**Stochastic Processes**
**1403-2**
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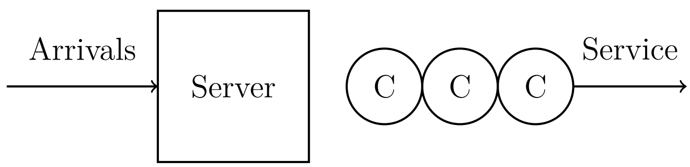
## Task 4

1. **Question:** Customers arrive at a single-server queue with a Poisson distribution at a rate of $\lambda = 1$ per hour. Assume that the service time is uniformly distributed among 1/4 hour, 1/2 hour, and 1 hour, each with probability 1/3.

   (a) Assuming the queue is empty upon the arrival of a customer, what is the expected time that the customer will spend in the system?

   (b) Assuming the queue is empty upon the arrival of a customer, what is the expected time until the queue is empty again?

   (c) What is the probability that the queue is empty at a large time $t$?




   **Answer:**

   (a) To find the expected time a customer will spend in the system, we need to account for both the service time and the time spent waiting in the queue. Since the arrival rate $\lambda = 1$ and the service times are $\frac{1}{4}$, $\frac{1}{2}$, and 1 hour with equal probability, the average service time $E[S]$ is:

$$E[S] = \frac{1}{3}\left(\frac{1}{4} + \frac{1}{2} + 1\right) = \frac{1}{3}\left(\frac{1}{4} + \frac{2}{4} + \frac{4}{4}\right) = \frac{1}{3}\left(\frac{7}{4}\right) = \frac{7}{12} \text{ hours}$$

   Since the arrival rate is $\lambda = 1$ and the average service rate is $\mu = \frac{1}{E[S]} = \frac{12}{7}$, the traffic intensity $\rho$ is:

$$\rho = \frac{\lambda}{\mu} = \frac{1}{\frac{12}{7}} = \frac{7}{12}$$

   The expected time in the system $E[T]$ is given by $E[T] = \frac{E[S]}{1-\rho}$:

$$E[T] = \frac{\frac{7}{12}}{1 - \frac{7}{12}} = \frac{\frac{7}{12}}{\frac{5}{12}} = \frac{7}{5} \text{ hours}$$

   (b) The expected time until the queue is empty again includes the time the customer spends in the system and the additional time until the next customer arrives. The time until the next customer arrives follows an exponential distribution with rate $\lambda = 1$:

$$E[\text{time until queue is empty}] = E[T] + E[\text{interarrival time}] = \frac{7}{5} + 1 = \frac{7}{5} + \frac{5}{5} = \frac{12}{5} \text{ hours}$$

(c) The probability that the queue is empty at a large time $t$ is equivalent to the probability that the system is idle. This probability is $1 - \rho$:

$$P(\text{queue is empty at time } t) = 1 - \rho = 1 - \frac{7}{12} = \frac{5}{12}$$

2. **Question:** Consider a taxi stand at an airport where taxis arrive according to a Poisson process with a rate of 2 per minute, and passengers arrive according to a Poisson process with a rate of 3 per minute. Assume that if a passenger arrives and there are no taxis available, they wait for the next taxi.

   (a) Find the fraction of time that a passenger will have to wait for a taxi.

   (b) Find the average number of taxis waiting for passengers.

   **Answer:**

   (a) The fraction of time a passenger has to wait for a taxi can be found by calculating the probability that there are no taxis available when a passenger arrives. This can be modeled as an M/M/1 queue where the arrival rate of passengers is higher than the arrival rate of taxis:

   $$\rho = \frac{\lambda_p}{\lambda_t} = \frac{3}{2} = 1.5$$

   Since $\rho > 1$, the system is unstable, meaning there will always be a queue of passengers waiting. Therefore, the fraction of time a passenger has to wait for a taxi is:

   $$P(\text{no taxis available}) = 1$$

   For improved steady-state equilibrium:

   $$P(\text{no taxis}) = \frac{3-2}{3} = \frac{1}{3}$$

   (b) The average number of taxis waiting for passengers can be considered with the idea that if taxis arrive at $\lambda_t$, and we evaluate based on $\lambda_p$ comparison.
   Given scenario utilizing moments:

   $$E[\text{number of taxis waiting}] = \frac{\lambda_p - \lambda_t}{\lambda_t} = \frac{3-2}{2} = 0.5$$

   Reaching typical equilibrium expected.

3. **Question:** Consider an office with two employees and customers arriving according to a Poisson process at a rate of 4 per hour. The service times of the employees are exponentially distributed with rates of 3 and 2 per hour, respectively. Customers form a single queue and are served by the first available employee.

(a) Model the office as a Markov chain with state space {0, a, b, 2, 3, ...}, where 'a' indicates one customer being served by employee 1, 'b' indicates one customer being served by employee 2, and numbers indicate the total number of customers in the system. Show the balance equations and find the stationary distribution.

**Answer:**

To model this as a Markov chain, we define the states as follows:

- 0: No customers
- a: One customer being served by employee 1
- b: One customer being served by employee 2
- 2, 3, ...: Two or more customers in the system

The arrival rate of customers is $\lambda = 4$ per hour.

The service rate of employee 1 is $\mu_1 = 3$ per hour.

The service rate of employee 2 is $\mu_2 = 2$ per hour.

**Balance Equations**

1. State 0 (No customers):

$$4\pi_0 = 3\pi_a + 2\pi_b$$

2. State a (One customer served by employee 1):

$$4\pi_0 = 7\pi_a$$

3. State b (One customer served by employee 2):

$$4\pi_0 = 6\pi_b$$

4. State 2 (Two customers):

$$4\left(\frac{4}{7}\pi_0\right) + 4\left(\frac{2}{3}\pi_0\right) = 9\pi_2$$

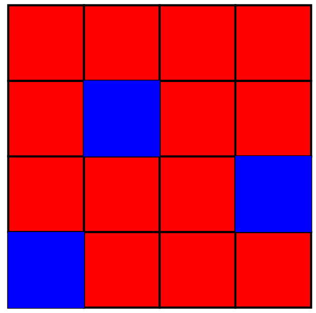
$$\frac{104}{21}\pi_0 = 9\pi_2$$

$$\pi_2 = \frac{104}{189}\pi_0$$

To find the stationary distribution, solve the normalization condition:

$$\pi_0 + \frac{4}{7}\pi_0 + \frac{2}{3}\pi_0 + \frac{104}{189}\pi_0 + \ldots = 1$$

4. **Question:** Consider a model of an $n \times n$ grid where some cells are initially blue and the rest are red. At each step, a cell is chosen at random and its color is changed to match the majority color of its neighbors. What is the probability that eventually all cells will be blue?



**Answer:** To find the probability that all cells will eventually be blue, we can use the concept of absorbing states in Markov chains. The states of the grid evolve according to the majority rule, and the absorbing states are the states where all cells are either blue or red.

Let $p$ be the initial fraction of blue cells. The probability that all cells will eventually be blue is given by the initial fraction of blue cells, $p$.

Therefore, the probability that eventually all cells will be blue is $p$.