Dimensions of Bedford–McMullen carpets

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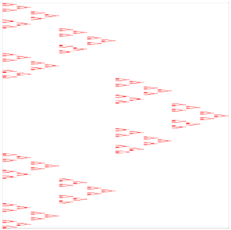


Based on work in 'Intermediate dimensions of Bedford-McMullen carpets with applications to Lipschitz equivalence' (with István Kolossváry), arXiv preprint (2021), https://arxiv.org/abs/2111.05625

I also thank István for many of the pictures in these slides.

Fractals and dimension

- Fractals are sets which typically have features such as fine structure at arbitrarily small scales, or some kind of self-similarity.
- Many different notions of dimension which attempt to quantify the 'thickness' of sets at small scales.
- Throughout, $F \subset \mathbb{R}^d$ will be non-empty and bounded.
- The (upper) box dimension is defined by

$$\overline{\dim}_B F := \limsup_{\delta \to 0^+} \frac{\log N_\delta(F)}{-\log \delta}$$

  where $N_\delta(F)$ is the smallest number of balls of radius δ needed to cover F.
- Intuitively, a disc has box dimension 2 because the number of discs of size r needed to cover it scales approximately like r^{-2} as $r \to 0$.

Hausdorff dimension

- Letting $|U_i| := \mathrm{diam}(U_i)$, for $s \geq 0$ define

$$H^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s \,\middle|\, \{U_i\} \text{ is a countable cover of } F \text{ with each } |U_i| \leq \delta \right\}.$$

- As δ decreases, the infimum increases, so converges to a limit

$$H^s_\delta(F) \to H^s(F) \in [0, \infty] \text{ as } \delta \to 0,$$

called the s-dimensional Hausdorff measure of F, whose restriction to the Borel sets is a measure.

Hausdorff dimension

- Straightforward to see that there is a unique $s \geq 0$, called the Hausdorff dimension of F, denoted $\dim_{\mathrm{H}} F$, such that if $0 \leq t < s$ then $H^t(F) = \infty$ and if $t > s$ then $H^t(F) = 0$.
- Intuitively, disc has Hausdorff dimension 2 because it has positive and finite area.
- Always $\dim_{\mathrm{H}} F \leq \dim_{\mathrm{B}} F$. For some fractals this is strict, indicating an inhomogeneity in space.

Alternative definitions

- (Upper) box dimension:

$$\overline{\dim}_{\mathrm{B}} F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0, \delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

- Hausdorff dimension:

$$\dim_{\mathrm{H}} F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \epsilon \,\}$$

- Falconer, Fraser and Kempton (2020) noted that these "may be regarded as two extreme cases of the same definition..."

Intermediate dimensions

- and defined the upper θ-intermediate dimension of F for $\theta \in (0,1)$ by

$$\overline{\dim}_\theta F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

- Indeed, for all sets F, $\dim_{\mathrm{H}} F \leq \overline{\dim}_\theta F \leq \overline{\dim}_{\mathrm{B}} F$ for all $\theta \in (0,1)$.
- Function $\theta \mapsto \overline{\dim}_\theta F$ increasing, continuous for $\theta \in (0,1]$ but not necessarily continuous at $\theta = 0$ (see Φ-intermediate dimensions, B. '20).
- Example of dimension interpolation. See also Assouad spectrum (Fraser-Yu, '18).

Simple example: polynomial sequence

Let $F = \{0\} \cup \{ 1/n : n \in \mathbb{N} \}$.
More 'dense' near the origin (cover with largest possible intervals) and more 'sparse elsewhere (cover with smallest possible intervals).

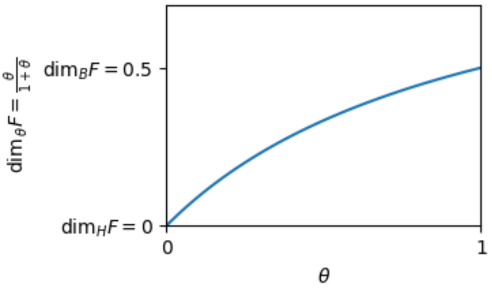


Figure: Intermediate dimensions of the set F, proved by Falconer, Fraser and Kempton ('20)

Iterated function systems (IFSs)

- A finite set of contractions $\{S_i \colon X \to X\}_{i \in I}$ where $X \subset \mathbb{R}^d$ is closed. By Hutchinson (1981) there is a unique non-empty compact attractor satisfying

$$F = \bigcup_{i \in I} S_i(F).$$

- Self-similar sets: for all i there exists c_i such that $||S_i(x) - S_i(y)|| = c_i ||x - y||$ for all $x, y \in X$. These have equal Hausdorff and box dimensions h. Under separation conditions, this satisfies Hutchinson's formula:

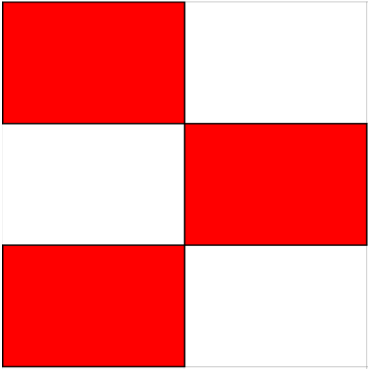
$$\sum_{i \in I} c_i^h = 1.$$

Self-affine sets

Now assume each contraction is affine: $S_i(\underline{x}) = A_i\underline{x} + \underline{b}_i$.
By Bárány-Hochman-Rapaport ('19), under separation conditions, planar self-affine sets "typically" have equal Hausdorff and box dimensions (if the group of matrices $\langle A_i \rangle$ does not preserve a finite union of lines and is compact in $GL_2(\mathbb{R})$), but not always...

Bedford–McMullen carpets (Bedford '84, McMullen '84)

- A widely-studied class of self-affine fractals in the plane.
- Divide a square into an $m \times n$ grid, $m < n$. Write $\gamma := \frac{\log n}{\log m}$. Parameters: $M :=$ #non-empty columns, $N_{\hat{\imath}} :=$ #maps in column $\hat{\imath}$, $N := N_1 + \ldots + N_M$.
- IFS: $\{S_1, \ldots, S_M\}$ where $S_i(\underline{x}) := \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} (\underline{x}) + \underline{t}_i$.



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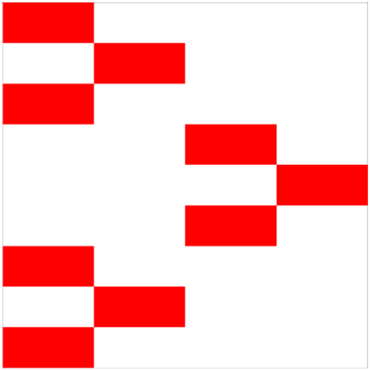

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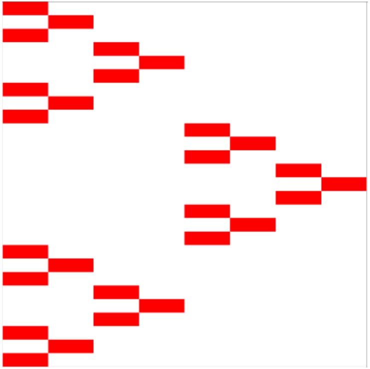

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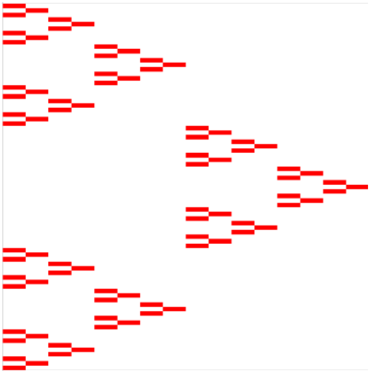

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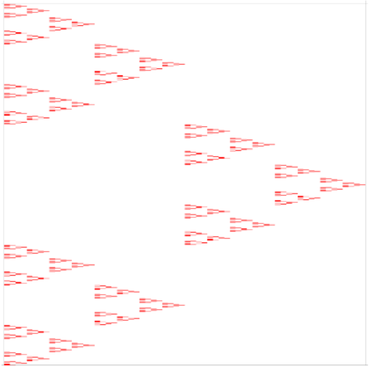


Hausdorff and box dimensions

Cylinders become elongated and line up, causing inhomogeneity in space.

Theorem (Bedford '84, McMullen '84)

$$\dim_{\mathrm{H}} \Lambda = \frac{1}{\log m} \log \left(\sum_{\hat{\imath}=1}^{M} N_{\hat{\imath}}^{\frac{\log m}{\log n}} \right); \qquad \dim_{\mathrm{B}} \Lambda = \frac{\log M}{\log m} + \frac{\log(N/M)}{\log n}.$$

In particular, $\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda$ if and only if Λ has uniform vertical fibres.
Throughout we assume this is not the case.

Intermediate dimensions: asymptotic behaviour at $\theta = 0$

Let Λ be a Bedford–McMullen carpet with non-uniform vertical fibres.

Proposition (Falconer–Fraser–Kempton, '20)

For small enough θ,

$$\overline{\dim}_\theta \Lambda \leq \dim_{\mathrm{H}} \Lambda + \frac{C}{-\log \theta}.$$

In particular, $\overline{\dim}_\theta \Lambda$ is continuous at $\theta = 0$.

Proposition (B.-Kolossváry, '21)

For small enough θ,

$$\dim_{\mathrm{H}} \Lambda + \frac{C_1}{(\log \theta)^2} \leq \dim_\theta \Lambda \leq \dim_{\mathrm{H}} \Lambda + \frac{C_2}{(\log \theta)^2}.$$

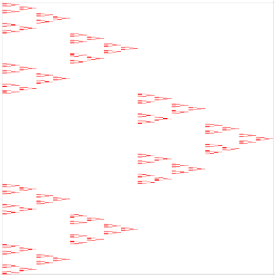
In particular, the slope of $\theta \mapsto \dim_\theta \Lambda$ tends to ∞ as $\theta \to 0$.

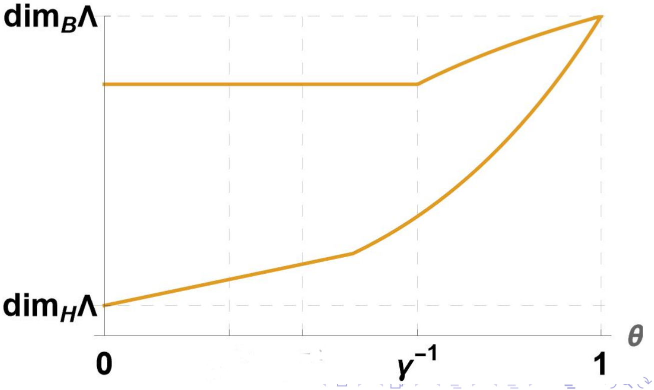
Consequences of continuity at $\theta = 0$

- If $\dim_{\mathrm{H}} \Lambda < 1 \leq \dim_{\mathrm{B}} \Lambda$ then $\overline{\dim}_{\mathrm{B}} \pi(\Lambda) < 1$ for every orthogonal projection π from $\mathbb{R}^2$ onto a 1-dimensional subspace (Burrell–Falconer–Fraser, '21).
- If $B_\alpha \colon \mathbb{R}^2 \to \mathbb{R}^2$ is index-α fractional Brownian motion, then if $\alpha > (\dim_{\mathrm{H}} \Lambda)/2$ then almost surely $\overline{\dim}_{\mathrm{B}} B_\alpha(\Lambda) < 2$ (Burrell, '20).

Further bounds

- Falconer, Fraser and Kempton ('20) proved a lower bound showing that $\dim_\theta \Lambda > \dim_{\mathrm{H}} \Lambda$ for all $\theta > 0$.
- Kolossváry ('20) proved upper and lower bounds near $\theta = 1$ showing that $\dim_\theta \Lambda < \dim_{\mathrm{B}} \Lambda$ for all $\theta < 1$, and that the graph is neither convex nor concave in general.
- A general lower bound near $\theta = 1$ was obtained in B. ('20).





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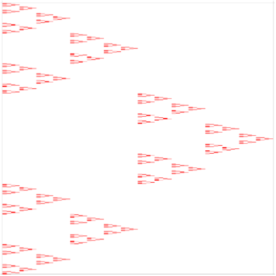
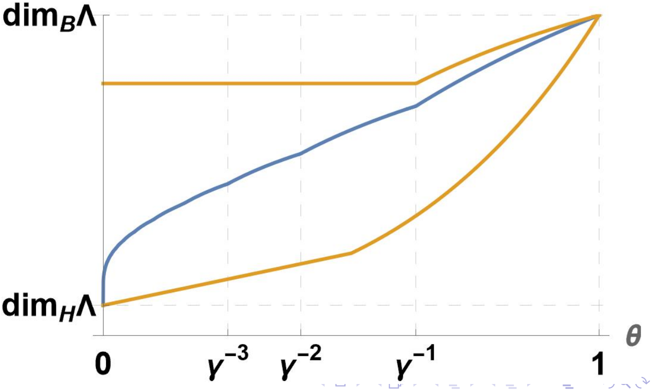




Rate function

- Define the function $I(t)$ as the Legendre transform

$$I(t) := \sup_{\lambda \in \mathbb{R}} \left\{ \lambda t - \log \left(\frac{1}{M} \sum_{\hat{\jmath}=1}^{M} N_{\hat{\jmath}}^{\lambda} \right) \right\}.$$

  This is a large deviation rate function, describing the exponential decline of the probability of certain extreme events - see Cramér's theorem.

- For $s \in \mathbb{R}$, define the function $T_s \colon \mathbb{R} \to \mathbb{R}$ by

$$T_s(t) := \left(s - \frac{\log M}{\log m} \right) \log n + \gamma I(t).$$

- For $\ell \in \mathbb{N}$, write $T_s^{\ell} := \underbrace{T_s \circ \cdots \circ T_s}_{\ell \text{ times}}$, and T_s^0 is the identity. Define

$$t_\ell(s) := T_s^{\ell-1} \left(\left(s - \frac{\log M}{\log m} \right) \log n \right).$$

Main result: formula for the intermediate dimensions

Theorem (B.-Kolossváry, '21)

Let Λ be any Bedford–McMullen carpet with non-uniform vertical fibres. For fixed $\theta \in (0,1)$ let $L = L(\theta) \in \mathbb{N}$ be such that $\gamma^{-L} < \theta \leq \gamma^{-(L-1)}$. Then there exists a unique solution $s = s(\theta) \in (\dim_{\mathrm{H}} \Lambda, \dim_{\mathrm{B}} \Lambda)$ to the equation

$$\gamma^L \theta \log N - (\gamma^L \theta - 1) t_L(s) + \gamma(1 - \gamma^{L-1}\theta)(\log M - I(t_L(s))) - s \log n = 0,$$

and $s(\theta) = \dim_{\theta} \Lambda$.

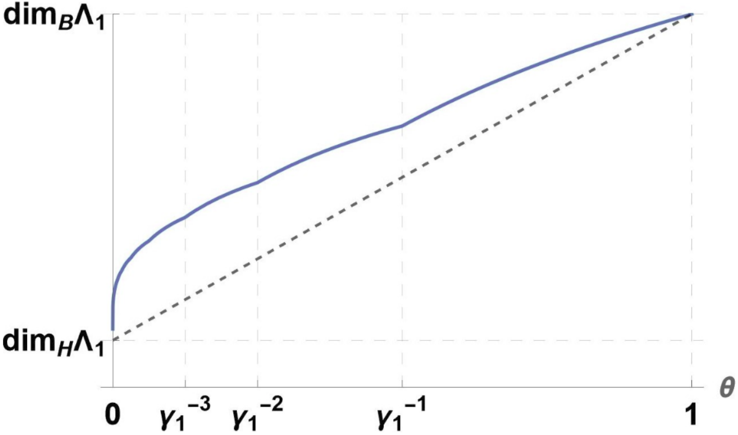
In particular the intermediate dimensions exist.
If the carpet has just two column types then we can calculate the rate function explicitly. We can always compute the intermediate dimensions numerically and draw plots.

Comments about the proof

- Cover uses an increasing number of scales as $\theta \to 0$. Simplest case: when $\theta \geq 1/\gamma$, just use largest and smallest scales.
- We can prove that we need to use more than two scales to achieve the intermediate dimension (when θ is small). The cover for continued fraction sets also uses many scales (B.-Fraser, '21).
- Covering strategy involves breaking the carpet into approximate squares and deciding how to cover each one. An approximate square of size δ is made of stacking cylinders of length $\approx \delta$ and height $\approx \delta^\gamma$ on top of each other.

Shape of the graph

Varying the parameters can lead to the graph having different shapes.



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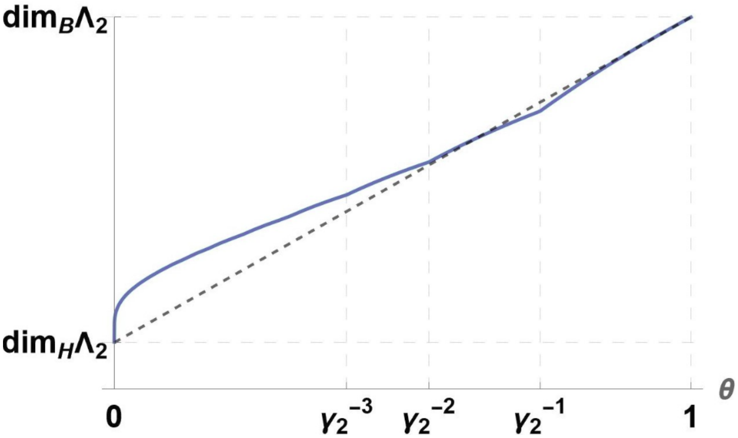

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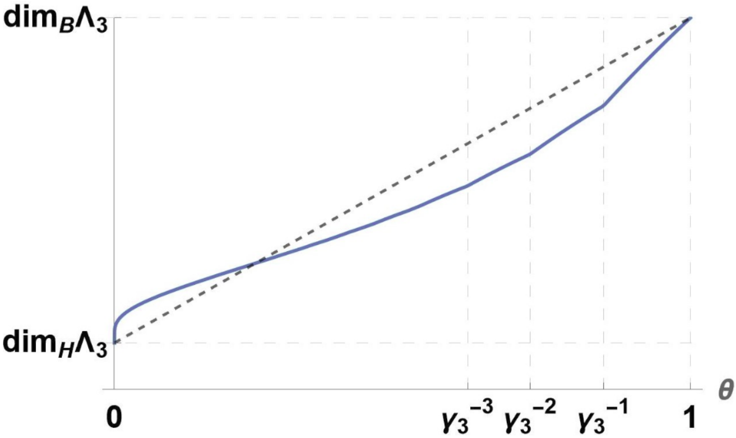

Properties of the graph: a form not seen in previous examples

- Intermediate dimensions are strictly increasing (using implicit differentiation).
- There is a phase transition (derivative jumps up) at every negative integer power of γ. Intuitively, this is because there is a discrete jump in the number of scales used.
- Graph is strictly concave between phase transitions.
- The graph is real analytic between phase transitions (using implicit function theorem methods).

Two carpets with the same intermediate dimensions

- We show that if two carpets with non-uniform vertical fibres have equal intermediate dimensions for all θ then they can be defined on the same grid, on which an explicit condition considered by Rao–Yang–Zhang ('20) on the parameters holds.
- In particular, if the number of non-empty columns is equal, then this condition says that the column sequence of one carpet must be a permutation of the other.
- There exist carpets, taken from an example of Rao, Yang and Zhang ('20), defined on the same grid, with different column sequences, different number of non-empty columns, but the same intermediate dimensions.

Example

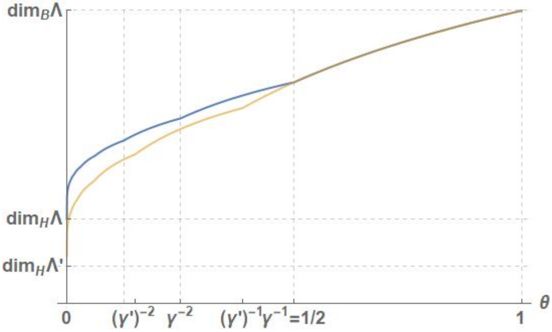

Figure: Intermediate dimensions of two particular carpets we construct on different grids. Equal on an interval $(\theta_0, 1)$ but not for the whole spectrum of θ.

Impossible for carpets defined on the same grid.

Bi-Lipschitz equivalence

- If $F, G \subset \mathbb{R}^d$ then $f \colon F \to G$ is bi-Lipschitz if there exists $C \in [1, \infty)$ such that

$$C^{-1}||x-y|| \leq ||f(x)-f(y)|| \leq C||x-y|| \qquad \text{for all } x, y \in F.$$

- Then straightforward to see that $\overline{\dim}_\theta F = \overline{\dim}_\theta G$.
- Difficult open problem: characterise when two Bedford–McMullen carpets are bi-Lipschitz equivalent.
- Being defined on the same grid and satisfying the explicit condition mentioned previously is therefore a necessary condition for two carpets with non-uniform fibres to be bi-Lipschitz equivalent. This improves a result of Rao, Yang and Zhang ('20).

Bi-Lipschitz equivalence

- Consider the two carpets with $m = M = 32$, $n = 243$, and parameters

$$\begin{aligned} \Lambda_1 &: \qquad N_1 = N_2 = 27, \quad N_3 = \cdots = N_{13} = 3, \quad N_{14} = \cdots = N_{32} = 1, \\ \Lambda_2 &: \qquad N_1 = 27, \quad N_2 = \cdots = N_7 = 9, \quad N_8 = \cdots = N_{32} = 1. \end{aligned}$$

- Then by the discussion above they have different intermediate dimensions. But all other (usual) dimensions are equal.
- The maps can be arranged so that the Rao–Yang–Zhang result does not apply. So only the intermediate dimensions can tell that they are not bi-Lipschitz equivalent.

Hölder distortion

- If $F \subset \mathbb{R}^d$ and $\alpha \in (0,1]$ then $f\colon F \to \mathbb{R}^d$ is α-Hölder if there exists $C \geq 0$

$$||f(x) - f(y)|| \leq C||x - y||^{\alpha} \qquad \text{for all } x, y \in F.$$

- Straightforward to see that if $f\colon \Lambda' \to \mathbb{R}^2$ is α-Hölder then

$$\dim_{\theta} f(\Lambda') \leq \alpha^{-1} \dim_{\theta} \Lambda'.$$

Hölder distortion

- So if $f(\Lambda') \supseteq \Lambda$ then using the optimal value $\theta \approx 0.40$,

$$\alpha \leq \frac{\dim_{\gamma^{-2}} \Lambda'}{\dim_{\gamma^{-2}} f(\Lambda')} \leq \frac{\dim_{\gamma^{-2}} \Lambda'}{\dim_{\gamma^{-2}} \Lambda} < 0.9995,$$

with the last inequality computed numerically.

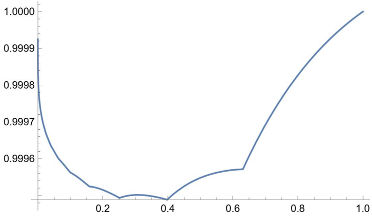

Figure: Ratio of intermediate dimensions

- Intermediate dimensions can also sometimes give better Hölder distortion estimates than the Hausdorff or box dimensions for continued fraction sets (B.-Fraser, '21).

Open problems

- Do there exist two carpets with non-uniform vertical fibres defined on the same grid with different column sequences that are bi-Lipschitz equivalent?
- Intermediate dimensions of other classes of self-affine sets, e.g. higher-dimensional sponges?

Thank you for listening! Questions welcome.