Stochastic Processes
1401-2
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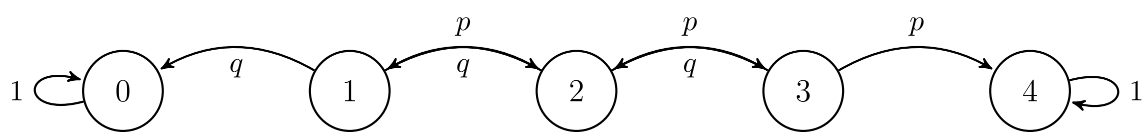
Homework 1

Consider a gambler who starts with \$2 and then on each successive gamble either wins \$1 with probability $p = 0.4$ or loses \$1 with probability $q = 1 - p = 0.6$. The gambler stops playing when either reaching \$0 (ruin) or \$4 (success)



1. **What is the probability that the gambler eventually wins?That is she gets to 4?**
   Let X_n be the amount of money the gambler has after n gambles, and let p_n be the probability that the gambler eventually wins given that $X_n = x$. Then, by the law of total probability, we have the following recursive equation for P_n:

$$P_n = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 4 \\ P_{(x+1)}p + P_{(x-1)}q & \text{if } 0 < x < 4 \end{cases} \tag{1}$$

   Solving this equation with the boundary conditions, we get:

$$P_n = \frac{1 - (q/p)^x}{1 - (q/p)^4} \tag{2}$$

   Plugging in $x = 2$, $p = 0.4$, and $q = 0.6$, we get:

$$P_2 = \frac{1 - (0.6/0.4)^2}{1 - (0.6/0.4)^4} \approx 0.36$$

   Therefore, the probability that the gambler eventually wins starting from \$2 is about **0.36**.

2. **Starting from 2, what is the expected number of steps until the gambler gets to either 0 (ruin) or to 4 (success)**

   Let E_n be the expected number of steps until the gambler reaches either \$0 or \$4 given that $X_n = x$. Then, by the law of total expectation, we have the following recursive equation for E_n:

$$E_x = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = 4 \\ 1 + pE_{(x+1)} + qE_{(x-1)} & \text{if } 0 < x < 4 \end{cases}$$

Unlike in the previous section, this is a heterogeneous equation. A solution to this equation can be written in the form: $E_n = w_n + v_n$. We first find a solution to the homogeneous equation (w_n) and a solution to the heterogeneous equation (v_n).

As before, let $w_n = A\lambda^n$, where A is a constant. We have:

$$pw_{n+1} - w_n + qw_{n-1} = 0$$
$$\implies \lambda^2 - \frac{1}{p}\lambda + \frac{q}{p} = 0$$

Giving solutions:

$$\lambda_{1,2} = (\frac{q}{p}, 1)$$

Our solution to the homogeneous equation is then:

$$w_n = A\lambda_1^n + B\lambda_2^n = A(\frac{q}{p})^n + B$$

For a particular solution (v_n) to the heterogeneous equation we try $v_n = Cn$:

$$pC(n+1) - Cn + qC(n-1) = -1$$
$$\implies pC - qC = -1$$

Giving:

$$C = \frac{-1}{p-q}$$

and our particular solution is

$$v_n = \frac{-n}{p-q}$$

Giving the full solution as

$$E_n = w_n + v_n = A(\frac{q}{p})^n + B - \frac{n}{p-q} \tag{3}$$

Using the boundary conditions $E_0 = 0, E_N = 0$

$$E_0 = A + B = 0 \implies B = -A$$

$$E_N = A(\frac{q}{p})^N + B - \frac{N}{p-q} = 0$$
$$\implies A((\frac{q}{p})^N - 1) = \frac{N}{p-q}$$

$$\implies A = \frac{N}{(p-q)((\frac{q}{p})^N - 1)}$$

Our final solution is then (for $p \neq \frac{1}{2}$)

$$E_n = \frac{N}{(p-q)((\frac{q}{p})^N - 1)}(\frac{q}{p})^n - \frac{N}{(p-q)((\frac{q}{p})^N - 1)} - \frac{n}{p-q} \tag{4}$$

Suppose $p = \frac{1}{2}$. We get a repeated solution $\lambda = 1$, meaning we try the next most complicated solution to the homogeneous equation

$$w_n = (An + B)\lambda^n = (An + B)$$

For the particular solution to the equation

$$E_{n+1} - 2E_n + E_{n-1} = -2$$

we try $v_n = Cn^2$ as the next most complicated particular solution. Substituting this into the heterogeneous equation above

$$C(n+1)^2 - 2Cn^2 + C(n-1)^2 = -1$$

giving

$$C = -1$$

The full equation is then

$$E_n = w_n + v_n = An + B - n^2$$

Applying the boundary conditions

$$E_0 = B = 0$$

$$E_N = AN - N^2 = 0 \implies A = N$$

Giving the final equation for E_n

$$E_n = w_n + v_n = Nn - n^2 \tag{5}$$

In summary

$$E_n = \begin{cases} Nn - n^2 & \text{if } p = 0.5 \\ \frac{N}{(p-q)((\frac{q}{p})^N-1)}(\frac{q}{p})^n - \frac{N}{(p-q)((\frac{q}{p})^N-1)} - \frac{n}{p-q} & \text{if } p \neq 0.5 \text{ and } p \neq 0 \\ n & \text{if } p = 0 \end{cases} \tag{6}$$

Note: The case where $p = 0$ is obtained by multiplying the numerator and denominator of the first to terms in the equation for the $p \neq 0.5$ case by p^N and simplifying.

Solving this equation with the boundary conditions, we get:

$$E_2 = \frac{-p - q - 1}{2pq - 1}$$

Plugging in $p = 0.4$, and $q = 0.6$, we get:

$$E_2 = \frac{-2}{2(0.4)(0.6) - 1} = 3.85$$

Therefore, the expected number of steps until the gambler reaches either \$0 or \$4 starting from \$2 is about **3.85**.