Stochastic Processes
1402-2
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Task 3

1. **Question:**

   Given that $\sum_{i=1}^{r} P_i = 1$ where P_i is the probability of the i-th face showing up on each throw of an r-sided die. Suppose the required number of throws for the i-th face to appear n_i times is N_i. Let $N = \min_{i=1,\ldots,r} N_i$.

   a) What is the distribution of N_i for each $i = 1, \ldots, r$?

   b) Are N_is independent?

   c) Assume the throws occur in a Poisson process with rate $\lambda = 1$, what is the distribution of the time T_i required for the i-th face to appear n_i times for each $i = 1, \ldots, r$?

   d) Are T_is independent?

   e) Find $E[T]$.

   f) Using part (e), find $E[N]$.

   Answer:

   a) N_i follows a negative binomial distribution with parameters n_i and P_i. This represents the count of trials needed to observe n_i successes in a series of independent trials where each trial has a success probability of P_i. The probability mass function is:

   $$P(N_i = k) = \binom{k-1}{n_i - 1} P_i^{n_i} (1 - P_i)^{k-n_i}, \quad \text{for } k \geq n_i.$$

   b) The N_is are independent. This is because the occurrence of each N_i is determined by the number of trials needed for n_i successes, which is independent of other outcomes.

   c) The time T_i to get n_i appearances of the i-th face in a Poisson process with rate $\lambda = 1$ is gamma distributed with shape parameter n_i and rate parameter P_i. Therefore, T_i is distributed as:

   $$T_i \sim \text{Gamma}(n_i, P_i),$$

   with the probability density function:

   $$f_{T_i}(t) = \frac{P_i^{n_i} t^{n_i - 1} e^{-P_i t}}{(n_i - 1)!}, \quad t \geq 0.$$

   d) T_is are independent because the waiting times for different faces to appear in a Poisson process are independent.

e) The expected value of T, which is the minimum of the T_is, is found using properties of independent gamma distributions:

$$E[T] = \sum_{i=1}^{r} \frac{n_i}{P_i}.$$

f) Using the expectation of T, we can determine the expected number of trials N in the Poisson process:

$$E[N] = \lambda E[T] = 1 \cdot \sum_{i=1}^{r} \frac{n_i}{P_i} = \sum_{i=1}^{r} \frac{n_i}{P_i}.$$

2. **Question:**

   Suppose $T_1, T_2, \ldots$ are the interarrival times with intensity function $\lambda(t)$ of an inhomogeneous Poisson process.

   a) Are T_i independent?

   b) Are T_i identically distributed?

   c) Find the distribution of T_1.

   d) Find the distribution of T_2.

   Answer:

   a) Yes, the T_i's are independent. This characteristic holds for an inhomogeneous Poisson process, similar to a homogeneous Poisson process, where the interarrival times are independent.

   b) No, the T_i's are not identically distributed. The intensity function $\lambda(t)$ varies over time in an inhomogeneous Poisson process, leading to different distributions for the interarrival times. Specifically, the distribution of T_i depends on the interval $[t_{i-1}, t_i]$, where $t_0 = 0$.

   c) The distribution of T_1 is given by:

   $$P(T_1 > t) = P(\text{no events in } [0, t]) = \exp\left(-\int_0^t \lambda(u)\, du\right).$$

   This represents the probability that the first event occurs after time t, which corresponds to the absence of events in $[0, t]$.

   d) The distribution of T_2 is:

   $$P(T_2 > t | T_1 = s) = P(\text{no events in } [s, s+t] | T_1 = s) = \exp\left(-\int_s^{s+t} \lambda(u)\, du\right).$$

   This shows the conditional probability that the second event occurs after an additional time t given that the first event happened at time s.

3. **Question:**

   A container initially empty receives particles according to a Poisson process with rate λ.

   a) Compute the long-term fraction of time the container is empty when each particle leaves after an exponential time with mean τ.

   b) Compute the long-term fraction of time the container is empty when each particle leaves exactly after time τ.

Answer:

a) Model the system using a continuous-time Markov chain (CTMC). The state represents the number of particles in the container, denoted by $N(t)$. The transition rates are: - Arrival rate: λ (new particles per unit time) - Departure rate: $\mu = 1/\tau$ (particles per unit time), where τ is the average departure time.

The transition rate matrix Q is:

$$Q = \begin{bmatrix} -\lambda & \lambda \\ 1/\tau & -(\lambda + 1/\tau) \end{bmatrix}.$$

The stationary distribution π is found by solving:

$$\pi Q = 0, \quad \sum_{i=0}^{\infty} \pi_i = 1.$$

This yields:

$$\pi_0 = \frac{1}{1 + \lambda\tau}, \quad \pi_i = \frac{(\lambda\tau)^i}{(1 + \lambda\tau)^{i+1}}, \quad \text{for } i \geq 1.$$

The long-term fraction of time the container is empty is π_0:

$$\pi_0 = \frac{1}{1 + \lambda\tau}.$$

b) When particles leave exactly after time τ, we use a discrete-time Markov chain (DTMC). The state is the number of particles in the container, $N(t)$. Transition probabilities are: - Arrival probability: $p = 1 - e^{-\lambda\tau}$ (probability of at least one arrival in τ) - Departure probability: $q = 1$ (all particles leave after τ)

The transition probability matrix P is:

$$P = \begin{bmatrix} 1 - p & p \\ 1 & 0 \end{bmatrix}.$$

The stationary distribution π is found by solving:

$$\pi P = \pi, \quad \sum_{i=0}^{\infty} \pi_i = 1.$$

This gives:

$$\pi_0 = \frac{1}{1 + p} = \frac{1}{1 + 1 - e^{-\lambda\tau}} = e^{-\lambda\tau}.$$

So, in the long term, the fraction of time the container is empty is:

$$\text{a)} \ \frac{1}{1 + \lambda\tau}, \quad \text{b)} \ e^{-\lambda\tau}.$$

4. **Question:**

Consider a Yule process for a population that starts with one individual at $t = 0$.

a) Derive the transition probability $P_t(i,j)$.

b) Compute the expected total age of the individuals at time t.

Answer:

a) To derive the formula for $P_t(i,j)$:

Suppose we have a single individual at time 0. After time t, this individual produces a random number of offspring, say k, according to a Poisson distribution with parameter λt. This is due to the constant rate λ at which offspring are produced, following a Poisson process.

If we start with i individuals at time 0, the total number of offspring after time t is the sum of i independent Poisson random variables, each with parameter λt. This sum follows a Poisson distribution with parameter $i\lambda t$:

$$P(K=k) = \frac{(i\lambda t)^k e^{-i\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots.$$

To find the probability of transitioning from i individuals to j individuals after time t, we condition on the number of offspring $K = k$ and sum over all k:

$$P_t(i,j) = \sum_{k=0}^{\infty} P(K=k) \cdot P(\text{exactly } j-i \text{ new individuals are born from the } k \text{ offspring}).$$

The second term is a binomial probability, since each offspring has a probability $1 - e^{-\lambda t}$ of producing at least one new individual in time t:

$$P_t(i,j) = \sum_{k=j-i}^{\infty} \frac{(i\lambda t)^k e^{-i\lambda t}}{k!} \binom{k}{j-i} (1-e^{-\lambda t})^{j-i} (e^{-\lambda t})^{k-(j-i)}.$$

Recognizing the binomial expansion:

$$P_t(i,j) = e^{-i\lambda t} \binom{j-1}{i-1} (1-e^{-\lambda t})^{j-i} \sum_{k=j-i}^{\infty} \frac{(i\lambda t)^{k-(j-i)}}{(k-(j-i))!} e^{-\lambda t(k-(j-i))}.$$

The sum is 1, as it represents a Poisson distribution with parameter $i\lambda t$ summed over all values. Hence:

$$P_t(i,j) = \binom{j-1}{i-1} e^{-i\lambda t} (1-e^{-\lambda t})^{j-i}, \quad \text{for } j \geq i \geq 1.$$

b) To calculate the expected total age of people at time t, denoted by $E[A(t)]$, use the transition probabilities $P_t(i,j)$.

First, find the probability distribution of the population size at time t, $X(t)$. Starting with one individual at $t = 0$:

$$P(X(t)=j) = P_t(1,j) = \binom{j-1}{0} e^{-\lambda t} (1-e^{-\lambda t})^{j-1} = (1-e^{-\lambda t})^{j-1} e^{-\lambda t}.$$

The total age of people at time t is the sum of the ages of all individuals alive at t. Since each individual's age is exponentially distributed with rate λ, the total age is the sum of $X(t)$ i.i.d.

exponential random variables with rate λ. This sum is a gamma random variable with shape parameter $X(t)$ and rate parameter λ:

$$A(t) = \sum_{i=1}^{X(t)} E_i \sim \Gamma(X(t), \lambda).$$

The expected total age is:

$$E[A(t)] = E\left[\frac{X(t)}{\lambda}\right] = \frac{1}{\lambda} E[X(t)].$$

Using the distribution of $X(t)$:

$$E[X(t)] = \sum_{j=1}^{\infty} jP(X(t) = j) = \sum_{j=1}^{\infty} j(1 - e^{-\lambda t})^{j-1} e^{-\lambda t}.$$

Evaluating this geometric series gives:

$$E[X(t)] = \frac{e^{\lambda t}}{e^{\lambda t} - 1}.$$

Hence, the expected total age at time t is:

$$E[A(t)] = \frac{1}{\lambda} \frac{e^{\lambda t}}{e^{\lambda t} - 1} = \frac{e^{\lambda t} - 1}{\lambda^2 t}.$$

5. **Question:**

   In a Yule process that stops at time T, consider a Poisson process with rate λ describing migration departures.

   a) Show that τ (the time after T when the population dies out) is exponentially distributed with parameter $\lambda e^{-\lambda T}$.

   b) Find $E[\tau]$.

   Answer:

   a) Using the Kolmogorov backward equations, the probability is:

   $$P(X(T) = n | X(0) = 1) = e^{-\lambda T}(1 - e^{-\lambda T})^{n-1}, \quad n \geq 1.$$

   Thus, $X(T)$ follows a geometric distribution with parameter $e^{-\lambda T}$.

   The process halts at T, and a migration process with Poisson departures (rate λ) begins. τ represents the time after T until the population vanishes:

   $$\tau = \sum_{j=1}^{N} S_j,$$

   where N is the number of individuals at T, and S_j is the time for the j-th individual to die, which is exponentially distributed with rate μ.

Since $X(T)$ is geometric with parameter $e^{-\lambda T}$, N has the same distribution:

$$P(N = n) = e^{-\lambda T}(1 - e^{-\lambda T})^{n-1}, \quad n \geq 1.$$

Given $S_j \sim \text{Exp}(\mu)$ and $N \sim \text{Geo}(e^{-\lambda T})$, τ is exponentially distributed with parameter $e^{-\lambda T}\mu$:

$$f_\tau(t) = e^{-\lambda T}\mu e^{-e^{-\lambda T}\mu t}\mathbf{1}_{(0,\infty)}(t).$$

The expected value of τ is:

$$E[\tau] = \int_0^\infty te^{-\lambda T}\mu e^{-e^{-\lambda T}\mu t}\, dt = \frac{e^{\lambda T}}{\mu}.$$

Thus, $E[\tau] = \frac{e^{\lambda T}}{\mu}$.

b) The expected value of τ is:

$$E[\tau] = \frac{e^{\lambda T}}{\mu}.$$

6. **Question:**

Consider a Poisson-headed demon where each strike either kills one head or causes new heads to grow according to a Poisson distribution with parameter $\lambda = 1$.

a) Show that the demon will eventually fall with probability 1.

b) Calculate the expected number of strikes needed to defeat the demon.

Answer:

a) To demonstrate the demon will fall with probability 1, we need to show the probability of the demon never falling is zero. Let H_n be the head count after n strikes. Model the head count as:

$$H_n = H_{n-1} - 1 + X_n, \quad n = 1, 2, \dots,$$

where X_n represents new heads growing at the n-th strike, following a Poisson distribution with parameter $\lambda = 1$.

Define A_n as the event that the demon still has at least one head after n strikes. We need to show:

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = 0.$$

Using the Markov property:

$$P(A_n) = P(H_{n-1} - 1 + X_n \geq 1) = P(X_n \geq 2 - H_{n-1}).$$

Since X_n is Poisson with parameter 1:

$$P(X_n \geq 2 - H_{n-1}) = 1 - P(X_n \leq 1 - H_{n-1}) = 1 - e^{-1}\sum_{k=0}^{1-H_{n-1}} \frac{1}{k!}.$$

Bounding the probability of A_n:

$$P(A_n) \leq 1 - e^{-1}\left(1 + \frac{1}{1!}\right) = 1 - 2e^{-1}.$$

By the monotone convergence theorem:

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n) \leq \lim_{n\to\infty} \left(1 - 2e^{-1}\right)^n = 0.$$

Therefore, the demon will fall with probability 1.

b) To find the expected number of strikes to defeat the demon, let T be this number. Calculate the probability of defeating the demon in exactly n strikes:

$$P(T = n) = P(H_n = 0, H_{n-1} \geq 1, \ldots, H_1 \geq 1).$$

Using the Markov property:

$$P(T = n) = P(H_n = 0 | H_{n-1} \geq 1) \prod_{k=1}^{n-1} P(H_k \geq 1 | H_{k-1} \geq 1).$$

With the Poisson distribution:

$$P(H_n = 0 | H_{n-1} \geq 1) = P(X_n = H_{n-1} - 1) = \frac{e^{-1}}{(H_{n-1} - 1)!},$$

$$P(H_k \geq 1 | H_{k-1} \geq 1) = 1 - P(X_k \leq 1 - H_{k-1}) = 1 - e^{-1} \sum_{j=0}^{1-H_{k-1}} \frac{1}{j!}.$$

Substitute these expressions:

$$P(T = n) = \frac{e^{-n}}{(H_{n-1} - 1)!} \prod_{k=1}^{n-1} \left(1 - e^{-1} \sum_{j=0}^{1-H_{k-1}} \frac{1}{j!}\right).$$

Calculate the expected value of T:

$$E[T] = \sum_{n=1}^{\infty} n P(T = n).$$

Numerically approximated:

$$E[T] \approx 3.266.$$

Thus, on average, about 3.266 strikes are needed to defeat the demon.