AM207 Final Project: Mastermind

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Abstract

Insert abstract text.

1 Introduction

The game Mastermind was invented in 1970 by Mordecai Meirowitz, and is similar to an earlier game called Bulls and Cows. There are many variations of the game[1], but they all follow a broadly consistent format. The game involves two players: a code-maker and a code-breaker. The code-maker selects a sequence of length L (usually 4), where each element in the sequence is chosen with replacement from one of C colors (usually 6). At each turn, the code-breaker guesses a sequence and receives feedback in the form of black pegs and white pegs, where the black pegs denote the number of correct guesses in the right position, and the white pegs denote the number of correct guesses in the wrong position. Based on this information, the code-breaker refines her guesses and attempts to crack the code within the maximum number of guesses (usually 10).

For our project, we set out to implement various strategies and algorithms for iteratively optimizing the code-breaker's guess at each turn. We compared the performance of these strategies based on the mean and standard deviation of the required number of guesses to win and the runtime across 20 random game initializations. In order to assess the extent to which the different solutions scale efficiently, we tested seven game setups of varying complexity.

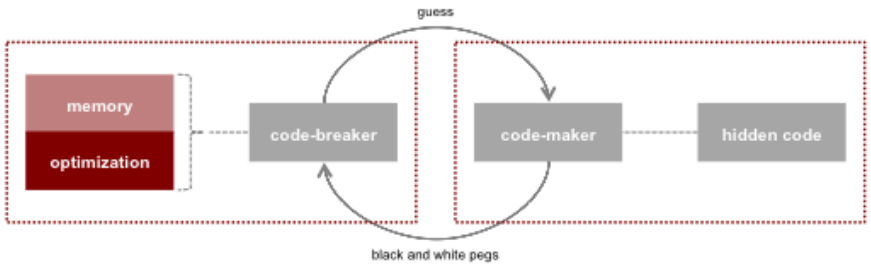


Figure 1: Mastermind Framework

[1] https://en.wikipedia.org/wiki/Mastermind_(board_game)#Variations

2 Optimization Methods

It is helpful at this stage to introduce some mathematical notation to describe the game, which we will use consistently when describing the various methods that we tested. For simplicity, we use one-indexing in the formulas below; however, we note that the Python code for the game interface that we built is zero-indexed.

We define:

- set of possible colors: $C = \{1, ..., C\}$
- set of positions in the code: $L = \{1, ..., L\}$
- hidden code: $H_i \; \forall i \in L$
- guess of the hidden code: $T_i \; \forall i \in L$
- indicator function $\mathbb{1}_{A=B}$, which equals 1 if A=B and equals 0 otherwise

Using the above notation, we can denote the responses at each turn as follows:

- correct guesses in the right position: $B = \sum_{i=1}^{L} \mathbb{1}_{T_i=H_i} \; \forall i \in L$
- correct guesses in the wrong position: $W = \sum_{i=1}^{C} \min(\sum_{j=1}^{L} \mathbb{1}_{H_j=i,G_i}, \sum_{j=1}^{L} \mathbb{1}_{T_j=i,G_i}) - B$

2.1 Knuth's Five-Guess Algorithm

The most commonly referenced optimization technique in the Mastermind literature is Knuth's five-guess algorithm [4] – sometimes also referred to as the "worst-case strategy" – which can always solve the classic game configuration in five moves or less. This strategy always begins with the same initial guess of 1122 (or 0011, when zero-indexed); this choice is motivated by examples of other initial guesses that do not always lead to a solution in five moves. Our implementation uses this deterministic initial guess for the classic game configuration, but uses a random initial guess for all other configurations as analogous "best" starting points are not defined in the literature.

After the initial guess, the algorithm determines the minimum number of codes that each guess would eliminate from the list of possibilities, and chooses one of the guesses that maximizes this number[2]. At each stage of the game, the set of possible codes is updated to only include codes that would have generated the same responses at each of the previous steps. This ensures that the state space shrinks with each subsequent guess.

2.2 Random Search with Constraints / Random Sampling From Posterior

This method is described in the Mastermind literature both as a constrained random search algorithm [2] and in terms of posterior distribution updates [10]. We follow the latter approach below.

We start by defining the joint probability distribution over all possible code sequences as $P(H_1, ..., H_L)$. As we have no information, our prior is uniformly distributed.

[2]Wikipedia refers to this as the minimax *technique*. We note that this one-off maximization differs from the recursive minimax *algorithm* that is also used in game theory.

$$P(H_1 = h_1, ..., H_L = h_l) = \frac{1}{C^L}, \quad \text{for all combinations of } (h_1, ..., h_l)$$

We can denote the evidence that obtain at each step as $e = (B, W)$, where B and W are defined as above, and use this to update the posterior joint distribution over code sequences as follows:

$$P(H_1 = h_1, ..., H_L = h_l | e) = \begin{cases} \frac{1}{|s(e)|}, & \text{if } (h_1, ..., h_l) \text{ is a possible code} \\ 0, & \text{otherwise} \end{cases}$$

where s(e) denotes the set of possible hidden codes, given the evidence, and $|s(e)|$ denotes the cardinality of this set.

We can define the posterior after multiple game steps analogously:

$$P(H_1 = h_1, ..., H_L = h_l | e_1, ..., e_n) = \begin{cases} \frac{1}{|s(e_1) \cap ... \cap s(e_n)|}, & \text{if } (h_1, ..., h_l) \text{ is a possible code} \\ 0, & \text{otherwise} \end{cases}$$

where $s(e_1) \cap ... \cap s(e_n)$ denotes the intersection of the sets of possible hidden codes, given the evidence at each step, and the entire denominator denotes the cardinality of this intersection.

We can use this framework to define the posterior updates at each round of the game, and then choose the next guess by sampling from the posterior distribution. Figure 2 shows how the number of possible guesses shrinks as the game progresses.

2.3 Maximizing Shannon Entropy

Entropy is a measure that is commonly used in information theory to quantify the average amount of information that is contained in a message. In the context of Mastermind, the "message" is the response of black and white pins that is returned by the code-maker at each turn. The goal of the code-breaker is to choose guesses that create as even a distribution as possible between the various responses[3], as it will allow her to discard more possible codes at the next step.

Let us denote r_i as the ith response category and R as the number of possible responses[4]. We can then define the entropy of the discrete response space $\{r_1, ..., r_R\}$ for a given guess as:

$$H(\text{guess}|\text{possible codes}) = \sum_{i=1}^{R} P(r_i) I(r_i) = -\sum_{i=1}^{R} P(r_i) \log_b P(r_i)$$

where $I(r_i)$ denotes the information content of the ith response category. We use $b = 2$, meaning that we are measuring entropy in shannons, but note that any other value of b would yield a consistent ranking between guesses.

Practically, we calculate the probability of each response category for a given guess by counting (and normalizing) the total number of possible responses in each category, given the hidden codes that are still possible at that particular stage in the game. The value will depend on the shape of

[3] http://www.geometer.org/mathcircles/mastermind.pdf

[4] For example, the classic version of the game with codes of length 4 has 14 possible responses: $(4,0), (3,0), (2,2), (2,1), (2,0), (1,3), (1,2), (1,1), (1,0), (0,4), (0,3), (0,2), (0,1), (0,0)$.

the probability distribution across the response categories, with the minimum entropy of 0 only achievable when there is certainty of a particular outcome (i.e. $log_2(1) = 0$).

In order to improve her performance, the code-breaker will want to pick the guess that results in the highest entropy – or, if there are ties, one of the best guesses – in order to be able to discard more possible codes at the next step. This can be achieved through an exhaustive calculation of the entropy of all possible guesses at each stage or, for larger state spaces, through a local search technique such as simulated annealing or genetic algorithms. Figure 3 illustrates how the distribution of entropy for the remaining possible guesses can change as the game progresses; this technique is particularly effective where there is a clear maximum value (e.g. at the third guess).

2.4 Simulated Annealing

Knuth's algorithm and maximizing shannon entropy are both examples of global optimization techniques. These approaches work well when optimizing across relatively small state spaces, but are not guaranteed to scale well as the problem size increases. For example, maximizing shannon entropy has complexity $O(|s(e_i^2)|)$ at the ith round of the game, i.e. the runtime scales quadratically with the number of possible codes. More generally, we can use the starting number of possible codes (C^L) as an indicator of the complexity of the puzzle – when this number becomes very large, approaches that allow us to approximate "good" guesses without an exhaustive search through all possible codes become increasingly valuable.

Simulated annealing (SA) is one example of a local optimization method that can be effective in larger state spaces[5]. The effectiveness of this technique depends upon the objective function that is used to evaluate potential guesses, and the method that is used to propose potential new guesses. As mentioned above, one application of simulated annealing is as an alternative to the global maximization of entropy across all possible guesses. In this framework, new guesses are proposed by randomly picking from the set of remaining possible guesses, and are evaluated by calculating their entropy.

An alternative SA approach is described by Bernier et al. [2]. Under this framework, new guesses are generated by introducing *permutations* (swaps of code element pairs) and/or *mutations* (changes to individual code elements) to the last code that was considered. The exact number of permutations and mutations is determined by the SA "temperature'": the code can change significantly when the temperature is high, encouraging exploration of the state of potential next guesses, but will only change slightly when the temperature is lower (by which point we hope to be close to guessing the correct code), encouraging convergence to a "good" solution.

In order to determine whether the proposed code is "better" than the current code, Bernier et al. define an objective function based on the difference between the black and white pegs obtained from all previous guesses and the number of black and white pegs that would have been obtained if the newly proposed guess were the right code. The function is defined as

[5]The classical SA formulation involves the minimization of a cost function. Potential solutions are proposed iteratively, and their cost is compared to that of previously accepted solutions. Solutions with lower cost are automatically accepted, while solutions with greater cost are accepted with a certain probability. This probability decreases as the "temperature" model parameter decreases, and varies inversely with the difference between the cost of the previously accepted solution and the cost of the newly proposed solution. In practice, SA can also be used to maximize objective functions, in addition to strictly minimizing cost functions.

$$\sum_{i=1}^{n} abs(\Delta n_w^i) + abs(\Delta n_w^i + \Delta n_b^i)$$

where n is the number of previous guesses made and Δn_w^i and Δn_b^i are the differences in white and black peg response when comparing the response that would be obtained if the proposed code were correct with the response it actually received when it had actually made the guess.

2.5 Genetic Algorithms

TODO

3 Experiments

TODO

4 Conclusion

TODO

A Figures

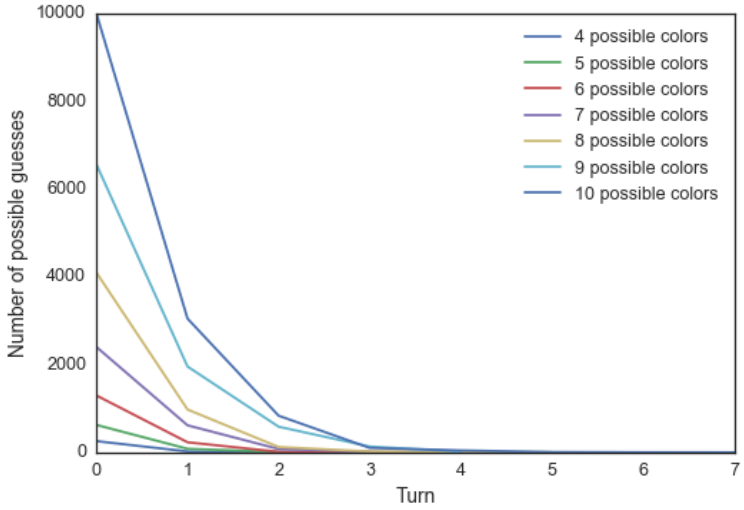


Figure 2: Evolution in number of possible guesses

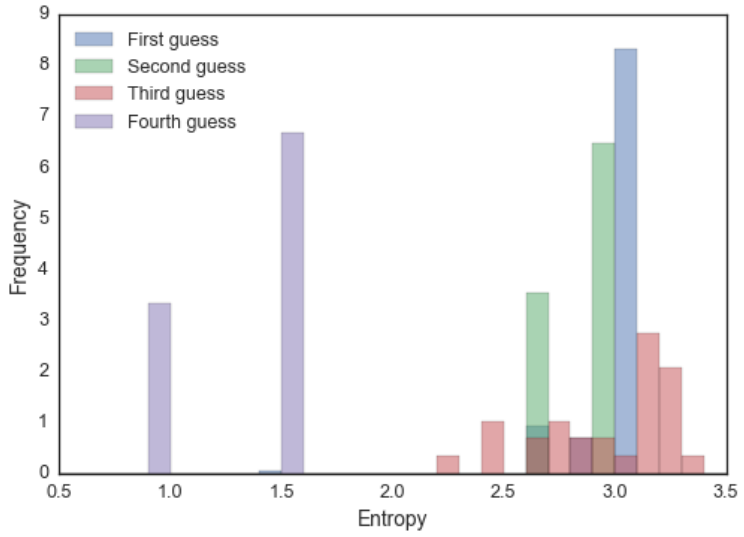
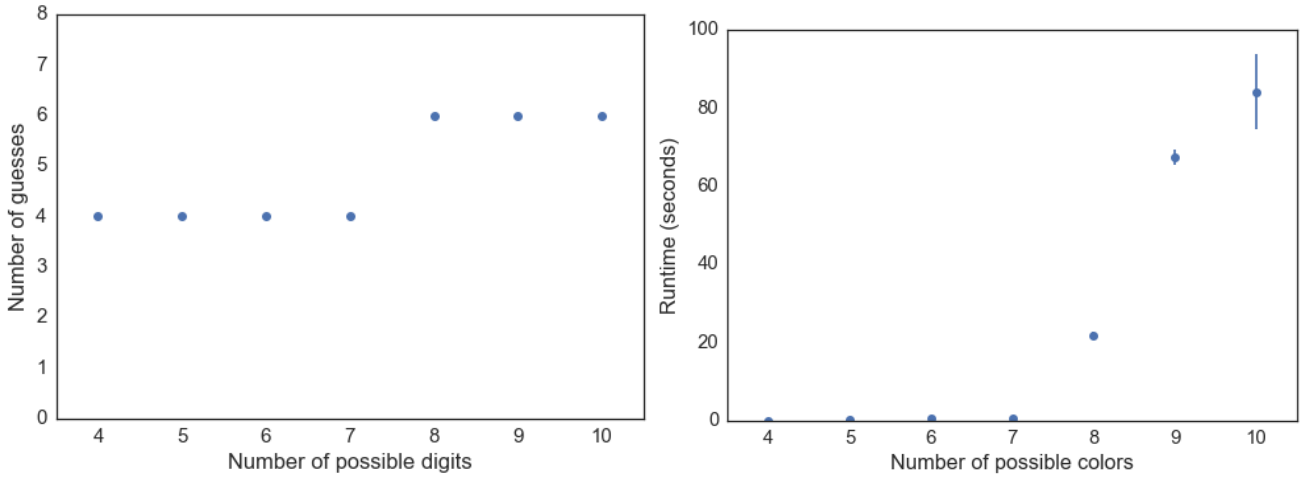


Figure 3: Entropy distributions for classic game parameters

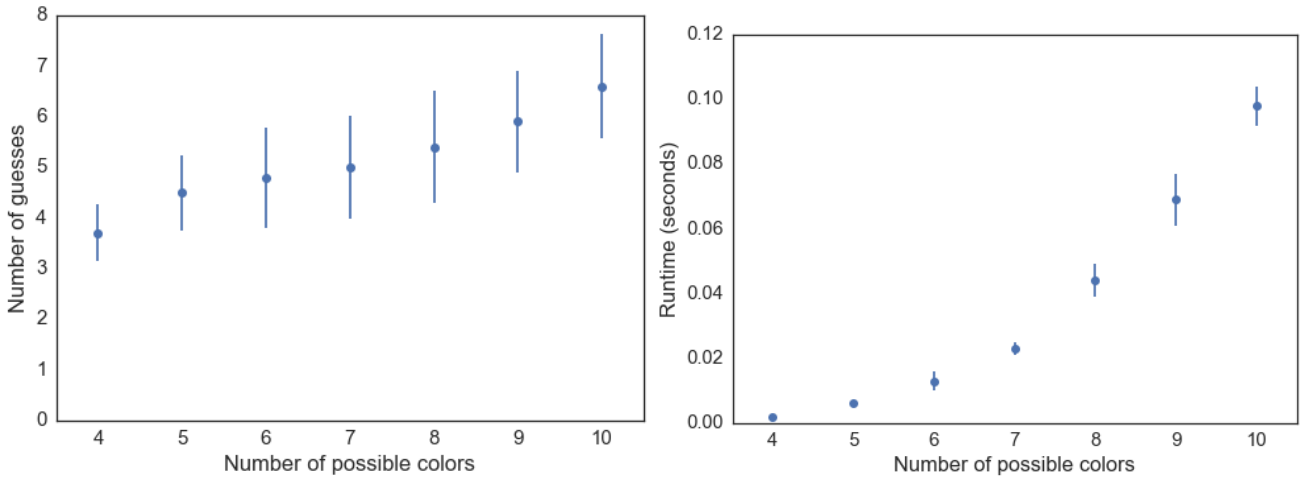
B Results by Optimization Method

Game configuration: 4 positions, varying number of possible colors



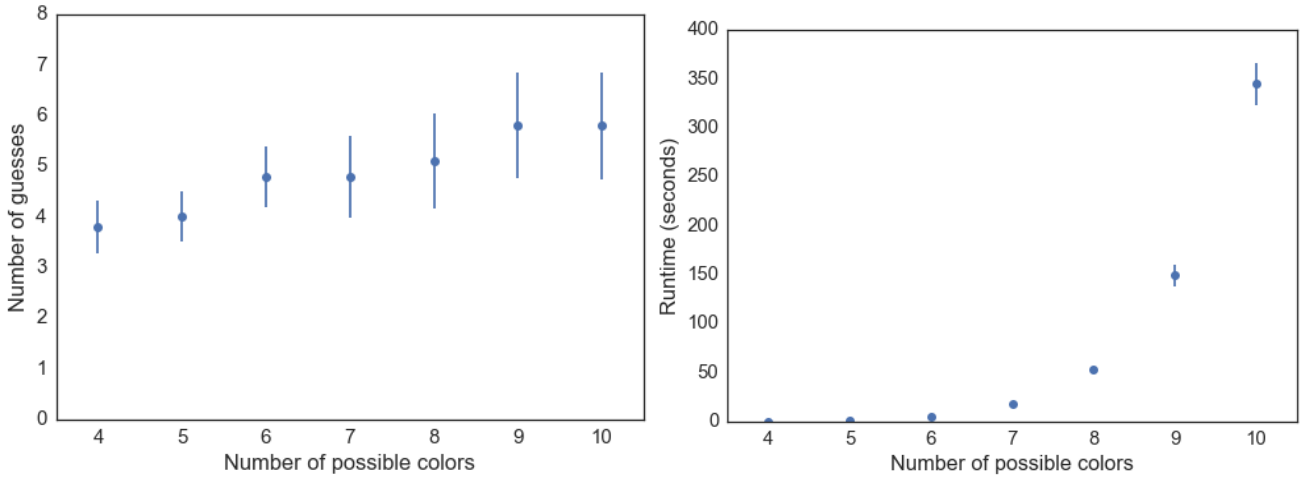
(a) Number of guesses (b) Runtime (seconds)

Figure 4: Knuth's algorithm



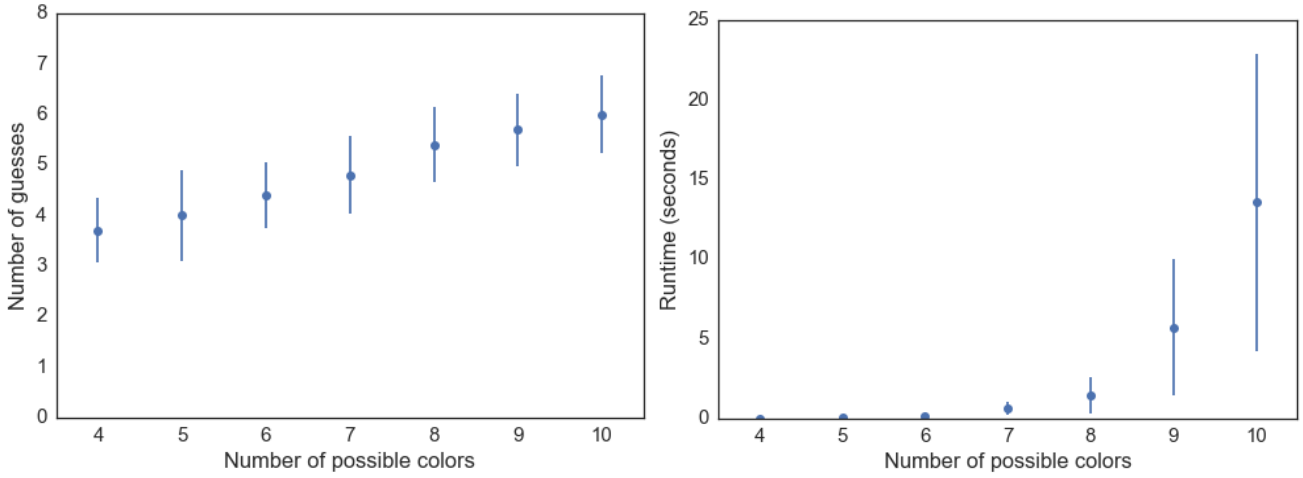
(a) Number of guesses (b) Runtime (seconds)

Figure 5: Random search / Random sampling from posterior



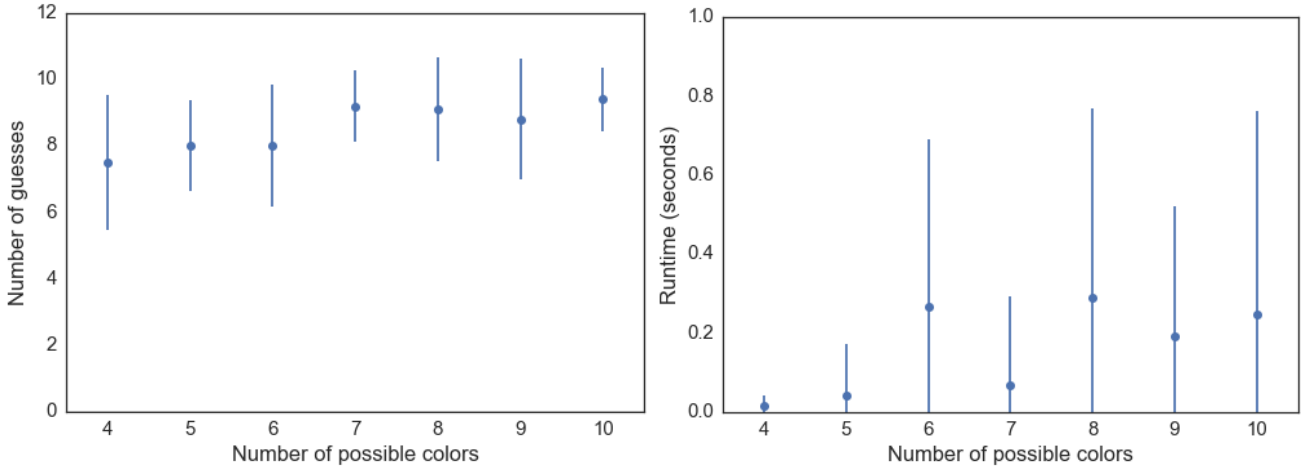
(a) Number of guesses (b) Runtime (seconds)

Figure 6: Maximizing entropy (all steps)



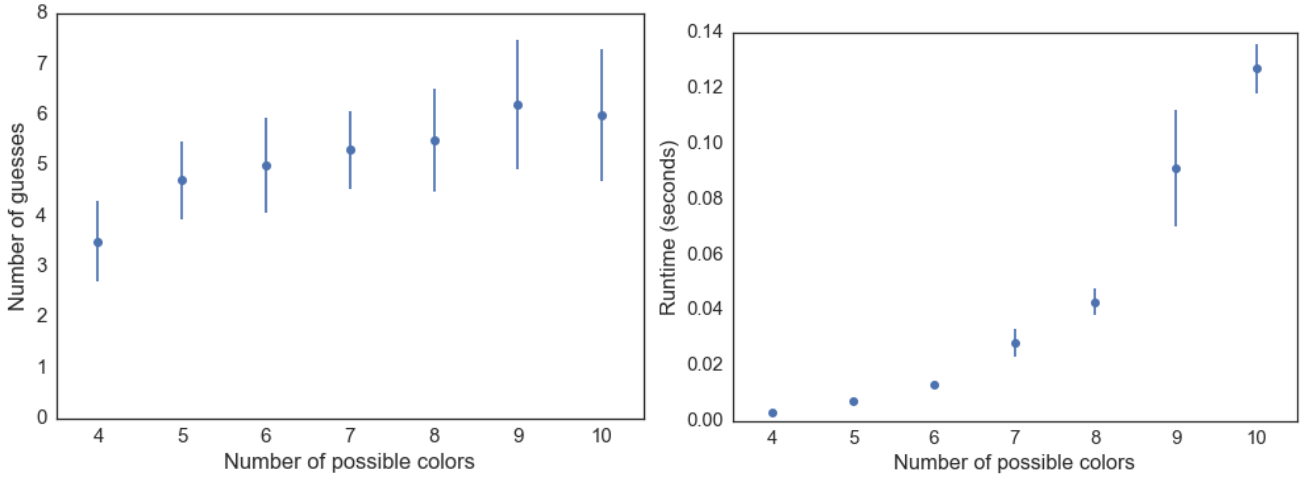
(a) Number of guesses (b) Runtime (seconds)

Figure 7: Maximizing entropy (except first step)



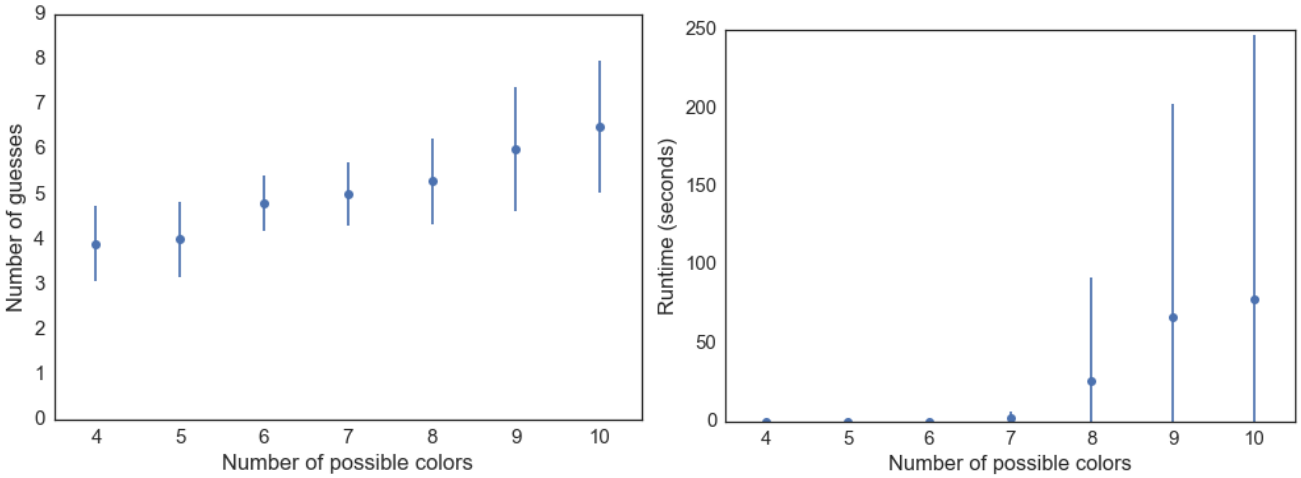
(a) Number of guesses (b) Runtime (seconds)

Figure 8: Simulated annealing (Bernier objective function)



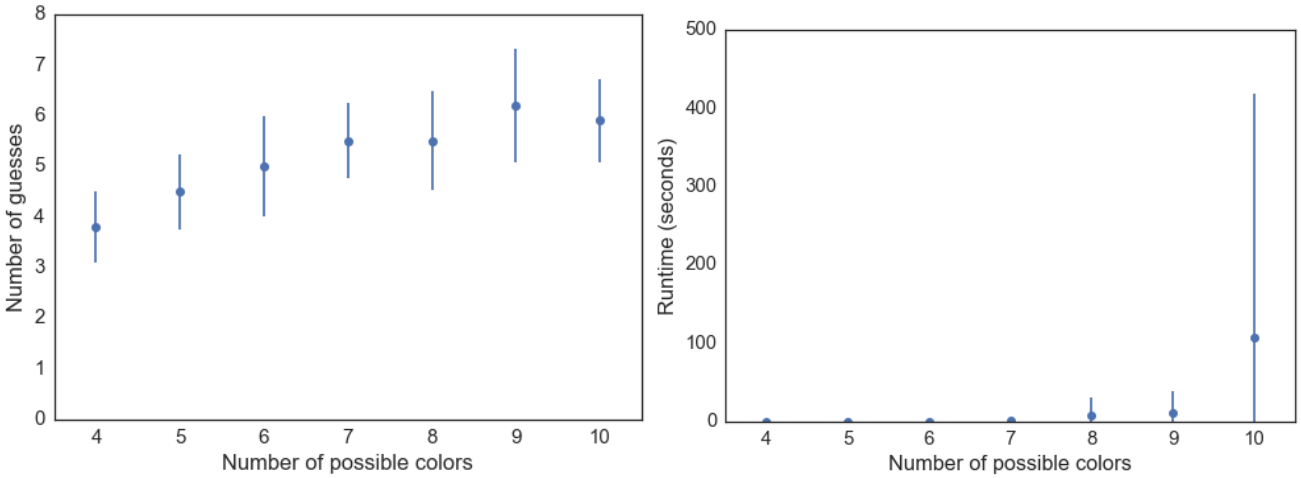
(a) Number of guesses (b) Runtime (seconds)

Figure 9: Simulated annealing (entropy objective function)



(a) Number of guesses (b) Runtime (seconds)

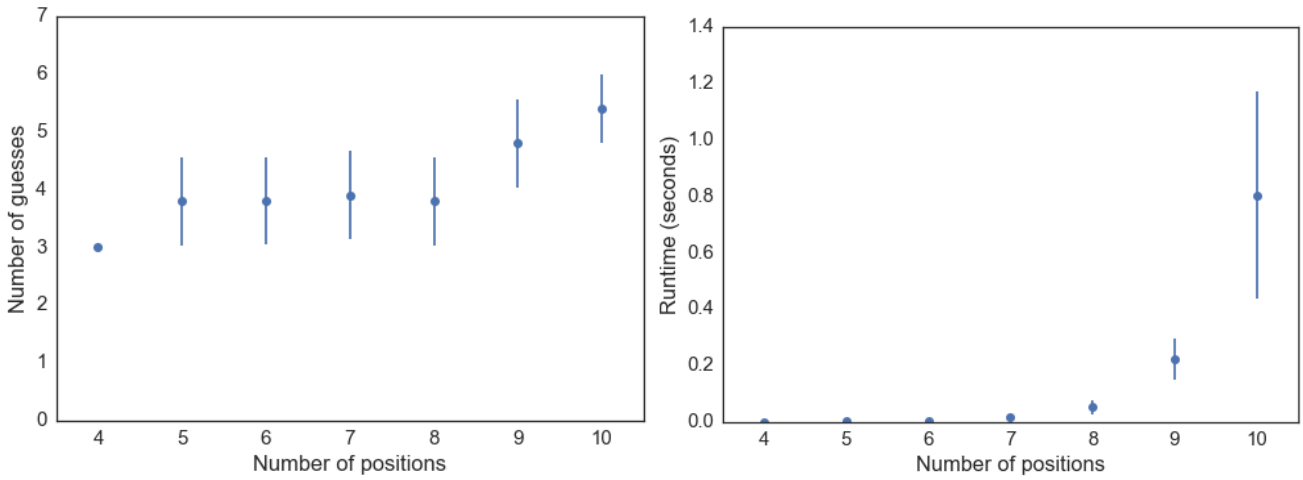
Figure 10: Genetic algorithms (Bernier objective function)



(a) Number of guesses (b) Runtime (seconds)

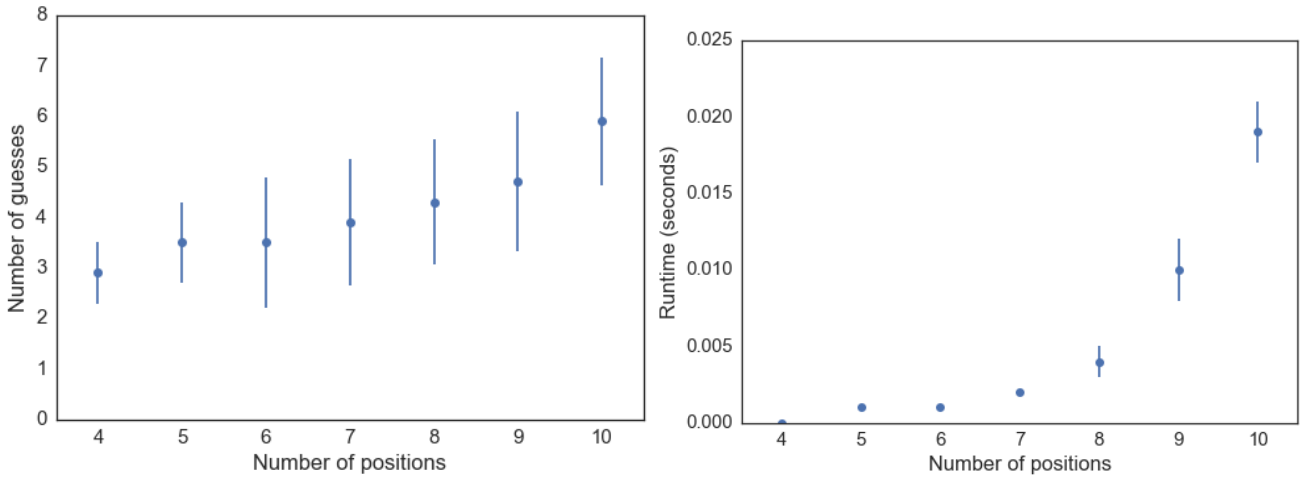
Figure 11: Genetic algorithms (entropy objective function)

Game configuration: 2 possible colors, varying number of positions



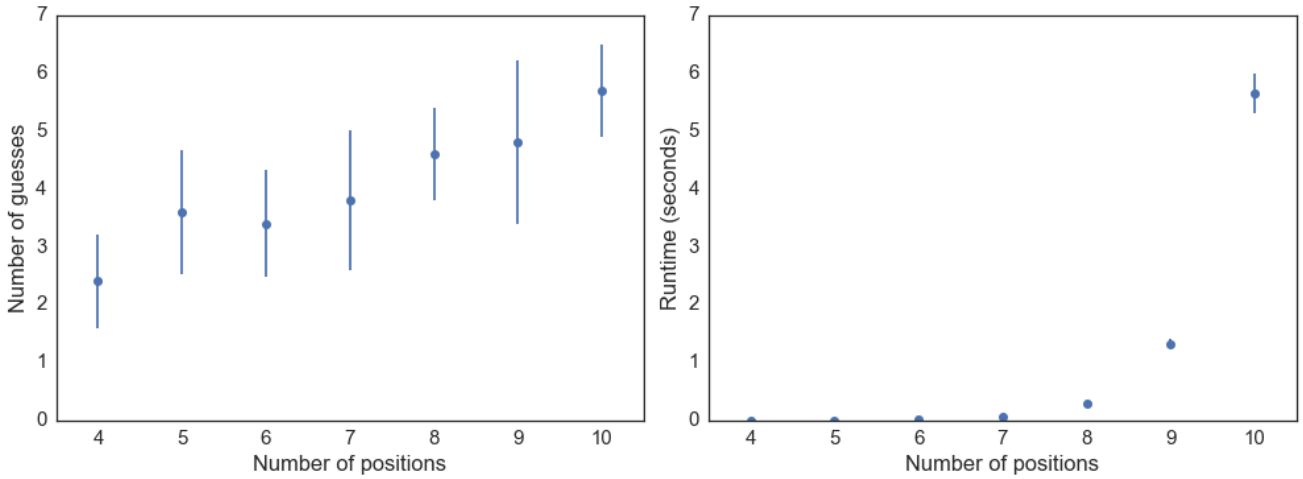
(a) Number of guesses (b) Runtime (seconds)

Figure 12: Knuth's algorithm2



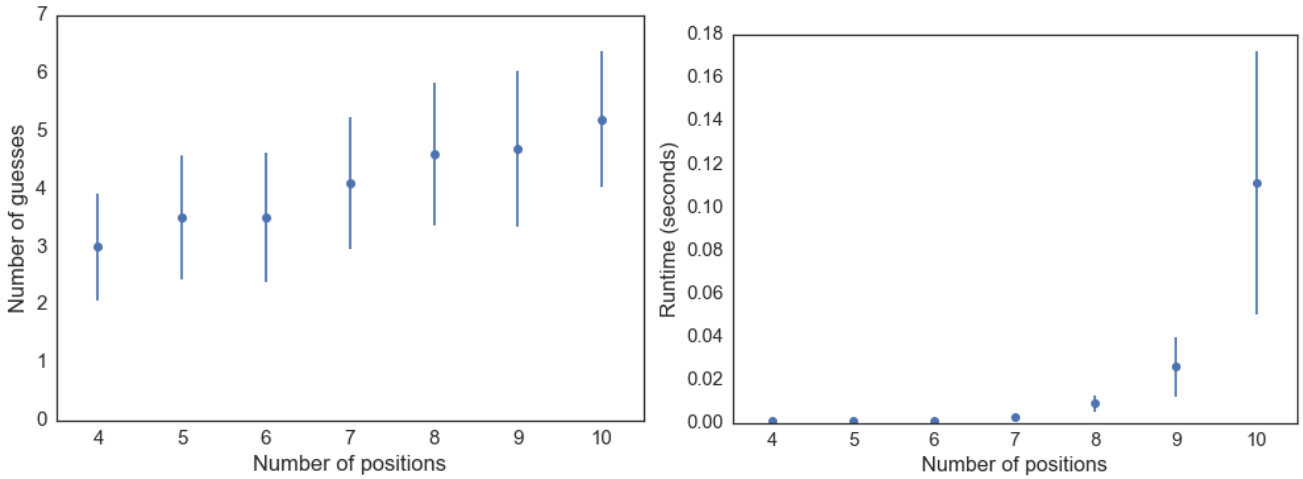
(a) Number of guesses (b) Runtime (seconds)

Figure 13: Random search / Random sampling from posterior



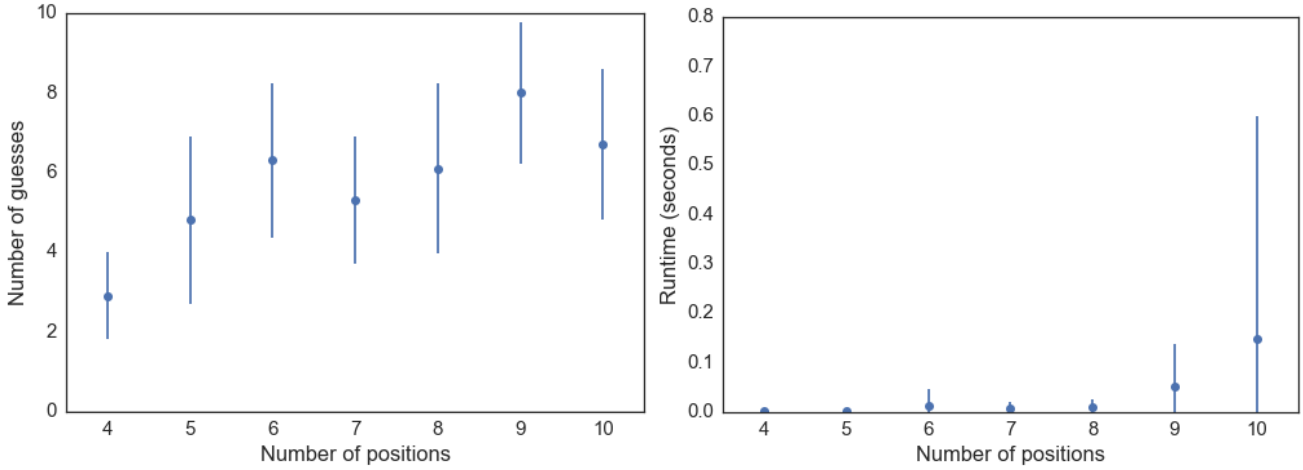
(a) Number of guesses (b) Runtime (seconds)

Figure 14: Maximizing entropy (all steps)



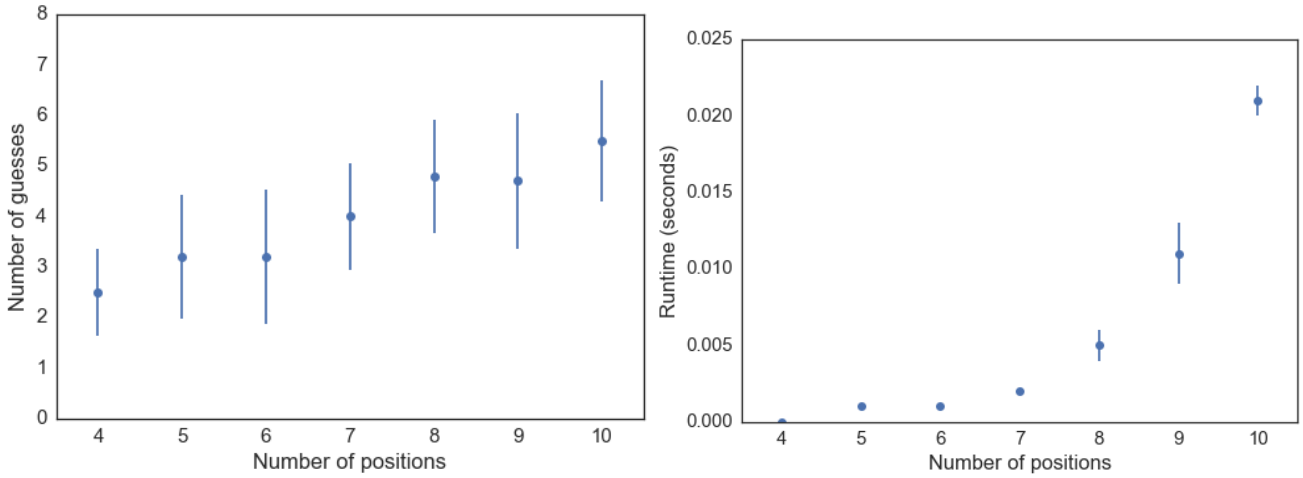
(a) Number of guesses (b) Runtime (seconds)

Figure 15: Maximizing entropy (except first step)



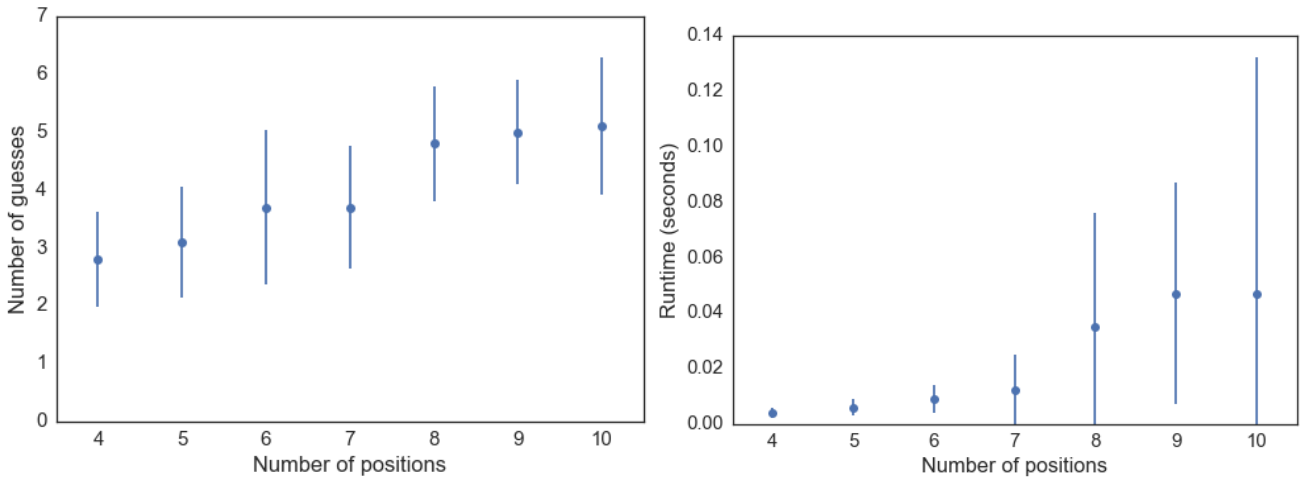
(a) Number of guesses (b) Runtime (seconds)

Figure 16: Simulated annealing (Bernier objective function)



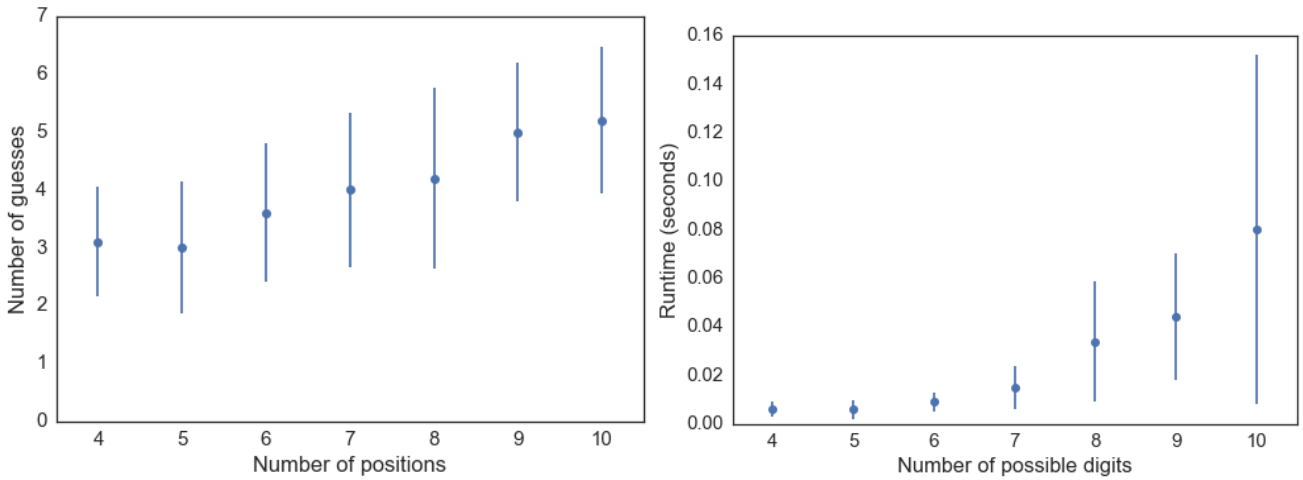
(a) Number of guesses (b) Runtime (seconds)

Figure 17: Simulated annealing (entropy objective function)



(a) Number of guesses (b) Runtime (seconds)

Figure 18: Genetic algorithms (Bernier objective function)



(a) Number of guesses (b) Runtime (seconds)

Figure 19: Genetic algorithms (entropy objective function)

C Results by Game Configuration

Game configuration: 4 positions, varying number of possible colors

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	4.0	0.000	0.021	0.002
Random search / Random sampling	3.7	0.557	0.002	0.000
Maximizing entropy (all steps)	3.8	0.510	0.193	0.010
Maximizing entropy (except first step)	3.7	0.640	0.007	0.004
Simulated annealing (Bernier objective function)	7.5	2.037	0.017	0.026
Simulated annealing (entropy objective function)	3.5	0.805	0.003	0.001
Genetic algorithms (Bernier objective function)	3.9	0.831	0.022	0.026
Genetic algorithms (entropy objective function)	3.8	0.698	0.027	0.032

Table 1: 4 positions; 4 possible colors

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	4.0	0.000	0.264	0.006
Random search / Random sampling	4.5	0.742	0.006	0.001
Maximizing entropy (all steps)	4.0	0.497	1.147	0.035
Maximizing entropy (except first step)	4.0	0.894	0.042	0.017
Simulated annealing (Bernier objective function)	8.0	1.359	0.042	0.129
Simulated annealing (entropy objective function)	4.7	0.781	0.007	0.001
Genetic algorithms (Bernier objective function)	4.0	0.837	0.160	0.289
Genetic algorithms (entropy objective function)	4.5	0.740	0.273	0.442

Table 2: 4 positions; 5 possible colors

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	4.0	0.000	0.606	0.017
Random search / Random sampling	4.8	0.994	0.013	0.003
Maximizing entropy (all steps)	4.8	0.600	5.225	0.207
Maximizing entropy (except first step)	4.4	0.663	0.145	0.099
Simulated annealing (Bernier objective function)	8.0	1.844	0.268	0.421
Simulated annealing (entropy objective function)	5.0	0.949	0.013	0.001
Genetic algorithms (Bernier objective function)	4.8	0.622	0.230	0.472
Genetic algorithms (entropy objective function)	5.0	1.000	0.376	0.660

Table 3: 4 positions; 6 possible colors

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	4.0	0.000	0.528	0.014
Random search / Random sampling	5.0	1.023	0.023	0.002
Maximizing entropy (all steps)	4.8	0.812	18.600	0.883
Maximizing entropy (except first step)	4.8	0.766	0.669	0.407
Simulated annealing (Bernier objective function)	9.2	1.077	0.069	0.224
Simulated annealing (entropy objective function)	5.3	0.781	0.028	0.005
Genetic algorithms (Bernier objective function)	5.0	0.707	2.432	3.576
Genetic algorithms (entropy objective function)	5.5	0.742	1.384	2.043

Table 4: 4 positions; 7 possible colors

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	6.0	0.000	21.711	0.269
Random search / Random sampling	5.4	1.114	0.044	0.005
Maximizing entropy (all steps)	5.1	0.943	53.689	1.944
Maximizing entropy (except first step)	5.4	0.735	1.464	1.162
Simulated annealing (Bernier objective function)	9.1	1.578	0.290	0.478
Simulated annealing (entropy objective function)	5.5	1.023	0.043	0.005
Genetic algorithms (Bernier objective function)	5.3	0.954	25.809	65.751
Genetic algorithms (entropy objective function)	5.5	0.973	7.866	22.396

Table 5: 4 positions; 8 possible colors

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	6.0	0.000	67.299	1.928
Random search / Random sampling	5.9	0.995	0.069	0.008
Maximizing entropy (all steps)	5.8	1.043	149.057	11.014
Maximizing entropy (except first step)	5.7	0.714	5.702	4.282
Simulated annealing (Bernier objective function)	8.8	1.824	0.191	0.331
Simulated annealing (entropy objective function)	6.2	1.288	0.091	0.021
Genetic algorithms (Bernier objective function)	6.0	1.378	66.406	135.881
Genetic algorithms (entropy objective function)	6.2	1.122	10.492	28.552

Table 6: 4 positions; 9 possible colors

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	6.0	0.000	84.160	9.606
Random search / Random sampling	6.6	1.020	0.098	0.006
Maximizing entropy (all steps)	5.8	1.062	344.603	21.921
Maximizing entropy (except first step)	6.0	0.775	13.557	9.339
Simulated annealing (Bernier objective function)	9.4	0.970	0.248	0.513
Simulated annealing (entropy objective function)	6.0	1.304	0.127	0.009
Genetic algorithms (Bernier objective function)	6.5	1.466	77.632	169.038
Genetic algorithms (entropy objective function)	5.9	0.831	107.604	310.685

Table 7: 4 positions; 10 possible colors

Game configuration: 2 possible colors, varying number of positions

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	3.0	0.000	0.001	0.000
Random search / Random sampling	2.9	0.624	0.000	0.000
Maximizing entropy (all steps)	2.4	0.800	0.001	0.000
Maximizing entropy (except first step)	3.0	0.921	0.001	0.000
Simulated annealing (Bernier objective function)	2.9	1.091	0.001	0.001
Simulated annealing (entropy objective function)	2.5	0.865	0.000	0.000
Genetic algorithms (Bernier objective function)	2.8	0.812	0.004	0.002
Genetic algorithms (entropy objective function)	3.1	0.943	0.006	0.003

Table 8: 2 possible colors; 4 positions

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	3.8	0.766	0.002	0.000
Random search / Random sampling	3.5	0.805	0.001	0.000
Maximizing entropy (all steps)	3.6	1.068	0.005	0.001
Maximizing entropy (except first step)	3.5	1.072	0.001	0.000
Simulated annealing (Bernier objective function)	4.8	2.112	0.002	0.002
Simulated annealing (entropy objective function)	3.2	1.220	0.001	0.000
Genetic algorithms (Bernier objective function)	3.1	0.963	0.006	0.003
Genetic algorithms (entropy objective function)	3.0	1.140	0.006	0.004

Table 9: 2 possible colors; 5 positions

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	3.8	0.748	0.004	0.002
Random search / Random sampling	3.5	1.284	0.001	0.000
Maximizing entropy (all steps)	3.4	0.917	0.016	0.001
Maximizing entropy (except first step)	3.5	1.118	0.001	0.000
Simulated annealing (Bernier objective function)	6.3	1.931	0.013	0.033
Simulated annealing (entropy objective function)	3.2	1.327	0.001	0.000
Genetic algorithms (Bernier objective function)	3.7	1.345	0.009	0.005
Genetic algorithms (entropy objective function)	3.6	1.200	0.009	0.004

Table 10: 2 possible colors; 6 positions

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	3.9	0.768	0.015	0.004
Random search / Random sampling	3.9	1.261	0.002	0.000
Maximizing entropy (all steps)	3.8	1.208	0.065	0.004
Maximizing entropy (except first step)	4.1	1.136	0.003	0.001
Simulated annealing (Bernier objective function)	5.3	1.590	0.008	0.012
Simulated annealing (entropy objective function)	4.0	1.049	0.002	0.000
Genetic algorithms (Bernier objective function)	3.7	1.054	0.012	0.013
Genetic algorithms (entropy objective function)	4.0	1.342	0.015	0.009

Table 11: 2 possible colors; 7 positions

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	3.8	0.766	0.052	0.024
Random search / Random sampling	4.3	1.236	0.004	0.001
Maximizing entropy (all steps)	4.6	0.800	0.289	0.014
Maximizing entropy (except first step)	4.6	1.241	0.009	0.004
Simulated annealing (Bernier objective function)	6.1	2.142	0.010	0.015
Simulated annealing (entropy objective function)	4.8	1.122	0.005	0.001
Genetic algorithms (Bernier objective function)	4.8	0.994	0.035	0.041
Genetic algorithms (entropy objective function)	4.2	1.568	0.034	0.025

Table 12: 2 possible colors; 8 positions

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	4.8	0.766	0.221	0.074
Random search / Random sampling	4.7	1.382	0.010	0.002
Maximizing entropy (all steps)	4.8	1.410	1.323	0.075
Maximizing entropy (except first step)	4.7	1.345	0.026	0.014
Simulated annealing (Bernier objective function)	8.0	1.774	0.052	0.086
Simulated annealing (entropy objective function)	4.7	1.345	0.011	0.002
Genetic algorithms (Bernier objective function)	5.0	0.894	0.047	0.040
Genetic algorithms (entropy objective function)	5.0	1.203	0.044	0.026

Table 13: 2 possible colors; 9 positions

Optimization method	Number of guesses		Runtime (seconds)	
	μ	σ	μ	σ
Knuth's algorithm	5.4	0.583	0.803	0.367
Random search / Random sampling	5.9	1.261	0.019	0.002
Maximizing entropy (all steps)	5.7	0.792	5.654	0.339
Maximizing entropy (except first step)	5.2	1.178	0.111	0.061
Simulated annealing (Bernier objective function)	6.7	1.878	0.147	0.453
Simulated annealing (entropy objective function)	5.5	1.203	0.021	0.001
Genetic algorithms (Bernier objective function)	5.1	1.179	0.047	0.085
Genetic algorithms (entropy objective function)	5.2	1.260	0.080	0.072

Table 14: 2 possible colors; 10 positions

D Additional Methods Tested

TODO

References

[1] Lotte Berghman, Dries Goossens, and Roel Leus. Efficient solutions for mastermind using genetic algorithms. *Computers & operations research*, 36(6):1880–1885, 2009.

[2] José Luis Bernier, C Ilia Herráiz, JJ Merelo, S Olmeda, and Alberto Prieto. Solving mastermind using gas and simulated annealing: a case of dynamic constraint optimization. In *Parallel Problem Solving from Nature?PPSN IV*, pages 553–563. Springer, 1996.

[3] Benjamin Doerr, Reto Spöhel, Henning Thomas, and Carola Winzen. Playing mastermind with many colors. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 695–704. SIAM, 2013.

[4] D.E. Knuth. The computer as a master mind. *Journal of Recreational Mathematics*, 1976.

[5] Barteld P Kooi. Yet another mastermind strategy. *ICGA Journal*, 28(1):13–20, 2005.

[6] Juan J Merelo-Guervós and Thomas Philip Runarsson. Finding better solutions to the mastermind puzzle using evolutionary algorithms. In *Applications of Evolutionary Computation*, pages 121–130. Springer, 2010.

[7] Juan Julián Merelo-Guervós, Pedro Castillo, Antonio M Mora García, and Anna I Esparcia-Alcázar. Improving evolutionary solutions to the game of mastermind using an entropy-based scoring method. In *Proceedings of the 15th annual conference on Genetic and evolutionary computation*, pages 829–836. ACM, 2013.

[8] E Neuwirth. Some strategies for mastermind. *Zeitschrift für Operations Research*, 26(1):B257–B278, 1982.

[9] Thomas Philip Runarsson and Juan J Merelo-Guervós. Adapting heuristic mastermind strategies to evolutionary algorithms. In *Nature Inspired Cooperative Strategies for Optimization (NICSO 2010)*, pages 255–267. Springer, 2010.

[10] Jırı Vomlel. Bayesian networks in mastermind. In *Proceedings of the 7th Czech-Japan Seminar*, 2004.