Overlapping iterated function systems

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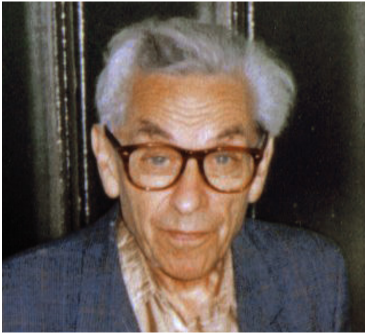
Paul Erdős



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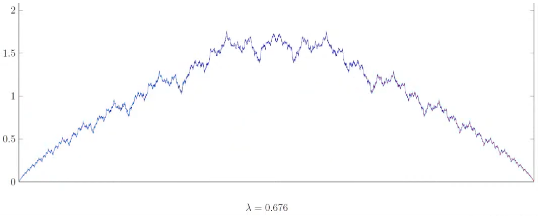
In the 1930s, Paul Erdős became fascinated by a problem involving random sums.
This problem turns out to be connected to number theory, harmonic analysis, probability theory, dynamical systems, and fractal geometry.

Random sums

- Fix $1/2 < \lambda < 1$. Repeatedly toss a fair coin.
- Start at 0. At the nth toss:
  If Heads, add λ^{n-1} to the running total; if Tails, subtract λ^{n-1}.
- This gives a random sum:

$$\sum_{n=0}^{\infty} \pm\lambda^n \in \left[-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}\right].$$



Picture by Péter Varjú

An open problem

Does there exist $A \subset \mathbb{R}$ with Lebesgue measure 0, but with

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \pm\lambda^n \in A\right) > 0?$$

Theorem (Erdős, 1939 & 1940)

For some λ (e.g. the reciprocal of the golden ratio $(1+\sqrt{5})/2$), this is possible, but for other λ this is impossible!

Determining for which λ this is possible remains a major open problem, and can be described using the language of fractal geometry.

Modelling the world

'Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.'

Benoit B. Mandelbrot.

What is a fractal?

There is no precise definition of a fractal.

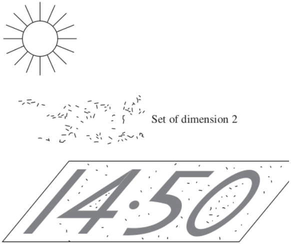
Romanesco broccoli exhibits fractal features. Picture by Lauren Lester, CC0

Typical features include:

- Fine structure at arbitrarily small scales
- Some form of self-similarity

Theoretical applications

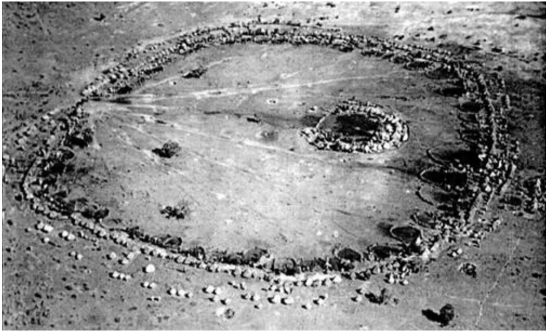
Kenneth Falconer showed that a fractal set can theoretically be used to make a digital sundial! But those that have been manufactured work by a different principle.



Pictures by Kenneth Falconer

Fractals in creative work

Fractal patterns have been used for centuries by many different cultures in creative work such as art and architecture.



A 2000-year-old settlement in Zambia.
Picture from African Fractals, CC BY-SA 4.0

Iterated function systems

- Let $D \subseteq \mathbb{R}^2$ be closed. An iterated function system (IFS) on D is a finite set of contractions $S_1, \ldots, S_m \colon D \to D$.
- Hutchinson (1981) showed there is a unique non-empty compact set, called the **attractor**, satisfying

$$F = \bigcup_{i=1}^{m} S_i(F).$$

Self-similar sets and measures

- Given weights $p_1, \ldots, p_m > 0$ with $\sum_i p_i = 1$, there is a unique measure μ, called the **stationary measure**, satisfying

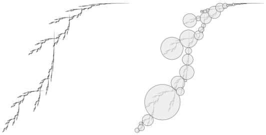
$$\mu(A) = \sum_i p_i \mu(S_i^{-1}(A)) \qquad \text{for all Borel } A \subset D.$$

- If for each i there exists c_i such that $||S_i(x) - S_i(y)|| = c_i ||x - y||$ for all $x, y \in D$, then we call the attractor a **self-similar set** and the stationary measure a **self-similar measure**.
- Important questions:
  - When are the fractal measures absolutely continuous? (Absolute continuity means $\mathrm{Leb}(A) = 0 \Rightarrow \mu(A) = 0$.)
  - What is the dimension of the fractal set?

Hausdorff dimension

- There are many different notions of dimension which attempt to quantify the 'thickness' of sets at small scales.
- Hausdorff dimension is defined by

$$\dim F = \inf\{s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ with } \sum_i (\mathrm{diam}(U_i))^s \leq \varepsilon\}.$$



Picture by Jonathan Fraser

- Intuitively, a disc has dimension 2 because it has positive and finite area.

Dimension of self-similar sets

- The natural guess for the dimension of a self-similar set is the unique $s \geq 0$ satisfying Hutchinson's formula

$$\sum_{i=1}^{m} c_i^s = 1.$$

- This is always an upper bound. Indeed just cover F by similar copies of it with diameter $\approx r$.
- Lower bounds are harder because you have to show something for **every** cover.
  The natural guess is correct if the images $S_i(F)$ are sufficiently separated, e.g. pairwise-disjoint. In this case, if μ is the measure with weights c_i^s, then $\mu(U) \leq (\mathrm{diam}(U))^s$ for all U, so we need approximately r^{-s} sets of diameter r to cover F.

Exact overlaps

From now on fix $d = 1$. The obstructions to the natural guess holding are:

- Always $\dim F \leq 1$.
- If there is an exact overlap i.e. the semigroup generated by the contractions is not free, then $\dim F < s$.

Exact overlaps conjecture (folklore / Simon 1996)

Consider an IFS of similarities on $[0, 1]$. If there is no exact overlap then

- $\dim F = \min\{s, 1\}$
- If $s > 1$ then the self-similar measure with weights c_i^s is absolutely continuous.

Recent progress

- Hochman (2014) has proved that the dimension part of the conjecture holds under the **exponential separation condition** (ESC), which says that there exists $c > 0$ such that for all distinct strings with $c_{i_1} \cdots c_{i_n} = c_{j_1} \cdots c_{j_{n'}}$ we have

$$|S_{i_1}(0) \circ \cdots \circ S_{i_n} - S_{j_1} \circ \cdots \circ S_{j_{n'}}(0)| \geq c^n.$$

- The set of parameters (contraction ratios and translations) for which this fails has dimension 0.
- Rapaport 2022: If the contraction ratios are **algebraic** then the conjecture holds.
- There exist IFSs which fail the ESC (Baker 2021, Bárány–Käenmäki 2021), and the conjecture remains open.

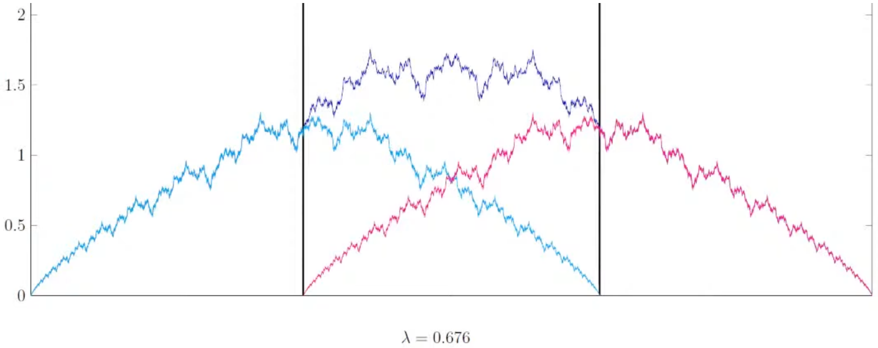
Bernoulli convolutions

- Let μ_λ be the law of the random sum $\sum_{n=1}^{\infty} \pm\lambda^n$:

$$\mathbb{P}\left(\sum_{n=1}^{\infty} \pm\lambda^n \in [a,b]\right) =: \mu_\lambda([a,b]) \qquad \text{for all } a<b.$$

- This measure is called a Bernoulli convolution; its support is $[-(1-\lambda)^{-1}, (1-\lambda)^{-1}]$, and Erdős asked for which λ it is absolutely continuous.
- But μ_λ is a self-similar measure! Specifically, for the IFS $S_-(x) = \lambda x - 1$, $S_+(x) = \lambda x + 1$ and equal weights.

Bernoulli convolutions



Picture by Péter Varjú

Shmerkin (2014) using Hochman:

$$\dim\{\lambda \in (1/2, 1) : \mu_\lambda \text{ is } \textbf{not} \text{ absolutely continuous}\} = 0.$$

Fourier decay

- Erdős found some λ where μ_λ is not absolutely continuous using Fourier transforms.
- The Fourier transform $\hat{\mu}\colon \mathbb{R} \to \mathbb{C}$ of a Borel probability measure μ on $\mathbb{R}$ is

$$\hat{\mu}(\xi) := \int_{-\infty}^{\infty} e^{-2\pi i \xi x} d\mu(x).$$

- Given a measure, one can ask:
  - Is μ Rajchman? Does $\hat{\mu}(\xi) \to 0$ as $|\xi| \to \infty$?
  - If so, does it have polynomial Fourier decay

$$|\hat{\mu}(\xi)| \leq C|\xi|^{-\varepsilon} \qquad \text{for all } \xi \neq 0?$$

- Absolutely continuous measures are Rajchman by the Riemann–Lebesgue lemma.

Fourier decay

- The support of $\mu_{1/3}$ is the middle-third Cantor set, and

$$\widehat{\mu_{1/3}}(1) = \widehat{\mu_{1/3}}(3) = \widehat{\mu_{1/3}}(9) = \cdots \not\to 0.$$

- A Pisot number α is a real algebraic integer greater than 1, all of whose Galois conjugates are less than 1 in absolute value. An example is the golden ratio $(1+\sqrt{5})/2$.
  It is known that they are the only algebraic numbers satisfying $\mathrm{dist}(\alpha^n, \mathbb{Z}) \to 0$ as $n \to \infty$.
- Erdős proved that if $\lambda \in (1/2, 1)$ and $1/\lambda$ is Pisot, then μ_λ is not Rajchman, so is not absolutely continuous.
- Are Pisot numbers the only parameters whose Bernoulli convolutions are not absolutely continuous? Non-Rajchman?

Nonlinear IFSs

- Fractal measures made of nonlinear copies of themself often have polynomial Fourier decay. Bourgain and Dyatlov (2017) proved this is the case for some measures supported on limit sets of certain Fuchsian groups.
- Let μ be any non-atomic self-similar measure on $\mathbb{R}$ (we impose no homogeneity or separation assumptions, and note μ may not be Rajchman).

Theorem (Baker–B. 2024+ / Algom–Chang–Wu–Wu 2024+)

Let $F\colon \mathbb{R} \to \mathbb{R}$ be C^2 with $F''(x) > 0$ for all $x \in \mathbb{R}$. There exists $\varepsilon = \varepsilon(\mu) > 0$ (independent of F) and $C = C(\mu, F) > 0$ such that

$$|\widehat{F\mu}(\xi)| \leq C|\xi|^{-\varepsilon} \qquad \text{for all } \xi \neq 0.$$

Proving and quantifying the decay

- Key tool in proof: disintegrate $\mu = \int_\Omega \mu_\omega \, dP(\omega)$, where each μ_ω is a convolution of finite sums of point masses. Recall that the distribution of the sum of two independent random variables is the convolution of their distributions.
- Fourier transform sends convolution to product, so $\widehat{\mu_\omega}$ is an infinite product of averages of finitely many points on the unit circle in $\mathbb{C}$.
- A general upper bound for the supremum of possible exponents $\varepsilon(\mu)$ is $\dim(\mathrm{supp}\mu)/2$. Not a single example of a stationary measure for an IFS on $\mathbb{R}$ with $0 < \dim(\mathrm{supp}\mu) < 1$ and maximum decay is known.
- For $\widehat{F\mu_{1/3}}$ the best known bounds are 0.02 (Mosquera–Shmerkin 2018) and 0.06 (B.–Yu, in progress), while one might expect the maximum possible of 0.32.

Analytic IFSs

Theorem

Consider an IFS of analytic maps $S_i\colon [0,1] \to [0,1]$, at least one of which is not affine. Then every non-atomic stationary measure has polynomial Fourier decay.

Proof idea:

- Case 1 (Baker–B. 2024+ / Algom–Chang–Wu–Wu 2024+ + Algom–Rodriguez Hertz–Wang 2023+):
  There is an analytic map F for which all $F \circ S_i \circ F^{-1}$ are similarities. Then μ is the pushforward of a self-similar measure by F, and a similar proof to the previous theorem works.
- Case 2 (Baker–Sahlsten 2023+ / Algom–Rodriguez Hertz–Wang 2023+): no such conjugacy exists, and different methods are needed.

Normal numbers

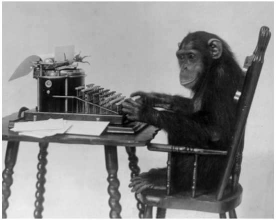
- An irrational number is normal if for all $b \geq 2$, $k \geq 1$, all strings of length k appear equally often in its base-b expansion.
- It is strongly conjectured that $\pi, e, \sqrt{2}$ are normal but no-one has a clue how to prove this!

$$\begin{aligned}\pi = 3.&14159265358979323846264338327950288419716\\&939937510582097494459230781640628620899\\&862803482534211706798214808651328230664\\&7093844609550582231725359408128481117450 2\ldots\end{aligned}$$

- Borel (1909) proved that Lebesgue-almost every number is normal using the Borel–Cantelli lemma.

The infinite monkey thought experiment

If a monkey types digits between 0 and 9 randomly, with probability 1 it will eventually type a million sevens in a row (or any other string of numbers you can think of).

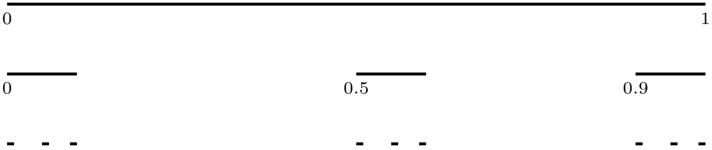


Picture by New York Zoological Society, public domain

- If the monkey types letters instead of numbers, with probability 1 it will eventually write Shakespeare's Hamlet without a single mistake!
- But the monkey would need a ridiculously long time.

Normal numbers in fractal sets

- The set of (non-normal) numbers between 0 and 1 whose decimal expansion consists of numbers $0, 5, 9$, for example, forms a self-similar set.



- However, combining a corollary of a theorem of Davenport–Erdős–LeVeque (1963) with the previous theorem gives that if μ is a measure arising from a nonlinear IFS as in the previous theorem, then μ-almost every number is normal.

Thank you for listening!