

# When does the box dimension of a fractal exist?

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<sup>1</sup>Based on joint work with Alex Rutar, <https://arxiv.org/abs/2406.12821> Picture by Prokofiev, CC BY-SA 3.0



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# Box dimension

- The least number of balls of radius  $r$  needed to cover the British coastline  $E$  scales like  $N_r(E) \approx r^{-1.2}$  for a range of  $r$ . The 'fractal dimension' of the British coastline is  $\approx 1.2$ .
- The lower and upper box (Minkowski) dimensions of a non-empty bounded set  $E \subset \mathbb{R}^d$  are

$$\underline{\dim}_B E = \liminf_{r \rightarrow 0} \frac{\log N_r(E)}{\log(1/r)}, \quad \overline{\dim}_B E = \limsup_{r \rightarrow 0} \frac{\log N_r(E)}{\log(1/r)}.$$

- Always  $\underline{\dim}_B E \leq \overline{\dim}_B E$ . If  $\underline{\dim}_B E = \overline{\dim}_B E$ , the common value  $\dim_B E$  is the box dimension of  $E$ . In this case the extent to which  $E$  'fills up space' does not vary much at different scales.
- **Question:** for which classes of sets does the box dimension exist?

# Dynamically invariant sets

## Theorem (Barreira 1996 / Gatzouras–Peres 1997)

If  $f: M \rightarrow M$  is an expanding **conformal**  $C^1$  map of a Riemannian manifold and a **compact**  $\Lambda \subseteq M$  satisfies  $f(\Lambda) = \Lambda$  and  $f^{-1}(\Lambda) \cap U \subseteq \Lambda$  for a neighbourhood  $U$  of  $\Lambda$ , then  $\dim_B \Lambda$  exists and coincides with Hausdorff dimension.

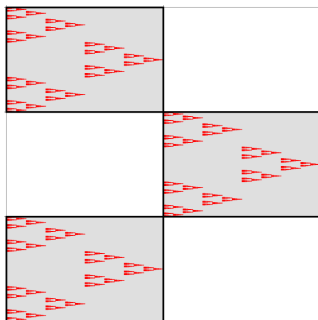
Self-similar and self-conformal sets  $\Lambda$  satisfy  $\dim_H \Lambda = \dim_B \Lambda$ .



The middle-third Cantor set is self-similar and invariant for  $f(x) = 3x \bmod 1$ . Picture based on one by Thefrettinghand, CC BY-SA 3.0

# Non-conformal dynamics

Bedford (1984) and McMullen (1984) constructed compact sets invariant under non-conformal toral endomorphisms such as  $(x, y) \mapsto (2x \bmod 1, 3y \bmod 1)$ , with distinct Hausdorff and box dimension.



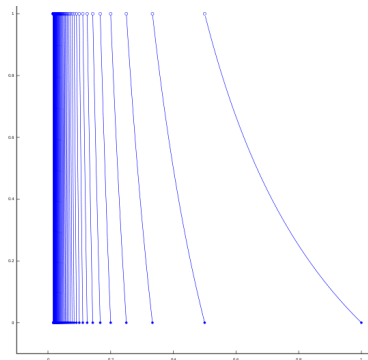
# Non-conformal dynamics

- Jurga (2023) constructed a compact set  $\Lambda$  invariant for  $(x, y) \mapsto (2x \bmod 1, 12y \bmod 1)$  with  $\underline{\dim}_B \Lambda < \overline{\dim}_B \Lambda$ .
- The example is a sub-self-affine set ( $\Lambda \subset \bigcup_i S_i(\Lambda)$  for finitely many affine contractions  $S_i$ ), whereas Bedford–McMullen carpets are self-affine sets ( $\Lambda = \bigcup_i S_i(\Lambda)$ ).
- **Folklore conjecture:** the box dimension of every self-affine set should exist.

# The Gauss map

The Gauss map  $\mathcal{G}: [0, 1) \rightarrow [0, 1)$  is defined by

$$\mathcal{G}(x) = \begin{cases} x^{-1} - \lfloor x^{-1} \rfloor & : 0 < x < 1 \\ 0 & : x = 0. \end{cases}$$



Picture by Adam majewski, CC BY-SA 4.0

# The Gauss map

Typical invariant sets are numbers whose continued fraction expansions are restricted to some  $I \subset \mathbb{N}$ :

$$\Lambda_I := \left\{ z \in (0, 1) \setminus \mathbb{Q} : z = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots}}} , b_n \in I \text{ for all } n \in \mathbb{N} \right\}$$

satisfies  $\mathcal{G}(\Lambda_I) = \Lambda_I$ . If  $I$  is infinite then  $F_I$  is non-compact.

## Theorem

- Mauldin & Urbański ('96, '99): there exists  $I \subset \mathbb{N}$  with  $\dim_{\text{H}} \Lambda_I < \dim_{\text{B}} \Lambda_I$ .
- B.-Rutar ('24+): there exists  $I \subset \mathbb{N}$  with  $\dim_{\text{H}} \Lambda_I < \underline{\dim}_{\text{B}} \Lambda_I < \overline{\dim}_{\text{B}} \Lambda_I$ . In particular, the box dimension of  $\Lambda_I$  does not exist.

# Infinite conformal IFS (Mauldin & Urbański, '96)

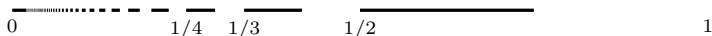
A conformal iterated function system is a countable family of uniformly contracting, conformal maps  $\{S_i: X \rightarrow X\}_{i \in I}$  on a 'nice' (e.g. non-empty convex compact) set  $X \subset \mathbb{R}^d$ . For continued fraction sets the maps are  $\{x \mapsto (b+x)^{-1} : b \in I\}$ . We always assume:

- **Open set condition:**  $\text{Int}(X) \neq \emptyset$  and  $\bigcup_{i \in I} S_i(\text{Int}(X)) \subseteq \text{Int}(X)$  with the union disjoint.
- **Bounded distortion**

The **limit set** is the largest set  $\Lambda \subseteq X$  satisfying

$$\Lambda = \bigcup_{i \in I} S_i(\Lambda)$$

(it is generally non-compact).





# Hausdorff and box dimensions

For  $w \in I^k$  let  $R_w$  be the smallest possible Lipschitz constant for  $S_w := S_{w_1} \circ \cdots \circ S_{w_k}$  and define the **pressure function**

$$P(t) := \lim_{k \rightarrow \infty} \frac{1}{k} \log \sum_{w \in I^k} R_w^t,$$

## Theorem (Mauldin–Urbański, '96, '99)

- $\dim_{\text{H}} \Lambda = \inf\{t > 0 : P(t) < 0\}$
- $\overline{\dim}_{\text{B}} \Lambda = \max\{\dim_{\text{H}} \Lambda, \overline{\dim}_{\text{B}} F\}$ , where  $F$  is the set of **fixed points** of the contractions (or for the continued fraction sets  $F_I$  we can take  $F = \{1/b : b \in I\}$ ).

# Bounds for lower box dimension

Bounds for  $\underline{\dim}_B \Lambda$  that are immediate from Mauldin–Urbański:

$$\max\{\dim_H \Lambda, \underline{\dim}_B F\} \leq \underline{\dim}_B \Lambda \leq \max\{\dim_H \Lambda, \overline{\dim}_B F\} = \overline{\dim}_B \Lambda.$$

## Theorem (B.–Rutar, '24+)

The box dimension of  $\Lambda$  exists if and only if these bounds coincide.

In fact  $\underline{\dim}_B \Lambda$  is **not** a function of  $\dim_H \Lambda$ ,  $\underline{\dim}_B F$ ,  $\overline{\dim}_B F$ :

## Theorem (B.–Rutar, '24+)

The trivial lower bound for  $\underline{\dim}_B \Lambda$  is sharp, and a (non-trivial) sharp upper bound is

$$\underline{\dim}_B \Lambda \leq \dim_H \Lambda + \frac{(\overline{\dim}_B F - \dim_H \Lambda)(d - \dim_H \Lambda) \underline{\dim}_B F}{d \overline{\dim}_B F - \dim_H \Lambda \underline{\dim}_B F}.$$

# An asymptotic formula

These results follow from a formula for  $\underline{\dim}_B \Lambda$  in terms of the whole function

$$r \mapsto s_F(r) := \frac{\log N_r(F)}{\log(1/r)}.$$

(and on the contraction ratios, but only via  $\dim_H \Lambda$ ).  
Define the weighted average

$$\Psi(r, \theta) := (1 - \theta) \dim_H \Lambda + \theta s_F(r^\theta)$$

and  $\psi(r) := \sup_{\theta \in (0,1]} \Psi(r, \theta)$ .

**Theorem (B.-Rutar, '24+)**

$$\frac{\log N_r(\Lambda)}{\log(1/r)} - \psi(r) \xrightarrow[r \rightarrow 0]{} 0, \quad \text{hence} \quad \underline{\dim}_B \Lambda = \liminf_{r \rightarrow 0} \psi(r).$$

Thank you for listening!