DISTINCT DIMENSIONS FOR ATTRACTORS OF ITERATED FUNCTION SYSTEMS

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ABSTRACT. In this paper, we construct an iterated function system on $\mathbb{R}$ consisting of two bi-Lipschitz contractions such that the Hausdorff dimension and lower and upper box dimensions of its attractor are all different.

1. INTRODUCTION

By an iterated function system (IFS) on $\mathbb{R}^d$, we mean a finite collection $\Phi = \{\phi_i\}_{i=1}^{\ell}$ of contraction mappings from $\mathbb{R}^d$ to $\mathbb{R}^d$. It is well known [15] that given an IFS there exists a unique non-empty compact set $E \subset \mathbb{R}^d$, called the **attractor** of Φ, such that

$$E = \bigcup_{i=1}^{\ell} \phi_i(E).$$

The attractor E is said to be **self-similar** if all ϕ_i are similitudes, and **self-affine** if all ϕ_i are affine maps. The IFS Φ is said to be a **bi-Lipschitz IFS** if the maps ϕ_i are bi-Lipschitz contractions on $\mathbb{R}^d$, i.e. there exists $0 < A \le B < 1$ such that

$$A|x-y| \le |\phi_i(x) - \phi_i(y)| \le B|x-y|, \quad \text{for all } x, y \in \mathbb{R}^d.$$

One of the fundamental problems in fractal geometry is to study the dimension theory of attractors of IFSs; see e.g. [4, 10, 17, 3]. In this paper, we are mainly concerned with the following folklore question in fractal geometry.

Question 1.1. *Does the box dimension of the attractor of an IFS on $\mathbb{R}^d$ always exist?*

It is well known that Hausdorff dimension, lower box dimension and upper box dimension satisfy $\dim_{\mathrm{H}} E \le \underline{\dim}_{\mathrm{B}} E \le \overline{\dim}_{\mathrm{B}} E$ for all non-empty bounded sets $E \subset \mathbb{R}^d$ and that these inequalities can be strict in general. The question of which classes of IFS allow these inequalities to be strict is much more subtle. In [8, Theorems 3-4] (see also [9, Chapter 3]), Falconer gave some sufficient conditions for these three notions of dimension to coincide for a compact metric space. As an application, he showed that, without requiring any separation conditions, the box dimension of a self-similar set in $\mathbb{R}^d$ always exists and coincides with its Hausdorff dimension, and moreover, this property (i.e. the coincidence of the Hausdorff, lower and upper box dimensions) also holds for all $C^{1+\delta}$ conformal repellers. Later, this property was further verified

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for all C^1 conformal repellers by Gatzouras and Peres [14] and Barreira [2], and for the attractors of all C^1 (weakly) conformal IFSs by Feng and Hu [12, Theorem 8.1]. However, the situation for non-conformal repellers or self-affine sets is quite different. It is well known [5, 20] that the Hausdorff and box dimensions of a special class of planar self-affine sets, known as Bedford–McMullen carpets, are generally distinct; however, the box dimension of these self-affine sets always exists.

Recently, Jurga [16] showed that there exist integers $n > m \geq 2$ and a compact subset E of the 2-torus $\mathbb{T}^2$ such that E is invariant under the expanding toral endomorphism $T(x, y) = (mx \pmod 1, ny \pmod 1)$, and the box dimension of F does not exist. However, this invariant set may not be an IFS attractor. It has been known since the classical work of Mauldin and Urbański [18, 19] that the attractor of an infinite conformal IFS (which is generally non-compact) can have distinct Hausdorff and upper box dimension. Very recently Banaji and Rutar [1] showed that for these sets the Hausdorff dimension and lower and upper box dimension can all be distinct. There are detailed discussions about different notions of dimension for different classes of fractals in [3]. In particular, in [3, Theorem 2.8.4], it was shown that the packing dimension is always equal to the upper box dimension for an attractor of a bi-Lipschitz IFS.

In this paper, we provide an explicit construction of an IFS consisting of two bi-Lipschitz contractions on $\mathbb{R}$, for which the box dimension of its attractor does not exist, and the Hausdorff and lower box dimension are also distinct. This gives a negative answer to Question 1.1.

Theorem 1.2. *There exists a bi-Lipschitz iterated function system on $\mathbb{R}$ such that the Hausdorff dimension, lower box dimension and upper box dimension of the attractor are all different.*

Theorem 1.2 will be proved as a consequence of a more statement (Theorem 2.6). The idea of the construction can be understood as follows: for a Cantor set E with $\{0, 1\} \subset E \subset [0, 1]$, through the standard Cantor construction of E, we are able to define two strictly increasing continuous functions ϕ_0, ϕ_1 from $[0, 1]$ to itself such that

$$E = \phi_0(E) \cup \phi_1(E), \tag{1.1}$$

with the union being disjoint. Furthermore, we provide a necessary and sufficient condition for ϕ_0, ϕ_1 to be bi-Lipschitz contraction maps on $[0, 1]$;[1] see Theorem 2.1.

Next, we will apply this theorem to a special Cantor set construction. In this construction, the generating intervals of the Cantor set have lengths equal to $b_\omega M^{-n}$ where M is a large number and b_ω is an oscillatory function depending on the orbit of iterations and lies in the interval $(2^{-n}, 2^n)$. The precise formula for b_ω was motivated by the calculation of the dimensions of Bedford–McMullen carpets [20]. We will show

[1] In this case, we can extend ϕ_0, ϕ_1 to bi-Lipschitz contraction maps on $\mathbb{R}$ by simply letting ϕ_0, ϕ_1 be piecewise linear on $(-\infty, 0] \cup [1, \infty)$ with slope $1/2$.

that the Cantor set is generated by a bi-Lipschitz IFS consisting of two maps and has distinct Hausdorff and box dimensions. Finally, to separate the upper and lower box dimensions, we oscillate the contraction ratios in between the two values $(M-\delta)^{-1}$ and $(M+\delta)^{-1}$ for all scales of iterations. With a suitable perturbation and small enough $\delta > 0$, all three notions of dimension will be distinct. Both the unperturbed and perturbed IFSs satisfy the strong separation condition, as in (1.1).

It is also possible to offer a simpler construction of Cantor sets generated by a bi-Lipschitz IFS whose box dimension does not exist using a class of symmetric Cantor sets. Due to the homogeneity of the length of the intervals in each iteration, we can obtain an easily checkable sufficient condition for a symmetric Cantor set to be the attractor of a bi-Lipschitz IFS; see Proposition 6.2. Meanwhile, it is well known that the box dimension of a symmetric Cantor set does not necessarily exist (see [13]). Hence by Proposition 6.2 and the dimensional results in [13], we can construct symmetric Cantor sets, which serve as the attractors of bi-Lipschitz IFSs, but their box dimensions do not exist. However, in this construction, the Hausdorff dimension is always equal to the lower box dimension, so it does not offer a proof for Theorem 1.2. Nonetheless, as an interesting part of this simpler construction, we will prove that the contraction maps are non-differentiable at all points inside this Cantor set (see Proposition 6.5). This leads to the following open question for future study:

Open Question 1. *Is there an IFS on $\mathbb{R}^d$ consisting of differentiable contractions for which the box dimension of its attractor does not exist?*

The intermediate dimensions are a family of dimensions (parameterized by $\theta \in (0,1)$ and introduced in [11]) which lie between Hausdorff and box dimension. Another possible direction for future study would be to investigate the intermediate dimensions of attractors of IFSs on $\mathbb{R}$ (such as the one from Theorem 1.2) with distinct Hausdorff and lower box dimension.

There is a natural dynamical interpretation to our results. Corollary 1.3 below is a straightforward consequence of Theorem 1.2. It illustrates that the dimension theory of dynamical systems is very different when the expanding map merely has Lipschitz regularity compared with C^1 regularity. Indeed, if in Corollary 1.3 one additionally assumes that the map f is C^1, then the conclusion for the invariant set E would be impossible by [14, 2].

Corollary 1.3. *There exists a Lipschitz map $f\colon \mathbb{R} \to \mathbb{R}$, and $c > 1$, $\delta > 0$, and $E \subset U \subset \mathbb{R}$ where E is non-empty and compact and U is bounded and open, such that the following three conditions hold:*

- *for all $x \in U$ and $y \in (x-\delta, x+\delta)$ we have $|f(x) - f(y)| \geq c|x-y|$,*
- *$E = f(E) = f^{-1}(E) \cap U$,*
- *$\dim_{\mathrm{H}} E < \underline{\dim}_{\mathrm{B}} E < \overline{\dim}_{\mathrm{B}} E$.*

Proof. From the proof of Theorem 1.2, there exist bi-Lipschitz contraction maps $\phi_0, \phi_1 \colon \mathbb{R} \to \mathbb{R}$ with $0 = \phi_0(0) < \phi_0(1) < \phi_1(0) < \phi_1(1) = 1$, and with the attractor $E \subset [0,1]$ of the IFS $\{\phi_0, \phi_1\}$ satisfying $\dim_{\mathrm{H}} E < \underline{\dim}_{\mathrm{B}} E < \overline{\dim}_{\mathrm{B}} E$. Fix $\delta > 0$ small enough that $\phi_0(1) + 2\delta < \phi_1(0) - 2\delta$. We now define f piece by piece. For $0 \le x \le \phi_0(1)$ let $f(x) = \phi_0^{-1}(x)$ and for $\phi_1(0) \le x \le 1$ let $f(x) = \phi_1^{-1}(x)$. On each of the four intervals

$$(-\infty, 0], \quad [\phi_0(1), \phi_0(1) + 2\delta], \quad [\phi_1(0) - 2\delta, \phi_1(0)], \quad [1, \infty),$$

let f be affine with slope 2. Finally, let f be affine on $[\phi_0(1) + 2\delta, \phi_1(0) - 2\delta]$ with the (negative) slope required to ensure f is continuous. Now let $U = (-\delta, \phi_0(1) + \delta) \cup (\phi_1(0) - \delta, 1 + \delta)$. Since ϕ_0 and ϕ_1 are contractions, the desired local uniform expansion for f on U holds. Finally, since f maps $\phi_0(F)$ bijectively onto F, and f maps $\phi_1(F)$ bijectively onto F, we have $E = f(E) = f^{-1}(E) \cap U$, as required. $\square$

The paper is organized as follows. We will present the precise main setup in Section 2. Theorems characterizing bi-Lipschitz contractions will be given in Section 3. In Section 4 and Section 5, we will give detailed arguments for calculating the dimensions. Finally, in Section 6, we will offer a simpler construction based on symmetric Cantor sets and study its differentiability.

2. Main setup and results

The main setup and results leading to Theorem 1.2 will be given in this section. Let us first recall the definition of lower and upper box dimensions of a set $E \subset \mathbb{R}$:

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}, \quad \overline{\dim}_{\mathrm{B}} E = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}$$

where $N_\delta(E)$ is the minimum number of intervals of diameter at most δ to cover E. We will not directly use the definition of Hausdorff dimension in this paper. Readers can refer to [10] for the precise definition.

We will adopt the standard multi-index notation for two digit alphabets: Set $\Omega = \{0, 1\}$. For $n \in \mathbb{N}$, let Ω^n be the n times Cartesian product of Ω. Write $\Omega^0 = \{\varnothing\}$ and $\Omega^* = \bigcup_{n=0}^{\infty} \Omega_n$. Let $\Omega^{\mathbb{N}}$ denote the countably infinite Cartesian product of Ω. The concatenation of two words w and v is denoted wv. If $w = w_1 \cdots w_n \in \Omega^n$, then $w^- = w_1 \cdots w_{n-1}$.

2.1. **General Cantor sets.**

Recall that a subset of $\mathbb{R}$ is called a **Cantor set** if it is compact, totally disconnected, and has no isolated points. We now review the standard process for constructing a Cantor set. Let $I_\varnothing = [0, 1]$. Suppose that an interval I_ω has been constructed for some $\omega \in \Omega^*$. We decompose

$$I_\omega = I_{\omega 0} \cup G_\omega \cup I_{\omega 1},$$

where the union is disjoint, $I_{\omega 0}$ and $I_{\omega 1}$ are closed intervals, G_w is an open interval, the left endpoint of $I_{\omega 0}$ coincides with the left endpoint of I_ω and the right endpoint of $I_{\omega 1}$ coincides with the right endpoint of I_ω. Moreover, we assume that

$$\lim_{n\to\infty} \max_{\omega\in\Omega^n} |I_\omega| = 0, \tag{2.1}$$

where we denote the length of an interval I by $|I|$. Define

$$E = \bigcap_{n=1}^{\infty} \bigcup_{\omega\in\Omega^n} I_\omega. \tag{2.2}$$

Then E is a Cantor set with $\{0,1\} \subset E \subset [0,1]$. Moreover,

$$[0,1] = E \cup \bigcup_{\omega\in\Omega^*} G_\omega. \tag{2.3}$$

We will call $\{I_\omega : \omega \in \Omega^*\}$ the **generating intervals** of the Cantor set E and $\{G_\omega : w \in \Omega^*\}$ the **gaps** of E. Notice that G_ω, $\omega \in \Omega^*$, are mutually disjoint. We remark that each Cantor subset of $[0,1]$ with $0,1$ contained in the set can be constructed this way.

Now suppose that E is the Cantor set defined by (2.2). We define the encoding map $\pi : \Omega^{\mathbb{N}} \to E$ by

$$\{\pi(\omega)\} = \bigcap_{n=1}^{\infty} I_{\omega_1\cdots\omega_n} \quad \text{for } \omega = (\omega_n)_{n=1}^{\infty} \in \Omega^{\mathbb{N}}. \tag{2.4}$$

The map π is one-to-one and surjective. Moreover, if $x = \pi(\omega)$ for the unique $\omega = (\omega_n)_{n=1}^{\infty} \in \Omega^{\mathbb{N}}$, we will write

$$I_n(x) = I_{\omega_1\cdots\omega_n}. \tag{2.5}$$

Next we introduce two maps ϕ_0, ϕ_1 from $[0,1]$ to itself. For $i = 0,1$, we define $\phi_i :\ I_\varnothing = [0,1] \to I_i$ so that ϕ_i maps G_ω onto $G_{i\omega}$ for every $\omega \in \Omega^*$ as an affine map. More precisely, we define

$$\phi_i(x) = \begin{cases} \frac{d-c}{b-a}(x-a)+c & \text{if } x \in G_\omega \text{ for some } \omega\in\Omega^*, \\ \pi(i\omega) & \text{if } x = \pi(w) \text{ for some } w \in \Omega^{\mathbb{N}}. \end{cases} \tag{2.6}$$

where we write $G_\omega := (a,b)$ and $G_{i\omega} := (c,d)$. By (2.3), ϕ_i is well-defined and it maps $I_\varnothing$ onto I_i for $i = 0,1$. Moreover, it is readily checked that ϕ_0 and ϕ_1 are strictly increasing and continuous. It is also straightforward to verify that

$$E = \phi_0(E) \cup \phi_1(E). \tag{2.7}$$

Below we provide a necessary and sufficient condition for ϕ_i, $i = 0,1$, to be bi-Lipschitz contraction mappings on $[0,1]$.

Theorem 2.1. *Let E be a Cantor subset of $[0,1]$ with generating intervals $\{I_\omega : \omega \in \Omega^*\}$ and gaps $\{G_\omega : \omega \in \Omega^*\}$. Let $\phi_i : [0,1] \to I_i$, $i = 0,1$, be defined as above. Then ϕ_i is a bi-Lipschitz contraction on $[0,1]$ if and only if*

$$0 < \theta_* := \inf_{\omega\in\Omega^*} \min\left\{\frac{|I_{i\omega}|}{|I_\omega|}, \frac{|G_{i\omega}|}{|G_\omega|}\right\} \quad \textit{and} \quad \theta^* := \sup_{\omega\in\Omega^*} \max\left\{\frac{|I_{i\omega}|}{|I_\omega|}, \frac{|G_{i\omega}|}{|G_\omega|}\right\} < 1.$$

2.2. **Special Cantor set construction.** We will begin our construction of a class of bi-Lipschitz IFSs for which the Hausdorff and box dimensions of their attractor are all different. This section will be devoted to introducing the setup, and the complete argument for computing the dimensions will be given in the next section. Throughout the rest of the paper, $[x]$ denotes the largest integer less than x.

Let $a_0 = 1$, $a_1 = 2$, $\beta \in (0,1)$ and M be a large positive integer (say $M \geq 100$). Define a length function $\ell\colon \Omega^* \to (0,1]$ by

$$\ell(\varnothing) = 1 \quad \text{and} \quad \ell(\omega) = \frac{a_{\omega_1} \cdots a_{\omega_{[\beta n]}}}{(a_{\omega_1} \cdots a_{\omega_n})^{\beta}} \cdot M^{-n} \tag{2.8}$$

Here we adopt the convention that $a_1 \cdots a_m = 1$ when $m = 0$. The product terms of a_i are motivated by the calculation of the dimensions of the Bedford–McMullen carpet. In the paper [20], a_i was the number of rectangles in the ith row. This product term was roughly the size of the "approximate square" in the Benford-McMullen carpet. This product has a huge range in the same level of the iterations. More precisely, we can see that

$$2^{-\beta(1-\beta)n-1} M^{-n} \leq \ell(\omega) \leq 2^{\beta(1-\beta)n} M^{-n}, \quad \text{for all } \omega \in \Omega^n. \tag{2.9}$$

However, this still has a very stable limiting behavior (See Lemma 4.2). In the end, similar to the carpet case, it will have a separation between Hausdorff dimension and box dimension.

Let $E = E_{\beta,M}$ be the Cantor set generated by the length function ℓ, i.e., the generating intervals $\{I_\omega\}_{\omega\in\Omega^*}$ of E_β satisfy $|I_\omega| = \ell(\omega)$. It follows from (2.9) that $\max_{\omega\in\Omega^n} |I_\omega|$ tends to zero as n tends to infinity. This shows that E is a well-defined Cantor set. The following shows that E is the attractor of a bi-Lipschitz IFS.

Proposition 2.2. *Let $\beta \in (0,1)$ and $M \geq 100$. Then $\Phi = \{\phi_0, \phi_1\}$, where ϕ_i is given by (2.6), is a bi-Lipschitz IFS whose attractor is the Cantor set E.*

The proofs of Theorem 2.1 and Proposition 2.2 will be given in Section 3.

2.3. **Dimension of the special Cantor sets.** We will now calculate the Hausdorff and the box dimensions of the Cantor sets E defined in the previous sections. We will need to set up some notations. The entropy function $H\colon [0,1] \to \mathbb{R}$ is defined to be

$$H(p) = -p\log p - (1-p)\log(1-p)$$

with $0\log 0 = 0$ by convention. For $\beta \in (0,1)$ and $M \geq 100$,

$$D(\lambda) = \sup_{(p,q)\in[0,1]^2} \frac{\beta H(p) + (1-\beta)H(q)}{\lambda + \beta(1-\beta)(q-p)\log 2} \tag{2.10}$$

where $\lambda > 0$. One can check that $D(\lambda)$ is a continuous and strictly decreasing function. Moreover,

Lemma 2.3.
$$D(\lambda) > \frac{\log 2}{\lambda}.$$

Proof. By taking $p = q = 1/2$, we know that $H(1/2) = \log 2$ and hence $D(\lambda) \geq \frac{\log 2}{\lambda}$. We now consider $p = 1/2$ and $q = 1/2 - \varepsilon$ for some small $\varepsilon > 0$. Note that
$$H(q) = -\left(\frac{1}{2} + \varepsilon\right) \log \left(\frac{1}{2} + \varepsilon\right) - \left(\frac{1}{2} - \varepsilon\right) \log \left(\frac{1}{2} - \varepsilon\right)$$
Expanding $\log\left(\frac{1}{2} + \varepsilon\right) = -\log 2 + 2\varepsilon - 2\varepsilon^2 + o(\varepsilon^2)$ as the Taylor series, we see that $H(q) = \log 2 - 2\varepsilon^2 + o(\epsilon^2)$. Hence,
$$D(\lambda) \geq \frac{\log 2 - 2(1-\beta)\varepsilon^2 + o(\epsilon^2)}{\lambda - \beta(1-\beta)\varepsilon \log 2}. \tag{2.11}$$
Hence when ε is sufficiently small we have $D(\lambda) > \frac{\log 2}{\lambda}$. $\square$

The following theorem computes the dimensions of E.

Theorem 2.4. *Let $\beta \in (0, 1)$ and $M \geq 100$, then*

(a) $\dim_{\mathrm{H}} E = \frac{\log 2}{\log M}$;
(b) $\dim_{\mathrm{B}} E = D(\log M)$.

Combining Theorem 2.2 and Lemma 2.3, Theorem 2.4 provides an example of a bi-Lipschitz IFS acting on $\mathbb{R}$ for which the Hausdorff dimension and the box dimension of the attractor are different. Theorem 2.4 will be proved in Section 4.

2.4. **Separating the lower and upper box dimensions.** In the following, we will separate the upper and lower box dimensions by perturbing the contraction ratios. Fix $\delta > 0$, we take
$$\varrho_n \in \left\{\frac{1}{M-\delta}, \frac{1}{M+\delta}\right\}, \quad \text{for all } n \in \mathbb{N}.$$
Let E^* be the Cantor set whose length function, denoted by ℓ^*, is given by
$$\ell^*(\omega_1 \cdots \omega_n) = \frac{a_{\omega_1} \cdots a_{\omega_{[\beta n]}}}{(a_{\omega_1} \cdots a_{\omega_n})^{\beta}} \cdot \varrho_1 \cdots \varrho_n.$$
Let $(J_\omega)_{\omega \in \Omega^*}$ be the corresponding generating intervals for E^*. Using the same proof in Proposition 2.2, we can see that E^* is generated by a bi-Lipschitz IFS for all sufficiently small δ. Moreover, we have

Proposition 2.5. *For all δ sufficiently small,* $\dim_{\mathrm{H}} E^* < \underline{\dim}_{\mathrm{B}} E^*$.

Finally, to separate the two box dimensions, we let K be a large constant and let $(r_n)_{n=1}^{\infty}$ be a K-lacunary sequence (i.e. $r_n > K r_{n-1}$ for all $n > 1$) with $r_1 = 1$. Define now
$$\varrho_n = \begin{cases} \frac{1}{M-\delta} & \text{if } n \in \{r_{2k-1}+1, \ldots, r_{2k}\} \\ \frac{1}{M+\delta} & \text{if } n \in \{r_{2k}+1, \ldots, r_{2k+1}\} \end{cases}.$$

Theorem 2.6. *There exists $\delta > 0$ and a K-lacunary sequence $(r_n)_{k=1}^{\infty}$ (for some large K) such that the Cantor set E^* satisfies*

$$\dim_{\mathrm{H}} E^* < \underline{\dim}_{\mathrm{B}} E^* < \overline{\dim}_{\mathrm{B}} E^*.$$

Theorem 1.2 is now a consequence of Theorem 2.6. Intuitively, the choice of ϱ_n allows us to make the Cantor set contracting on the scale $1/(M-\delta)$ for a long period, so the upper box dimension gets closer to the Cantor set $E_{\beta,M-\delta}$. However, we change the scale to $1/(M+\delta)$ for another long period so that the Cantor set becomes close to $E_{\beta,M+\delta}$. As this happens in an alternating way, the upper and lower box dimensions are separated. Proposition 2.5 and Theorem 2.6 will be proved in Section 5.

3. Bi-Lipschitz mappings

We will prove our bi-Lipschitz characterization in this section. Then we apply it to the Cantor sets of our interest.

3.1. **The proof of Theorem 2.1.** We first prove the necessity part of Theorem 2.1. Fix $i \in \{0,1\}$. Notice that ϕ_i maps $G_\omega =: (a,b)$ onto $G_{i\omega} = (\phi_i(a), \phi_i(b))$ for each $\omega \in \Omega^*$. Applying the bi-Lipschitz assumption to the points a and b, we see that

$$0 < \inf_{\omega\in\Omega^*} \left\{ \frac{|G_{i\omega}|}{|G_\omega|} \right\} \le \sup_{\omega\in\Omega^*} \left\{ \frac{|G_{i\omega}|}{|G_\omega|} \right\} < 1. \tag{3.1}$$

On the other hand, for each $\omega \in \Omega^*$, $I_\omega = [\pi(\omega 0^\infty), \pi(\omega 1^\infty)]$, where $j^\infty = jjj\ldots$ for $j \in \{0,1\}$ and π is defined as in (2.4). Therefore,

$$I_{i\omega} = [\pi(i\omega 0^\infty), \pi(i\omega 1^\infty)] = \phi_i(I_\omega).$$

Applying the bi-Lipschitz assumption to the points $\pi(\omega 0^\infty)$ and $\pi(\omega 1^\infty)$, we see that (3.1) also holds with G_w replaced by I_ω. Hence, we have $0 < \theta_* \le \theta^* < 1$.

In what follows, we prove the sufficiency part of Theorem 2.1. Our goal is to prove that

$$0 < \theta_* \le \frac{\phi_i(y) - \phi_i(x)}{y - x} \le \theta^* < 1 \quad \text{for all } x, y \in [0,1], \tag{3.2}$$

To justify (3.2), we let $0 \le x < y \le 1$ and $J = (x,y)$. First, notice that if there exists $\omega \in \Omega^*$ such that $x, y \in G_\omega$, then by the linearity of ϕ_i on G_w and our assumption, it is clear that (3.2) holds. Moreover, it is also clear that the statement is true for $x = 0$ and $y = 1$.

We now assume that x, y are not in the same gap and $0 < x < y < 1$. Then $J \cap E \ne \varnothing$. Hence, J contains at least one generating interval that is not $I_\varnothing$ since J is open. We now let

$$\mathcal{U} = \{w \in \Omega^* : I_\omega \subset J, \text{ and } I_{\omega^-} \not\subset J\}.$$

(recall that ω^- means the word obtained by removing the last letter of ω) and

$$J' = J \setminus \left(\bigcup_{w \in \mathcal{U}} I_\omega \right)$$

The following lemma records some important properties of the sets $\mathcal{U}$ and J'.

Lemma 3.1. *(1) $I_w \cap I_v = \varnothing$ for all $w, v \in \mathcal{U}$ and $w \neq v$.*

(2) $E \cap J' = \varnothing$.

(3) Suppose that $x, y \in E$. Then for every $w \in \Omega^*$, either $G_w \subset J'$ or $G_w \cap J' = \varnothing$.

(4) Suppose that $x, y \in E$. Then there exists $\mathcal{V} \subset \Omega^*$ ($\mathcal{V}$ may be empty) such that

$$J = \left(\bigcup_{w \in \mathcal{U}} I_w \right) \cup \left(\bigcup_{w \in \mathcal{V}} G_w \right).$$

Proof. We first prove (1). By the construction of generating intervals, the collection of all generating intervals $\{I_w : w \in \Omega^*\}$ has the following net property: for all $w, w' \in \Omega^*$, either

$$I_{w'} \subset I_w, \text{ or } I_{w'} \cap I_w = \varnothing. \tag{3.3}$$

Moreover, $I_{w'} \subset I_w$ occurs if and only if w' is a concatenation of w. Suppose that $I_w \cap I_v \neq \varnothing$ for some $w, v \in \mathcal{U}$ with $w \neq v$. Let $w \in \Omega^m$ and $v \in \Omega^n$ with $m > n$. Then the net property (3.3) implies that $I_w \subset I_v$ and v is a prefix of w. But then $I_{w^-} \subset I_v \subset J$ since w^- is also a prefix of v. This contradicts the fact of $w \in \mathcal{U}$. Hence, (1) holds.

To prove (2), suppose on the contrary that $E \cap J' \neq \varnothing$. Take $z \in E \cap J'$ and set $\pi^{-1}(z) = (v_n)_{n=1}^{\infty}$, where π is defined as in (2.4). Then

$$\{z\} = \bigcap_{n=1}^{\infty} I_{v_1 \cdots v_n}.$$

Since J is open, $I_{v_1 \cdots v_n} \subset J$ when n is large enough. Let n_0 be the smallest integer so that $I_{v_1 \cdots v_{n_0}} \subset J$. Then $v_1 \cdots v_{n_0} \in \mathcal{U}$. This means that $I_{v_1 \cdots v_{n_0}} \cap J' = \varnothing$ and thus $z \notin J'$. This results in a contradiction.

We now prove (3). Let $w \in \Omega^*$. Since the endpoints of J are in E and G_w does not intersect E since they are the gaps. We have either $G_w \subset J$ or $G_w \cap J = \varnothing$. We only need to consider the case when $G_w \subset J$ and show that either $G_w \subset J'$ or $G_w \cap J' = \varnothing$. Suppose the conclusion is false, i.e., $G_w \not\subset J'$ and $G_w \cap J' \neq \varnothing$. Since $G_w \not\subset J'$, it follows that

$$G_w \cap \left(\bigcup_{v \in \mathcal{U}} I_v \right) \neq \varnothing.$$

Hence, there exists $v \in \mathcal{U}$ such that $G_w \cap I_v \neq \varnothing$. However, since the two endpoints of I_v are in E and $G_w \cap E = \varnothing$, we have $G_w \subset I_v$, which implies that $G_w \cap J' = \varnothing$, and this leads to a contradiction.

Finally, we prove (4). Since $[0,1] = E \cup \bigcup_{w\in\Omega^*} G_w$ statement (2) implies that $J' \subset \bigcup_{w\in\Omega^*} G_w$. Set

$$\mathcal{V} = \{\omega \in \Omega^* : G_w \subset J'\}.$$

Now using (2) and (3), our assumption $x, y \in E$, and the fact $J' \subset \bigcup_{w\in\Omega^*} G_w$, we must have $J' = \bigcup_{w\in\mathcal{V}} G_w$. Hence

$$J = \left(\bigcup_{w\in\mathcal{U}} I_w\right) \cup J' = \left(\bigcup_{w\in\mathcal{U}} I_w\right) \cup \left(\bigcup_{w\in\mathcal{V}} G_w\right).$$

This completes the proof of the lemma. □

We now return to the proof of Theorem 2.1. Let

$$x' = \min\{t \in E : t \geq x\} \quad \text{and} \quad y' = \max\{t \in E : t \leq y\}.$$

Then $(x, x') \cap E = \varnothing$, $(y', y) \cap E = \varnothing$ and $x', y' \in E$. By Lemma 3.1(4), we know that

$$(x', y') = \left(\bigcup_{w\in\mathcal{U}} I_w\right) \cup \left(\bigcup_{w\in\mathcal{V}} G_w\right)$$

and the union is disjoint by Lemma 3.1(1). Therefore,

$$\phi_i(x', y') = \left(\bigcup_{w\in\mathcal{U}} I_{iw}\right) \cup \left(\bigcup_{w\in\mathcal{V}} G_{iw}\right).$$

Decomposing $(x, y) = (x, x'] \cup (x', y') \cup [y', y)$ as a disjoint union and using the fact each ϕ_i is increasing, we have

$$\phi_i(y) - \phi_i(x) = \phi_i(y) - \phi_i(y') + \sum_{w\in\mathcal{U}} |I_{iw}| + \sum_{w\in\mathcal{V}} |G_{iw}| + \phi_i(x) - \phi_i(x')$$

We remark that either $x = x'$ or (x, x') is contained in a gap. Similarly, $y = y'$ or (y, y') is contained in a gap. Using these observations together with our assumption we have

$$\phi_i(y) - \phi_i(x) \geq \theta_* \left((y - y') + \sum_{w\in\mathcal{U}} |I_w| + \sum_{w\in\mathcal{V}} |G_w| + (x' - x) \right) = \theta_*(y - x).$$

The same argument also works for the upper bound. This completes the proof of Theorem 2.1.

3.2. **The proof of Proposition 2.2.** We now prove that the Cantor set $E = E_{\beta,M}$ with length given by (2.8) is generated by a bi-Lipschitz IFS. To this end we will show that the assumptions in Theorem 2.1 are satisfied. For our record, we use (2.8) to compute the ratios of the intervals:

$$\frac{|I_{i\omega}|}{|I_\omega|} = \begin{cases} u_i M^{-1} & \text{if } [\beta(n+1)] = [\beta n] = 0, \\ v_i a^{-1}_{\omega_{[\beta n]}} M^{-1} & \text{if } [\beta(n+1)] = [\beta n] \geq 1, \\ v_i M^{-1} & \text{if } [\beta(n+1)] = [\beta n] + 1, \end{cases} \tag{3.4}$$

where $u_0 = v_0 = 1$, $u_1 = 2^{-\beta}$ and $v_1 = 2^{1-\beta}$. It is also useful to compute

$$\frac{|I_{\omega i}|}{|I_\omega|} = \begin{cases} s_i \frac{1}{M} & \text{if } [\beta(n+1)] = [\beta n], \\ s_i a_{\omega_{[\beta n]+1}} \frac{1}{M} & \text{if } [\beta(n+1)] = [\beta n] + 1, \end{cases} \tag{3.5}$$

where $s_0 = 1$ and $s_1 = 2^{-\beta}$. Therefore,

$$\frac{1}{2^\beta M} \leq \frac{|I_{\omega i}|}{|I_w|} \leq \frac{2}{M} \quad \text{for all } \omega \in \Omega^*. \tag{3.6}$$

We are now ready to compute the ratios in Theorem 2.1. For each $\omega = \omega_1 \dots \omega_n \in \Omega^*$ and $i = 0, 1$, by (3.4),

$$\begin{aligned} \inf_{\omega \in \Omega^*} \min \left\{ \frac{|I_{0\omega}|}{|I_\omega|}, \frac{|I_{1\omega}|}{|I_\omega|} \right\} &\geq \frac{1}{2M} > 0, \\ \sup_{\omega \in \Omega^*} \max \left\{ \frac{|I_{0\omega}|}{|I_\omega|}, \frac{|I_{1\omega}|}{|I_\omega|} \right\} &\leq \frac{2^{1-\beta}}{M} < 1. \end{aligned} \tag{3.7}$$

It remains to show that the ratios of all the gaps are also strictly bounded between 0 and 1. To this end, we note that $|G_\omega| = |I_\omega| - |I_{\omega 0}| - |I_{\omega 1}|$ and

$$\frac{|G_{i\omega}|}{|G_\omega|} = \frac{|I_{i\omega}|}{|I_\omega|} \cdot \frac{1 - |I_{i\omega 0}|/|I_{i\omega}| - |I_{i\omega 1}|/|I_{i\omega}|}{1 - |I_{\omega 0}|/|I_\omega| - |I_{\omega 1}|/|I_\omega|}.$$

Combining this with (3.6), we obtain that

$$\begin{aligned} \inf_{\omega \in \Omega^*} \min \left\{ \frac{|G_{0\omega}|}{|G_\omega|}, \frac{|G_{1\omega}|}{|G_\omega|} \right\} &\geq \frac{1}{2M} \cdot \frac{M - 2^{2-\beta}}{M - 2} > 0, \\ \sup_{\omega \in \Omega^*} \max \left\{ \frac{|G_{0\omega}|}{|G_\omega|}, \frac{|G_{1\omega}|}{|G_\omega|} \right\} &\leq \frac{2^{1-\beta}}{M} \cdot \frac{M - 2}{M - 2^{2-\beta}} < 1 \end{aligned} \tag{3.8}$$

when M is large enough. This completes the proof.

4. The proof of Theorem 2.4

4.1. **Hausdorff dimension.** We will prove Theorem 2.4 (1) using the following version of Billingsley's lemma [3, Lemma 3.1] (from [3, Remark 3.2(2)], we know that the lemma also works using the generating intervals instead of dyadic intervals).

Lemma 4.1 (Billingsley's Lemma)**.** *Let μ be a finite Borel measure on a Cantor set $E \subset [0, 1]$ whose generating intervals are given by $\{I_w : w \in \Omega^*\}$. Let $A \subset E$ with $\mu(A) > 0$. Suppose that for all $x \in A$, we have*

$$\liminf_{n \to \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \leq \alpha$$

where $I_n(x)$ is defined as in (2.5). Then $\dim_{\mathrm{H}}(A) \leq \alpha$. The same also holds if $\leq$ is replaced by $\geq$ in both inequalities.

Apart from Billingsley's lemma, we need a simple observation due to McMullen in Lemma 4 of [20]. Let ν be the product measure on $\Omega^{\mathbb{N}}$ which assigns measure 2^{-n} on every cylinder set of length n

Lemma 4.2. [20, Lemma 4] *Let $a_0 = 1$ and $a_1 = 2$ and $\beta \in (0,1)$. Then for all $\omega = (\omega_n)_{n=1}^{\infty} \in \Omega^{\mathbb{N}}$, we have*

$$\limsup_{n\to\infty}\left(\frac{(a_{\omega_1}\cdots a_{\omega_n})^{\beta}}{a_{\omega_1}\cdots a_{\omega_{[\beta n]}}}\right)^{1/n} \geq 1.$$

Moreover,

$$\limsup_{n\to\infty}\left(\frac{(a_{\omega_1}\cdots a_{\omega_n})^{\beta}}{a_{\omega_1}\cdots a_{\omega_{[\beta n]}}}\right)^{1/n} = 1$$

for ν-almost all $\omega \in \Omega^{\mathbb{N}}$.

Proof of Theorem 2.4(1) Let μ be the push forward measure of ν under π. Then μ is supported on E and

$$\mu(I_\omega) = 2^{-n}, \quad \text{for all } \omega \in \Omega^n,\ n \in \mathbb{N}.$$

For each $x \in E$, let $\pi^{-1}\{x\} = \omega_1 \cdots \omega_n \cdots$. Recalling the definition of $\ell(\omega)$ in (2.8),

$$\limsup_{n\to\infty}\frac{\log|I_n(x)|}{-n} = \limsup_{n\to\infty}\log\left(\frac{(a_{\omega_1}\cdots a_{\omega_n})^{\beta}}{a_{\omega_1}\cdots a_{\omega_{[\beta n]}}}\right)^{1/n} + \log M \geq \log M \tag{4.1}$$

since the first term is greater than or equal to zero by the first part of Lemma 4.2, Hence, for all $x \in E$ we have

$$\liminf_{n\to\infty}\frac{\log\mu(I_n(x))}{\log|I_n(x)|} \leq \frac{\log 2}{\log M}. \tag{4.2}$$

By the upper bound of Billingsley's lemma, we have $\dim_{\mathrm{H}}(E) \leq \frac{\log 2}{\log M}$. On the other hand, the second part of Lemma 4.2 implies that (4.1) is an equality for μ almost all x. Hence, by the lower bound of Billingsley's lemma, there exists a set $A \subset E$ with $\mu(A) = \mu(E)$ attaining the Hausdorff dimension $\log 2/\log M$. Therefore $\dim_{\mathrm{H}}(E) = \log 2/\log M$. This completes the proof of Theorem 2.4(1). $\square$

4.2. **Box dimensions.** We will need to set up some notation and prove three lemmas to prepare for our proof of Theorem 2.4 (2) about the box dimension. First, the following approximation of binomial coefficients is well-known and can be proved using Stirling's approximation. Here we denote by $a_n \sim_c b_n$ if $a_n/b_n \to c$ as $n \to \infty$ and $\binom{n}{k}$ denotes the standard binomial coefficients with $0 \leq k \leq n$.

Lemma 4.3. *Let $\delta > 0$ and let $p \in (\delta, 1-\delta)$. Then we can find N depending only on δ such that if $n > N$, then*

$$\binom{n}{[p\cdot n]} \sim_{c_p} \frac{1}{\sqrt{2\pi np(1-p)}}\cdot e^{nH(p)} \tag{4.3}$$

where c_p is a uniformly bounded constant for all $p \in (\delta, 1-\delta)$. Furthermore, if $\delta < 1/2$, we can find a constant $c > 0$ depending only on δ such that

$$\sum_{k=0}^{[\delta n]}\binom{n}{k} \leq c\cdot\sqrt{n}\cdot e^{nH(\delta)}. \tag{4.4}$$

Proof. First note that by a quick calculation, $\frac{n^n}{k^k(n-k)^{n-k}} = e^{nH(k/n)}$. We now use Striling's approximation, $n! \sim_1 \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ and take N_0 so that the Stirling's approximation holds for all $n > N_0$ with an error ε fixed at the beginning. Hence, if we have $n > N_0/\delta$, then $k = [pn] > \delta n > N_0$. Therefore,

$$\begin{aligned}\binom{n}{[pn]} &\sim_1 \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi k}\left(\frac{k}{e}\right)^n \cdot \sqrt{2\pi(n-k)}\left(\frac{n-k}{e}\right)^{n-k}} \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{n^n}{k^k(n-k)^{n-k}} \cdot \sqrt{\frac{n}{k(n-k)}} \\ &\sim_1 \frac{1}{\sqrt{2\pi np(1-p)}} e^{nH(\frac{k}{n})}.\end{aligned}$$

Furthermore, H is a uniformly continuous function on $[0,1]$ and $|\frac{k}{n} - p| < \frac{1}{n}$, so we have our desired estimation (4.3) with $c_p = e^{H'(p)}$. Note that p is bounded away from 0 and 1. The derivative $H'(p)$ is uniformly bounded on $(\delta, 1-\delta)$. To prove (4.4), we simply note that the binomial coefficient is strictly increasing for the first half.

$$\sum_{k=0}^{[\delta n]} \binom{n}{k} \le ([\delta n]+1)\binom{n}{[\delta n]+1} \le \frac{[\delta n]+1}{\sqrt{2\pi n\delta(1-\delta)}} e^{nH(\delta)} \le c \cdot \sqrt{\delta n} \cdot e^{nH(\delta)}$$

after absorbing all constants into c. This completes the proof. □

Let $0 < \rho < 1$. We now define below the Moran cut-set as usual:

$$\Lambda(\rho) := \left\{\omega \in \Omega^* : \ell(\omega) \le \rho < \ell(\omega^-)\right\}.$$

We also define

$$\mathcal{N}_\rho = \{k \in \mathbb{N} : \exists\, \omega \in \Omega^k \text{ such that } \omega \in \Lambda(\rho)\}$$

We have the following lemma.

Lemma 4.4. *There exists $c_1, c_2, C > 0$ such that*

$$c_1 \cdot \log\frac{1}{\rho} \le n \le c_2 \cdot \log\frac{1}{\rho} \tag{4.5}$$

whenever $n \in \mathcal{N}_\rho$. Consequently,

$$\#\mathcal{N}_\rho \le C \cdot \log\frac{1}{\rho}.$$

Proof. It suffices to show the first statement. Then the second statement follows by taking the difference of the bounds. We now take $n \in \mathcal{N}_\rho$ and take $\omega \in \Omega^n$ such that $\omega \in \Lambda(\rho)$. As $\beta < 1$, (2.9) and the definition of $\Lambda(\rho)$ implies that we have

$$2^{-n}M^{-n} \le \ell(\omega) \le \rho.$$

This implies that $n \ge c_1 \log(1/\rho)$ where $c_1 = (\log 2M)^{-1}$. On the other hand, $\ell(\omega^-) \le (M/2) \cdot \ell(\omega)$ by (3.6). Therefore, combining this inequality with (2.9), we have

$$\rho \le \ell(\omega_1 \cdots \omega_{n-1}) \le \frac{M}{2} \cdot 2^n \cdot M^{-n} = \left(\frac{2}{M}\right)^{n-1}.$$

Taking the logarithm, we obtain the upper bound by taking $c_2 = \log(M/2) + 1$. $\square$

For all $\omega = \omega_1 \cdots \omega_n$, we will rewrite (2.8) as

$$\ell(\omega) = \frac{a_{\omega_1} \cdots a_{\omega_{[\beta n]}}}{(a_{\omega_1} \cdots a_{\omega_n})^\beta} \cdot M^{-n} = 2^{(1-\beta)N_1 - \beta N_2} \cdot M^{-n}, \tag{4.6}$$

where

$$N_1 = N_1(\omega_1 \cdots \omega_n) = \#\{1 \le j \le [\beta n] : \omega_j = 1\},$$
$$N_2 = N_2(\omega_1 \cdots \omega_n) = \#\{[\beta n] + 1 \le j \le n : \omega_j = 1\}.$$

Let

$$\gamma = \gamma_{p,q} = \log M + \beta(1-\beta)(q-p)\log 2$$

be denominator in the expression of $D(\log M)$. We have the following lemma.

Lemma 4.5. *Suppose that $0 < p, q < 1$. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \ge N$,*

$$\#\left\{\omega_1 \cdots \omega_n \in \Omega^n : \ell(\omega_1 \cdots \omega_n) \in [e^{-n\gamma(1-\varepsilon)}, e^{-n\gamma(1+\varepsilon)}]\right\} \ge c_{p,q,\beta} \cdot \frac{1}{n} \cdot e^{n(\beta H(p) + (1-\beta)H(q))}$$

for some constant $c_{p,q,\beta} > 0$ depending only on p, q, β.

Proof. Denote by $F_{n,\varepsilon}$ the set on the left of the lemma. Fix $\varepsilon > 0$, we take $N > \frac{\log 2}{\varepsilon}$. We now consider $\omega = \omega_1 \cdots \omega_n \in \Omega^n$ such that there are exactly $N_1 = [p\beta n]$ many 1s in the first $[\beta n]$ positions and $N_2 = [q(1-\beta)n]$ many 1s in the other $[(1-\beta)n]$ positions. We now denote by G_n the set of all such $\omega \in \Omega^n$. Using (4.6), if $n > N$,

$$\ell(\omega_1 \cdots \omega_n) = e^{-n\left(\log M - (1-\beta)\frac{N_1}{n}\log 2 + \beta\frac{N_2}{n}\log 2\right)} = e^{-n(\gamma + \delta_n)}$$

where

$$\delta_n = \beta(1-\beta)\log 2 \cdot \left(\left(\frac{N_2}{(1-\beta)n} - q\right) - \left(\frac{N_1}{\beta n} - p\right) \right).$$

As $|\frac{N_1}{\beta n} - p| \le \frac{1}{\beta n}$ and $|\frac{N_2}{(1-\beta)n} - q| \le \frac{1}{(1-\beta)n}$, using our choice of N, $|\delta_n| < \varepsilon$. This shows that $F_{n,\varepsilon}$ contains all $\omega \in G_n$. Therefore,

$$\#F_{n,\varepsilon} \ge \#G_n = \binom{[\beta n]}{[p\beta n]} \cdot \binom{[(1-\beta)n]}{[q(1-\beta)n]}.$$

By Lemma 4.4, we can take N sufficiently large so that

$$\binom{[\beta n]}{[p\beta n]} \cdot \binom{[(1-\beta)n]}{[q(1-\beta)n]} \ge \frac{c_{p,q,\beta}}{n} \cdot e^{n(\beta H(p) + (1-\beta)H(q))}$$

for some constant $c_{p,q,\beta} > 0$ depending only on p, q, β. This completes the proof. $\square$

We are now ready to prove the box dimension formula.

Proof of Theorem 2.4(2). The proof will be divided into two parts:

(a) $\overline{\dim}_{\mathrm{B}} E \le D(\log M)$ and
(b) $\underline{\dim}_{\mathrm{B}} E \ge D(\log M)$.

(a) We notice that since $N_\rho(E) \le \#\Lambda(\rho)$, we have

$$\overline{\dim}_{\mathrm{B}} E \le \limsup_{\rho\to 0} \frac{\#\Lambda(\rho)}{\log(1/\rho)}. \tag{4.7}$$

Let $U_n = \{0, 1, \dots, [\beta n]\} \times \{0, \dots, n - [\beta n]\}$. We note that

$$\#\Lambda(\rho) = \sum_{n\in\mathcal{N}_\rho} \sum_{(N_1,N_2)\in U_n} \#\left(\Omega^n \cap \{\omega \in \Lambda(\rho) : N_1(\omega) = N_1, N_2(\omega) = N_2\}\right). \tag{4.8}$$

We now estimate the cardinality on the right. Fix $\delta > 0$ so that $H(\delta) < D(\log M)$. Using Lemma 4.4, for all n sufficiently large

$$\binom{n}{[pn]} \le \frac{c}{\sqrt{2\pi n p(1-p)}} e^{nH(p)} \tag{4.9}$$

holds for all $p \in (\delta, 1-\delta)$ and $c = \sup c_p$. We now divide U_n into sets:

$$U_n'(\delta) = \left\{(N_1, N_2) \in U_n : \delta < \frac{N_1}{[\beta n]} < 1 - \delta, \text{ and } \delta < \frac{N_2}{n - [\beta n]} < 1 - \delta\right\}$$

$$U_n''(\delta) = U_n \setminus U_n'(\delta).$$

We will now estimate the sum in (4.8) by splitting $U_n = U_n'(\delta) \cup U_n''(\delta)$. In the first case, for each $(N_1, N_2) \in U_n'(\delta)$ and $\omega \in \Lambda(\rho)$ such that $N_1(\omega) = N_1$ and $N_2(\omega) = N_2$, we let $p = \frac{N_1}{[\beta n]}$ and $q = \frac{N_2}{n-[\beta n]}$ and they all belong to $(\delta, 1-\delta)$, by (4.9), we have

$$\begin{aligned}\#\left(\{\omega \in \Lambda(\rho) : N_1(\omega) = N_1, N_2(\omega) = N_2\} \cap \Omega^n\right) &\le \binom{[\beta n]}{N_1} \cdot \binom{n - [\beta n]}{N_2} \\ &\le C_1 \cdot \frac{1}{n} \cdot e^{n\cdot(\beta H(p) + (1-\beta)H(q))}\end{aligned}$$

where C_1 is a constant depending only on δ and β. Invoking the definition of $D := D(\log M)$ in (2.10) and length function in (4.6), we have

$$\begin{aligned}\beta H(p) + (1-\beta)H(q) &\le D \cdot (\log M + \beta(1-\beta)(q-p)\log 2) \\ &= -D \cdot \frac{\log \ell(\omega)}{n} \\ &\le D \cdot \frac{\log(1/\rho)}{n}.\end{aligned}$$

This implies that

$$\#\left(\{\omega \in \Lambda(\rho) : N_1(\omega) = N_1, N_2(\omega) = N_2\} \cap \Omega^n\right) \le C_1 \cdot \frac{1}{n} \cdot \left(\frac{1}{\rho}\right)^D \tag{4.10}$$

On the other hand, if $(N_1, N_2) \in U_n''(\delta)$, we use (4.4) in Lemma 4.3,

$$\begin{aligned}\sum_{(N_1,N_2)\in U_n''(\delta)} \#\left(\Omega^n \cap \{\omega \in \Lambda(\rho) : N_1(\omega) = N_1, N_2(\omega) = N_2\}\right) &\le \sum_{N_1=0}^{[\delta\beta n]} \sum_{N_2=0}^{[\delta(n-[\beta n])]} \binom{[\beta n]}{N_1} \cdot \binom{n-[\beta n]}{N_2} \\ &\le C_2 \cdot n \cdot e^{nH(\delta)}.\end{aligned}$$

Again, C_2 is a constant depending only on δ and β. We now put this estimate and (4.10) into (4.8), we have

$$\#\Lambda(\rho) \leq \sum_{n\in\mathcal{N}_\rho} \left(\#U'_n(\delta)\cdot C_1 \cdot \frac{1}{n}\cdot \left(\frac{1}{\rho}\right)^D + C_2\cdot n\cdot e^{nH(\delta)} \right).$$

Note that $\#U'_n(\delta) \leq n^2$ From Lemma 4.3, $n \leq c_2 \left(\log\frac{1}{\rho}\right)$ whenever $n\in\mathcal{N}_\rho$ and $\#\mathcal{N}_\rho \leq C\left(\log\frac{1}{\rho}\right)$. We now obtain

$$\#\Lambda(\rho) \leq C' \left(\log\frac{1}{\rho}\right)^2 \left(\left(\frac{1}{\rho}\right)^D + \left(\frac{1}{\rho}\right)^{H(\delta)} \right).$$

for some positive constant C' depending only on δ and β. But we picked δ so that $H(\delta) < D$, then

$$\#\Lambda(\rho) \leq 2C' \left(\log\frac{1}{\rho}\right)^2 \cdot \left(\frac{1}{\rho}\right)^D .$$

Substituting this bound back into (4.7), we have

$$\overline{\dim}_{\mathrm{B}} E \leq D = D(\log M).$$

This completes the proof of (a).

(b). Let $p, q\in(0,1)$ and $\gamma = \gamma_{p,q}$. Furthermore let $\varepsilon > 0$, $\delta > 0$ and let $\eta = e^{-\gamma(1-\varepsilon)}$. Choose n to be the unique integer such that $\delta\in(\eta^{n+1},\eta^n]$. For all δ sufficiently small, by Lemma 4.5,

$$N_{\eta^n}(E) \geq \frac{c}{n}\cdot e^{n(\beta H(p)+(1-\beta)H(q))}$$

where c depends only on p, q, β. Then

$$\frac{\log N_\delta(E)}{-\log\delta} \geq \frac{\log N_{\eta^n}(E)}{-\log\eta^{n+1}} \geq \frac{\log c - \log n + n(\beta H(p)+(1-\beta)H(q))}{(n+1)\gamma(1-\varepsilon)}.$$

Taking the lower limit implies that

$$\underline{\dim}_{\mathrm{B}} E \geq \frac{\beta H(p)+(1-\beta)H(q)}{\gamma(1-\varepsilon)} = \frac{\beta H(p)+(1-\beta)H(q)}{\log M + \beta(1-\beta)(q-p)\log 2}\cdot\frac{1}{1-\epsilon}.$$

By taking $\varepsilon\to 0$, we have $\underline{\dim}_{\mathrm{B}} E \geq \Phi(p,q)$ for all $(p,q)\in(0,1)^2$, where

$$\Phi(p,q) = \frac{\beta H(p)+(1-\beta)H(q)}{\log M + \beta(1-\beta)(q-p)\log 2}.$$

Note that Φ is a continuous function on $[0,1]^2$ and $D(\log M) = \sup_{[0,1]^2}\Phi(p,q)$, we obtain that $\underline{\dim}_{\mathrm{B}} E \geq D(\log M)$. This completes our proof. $\square$

5. The proofs of Proposition 2.5 and Theorem 2.6

5.1. **Small perturbation of generating intervals.** In this section, we will construct a Cantor set from a bi-Lipschitz IFS such that its Hausdorff, upper, and lower box dimensions are all different. We first fix $E = E_{\beta,M}$ to be the Cantor set obtained in the previous section that has distinct Hausdorff dimension and box dimension (which exists). In the same notation, $(I_\omega)_{\omega\in\Omega^*}$ are the generating intervals of E. Fix $\delta > 0$, and take

$$\varrho_n \in \left\{ \frac{1}{M-\delta}, \frac{1}{M+\delta} \right\}, \quad \text{for all } n \in \mathbb{N}.$$

Let E^* be the Cantor set whose length function, denoted by ℓ^*, is given by

$$\ell^*(\omega_1\cdots\omega_n) = \frac{a_{\omega_1}\cdots a_{\omega_{[\beta n]}}}{(a_{\omega_1}\cdots a_{\omega_n})^\beta} \cdot \ \varrho_1\cdots\varrho_n.$$

Let $(J_\omega)_{\omega\in\Omega^*}$ be the corresponding generating intervals for E^*. We also let $\pi^* : \Omega^{\mathbb{N}} \to E^*$ be defined analogously to π (see (2.4)) where E is replaced by E^*. Given $x \in E^*$ and $n \in \mathbb{N}$ we let $J_n(x) = J_{w_1\cdots w_n}$ where $(w_n)_{n=1}^\infty \in \Omega^{\mathbb{N}}$ is the unique sequence satisfying $\pi^*((w_n)) = x$. Using the same proof in Proposition 2.2, we can see that E^* is generated by a bi-Lipschitz IFS for all sufficiently small δ.

We will need a calculus lemma that we will use several times throughout this section. The proof is simple and will be omitted.

Lemma 5.1. *Let $A > B > 0$ and let $Q = [-b, b]$ where $b < B$. Then*

$$\max_{x\in Q} \frac{A-x}{B-x} = 1 + \frac{A-B}{B-b}, \ \textit{and} \ \min_{x\in Q} \frac{A-x}{B-x} = 1 + \frac{A-B}{B+b}$$

On the other hand, if $0 < A < B$, then

$$\min_{x\in Q} \frac{A-x}{B-x} = 1 - \frac{B-A}{B-b}, \ \textit{and} \ \max_{x\in Q} \frac{A-x}{B-x} = 1 - \frac{B-A}{B+b}.$$

The following proposition gives an estimate for the change of the dimensions. In particular, the last statement shows that the Hausdorff and lower box dimensions are still different for all sufficiently small δ.

Proposition 5.2. *The following statements are true:*

(1) There exists $0 < A_\delta < 1 < B_\delta$ such that $A_\delta, B_\delta \to 1$ as $\delta \to 0$ and

$$\frac{\log |J_\omega|}{\log |I_\omega|} \in [A_\delta, B_\delta]\,.$$

for all $\omega \in \Omega^*$.

(2) $\dim_{\mathrm{H}} E^* \le A_\delta^{-1} \cdot \dim_{\mathrm{H}} E$.

(3) $\underline{\dim}_{\mathrm{B}} E^* \ge B_\delta^{-1} \cdot \dim_B E$.

(4) For all sufficiently small $\delta > 0$, $\dim_{\mathrm{H}} E^* < \underline{\dim}_{\mathrm{B}} E^*$.

Proof. (1). For each $\omega = \omega_1 \cdots \omega_n$, let $b_\omega = \frac{a_{\omega_1} \cdots a_{\omega_{[\beta\omega n]}}}{(a_{\omega_1} \cdots a_{\omega_n})^\beta}$. The definition of ϱ_n implies that

$$\frac{\log |J_\omega|}{\log |I_\omega|} = \frac{-\log \varrho_1 \cdots \varrho_n - \log b_\omega}{n \log M - \log b_\omega} \leq \frac{\log(M+\delta) - (\log b_\omega)/n}{\log M - (\log b_\omega)/n}.$$

Notice that $2^{-n} < 2^{-\beta(1-\beta)n} \leq b_\omega \leq 2^{\beta(1-\beta)n} < 2^n$, so $(\log b_\omega)/n \in Q_2 := [-\log 2, \log 2]$ for all $\omega \in \Omega^*$. We now see that the ratios are bounded above by

$$B_\delta := \max_{x \in Q_2} \frac{\log(M+\delta) - x}{\log M - x}.$$

Plugging in the respective constants that appeared in Lemma 5.1, we see that B_δ tends to 1 as δ approaches zero. The lower bound is also proved similarly.

(2). Let μ and μ^* be respectively the measures supported on E and E^* defined by $\mu(I_\omega) = \mu^*(J_\omega) = 2^{-n}$ for all $\omega \in \Omega^n$ and $n \in \mathbb{N}$. Then for all $\omega \in \Omega^*$,

$$\frac{\log \mu^*(J_\omega)}{\log |J_\omega|} = \frac{\log \mu(I_\omega)}{\log |I_\omega|} \cdot \frac{\log |I_\omega|}{\log |J_\omega|} \leq A_\delta^{-1} \cdot \frac{\log \mu(I_\omega)}{\log |I_\omega|}.$$

Let $x^* \in E^*$ be arbitrary and $x \in E$ be such that $\pi((\pi^*)^{-1}(x^*)) = x$. Then using the above, (4.2) and Theorem 2.4, we have the following

$$\liminf_{n\to\infty} \frac{\log \mu^*(J_n(x^*))}{\log |J_n(x^*)|} \leq A_\delta^{-1} \cdot \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \leq A_\delta^{-1} \frac{\log 2}{\log M} = A_\delta^{-1} \dim_H E.$$

Since $x^* \in E$ was arbitrary the desired bound follows now from Lemma 4.1.

(3). Let $\varepsilon > 0$ and let $a = \dim_{\mathrm{B}} E$. By the definition of box dimension, for all sufficiently small γ, $N_\gamma(E) \geq \gamma^{-a+\varepsilon}$. Let

$$\Lambda(\gamma) = \{\omega : \ell(\omega) \leq \gamma < \ell(\omega^-)\}.$$

Then the intervals I_ω corresponding to $\Lambda(\gamma)$ forms a covering of E, so $\#\Lambda(\gamma) \geq \gamma^{-a+\varepsilon}$. Now $\{J_\omega : \omega \in \Lambda(\gamma)\}$ also forms covering of E^*. But by (1),

$$|J_\omega| \geq |I_\omega|^{B_\delta} \geq c \cdot |I_{\omega^-}|^{B_\delta} \geq c \cdot \gamma^{B_\delta}$$

where c is some constant independent of ω as found in (3.6). Thus,

$$N_{c\cdot\gamma^{B_\delta}}(E^*) \geq \#\Lambda(\gamma) \geq \gamma^{-a+\varepsilon},$$

or equivalently, $N_\gamma(E^*) \geq \left(\frac{\gamma}{c}\right)^{B_\delta^{-1}\cdot(-a+\varepsilon)}$. Taking logarithms and letting $\gamma \to 0$, we obtain

$$\underline{\dim}_{\mathrm{B}} E^* \geq B_\delta^{-1}(a - \varepsilon).$$

Finally letting $\varepsilon \to 0$ we complete the proof.

(4). We let $\eta > 0$ and $\delta > 0$ be such that

$$\dim_{\mathrm{B}} E > (1+\eta) \cdot \dim_{\mathrm{H}} E \quad \text{and} \quad \frac{B_\delta}{A_\delta} < 1 + \eta.$$

The second requirement on δ is possible since $B_\delta/A_\delta \to 1$ as $\delta \to 0$ by (1). Hence, combining (2) and (3), when δ is sufficiently small, we have

$$\begin{aligned}\underline{\dim}_{\mathrm{B}} E^* &\geq \frac{1}{B_\delta} \cdot \dim_{\mathrm{B}} E \\ &> \frac{1+\eta}{B_\delta} \cdot \dim_{\mathrm{H}} E \\ &\geq \frac{A_\delta}{B_\delta} \cdot (1+\eta) \cdot \dim_{\mathrm{H}} E^* > \ \dim_{\mathrm{H}} E^*.\end{aligned}$$

□

Proposition 2.5 now follows from Proposition 5.2(4) above.

5.2. **Separating upper and lower box dimension.** We now construct a particular choice of (ϱ_n) so that the upper and lower box dimensions are all different. First, recall the following sets:

$$\Lambda(\rho) = \{\omega \in \Omega^* : \ell(\omega) \leq \rho < \ell(\omega^-)\}, \quad \mathcal{N}_\rho = \{n \in \mathbb{N} : \exists \omega \in \Omega^n \text{ s.t. } \omega \in \Lambda(\rho)\}.$$

By Lemma 4.4, there exists $c_1 < c_2$ such that

$$c_1 \log\left(\frac{1}{\rho}\right) \leq n \leq c_2 \log\left(\frac{1}{\rho}\right) \tag{5.1}$$

if $n \in \mathcal{N}_\rho$. We let

$$u_k = 2^{-k}M^{-k}, \quad v_k = 2^k M^{-k}.$$

and recall that $u_k \leq \ell(\omega) \leq v_k$ for all $\omega \in \Omega^k$. Moreover, we let

$$P = [c_2 \log(2M)] + 1, \quad P' = c_1 \log\left(\frac{M}{2}\right).$$

Lemma 5.3. *(1)* $\mathcal{N}_{u_k} \subset [k, Pk]$*;*
(2) $\mathcal{N}_{v_k} \subset [P'k, k]$.

Proof. As $\ell(\omega) \geq u_k$ for all $\omega \in \Omega^k$, we must have that $n \geq k$ if $n \in \mathcal{N}_{u_k}$. Moreover, by (5.1),

$$n \leq c_2 \log\left(\frac{1}{u_k}\right) \leq P \cdot k.$$

Similarly, to see (2), we note that $\ell(\omega) \leq v_k$ for all $\omega \in \Omega^k$, so $n \leq k$ if $n \in \mathcal{N}_{v_k}$. By (5.1),

$$n \geq c_1 \log\left(\frac{1}{v_k}\right) = P' \cdot k.$$

This completes the proof of the lemma. □

Let $\varepsilon > 0$ and (r_k) be a rapidly increasing lacunary sequence of positive integers such that $r_1 = 1$ and

$$\frac{r_{k+1}}{r_k} > \max\left\{\frac{1}{\varepsilon}, \frac{P}{P'}\right\} := K' \quad \text{for all } k \geq 1.$$

Define

$$\varrho_n = \begin{cases} \frac{1}{M-\delta} & \text{if } n \in \{r_{2k-1}+1, \ldots, Pr_{2k}\} \\ \frac{1}{M+\delta} & \text{if } n \in \{Pr_{2k}+1, \ldots, r_{2k+1}\} \end{cases} \tag{5.2}$$

Theorem 2.6 will follow from the theorem below by considering $K = PK'$ and the sequence $\widetilde{r_{2k}} = Pr_{2k}$ and $\widetilde{r_{2k-1}} = r_{2k-1}$.

Theorem 5.4. *There exists a sequence $(r_k)_{k=1}^{\infty}$ such that the Cantor set E^* satisfies*

$$\dim_{\mathrm{H}} E^* < \underline{\dim}_{\mathrm{B}} E^* < \overline{\dim}_{\mathrm{B}} E^*.$$

Proof. Let E be the Cantor set determined in Theorem 2.4, whose Hausdorff and box dimensions are different and its box dimension exists. By Proposition 5.2(4), we choose a sufficiently small $\delta > 0$ so that $\dim_{\mathrm{H}} E < \underline{\dim}_{\mathrm{B}} E^*$ and this holds for all choices of r_k. We now choose a sufficiently small $\varepsilon > 0$ so that the corresponding ϱ_n in (5.2) will give rise to an E^* with different upper and lower box dimensions. To this end, we have two cases:

Case (1). For all ω whose $|\omega| \in \{r_{2k}, \ldots, Pr_{2k}\}$, splitting

$$\log(\varrho_1 \cdots \varrho_{|\omega|}) = \sum_{j=1}^{r_{2k-1}} \log \varrho_j + \sum_{j=r_{2k-1}+1}^{|\omega|} \log \varrho_j$$

and noticing that the first summand is at most r_{2k-1} times $\log(M+\delta)$ and the later summand are all equal to $\log(M-\delta)$, we have

$$\frac{\log |J_\omega|}{\log |I_\omega|} = \frac{r_{2k-1} \log(M+\delta) + (|\omega| - r_{2k-1}) \log(M-\delta) - \log b_\omega}{|\omega| \log M - \log b_\omega}.$$

Here we are reusing the notation $b_\omega = \frac{a_{\omega_1} \cdots a_{\omega_{[\beta \omega_n]}}}{(a_{\omega_1} \cdots a_{\omega_n})^{\beta}}$. As $(\log b_\omega)/|\omega| \in Q_2 := [-\log 2, \log 2]$, dividing by $|\omega|$ and using that $r_{2k-1}/|\omega| \le r_{2k-1}/r_{2k} < \varepsilon$, we have

$$\frac{\log |J_\omega|}{\log |I_\omega|} \le \max_{x \in Q_2} \frac{\varepsilon \cdot \log(M+\delta) + \log(M-\delta) - x}{\log M - x} := \gamma_1 \tag{5.3}$$

Choosing ε sufficiently small, we will have $\gamma_1 < 1$ by the second part of Lemma 5.1.

Case (2). For all ω whose $|\omega| \in \{P' r_{2k+1}, \ldots, r_{2k+1}\}$, we have, by a similar splitting in case (1)

$$\begin{aligned} \frac{\log |J_\omega|}{\log |I_\omega|} &= \frac{Pr_{2k} \log(M-\delta) + (|\omega| - Pr_{2k}) \log(M+\delta) - \log b_\omega}{|\omega| \log M - \log b_\omega} \\ &\ge \frac{(|\omega| - Pr_{2k}) \log(M+\delta) - \log b_\omega}{|\omega| \log M - \log b_\omega} \end{aligned}$$

Using the inequality $|\omega| \ge P' r_{2k+1}$ and our assumption $P' r_{2k+1} > Pr_{2k}$, dividing by $|\omega|$ yields

$$\frac{\log |J_\omega|}{\log |I_\omega|} \ge \min_{x \in Q_2} \frac{(1 - \frac{P}{P'}\varepsilon) \log(M+\delta) - x}{\log M - x} := \gamma_2 \tag{5.4}$$

Choosing ε sufficiently small, we will have $\gamma_2 > 1$ by the first part of Lemma 5.1.

These two cases will result in the following two claims:

(a)
$$\overline{\dim}_{\mathrm{B}} E^* \geq \frac{1}{\gamma_1} \dim_{\mathrm{B}} E.$$
(b)
$$\underline{\dim}_{\mathrm{B}} E^* \leq \frac{1}{\gamma_2} \dim_{\mathrm{B}} E.$$

As we have chosen ε sufficiently small so that $\gamma_1 < 1 < \gamma_2$, (a) and (b) imply that $\underline{\dim}_{\mathrm{B}} E^* < \overline{\dim}_{\mathrm{B}} E^*$. Thus the proof will be complete if we justify the claims.

Proof of Claim (a): From Case (1), we have that whenever $|\omega| \in \{r_{2k}, \dots, Pr_{2k}\}$, by (5.3),

$$|J_\omega| \geq |I_\omega|^{\gamma_1}. \tag{5.5}$$

Let
$$s_k = 2^{-r_{2k}} M^{-r_{2k}} = u_{r_{2k}}.$$

If $\omega \in \Lambda(s_k)$, then by (3.6) we have

$$|I_\omega| \geq \frac{M}{2} s_k. \tag{5.6}$$

By Lemma 5.3(1) we know that $\mathcal{N}_{s_k} \subset [r_{2k}, Pr_{2k}]$. Therefore (5.5) and (5.6) imply that

$$|J_\omega| \geq \left(\frac{M}{2} s_k\right)^{\gamma_1}. \tag{5.7}$$

for all $\omega \in \Lambda(s_k)$. (5.7) and the definition of $\Lambda(s_k)$ now imply the following inequalities
$$N_{\left(\frac{M}{2} s_k\right)^{\gamma_1}}(E^*) \geq \#\Lambda(s_k) \geq N_{s_k}(E).$$

Therefore,
$$\overline{\dim}_{\mathrm{B}} E \geq \limsup_{k\to\infty} \frac{\log N_{\left(\frac{M}{2} s_k\right)^{\gamma_1}}(E^*)}{-\gamma_1 \log \frac{M}{2} - \gamma_1 \log s_k} \geq \frac{1}{\gamma_1} \cdot \limsup_{k\to\infty} \frac{\log N_{s_k}(E)}{-\log s_k} = \frac{1}{\gamma_1} \dim_B(E).$$

Thus we have completed our proof of Claim (a).

Proof of Claim (b): By Case (2) the following holds whenever $|\omega| \in \{P' r_{2k+1}, \dots, r_{2k+1}\}$:

$$|J_\omega| \leq |I_\omega|^{\gamma_2}. \tag{5.8}$$

We now let $t_k := 2^{r_{2k+1}} M^{-r_{2k+1}}$. By Lemma 5.3(2), ω satisfies (5.8) for all $\omega \in \Lambda(t_k)$. Hence,
$$N_{t_k^{\gamma_2}}(E^*) \leq \#\Lambda(t_k)$$
since $(J_\omega)_{\omega\in\Lambda(t_k)}$ covers E^* and they all have length less than $t_k^{\gamma_2}$. It is clear that $N_{t_k}(E) \leq \#\Lambda(t_k)$. However, we can show that they are equal. Indeed, if we let $\Lambda(t_k) = \{\omega_1, \dots, \omega_p\}$. By (3.6), the gaps
$$|G_{\omega_i^-}| \geq \left(1 - \frac{4}{M}\right) |I_{\omega_i^-}| \geq \left(1 - \frac{4}{M}\right) \cdot t_k.$$

Hence, any interval I with $|I| \le t_k$ intersects at most two I_{ω_i} and it cannot cover two I_{ω_i}. This means that $N_{t_k}(E) \ge \frac{1}{2}\#\Lambda(t_k)$. Hence,

$$\frac{\log N_{t_k^{\gamma_2}}(E^*)}{-\gamma_2 \log t_k} \le \frac{1}{\gamma_2} \cdot \frac{\log N_{t_k}(E)}{-\log t_k}$$

The inequality in Claim (b) follows by taking $k \to \infty$. □

6. A simpler construction

In this section we give a simpler construction of a bi-Lipschitz IFS for which the lower and upper box dimensions are distinct. However in this case, the Hausdorff dimension cannot be separated from the lower box dimensions. Let us first introduce the following.

Definition 6.1. Let $\mathbf{c} = (c_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $c_n < 1/2$ for all n. The **symmetric Cantor set** $E_{\mathbf{c}}$ is the Cantor set whose generating intervals satisfy

$$|I_{w0}| = |I_{\omega 1}| = c_n |I_w| \quad \text{for every } n \ge 1 \text{ and } \omega \in \Omega^{n-1}.$$

In what follows, we let $E_{\mathbf{c}}$ be the symmetric Cantor set associated with a sequence $\mathbf{c} = (c_n)_{n=1}^{\infty}$ of real numbers satisfying $0 < c_n < 1/2$. By the Cantor construction, the gap intervals of $E_{\mathbf{c}}$ satisfy

$$|G_w| = (1 - 2c_n)|I_w|$$

for each $\omega \in \Omega^{n-1}$. Moreover, it is standard to see that $E_{\mathbf{c}}$ admits the following additive representation:

$$E_{\mathbf{c}} = \left\{ \sum_{n=1}^{\infty} \omega_n u_n : \omega_n \in \{0,1\} \right\}, \tag{6.1}$$

where $u_1 = 1 - c_1$ and $u_n = \left(\prod_{j=1}^{n-1} c_j\right)(1 - c_n)$ for $n \ge 2$.

It is known (see [13]) that if $\inf c_n > 0$, then

$$\begin{aligned} \underline{\dim}_{\mathrm{B}} E_{\mathbf{c}} = \dim_{\mathrm{H}} E_{\mathbf{c}} &= \liminf_{n\to\infty} \frac{n \log 2}{-\log(c_1 c_2 \cdots c_n)}, \\ \overline{\dim}_{\mathrm{B}} E_{\mathbf{c}} = \dim_{\mathrm{P}} E_{\mathbf{c}} &= \limsup_{n\to\infty} \frac{n \log 2}{-\log(c_1 c_2 \cdots c_n)}. \end{aligned} \tag{6.2}$$

Now we can use Theorem 2.1 to obtain a sufficient condition for $E_{\mathbf{c}}$ to be the attractor of a bi-Lipschitz IFS.

Proposition 6.2. *Let $\mathbf{c} = (c_n)_{n\ge 1}$ be a sequence such that $0 < c_n < 1/2$ for all $n \in \mathbb{N}$ and let $E_{\mathbf{c}}$ be the corresponding symmetric Cantor set, with ϕ_0, ϕ_1 being defined in (2.6), respectively. Then ϕ_0 and ϕ_1 are bi-Lipschitz contractions on $[0,1]$ if and only if*

$$0 < \inf_{n\ge 1} \frac{c_n(1 - 2c_{n+1})}{1 - 2c_n}, \qquad \sup_{n\ge 1} \frac{c_n(1 - 2c_{n+1})}{1 - 2c_n} < 1. \tag{6.3}$$

Proof. By a direct calculation, for each $w \in \Omega^n$ and $i \in \{0,1\}$ we have

$$\frac{|G_{iw}|}{|G_w|} = \frac{c_n(1-2c_{n+1})}{1-2c_n} \quad \text{and} \quad \frac{|I_{iw}|}{|I_w|} = c_{n+1}.$$

Notice that the condition $\inf c_n = 0$ implies that $\inf \frac{c_n(1-2c_{n+1})}{1-2c_n} = 0$. Therefore, this proposition follows immediately from Theorem 2.1. □

Remark 6.3. By taking $\mathbf{c} = (c_n)_{n\geq 1}$ with $c_n = \frac{1}{2} - \frac{1}{2^n}$, our construction also yields a bi-Lipschitz IFS whose attractor is a symmetric Cantor set of positive Lebesgue measure. Such a Cantor set is known to exist, as shown by Bowen [6], who actually constructed a C^1 horseshoe of positive Lebesgue measure for the first time. The construction of these Cantor sets was also similar to the idea given in Section 3. It is also straightforward to observe that the attractor of this IFS has empty-interior. It is an open problem whether there exists a self-similar set in $\mathbb{R}$ with empty interior and positive Lebesgue measure. Self-similar sets satisfying this property are known to exist in higher dimensions by work of Csörnyei et al [7]. This example shows that there exist attractors of bi-Lipschitz IFSs in $\mathbb{R}$ with positive Lebesgue measure and empty interior.

We now construct a symmetric Cantor set, which is the attractor of a bi-Lipschitz IFS but its box dimension does not exist. Then we show that the maps in this IFS are non-differentiable at every point in the Cantor set.

Proposition 6.4. *There exists a symmetric Cantor set generated by a bi-Lipschitz IFS and its box dimension does not exist.*

Proof. Choose a sequence $\mathbf{c} = (c_n)_{n\geq 1}$ with $c_n \in \{1/3, 1/4\}$ such that

$$\lim_{n\to\infty} \frac{\log c_1 + \cdots + \log c_n}{n} \quad \text{does not exist.}$$

Let $E_{\mathbf{c}}$ be the symmetric Cantor set associated with $\mathbf{c}$. By (6.2), the box dimension of $E_{\mathbf{c}}$ does not exist. On the other hand, the sequence $\mathbf{c}$ satisfies the condition (6.3). Let ϕ_0, ϕ_1 be the maps given by (2.6), in which E is replaced by $E_{\mathbf{c}}$. Then (2.7) holds with E replaced by $E_{\mathbf{c}}$. So by Proposition 6.2, $\{\phi_0, \phi_1\}$ is a bi-Lipschitz IFS on $[0,1]$, having E_c as its attractor. □

6.1. **Differentiability of the maps.** The differentiability of the maps ϕ_0 and ϕ_1 associated with a given symmetric Cantor set $E_{\mathbf{c}}$ can be studied relatively easily. Notice that the two maps are of course differentiable on $[0,1] \setminus E_{\mathbf{c}}$. However, we will show that they are non-differentiable at every point in $E_{\mathbf{c}}$ if the box dimension of $E_{\mathbf{c}}$ does not exist.

Proposition 6.5. *Let $E_{\mathbf{c}}$ be a symmetric Cantor set given by (6.1) and ϕ_0, ϕ_1 the maps given by (2.6), in which E is replaced by $E_{\mathbf{c}}$. If there exists $i \in \{0,1\}$ and $x \in E_{\mathbf{c}}$ such that ϕ_i is differentiable at x. Then the box dimension of $E_{\mathbf{c}}$ exists.*

Proof. Suppose that the derivative $\phi_i'(x)$ exists for some $i \in \{0,1\}$ and $x \in E_{\mathbf{c}}$. Since $x \in E_{\mathbf{c}}$, there exists $(\omega_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ such that

$$x = \omega_1(1-c_1) + \omega_2 c_1(1-c_2) + \cdots + \omega_n c_1 c_2 \cdots c_{n-1}(1-c_n) + \cdots.$$

For $n \in \mathbb{N}$, let $y_n \in E_{\mathbf{c}}$ be given as

$$y_n = \omega_1(1-c_1) + \omega_2 c_1(1-c_2) + \cdots + (1-\omega_n) c_1 c_2 \cdots c_{n-1}(1-c_n) + \cdots.$$

That is, y_n is obtained from x by replacing ω_n in the nth term in the summation for x with $(1-\omega_n)$. By the definition of ϕ_i (see (2.6)), we have

$$\phi_i(x) = \pi(i0^\infty) + \omega_1 c_1(1-c_2) + \cdots + \omega_n c_1 c_2 \ldots c_n(1-c_{n+1}) + \cdots$$

and

$$\phi_i(y_n) = \pi(i0^\infty) + \omega_1 c_1(1-c_2) + \cdots + (1-\omega_n) c_1 c_2 \ldots c_n(1-c_{n+1}) + \cdots.$$

Consequently,

$$\frac{\phi_i(x) - \phi_i(y_n)}{x - y_n} = \frac{(2\omega_n - 1)c_1 \cdots c_n(1-c_{n+1})}{(2\omega_n - 1)c_1 \cdots c_{n-1}(1-c_n)} = \frac{c_n(1-c_{n+1})}{1-c_n}.$$

Since y_n converges to x, the existence of $\phi_i'(x)$ implies the existence of

$$\lim_{n\to\infty} \frac{c_n(1-c_{n+1})}{1-c_n}.$$

Taking logarithms and averaging, it follows that the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \frac{c_j(1-c_{j+1})}{1-c_j}$$

exists. But

$$\frac{1}{n} \sum_{j=1}^{n} \log \frac{c_j(1-c_{j+1})}{1-c_j} = \frac{1}{n} \sum_{j=1}^{n} \log c_j + \frac{1}{n} \log \frac{1-c_{n+1}}{1-c_1}.$$

Since $c_n \in (0, 1/2)$ for all $n \geq 1$, $\frac{1}{n} \log \frac{1-c_{n+1}}{1-c_1} \to 0$ as $n \to \infty$. It follows that the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log c_j$$

exists. By (6.2), the box dimension of $E_{\mathbf{c}}$ exists. □

It is a consequence of the above that if the box dimension of the symmetric Cantor set $E_{\mathbf{c}}$ does not exist, then the contraction maps ϕ_i we defined must be non-differentiable at every point in $E_{\mathbf{c}}$.

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