Dimensions of self-affine carpets

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[1]Based on joint work with Jonathan Fraser, István Kolossváry, and Alex Rutar
https://arxiv.org/abs/2111.05625
https://arxiv.org/abs/2401.07168

Iterated function systems

An iterated function system (IFS) is a finite set of contractions $\{S_i \colon D \to D\}_{i \in I}$, where $D \subseteq \mathbb{R}^d$ is closed. Hutchinson (1981) showed there is a unique non-empty compact attractor satisfying

$$F = \bigcup_{i \in I} S_i(F).$$

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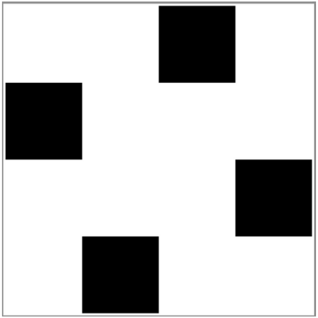

Similarity maps ($S_i(x) = c_i O_i(x) + b_i$ with O_i orthogonal linear). Pictures by Kenneth Falconer.

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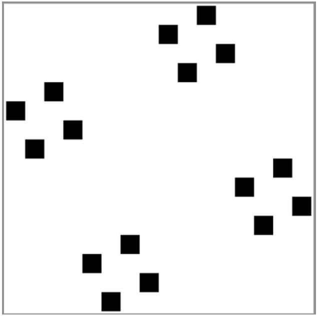

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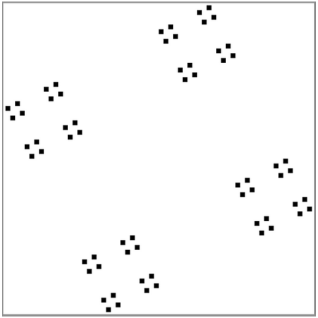

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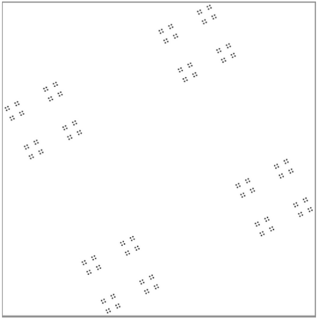

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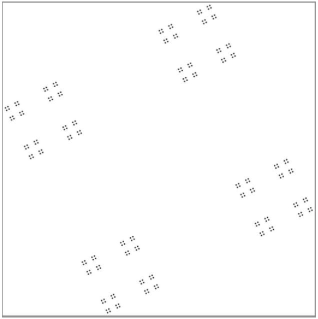

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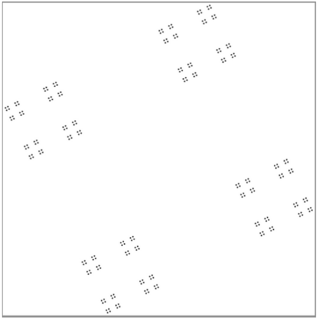

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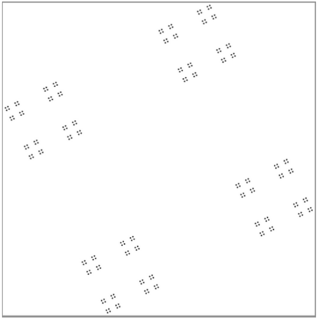

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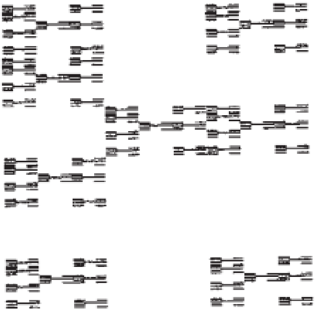


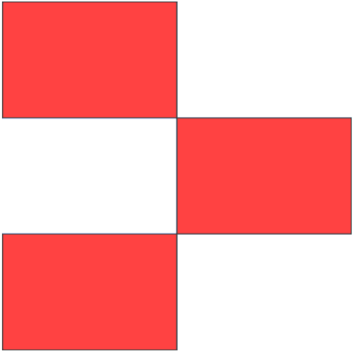
Affine maps ($S_i(x) = T_i(x) + b_i$ with T_i linear). Pictures by Kenneth Falconer.

Dimensions of attractors

- Throughout, we assume the strong open set condition: there is a non-empty bounded open set V intersecting F such that $\bigcup_{i \in I} S_i(V) \subseteq V$ with the union disjoint.
- Then for self-similar sets the Hausdorff, box and Assouad dimensions coincide and satisfy Hutchinson's formula $\sum_i c_i^{\dim F} = 1$.
- Self-affine sets:
  Generic theory: Hausdorff and box dimension often coincide with Falconer's affinity dimension, for example if the translates are randomised (and the singular values of the matrices are less than $1/2$) or if the matrices satisfy mild non-compactness and irreducibility assumptions.
  Exceptional theory: for carpet-like sets, cylinders can line up at small scales, and $\dim_{\mathrm{H}} F < \dim_{\mathrm{B}} F < \dim_{\mathrm{A}} F$ is possible.

Bedford–McMullen carpets

Divide a square into an $m \times n$ grid with $2 \leq m < n$.
$\gamma := \log n / \log m$.
$M :=$ # non-empty columns.
$N_{\hat{\imath}} :=$ # maps in $\hat{\imath}$th non-empty column.
$N := N_1 + \cdots + N_M$.



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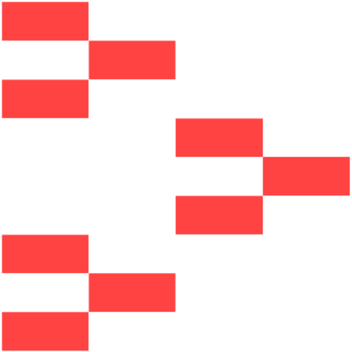

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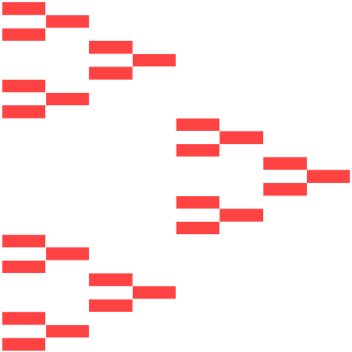

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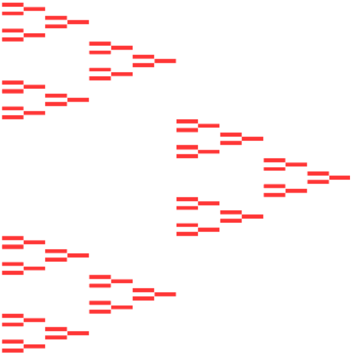

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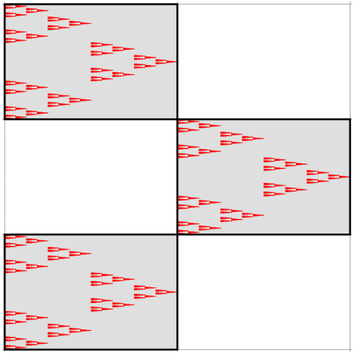

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$N_{\hat{\imath}} := \#$ maps in $\hat{\imath}$th non-empty column.
$N := N_1 + \cdots + N_M$.

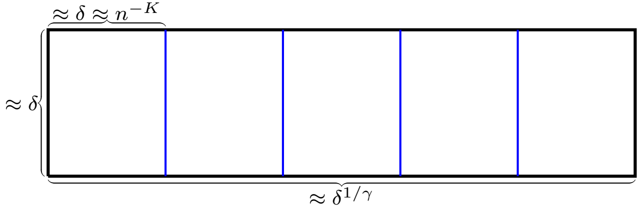
Bedford (1984) and McMullen (1984) proved

$$\dim_{\mathrm{H}} \Lambda = \frac{1}{\log m} \log \left(\sum_{\hat{\imath}=1}^{M} N_{\hat{\imath}}^{\gamma^{-1}} \right)$$

Approximate squares and box dimension

If $0 < \delta \ll 1$, let K be such that $\delta \approx n^{-K}$. An approximate square of size δ can be represented by a string

$$\mathtt{i} = (\underbrace{i_1, \ldots, i_K}_{\in\{1,\ldots,N\}^K}, \underbrace{\hat{\imath}_{K+1}, \ldots, \hat{\imath}_{\lfloor \gamma K \rfloor}}_{\in\{1,\ldots,M\}^{\lfloor \gamma K \rfloor - K}}).$$



$$N_\delta(\Lambda) \approx N^K M^{\gamma K - K} \approx \delta^{-\left(\frac{\log M}{\log m} + \frac{\log(N/M)}{\log n}\right)},$$

so

$$\dim_{\mathrm{B}} \Lambda = \frac{\log M}{\log m} + \frac{\log(N/M)}{\log n}.$$

Assouad type dimensions

Assouad dimension captures the extremal scaling behaviour of the 'thickest' parts of the set.

$$\dim_{\mathrm{A}} F := \inf \{ \alpha : \text{ there exists } C > 0 \text{ such that for all } x \in F \text{ and } 0 < r < R, \text{ we have } N_r(B(x,R) \cap F) \leq C(R/r)^{\alpha} \}.$$

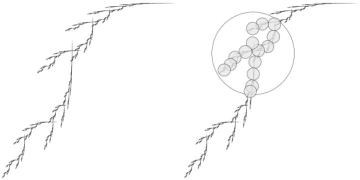


Figure: Picture by Jonathan Fraser

Assouad spectrum (Fraser–Yu, 2018) satisfies $\overline{\dim}_{\mathrm{B}} F \leq \dim_{\mathrm{A}}^{\theta} F \leq \dim_{\mathrm{A}} F$:

$$\dim_{\mathrm{A}}^{\theta} F := \inf \Big\{ \alpha : \text{ there exists } C > 0 \text{ such that for all } x \in F \text{ and } 0 < R < 1, r = R^{1/\theta}, \text{ we have } N_r(B(x,R) \cap F) \leq C(R/r)^{\alpha} \Big\}.$$

Assoud type dimensions

Inside an approximate R-square using only the N_{max} column, up to 'resolution' R^{γ}, Λ looks like the product of self-similar sets $\text{proj}_x \Lambda \times E$, where E is generated by N_{max} contractions with ratio $1/n$. More formally, this product is a weak tangent to Λ:

$$d_{\mathcal{H}}((\text{proj}_x \Lambda \times E), T_R \Lambda) \approx R^{\gamma - 1} \to_R 0.$$

This shows that for $\theta \geq 1/\gamma$,

$$\dim_{\mathrm{A}} \Lambda = \dim_{\mathrm{A}}^{\theta} \Lambda = \dim \text{proj}_x \Lambda + \dim E = \frac{\log M}{\log m} + \frac{\log N_{\text{max}}}{\log n}$$

(Mackay, 2011, and Fraser–Yu, 2018).

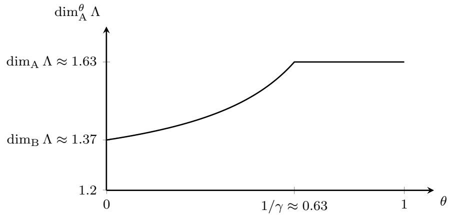
Assouad spectrum when $\theta < \gamma^{-1}$

Now cylinder of height $R^{1/\theta}$ have length $\ll R$. Suppose $R \approx n^{-K}$ and $R^{1/\theta} \approx n^{-k}$.

$$\begin{aligned} R: &\quad (i_1, \ldots, i_K, \hat{\imath}_{K+1}, \ldots, \hat{\imath}_{\lfloor \gamma K \rfloor}) \\ R^{1/\theta}: &\quad (\underbrace{j_1, \ldots, j_K}_{\text{equal}}, \underbrace{j_{K+1}, \ldots, j_{\lfloor \gamma K \rfloor}}_{\text{same column}}, \underbrace{j_{\lfloor \gamma K \rfloor + 1}, \ldots, j_k}_{\text{any map}}, \underbrace{\hat{\jmath}_{k+1}, \ldots, \hat{\jmath}_{\lfloor \gamma k \rfloor}}_{\text{any column}}) \end{aligned}$$

The number of $R^{1/\theta}$ approximate squares inside an approximate R-square is

$$N_{R^{1/\theta}}(B_{\mathtt{i}} \cap \Lambda) \lesssim N_{\max}^{\gamma K - K} N^{k - \gamma K} M^{\gamma k - k} \approx R^{(1-1/\theta)\left(\frac{\dim_{\mathrm{B}} \Lambda - \theta(\log(N/N_{\max})/\log m + \log N_{\max}/\log n)}{1-\theta}\right)}.$$



Equality of dimensions

All three dimensions coincide if the carpet has uniform fibres. Otherwise $\dim_{\mathrm{H}} \Lambda < \dim_{\mathrm{B}} \Lambda < \dim_{\mathrm{A}} \Lambda$.

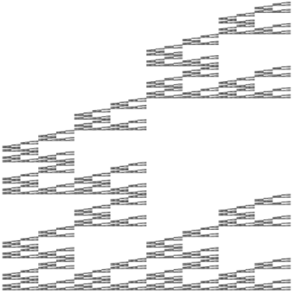


Figure: Uniform fibres

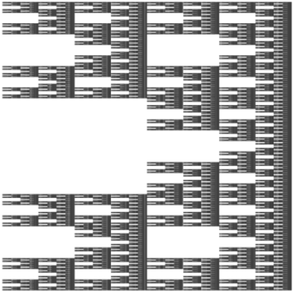
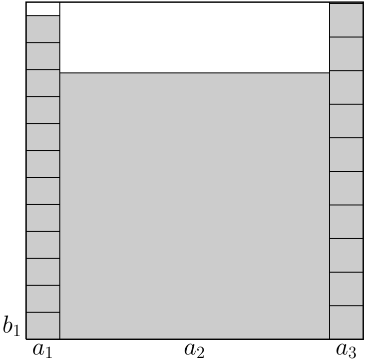


Figure: Non-uniform fibres

Pictures by Jonathan Fraser

Lalley–Gatzouras carpets



For simplicity assume there is only one type of map in each column. Let $\theta_i := \log a_i / \log b_i$, so $\theta_1 < \theta_2 < \theta_3$. Suffices to consider cylinders of height R formed using only maps in the first column.
Mackay (2011) and B.–Fraser–Kolossváry (in progress): using the approximate square formed using only the third column, for $\theta \geq \theta_{\max}/(1 + \theta_{\max} - \theta_{\min})$,

$$\dim_{\mathrm{A}} \Lambda = \dim_{\mathrm{A}}^{\theta} \Lambda = \dim \mathrm{proj}_x \Lambda + \dim E_3.$$

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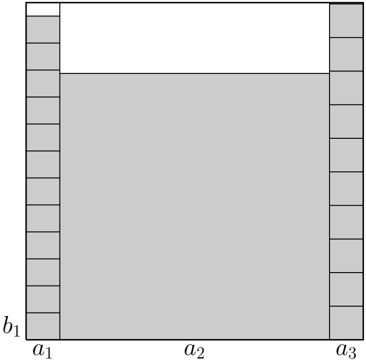

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When $\theta_{\min} \leq \theta < \theta_{\max}/(1 + \theta_{\max} - \theta_{\min})$, use columns 1 and 3 in the ratio such that cylinders of height $R^{1/\theta}$ have length R.

There is another phase transition at $\theta_{\min}$.

Hausdorff dimension

Alternative definition of Hausdorff dimension:

$$\dim_{\mathrm{H}} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \varepsilon \,\}$$

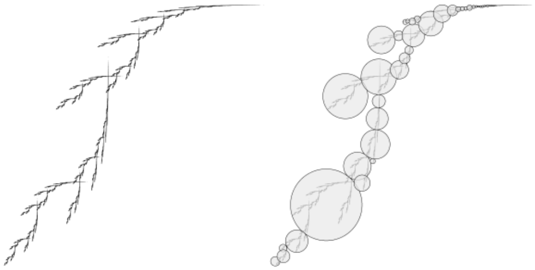


Figure: A cover using sets of different sizes. Picture by Jonathan Fraser.

Box dimension

Alternative definition of box dimension:

$$\overline{\dim}_{\mathrm{B}} F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \varepsilon \,\}.$$

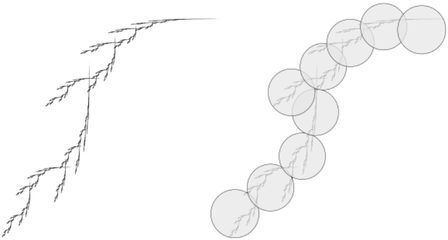


Figure: A cover using sets of the same size. Picture by Jonathan Fraser.

Intermediate dimensions

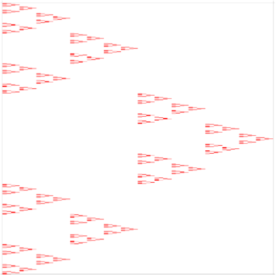
Upper θ-intermediate dimension for $\theta \in (0,1)$ (Falconer–Fraser–Kempton, 2020):

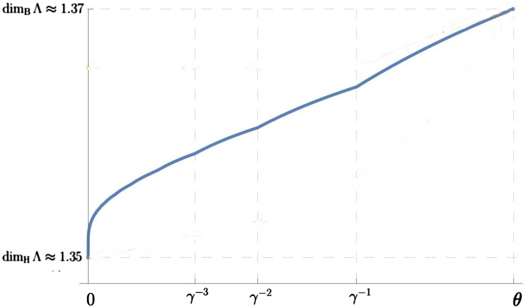
$$\overline{\dim}_\theta F = \inf\{\, s \geq 0 : \text{for all } \varepsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \varepsilon \,\}.$$

Always $\dim_{\mathrm{H}} F \leq \overline{\dim}_\theta F \leq \overline{\dim}_{\mathrm{B}} F$, and $\theta \mapsto \overline{\dim}_\theta F$ is increasing.

Intermediate dimensions

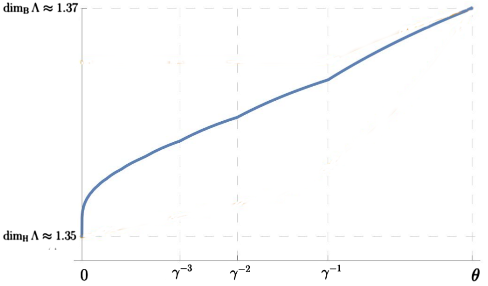
- We work with Bedford–McMullen carpets, and leave the intermediate dimensions of more general self-affine carpets as an open problem.
- Falconer, Fraser and Kempton (2020) proved that $\dim_\theta \Lambda \to \dim_{\mathrm{H}} \Lambda$ as $\theta \to 0$.
  Burrell, Falconer and Fraser (2021) deduced that if $\dim_{\mathrm{H}} \Lambda < 1$ then $\overline{\dim}_{\mathrm{B}} \pi(\Lambda) < 1$ for every orthogonal projection π.
- B. and Kolossváry (2021+) have calculated a precise formula.



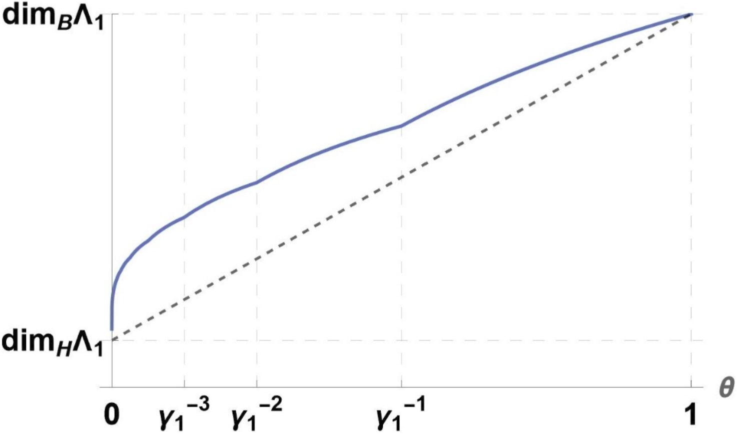


Intermediate dimensions

- Phase transitions at negative integer powers of γ.
- Real analytic and strictly concave between phase transitions
- Strictly increasing
- Right derivative tends to ∞ as $\theta \to 0$



Different possible shapes of the graph



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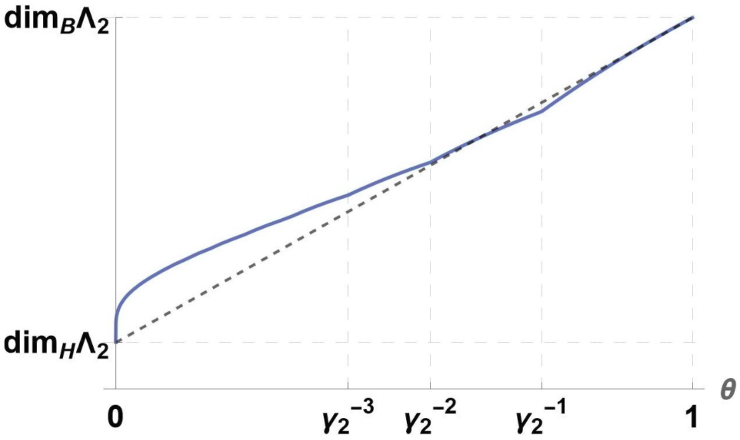

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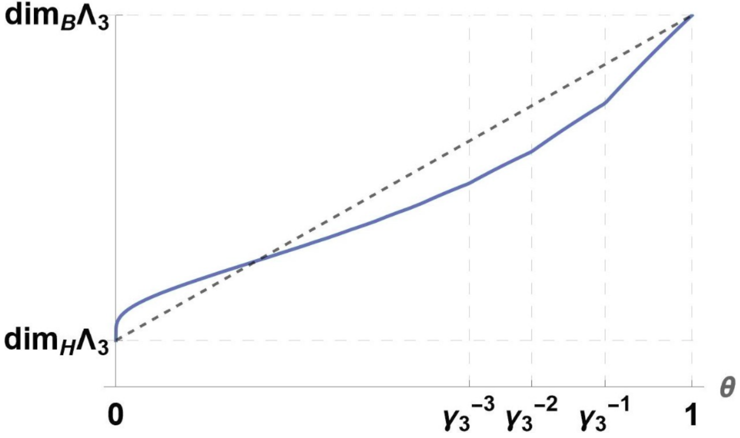

Proof outline when $\theta \geq 1/\gamma$

For $\theta \geq 1/\gamma$ our cover uses only the two extreme scales. Letting $n^{-K} \approx \delta$ and comparing s-costs ($\sum_i |U_i|^s$), we keep a square at size δ iff

$$\delta^s \leq \delta^{s/\theta} M^{\gamma K/\theta - \gamma K} \prod_{\ell=K+1}^{\lfloor K/\theta \rfloor} N_{\hat{\imath}_\ell},$$

i.e.

$$\frac{1}{\lfloor K/\theta \rfloor - K} \sum_{\ell=K+1}^{\lfloor K/\theta \rfloor} \log N_{\hat{\imath}_\ell} \geq \left(s - \frac{\log M}{\log m} \right) \log n =: t_1(s).$$

Let $X_1, X_2, \ldots$ be i.i.d random variables, uniformly distributed on $\{\log N_1, \ldots, \log N_M\}$. The number of squares which we keep at scale δ is

$$M^{\lfloor K/\theta \rfloor - K} \mathbb{P} \left(\frac{1}{\lfloor K/\theta \rfloor - K} \sum_{\ell=1}^{\lfloor K/\theta \rfloor - K} X_\ell \geq t_1(s) \right) \approx M^{K/\theta - K} e^{-(K/\theta - K) I(t_1(s))}$$

by Cramér's theorem.

Proof outline when $\theta \geq 1/\gamma$

- The function $I(t)$ is the rate function given by the Legendre transform

$$I(t) := \sup_{\lambda \in \mathbb{R}} \left\{ \lambda t - \log \left(\frac{1}{M} \sum_{\hat{\jmath}=1}^{M} N_{\hat{\jmath}}^{\lambda} \right) \right\}.$$

- To show that this is the true s-cost of this cover, group the exponentially (in K) many δ-squares into polynomially many type classes based on digit frequency.
  The critical type class has the same s-cost whether we subdivide or not.
- Lower bound: put a measure on the critical type class and apply a mass distribution principle.
  The value of $s(\theta) = \dim_\theta \Lambda$ for which this cost neither blows up nor decays exponentially in K solves

$$s = \dim_{\mathrm{B}} \Lambda - \frac{1}{\log n} \left(\frac{1}{\theta} - 1 \right) I(t_1(s)).$$

Case when $\theta = \gamma^{-(L-1)}$

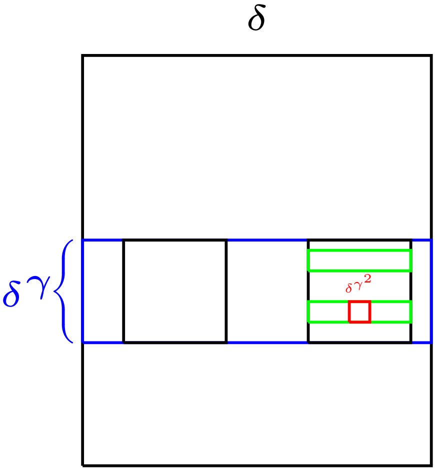

Figure: Approximate squares of size δ^γ and δ^{γ^2} inside an approximate square of size δ.

Proof strategy when $\theta = \gamma^{-(L-1)}$

Keep an approximate δ-square at size δ iff

$$\frac{1}{\gamma K - K} \sum_{j=K}^{\lfloor \gamma K \rfloor} \log N_{\hat{\imath}_j} \geq t_1(s).$$

If we subdivide, then keep the δ^γ-squares at this level iff

$$\frac{1}{\gamma^2 K - \gamma K} \sum_{j=\lfloor \gamma K \rfloor}^{\lfloor \gamma^2 K \rfloor} \log N_{\hat{\imath}_j} \geq t_2(s)$$

for some threshold $t_2(s)$. Repeating gives a cover using scales $\delta, \delta^\gamma, \ldots, \delta^{\gamma^{L-1}}$. Optimising the $t_i(s)$ so the s-cost of each part of the cover is equal gives $t_i(s) = T_s(t_{i-1}(s))$, where

$$T_s(t) \coloneqq t_1(s) + \gamma I(t).$$

Deduce that the critical exponent $s = \dim_\theta \Lambda$ solves

$$s = \dim_{\mathrm{B}} \Lambda - \frac{1}{\log m} \left(1 - \frac{1}{\gamma}\right) I(t_{L-1}(s)).$$

Proof strategy in general case

When $\gamma^{-L} < \theta < \gamma^{-(L-1)}$ the cover uses scales $\delta, \delta^{\gamma}, \delta^{\gamma^2}, \ldots, \delta^{\gamma^{L-1}}$ and $\delta^{1/\theta}, \delta^{1/(\gamma\theta)}, \ldots, \delta^{1/(\gamma^{L-1}\theta)}$. Which scales to cover a square with depends how the different parts of the symbolic representation relate to each other.
The dimension $s = \dim_\theta \Lambda$ solves

$$\gamma^L \theta \log N - (\gamma^L \theta - 1) t_L(s) + \gamma(1 - \gamma^{L-1}\theta)(\log M - I(t_L(s))) - s \log n = 0.$$

Thank you for listening!

Questions welcome