AM207 Final Project: Mastermind

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Abstract

Insert abstract text.

1 Introduction

The game Mastermind was invented in 1970 by Mordecai Meirowitz, and is similar to an earlier game called Bulls and Cows. There are many variations of the game[1], but they all follow a broadly consistent format. The game involves two players: a code-maker and a code-breaker. The code-maker selects a sequence of length L (usually 4), where each element in the sequence is chosen with replacement from one of C colors (usually 6). At each turn, the code-breaker guesses a sequence and receives feedback in the form of black pegs and white pegs, where the black pegs denote the number of correct guesses in the right position, and the white pegs denote the number of correct guesses in the wrong position. Based on this information, the code-breaker refines her guesses and attempts to crack the code within the maximum number of guesses (usually 10).

For our project, we set out to implement various strategies and algorithms for iteratively optimizing the code-breaker's guess at each turn. We compared the performance of these strategies based on the mean and standard deviation of the required number of guesses to win and the runtime across 20 random game initializations. In order to assess the extent to which the different solutions scale efficiently, we tested seven game setups of varying complexity.

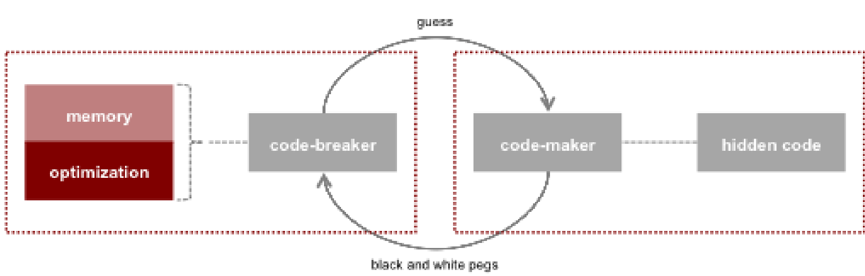


Figure 1: Mastermind Framework

[1]https://en.wikipedia.org/wiki/Mastermind_(board_game)#Variations

2 Optimization Methods

It is helpful at this stage to introduce some mathematical notation to describe the game, which we will use consistently when describing the various methods that we tested. For simplicity, we use one-indexing in the formulas below; however, we note that the Python code for the game interface that we built is zero-indexed.

We define:

- set of possible colors: $C = \{1, ..., C\}$
- set of positions in the code: $L = \{1, ..., L\}$
- hidden code: $H_i \ \forall i \in L$
- guess of the hidden code: $T_i \ \forall i \in L$
- indicator function $\mathbb{1}_{A=B}$, which equals 1 if A=B and equals 0 otherwise

Using the above notation, we can denote the responses at each turn as follows:

- correct guesses in the right position: $B = \sum_{i=1}^{L} \mathbb{1}_{T_i=H_i} \ \forall i \in L$
- correct guesses in the wrong position: $W = \sum_{i=1}^{C} \min(\sum_{j=1}^{L} \mathbb{1}_{H_j=i,G_i}, \sum_{j=1}^{L} \mathbb{1}_{T_j=i,G_i}) - B$

2.1 Knuth's Five-Guess Algorithm

The most commonly referenced optimization technique in the Mastermind literature is Knuth's five-guess algorithm [3] – sometimes also referred to as the "worst-case strategy" – which can always solve the classic game configuration in five moves or less. This strategy always begins with the same initial guess of 1122 (or 0011, when zero-indexed); this choice is motivated by examples of other initial guesses that do not always lead to a solution in five moves. Our implementation uses this deterministic initial guess for the classic game configuration, but uses a random initial guess for all other configurations as analogous "best" starting points are not defined in the literature.

After the initial guess, the algorithm determines the minimum number of codes that each guess would eliminate from the list of possibilities, and chooses one of the guesses that maximizes this number[2]. At each stage of the game, the set of possible codes is updated to only include codes that would have generated the same responses at each of the previous steps. This ensures that the state space shrinks with each subsequent guess.

2.2 Random Search with Constraints / Random Sampling From Posterior

This method is described in the Mastermind literature both as a constrained random search algorithm [1] and in terms of posterior distribution updates [9]. We follow the latter approach below.

We start by defining the joint probability distribution over all possible code sequences as $P(H_1, ..., H_L)$. As we have no information, our prior is uniformly distributed.

[2]Wikipedia refers to this as the minimax *technique*. We note that this one-off maximization differs from the recursive minimax *algorithm* that is also used in game theory.

$$P(H_1 = h_1, ..., H_L = h_l) = \frac{1}{C^L}, \quad \text{for all combinations of } (h_1, ..., h_l)$$

We can denote the evidence that obtain at each step as $e = (B, W)$, where B and W are defined as above, and use this to update the posterior joint distribution over code sequences as follows:

$$P(H_1 = h_1, ..., H_L = h_l | e) = \begin{cases} \frac{1}{|s(e)|}, & \text{if } (h_1, ..., h_l) \text{ is a possible code} \\ 0, & \text{otherwise} \end{cases}$$

where s(e) denotes the set of possible hidden codes, given the evidence, and $|s(e)|$ denotes the cardinality of this set.

We can define the posterior after multiple game steps analogously:

$$P(H_1 = h_1, ..., H_L = h_l | e_1, ..., e_n) = \begin{cases} \frac{1}{|s(e_1) \cap ... \cap s(e_n)|}, & \text{if } (h_1, ..., h_l) \text{ is a possible code} \\ 0, & \text{otherwise} \end{cases}$$

where $s(e_1) \cap ... \cap s(e_n)$ denotes the intersection of the sets of possible hidden codes, given the evidence at each step, and the entire denominator denotes the cardinality of this intersection.

We can use this framework to define the posterior updates at each round of the game, and then choose the next guess by sampling from the posterior distribution. Figure 2 shows how the number of possible guesses shrinks as the game progresses.

2.3 Maximizing Shannon Entropy

Entropy is a measure that is commonly used in information theory to quantify the average amount of information that is contained in a message. In the context of Mastermind, the "message" is the response of black and white pins that is returned by the code-maker at each turn. The goal of the code-breaker is to choose guesses that create as even a distribution as possible between the various responses[3], as it will allow her to discard more possible codes at the next step.

Let us denote r_i as the ith response category and R as the number of possible responses[4]. We can then define the entropy of the discrete response space $\{r_1, ..., r_R\}$ for a given guess as:

$$H(\text{guess}|\text{possible codes}) = \sum_{i=1}^{R} P(r_i) I(r_i) = -\sum_{i=1}^{R} P(r_i) \log_b P(r_i)$$

where $I(r_i)$ denotes the information content of the ith response category. We use $b = 2$, meaning that we are measuring entropy in shannons, but note that any other value of b would yield a consistent ranking between guesses.

Practically, we calculate the probability of each response category for a given guess by counting (and normalizing) the total number of possible responses in each category, given the hidden codes that are still possible at that particular stage in the game. The value will depend on the shape of

[3] http://www.geometer.org/mathcircles/mastermind.pdf

[4] For example, the classic version of the game with codes of length 4 has 14 possible responses: $(4,0), (3,0), (2,2), (2,1), (2,0), (1,3), (1,2), (1,1), (1,0), (0,4), (0,3), (0,2), (0,1), (0,0)$.

the probability distribution across the response categories, with the minimum entropy of 0 only achievable when there is certainty of a particular outcome (i.e. $log_2(1) = 0$).

In order to improve her performance, the code-breaker will want to pick the guess that results in the highest entropy – or, if there are ties, one of the best guesses – in order to be able to discard more possible codes at the next step. This can be achieved through an exhaustive calculation of the entropy of all possible guesses at each stage or, for larger state spaces, through a local search technique such as simulated annealing or genetic algorithms. Figure 3 illustrates how the distribution of entropy for the remaining possible guesses can change as the game progresses; this technique is particularly effective where there is a clear maximum value (e.g. at the third guess).

2.4 Simulated Annealing

Knuth's algorithm and maximizing shannon entropy are both examples of global optimization techniques. These approaches work well when optimizing across relatively small state spaces, but they are not guaranteed to scale well as the problem size increases. For example, maximizing shannon entropy has complexity $O(|s(e_i^2)|)$ at the ith round of the game, i.e. the runtime scales quadratically with the number of possible codes.

Simulated annealing provides an alternative optimization approach that does not require an exhaustive search of the state space[5]. The effectiveness and runtime of this technique will depend upon the objective function that is used to evaluate potential guesses, and the method that is used to propose potential new guesses. As mentioned above, we can use simulated annealing as a faster alternative to calculating the shannon entropy of all guesses. In this case, new guesses will be proposed by randomly picking from the set of remaining possible guesses, and will be evaluated by calculating their entropy.

An alternative approach is described by Bernier et al. [1]. Under this framework, new guesses are proposed by randomly picking from the set of all guesses that have not already been used. Not updating the set of possible guesses at each turn reduces the runtime complexity of the algorithm. The proposed guess's objective function is calculated by comparing it to all the previously played guesses. Concretely, the cost function is defined as:

$$C = \sum_{i=1}^{n} |\Delta n_W^i| + |\Delta n_B^i + \Delta n_W^i|$$

where n represents the number of guesses played until that point in the game, and Δn_B and Δn_W denote the difference between the black and white pegs obtained from previous guesses and the number of black and white pegs that would have been obtained if the newly proposed guess was the right code. Once again, this approach has the advantage of being computationally inexpensive despite the changing game environment: the number of white pegs are simply computed and added to the cost so that it increases with the number of unsatisfied rules (or pegs).

2.5 Genetic Algorithms

TODO

[5]For the sake of brevity, the simulated annealing algorithm is not described here.

3 Experiments

TODO

4 Conclusion

TODO

A Figures

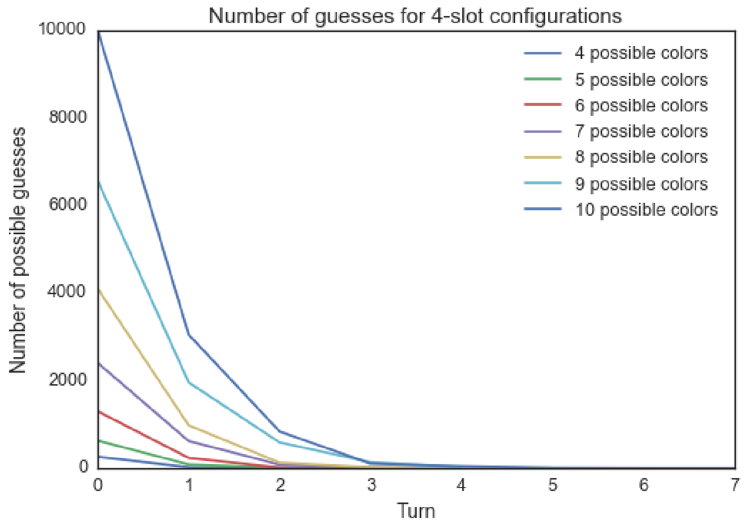


Figure 2: Evolution in number of possible guesses

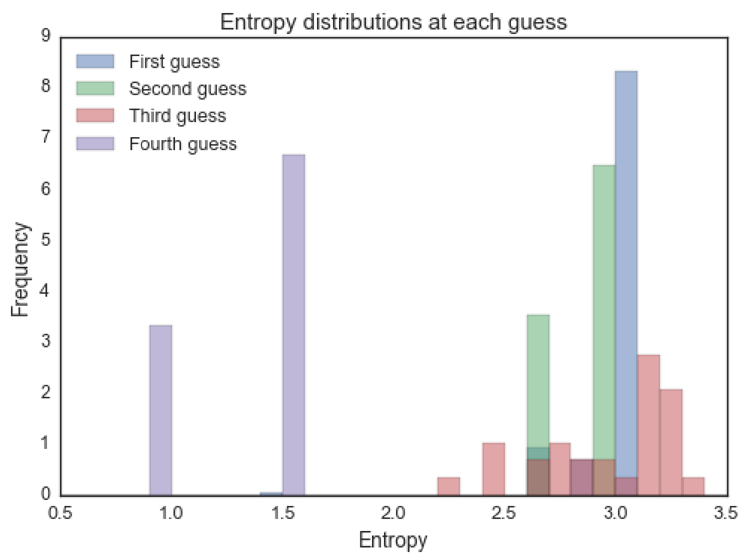
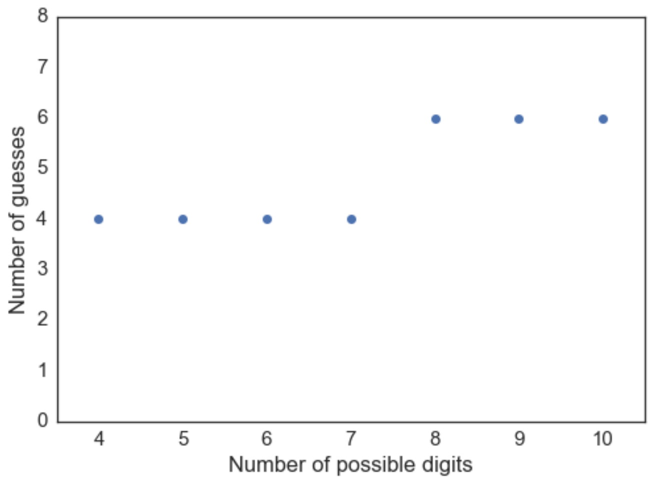
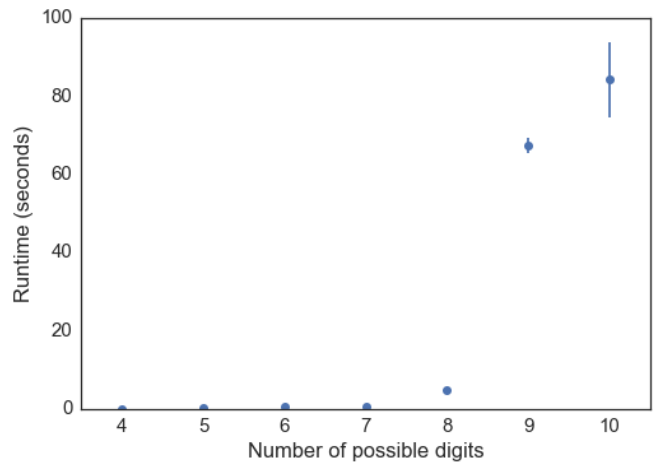


Figure 3: Entropy distributions for classic game parameters

B Results by Optimization Method

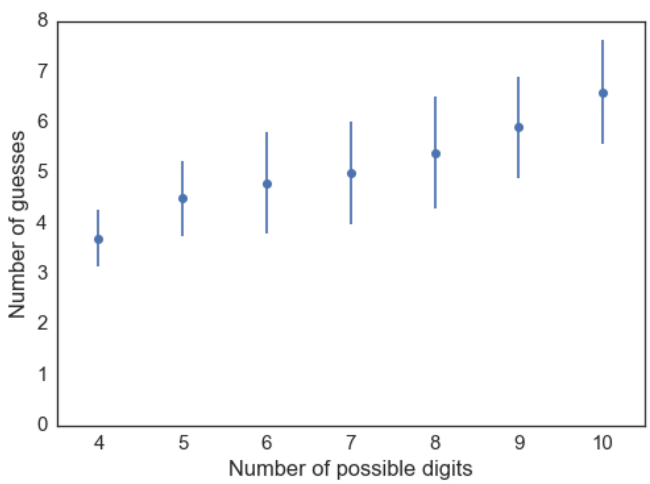


(a) Number of guesses

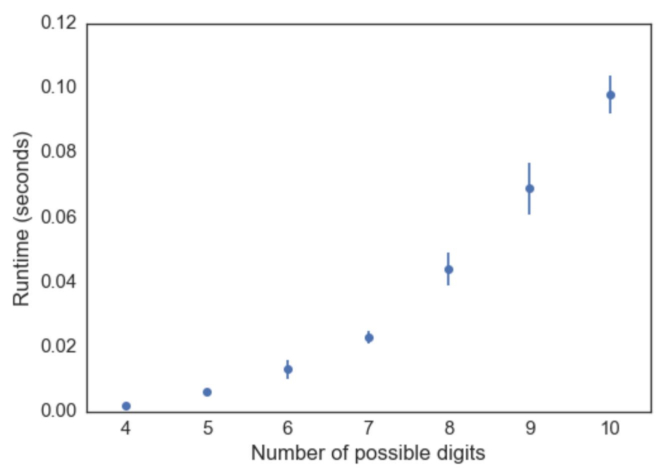


(b) Runtime (seconds)

Figure 4: Knuth's algorithm

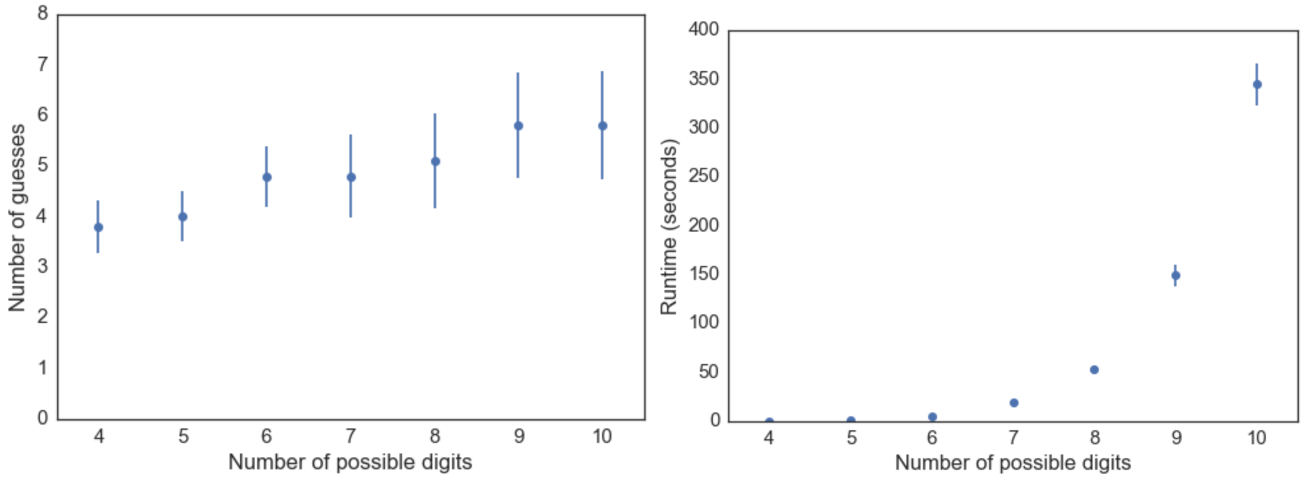


(a) Number of guesses



(b) Runtime (seconds)

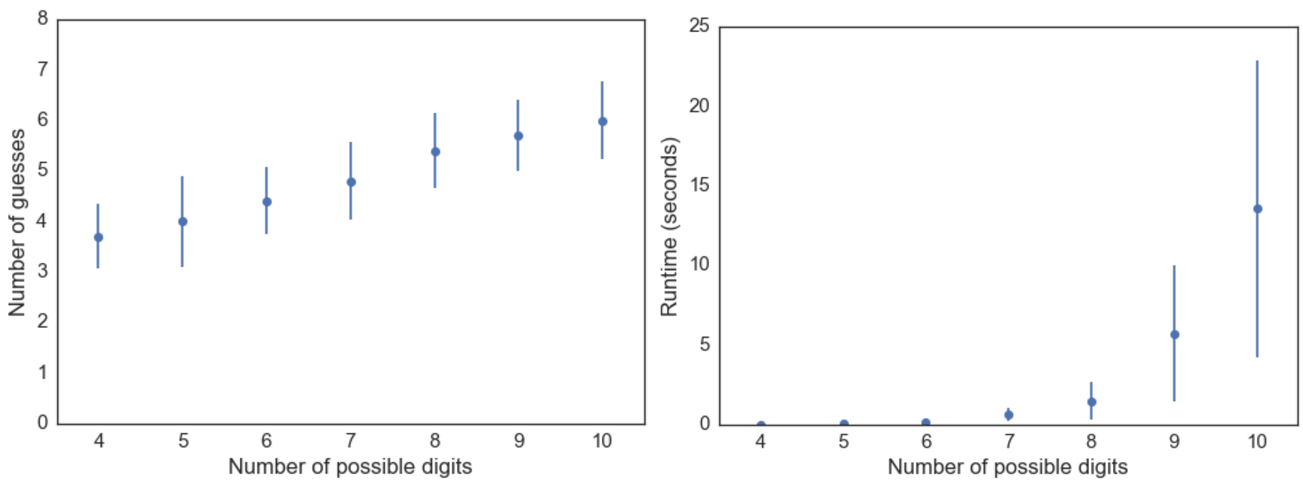
Figure 5: Random search / Random sampling from posterior



(a) Number of guesses

(b) Runtime (seconds)

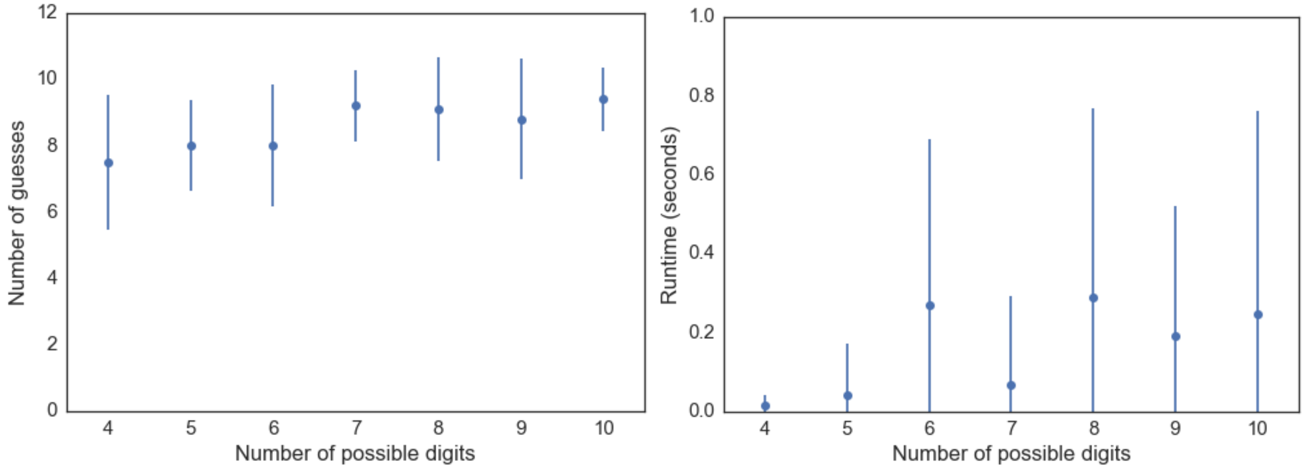
Figure 6: Maximizing entropy (all steps)



(a) Number of guesses

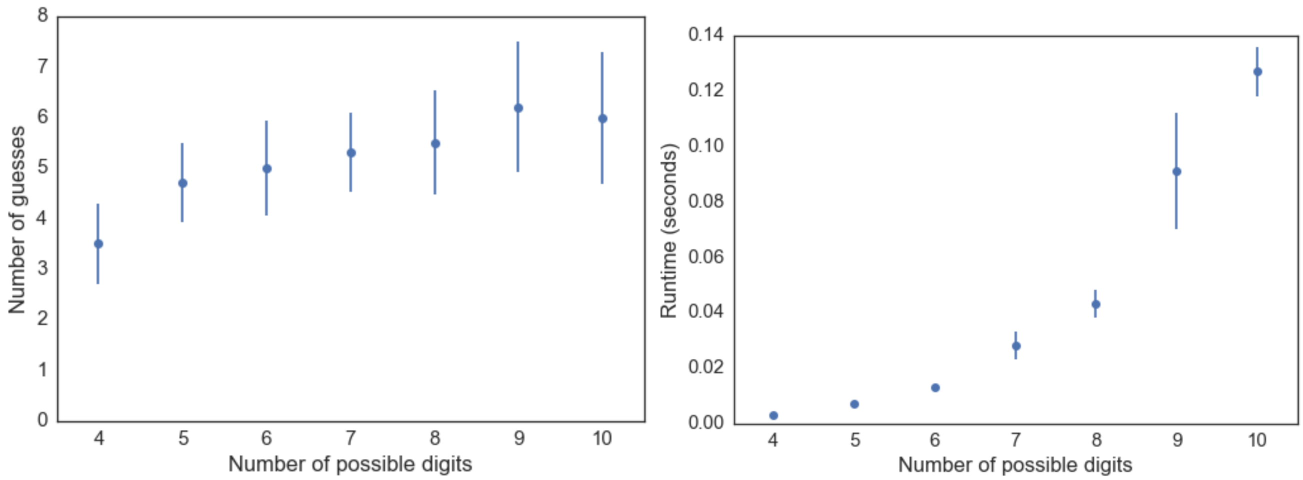
(b) Runtime (seconds)

Figure 7: Maximizing entropy (except first step)



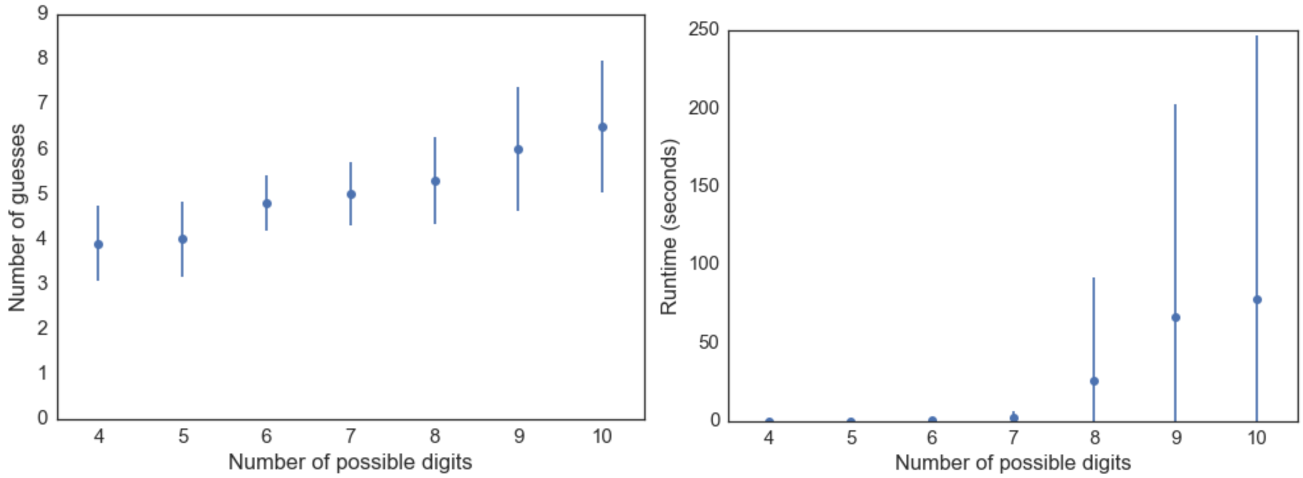
(a) Number of guesses (b) Runtime (seconds)

Figure 8: Simulated annealing (Bernier objective function)



(a) Number of guesses (b) Runtime (seconds)

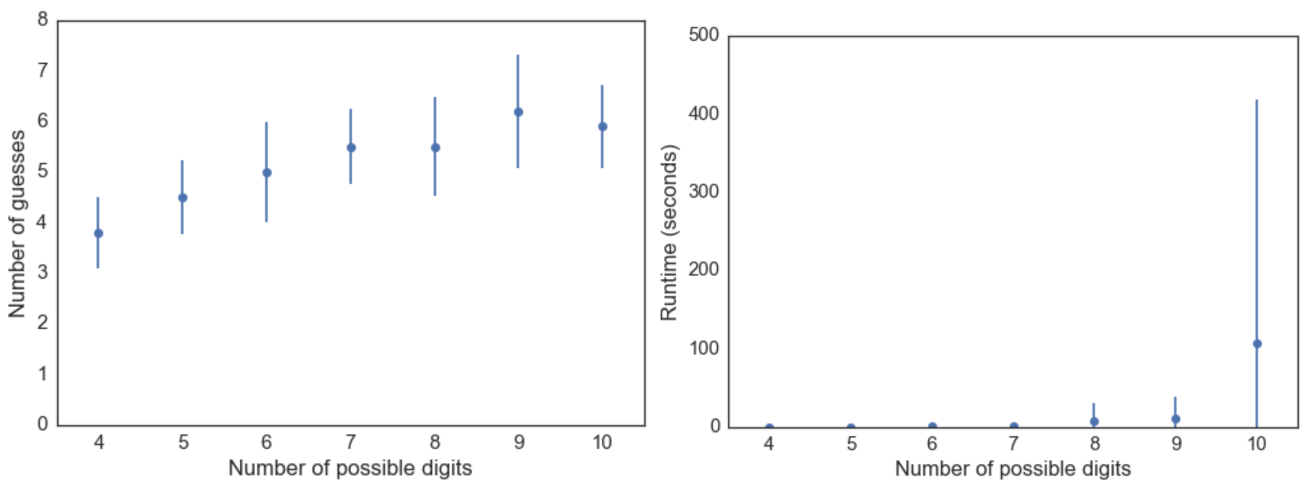
Figure 9: Simulated annealing (entropy objective function)



(a) Number of guesses

(b) Runtime (seconds)

Figure 10: Genetic algorithms (Bernier objective function)



(a) Number of guesses

(b) Runtime (seconds)

Figure 11: Genetic algorithms (entropy objective function)

C Results by Game Configuration

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	4.0	0.000	0.021	0.002
Random search / Random sampling	3.7	0.557	0.002	0.000
Maximizing entropy (all steps)	3.8	0.510	0.193	0.010
Maximizing entropy (except first step)	3.7	0.640	0.007	0.004
Simulated annealing (Bernier objective function)	7.5	2.037	0.017	0.026
Simulated annealing (entropy objective function)	3.5	0.805	0.003	0.001
Genetic algorithms (Bernier objective function)	3.9	0.831	0.022	0.026
Genetic algorithms (entropy objective function)	3.8	0.698	0.027	0.032

Table 1: 4 positions; 4 possible digits

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	4.0	0.000	0.264	0.006
Random search / Random sampling	4.5	0.742	0.006	0.001
Maximizing entropy (all steps)	4.0	0.497	1.147	0.035
Maximizing entropy (except first step)	4.0	0.894	0.042	0.017
Simulated annealing (Bernier objective function)	8.0	1.359	0.042	0.129
Simulated annealing (entropy objective function)	4.7	0.781	0.007	0.001
Genetic algorithms (Bernier objective function)	4.0	0.837	0.160	0.289
Genetic algorithms (entropy objective function)	4.5	0.740	0.273	0.442

Table 2: 4 positions; 5 possible digits

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	4.0	0.000	0.606	0.017
Random search / Random sampling	4.8	0.994	0.013	0.003
Maximizing entropy (all steps)	4.8	0.600	5.225	0.207
Maximizing entropy (except first step)	4.4	0.663	0.145	0.099
Simulated annealing (Bernier objective function)	8.0	1.844	0.268	0.421
Simulated annealing (entropy objective function)	5.0	0.949	0.013	0.001
Genetic algorithms (Bernier objective function)	4.8	0.622	0.230	0.472
Genetic algorithms (entropy objective function)	5.0	1.000	0.376	0.660

Table 3: 4 positions; 6 possible digits

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	4.0	0.000	0.528	0.014
Random search / Random sampling	5.0	1.023	0.023	0.002
Maximizing entropy (all steps)	4.8	0.812	18.600	0.883
Maximizing entropy (except first step)	4.8	0.766	0.669	0.407
Simulated annealing (Bernier objective function)	9.2	1.077	0.069	0.224
Simulated annealing (entropy objective function)	5.3	0.781	0.028	0.005
Genetic algorithms (Bernier objective function)	5.0	0.707	2.432	3.576
Genetic algorithms (entropy objective function)	5.5	0.742	1.384	2.043

Table 4: 4 positions; 7 possible digits

Optimization method	**Number of guesses**		**Runtime (seconds)**	
	μ	σ	μ	σ
Knuth's algorithm	6.0	0.000	21.711	0.269
Random search / Random sampling	5.4	1.114	0.044	0.005
Maximizing entropy (all steps)	5.1	0.943	53.689	1.944
Maximizing entropy (except first step)	5.4	0.735	1.464	1.162
Simulated annealing (Bernier objective function)	9.1	1.578	0.290	0.478
Simulated annealing (entropy objective function)	5.5	1.023	0.043	0.005
Genetic algorithms (Bernier objective function)	5.3	0.954	25.809	65.751
Genetic algorithms (entropy objective function)	5.5	0.973	7.866	22.396

Table 5: 4 positions; 8 possible digits

Optimization method	**Number of guesses**		**Runtime (seconds)**	
	μ	σ	μ	σ
Knuth's algorithm	6.0	0.000	67.299	1.928
Random search / Random sampling	5.9	0.995	0.069	0.008
Maximizing entropy (all steps)	5.8	1.043	149.057	11.014
Maximizing entropy (except first step)	5.7	0.714	5.702	4.282
Simulated annealing (Bernier objective function)	8.8	1.824	0.191	0.331
Simulated annealing (entropy objective function)	6.2	1.288	0.091	0.021
Genetic algorithms (Bernier objective function)	6.0	1.378	66.406	135.881
Genetic algorithms (entropy objective function)	6.2	1.122	10.492	28.552

Table 6: 4 positions; 9 possible digits

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	6.0	0.000	84.160	9.606
Random search / Random sampling	6.6	1.020	0.098	0.006
Maximizing entropy (all steps)	5.8	1.062	344.603	21.921
Maximizing entropy (except first step)	6.0	0.775	13.557	9.339
Simulated annealing (Bernier objective function)	9.4	0.970	0.248	0.513
Simulated annealing (entropy objective function)	6.0	1.304	0.127	0.009
Genetic algorithms (Bernier objective function)	6.5	1.466	77.632	169.038
Genetic algorithms (entropy objective function)	5.9	0.831	107.604	310.685

Table 7: 4 positions; 10 possible digits

D Additional Methods Tested

TODO

References

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