Homework No. 5 (Due on 02/27/14)

Problem 1. Let X and Y be discrete random variables with mean 0 and variance 1. Use the Cauchy-Schwartz inequality and the identity

$$\max\{a,b\} = \frac{1}{2}(a+b) + \frac{1}{2}|a-b|$$

to show that $\mathbb{E}\left[\max\{X^2, Y^2\}\right] \le 1 + \sqrt{1-\rho^2}$, where ρ is the correlation coefficient of X and Y.

Problem 2. A die is rolled 4 times. Let X and Y be the number of appearances of 1 and 6 respectively. Find the joint probability mass function of (X, Y).

Problem 3. Let X and Y be independent Bernoulli random variables with parameter $\frac{1}{2}$. Show that $X+Y$ and $|X-Y|$ are dependent though uncorrelated.

Problem 4. Let X_1, X_2 and X_3, be independent dr.v.s having a geometric distribution with parameters p_1, p_2 and p_3, respectively. Show that

$$\mathbb{P}(X_1 < X_2 < X_3) = \frac{(1-p_1)(1-p_2)p_2p_3^2}{(1-p_2p_3)(1-p_1p_2p_3)}.$$

(hint: use the joint probability mass function to compute the probability via an appropriate summation).

Problem 5. Let X and Y be independent discrete random variables satisfying

$$\mathbb{P}(X=k) = \mathbb{P}(Y=k) = pq^k, \quad k = 0, 1, 2, \ldots,$$

where $0 < p = 1-q < 1$. Calculate $\mathbb{P}(X = k \mid X+Y = n)$.

Problem 6. X and Y are discrete random variables, each taking only two distinct values. Suppose X and Y are uncorrelated. Are they independent?

Problem 7. You roll a fair six-sided die, and then you flip a fair coin the number of times shown by the die. Find the expected value and the variance of the number of heads obtained. Repeat part (a) for the case where you roll two dice, instead of one.