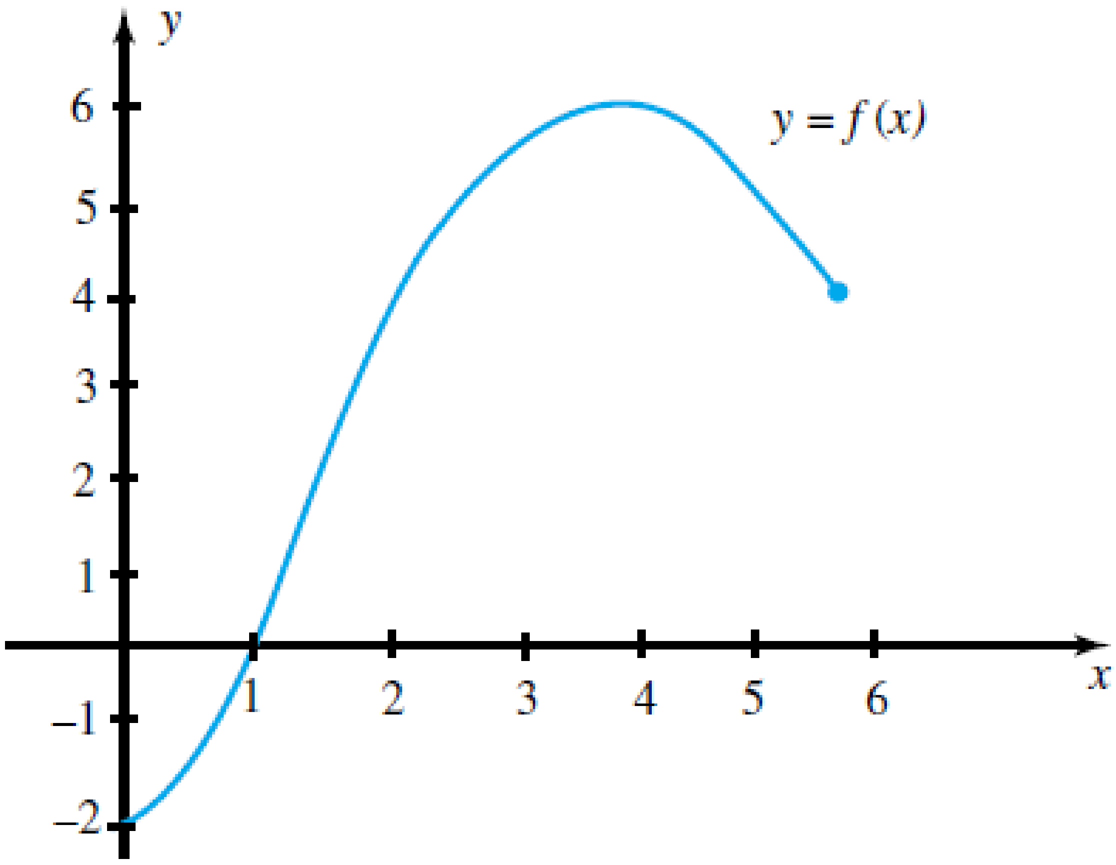


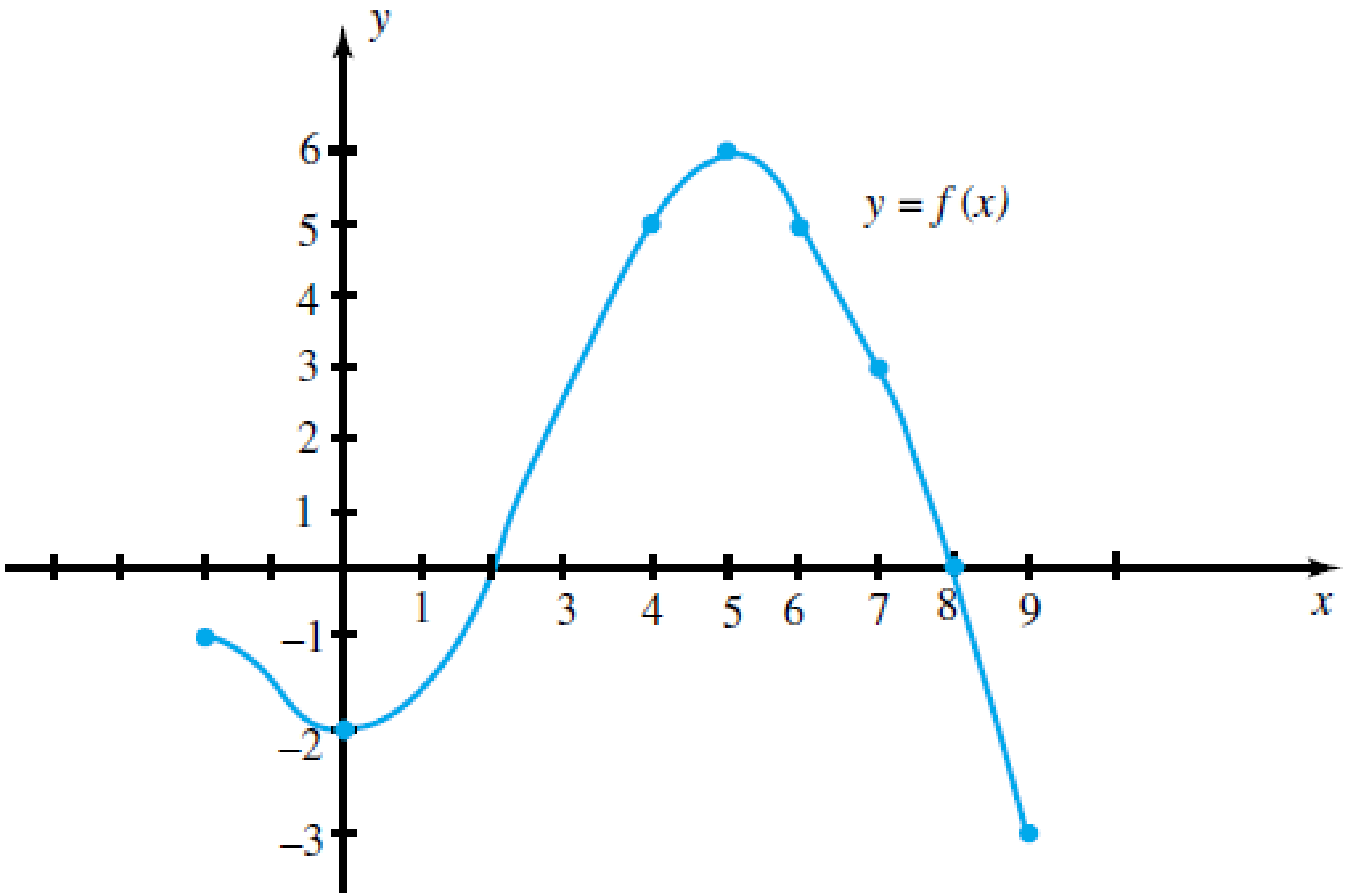
Self Practice:

a. Find the value of $f(0)$.

b. Find the value of x for which (i) $f(x) = 3$ and (ii) $f(x) = 0$.

c. Find the domain of f.

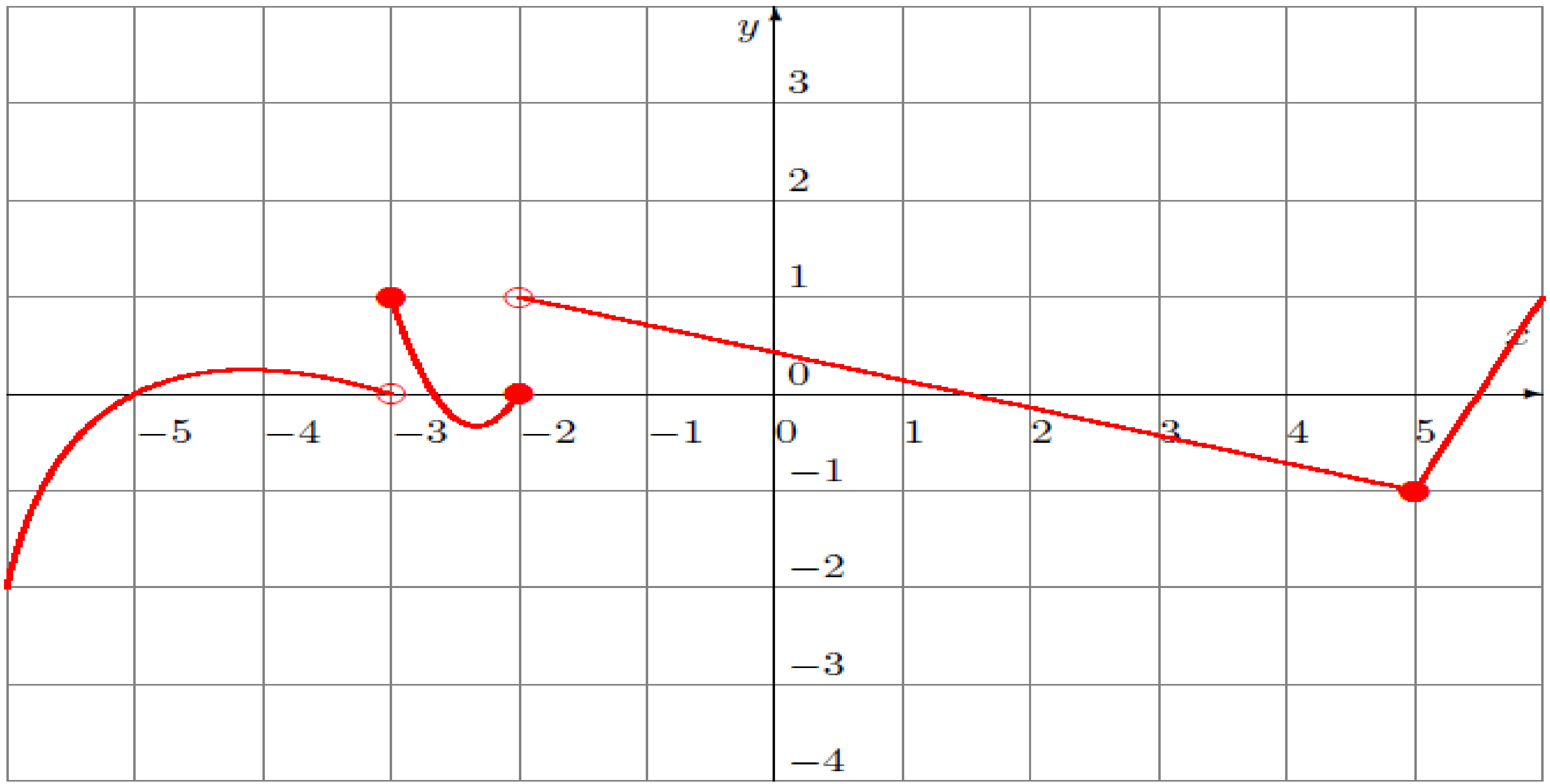
d. Find the range of f.



Self Practice:

a. Find the value of $f(7)$.
b. Find the values of x corresponding to the point on the graph of f located at a height of 5 units from the x-axis.
c. Find the points on the x-axis at which the graph of f crosses it. What are the values of $f(x)$ at those points?
d. Find the domain and range of f.

Consider the following function defined by its graph:



P
R
A
C
T
I
C
E

Find the following limits:

$\lim\limits_{x \to -2^-} f(x)$ b) $\lim\limits_{x \to -2^+} f(x)$ c) $\lim\limits_{x \to -2} f(x)$ d) $\lim\limits_{x \to -3} f(x)$ e) $\lim\limits_{x \to 5} f(x)$

Practice:

1. Consider the following piece-wise defined function: $f(x) = \begin{cases} 1 - 3x + 2x^2 & \text{if } x < -2 \\ 25 + 3x - x^2 & \text{if } x > -2 \end{cases}$

Find the following limits:

$a)\ \lim_{x\to -2^-} f(x)$ $\quad b)\ \lim_{x\to -2^+} f(x)$ $\quad c)\ \lim_{x\to -2} f(x)$ $\quad d)\ \lim_{x\to -3} f(x)$ $\quad e)\ \lim_{x\to -1} f(x)$

5. Consider the following piece-wise defined function: $f(x) = \begin{cases} 3 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 8 - x - x^2 & \text{if } x > 1 \end{cases}$

Find the following limits:

$a)\ \lim_{x\to 1^-} f(x)$ $\quad b)\ \lim_{x\to 1^+} f(x)$ $\quad c)\ \lim_{x\to 1} f(x)$ $\quad d)\ \lim_{x\to 0} f(x)$ $\quad e)\ \lim_{x\to 2} f(x)$

$$\lim_{x\to 3}\frac{x-3}{|x-3|}$$

Practice:

5. Analyse the continuity of the following function. Graph the function.

$$f(x)=\begin{cases} x^2-2x+1 & , \quad x\le 0 \\ 1 & , \quad 0<x<4 \\ \sqrt{x}, & x\ge 4 \end{cases}$$

9. Analyse the continuity of the function. Graph the function.

$$f(x)=\begin{cases} \dfrac{x|x-1|}{x-1} & , x\ne 1 \\ 0 & , x=1 \end{cases}$$

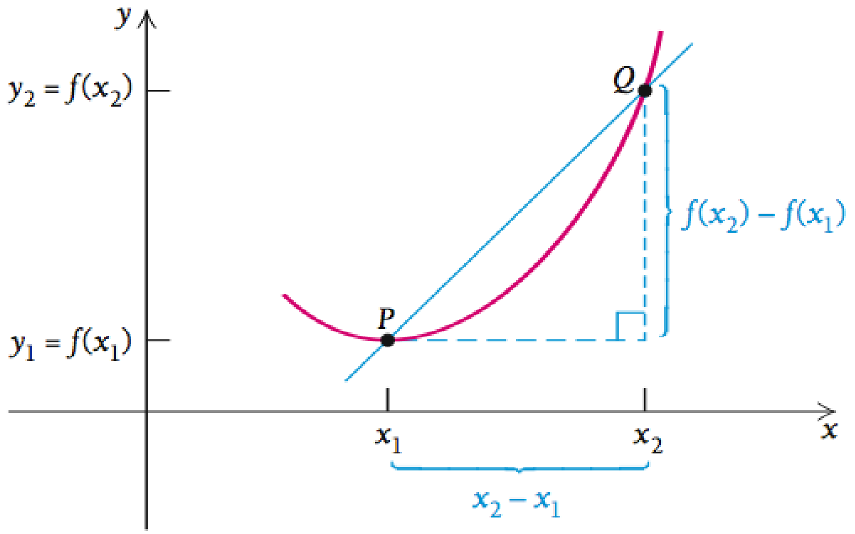
Chap-2

THE DERIVATIVE

Average Rates of Change

Definition:

The **average rate of change of y with respect to x**, as x changes from x_1 to x_2, is the ratio of the change in output to the change in input: $\dfrac{y_2 - y_1}{x_2 - x_1}$, where $x_2 \neq x_1$.



$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

Examining the graph of the function, the average rate of change *and* the slope of the line from P(x_1, y_1) to Q(x_2, y_2) are the same. The line through P and Q, is called a **secant line**.

2.1.1 DEFINITION Suppose that x_0 is in the domain of the function f. The ***tangent line*** to the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the line with equation

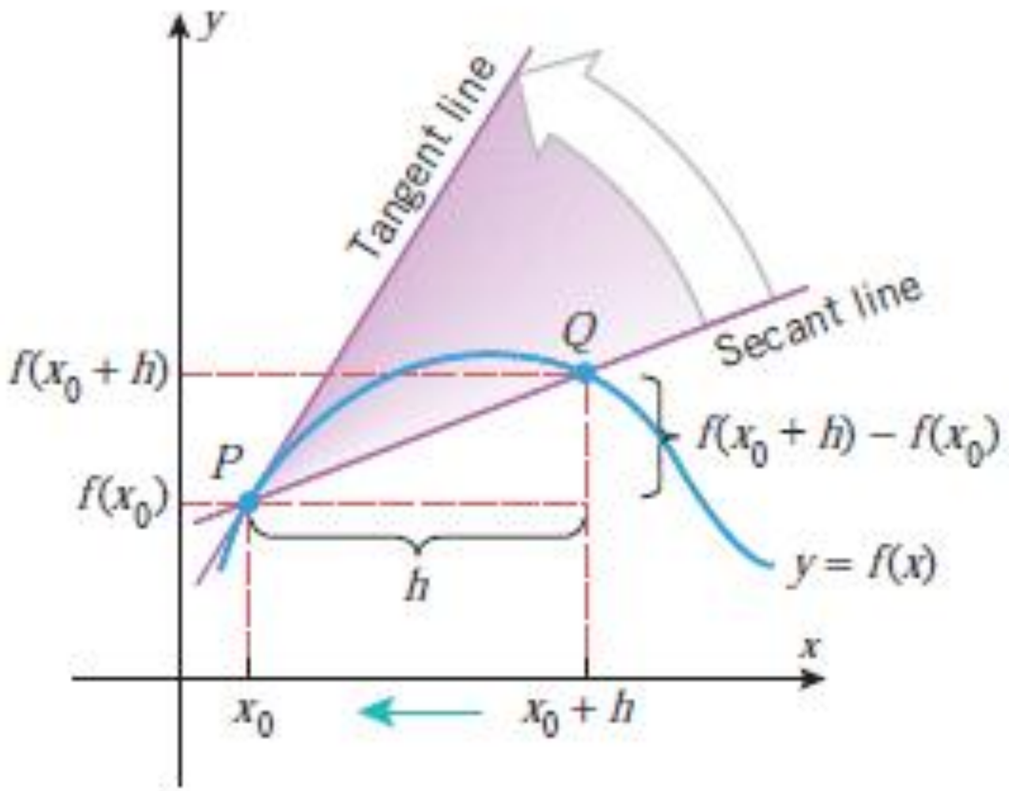
$$y - f(x_0) = m_{\tan}(x - x_0)$$

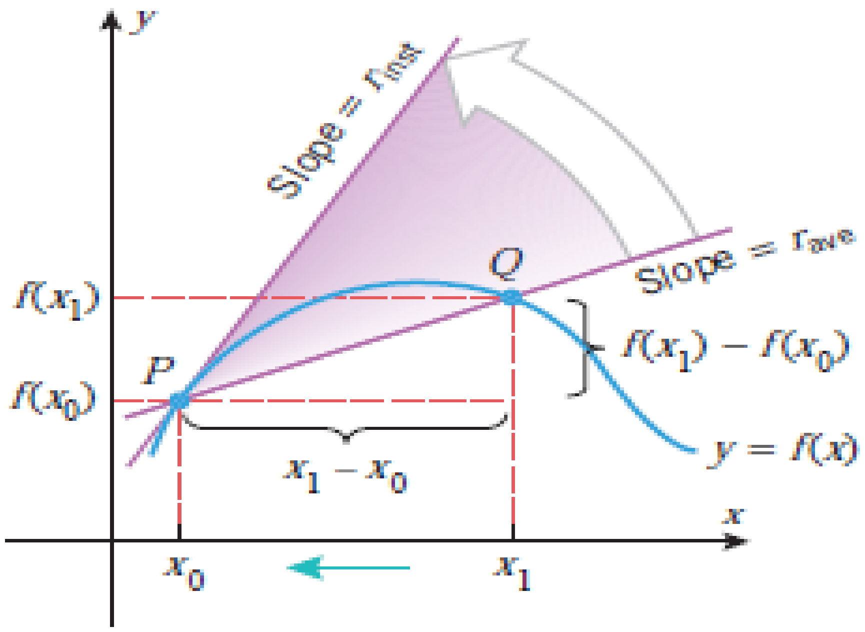
where

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

provided the limit exists. For simplicity, we will also call this the tangent line to $y = f(x)$ at x_0.

Slope of the Secant Line





► **Figure 2.1.11**

If desired, we can let $h = x_1 - x_0$, and rewrite (8) and (9) as

$$\boxed{r_{\text{ave}} = \frac{f(x_0 + h) - f(x_0)}{h}} \tag{10}$$

$$\boxed{r_{\text{inst}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}} \tag{11}$$

Slope of secant / tangent $\longrightarrow$ Derivative

The **average rate of change** of f over the interval $[x, x + h]$ or **slope of the secant line** to the graph of f through the points $(x, f(x))$ and $(x + h, f(x + h))$ is

$$\frac{f(x+h) - f(x)}{h} \tag{9}$$

The **instantaneous rate of change** of f at x or **slope of the tangent line** to the graph of f at $(x, f(x))$ is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{10}$$

2.2.1 DEFINITION The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

is called the ***derivative of f with respect to x***. The domain of f' consists of all x in the domain of f for which the limit exists.

The term "derivative" is used because the function f' is *derived* from the function f by a limiting process.

Constant Rule : $\frac{d}{dx}[c] = 0$

Power Rule : $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$

Constant Multiple Rule : $\frac{d}{dx}[c \cdot u] = c \cdot \frac{du}{dx}$

Sum and Difference Rule : $\frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx}$

Product Rule : $\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

Quotient Rule : $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

Practice:

EXAMPLE If $f(x) = \sqrt{x}\, g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

Example: Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$. Find $h'(2)$.

(a) $h(x) = 5f(x) - 4g(x)$ (b) $h(x) = f(x)g(x)$

(c) $h(x) = \dfrac{f(x)}{g(x)}$ (d) $h(x) = \dfrac{g(x)}{1 + f(x)}$

Example: If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

Example: Find y' if $\sin(x + y) = y^2 \cos x$.

Table 2.6.1

GENERALIZED DERIVATIVE FORMULAS

$$\frac{d}{dx}[u^r] = ru^{r-1}\frac{du}{dx}$$

$$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx} \qquad \frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = -\csc^2 u\frac{du}{dx}$$

$$\frac{d}{dx}[\sec u] = \sec u \tan u\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = -\csc u \cot u\frac{du}{dx}$$

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}[\cos^{-1} u] = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad \frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

OTHER DERIVATIVE NOTATIONS

$$f'(x) = \frac{d}{dx}[f(x)] \quad \text{or} \quad f'(x) = D_x[f(x)]$$

$$f'(x) = y'(x) \quad \text{or} \quad f'(x) = \frac{dy}{dx}$$

$$f'(x_0) = \frac{d}{dx}[f(x)]\bigg|_{x=x_0}, \quad f'(x_0) = D_x[f(x)]\big|_{x=x_0}, \quad f'(x_0) = y'(x_0), \quad f'(x_0) = \frac{dy}{dx}\bigg|_{x=x_0}$$

$$\boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

▶ **Example 6** The graph of $y = |x|$ in Figure 2.2.10 has a corner at $x = 0$, which implies that $f(x) = |x|$ is not differentiable at $x = 0$.

(a) Prove that $f(x) = |x|$ is not differentiable at $x = 0$ by showing that the limit in Definition 2.2.2 does not exist at $x = 0$.

(b) Find a formula for $f'(x)$.

2.2.3 THEOREM *If a function f is differentiable at x_0, then f is continuous at x_0.*

If $y = f(x)$, then we define the ***average rate of change of y with respect to x over the interval*** $[x_0, x_1]$ to be

$$r_{ave} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{8}$$

and we define the ***instantaneous rate of change of y with respect to x at*** x_0 to be

$$r_{inst} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Exercise 2.1 (class work)

11–14 A function $y = f(x)$ and values of x_0 and x_1 are given.

(a) Find the average rate of change of y with respect to x over the interval $[x_0, x_1]$.

(b) Find the instantaneous rate of change of y with respect to x at the specified value of x_0.

(c) Find the instantaneous rate of change of y with respect to x at an arbitrary value of x_0.

(d) The average rate of change in part (a) is the slope of a certain secant line, and the instantaneous rate of change in part (b) is the slope of a certain tangent line. Sketch the graph of $y = f(x)$ together with those two lines. ■

11. $y = 2x^2;\ x_0 = 0,\ x_1 = 1$ **12.** $y = x^3;\ x_0 = 1,\ x_1 = 2$

13. $y = 1/x;\ x_0 = 2,\ x_1 = 3$ **14.** $y = 1/x^2;\ x_0 = 1,\ x_1 = 2$

Exercise 2.2 (Home work)

9–14 Use Definition 2.2.1 to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = a$. ■

9. $f(x) = 2x^2;\ a = 1$

10. $f(x) = 1/x^2;\ a = -1$

11. $f(x) = x^3;\ a = 0$

12. $f(x) = 2x^3 + 1;\ a = -1$

13. $f(x) = \sqrt{x+1};\ a = 8$

14. $f(x) = \sqrt{2x+1};\ a = 4$

15–20 Use Formula (12) to find dy/dx. ■

15. $y = \dfrac{1}{x}$

16. $y = \dfrac{1}{x+1}$

17. $y = x^2 - x$

18. $y = x^4$

19. $y = \dfrac{1}{\sqrt{x}}$

20. $y = \dfrac{1}{\sqrt{x-1}}$

Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable** on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } a$$

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } b$$

Example 2: Given $f(x) = \begin{cases} x^2, if \quad x < 1 \\ 4 - 3x, if \quad x \geq 1 \end{cases}$. Is f continuous at $x = 1$? Is f differentiable at $x = 1$? Sketch the graph of f.

LHD= $\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$

RHD = $\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$

Solution:

Given $f(x)=\begin{cases} x^2, \text{if} \quad x<1 \\ 4-3x, \text{if} \quad x\geq 1 \end{cases}$ Is f continuous at $x=1$?

(i) $f(1)=4-3(1)=1$

(ii) $\left.\begin{cases} \lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} x^2=1^2=1 \\ \lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+} (4-3x)=4-3(1)=1 \end{cases}\right\} \Rightarrow \lim\limits_{x\to 1} f(x)=1$

(iii) $\lim\limits_{x\to 1} f(x)=1=f(1) \Rightarrow$ yes, f is continuous at $x=1$.

$\boxed{\text{Is } f \text{ differentiable at } x=1?}$

$$LHD=\lim_{b\to 1^-}\frac{f(b)-f(1)}{b-1}=\lim_{b\to 1^-}\frac{b^2-1}{b-1}=\lim_{b\to 1^-}\frac{\cancel{(b-1)}^{1}(b+1)}{\cancel{b-1}_{1}}= \ 2$$

$$RHD=\lim_{b\to 1^+}\frac{f(b)-f(1)}{b-1}=\lim_{b\to 1^+}\frac{(4-3b)-1}{b-1}=\lim_{b\to 1^+}\frac{-3b+3}{b-1}= \ -3$$

Exercise 2.2

47. Show that

$$f(x) = \begin{cases} x^2 + 1, & x \le 1 \\ 2x, & x > 1 \end{cases}$$

is continuous and differentiable at $x = 1$. Sketch the graph of f.

48. Show that

$$f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ x + 2, & x > 1 \end{cases}$$

is continuous but not differentiable at $x = 1$. Sketch the graph of f.

3.6 L'HÔPITAL'S RULE; INDETERMINATE FORMS

$$\frac{0}{0} \quad \frac{\infty}{\infty} \quad \infty - \infty$$

$$0 \cdot \infty \quad 1^{\infty} \quad 0^{0} \quad \infty^{0}$$

3.6.1 THEOREM (L'Hôpital's Rule for Form 0/0) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = 0 \quad \textit{and} \quad \lim_{x \to a} g(x) = 0$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

▸ **Example 2** In each part confirm that the limit is an indeterminate form of type $0/0$, and evaluate it using L'Hôpital's rule.

$$\text{(a)}\ \lim_{x\to 0}\frac{\sin 2x}{x} \qquad \text{(b)}\ \lim_{x\to \pi/2}\frac{1-\sin x}{\cos x} \qquad \text{(c)}\ \lim_{x\to 0}\frac{e^x-1}{x^3}$$

$$\text{(d)}\ \lim_{x\to 0^-}\frac{\tan x}{x^2} \qquad \text{(e)}\ \lim_{x\to 0}\frac{1-\cos x}{x^2} \qquad \text{(f)}\ \lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}$$

▸ **Example 3** In each part confirm that the limit is an indeterminate form of type ∞/∞ and apply L'Hôpital's rule.

$$\text{(a)}\ \lim_{x\to +\infty}\frac{x}{e^x} \qquad \text{(b)}\ \lim_{x\to 0^+}\frac{\ln x}{\csc x}$$

$$\boxed{\lim_{x\to +\infty}\frac{x^n}{e^x}=0} \quad \text{and} \quad \boxed{\lim_{x\to +\infty}\frac{e^x}{x^n}=+\infty}$$

$$\frac{0}{0} \quad \frac{\infty}{\infty} \quad \infty - \infty$$

$$0 \cdot \infty \quad 1^{\infty} \quad 0^{0} \quad \infty^{0}$$

▸ **Example 4** Evaluate

(a) $\lim_{x \to 0^+} x \ln x$ (b) $\lim_{x \to \pi/4} (1 - \tan x) \sec 2x$

▸ **Example 5** Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

▸ **Example 6** Find $\lim_{x \to 0} (1 + \sin x)^{1/x}$.

Exercise : 3.6

9. $\lim_{\theta \to 0} \dfrac{\tan \theta}{\theta}$

10. $\lim_{t \to 0} \dfrac{te^t}{1 - e^t}$

11. $\lim_{x \to \pi^+} \dfrac{\sin x}{x - \pi}$

12. $\lim_{x \to 0^+} \dfrac{\sin x}{x^2}$

13. $\lim_{x \to +\infty} \dfrac{\ln x}{x}$

14. $\lim_{x \to +\infty} \dfrac{e^{3x}}{x^2}$

15. $\lim_{x \to 0^+} \dfrac{\cot x}{\ln x}$

16. $\lim_{x \to 0^+} \dfrac{1 - \ln x}{e^{1/x}}$

17. $\lim_{x \to +\infty} \dfrac{x^{100}}{e^x}$

18. $\lim_{x \to 0^+} \dfrac{\ln(\sin x)}{\ln(\tan x)}$

19. $\lim_{x \to 0} \dfrac{\sin^{-1} 2x}{x}$

20. $\lim_{x \to 0} \dfrac{x - \tan^{-1} x}{x^3}$

21. $\lim_{x \to +\infty} xe^{-x}$

22. $\lim_{x \to \pi^-} (x - \pi) \tan \frac{1}{2}x$

23. $\lim_{x \to +\infty} x \sin \dfrac{\pi}{x}$

24. $\lim_{x \to 0^+} \tan x \ln x$

25. $\lim_{x \to \pi/2^-} \sec 3x \cos 5x$

26. $\lim_{x \to \pi} (x - \pi) \cot x$

27. $\lim_{x \to +\infty} (1 - 3/x)^x$

28. $\lim_{x \to 0} (1 + 2x)^{-3/x}$

Section:A

QUIZ-1

1- Solve the inequality (any one)

a) $x^2 - 5x > 6$ b) $\frac{3}{|2x-1|} \geq 4$

2- Find domain and range of the function $f(x) = \sqrt{3-x}$

3- Complete the table then Evaluate the following limit

$$f(x) = \frac{\sin^{-1} 2x}{x}; \lim_{x \to 0} f(x)$$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

4- $a)\ \lim_{x \to 2-} \frac{x}{x^2-4}$ $b)\ \lim_{x \to 2+} \frac{x}{x^2-4}$ $c)\ \lim_{x \to 2} \frac{x}{x^2-4}$

5- Consider $f(x) = \begin{cases} 2x+3, & if \quad x \leq 4 \\ 7 + \frac{16}{x}, & if \quad x > 4 \end{cases}$ Find $\lim_{x \to 4} f(x)$

Skecth the graph , Discuss type of discountinuity

QUIZ-1

Section:C

1- Solve the inequality

a) $x^2 - 9x + 20 \leq 0$

2- Find domain and range of the function $f(x) = 2 + \sqrt{x^2 - 4}$

3- Consider $f(x) = \begin{cases} 2x - 3, & if \quad x \leq 2 \\ x^2, & if \quad x > 2 \end{cases}$ Find $\lim_{x \to 2} f(x)$

Is f(x) continuous at x=2 if not then write discontinuity type

also write interval on which f(x) is continuous , Skecth the graph

Section:F

1- Check continuity at x=0 , write type of discontinuity if exist

$$f(x) = \begin{cases} 2x \mp 1, & if \quad x \leq 0 \\ x^2 - x, & if \quad x > 0 \end{cases}$$

write interval on which f(x) is continuous, Skecth the graph

2- Solve the inequality (any one)

a) $x^2 - 3x < 10$ b) $\frac{3x+1}{x-2} < 1$

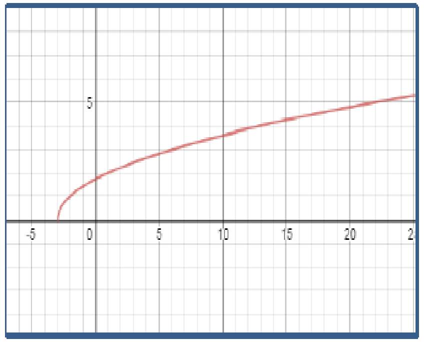
3- a) Find domain and range of the function $f(x) = \sqrt{x^2 - 9}$

4- Let R={(−2, −1), (0,3), (5,4), (−2,3)} write domain / Range

Check the above relation is function or not ?

5- See the graph and write domain/Range

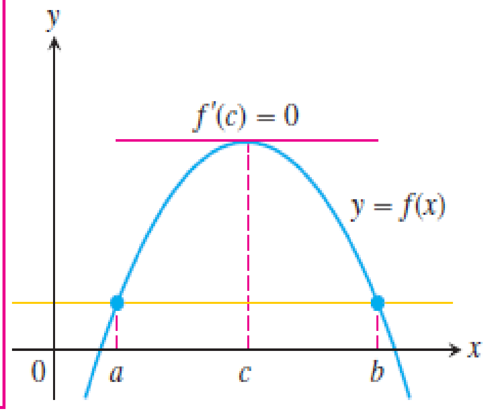
in term of interval notation



ROLLE'S THEOREM Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.



THE MEAN VALUE THEOREM Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

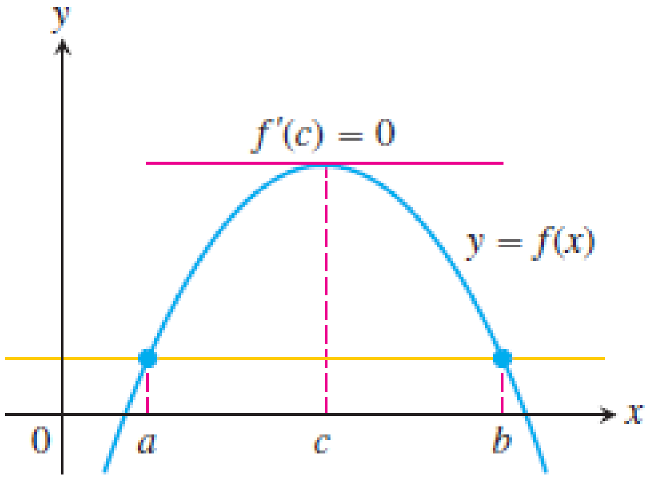
$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

4.8 ROLLE'S THEOREM; MEAN-VALUE THEOREM

4.8.1 THEOREM (*Rolle's Theorem*) *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If*

$$f(a) = 0 \quad \text{and} \quad f(b) = 0$$

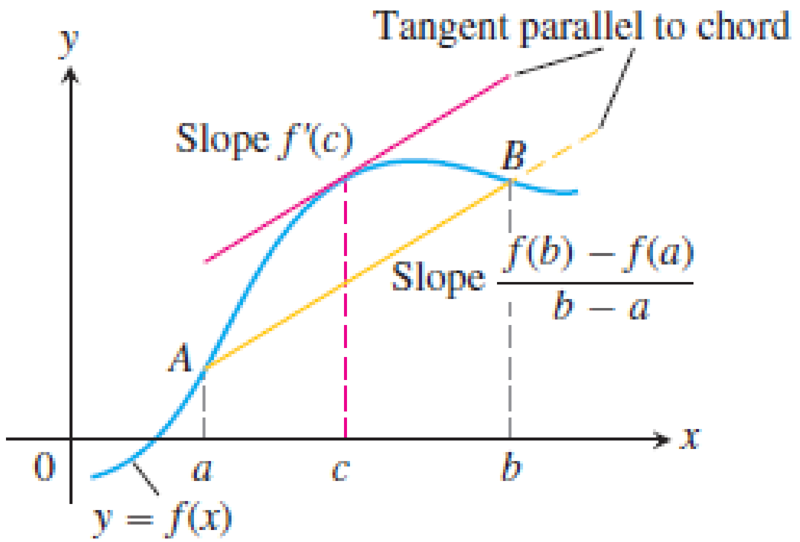
then there is at least one point c in the interval (a, b) such that $f'(c) = 0$.



▶ **Example 1** Find the two x-intercepts of the function $f(x) = x^2 - 5x + 4$ and confirm that $f'(c) = 0$ at some point c between those intercepts.

4.8.2 THEOREM (Mean-Value Theorem) *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$



► **Example 4** Show that the function $f(x) = \frac{1}{4}x^3 + 1$ satisfies the hypotheses of the Mean-Value Theorem over the interval $[0, 2]$, and find all values of c in the interval $(0, 2)$ at which the tangent line to the graph of f is parallel to the secant line joining the points $(0, f(0))$ and $(2, f(2))$.

Example

$$f(x) = x^3 - x^2 - 2x \quad \text{on } [-1,1]$$

(f is continuous and differentiable)

MVT applies

$$f'(x) = 3x^2 - 2x - 2$$

$$f'(c) = \frac{-2-0}{1-(-1)} = -1$$

$$3c^2 - 2c - 2 = -1$$

$$(3c+1)(c-1) = 0$$

$$c = -\frac{1}{3}, \quad c = 1$$

EXERCISE SET 4.8 Graphing Utility

1–4 Verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem. ■

1. $f(x) = x^2 - 8x + 15$; $[3, 5]$

2. $f(x) = \frac{1}{2}x - \sqrt{x}$; $[0, 4]$

3. $f(x) = \cos x$; $[\pi/2, 3\pi/2]$

4. $f(x) = \ln(4 + 2x - x^2)$; $[-1, 3]$

5–8 Verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem. ■

5. $f(x) = x^2 - x$; $[-3, 5]$

6. $f(x) = x^3 + x - 4$; $[-1, 2]$

7. $f(x) = \sqrt{25 - x^2}$; $[-5, 3]$

8. $f(x) = x - \dfrac{1}{x}$; $[3, 4]$

3.5 LOCAL LINEAR APPROXIMATION; DIFFERENTIALS

a function that is differentiable at x_0 is sometimes said to be ***locally linear*** at x_0.

The line that best approximates the graph of f in the vicinity of $P(x_0, f(x_0))$ is the tangent line to the graph of f at x_0, given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{1}$$

This is called the ***local linear approximation*** of f at x_0. This formula can also be expressed in terms of the increment $\Delta x = x - x_0$ as

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x \tag{2}$$

An important linear approximation for roots and powers is

$$(1+x)^k \approx 1 + kx \qquad (x \text{ near } 0;\ \text{any number } k)$$

$$\begin{aligned}
\sqrt{1+x} &\approx \quad 1 + \frac{1}{2}x \\
\frac{1}{1-x} &= \quad (1-x)^{-1} \approx 1 + (-1)(-x) = 1 + x \\
\sqrt[3]{1+5x^4} &= \quad (1+5x^4)^{1/3} \approx 1 + \frac{1}{3}\left(5x^4\right) = 1 + \frac{5}{3}x^4 \\
\frac{1}{\sqrt{1-x^2}} &= \quad (1-x^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2
\end{aligned}$$

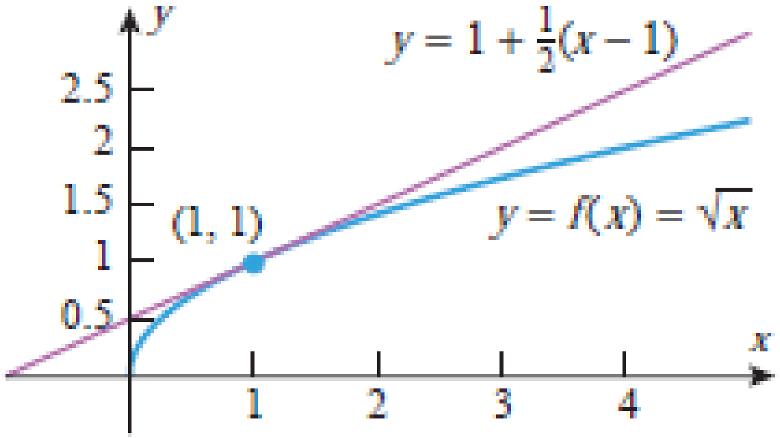
▶ **Example 1**

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$, and compare your approximation to the result produced directly by a calculating utility.

Rule:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$



▲ **Figure 3.5.2**

DIFFERENTIALS

$$\frac{dy}{dx} = f'(x)$$

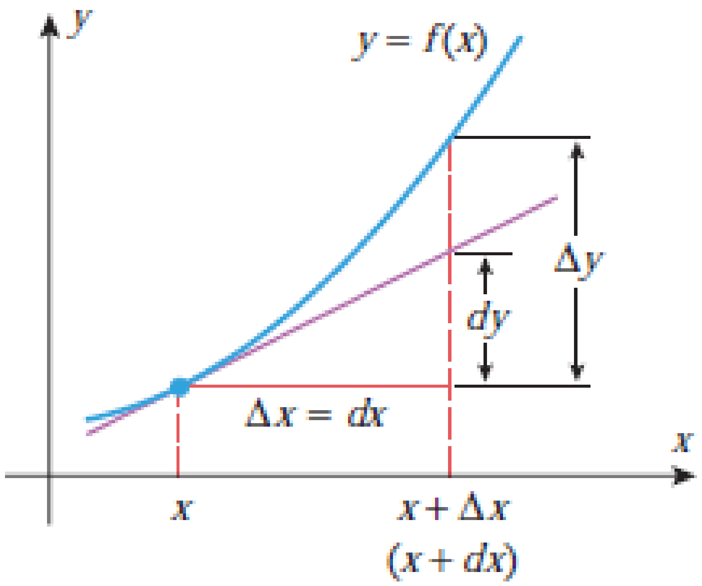
the symbols "dy" and "dx," which are called *differentials*,

$$dy = f'(x)\,dx$$

DEFINITION **Differential**

Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$dy = f'(x)\,dx.$$



▲ Figure 3.5.7

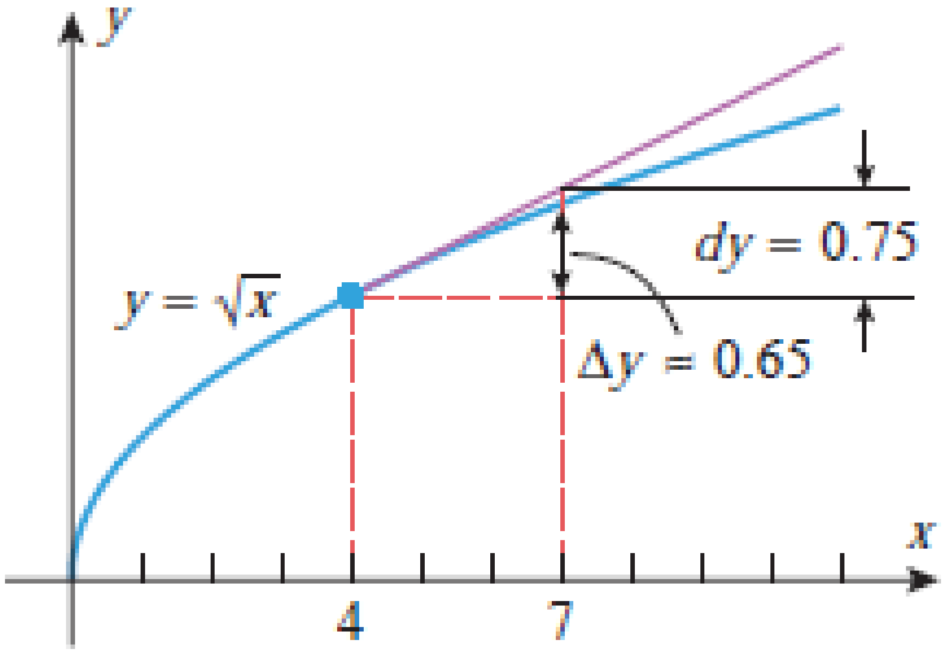
▶ **Example 4** Let $y = \sqrt{x}$.

(a) Find formulas for Δy and dy.

(b) Evaluate Δy and dy at $x = 4$ with $dx = \Delta x = 3$. Then make a sketch of $y = \sqrt{x}$, showing the values of Δy and dy in the picture.

Rule:

$$\Delta y = f(x + \Delta x) - f(x)$$



▲ **Figure 3.5.8**

Exercise 3.5:

11–16 Confirm that the stated formula is the local linear approximation of f at $x_0 = 1$, where $\Delta x = x - 1$.

11. $f(x) = x^4;\ (1+\Delta x)^4 \approx 1 + 4\Delta x$

12. $f(x) = \sqrt{x};\ \sqrt{1+\Delta x} \approx 1 + \frac{1}{2}\Delta x$

13. $f(x) = \dfrac{1}{2+x};\ \dfrac{1}{3+\Delta x} \approx \dfrac{1}{3} - \dfrac{1}{9}\Delta x$

14. $f(x) = (4+x)^3;\ (5+\Delta x)^3 \approx 125 + 75\Delta x$

15. $\tan^{-1} x;\ \tan^{-1}(1+\Delta x) \approx \dfrac{\pi}{4} + \dfrac{1}{2}\Delta x$

Rule:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

23–33 Use an appropriate local linear approximation to estimate the value of the given quantity.

23. $(3.02)^4$ **24.** $(1.97)^3$ **25.** $\sqrt{65}$

26. $\sqrt{24}$ **27.** $\sqrt{80.9}$ **28.** $\sqrt{36.03}$

29. $\sin 0.1$ **30.** $\tan 0.2$ **31.** $\cos 31°$

Exercise 3.5:

39–42 Find formulas for dy and Δy.

39. $y = x^3$

40. $y = 8x - 4$

41. $y = x^2 - 2x + 1$

42. $y = \sin x$

43–46 Find the differential dy.

43. (a) $y = 4x^3 - 7x^2$ (b) $y = x \cos x$

44. (a) $y = 1/x$ (b) $y = 5 \tan x$

45. (a) $y = x\sqrt{1 - x}$ (b) $y = (1 + x)^{-17}$

46. (a) $y = \dfrac{1}{x^3 - 1}$ (b) $y = \dfrac{1 - x^3}{2 - x}$

51–54 Use the differential dy to approximate Δy when x changes as indicated.

51. $y = \sqrt{3x - 2}$; from $x = 2$ to $x = 2.03$

52. $y = \sqrt{x^2 + 8}$; from $x = 1$ to $x = 0.97$

53. $y = \dfrac{x}{x^2 + 1}$; from $x = 2$ to $x = 1.96$

54. $y = x\sqrt{8x + 1}$; from $x = 3$ to $x = 3.05$