AM207 Final Project: Mastermind

Gioia Dominedò   Amy Lee   Kendrick Lo   Reiner Maat

April 29, 2016

Abstract

Insert abstract text.

1   Introduction

The game Mastermind was invented in 1970 by Mordecai Meirowitz, and is similar to an earlier game called Bulls and Cows. There are many variations of the game[1], but they all follow a broadly consistent format. The game involves two players: a code-maker and a code-breaker. The code-maker selects a sequence of length L (usually 4), where each element in the sequence is chosen with replacement from one of C colors (usually 6). At each turn, the code-breaker guesses a sequence and receives feedback in the form of black pegs and white pegs, where the black pegs denote the number of correct guesses in the right position, and the white pegs denote the number of correct guesses in the wrong position. Based on this information, the code-breaker refines her guesses and attempts to crack the code within the maximum number of guesses (usually 10).

For our project, we set out to implement various strategies and algorithms for iteratively optimizing the code-breaker's guess at each turn. We compared the performance of these strategies based on the mean and standard deviation of the required number of guesses to win and the run-time across 20 random game initializations. In order to assess the extent to which the different solutions scale efficiently, we tested seven game setups of varying complexity.
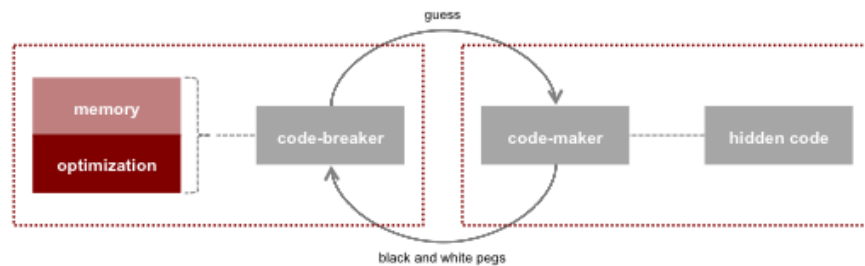

Figure 1: Mastermind Framework

---

[1]https://en.wikipedia.org/wiki/Mastermind_(board_game)#Variations

1

2 Optimization Methods

It is helpful at this stage to introduce some mathematical notation to describe the game, which we will use consistently when describing the various methods that we tested. For simplicity, we use one-indexing in the formulas below; however, we note that the Python code for the game interface that we built is zero-indexed.

We define:

- set of possible colors: $C = \{1, ..., C\}$
- set of positions in the code: $L = \{1, ..., L\}$
- hidden code: $H_i \; \forall i \in L$
- guess of the hidden code: $T_i \; \forall i \in L$
- indicator function $\mathbb{1}_{A=B}$, which equals 1 if A=B and equals 0 otherwise

Using the above notation, we can denote the responses at each turn as follows:

- correct guesses in the right position: $B = \sum_{i=1}^{L} \mathbb{1}_{T_i = H_i} \; \forall i \in L$
- correct guesses in the wrong position: $W = \sum_{i=1}^{C} \min(\sum_{j=1}^{L} \mathbb{1}_{H_j=i,G_i}, \sum_{j=1}^{L} \mathbb{1}_{T_j=i,G_i}) - B$

2.1 Knuth's Five-Guess Algorithm

The most commonly referenced optimization technique in the Mastermind literature is Knuth's five-guess algorithm [3] – sometimes also referred to as the "worst-case strategy" – which can always solve the classic game configuration in five moves or less. This strategy always begins with the same initial guess of 1122 (or 0011, when zero-indexed); this choice is motivated by examples of other initial guesses that do not always lead to a solution in five moves. Our implementation uses this deterministic initial guess for the classic game configuration, but uses a random initial guess for all other configurations as analogous "best" starting points are not defined in the literature.

After the initial guess, the algorithm determines the minimum number of codes that each guess would eliminate from the list of possibilities, and chooses one of the guesses that maximizes this number[2]. At each stage of the game, the set of possible codes is updated to only include codes that would have generated the same responses at each of the previous steps. This ensures that the state space shrinks with each subsequent guess.

2.2 Random Search with Constraints / Random Sampling From Posterior

This method is described in the Mastermind literature both as a constrained random search algorithm [1] and in terms of posterior distribution updates [9]. We follow the latter approach below.

We start by defining the joint probability distribution over all possible code sequences as $P(H_1, ..., H_L)$. As we have no information, our prior is uniformly distributed.

---

[2] Wikipedia refers to this as the minimax technique. We note that this one-off maximization differs from the recursive minimax *algorithm* that is also used in game theory.

$$P(H_1 = h_1, ..., H_L = h_l) = \frac{1}{C^L}, \quad \text{for all combinations of } (h_1, ..., h_l)$$

We can denote the evidence that obtain at each step as $e = (B, W)$, where B and W are defined as above, and use this to update the posterior joint distribution over code sequences as follows:

$$P(H_1 = h_1, ..., H_L = h_l | e) = \begin{cases} \frac{1}{|s(e)|}, & \text{if } (h_1, ..., h_l) \text{ is a possible code} \\ 0, & \text{otherwise} \end{cases}$$

where s(e) denotes the set of possible hidden codes, given the evidence, and $|s(e)|$ denotes the cardinality of this set.

We can define the posterior after multiple game steps analogously:

$$P(H_1 = h_1, ..., H_L = h_l | e_1, ..., e_n) = \begin{cases} \frac{1}{|s(e_1) \cap ... \cap s(e_n)|}, & \text{if } (h_1, ..., h_l) \text{ is a possible code} \\ 0, & \text{otherwise} \end{cases}$$

where $s(e_1) \cap ... \cap s(e_n)$ denotes the intersection of the sets of possible hidden codes, given the evidence at each step, and the entire denominator denotes the cardinality of this intersection.

We can use this framework to define the posterior updates at each round of the game, and then choose the next guess by sampling from the posterior distribution.

2.3   Maximizing Shannon Entropy

Entropy is a measure that is commonly used in information theory to quantify the average amount of information that is contained in a message. In the context of Mastermind, the "message" is the response of black and white pins that is returned by the code-maker at each turn. The goal of the code-breaker is to choose guesses that create as even a distribution as possible between the various responses[3], as it will allow her to discard more possible codes at the next step.

Let us denote r_i as the ith response category and R as the number of possible responses[4]. We can then define the entropy of the discrete response space $\{r_1, ..., r_R\}$ for a given guess as:

$$H(\text{guess}|\text{possible codes}) = \sum_{i=1}^{R} P(r_i) I(r_i) = - \sum_{i=1}^{R} P(r_i) \log_b P(r_i)$$

where $I(r_i)$ denotes the information content of the ith response category. We use $b = 2$, meaning that we are measuring entropy in shannons, but note that any other value of b would yield a consistent ranking between guesses.

Practically, we calculate the probability of each response category for a given guess by counting (and normalizing) the total number of possible responses in each category, given the hidden codes that are still possible at that particular stage in the game. The value will depend on the shape of the probability distribution across the response categories, with the minimum entropy of 0 only achievable when there is certainty of a particular outcome (i.e. $log_2(1) = 0$).

---

[3]http://www.geometer.org/mathcircles/mastermind.pdf
[4]For example, the classic version of the game with codes of length 4 has 14 possible responses: $(4,0), (3,0), (2,2), (2,1), (2,0), (1,3), (1,2), (1,1), (1,0), (0,4), (0,3), (0,2), (0,1), (0,0)$.

In order to improve her performance, the code-breaker will want to pick the guess that results in the highest entropy – or, if there are ties, one of the best guesses – in order to be able to discard more possible codes at the next step. This can be achieved through an exhaustive calculation of the entropy of all possible guesses at each stage or, for larger state spaces, through a local search technique such as simulated annealing or genetic algorithms. Figure 2 illustrates how the distribution of entropy for the remaining possible guesses can change as the game progresses; this technique is particularly effective where there is a clear maximum value (e.g. at the third guess).

2.4 Simulated Annealing

TODO

2.5 Genetic Algorithms

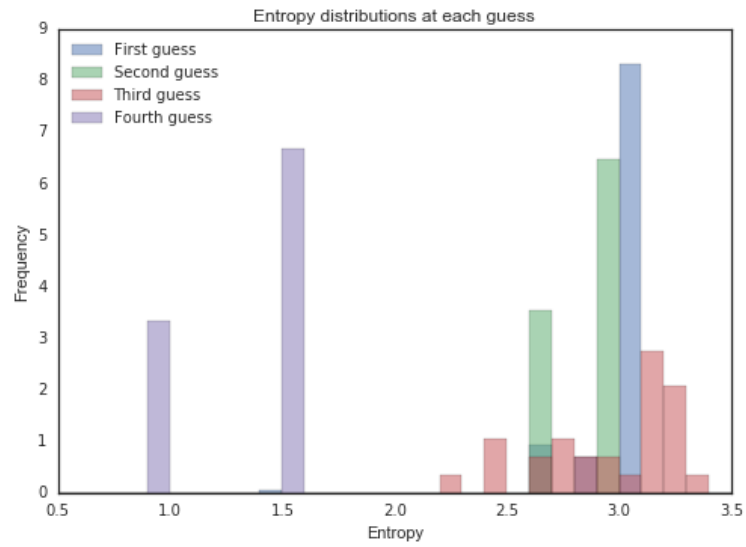TODO

3 Experiments

TODO

4 Conclusion

TODO

A Figures

Figure 2: Entropy distributions for classic game parameters

B   Results

Optimization Method	Number of Guesses		Runtime (Seconds)	
	μ	σ	μ	σ
Knuth's Algorithm	4.0	0.000	0.021	0.002
Random Search/Random Sampling	3.7	0.557	0.002	0.000
Maximizing Entropy (All Steps)	3.8	0.510	0.193	0.010
Maximizing Entropy (Except First Step)	3.7	0.640	0.007	0.004
Simulated Annealing (Bernier Objective Function)	7.5	2.037	0.017	0.026
Simulated Annealing (Entropy Objective Function)	3.5	0.805	0.003	0.001
Genetic Algorithms (XX Objective Function)				

Table 1: 4 Positions; 4 Possible Digits

Optimization Method	Number of Guesses		Runtime (Seconds)	
	μ	σ	μ	σ
Knuth's Algorithm	4.0	0.000	0.264	0.006
Random Search/Random Sampling	4.5	0.742	0.006	0.001
Maximizing Entropy (All Steps)	4.0	0.497	1.147	0.035
Maximizing Entropy (Except First Step)	4.0	0.894	0.042	0.017
Simulated Annealing (Bernier Objective Function)	8.0	1.359	0.042	0.129
Simulated Annealing (Entropy Objective Function)	4.7	0.781	0.007	0.001
Genetic Algorithms (XX Objective Function)				

Table 2: 4 Positions; 5 Possible Digits

Optimization Method	Number of Guesses		Runtime (Seconds)	
	μ	σ	μ	σ
Knuth's Algorithm	4.0	0.000	0.606	0.017
Random Search/Random Sampling	4.8	0.994	0.013	0.003
Maximizing Entropy (All Steps)	4.8	0.600	5.225	0.207
Maximizing Entropy (Except First Step)	4.4	0.663	0.145	0.099
Simulated Annealing (Bernier Objective Function)	8.0	1.844	0.268	0.421
Simulated Annealing (Entropy Objective Function)	5.0	0.949	0.013	0.001
Genetic Algorithms (XX Objective Function)				

Table 3: 4 Positions; 6 Possible Digits

Optimization Method	Number of Guesses		Runtime (Seconds)	
	μ	σ	μ	σ
Knuth's Algorithm	4.0	0.000	0.528	0.014
Random Search/Random Sampling	5.0	1.023	0.023	0.002
Maximizing Entropy (All Steps)	4.8	0.812	18.600	0.883
Maximizing Entropy (Except First Step)	4.8	0.766	0.669	0.407
Simulated Annealing (Bernier Objective Function)	9.2	1.077	0.069	0.224
Simulated Annealing (Entropy Objective Function)	5.3	0.781	0.028	0.005
Genetic Algorithms (XX Objective Function)				

Table 4: 4 Positions; 7 Possible Digits

Optimization Method	Number of Guesses		Runtime (Seconds)	
	μ	σ	μ	σ
Knuth's Algorithm	6.0	0.000	4.663	0.269
Random Search/Random Sampling	5.4	1.114	0.044	0.005
Maximizing Entropy (All Steps)	5.1	0.943	53.689	1.9444
Maximizing Entropy (Except First Step)	5.4	0.735	1.464	1.162
Simulated Annealing (Bernier Objective Function)	9.1	1.578	0.290	0.478
Simulated Annealing (Entropy Objective Function)	5.5	1.023	0.043	0.005
Genetic Algorithms (XX Objective Function)				

Table 5: 4 Positions; 8 Possible Digits

Optimization Method	Number of Guesses		Runtime (Seconds)	
	μ	σ	μ	σ
Knuth's Algorithm	7.0	0.000	67.299	1.928
Random Search/Random Sampling	5.9	0.995	0.069	0.008
Maximizing Entropy (All Steps)	5.8	1.043	149.057	11.014
Maximizing Entropy (Except First Step)	5.7	0.714	5.702	4.282
Simulated Annealing (Bernier Objective Function)	8.8	1.824	0.191	0.331
Simulated Annealing (Entropy Objective Function)	6.2	1.288	0.091	0.021
Genetic Algorithms (XX Objective Function)				

Table 6: 4 Positions; 9 Possible Digits

Optimization Method	Number of Guesses		Runtime (Seconds)	
	μ	σ	μ	σ
Knuth's Algorithm	6.0	0.000	84.160	9.606
Random Search/Random Sampling	6.6	1.020	0.098	0.006
Maximizing Entropy (All Steps)	5.8	1.062	344.603	21.921
Maximizing Entropy (Except First Step)	6.0	0.775	13.557	9.339
Simulated Annealing (Bernier Objective Function)	9.4	0.970	0.248	0.513
Simulated Annealing (Entropy Objective Function)	6.0	1.304	0.127	0.009
Genetic Algorithms (XX Objective Function)				

Table 7: 4 Positions; 10 Possible Digits

References

[1] José Luis Bernier, C Ilia Herráiz, JJ Merelo, S Olmeda, and Alberto Prieto. Solving mastermind using gas and simulated annealing: a case of dynamic constraint optimization. In *Parallel Problem Solving from Nature?PPSN IV*, pages 553–563. Springer, 1996.

[2] Benjamin Doerr, Reto Spöhel, Henning Thomas, and Carola Winzen. Playing mastermind with many colors. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 695–704. SIAM, 2013.

[3] D.E. Knuth. The computer as a master mind. *Journal of Recreational Mathematics*, 1976.

[4] Barteld P Kooi. Yet another mastermind strategy. *ICGA Journal*, 28(1):13–20, 2005.

[5] Juan J Merelo-Guervós and Thomas Philip Runarsson. Finding better solutions to the mastermind puzzle using evolutionary algorithms. In *Applications of Evolutionary Computation*, pages 121–130. Springer, 2010.

[6] Juan Julián Merelo-Guervós, Pedro Castillo, Antonio M Mora García, and Anna I Esparcia-Alcázar. Improving evolutionary solutions to the game of mastermind using an entropy-based scoring method. In *Proceedings of the 15th annual conference on Genetic and evolutionary computation*, pages 829–836. ACM, 2013.

[7] E Neuwirth. Some strategies for mastermind. *Zeitschrift für Operations Research*, 26(1):B257–B278, 1982.

[8] Thomas Philip Runarsson and Juan J Merelo-Guervós. Adapting heuristic mastermind strategies to evolutionary algorithms. In *Nature Inspired Cooperative Strategies for Optimization (NICSO 2010)*, pages 255–267. Springer, 2010.

[9] Jirı Vomlel. Bayesian networks in mastermind. In *Proceedings of the 7th Czech-Japan Seminar*, 2004.