2c

2. Show that each of the following initial-value problems has a unique solution and find the solution. Can Theorem 5.4 be applied in each case?

a. $y' = e^{t-y},\ 0 \le t \le 1,\ y(0) = 1.$

b. $y' = t^{-2}(\sin 2t - 2ty),\ 1 \le t \le 2,\ y(1) = 2.$

c. $y' = -y + ty^{1/2},\ 2 \le t \le 3,\ y(2) = 2.$

d. $y' = \dfrac{ty+y}{ty+t},\ 2 \le t \le 4,\ dy(2) = 4.$

با استفاده از تغییر متغیر داریم:

$$y(t) = \left(z(t)\right)^2$$

$$y'(t) = 2z(t)z'(t)$$

$$\rightarrow 2z(t)z'(t) = -z^2(t) + tz(t), 2 \le t \le 3, z(2) = \sqrt{2}$$

$z(t) = 0$ نمی تواند جواب باشد چون در شرایط اولیه صدق نمی کند پس عبارت را بر $z(t)$ تقسیم می کنیم:

$$\rightarrow z'(t) = -\frac{z(t)}{2} + \frac{t}{2}$$

حال بررسی می کنیم که آیا تابع لیبشیتز هست یا خیر.

$$z'(t) = -\frac{z(t)}{2} + \frac{t}{2} = f(t,z) \rightarrow \left|\frac{\partial f}{\partial z}\right| = \frac{1}{2}$$

تابع ما لیبشیتز است پس معادله جواب یکتا دارد.

حال برای محاسبه جواب داریم:

$$z'(t) + \frac{z(t)}{2} = \frac{t}{2}$$

$$e^{\frac{t}{2}}\left(z'(t) + \frac{z(t)}{2}\right) = e^{\frac{t}{2}}\frac{t}{2} \rightarrow \left(e^{\frac{t}{2}}z(t)\right)' = e^{\frac{t}{2}}\frac{t}{2}$$

$$e^{\frac{t}{2}}z(t) = \int e^{\frac{t_0}{2}}\frac{t_0}{2}\,dt_0 = (t-2)e^{\frac{t}{2}} + c \rightarrow z(2) = \sqrt{2} \rightarrow c = e\sqrt{2}$$

$$\rightarrow z(t) = t - 2 + e^{1-\frac{t}{2}}\sqrt{2} \rightarrow y(t) = (t-2)^2 + 2(t-2)e^{1-\frac{t}{2}}\sqrt{2} + 2e^{2-t}$$

3d

3. For each choice of $f(t, y)$ given in parts (a)–(d):

i. Does f satisfy a Lipschitz condition on $D = \{ (t, y) \mid 0 \le t \le 1, -\infty < y < \infty \}$?

ii. Can Theorem 5.6 be used to show that the initial-value problem

$$y' = f(t, y), \quad 0 \le t \le 1, \quad y(0) = 1,$$

is well posed?

a. $f(t, y) = t^2 y + 1$ **b.** $f(t, y) = ty$ **c.** $f(t, y) = 1 - y$ **d.** $f(t, y) = -ty + \dfrac{4t}{y}$

Theorem 5.6 Suppose $D = \{ (t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty \}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

is well posed. ■

الف

$$\exists L : |f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$$

$$\rightarrow \left|\frac{4t}{y_1} - ty_1 + ty_2 - \frac{4t}{y_2}\right| \le |t||y_1 - y_2| \left|1 + \frac{4}{y_1 y_2}\right| \le L|y_1 - y_2|$$

نشان می دهیم که چنین L پیدا نمی شود:

$$|t| \left|1 + \frac{4}{y_1 y_2}\right| \le L$$

اگر y_1 , y_2 را به صفر میل بدهیم عبارت سمت چپ به بی نهایت میل کرده و از هر L بیشتر خواهد شد. در نتیجه چنین چیزی وجود ندارد و تابع ما لیپشیتز نیست.

ب

چون تابع لیبشیتز نیست از قضیه 5.6 نمی توان استفاده کرد.

6c

6. Suppose the perturbation $\delta(t)$ is proportional to t, that is, $\delta(t) = \delta t$ for some constant δ. Show directly that the following initial-value problems are well posed.

a. $y' = 1 - y, 0 \le t \le 2, y(0) = 0$

b. $y' = t + y, 0 \le t \le 2, y(0) = -1$

c. $y' = \frac{2}{t}y + t^2e^t, 1 \le t \le 2, y(1) = 0$

d. $y' = -\frac{2}{t}y + t^2e^t, 1 \le t \le 2, y(1) = \sqrt{2}e$

$f(t, y) = \frac{2}{t}y + t^2e^t \rightarrow \left|\frac{\partial f}{\partial t}\right| = \left|\frac{2}{t}\right| \le 2$

حال طبق قضیه 5.6 می توان گفت که این مسئله خوش وضع است.

10

10. *Picard's method* for solving the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is described as follows: Let $y_0(t) = \alpha$ for each t in $[a, b]$. Define a sequence $\{y_k(t)\}$ of functions by

$$y_k(t) = \alpha + \int_a^t f(\tau, y_{k-1}(\tau))\, d\tau, \quad k = 1, 2, \ldots .$$

a. Integrate $y' = f(t, y(t))$ and use the initial condition to derive Picard's method.

b. Generate $y_0(t)$, $y_1(t)$, $y_2(t)$, and $y_3(t)$ for the initial-value problem

$$y' = -y + t + 1, \quad 0 \le t \le 1, \quad y(0) = 1.$$

c. Compare the result in part (b) to the Maclaurin series of the actual solution $y(t) = t + e^{-t}$.

a)

$$\int_a^t y'(t_p)\ dt_p = y(t) - y(a) = y(t) - \alpha = \int_a^t f(t_p, y)\, dt_p$$

$$\rightarrow y(t) = \alpha + \int_a^t f(t_p, y)\, dt_p$$

b)

$$y_o(t) = 1$$

$$y_1(t) = 1 + \int_0^t f(\tau, y_0(\tau))\ d\tau = 1 + \int_0^t f(\tau, 1)\ d\tau = 1 + \int_0^t \tau\, d\tau = 1 + \frac{t^2}{2}$$

$$y_2(t) = 1 + \int_0^t f(\tau, y_1(\tau))\ d\tau = 1 + \int_0^t f\left(\tau, 1 + \frac{\tau^2}{2}\right)\ d\tau = 1 + \int_0^t \tau - \frac{\tau^2}{2}\, d\tau = 1 + \frac{t^2}{2} - \frac{t^3}{6}$$

$$y_3(t) = 1 + \int_0^t f(\tau, y_2(\tau))\ d\tau = 1 + \int_0^t f\left(\tau, 1 + \frac{t^2}{2} - \frac{t^3}{6}\right)\ d\tau = 1 + \int_0^t \tau - \frac{\tau^2}{2} + \frac{\tau^3}{6}\, d\tau$$

$$= 1 + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24}$$

c)

$$t + e^{-t} = t + 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \cdots$$

در نتیجه $y_3(t)$ ۴ جمله اول بسط مکلورن را به ما می دهد.

5.2.9

9. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

with exact solution $y(t) = t^2(e^t - e)$:

a. Use Euler's method with $h = 0.1$ to approximate the solution and compare it with the actual values of y.

b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y and compare them to the actual values.

i. $y(1.04)$ **ii.** $y(1.55)$ **iii.** $y(1.97)$

c. Compute the value of h necessary for $|y(t_i) - w_i| \le 0.1$, using Eq. (5.10).

Theorem 5.9 Suppose f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{ (t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty \}$$

and that a constant M exists with

$$|y''(t)| \le M, \quad \text{for all } t \in [a, b],$$

where $y(t)$ denotes the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Let $w_0, w_1, \ldots, w_N$ be the approximations generated by Euler's method for some positive integer N. Then, for each $i = 0, 1, 2, \ldots, N$,

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right]. \tag{5.10}$$

بخش الف و ب به شکل کد پیوست شده است.

a)

	x	y_euler	y_exact	error
0	1.0	0.000000	0.000000	0.000000
1	1.1	0.271828	0.345920	0.074092
2	1.2	0.684756	0.866643	0.181887
3	1.3	1.276978	1.607215	0.330237
4	1.4	2.093548	2.620360	0.526812
5	1.5	3.187445	3.967666	0.780221
6	1.6	4.620818	5.720962	1.100144
7	1.7	6.466396	7.963873	1.497477
8	1.8	8.809120	10.793625	1.984505
9	1.9	11.747997	14.323082	2.575085
10	2.0	15.398236	18.683097	3.284861

b)

```
i=np.interp(1.04,[1.0,1.1],table.iloc[0:2,1].values)
ii=np.interp(1.55,[1.5,1.6],table.iloc[5:7,1].values)
iii=np.interp(1.97,[1.9,2],table.iloc[9:,1].values)
print(i,ii,iii)
```

✓ 0.0s

```
0.10873127313836181 3.9041314843692168 14.303163920134224
```

c)

$$f(t,y)=\frac{2}{t}\,y+t^2e^t,\left|\frac{\partial f}{\partial y}\right|=\frac{2}{t}\rightarrow L=\frac{2}{t}$$

$$y'=2t(e^t-e)+t^2e^t\,,$$

$$y''=2(e^t-e)+2te^t+2te^t+t^2e^t=(2+4t+t^2)e^t-2e\rightarrow|y''|<14e^2-2e$$

$$|y_i-w_i|\leq\frac{h(14e^2-2e)}{2\left(\frac{2}{t}\right)}\left(e^{\frac{2}{t}(t-1)}-1\right)=0.1\rightarrow h=\frac{0.1\left(\frac{4}{t}\right)\left(e^{2-\frac{2}{t}}-1\right)}{14e^2-2e}$$

12

12. Consider the initial-value problem

$$y' = -10y, \quad 0 \le t \le 2, \quad y(0) = 1,$$

which has solution $y(t) = e^{-10t}$. What happens when Euler's method is applied to this problem with $h = 0.1$? Does this behavior violate Theorem 5.9?

$$w_{i+1} = w_i + hf(t_i, w_i) = w_i + 0.1(-10w_i) = w_i - w_i = 0$$

اگر برای دو گام اول بررسی کنیم به نتایج زیر می رسیم:

$$y_1 = 0.359, y_{20} = 2.06 \cdot 10^{-9}$$

پس به ازای همه مقادیر صفر است. حال شرایط قضیه 5.9 را بررسی می کنیم:

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left(e^{L(t_i-a} - 1\right)$$

$$h = 0.1$$

$$y''(t) = 100e^{-10t} \to |y''(t)| \le 100 = M$$

$$\left|\frac{\partial f}{\partial y}\right| = 10 = L$$

$$a = 0$$

در گام اول و آخر قضیه را بررسی می کنیم:

$$|0.369 - 0| \le \frac{10}{20}\left(e^{10(0.1-0)} - 1\right) - 0.859$$

$$|2.06 \cdot 10^{-9} - 0| \le \frac{10}{20}\left(e^{10(2-0)} - 1\right) - 2.43 \cdot 10^{8}$$

که در هر دو حالت صحیح است.

5.3.10

10. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \le t \le 2, \quad y(1) = -1,$$

with exact solution $y(t) = -1/t$:

a. Use Taylor's method of order two with $h = 0.05$ to approximate the solution and compare it with the actual values of y.

b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y and compare them to the actual values.

i. $y(1.052)$ **ii.** $y(1.555)$ **iii.** $y(1.978)$

c. Use Taylor's method of order four with $h = 0.05$ to approximate the solution and compare it with the actual values of y.

d. Use the answers generated in part (c) and piecewise cubic Hermite interpolation to approximate the following values of y and compare them to the actual values.

i. $y(1.052)$ **ii.** $y(1.555)$ **iii.** $y(1.978)$

الف)

$$y' = f(t,y), y''(t) = f'(t,y) = \frac{3}{t^3} + \frac{3y^2}{t} + 2y^3$$

$$a = t_0 = 1, \alpha = y_0 = w_o = -1$$

$$t_i = a + ih = 1 + 0.05i$$

$$T^2(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) = \frac{1}{t_i^2} - \frac{w_i}{t_i} - w_i^2 + \frac{h}{2}\left(\frac{3}{t_i^3} + \frac{3w_i^2}{t_i} + 2w_i^3\right)$$

$$= \frac{1}{(1+0.05i)^2} - \frac{w_i}{(1+0.05i)} - w_i^2 + \frac{0.05}{2}\left(\frac{3}{(1+0.05i)^3} + \frac{3w_i^2}{(1+0.05i)} + 2w_i^3\right)$$

$$\rightarrow w_{i+1} = w_i + hT^2(t_i, w_i)$$

$$= w_i + 0.05\left(\frac{1}{(1+0.05i)^2} - \frac{w_i}{(1+0.05i)} - w_i^2 + \frac{0.05}{2}\left(\frac{3}{(1+0.05i)^3} + \frac{3w_i^2}{(1+0.05i)} + 2w_i^3\right)\right)$$

i	t_i	w_i (Approximate values)	$y[t_i]$ (Exact Values)	$Error[i] = \|y[t_i] - w_i\|$ (Actual Error)
11	1.55	-0.645979	-0.645161	0.000817493
12	1.6	-0.625865	-0.625	0.000864877
13	1.65	-0.606971	-0.606061	0.000910437
14	1.7	-0.58919	-0.588235	0.000954436
15	1.75	-0.572426	-0.571429	0.000997092
16	1.8	-0.556594	-0.555556	0.00103859
17	1.85	-0.54162	-0.540541	0.00107908
18	1.9	-0.527434	-0.526316	0.00111869
19	1.95	-0.513978	-0.512821	0.00115754
20	2.	-0.501196	-0.5	0.00119571

i	t_i	w_i (Approximate values)	$y[t_i]$ (Exact Values)	$Error[i] = \|y[t_i] - w_i\|$ (Actual Error)
1	1.05	-0.9525	-0.952381	0.000119048
2	1.1	-0.909314	-0.909091	0.000222881
3	1.15	-0.86988	-0.869565	0.000314679
4	1.2	-0.83373	-0.833333	0.000396857
5	1.25	-0.800471	-0.8	0.000471271
6	1.3	-0.76977	-0.769231	0.000539363
7	1.35	-0.741343	-0.740741	0.000602268
8	1.4	-0.714947	-0.714286	0.000660881
9	1.45	-0.690371	-0.689655	0.000715923
10	1.5	-0.667435	-0.666667	0.00076797

ب)

ب.۱)

$$\hat{y}(1.052) = (-0.9525) + \left(\frac{-0.909314 + 0.9525}{1.1 - 1.05}\right)(1.052 - 1.05) = -0.9507726$$

$$y(1.052) = -\frac{1}{1.052} = -0.9505703422$$

$$Error = |\,\hat{y} - y| = 0.00020226$$

ب.۲)

$$\hat{y}(1.555) = (-0.645979) + \left(\frac{-0.625865 + 0.645979}{1.6 - 1.55}\right)(1.555 - 1.55) = -0.6439676$$

$$y(1.555) = -\frac{1}{1.555} = -0.6430868167$$

$$Error = |\,\hat{y} - y| = 0.000808$$

ب.۳)

$$\hat{y}(1.978) = (-0.513978) + \left(\frac{-0.501196 + 0.513978}{2 - 1.95}\right)(1.978 - 1.95) = -0.50682008$$

$$y(1.978) = -\frac{1}{1.978} = -0.5055611729$$

$$Error = |\,\hat{y} - y| = 0.001258907$$

پ)

$$y' = f(t,y), y''^{(t)} = f'(t,y) = \frac{3}{t^3} + \frac{3y^2}{t} + 2y^3, y'''(t) = f''(t,y)$$

$$= \frac{3\left(-3 - 2ty + t^2{y^2}^{3}_{4t}y^3 + 2t^4y^4\right)}{t^4}, y^4(t) = f'''(t,y)$$

$$= \frac{6(-5 - 5ty - 5t^2y^2 + 5t^3y^3 + 10t^4y^4 + 4t^5y^5)}{t^5}$$

$$a = t_0 = 1, \alpha = y_0 = w_o = -1$$

$$t_i = a + ih = 1 + 0.05i$$

$$T^4(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \frac{h^2}{6}f''(t_i, w_i) + \frac{h^3}{24}f'''(t_i, w_i)$$

$$= \frac{1}{(1+0.05i)^2} - \frac{w_i}{(1+0.05i)} - w_i^2 + \frac{0.05}{2}\left(\frac{3}{(1+0.05i)^3} + \frac{3w_i^2}{(1+0.05i)} + 2w_i^3\right)$$

$$+ \frac{(0.05)^2}{6}\frac{\left(-3\left(-3 - 2(1+0.05i)w_i + (1+0.05i)^2w_i^2 + 4(1+0.05i)^3 + 2(1+0.05i)^4w_i^4\right)\right)}{(1+0.05i)^4}$$

$$+ \frac{(0.05)^3}{24}\left(\frac{6\left(-5 - 5(1+0.05i)w_i - 5(1+0.05i)^2w_i^2 + 5(1+0.05i)^3w_i^3 + 10(1+0.05i)^4w_i^4 + 4(1+0.05i)^4w_i^4 + 4(1+0.05i)^5w_i^5\right)}{(1+0.05i)^5}\right)$$

$$\rightarrow w_{i+1} = w_i + hT^4(t_i, w_i)$$

i	t_i	w_i (Approximate values)	$y[t_i]$ (Exact Values)	$Error[i] = \lvert y[t_i] - w_i \rvert$ (Actual Error)
1	1.05	-0.952381	-0.952381	2.97619*10^-7
2	1.1	-0.909091	-0.909091	5.34384*10^-7
3	1.15	-0.869566	-0.869565	7.27403*10^-7
4	1.2	-0.833334	-0.833333	8.88502*10^-7
5	1.25	-0.800001	-0.8	1.02599*10^-6
6	1.3	-0.769232	-0.769231	1.1458*10^-6
7	1.35	-0.740742	-0.740741	1.25222*10^-6
8	1.4	-0.714287	-0.714286	1.34837*10^-6
9	1.45	-0.689657	-0.689655	1.4366*10^-6
10	1.5	-0.666668	-0.666667	1.51864*10^-6
11	1.55	-0.645163	-0.645161	1.59581*10^-6
12	1.6	-0.625002	-0.625	1.66912*10^-6
13	1.65	-0.606062	-0.606061	1.73935*10^-6
14	1.7	-0.588237	-0.588235	1.80708*10^-6
15	1.75	-0.57143	-0.571429	1.87281*10^-6
16	1.8	-0.555557	-0.555556	1.9369*10^-6
17	1.85	-0.540543	-0.540541	1.99964*10^-6
18	1.9	-0.526318	-0.526316	2.06127*10^-6
19	1.95	-0.512823	-0.512821	2.12199*10^-6
20	2.	-0.500002	-0.5	2.18194*10^-6

ت)

$$H_3(t) = \sum_{i=0}^{1}(1 - 2(t - t_i)I_i'^2 y_i) + \sum_{i=0}^{1}(t - t_i)I_i^2 y'_i$$

$$I_0(t) = \frac{t - t_1}{t_0 - t_1}, I_1(t) = \frac{t - t_0}{t_1 - t_0}$$

ت.۱)

$$\begin{aligned} H_3(1.052) = {} & \big(1 - 2(1.052)(-20)\big)\left(\frac{1.1 - 1.052}{0.05}\right)^2 (-0.9525) \\ & + \big(1 - 2(1.052 - 1.1)(20)\big)\left(\frac{1.052 - 1.05}{0.05}\right)^2 (-0.909314) \\ & + (1.052 - 1.05)\left(\frac{1.1 - 1.052}{0.05}\right)^2 (0.906916) \\ & + (1.052 - 1.1)\left(\frac{1.052 - 1.05}{0.05}\right)^2 (0.82624342)) = -0.9506900629 \end{aligned}$$

$$y(1.052) = -0.9505703422$$

$$Error = |H_3 - y| = 0.0001972073$$

ت.۲)

$$H_3(1.555) = -0.6430884$$

$$y(1.555) = -0.6430868167$$

$$Error = |H_3 - y| = 0.0000015833$$

ت.۳)

$$H_3(1.978) = -0.5055633$$

$$y(1.978) = -0.5055611729$$

$$Error = |H_3 - y| = 0.0000021271$$

29

29. Show that the Midpoint method and the Modified Euler method give the same approximations to the initial-value problem

$$y' = -y + t + 1, \quad 0 \le t \le 1, \quad y(0) = 1,$$

for any choice of h. Why is this true?

Midpoint method

$$y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right)$$

Modified Euler method:

$$y_{n+1} = y_n + \frac{h}{2}\Big(f(t_n, y_n) + f\big(t_{n+1}, y_n + hf(t_n, y_n)\big)\Big)$$

با مساوی قرار دادن دو طرف داریم:

$$y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right) = y_n + \frac{h}{2}\Big(f(t_n, y_n) + f\big(t_{n+1}, y_n + hf(t_n, y_n)\big)\Big)$$

$$f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right) = \frac{1}{2}\Big(f(t_n, y_n) + f\big(t_{n+1}, y_n + hf(t_n, y_n)\big)\Big)$$

$$1 + \left(t_n - \frac{h}{2}\right) - \left(y_n + \frac{h}{2}\big(f(t_i, w_i)\big)\right) = \frac{1}{2}(1 + t_n - y_n) + \frac{1}{2}\Big(1 + t_n - y_n + 1 + t_{n+1} - \big(y_n + hf(t_n, y_n)\big)\Big)$$

$$\rightarrow t_n = \frac{1}{2} t_n + \frac{1}{2} t_{n+1} - \frac{h}{2} \rightarrow h = t_{n+1} - t_n$$

در نتیجه این دو عبارت برای هر h مقدار یکسانی دارند.

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31. Show that Heun's method can be expressed in difference form, similar to that of the Runge-Kutta method of order four, as

$$w_0 = \alpha,$$
$$k_1 = hf(t_i, w_i),$$
$$k_2 = hf\left(t_i + \frac{h}{3}, w_i + \frac{1}{3}k_1\right),$$
$$k_3 = hf\left(t_i + \frac{2h}{3}, w_i + \frac{2}{3}k_2\right),$$
$$w_{i+1} = w_i + \frac{1}{4}(k_1 + 3k_3),$$

for each $i = 0, 1, \ldots, N-1$.

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{4}\left(f(t_i, w_i) + 3\left(f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3}f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3}f(t_i, w_i)\right)\right)\right)\right)$$

k_i ها را به شکل زیر تعریف می کنیم:

$$k_1 = hf(t_i. w_i)$$
$$k_2 = hf\left(t_i + \frac{h}{3}, w_i + \frac{k_1}{3}\right)$$
$$k_3 = hf\left(t_i + \frac{2h}{3}, w_i + \frac{2k_2}{3}\right)$$

حال داریم:

$$w_{i+1} = w_i + \frac{1}{4}(k_1 + 3k_3)$$

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7. The Runge-Kutta-Verner method (see [Ve]) is based on the formulas

$$w_{i+1} = w_i + \frac{13}{160}k_1 + \frac{2375}{5984}k_3 + \frac{5}{16}k_4 + \frac{12}{85}k_5 + \frac{3}{44}k_6 \quad \text{and}$$

$$\tilde{w}_{i+1} = w_i + \frac{3}{40}k_1 + \frac{875}{2244}k_3 + \frac{23}{72}k_4 + \frac{264}{1955}k_5 + \frac{125}{11592}k_7 + \frac{43}{616}k_8,$$

where

$$k_1 = hf(t_i, w_i),$$

$$k_2 = hf\left(t_i + \frac{h}{6}, w_i + \frac{1}{6}k_1\right),$$

$$k_3 = hf\left(t_i + \frac{4h}{15}, w_i + \frac{4}{75}k_1 + \frac{16}{75}k_2\right),$$

$$k_4 = hf\left(t_i + \frac{2h}{3}, w_i + \frac{5}{6}k_1 - \frac{8}{3}k_2 + \frac{5}{2}k_3\right),$$

$$k_5 = hf\left(t_i + \frac{5h}{6}, w_i - \frac{165}{64}k_1 + \frac{55}{6}k_2 - \frac{425}{64}k_3 + \frac{85}{96}k_4\right),$$

$$k_6 = hf\left(t_i + h, w_i + \frac{12}{5}k_1 - 8k_2 + \frac{4015}{612}k_3 - \frac{11}{36}k_4 + \frac{88}{255}k_5\right),$$

$$k_7 = hf\left(t_i + \frac{h}{15}, w_i - \frac{8263}{15000}k_1 + \frac{124}{75}k_2 - \frac{643}{680}k_3 - \frac{81}{250}k_4 + \frac{2484}{10625}k_5\right),$$

and

$$k_8 = hf\left(t_i + h, w_i + \frac{3501}{1720}k_1 - \frac{300}{43}k_2 + \frac{297275}{52632}k_3 - \frac{319}{2322}k_4 + \frac{24068}{84065}k_5 + \frac{3850}{26703}k_7\right).$$

The sixth-order method $\tilde{w}_{i+1}$ is used to estimate the error in the fifth-order method w_{i+1}. Construct an algorithm similar to the Runge-Kutta-Fehlberg Algorithm and repeat Exercise 3 using this new method.

...

	t	h	R	rkv	exact
0	1.000000	0.000000	0.000000e+00	1.000000	1.000000
1	1.420876	0.420876	9.826040e-07	1.051498	1.051509
2	1.788747	0.367872	8.242650e-08	1.131024	1.131034
3	2.288747	0.500000	2.912426e-08	1.252037	1.252047
4	2.788747	0.500000	5.224625e-10	1.376747	1.376756
5	3.288747	0.500000	5.352159e-10	1.501354	1.501364
6	3.788747	0.500000	2.829981e-10	1.624643	1.624653
7	4.000000	0.211253	2.128870e-12	1.676229	1.676239

فایل کد پیوست شده است.