Stochastic Processes
1401-2
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Task 2

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1. **Question 1:** In a complete graph with n vertices, calculate the expected hitting time from vertex u to vertex v starting from u.

   Answer: We divide the complete graph into several sections. Assume the total number of vertices is n and each section has $m = n + 1$ vertices. The expected hitting time from u to v can be calculated as follows:
   $$E[u, v] = m - 1 + E[k, v]$$
   where k is a vertex in the section connected to the path from u to v. Simplifying, we find:
   $$E[u, v] = m - 1 + (n - m)(m^2 - 2m + n) = m^2 - m + 1$$

2. **Question 2:** Consider a Markov chain on the state space $\{1, 2, 3, 4, 5\}$ with the following transition matrix:
   $$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$
   a) Is this Markov chain irreducible?
   b) Find all the stationary distributions.
   c) Starting from state 1, what is the expected time to return to state 1?
   d) Starting from state 1, what is the expected time to reach state 4?
   e) Starting from state 1, what is the probability that the chain visits state 5 before state 3?

   Answer:

   a) The Markov chain is irreducible because it is possible to transition from any state to any other state, albeit indirectly.
   b) The stationary distribution for this chain is:
   $$\pi = \left[\frac{10}{37}, \frac{5}{37}, \frac{5}{37}, \frac{3}{37}, \frac{14}{37}\right]$$
   c) The expected return time to state 1 is 3.7 steps.
   d) The expected time to reach state 4 starting from state 1 is:
   $$E_{1,4} = \frac{34}{3}$$

e) The probability that the chain visits state 5 before state 3 when starting from state 1 is:

$$P = \frac{4}{9}$$

3. **Question 3:** In a 3x3 grid with black and white cells, each step randomly chooses a row or a column and recolors all cells in it to the majority color. Show that eventually, all cells will become white.

   Answer: Regardless of the initial configuration, each step either maintains the current state or increases the number of white cells. Considering this as a Markov chain, each state transitions towards the absorbing state where all cells are white. Using the following recurrence relations, we calculate the expected time to reach this absorbing state:

$$\begin{aligned} E_1 &= 1 + \frac{1}{2}E_1 + \frac{1}{2}E_2 \\ E_2 &= 1 + \frac{2}{6}E_3 + \frac{1}{6}E_1 + \frac{3}{6}E_2 \\ E_3 &= 1 + \frac{4}{6}E_3 \\ E_1 &= 9 \end{aligned}$$

   Thus, the expected time for all cells to become white is 9 steps.

4. **Question 4:** In a random walk on the integers, each particle independently moves left or right with equal probability $\frac{1}{2}$. There is a probability ϵ that a particle splits into two particles at each step. Show that if ϵ is positive, there is a positive probability that the number of particles grows indefinitely.

   Answer: We model this problem using a branching process, where each particle at each step can either split into two particles or move left or right.

   Let $E[X_n]$ denote the expected number of particles at step n.

   1. At each step, each particle: - Moves left or right with probability $\frac{1}{2}$. - Splits into two particles with probability ϵ. - Does not split with probability $1 - \epsilon$.

   2. The expected number of particles at step $n + 1$ is:

$$E[X_{n+1}] = (1 + \epsilon)E[X_n]$$

   Simplifying, we get:

$$E[X_{n+1}] = (1 + \epsilon)E[X_n]$$

   3. Starting with $E[X_0] = 1$:

$$E[X_n] = (1 + \epsilon)^n$$

   4. If ϵ is positive, the factor $(1 + \epsilon)$ is greater than 1, leading to exponential growth in $E[X_n]$.

   Since the expected number of particles grows indefinitely when $\epsilon > 0$, there is a positive probability that the actual number of particles will also grow indefinitely. This characteristic indicates that the number of particles can grow without bound.

Therefore, we have shown that if ϵ is positive, there is a positive probability that the number of particles grows indefinitely.

5. **Question 5:** In a symmetric random walk starting from zero, what is the expected time to return to zero? Let X_n be the position at step n, and $T_y = \min\{n \geq 1 : X_n = y\}$ be the first return time to y. Calculate the expected number of returns to zero by time n.

   Answer: To calculate the expected return time, we solve the following recurrence relations:

$$E_0 = 1 + E_1 + E_{-1} = 1 + E_1$$
$$E_1 = \frac{1}{2} + \frac{1}{2}(1 + E_2)$$

   By noting the symmetry, we find:

$$E_0 = 1 + 2E_1 \quad \text{and} \quad E_1 = 1 + E_1$$

   Thus, $E_0 = E_1$.

   For the expected number of returns to zero by time $2n$:

$$E[N_{2n}] = (2n+1)\binom{2n}{n}\frac{1}{4^n} - 1$$

   Using Stirling's approximation, we find that $E[N_{2n}]$ is proportional to $\sqrt{n}$ for large n.

6. **Question 6:** Consider the process of repeatedly flipping a fair coin until the sequence HTT is observed. Calculate the expected time to see the sequence HTT.

   Answer: To calculate the expected time, we set up the following equations:

$$E[HTT] = 1 + 0.5E[HTT|X_0 = T] + 0.5E[HTT|X_0 = H]$$

   Using the independence property:

$$E[HTT|H] = E[HTT] \implies E[HTT] = 2 + E[HTT|T]$$

   Solving these, we find:

$$E[HTT|T] = 1 + 0.5E[HTT|TT] + 0.5E[HTT]$$
$$\implies E[HTT|TT] = 1 + 0.5E[HTT|TT]$$
$$\implies E[HTT|TT] = 2$$

   Thus,

$$E[HTT|T] = 2 + 0.5E[HTT] \implies E[HTT] - 2 = 2 + 0.5E[HTT] \implies E[HTT] = 8$$

7. **Question 7:** Bob and Alice play a game where they each choose a sequence of coin flips. Alice wins if her sequence appears before Bob's sequence in an infinite series of coin flips. Show that the game is unfair if Alice can always choose her sequence after seeing Bob's sequence.

Answer: To demonstrate the unfairness, consider Alice's strategy based on Bob's choice. If Bob chooses HTT, Alice can choose TTH. The probability that TTH appears before HTT is:

$$P = 0.5Q + 0.5(0.5Q + 0.5 \times 0.5Q) = \frac{7}{8}$$

Similarly, for other combinations, Alice can always choose a sequence with a higher probability of appearing first. Thus, the game is unfair as Alice has a higher chance of winning.