all hyperplanes L. Let $\delta_2 = \min_{i \neq j} |x_i - x_j|$, and

$\delta_3 \coloneqq \min\{\delta_1, \delta_2\}/3$. Let D be a bounded closed set which is mapped into itself by each element of Φ_1. We iterate Φ_1 to a large enough level that the image of D under each element of the iterate has diameter at most δ_3. Fix a subset $\Phi_2 \coloneqq \{\phi_1, \ldots, \phi_{k+1}\}$ of this iterate with $x_i \in \phi_i(D)$ for all i; Φ_2 satisfies the strong separation condition by the definition of δ_3. Moreover, since $\delta_3 \leqslant \delta_1$, the attractor of Φ_2 is not contained in any hyperplane, so Φ_2 is irreducible.

Write h for the box dimension of the attractor of Φ_2. Let ε be the constant from Lemma 6.4 for Φ_2. Now, given $w \in [k] \coloneqq \{1, \ldots, k+1\}^n$, consider the map $\phi_{w_1} \circ \cdots \circ \phi_{w_n}$, and let r_w, p_w, O_w denote the contraction ratio, associated probability weight, and linear part respectively. We see that the number of possibilities for r_w grows polynomially in n by the commutativity of real numbers, and similarly for the number of possibilities for p_w; the number of possible O_w is polynomial by the non-expanding hypothesis. It follows that number of triples $\{(r_w, p_w, O_w) : w \in [k]^n\}$ is polynomial in n. Fix $\delta \coloneqq -\varepsilon(\log r_{\min})/2$. Since $\sum_{w \in [k]} r_w^h = 1$ by the strong separation condition, there exists $n \in \mathbb{N}$ and a triple (r, p, O) such that if Φ_3 is the subset of the nth iterate of Φ_2 consisting of maps with triple (r, p, O), then $\#\Phi_3 \cdot r^h \geqslant e^{-\delta n}$. Therefore $\#\Phi_3 \cdot r^{h - \delta/(-\log r_{\min})} \geqslant \#\Phi_3 \cdot r^h r_{\min}^{-n\delta/(-\log r_{\min})} \geqslant 1$, where $r_{\min}$ is the minimum contraction ratio for Φ_2. Therefore the dimension of the attractor of Φ_3 is at least $h - \delta/(-\log r_{\min}) > h - \varepsilon$.

Write $\Phi_3 = \{r_* O_*(\cdot) + t_i\}_{1 \leqslant i \leqslant M}$ (the order of the translations does not matter). By Lemma 6.4, Φ_3 is irreducible, so $\{t_1 - t_j : 1 < j \leqslant M\}$ spans $\mathbb{R}^k$. We can therefore find a subset of this set which forms a vector space basis of $\mathbb{R}^k$, and relabelling this subset gives the result. $\square$

In light of Lemma 6.5, we henceforth work with the iterated IFS, which we denote $\Lambda = \{f_i(\cdot) = r_i O_i(\cdot) + t_i\}_{i \in \mathcal{A}}$, where $\mathcal{A} = \{1, \ldots, N\}$. Let $J \coloneqq \{1, \ldots, k+1\}$ index the subsystem from Lemma 6.5. Let $p_1, \ldots, p_N$ be the weights for the measure μ, and let $p_* = p_i$ for $1 \leqslant i \leqslant k+1$. We may assume without loss of generality that our contractions send the unit cube $[0,1]^k$ into itself, so the attractor is a subset of the cube.

Fix $\varepsilon > 0$ and fix a large integer l to be determined later. Given $\mathbf{a}, \mathbf{b} \in \mathcal{A}^l$, define the equivalence relation $\mathbf{a} \sim \mathbf{b}$ if and only if for all i such that $a_i \in J$ we have $b_i \in J$, and if i is such that $a_i \notin J$ then $a_i = b_i$. Denote the equivalence class of $\mathbf{a}$ by $[\mathbf{a}]$; we think of each equivalence class as an IFS. Let $I = \{[\mathbf{a}] : \mathbf{a} \in \mathcal{A}^l\}$ and $\Omega = I^{\mathbb{N}}$. Given $x \in I$, let $q_x = \sum_{\mathbf{a} \in x} p_{\mathbf{a}}$ (where $p_{\mathbf{a}} = \prod_{i=1}^l p_{a_i}$), and let P be the infinite product measure on Ω corresponding to these weights. For all $\mathbf{a} \in \mathcal{A}^l$, write $f_{\mathbf{a}}(\cdot) = r_{[\mathbf{a}]} O_{[\mathbf{a}]}(\cdot) + t_{\mathbf{a}}$, where $r_{[\mathbf{a}]} = \prod_{i=1}^l r_{a_i}$ and

$O_{[\mathbf{a}]} = O_{a_1} \circ \cdots \circ O_{a_l} \in O_k(\mathbb{R})$. Given $\omega = (\omega_1, \omega_2, \dots) \in \Omega$, let μ_ω be the measure defined by the infinite convolution

e:infconv

$$*_{n=1}^{\infty} \frac{1}{\#\omega_n} \sum_{\mathbf{a} \in \omega_n} \delta_{\left(\prod_{j=1}^{n-1} r_{\omega_j}\right) \cdot O_{\omega_1} \circ \cdots \circ O_{\omega_{n-1}}(t_{\mathbf{a}})} \tag{15}$$

where δ_v is a Dirac mass at $v \in \mathbb{R}^k$. Then as in [7, Proposition 4.2], for example,

e:disintegration

$$\mu = \int_{\omega \in \Omega} \mu_\omega dP(\omega). \tag{16}$$

The infinite convolution structure in (15) is useful, because it allows us to write the Fourier transform of μ_ω as the infinite product of averages of points on the unit circle in the complex plane:

e:infprod

$$\widehat{\mu_\omega}(\xi) = \prod_{n=1}^{\infty} \frac{1}{\#\omega_n} \sum_{\mathbf{a} \in \omega_n} \exp\left(-2\pi i \left\langle \xi, \left(\prod_{j=1}^{n-1} r_{\omega_j}\right) \cdot O_{\omega_1} \circ \cdots \circ O_{\omega_{n-1}}(t_{\mathbf{a}}) \right\rangle\right) \tag{17}$$

We record the following basic proposition ([7, Lemma 4.7]), which is straightforward to verify.

l:ptsoncircle

Lemma 6.6. *Let* $\mathbf{p} = (p_1, \dots, p_n)$ *satisfy* $p_i > 0$ *for all* i *and* $\sum_i p_i = 1$. *Then for all* $\delta > 0$, *there exists* $\upsilon \in (0,1)$ *(depending upon* $\mathbf{p}$ *and* δ*) such that if the points* $z_1, \dots, z_n \in \mathbb{C}$ *each satisfy* $|z_i| = 1$, *and if there exist* j, k *such that* $|arg(z_j) - arg(z_k)| \geqslant \delta$, *then* $|\sum_i p_i z_i| \leqslant \upsilon$.

We can now state the key technical result that with high P-probability, $|\widehat{\mu}(\xi)|$ decays polynomially outside a sparse set of frequencies. The proof uses an Erdős–Kahane type argument; unlike the one-dimensional case [7, Proposition 4.8], we need to use lattices in $\mathbb{R}^k$ for this argument.

p:ek

Proposition 6.7. *Let* μ *be a stationary measure (with positive weights) for an IFS of similarities such that there exists some iterate with an irreducible subsystem of maps whose linear parts are all equal. We disintegrate* μ *as described above. Then for all* $\varepsilon > 0$ *there exists* l_0 *such that for all* $l \geqslant l_0$ *there exist* $\delta, C, \beta > 0$ *such that for all* $T' > 0$ *there exists* $\Omega_{T'}$ *with* $P(\Omega \setminus \Omega_{T'}) \leqslant C(T')^{-\beta}$ *and such that for all* $T \geqslant T'$ *and* $\omega \in \Omega_{T'}$,

$$|\{\xi \in \mathbb{R}^k : |\xi| \leqslant T, |\widehat{\mu_\omega}(\xi)| \geqslant T^{-\delta}\}| \leqslant CT^{\varepsilon}$$

Proof. Let $r_{\min}$ and $r_{\max}$ be the minimum and maximum contraction ratios in the iterated IFS. Fix arbitrary $\varepsilon > 0$ and fix a large integer l (we will see during the course of the proof that as long as we choose l sufficiently large in a way that depends only on the iterated IFS and ε, the rest of the proof will work). It suffices to consider the case $T' > r_{\min}^{-l}$

(or else we can just make C large and $\Omega_{T'} = \varnothing$), so we fix such a T' for the rest of the proof.

We first define the set $\Omega_{T'}$. Let $N' \in \mathbb{N}$ be such that $r_{\min}^{-lN'} < T' \leqslant r_{\min}^{-l(N'+1)}$. Let P_{seq} be the Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$ with weights $p_1, \ldots, p_N$. By the strong law of large numbers, P_{seq}-almost surely, $l^{-1}\#\{1 \leqslant i \leqslant l : a_i \leqslant k+1\} \to (k+1)p_*$ as $l \to \infty$. Therefore by large deviations theory (see Hoeffding's inequality, for example), there exist some small $\alpha > 0$ (depending only on $p_1, \ldots, p_N$) such that for all $l \in \mathbb{N}$,

$$\sum_{\omega \in I, \#\omega \leqslant 2^{p_* l}} q_\omega = P_{\text{seq}}(\#\{1 \leqslant i \leqslant l : a_i \leqslant p_* l\}) \leqslant e^{-\alpha l}.$$

Again applying Hoeffding's inequality, for all $l \in \mathbb{N}$ there exists $\beta' > 0$ such that for all $N \in \mathbb{N}$, if Ω_N denotes the set of $\omega \in \Omega$ for which the number of $n \in \{1, \ldots, N\}$ for which $\#\omega_n > 2^{p_* l}$ is at least $N(1-2e^{-\alpha l})$, then $P(\Omega_N) \geqslant 1 - e^{-2\beta' N}$. Let $\Omega_{T'} := \bigcap_{N=N'}^{\infty} \Omega_N$. Then

$$P(\Omega \backslash \Omega_{T'}) \leqslant \sum_{N=N'}^{\infty} P(\Omega \backslash \Omega_N) \leqslant \sum_{N=N'}^{\infty} e^{-2\beta' N} \leqslant \frac{e^{-2\beta' N'}}{1 - e^{-2\beta'}} \leqslant \frac{(T')^{\beta'/l \log r_{\min}}}{1 - e^{-2\beta'}}.$$

We can therefore set $\beta := -\beta'/l \log r_{\min}$.

Fix $T \geqslant T'$ and define N_ω by (18). Fix $\xi \in [-T, T]^k \setminus \{0\}$. Let N_ω be the unique integer depending on T satisfying

$$\left| T \prod_{j=1}^{N_\omega + 1} r_{\omega_j} \right| < 1 \leqslant \left| T \prod_{j=1}^{N_\omega} r_{\omega_j} \right|. \tag{18}$$

e:definenomega

Then since each $r_{\omega_j} \in [r_{\min}^l, r_{\max}^l]$, N_ω is comparable to N' up to multiplicative constants depending only on μ, so there exists $C_1(\mu) \geqslant 1$ such that $lN_\omega / C_1(\mu) \leqslant \log T \leqslant C_1(\mu) l N_\omega$, and $\left| T \prod_{j=1}^{N_\omega} r_{\omega_j} \right| \in [1, r_{\min}^{-l}]$. We will use the following bound which holds for all $\xi \in [-T, T]^k \setminus \{0\}$ by (17):

$$|\widehat{\mu_\omega}(\xi)| \leqslant \prod_{n=1}^{N_\omega} \frac{1}{\#\omega_n} \left| \sum_{\mathbf{a} \in \omega_n} \exp\left(-2\pi i \left\langle \xi, \left(\prod_{j=1}^{n-1} r_{\omega_j} \right) \cdot O_{\omega_1} \circ \cdots \circ O_{\omega_{n-1}}(t_{\mathbf{a}}) \right\rangle \right) \right|. \tag{19}$$

e:productbound

Define the set of *decay levels* to be

$$G_\omega := \{1 \leqslant n \leqslant N_\omega : \#\omega_n \geqslant 2^{p_* l}\}.$$

Since $\omega \in \Omega_{T'}$ and $N_\omega \geqslant N'$, we have $\#G_\omega = N_\omega(1-2e^{-\alpha l})$. Enumerate the decay levels by $i_1, \ldots, i_{\#G_\omega}$. For each decay level i_s choose distinct words $\mathbf{a}_s^{(1)},\dots,\mathbf{a}_s^{(k+1)} \in \omega_{i_s}$ and $j \in \{1,\dots,l\}$ such that $(\mathbf{a}_s^{(q)})_j = q$, and $(\mathbf{a}_s^{(q)})_r = (\mathbf{a}_p^{(q')})_r$ for all $r \neq l$, for all $q, q' \in \{1,\dots,k+1\}$.

For each decay level i_s and $q \in \{2,\dots,k+1\}$, write

$$\left\langle \xi, \left(\prod_{j=1}^{i_s-1} r_{\omega_j} \right) \cdot O_{\omega_1} \circ \cdots \circ O_{\omega_{n-1}}(t_{\mathbf{a}_p^{(1)}} - t_{\mathbf{a}_p^{(q)}}) \right\rangle = p_s^{(q)} + \varepsilon_s^{(q)} \tag{20}$$

where $p_s^{(q)} \in \mathbb{Z}$ and $\varepsilon_s^{(q)} \in (-1/2, 1/2]$ depend on ξ. Fix $\chi > 0$ small enough that

$$\{\xi \in \mathbb{R}^k : \langle \xi, t_1 - t_q \rangle \in \mathbb{Z} + [-\chi, \chi] \text{ for } 2 \leqslant q \leqslant k+1\}$$

is a countable union of disjoint k-dimensional parallelotopes centred at points in the lattice given by $\langle \xi, t_1 - t_q \rangle \in \mathbb{Z}$ for all q. Making χ smaller still, we can assume that the diameter of each parallelotope is smaller than one fifth of the smallest distance between two elements of this lattice. Let

$$\varepsilon^* := \chi r_{\min}^{2l}. \tag{21}$$

We now apply Lemma 6.6 to the i_s term of the product (19). The $\mathbf{p}$ in Lemma 6.6 can be the uniform probability vector on k^q points for some $1 \leqslant q \leqslant l$, and each such $\mathbf{p}$ gives some $\upsilon^{(\mathbf{p})} < 1$. Taking the maximum of these gives $\upsilon \in (0,1)$ such that if $\varepsilon_s \geqslant \varepsilon^*$ then

$$\frac{1}{\#\omega_{i_s}} \left| \sum_{\mathbf{a} \in \omega_{i_s}} \exp\left(-2\pi i \left\langle \xi, \left(\prod_{j=1}^{i_s-1} r_{\omega_j} \right) \cdot O_{\omega_1} \circ \cdots \circ O_{\omega_{i_s-1}}(t_{\mathbf{a}}) \right\rangle \right) \right| \leqslant \upsilon.$$

The rest of the proof uses similar ideas to the proof of [7, Proposition 4.8]. For each $\xi \in [-T,T]^k \setminus \{0\}$ we consider the set

$$\mathrm{Bad}(\xi) := \{ i_s \in G_\omega : \varepsilon_s \in [-\varepsilon^*, \varepsilon^*] \}.$$

If ξ is such that $\#\mathrm{Bad}(\xi) \leqslant \#G_\omega(1 - (l!)^{-1})$ then

$$|\widehat{\mu_\omega}(\xi)| \leqslant \upsilon^{\#G_\omega / l!} \leqslant T^{-\delta}$$

for some small $\delta > 0$. Note that we can assume υ depends only on μ and l, so we can assume the same about δ.

For the remainder of the proof, we assume that $\xi \in [-T,T]^k \setminus \{0\}$ is such that $\#\mathrm{Bad}(\xi) \geqslant \#G_\omega(1-(l!)^{-1})$ and determine how many possible choices of $\mathbf{p}_1$ there are. Using Stirling's formula as in [7, (4.22)], for all l sufficiently large (where 'sufficiently large' depends only on μ and ε) there exists a constant $C_1(\mu, \varepsilon, l) > 0$ such that the following bound for the number of possible sets $\mathrm{Bad}(\xi)$ holds:

$$\#\{\mathrm{Bad}(\xi) : \xi \in [-T,T]^k \setminus \{0\}\} \leqslant \frac{\#G_\omega}{k!}\binom{\#G_\omega}{\lceil \#G_\omega(1-1/k!))\rceil} \leqslant C_1(\mu,\varepsilon,l)T^{\varepsilon/4}.$$

Now fix a specific choice $\mathcal{I} \subseteq G_\omega$ of Bad and write the elements of $\mathcal{I}$ as $i_{\phi(1)},\ldots,i_{\phi(\#\mathcal{I})}$. Note that for some $C_2(\mu) > 0$ and $\zeta_l \coloneqq C_2(\mu)e^{-\alpha l}$ we have

$$\#\mathcal{I} \geqslant \#G_\omega(1-1/l!) \geqslant N_\omega(1-2e^{-\alpha l})(1-1/l!) \geqslant N_\omega(1-\zeta_l).$$

Now we will bound the number of choices for the vector $\mathbf{p}_1$ given $\xi \in [-T,T]^k\setminus\{0\}$ satisfying $\mathrm{Bad}(\xi) = \mathcal{I}$. By (20) there exists $C_3(\mu) > 0$ such that for all $1 \leqslant s < s' \leqslant \#G_\omega$, given a particular value of the vector $\mathbf{p}_{s'} \coloneqq (p_{s'}^{(2)},\ldots,p_{s'}^{(k+1)}) \in \mathbb{Z}^k$ there are at most $C_3(\mu)r_{\min}^{-3kl(i_{s'}-i_s)}$ choices for $\mathbf{p}_s \coloneqq (p_s^{(2)},\ldots,p_s^{(k+1)}) \in \mathbb{Z}^k$. Therefore given ξ there are at most $C_3(\mu)r_{\min}^{-3klN_\omega\zeta_l}$ choices for $\mathbf{p}_{\phi(\#\mathcal{I})}$. Let $J \coloneqq \{\phi(n) \in \mathcal{I} : i_{\phi(n)}+1 < i_{\phi(n+1)}\}$. Recall the definition of ε^* from (21), noting that it has the property that if $i_{\phi(n)} \in \mathcal{I}\setminus\mathcal{J}$ then given $\mathbf{p}_{\phi(n)+1} = \mathbf{p}_{\phi(n+1)}$, there is only one possible choice for $\mathbf{p}_{\phi(n)}$. Therefore given $\mathbf{p}_{\phi(\#\mathcal{I})}$, the number of choices for $\mathbf{p}_{\phi(1)}$ is at most

$$\prod_{i_{\phi(n)}\in\mathcal{J}} C_3(\mu)r_{\min}^{-3kl(i_{\phi(n+1)}-i_{\phi(n)})} \leqslant C_3(\mu)^{2N_\omega\zeta_l}r_{\min}^{-6klN_\omega\zeta_l},$$

where the last inequality is since $\#\mathcal{I} \geqslant N_\omega(1-\zeta_l)$. Finally, we note that given $\mathbf{p}_{\phi(1)}$ there are at most $C_3(\mu)r_{\min}^{-3klN_\omega\zeta_l}$ choices for $\mathbf{p}_1$.

Therefore given ξ, the number of choices for $\mathbf{p}_1$ is at most

$$C_2(\mu)^{2N_\omega+2}r_{\min}^{-12klN_\omega\zeta_l}.$$

Since $\zeta_l = C_2(\mu)e^{-\alpha l}$, for all l sufficiently large (where 'sufficiently large' depends only on μ and ε) there exists $C_2(\mu,\varepsilon,l) > 0$ such that, given ξ, the number of choices for $\mathbf{p}_1$ is at most $C_2(\mu,\varepsilon,l)T^{\varepsilon/4}$. Combining this with the bound from Stirling's formula we see that the total possible number of choices for $\mathbf{p}_1$ (across all possible choices of $\mathcal{I}$) is at most $(C_1(\mu,\varepsilon,l)+C_2(\mu,\varepsilon,l))T^{\varepsilon/2}$. Recall the formula (20) in the $s=1$ case, and note that there are at most $\zeta_l N_\omega$ terms in the product. Therefore by a similar calculation to above, there exists $C_3(\mu,\varepsilon,l) > 0$ such that if $\xi \in [-T,T]^k \setminus \{0\}$ satisfies $|\widehat{\mu_\omega}(\xi)| > T^{-\delta}$ for the choice of δ given above, then ξ can belong to at most $(C_1(\mu,\varepsilon,l)+C_2(\mu,\varepsilon,l))T^{\varepsilon/2}$ sets, each of which can be covered by $C_3(\mu,\varepsilon,l)T^{\varepsilon/4}$ hypercubes of sidelength 1. This completes the proof. $\square$

We are now ready prove that a wide class of self-similar measures in $\mathbb{R}^k$ have polynomial Fourier decay on average, without any separation condition. In particular, if all linear parts commuting (or equal), then the non-expanding condition is automatically satisfied. Moreover, it is straightforward to see that every IFS of similarities on $\mathbb{R}^2$ is non-expanding, so the following theorem recovers Theorem 1.11.

thm:afd

Theorem 6.8. *Let μ be a self-similar measure (with positive weights) for an IFS of similarities on $\mathbb{R}^k$ such that there exists some iterate with an non-expanding and irreducible subsystem. Then μ has polynomial Fourier decay on average.*

Proof. The proof is an application of Proposition 6.7, and is rather similar to the deduction of [7, Corollary 4.9] from [7, Proposition 4.8], but we provide the details for completeness. The idea is as follows. Assuming the result is false, we can find frequencies in $\mathbb{R}^k$ which are separated in the Euclidean distance by at least 1 and where $|\widehat{\mu}|$ is large. But Proposition 6.7 implies that the mean average of $|\widehat{\mu}|$ over these frequencies must be small, which will give a contradiction. This averaging argument is used to overcome the fact that in Proposition 6.7, different ω may give rise to different bad frequencies.

First, since we have assumed the result is false, there exists $\gamma > 0$ such that for all $\eta > 0$ there is a sequence $T_n \to \infty$ and frequencies $\{\xi_{n,m}\}_{n \geqslant 1, 1 \leqslant m \leqslant \lceil T_n^\gamma \rceil} \in [-T,T]^k$, with the Euclidean distance between every pair of frequencies bounded below by 1, such that $\widehat{\mu}(\xi_{n,m}) \leqslant T_n^{-\eta}$. Applying Proposition 6.7 with $\varepsilon = \gamma/5$, there exists a large l and $\delta, \beta > 0$ such that (setting $T' = T$) for all $T > 0$ sufficiently large there exists Ω_T with $P(\Omega \setminus \Omega_T) \leqslant T^{-\beta}$ and such that for all $\omega \in \Omega_T$, the set $\{\xi \in \mathbb{R}^k : |\xi| \leqslant T, |\widehat{\mu_\omega}(\xi)| \geqslant T^{\delta'}\}$ can be covered by at most $T^{\gamma/4}$ intervals of length 1, where $\delta = \min\{\delta', \gamma/20\}$. Now fix $\eta \coloneqq \min\{\varepsilon/4, \gamma/100\}$ and let $\{\xi_{n,m}\}$ be the set of frequencies as above corresponding to this η. For all n sufficiently large and all m, our disintegration (16) gives

$$T_n^{-\eta} \leqslant |\widehat{\mu}(\xi_{n,m})| \leqslant \int_\Omega |\widehat{\mu_\omega}(\xi_{n,m})| \leqslant T_n^{-4\eta} + \int_{\Omega_{T_n}} |\widehat{\mu_\omega}(\xi_{n,m})| dP(\omega).$$

But for all large enough n and $\omega \in \Omega_{T_n}$, since the gaps between frequencies are at least 1, Proposition 6.7 gives $\#\{1 \leqslant m \leqslant \lceil T_n^\gamma \rceil :$

$|\widehat{\mu_\omega}(\xi_{n,m})| \geqslant T_n^{-4\eta}\} \leqslant T_n^{\gamma/4}$. Therefore for all n sufficiently large,

$$\begin{aligned}
T_n^{-2\eta} &\leqslant (\lceil T_n^\gamma \rceil)^{-1} \sum_{m=1}^{\lceil T_n^\gamma \rceil} \int_{\Omega_{T_n}} |\widehat{\mu_\omega}(\xi_{n,m})| dP(\omega) \\
&= \int_{\Omega_{T_n}} (\lceil T_n^\gamma \rceil)^{-1} \sum_{m=1}^{\lceil T_n^\gamma \rceil} |\widehat{\mu_\omega}(\xi_{n,m})| dP(\omega) \\
&\leqslant \int_\Omega ((\lceil T_n^\gamma \rceil)^{-1} \cdot T_n^{\gamma/4} + T_n^{-4\eta}) dP(\omega) \\
&\leqslant T_n^{-3\eta},
\end{aligned}$$

which is a contradiction. □

Remark 6.9. *Theorem 6.8 can be generalised to certain countably infinite IFSs. Indeed, let μ be the stationary measure (with positive weights p_i) of a countable IFS $\{\Phi_i\}_i$ of similarities with contraction ratios r_i, and we assume that some iterate of which has a finite, irreducible, non-expanding subsystem. Then if we assume that there exists $\tau > 0$ such that $\sum_i p_i r_i^{-\tau} < \infty$ then one can prove that μ has polynomial Fourier decay on average. The proof is an adaptation of the one-dimensional case in* [7, Proposition 4.8]*, and the condition $\sum_i p_i r_i^{-\tau} < \infty$ ensures that certain large deviation bounds that are needed in the Erdős–Kahane argument; we leave the details to the interested reader.*

7. Proof of consequences of the main results

sec:consequencesarithproofs

7.1. Proofs of results in Section 2.1 on nonlinear arithmetic.

In this section we prove consequences of our main Fourier decay results which we have already stated. First, we prove the results from Section 2.1 on nonlinear arithmetic, and then we prove Theorem 1.13 related to self-conformal measures on $\mathbb{C}$.

Proof of Theorem 2.8. We first prove the symmetric results, i.e. $\mu = \nu$ and $E = F$. Consider the function $L\colon x \mapsto \log x$. Without loss of generality, we can assume that μ is supported on $[1/2, 1]$. Since μ has the ESC, we know from [67] that

$$\kappa_F = \dim_s \mu = \kappa_2.$$

If μ is also AD-regular, we can use Theorem 1.1 with $\kappa_F = \kappa_* = \kappa_2$ and obtain that

$$|\widehat{L_*\mu}(\xi)| \ll |\xi|^{-\sigma}$$

for

$$\sigma = \frac{-\kappa_2 + 1/2}{-2 - \kappa_2 + 1/2}.$$

Again, because μ is AD-regular, we see that $L_*(\mu)$ is AD-regular with the same exponent. Then we see that

$$\int |\widehat{L_*\mu}(\xi)|^2 \frac{1}{|\xi|^\rho} d\xi < \infty$$

whenever $\rho > 1 - \kappa_2$. This implies that as long as

eqn:condition

$$2\sigma + \kappa_2 > 1 \tag{22}$$

the measure $L_*(\mu) * L_*(\mu)$ has an L^2-density function. We can then apply the map $e\colon x \mapsto e^x$ to conclude the corresponding result for $f_*(\mu, \mu)$. The threshold $(\sqrt{65} - 5)/4$ sits precisely on the boundary of Condition 22.

If μ only has the ESC, then we only have that $\kappa_F = \dim_s \mu = \kappa_2$. In this case, we can trivially bound $\kappa_* \leqslant 1$. The threshold $7/9$ is the the boundary case for Condition 22 in this situation.

For asymmetric results, notice what we need is to replace (22) with

$$2\sigma + \kappa_2(\nu) > 1.$$

From here, we see that the condition is then

$$\kappa_2(\mu)\kappa_2(\nu) + \kappa_2(\mu) + 1.5\kappa_2(\nu) - 2.5 > 0.$$

Of course, by switching the roles of μ, ν we also have

$$\kappa_2(\mu)\kappa_2(\nu) + \kappa_2(\nu) + 1.5\kappa_2(\mu) - 2.5 > 0.$$

We only need one of the above conditions to hold. This proves Theorem 2.8. □

Proof of Theorem 2.10. Again, we prove the symmetric case first. Given a self-similar system Λ. We can iterate this IFS. Let $\dim_H F = s$ where F is the attractor of Λ. For each $\varepsilon > 0$, it is possible to find an integer $N > 0$ and a subsystem Λ' of the N-fold iterated system Λ^N so that Λ' has the OSC and it is homogeneous and moreover its attractor $F' \subset F$ satisfies

$$\dim_H F' \geqslant \dim_H F - \varepsilon.$$

We can then apply Theorem 1.1 for a suitable self-similar measure on F' and finish the proof.

The asymmetric case follows similarly by following the end part of the proof of Theorem 2.8. □

We note that Theorem 2.14 follows similarly by following the above proofs. We omit the details. We have thus proved the two-sets part of Theorem 2.6. We now finish the proof of this theorem by proving the three-sets part.

Proof of the three sets part of Theorem 2.6. The idea is again to first look at AD-regular self-similar measures μ_1, μ_2, μ_2 on $[1/2, 1]$. Recall the map $L\colon x \mapsto \log x$. For convenience, write μ^L for the pushforward measure $L_*(\mu)$, and consider the convolution of three measures

$$\nu = *_{i=1}^{3}(\mu_i^L).$$

We can now consider its Fourier coefficients,

$$\widehat{\nu}(\xi) = \widehat{\mu_1^L}(\xi)\widehat{\mu_2^L}(\xi)\widehat{\mu_3^L}(\xi).$$

We can use Cauchy–Schwartz for two of the three factors and then use the Fourier decay for the last factor[8] in the following sum for integers $N > 1$,

$$\begin{aligned}\sum_{N/2\leqslant|\xi|\leqslant N} |\widehat{\nu}(\xi)| \leqslant |N|^{-\sigma}\Big(\sum_{|\xi|\leqslant N} |\widehat{\mu_1^L}(\xi)|^2 \sum_{|\xi|\leqslant N} |\widehat{\mu_2^L}(\xi)|^2\Big)^{1/2} \\ \ll N^{-\sigma} N^{1-(\kappa_2(\mu_1^L)+\kappa_2(\mu_2^L))/2}\end{aligned}$$

where $\sigma > 0$ is related to $\kappa_2(\mu_3)$ as in Theorem 1.1. We have used the fact that

$$\sum_{|\xi|\leqslant N} |\widehat{\mu_i^L}(\xi)|^2 \ll N^{1-\kappa_2(\mu_i^L)}$$

for $i \in \{1, 2\}$ because $\mu_{1,2}^L$ are AD-regular. We see from above that as long as

$$\sigma + \frac{\kappa_2(\mu_1^L) + \kappa_2(\mu_2^L)}{2} > 1, \tag{23}$$

con: three

we have

$$\sum_{\xi} |\widehat{\nu}(\xi)| < \infty.$$

This shows that ν is absolutely continuous with a continuous (not only measurable) density function. Therefore $\mathrm{supp}(\nu)$ has non-empty interior. To pass from the measure-theoretic result to the set-theoretic result, we only need to follow the proof of Theorem 2.10.

[8] This three factors "curse" is prevalent in modern analytic/additive number theory. There, a lot of results are stated with three or more variables and it is extremely difficult to push them down to two variables, see Tao's book.

To see explicitly what Condition 23 says, we use Theorem 1.1 and the fact that L is smooth and see that (after writing $\kappa_2(\mu_{1,2,3}) > \kappa$)

$$\frac{\frac{\kappa-1/2}{\kappa+1/2}}{2-\frac{\kappa-1/2}{\kappa+1/2}}+\kappa>1.$$

From here we see that it is sufficient to require that

$$\kappa>\frac{-3+\sqrt{41}}{4}.$$

This proves the three-sets part of Theorem 2.6 and therefore finishes the entire proof. ☐

Proof of Theorem 2.11. Without loss of generality, let E_k be such that $\dim_{\mathrm{H}} E_k$ is the smallest. If $\dim_{\mathrm{H}} E_k < 1/(k-1)$, then we see that

$$\sum_{i=1}^{k-1}\dim_{\mathrm{H}} E_i > 1+(k-1)^{-1}-(k-1)^{-1}=1.$$

From Hochman:1008.3731 Theorem 1.23 we see that $\dim_{\mathrm{H}}\{E_1\cdot E_2\cdots E_{k-1}\}=1$. From here we see that

$$\log E_1+\cdots+\log E_{k-1}$$

has dimension one. Then from Theorem 5.4 we know that $\log E_k$ supports a measure with a polynomial Fourier decay. The rest of the argument is similar to the proofs of Theorems 2.8 and 2.10.

If $\dim_{\mathrm{H}} E_k \geqslant 1/(k-1)$. This implies that $\sum_{i=1}^{k-1}\dim_{\mathrm{H}} E_i \geqslant 1$. The above argument again applies. ☐

sec:analyticconf

7.2. **Self-conformal measures and proof of Theorem 1.13.** Given an open set $U\subset\mathbb{R}^k$, recall that a real analytic map $f\colon U\to\mathbb{R}^k$ is called conformal if its derivative is a similarity map at each point $x\in U$, i.e. $|\nabla f(x)y| = c_x|y|$ for some $c_x>0$ and all $y\in\mathbb{R}^k$. Let D be closed and U open with $D\subset U$, and consider an IFS $\Phi \coloneqq \{\phi_i\colon D\to D\}_{i\in I}$ of contractions which extend to conformal maps $U\to\mathbb{R}^k$. A stationary measure (recall Section 3.1) for such an IFS is called a self-conformal measure. In this section we are interested in *nonlinear self-conformal measures*, by which we mean non-atomic stationary measures for an IFS of conformal contractions which are *not* all affine (so μ is not self-similar), because self-similar measures like the Cantor–Lebesgue measure (or products of this measure with itself) may not be Rajchman. We say that a self-conformal measure μ (or the IFS generating μ) is *conjugate to linear* if there is some closed D' contained in an open U' and a conformal diffeomorphism $f\colon U'\to U$ and some self-similar IFS

$\{\psi_i \colon D' \to D'\}_{i\in I}$ so that $\Phi = \{f \circ \psi_i \circ f^{-1}\}_{i\in I}$. If μ is conjugate to linear then a straightforward application of the definition of a stationary measure shows that $\mu = f_*\nu$ for some self-similar measure ν on $\mathbb{C} \simeq \mathbb{R}^2$ (see the proof of [7, Theorem 1.1], for instance); we say also that μ and ν are conjugate.

In the case $k = 1$ it is known that non-atomic self-conformal measures for an IFS of analytic contractions which are not all affine have polynomial Fourier decay [4, 9, 1, 7].

Next, we consider the $k = 2$ case. Here it is convenient to work in $\mathbb{C}$ because conformal maps are just the same as complex analytic (holomorphic) maps with non-vanishing derivative. One needs to take care of a possible obstruction that can prevent nonlinear self-conformal measures from having Fourier decay, namely the possibility that it is contained in an analytic curve (i.e. the image of an analytic map $(0, 1) \to \mathbb{C}$ with non-vanishing derivative). If a self-conformal measure is contained in a line then it certainly is non-Rajchman. Slightly less trivially, recall from the discussion at the start of Section 5 that if a conformal map $f \colon \mathbb{C} \to \mathbb{C}$ degenerates along some line then f will not twist non-Rajchman self-similar measures supported on that line. This is what is happening in the following example.

ex:pushtoparabola

Example 7.1. *Let μ be the Cantor–Lebesgue measure supported in $[0, 1] \subset \mathbb{C}$, i.e. the stationary measure for $\{\phi_1, \phi_2$ where $\phi_1(z) = z/3$ and $\phi_2(z) = (z + 1)/3$. Let $f \colon \mathbb{C} \to \mathbb{C}$ be the conformal map that is the quadratic polynomial $f(z) = z + iz^2$, noting that f lifts the real axis onto a parabola. Then $f_*\mu$ is the self-conformal measure for the IFS $\{f \circ \phi_1 \circ f^{-1}, f \circ \phi_2 \circ f^{-1}\}$, and $f_*\mu$ has no Fourier decay at powers of 3 along the real axis.*

Note that Example 7.1 is in the setting of Corollary 1.14, so if we fix any dorection that is not parallel to the real axis then we will eventually see polynomial Fourier decay in that direction. Our next result says that the obstruction witnessed in Example 7.1 is essentially the only obstruction to polynomial Fourier decay.

thm:conformalconjugate

Theorem 7.2. *Let μ be a nonlinear self-conformal measure on $\mathbb{C}$ that is conjugate to linear. Then if μ does* not *have polynomial Fourier decay then there exists a self-similar measure ν supported in $[0, 1] \subset \mathbb{C}$, some open neighbourhood U of $[0, 1] \subset \mathbb{C}$, and a conformal map $g \colon U \to \mathbb{C}$ which degenerates on $[0, 1]$, such that $\mu = g_*\nu$.*

Proof. Write $\mu = f_*\nu'$ for a self-similar measure ν'. If ν is supported in a line then by Theorem 5.3 the only way μ can fail to have polynomial Fourier decay is if f degenerates along the line supporting ν', so we henceforth assume ν is irreducible. Again by Theorem 5.3, μ has polynomial Fourier decay unless f is degenerate (on its domain) so we assume for contradiction that f is degenerate. we claim that f is a linear function. Indeed, there is some non-trivial linear form L so that $L(x, y, u(x, y), v(x, y)) = 0$ constantly. Here u, v are the real and imaginary parts of f. We write out the linear form with real coefficients as

$$c_0 + c_x x + c_y y + c_u u + c_v v = 0.$$

We can make partial derivatives against x, y and see that

$$\begin{bmatrix} c_u & -c_v \\ c_v & c_u \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} -c_x \\ -c_y \end{bmatrix},$$

where we have used the Cauchy–Riemann relation for complex analytic functions to deal with partial derivatives of u, v. The determinant of the matrix cannot be zero unless $c_u = c_v = 0$. In this case, L indicates a linear relation between x, y which should not exist. Therefore we can solve the above linear equations and see that u_x, u_y are constant functions. This implies that u is linear. Similarly, v is linear. This implies that f is a linear function as claimed. This implies that μ is itself a self-similar measure, which contradicts our assumption. $\square$

From the recent work of Algom, Rodriguez Hertz, and Wang, [5, Theorem 1.1] we know that μ has polynomial Fourier decay unless one of the following situations happens:

- μ is supported inside an analytic curve.
- μ is conjugate to linear.

Proof of Theorem 1.13. This is immediate from combining Theorem 7.2 with [5, Theorem 1.1]. $\square$

We make the following conjecture regarding a class of measures. An example of such a measure is the pushforward of a non-conjugate self-conformal measure on $[0, 1]$ (e.g. the measures studied in [9, 4]) under $z \mapsto z + iz^2$.

Conjecture 7.3. *Let μ be a non-atomic self-conformal measure which is* not *conjugate to linear and is supported inside an analytic curve in $\mathbb{C}$ which is not a line. Then μ has polynomial Fourier decay.*

Finally, we briefly consider self-conformal measures in higher dimensions $k \geqslant 3$. Now conformal maps have a very restricted form by Liouville's theorem: they can be expressed as compositions of translations, similarities, orthogonal transformations and inversions in spheres.

thm:higherconformal

Theorem 7.4. *Let μ be an irreducible nonlinear self-conformal measure on $\mathbb{R}^k$ for $k \geqslant 3$ which is conjugate to to a self-similar measure ν which has polynomial Fourier decay on average. Then μ has polynomial Fourier decay.*

Proof. Write $\mu = f_*\nu$ for a conformal map f. By the nonlinearity of μ and by Liouville's theorem, f satisfies the non-constant $|P_v|$ assumption in the statement of Theorem 5.4. Also by Liouville's theorem, one can check that if f fails the $|P_v|$ constant assumption on some proper subspace L, then $f(L)$ lies in a proper subspace of $\mathbb{R}^k$, so since μ is irreducible, μ cannot be supported on a proper subspace on which f fails this assumption. Therefore by Theorem 5.3, μ has polynomial Fourier decay. □

Theorem 7.4 applies also to certain self-conformal measures supported in proper submanifolds of $\mathbb{R}^k$, for example pushforwards of self-similar measures supported in planes under inversions in spheres whose centre does not lie in that plane. We doubt that examples like Example 7.1 are possible in higher dimensions because of Liouville's theorem, and finish with the following conjecture.

Conjecture 7.5. *Every irreducible nonlinear self-conformal measure in $\mathbb{R}^k$ where $k \geqslant 3$ has polynomial Fourier decay.*

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