

# Which numbers are normal?

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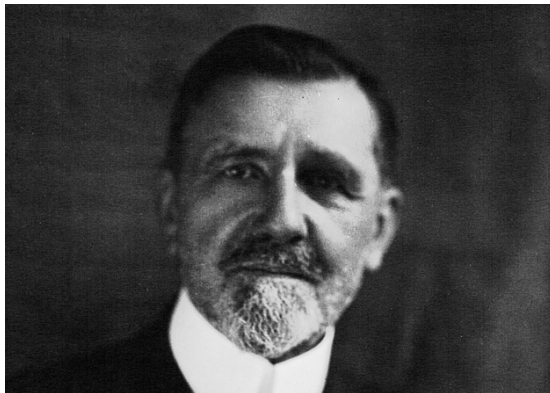
<sup>1</sup>Includes joint work with Simon Baker <https://arxiv.org/abs/2401.01241>



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# Émile Borel

More than 100 years ago, French mathematician Émile Borel began a quest to find certain elusive numbers.



[https://commons.wikimedia.org/wiki/File:Emile\\_Borel-1932.jpg](https://commons.wikimedia.org/wiki/File:Emile_Borel-1932.jpg)

# Normal number

Every number has an infinite decimal expansion, e.g.  $\frac{1}{2} = 0.50000000\dots$

In 1909, Borel introduced the concept of a **normal number**. A number is normal if

- Each digit from 0 to 9 appears equally often in its decimal expansion (each appears roughly 10% of the time)

For example, in the first million digits of a normal number, we expect to find roughly 100,000 of each digit.

- Each 2-digit number appears equally often
- Each 3-digit number appears equally often

And so on

# Probability of normal numbers

Borel proved the following theorem:

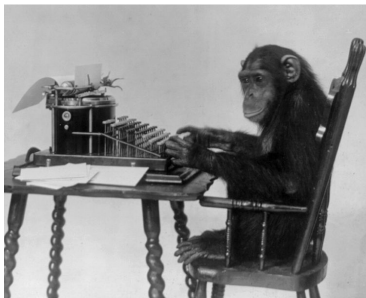
## Theorem (Borel)

Normal numbers exist! In fact, if you pick a random number between 0 and 1, the probability that it will be normal is 100%.

- This means if you pick a million different numbers between 0 and 1 independently at random, almost certainly all will be normal.
- This does **not** mean every number is normal!

# The infinite monkey thought experiment

If a monkey types digits between 0 and 9 randomly, there is a 100% chance it would eventually type a million sevens in a row (or any other string of numbers you can think of).



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- If the monkey types letters instead of numbers, there is a 100% chance that it will eventually write Shakespeare's Hamlet without a single mistake!
- But the monkey would need a ridiculously long time.

# Where are the normal numbers?

- Every fraction, like  $\frac{5}{7} = 0.714285714285\dots$ , has a repeating decimal expansion, so it **cannot** be normal.
- Plenty of numbers which are not fractions are also not normal, for example if the decimal expansion is non-repeating but only contain digits 0,5,9.
- Hence began the quest to write down examples of normal numbers. It took many years to find a single one.
- Normal numbers are everywhere, but elusive!

# Champernowne's number

The first person to write down the digits of a normal number was D. G. Champernowne in 1933.



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His number is

0.12345678910111213141516...

(its decimal expansion is made by writing every whole number in order).

# $\pi$ and $\sqrt{2}$

$\pi$  is the number of times a circle's diameter fits around its circumference:

$$\pi = 3.14159265358979323846264338327950288419716939937510$$
$$58209749445923078164062862089986280348253421170679$$
$$82148086513282306647093844609550582231725359408128$$
$$48111745028410270193852110555964462294895493038196...$$

$\sqrt{2}$  is the number that gives 2 when multiplied by itself:

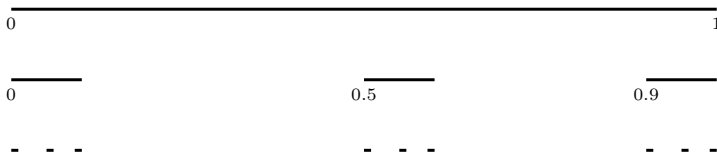
$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694$$
$$80731766797379907324784621070388503875343276415727$$
$$35013846230912297024924836055850737212644121497099$$
$$93583141322266592750559275579995050115278206057147...$$

These sequences look very 'random' (though of course they are not). Most mathematicians would be astonished if these numbers turn out not to be normal, but no-one has a clue how to **prove** it!



# Fractals

The set of (non-normal) numbers between 0 and 1 whose decimal expansion consists of numbers 0, 5, 9, for example, forms a **fractal** set.



Fractals are geometric objects which display intricate and often beautiful structure at small scales. They are often formed of scaled-down images of themselves.

# Fractals in nature

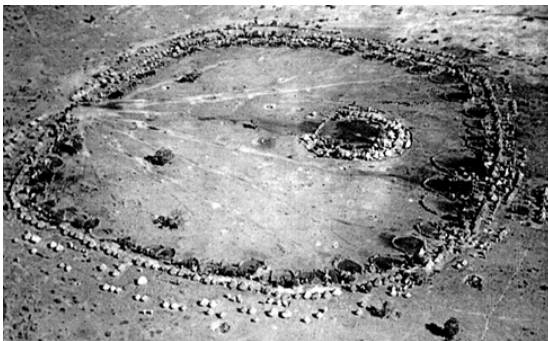


Romanesco broccoli. CC0,  
<https://stocksnap.io/photo/plant-green-XH1IME1TH4>



The Barnsley Fern. CC0,  
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# Fractals in architecture

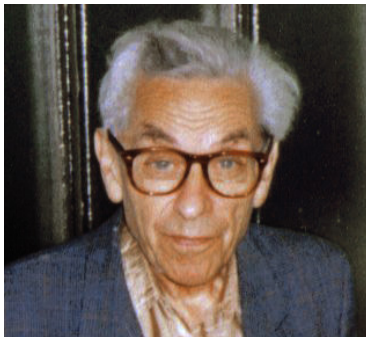


A settlement in Zambia with a fractal design.

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# Nonlinear fractals

- **Question:** Which fractals contain normal numbers?
- Not all of them (e.g. not missing digit fractals, which are made of strictly scaled copies of the whole fractal).
- In 1963, this problem caught the attention of Paul Erdős.



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# Fourier transforms

Davenport, Erdős and LeVeque showed that if the fractal admits a weighting whose Fourier transform satisfies certain properties, then the fractal contains normal numbers.

# Lots of fractals contain normal numbers

**Theorem** (Davenport–Erdős–LeVeque + Algom–Rodriguez Hertz–Wang / Baker–Sahlsten + Baker–B. / Algom–Chang–Wu–Wu)

If a fractal of real numbers is made of copies of itself which are scaled down in a very smooth (analytic) way, and at least one copy has some distortion (nonlinearity), then it contains normal numbers.

In fact, if we choose a ‘random’ point in such a fractal, then the probability of choosing a normal number is 100%.

- This is an extension of Borel’s result to non-linear fractals, and shows that these fractals are ‘unstructured.’
- Borel went on to study Einstein’s theory of special relativity, and was later active in the French resistance to Nazi occupation.
- The quest to show that ‘nice’ numbers like  $\pi$  or  $\sqrt{2}$  are normal may yet last another hundred years.

# Thank you for listening!