Gauss's Law and its Application

Course Title : Applied Physics

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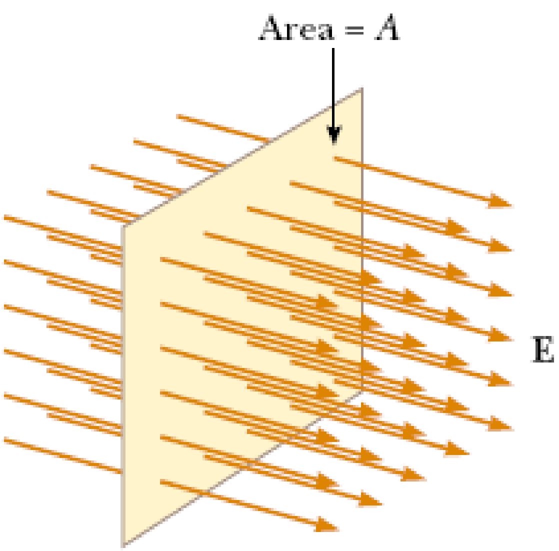
Electric Flux

This product of the magnitude of the electric field E and surface area A perpendicular to the field is called the electric flux φ_E .

$$\Phi_E = EA$$

From the SI units of E and A, we see that Φ_E has units of newton-meters squared per coulomb $(\mathrm{N \cdot m^2/C})$. **Electric flux is proportional to the number of electric field lines penetrating some surface.**



Electric Flux through a Sphere

What is the electric flux through a sphere that has a radius of 1.00 m and carries a charge of $+1.00\ \mu\text{C}$ at its center?

Solution The magnitude of the electric field 1.00 m from this charge is found using Equation 23.9:

$$E = k_e \frac{q}{r^2} = (8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\ \frac{1.00 \times 10^{-6}\ \text{C}}{(1.00\ \text{m})^2}$$

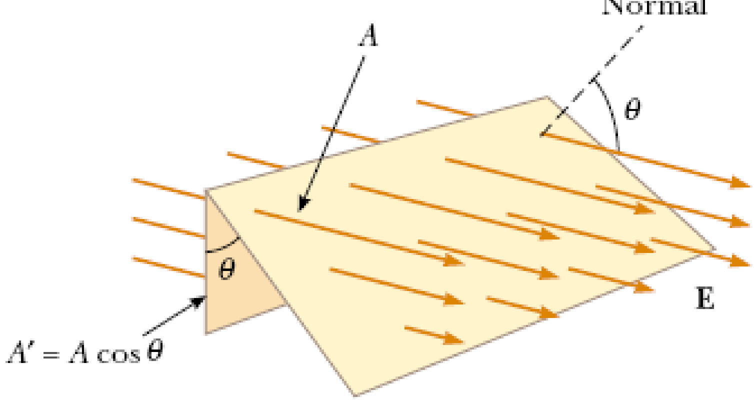
$$= 8.99 \times 10^3\ \text{N/C}$$

The field points radially outward and is therefore everywhere perpendicular to the surface of the sphere. The flux through the sphere (whose surface area $A = 4\pi r^2 = 12.6\ \text{m}^2$) is thus

$$\Phi_E = EA = (8.99 \times 10^3\ \text{N/C})(12.6\ \text{m}^2)$$

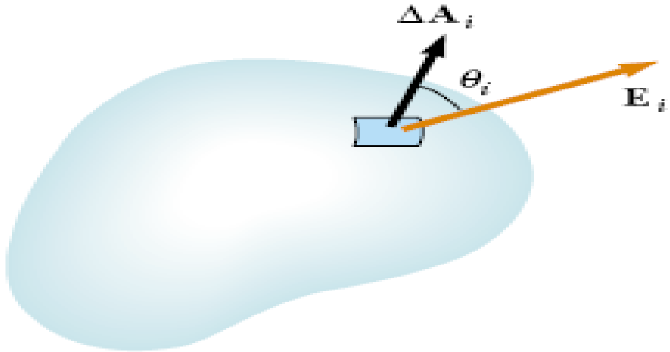
$$= \boxed{1.13 \times 10^5\ \text{N}\cdot\text{m}^2/\text{C}}$$

When the surface is not perpendicular to the field



$$\Phi_E = EA' = EA\cos\theta$$

Flux is maximum when surface is perpendicular to the field i.e. θ=0, and Flux is zero when surface is parallel to the field i.e. θ=90˚.



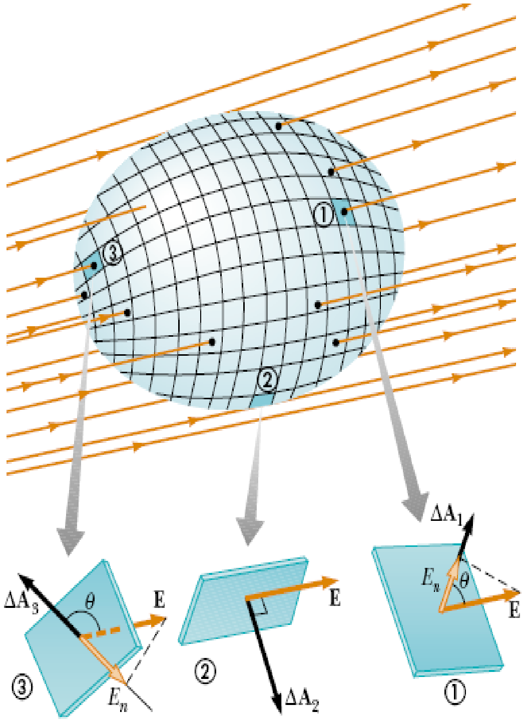
$$\Phi_E = \lim_{\Delta A_i \to 0} \sum \mathbf{E}_i \cdot \Delta \mathbf{A}_i = \int_{\text{surface}} \mathbf{E} \cdot d\mathbf{A}$$

Figure 24.3 A small element of surface area ΔA_i. The electric field makes an angle θ_i with the vector $\Delta \mathbf{A}_i$, defined as being normal to the surface element, and the flux through the element is equal to $E_i\, \Delta A_i \cos\theta_i$.

Flux Through a Closed surface

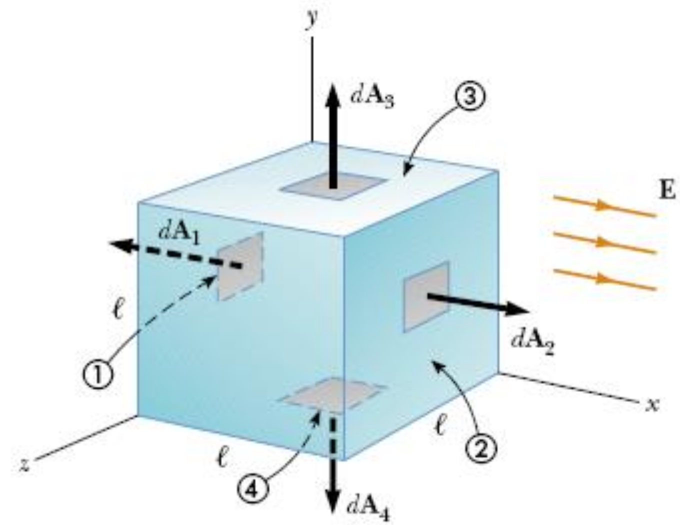
Consider the closed surface in Figure 24.4. The vectors $\Delta\mathbf{A}_i$ point in different directions for the various surface elements, but at each point they are normal to the surface and, by convention, always point outward. At the element labeled ①, the field lines are crossing the surface from the inside to the outside and $\theta < 90°$; hence, the flux $\Delta\Phi_E = \mathbf{E}\cdot\Delta\mathbf{A}_1$ through this element is positive. For element ②, the field lines graze the surface (perpendicular to the vector $\Delta\mathbf{A}_2$); thus, $\theta = 90°$ and the flux is zero. For elements such as ③, where the field lines are crossing the surface from outside to inside, $180° > \theta > 90°$ and the flux is negative because $\cos\theta$ is negative. The *net* flux through the surface is proportional to the net number of lines leaving the surface, where the net number means *the number leaving the surface minus the number entering the surface.* If more lines are leaving than entering, the net flux is positive. If more lines are entering than leaving, the net flux is negative. Using the symbol $\oint$ to represent an integral over a closed surface, we can write the net flux Φ_E through a closed surface as

$$\Phi_E = \oint \mathbf{E}\cdot d\mathbf{A} = \oint E_n\, dA$$



Flux Through a Cube

Consider a uniform electric field $\mathbf{E}$ oriented in the x direction. Find the net electric flux through the surface of a cube of edge length ℓ, oriented as shown in Figure 24.5.



Solution The net flux is the sum of the fluxes through all faces of the cube. First, note that the flux through four of the faces (③, ④, and the unnumbered ones) is zero because $\mathbf{E}$ is perpendicular to $d\mathbf{A}$ on these faces.

The net flux through faces ① and ② is

$$\Phi_E = \int_1 \mathbf{E} \cdot d\mathbf{A} + \int_2 \mathbf{E} \cdot d\mathbf{A}$$

For face ①, $\mathbf{E}$ is constant and directed inward but $d\mathbf{A}_1$ is directed outward ($\theta = 180°$); thus, the flux through this face is

$$\int_1 \mathbf{E} \cdot d\mathbf{A} = \int_1 E(\cos 180°)\, dA = -E \int_1 dA = -EA = -E\ell^2$$

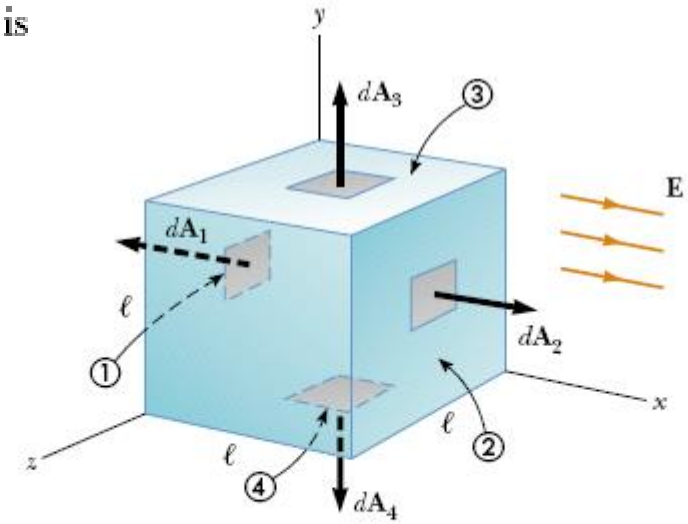
because the area of each face is $A = \ell^2$.

For face ②, $\mathbf{E}$ is constant and outward and in the same direction as $d\mathbf{A}_2$ ($\theta = 0°$); hence, the flux through this face is

$$\int_2 \mathbf{E}\cdot d\mathbf{A} = \int_2 E(\cos 0°)\, dA = E\int_2 dA = +EA = E\ell^2$$

Therefore, the net flux over all six faces is

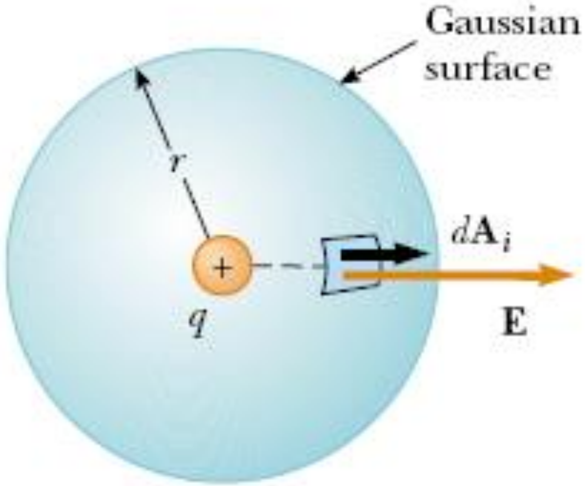
$$\Phi_E = -E\ell^2 + E\ell^2 + 0 + 0 + 0 + 0 = \boxed{0}$$



Gauss's Law

Let us consider a positive point charge q located at the center of a sphere of radius r, as shown in Figure.

$$\mathbf{E} \cdot \Delta \mathbf{A}_i = E \, \Delta A_i$$



the net flux through the Gaussian surface is,

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E \, dA = E \oint dA$$

where we have moved E outside of the integral because, by symmetry, E is constant over the surface and given by $E = k_e q / r^2$. Furthermore, because the surface is spherical, $\oint dA = A = 4\pi r^2$. Hence, the net flux through the gaussian surface is

$$\Phi_E = \frac{k_e q}{r^2} (4\pi r^2) = 4\pi k_e q$$

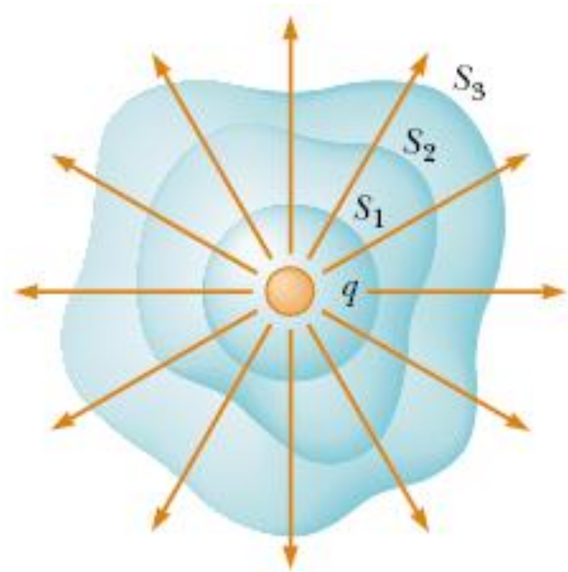
$k_e = 1/4\pi\epsilon_0$, we can write this equation in the form

$$\Phi_E = \frac{q}{\epsilon_0}$$

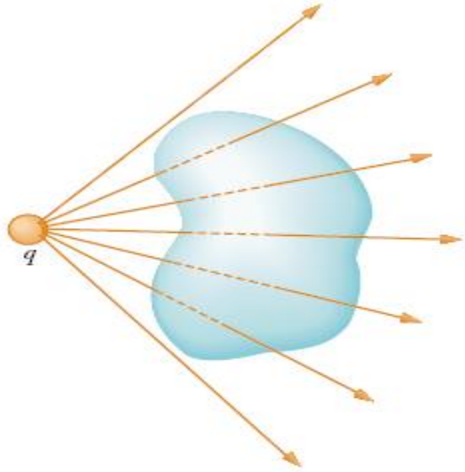
Note from above equation that the net flux through the spherical surface is proportional to he charge inside. The flux is independent of the radius r because the area of the spherical surface is proportional to r^2, whereas the electric field is proportional to $1/r^2$. Thus, in the product of area and electric field, the dependence on r cancels.

Now consider several closed surfaces surrounding a charge q,

Figure shows that the number of lines through S_1 *is equal to* the number of lines through the non spherical surfaces S_2 *and* S_3.
Therefore, we conclude that the net flux through any closed surface surrounding a point charge q is given by and q/ϵ_0 is independent of the shape of that surface.



Now consider a point charge located outside a closed surface of arbitrary shape, as shown in Figure,



the net electric flux through a closed surface that surrounds no charge is zero.

Let us extend these arguments to two generalized cases: (1) that of many point charges and (2) that of a continuous distribution of charge. We once again use the superposition principle, which states that **the electric field due to many charges is the vector sum of the electric fields produced by the individual charges.** Therefore, we can express the flux through any closed surface as

$$\oint \mathbf{E} \cdot d\mathbf{A} = \oint (\mathbf{E}_1 + \mathbf{E}_2 + \cdots) \cdot d\mathbf{A}$$

Gauss's law, which is a generalization of what we have just described, states that the net flux through *any* closed surface is

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{in}}}{\epsilon_0}$$

Flux Due to a Point Charge

A spherical gaussian surface surrounds a point charge q. Describe what happens to the total flux through the surface if

(A) the charge is tripled,

(A) The flux through the surface is tripled because flux is proportional to the amount of charge inside the surface.

(B) the radius of the sphere is doubled,

(B) The flux does not change because all electric field lines from the charge pass through the sphere, regardless of its radius.

(C) the surface is changed to a cube,

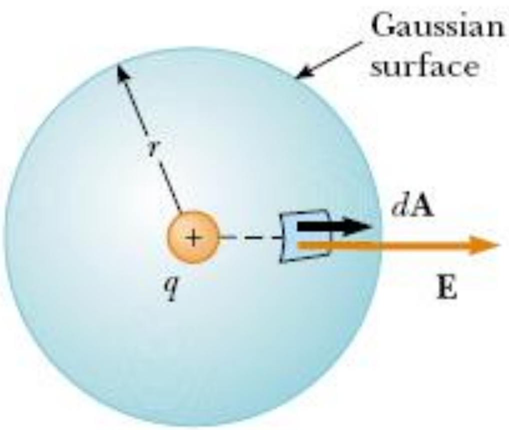
(C) The flux does not change when the shape of the gaussian surface changes because all electric field lines from the charge pass through the surface, regardless of its shape.

Electric Field due to a Point Charge

Starting with Gauss's law, calculate the electric field due to an isolated point charge q.

Solution A single charge represents the simplest possible charge distribution, and we use this familiar case to show how to solve for the electric field with Gauss's law. Figure 24.10 and our discussion of the electric field due to a point charge in Chapter 23 help us to conceptualize the physical situation. Because the space around the single charge has spherical symmetry, we categorize this problem as one in which there is enough symmetry to apply Gauss's law. To analyze any Gauss's law problem, we consider the details of the electric field and choose a gaussian surface that satisfies some or all of the conditions that we have listed above. We choose a spherical gaussian surface of radius r centered on the point charge, as shown in Figure 24.10. The electric field due to a positive point charge is directed radially outward by symmetry and is therefore normal to the surface at every point. Thus, as in condition (2), $\mathbf{E}$ is parallel to $d\mathbf{A}$ at each point. Therefore, $\mathbf{E} \cdot d\mathbf{A} = E\,dA$ and Gauss's law gives

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E\,dA = \frac{q}{\epsilon_0}$$

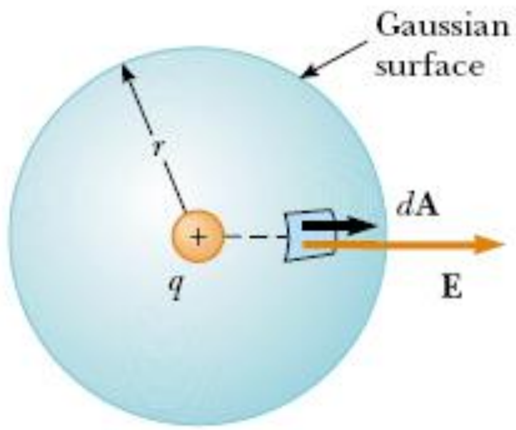


By symmetry, E is constant everywhere on the surface, which satisfies condition (1), so it can be removed from the integral. Therefore,

$$\oint E\,dA = E \oint dA = E(4\pi r^2) = \frac{q}{\epsilon_0}$$

where we have used the fact that the surface area of a sphere is $4\pi r^2$. Now, we solve for the electric field:

$$E = \frac{q}{4\pi\epsilon_0 r^2} = \boxed{k_e \frac{q}{r^2}}$$



What If? **What if the charge in Figure 24.10 were not at the center of the spherical gaussian surface?**

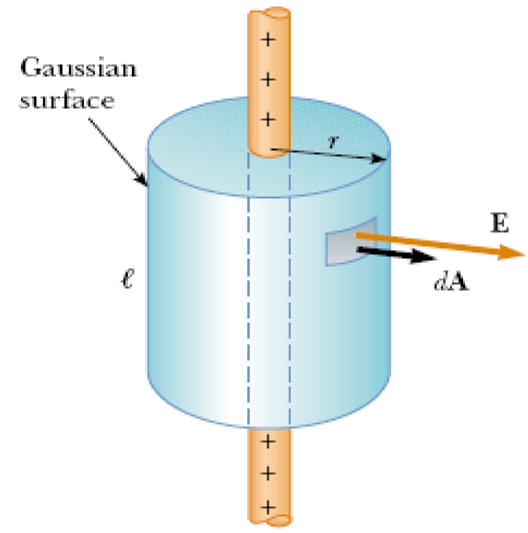
Answer In this case, while Gauss's law would still be valid, the situation would not possess enough symmetry to evaluate the electric field. Because the charge is not at the center, the magnitude of **E** would vary over the surface of the sphere and the vector **E** would not be everywhere perpendicular to the surface.

Electric Field due to a Cylindrically Symmetric Charge Distribution

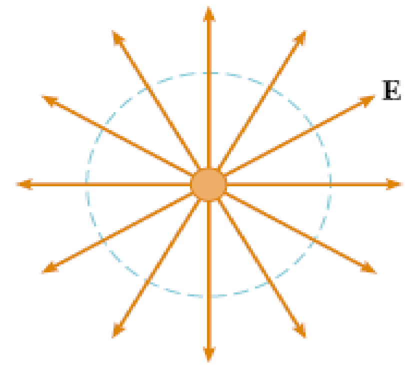
Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ (Fig. 24.14a).

Solution The symmetry of the charge distribution requires that $\mathbf{E}$ be perpendicular to the line charge and directed outward, as shown in Figure 24.14a and b. To reflect the symmetry of the charge distribution, we select a cylindrical gaussian surface of radius r and length ℓ that is coaxial with the line charge. For the curved part of this surface, $\mathbf{E}$ is constant in magnitude and perpendicular to the surface at each point—satisfaction of conditions (1) and (2). Furthermore, the flux through the ends of the gaussian cylinder is zero because $\mathbf{E}$ is parallel to these surfaces—the first application we have seen of condition (3).

We take the surface integral in Gauss's law over the entire gaussian surface. Because of the zero value of $\mathbf{E} \cdot d\mathbf{A}$ for the ends of the cylinder, however, we can restrict our attention to only the curved surface of the cylinder.



(a)



(b)

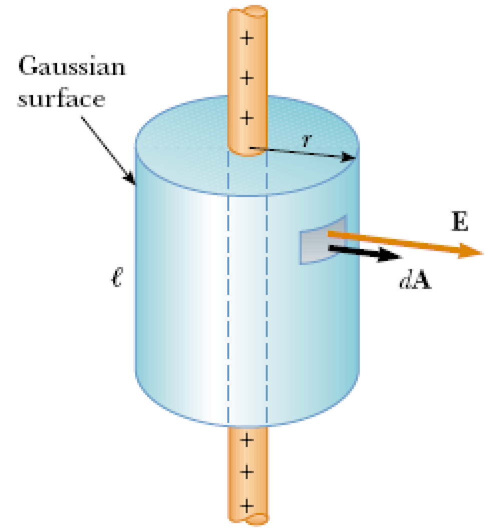
The total charge inside our gaussian surface is $\lambda\ell$. Applying Gauss's law and conditions (1) and (2), we find that for the curved surface

$$\Phi_E = \oint \mathbf{E}\cdot d\mathbf{A} = E\oint dA = EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\lambda\ell}{\epsilon_0}$$

The area of the curved surface is $A = 2\pi r\ell$; therefore,

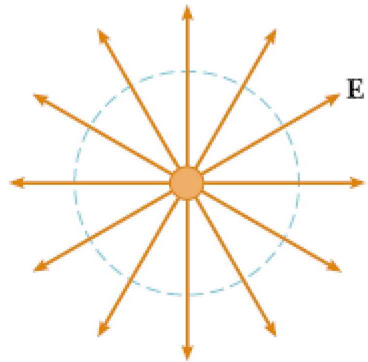
$$E(2\pi r\ell) = \frac{\lambda\ell}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = \boxed{2k_e\frac{\lambda}{r}} \qquad (24.7)$$



(a)

Thus, we see that the electric field due to a cylindrically symmetric charge distribution varies as 1/r,
whereas the field external to a spherically symmetric charge distribution varies as $1/r^2$.

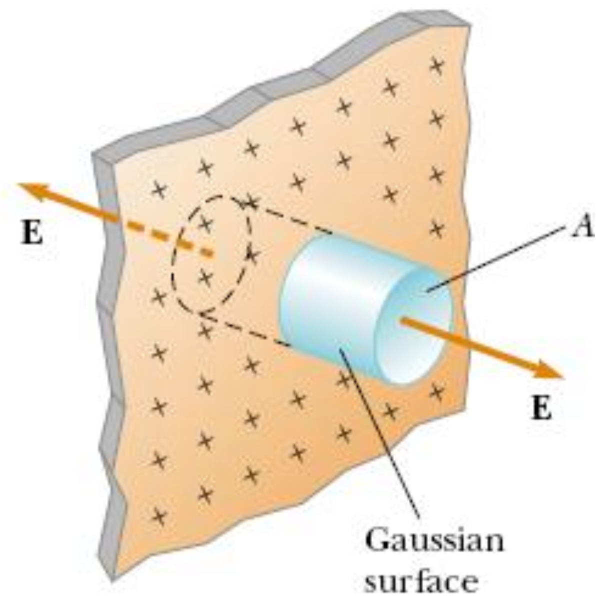


(b)

Electric Field due to a Plane of Charge

Find the electric field due to an infinite plane of positive charge with uniform surface charge density σ.

Solution By symmetry, **E** must be perpendicular to the plane and must have the same magnitude at all points equidistant from the plane. The fact that the direction of **E** is away from positive charges indicates that the direction of **E** on one side of the plane must be opposite its direction on the other side, as shown in Figure 24.15. A gaussian surface that reflects the symmetry is a small cylinder whose axis is perpendicular to the plane and whose ends each have an area A and are equidistant from the plane. Because **E** is parallel to the curved surface—and, therefore, perpendicular to $d\mathbf{A}$ everywhere on the surface—condition (3) is satisfied and there is no contribution to the surface integral from this surface. For the flat ends of the cylinder, conditions (1) and (2) are satisfied. The flux through each end of the cylinder is EA; hence, the total flux through the entire gaussian surface is just that through the ends, $\Phi_E = 2EA$.



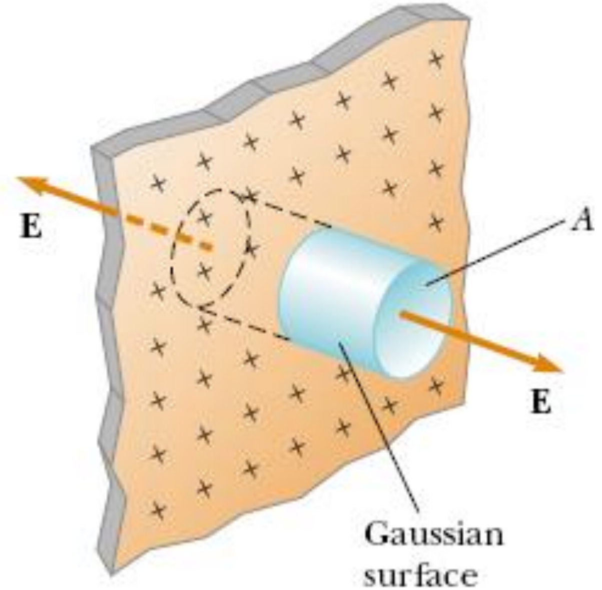
Noting that the total charge inside the surface is $q_{in} = \sigma A$, we use Gauss's law and find that the total flux through the gaussian surface is

$$\Phi_E = 2EA = \frac{q_{in}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

leading to

$$\boxed{E = \frac{\sigma}{2\epsilon_0}} \qquad (24.8)$$

Because the distance from each flat end of the cylinder to the plane does not appear in Equation 24.8, we conclude that $E = \sigma/2\epsilon_0$ at *any* distance from the plane. That is, the field is uniform everywhere.



What if two sheets are placed together parallel to each other?

Ans: the field made by each of the surface is same so they add up to give the magnitude $\frac{\delta}{\epsilon_0}$

A spherically symmetric Charge Distribution

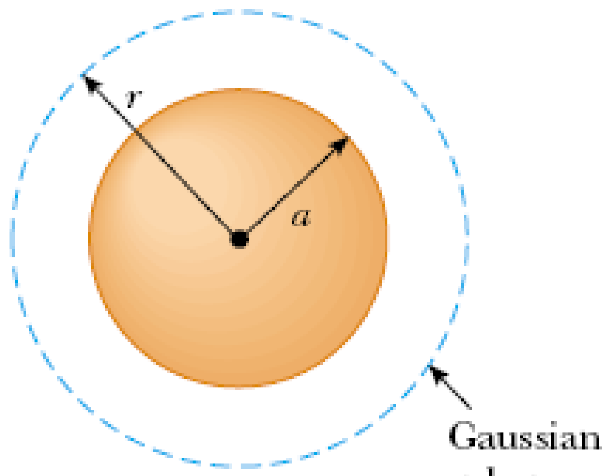
An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q (Fig. 24.11).

(A) Calculate the magnitude of the electric field at a point outside the sphere.

Solution Because the charge distribution is spherically symmetric, we again select a spherical gaussian surface of radius r, concentric with the sphere, as shown in Figure 24.11a. For this choice, conditions (1) and (2) are satisfied, as they were for the point charge in Example 24.4. Following the line of reasoning given in Example 24.4, we find that

$$(1) \qquad E = k_e \frac{Q}{r^2} \qquad (\text{for } r > a)$$

Note that this result is identical to the one we obtained for a point charge. Therefore, we conclude that, **for a uniformly charged sphere, the field in the region external to the sphere is *equivalent* to that of a point charge located at the center of the sphere.**



(a)

(B) Find the magnitude of the electric field at a point inside the sphere.

Solution In this case we select a spherical gaussian surface having radius $r < a$, concentric with the insulating sphere (Fig. 24.11b). Let us denote the volume of this smaller sphere by V'. To apply Gauss's law in this situation, it is important to recognize that the charge q_{in} within the gaussian surface of volume V' is less than Q. To calculate q_{in}, we use the fact that $q_{in} = \rho V'$:

$$q_{in} = \rho V' = \rho\left(\tfrac{4}{3}\pi r^3\right)$$

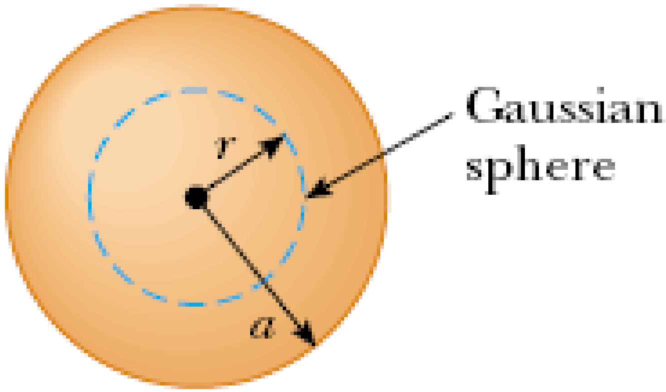
By symmetry, the magnitude of the electric field is constant everywhere on the spherical gaussian surface and is normal to the surface at each point—both conditions

(1) and (2) are satisfied. Therefore, Gauss's law in the region $r < a$ gives

$$\oint E\,dA = E\oint dA = E\,(4\pi r^2) = \frac{q_{in}}{\epsilon_0}$$

Solving for E gives

$$E = \frac{q_{in}}{4\pi\epsilon_0 r^2} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{4\pi\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0}\,r$$

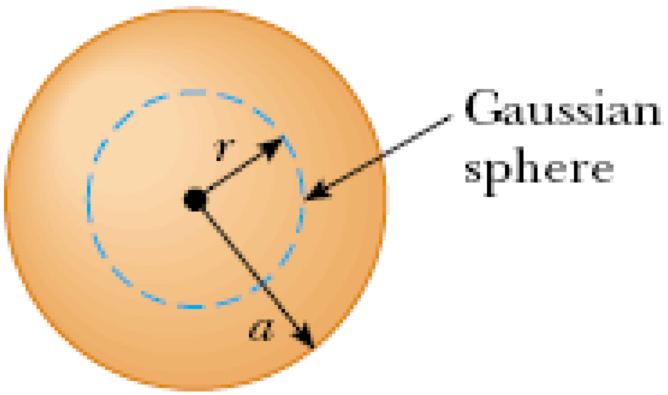


(b)

Because $\rho = Q/\frac{4}{3}\pi a^3$ by definition and because $k_e = 1/4\pi\epsilon_0$, this expression for E can be written as

$$(2) \qquad E = \frac{Qr}{4\pi\epsilon_0 a^3} = \boxed{k_e \frac{Q}{a^3} r} \qquad (\text{for } r < a)$$

Note that this result for E differs from the one we obtained in part (A). It shows that $E \rightarrow 0$ as $r \rightarrow 0$. Therefore, the result eliminates the problem that would exist at $r = 0$ if E varied as $1/r^2$ inside the sphere as it does outside the sphere. That is, if $E \propto 1/r^2$ for $r < a$, the field would be infinite at $r = 0$, which is physically impossible.



(b)

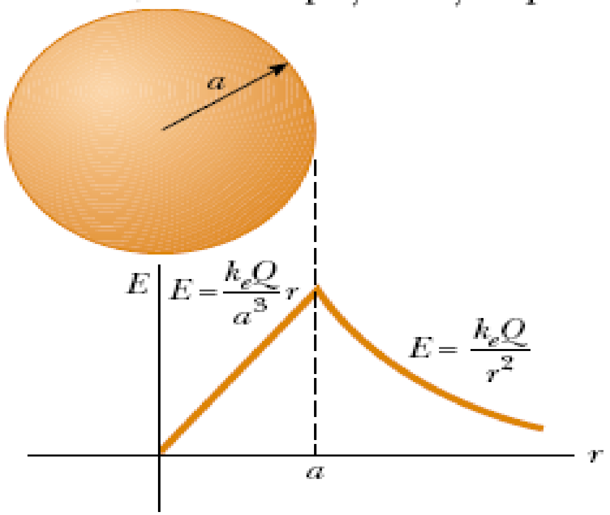


Figure 24.12 (Example 24.5) A plot of E versus r for a uniformly charged insulating sphere. The electric field inside the sphere ($r < a$) varies linearly with r. The field outside the sphere ($r > a$) is the same as that of a point charge Q located at $r = 0$.

What If? Suppose we approach the radial position $r = a$ from inside the sphere and from outside. Do we measure the same value of the electric field from both directions?

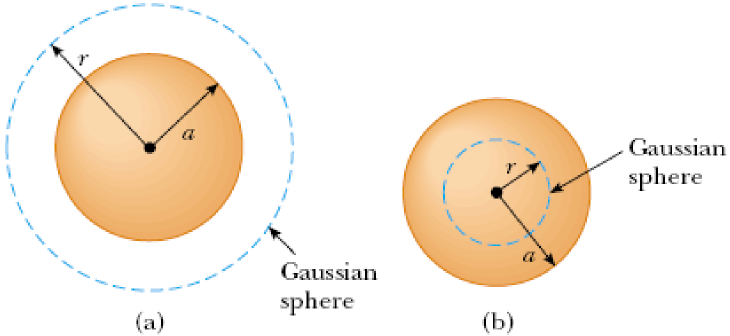
Answer From Equation (1), we see that the field approaches a value from the outside given by

$$E = \lim_{r \to a} \left(k_e \frac{Q}{r^2} \right) = k_e \frac{Q}{a^2}$$

From the inside, Equation (2) gives us

$$E = \lim_{r \to a} \left(k_e \frac{Q}{a^3} r \right) = k_e \frac{Q}{a^3} a = k_e \frac{Q}{a^2}$$

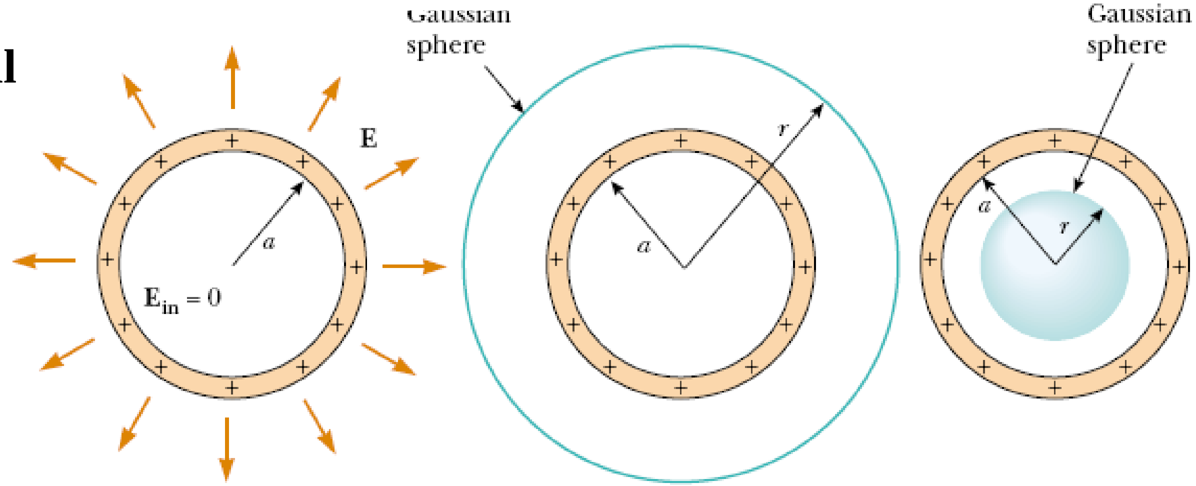
Thus, the value of the field is the same as we approach the surface from both directions. A plot of E versus r is shown in Figure 24.12. Note that the magnitude of the field is continuous, but the derivative of the field magnitude is not.



Electric Field due to Thin Spherical Shell

A thin spherical shell of radius a has a total charge Q distributed uniformly over its surface (Fig. 24.13a). Find the electric field at points

(A) Outside the Shell



(a) (b) (c)

Solution

(A) The calculation for the field outside the shell is identical to that for the solid sphere shown in Example 24.5a. If we construct a spherical gaussian surface of radius $r > a$ concentric with the shell (Fig. 24.13b), the charge inside this surface is Q. Therefore, the field at a point outside the shell is equivalent to that due to a point charge Q located at the center:

$$E = \boxed{k_e \frac{Q}{r^2}} \qquad (\text{for } r > a)$$

(B) inside the shell.

(B) The electric field inside the spherical shell is zero. This follows from Gauss's law applied to a spherical surface of radius $r < a$ concentric with the shell (Fig. 24.13c). Because of the spherical symmetry of the charge distribution and because the net charge inside the surface is zero—satisfaction of conditions (1) and (2) again—application of Gauss's law shows that $E = 0$ in the region $r < a$. We obtain the same results using Equation 23.11 and integrating over the charge distribution. This calculation is rather complicated. Gauss's law allows us to determine these results in a much simpler way.

$$\mathbf{E} = k_e \lim_{\Delta q_i \to 0} \sum_i \frac{\Delta q_i}{r_i^{\,2}} \hat{\mathbf{r}}_i = k_e \int \frac{dq}{r^2} \hat{\mathbf{r}} \qquad (23.11)$$