Limits and Countinuity

Chapter -1

Topics:

- Limits (An Intuitive Approach)
- Computing Limits
- One sided and two-sided limit
- Limits at Infinity;

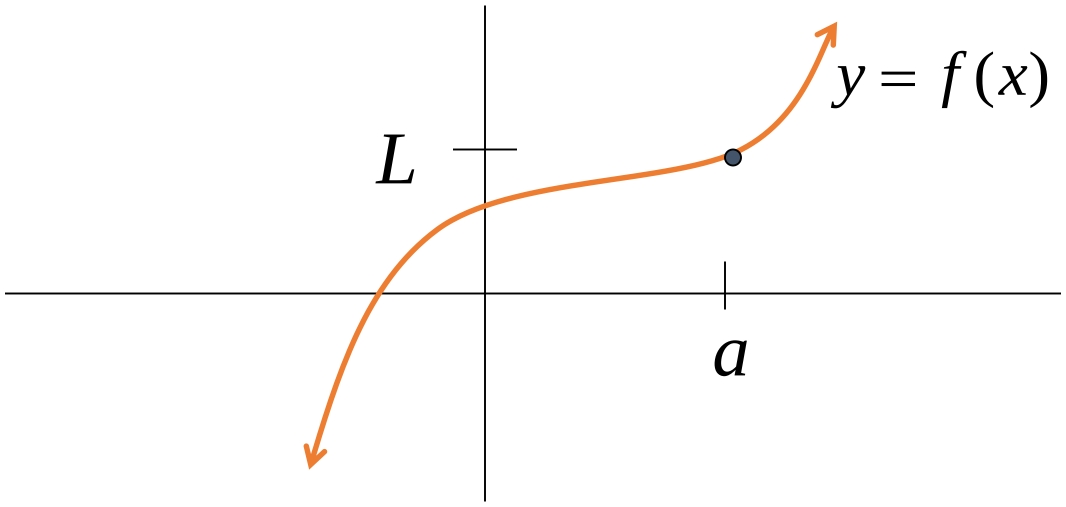
  End Behavior of a Function
- Indeterminant form
- Continuity
- Asymptotes

Limit

We say that the limit of $f(x)$ as x approaches a is L and write

$$\lim_{x \to a} f(x) = L$$

if the values of $f(x)$ approach L as x approaches a.



An Intuitive Approach

Example 1: $f(x) = 2x - 1$
Discuss the behavior of the values of f(x) when x is close to 2 using table

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	2	2.8	2.98	2.998	?	3.002	3.02	3.2	4

We see that as x approaches 2, $f(x)$ approaches 3.

$$\lim_{x \to 2}(2x-1) = 3 = f(2)$$

AN INTUITIVE APPROACH

Example

Let $f(x) = (\sin x)/x$. If we try to evaluate f at 0, we get the meaningless ratio 0/0; f is not defined at x = 0. However, f is defined for all $x \neq 0$, and so we can consider

$$\lim_{x \to 0} \frac{\sin x}{x}$$

■ Table 2.1.1

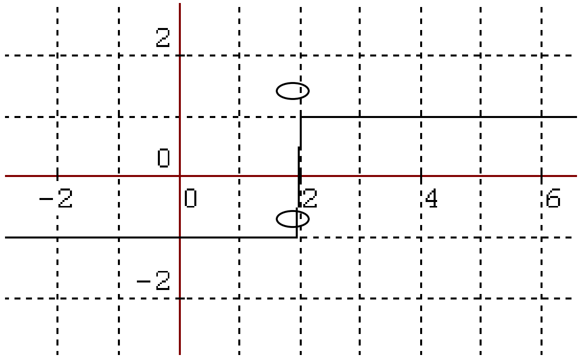
(Left side)		(Right side)	
x (radians)	$\frac{\sin x}{x}$	x (radians)	$\frac{\sin x}{x}$
−1	0.84147	1	0.84147
−0.5	0.95885	0.5	0.95885
−0.1	0.99833	0.1	0.99833
−0.01	0.99998	0.01	0.99998
−0.001	0.99999	0.001	0.99999

Example :

$$f(x) = \frac{x-2}{|x-2|}$$

Discuss the behavior of the values of f(x) when x is closer to 2.
Does the limit exist?

x	0	1	1.9	1.99	2	2.001	2.01	2.1	2.5
f (x)	-1	-1	-1	-1	?	1	1	1	1



* This function is not defined when x = 2.

- The limit does not exist because the limit on the left and the limit on the right are not same

$\lim_{x \to 2^-} f(x) = -1$

$\lim_{x \to 2^+} f(x) = 1$

Example (Limit by graphically)

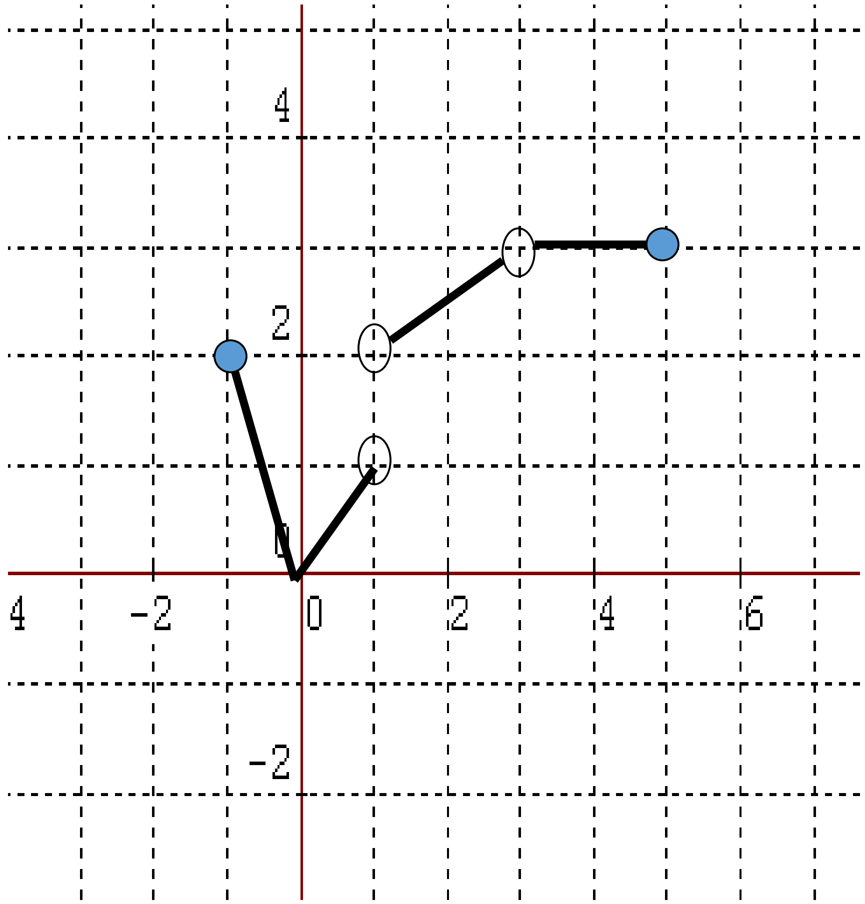
(A) Discuss the behavior of f(x) for x near 0

$\lim_{x \to 0^-} f(x) =$ 0

$\lim_{x \to 0^+} f(x) =$ 0

$\lim_{x \to 0} f(x) =$ 0

F(0) = 0



Example - continue

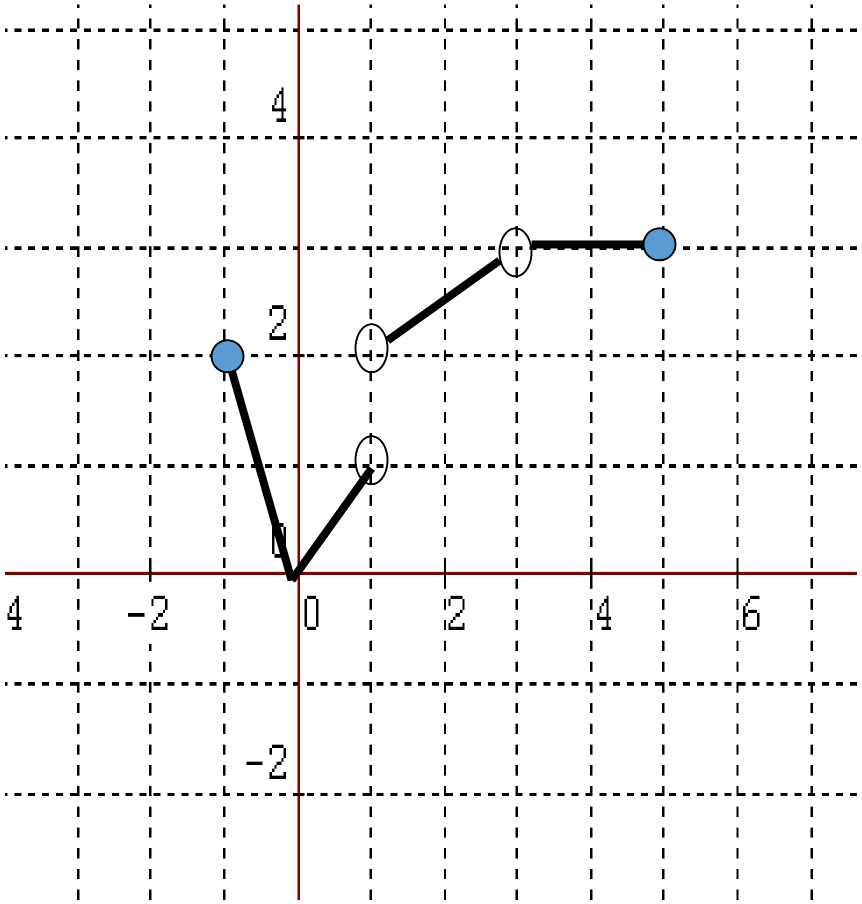
(B) Discuss the behavior of f(x) for x near **1**

$\lim_{x \to 1^-} f(x) =$ 1

$\lim_{x \to 1^+} f(x) =$ 2

$\lim_{x \to 1} f(x) =$ does not exist

F(1) = not defined



Example - continue

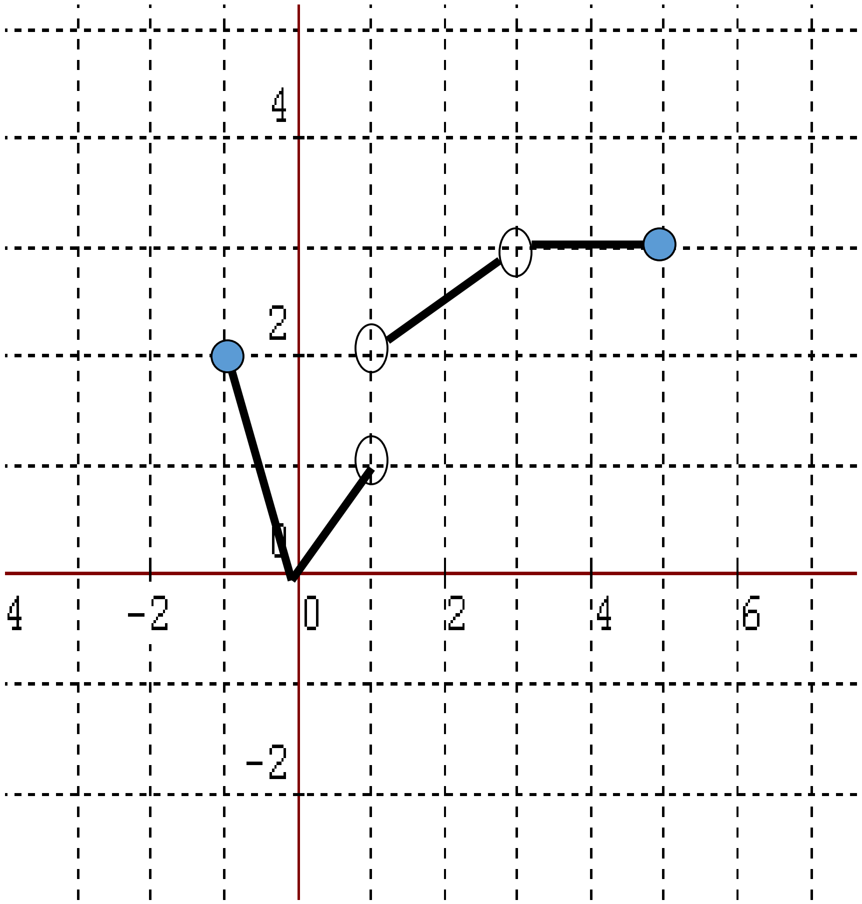
(C) Discuss the behavior of f(x) for x near 3

$\lim_{x \to 3^-} f(x) =$ 3

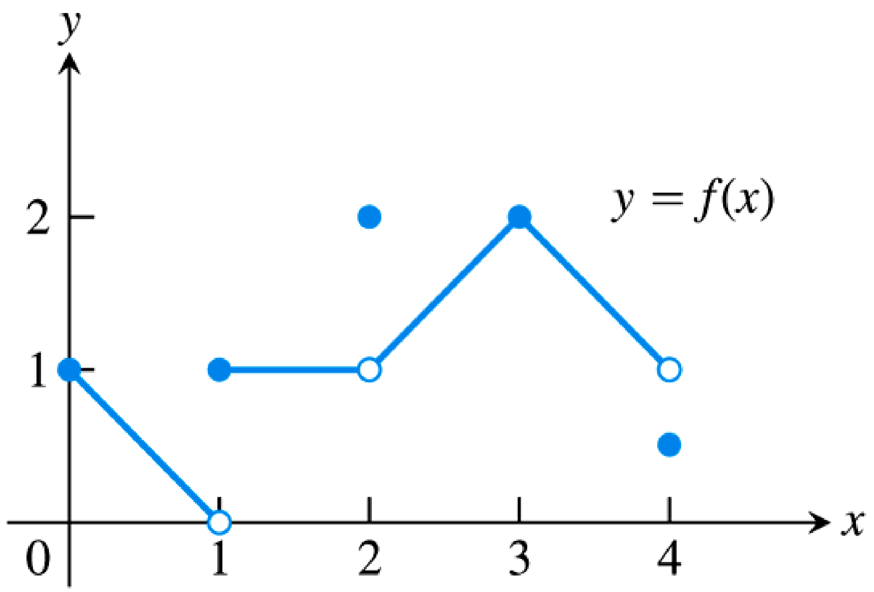
$\lim_{x \to 3^+} f(x) =$ 3

$\lim_{x \to 3} f(x) =$ 3

F(3) = not defined



One-sided limit by Graph



$\lim_{x \to 3^-} f(x) = 2$ $\lim_{x \to 3^+} f(x) = 2$

$\lim_{x \to 3} f(x) = 2$ $f(3) = 2$

$\lim_{x \to 1^-} f(x) = 0$ $\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 1} f(x) = DNE$ $f(1) = 1$

$\lim_{x \to 2^-} f(x) = 1$ $\lim_{x \to 2^+} f(x) = 1$

$\lim_{x \to 2} f(x) = 1$ $f(2) = 2$

$\lim_{x \to 4^-} f(x) = 1$ $\lim_{x \to 4^+} f(x) = not\ defined$

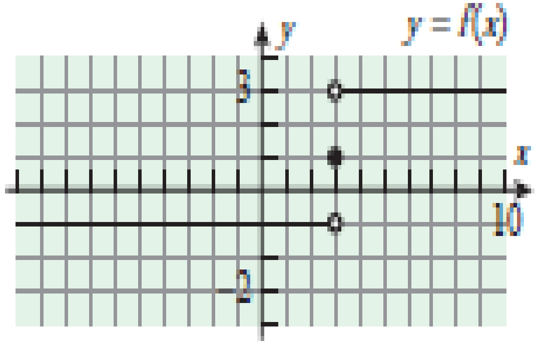
$\lim_{x \to 4} f(x) = none$ $f(4) = 0.5$

Exercise 1.1

3. For the function f graphed in the accompanying figure, find

(a) $\lim_{x \to 3^-} f(x)$ (b) $\lim_{x \to 3^+} f(x)$

(c) $\lim_{x \to 3} f(x)$ (d) $f(3)$.

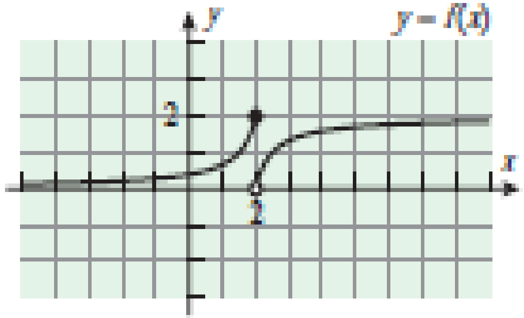


◀ **Figure Ex-3**

4. For the function f graphed in the accompanying figure, find

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$

(c) $\lim_{x \to 2} f(x)$ (d) $f(2)$.

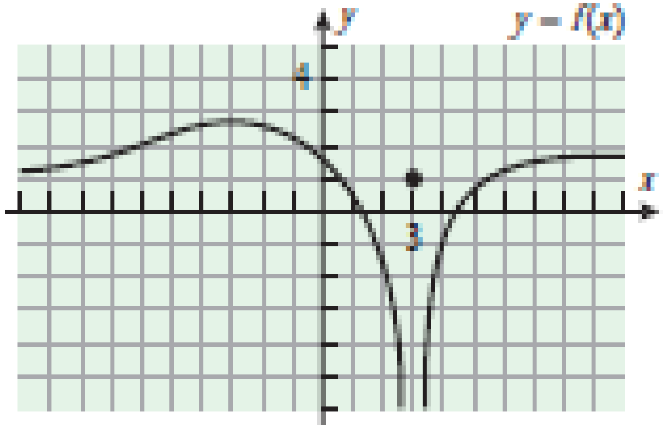


◀ **Figure Ex-4**

7. For the function f graphed in the accompanying figure, find

(a) $\lim_{x \to 3^-} f(x)$ (b) $\lim_{x \to 3^+} f(x)$

(c) $\lim_{x \to 3} f(x)$ (d) $f(3)$.



◀ **Figure Ex-7**

10. For the function f graphed in the accompanying figure, find

(a) $\lim_{x \to -2^-} f(x)$ (b) $\lim_{x \to -2^+} f(x)$ (c) $\lim_{x \to 0^-} f(x)$

(d) $\lim_{x \to 0^+} f(x)$ (e) $\lim_{x \to 2^-} f(x)$ (f) $\lim_{x \to 2^+} f(x)$

(g) the vertical asymptotes of the graph of f.

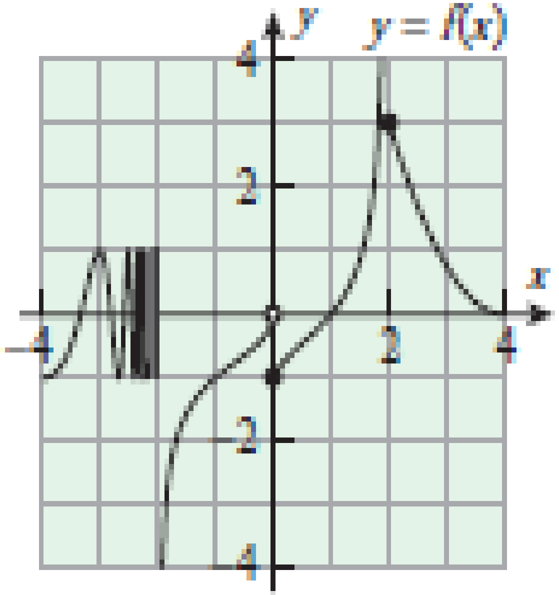


Figure Ex-10

Exercise 1.1

11. $f(x) = \dfrac{e^x - 1}{x}$; $\lim\limits_{x \to 0} f(x)$

x	−0.01	−0.001	−0.0001	0.0001	0.001	0.01
$f(x)$						

▲ Table Ex-11

12. $f(x) = \dfrac{\sin^{-1} 2x}{x}$; $\lim\limits_{x \to 0} f(x)$

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

▲ Table Ex-12

14. (a) $\lim\limits_{x \to 0} \dfrac{\sqrt{x+1} - 1}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

(b) $\lim\limits_{x \to 0^+} \dfrac{\sqrt{x+1} + 1}{x}$; $x = 0.25, 0.1, 0.001, 0.0001$

(c) $\lim\limits_{x \to 0^-} \dfrac{\sqrt{x+1} + 1}{x}$; $x = -0.25, -0.1, -0.001, -0.0001$

$$x \to a^+$$

means x approaches a from the right

$$x \to a^-$$

means x approaches a from the left

Examples

One-Sided Limit

1. Given $$f(x)=\begin{cases} x^2 & \text{if } x\le 3 \\ 2x & \text{if } x>3 \end{cases}$$

Find $\lim\limits_{x\to 3^+} f(x)$

$$\boxed{\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+} 2x=6}$$

Find $\lim\limits_{x\to 3^-} f(x)$

$$\boxed{\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-} x^2=9}$$

Definition:

1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Example:

$$f(x) = \begin{cases} x+1, & \text{if } x > 0 \\ x-1, & \text{if } x \le 0. \end{cases}$$

Draw the graph

$\lim_{x \to 0} f(x)$ does not exist because $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1$.

$\lim_{x \to 1} f(x) = 2$ because $\lim_{x \to 1^+} f(x) = 2$ and $\lim_{x \to 1^-} f(x) = 2$.

Example

For the function defined by then show that

$$f(x)=\begin{cases}2x+1, & x\le 0\\ x^2-x, & x>0\end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist.(DNE)

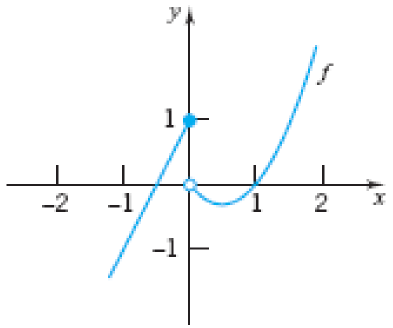


Figure 2.2.14

Solution

The left- and right-hand limits at 0 are as follows:

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(2x+1)=1, \qquad \lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\left(x^2-x\right)=0$$

Since these one-sided limits are different,

$\lim_{x\to 0} f(x)$ does not exist.

THEOREM:

As x approaches a, the **limit** of $f(x)$ is L, if the limit from the left exists and the limit from the right exists and both limits are L. That is,

If

$$\text{and} \quad \left\{ \begin{array}{l} 1) \quad \lim_{x \to a^-} f(x) = L, \\ \\ 2) \quad \lim_{x \to a^+} f(x) = L, \end{array} \right.$$

one sided limit

Then $\lim_{x \to a} f(x) = L,$

1.2 – Algebraic Limits and Continuity

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

Examples Using Limit Rule

Ex. $$\lim_{x\to3}\left(x^2+1\right)=\lim_{x\to3}x^2+\lim_{x\to3}1$$

$$=\left(\lim_{x\to3}x\right)^2+\lim_{x\to3}1$$

$$=3^2+1=10$$

Ex. $$\lim_{x\to1}\frac{\sqrt{2x-1}}{3x+5}=\frac{\sqrt{\lim_{x\to1}(2x-1)}}{\lim_{x\to1}(3x+5)}=\frac{\sqrt{2\lim_{x\to1}x-\lim_{x\to1}1}}{3\lim_{x\to1}x+\lim_{x\to1}5}$$

$$=\frac{\sqrt{2-1}}{3+5}=\frac{1}{8}$$

More Examples

1. Suppose $\lim_{x\to 3} f(x) = 4$ and $\lim_{x\to 3} g(x) = -2$. Find

a) $\lim_{x\to 3}\left(f(x)+g(x)\right) = \lim_{x\to 3} f(x) + \lim_{x\to 3} g(x)$

$= 4 + (-2) = 2$

b) $\lim_{x\to 3}\left(f(x)-g(x)\right) = \lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)$

$= 4 - (-2) = 6$

c) $\lim_{x\to 3}\left(\dfrac{2f(x)-g(x)}{f(x)g(x)}\right) = \dfrac{\lim_{x\to 3} 2f(x) - \lim_{x\to 3} g(x)}{\lim_{x\to 3} f(x) \cdot \lim_{x\to 3} g(x)} = \dfrac{2\cdot 4-(-2)}{4\cdot(-2)} = \dfrac{-5}{4}$

List of Indeterminate Forms

$$\frac{0}{0} \quad \frac{\infty}{\infty} \quad \infty - \infty$$

$$0 \bullet \infty \quad 1^{\infty} \quad 0^{0} \quad \infty^{0}$$

Indeterminate Forms

Indeterminate forms occur when substitution in the limit results in 0/0. In such cases either factor or rationalize the expressions.

Ex. $$\lim_{x \to -5} \frac{x+5}{x^2-25}$$ Notice $\frac{0}{0}$ form

$$= \lim_{x \to -5} \frac{x+5}{(x-5)(x+5)}$$

Factor and cancel common factors

$$= \lim_{x \to -5} \frac{1}{(x-5)} = \frac{1}{-10}$$

Example : Use algebraic and/or graphical techniques to analyze each of the following indeterminate forms

A) $$\lim_{x\to 1}\frac{x-1}{x^2-1}$$

B) $$\lim_{x\to 1}\frac{(x-1)^2}{x^2-1}$$

C) $$\lim_{x\to 1}\frac{x^2-1}{(x-1)^2}$$

See next page for step by step instruction

Example - Solutions

$$A)\lim_{x\to 1}\frac{x-1}{x^2-1}=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$$

$$B)\lim_{x\to 1}\frac{(x-1)^2}{x^2-1}=\lim_{x\to 1}\frac{(x-1)(x-1)}{(x-1)(x+1)}=\lim_{x\to 1}\frac{(x-1)}{(x+1)}=\frac{0}{2}=0$$

$$C)\lim_{x\to 1}\frac{x^2-1}{(x-1)^2}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x-1)}=\lim_{x\to 1}\frac{(x+1)}{(x-1)}=$$

Note: when you find the limits of the above problems, you must factor first and then simplify before you substitute the number for x

More Examples

a) $$\lim_{x\to 9}\left(\frac{\sqrt{x}-3}{x-9}\right) = \lim_{x\to 9}\left(\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}\right)$$

$$= \lim_{x\to 9}\left(\frac{x-9}{(x-9)(\sqrt{x}+3)}\right) = \lim_{x\to 9}\left(\frac{1}{\sqrt{x}+3}\right) = \frac{1}{6}$$

b) $$\lim_{x\to -2}\left(\frac{4-x^2}{2x^2+x^3}\right) = \lim_{x\to -2}\left(\frac{(2-x)(2+x)}{x^2(2+x)}\right)$$

$$= \lim_{x\to -2}\left(\frac{2-x}{x^2}\right) = \frac{2-(-2)}{(-2)^2} = \frac{4}{4} = 1$$

Exercise 1.2

3. $\lim_{x \to 2} x(x-1)(x+1)$

4. $\lim_{x \to 3} x^3 - 3x^2 + 9x$

5. $\lim_{x \to 3} \dfrac{x^2 - 2x}{x+1}$

6. $\lim_{x \to 0} \dfrac{6x - 9}{x^3 - 12x + 3}$

7. $\lim_{x \to 1^+} \dfrac{x^4 - 1}{x - 1}$

8. $\lim_{t \to -2} \dfrac{t^3 + 8}{t + 2}$

9. $\lim_{x \to -1} \dfrac{x^2 + 6x + 5}{x^2 - 3x - 4}$

10. $\lim_{x \to 2} \dfrac{x^2 - 4x + 4}{x^2 + x - 6}$

11. $\lim_{x \to -1} \dfrac{2x^2 + x - 1}{x + 1}$

12. $\lim_{x \to 1} \dfrac{3x^2 - x - 2}{2x^2 + x - 3}$

13. $\lim_{t \to 2} \dfrac{t^3 + 3t^2 - 12t + 4}{t^3 - 4t}$

14. $\lim_{t \to 1} \dfrac{t^3 + t^2 - 5t + 3}{t^3 - 3t + 2}$

15. $\lim_{x \to 3^+} \dfrac{x}{x - 3}$

16. $\lim_{x \to 3^-} \dfrac{x}{x - 3}$

17. $\lim_{x \to 3} \dfrac{x}{x - 3}$

18. $\lim_{x \to 2^+} \dfrac{x}{x^2 - 4}$

19. $\lim_{x \to 2^-} \dfrac{x}{x^2 - 4}$

20. $\lim_{x \to 2} \dfrac{x}{x^2 - 4}$

21. $\lim_{y \to 6^+} \dfrac{y + 6}{y^2 - 36}$

22. $\lim_{y \to 6^-} \dfrac{y + 6}{y^2 - 36}$

23. $\lim_{y \to 6} \dfrac{y + 6}{y^2 - 36}$

24. $\lim_{x \to 4^+} \dfrac{3 - x}{x^2 - 2x - 8}$

25. $\lim_{x \to 4^-} \dfrac{3 - x}{x^2 - 2x - 8}$

26. $\lim_{x \to 4} \dfrac{3 - x}{x^2 - 2x - 8}$

27. $\lim_{x \to 2^+} \dfrac{1}{|2 - x|}$

28. $\lim_{x \to 3^-} \dfrac{1}{|x - 3|}$

29. $\lim_{x \to 9} \dfrac{x - 9}{\sqrt{x} - 3}$

30. $\lim_{y \to 4} \dfrac{4 - y}{2 - \sqrt{y}}$

37. $\lim_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}$

38. $\lim_{x \to 0} \dfrac{\sqrt{x^2 + 4} - 2}{x}$

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17. $\lim \dfrac{1}{x^2 - 4}$ as

a. $x \to 2^+$

b. $x \to 2^-$

c. $x \to -2^+$

d. $x \to -2^-$

18. $\lim \dfrac{x}{x^2 - 1}$ as

a. $x \to 1^+$

b. $x \to 1^-$

c. $x \to -1^+$

d. $x \to -1^-$

19. $\lim \left(\dfrac{x^2}{2} - \dfrac{1}{x}\right)$ as

a. $x \to 0^+$

b. $x \to 0^-$

c. $x \to \sqrt[3]{2}$

d. $x \to -1$

20. $\lim \dfrac{x^2 - 1}{2x + 4}$ as

a. $x \to -2^+$

b. $x \to -2^-$

c. $x \to 1^+$

d. $x \to 0^-$

1.2.4 THEOREM *Let*

$$f(x) = \frac{p(x)}{q(x)}$$

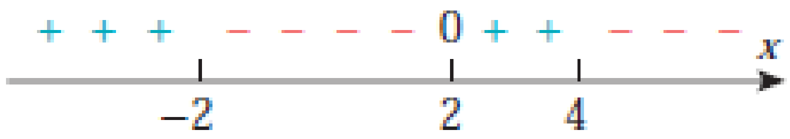
be a rational function, and let a be any real number.

(*a*) *If* $q(a) \neq 0$, *then* $\lim_{x \to a} f(x) = f(a)$.

(*b*) *If* $q(a) = 0$ *but* $p(a) \neq 0$, *then* $\lim_{x \to a} f(x)$ *does not exist.*

Example 8 Find

(a) $\lim_{x \to 4^+} \frac{2 - x}{(x-4)(x+2)}$ (b) $\lim_{x \to 4^-} \frac{2 - x}{(x-4)(x+2)}$ (c) $\lim_{x \to 4} \frac{2 - x}{(x-4)(x+2)}$



Sign of $\frac{2 - x}{(x-4)(x+2)}$

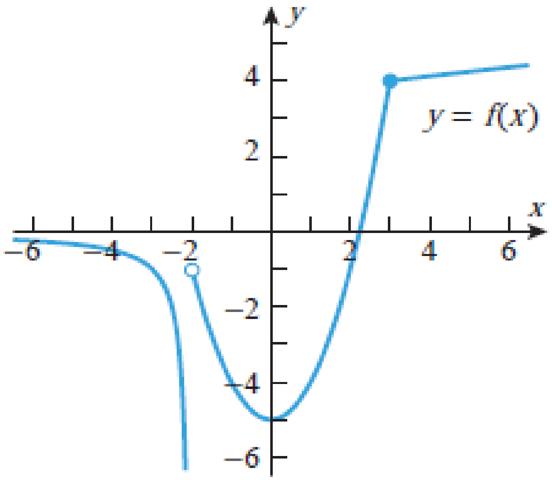
Example 9 Find

$$\text{(a)}\ \lim_{x\to 3}\frac{x^2-6x+9}{x-3} \qquad \text{(b)}\ \lim_{x\to -4}\frac{2x+8}{x^2+x-12} \qquad \text{(c)}\ \lim_{x\to 5}\frac{x^2-3x-10}{x^2-10x+25}$$

▶ Example 10 Find $\displaystyle\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$.

▶ Example 11 Let

$$f(x)=\begin{cases}1/(x+2), & x<-2\\ x^2-5, & -2<x\le 3\\ \sqrt{x+13}, & x>3\end{cases}$$



Find

$$\text{(a)}\ \lim_{x\to -2} f(x) \qquad \text{(b)}\ \lim_{x\to 0} f(x) \qquad \text{(c)}\ \lim_{x\to 3} f(x)$$

Exercise 1.2

40. Let

$$f(x)=\begin{cases}\dfrac{x^2-9}{x+3}, & x\neq -3\\ k, & x=-3\end{cases}$$

(a) Find k so that $f(-3)=\lim_{x\to -3} f(x)$.

FOCUS ON CONCEPTS

41. (a) Explain why the following calculation is incorrect.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=\lim_{x\to 0^+}\frac{1}{x}-\lim_{x\to 0^+}\frac{1}{x^2}$$
$$=+\infty-(+\infty)=0$$

(b) Show that $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=-\infty$.

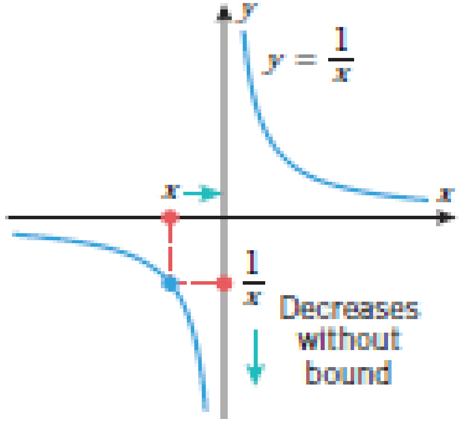
42. (a) Explain why the following argument is incorrect.

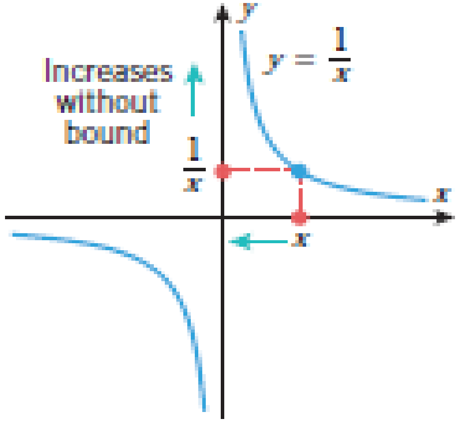
$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{x^2+2x}\right)=\lim_{x\to 0}\frac{1}{x}\left(1-\frac{2}{x+2}\right)$$
$$=\infty\cdot 0=0$$

(b) Show that $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{x^2+2x}\right)=\frac{1}{2}$.

INFINITE LIMITS

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

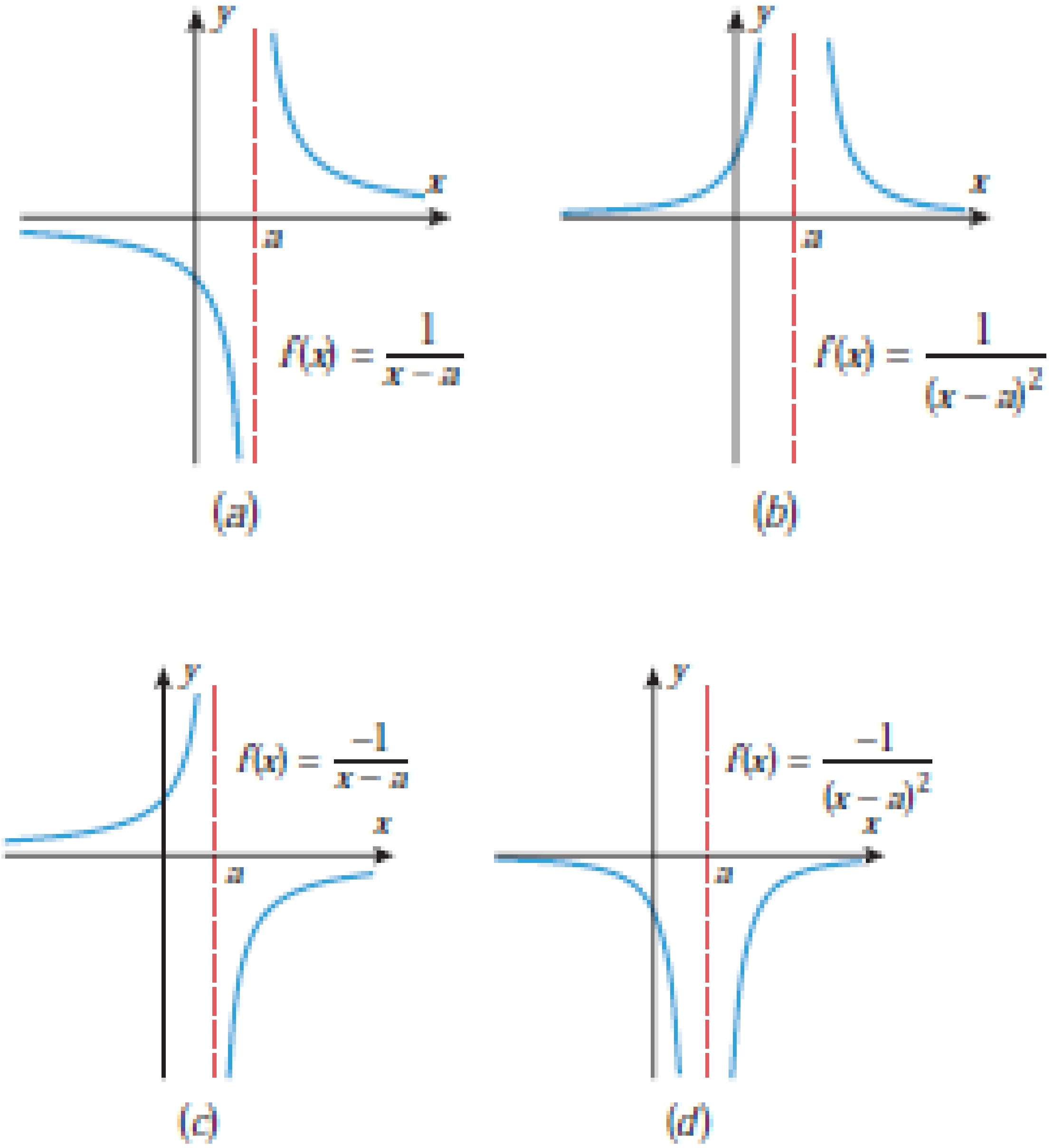




x	−1	−0.1	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	−1	−10	−100	−1000	−10,000		10,000	1000	100	10	1

Left side → ← Right side

Graph Analysis

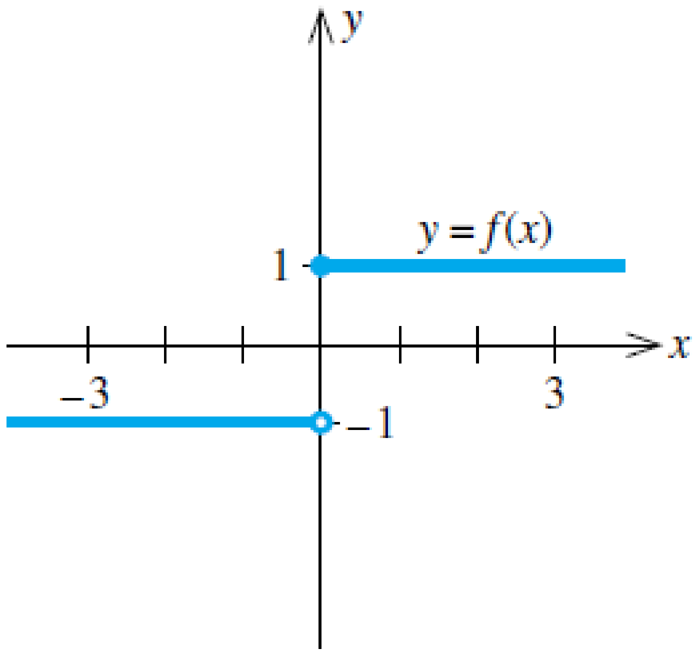


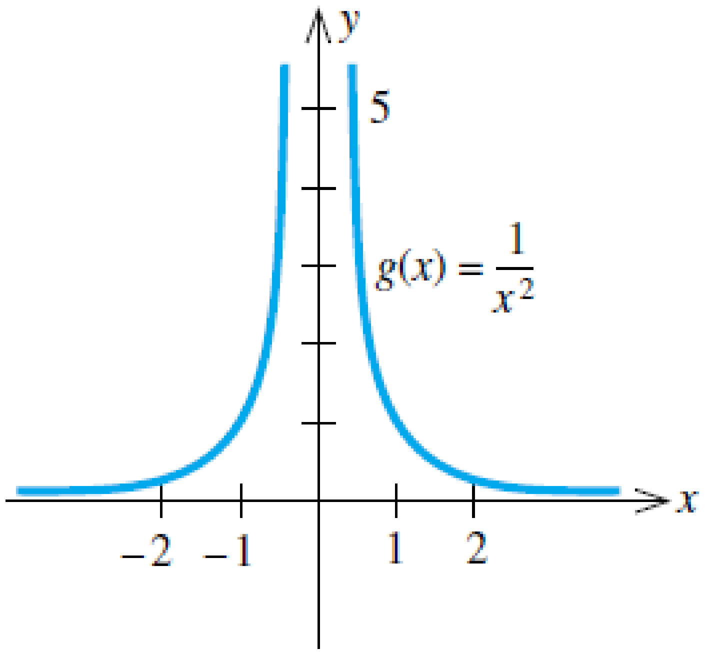
Let f and g be the functions

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \qquad \text{and} \qquad g(x) = \frac{1}{x^2}$$

Evaluate:

a. $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ **b.** $\lim_{x\to\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$





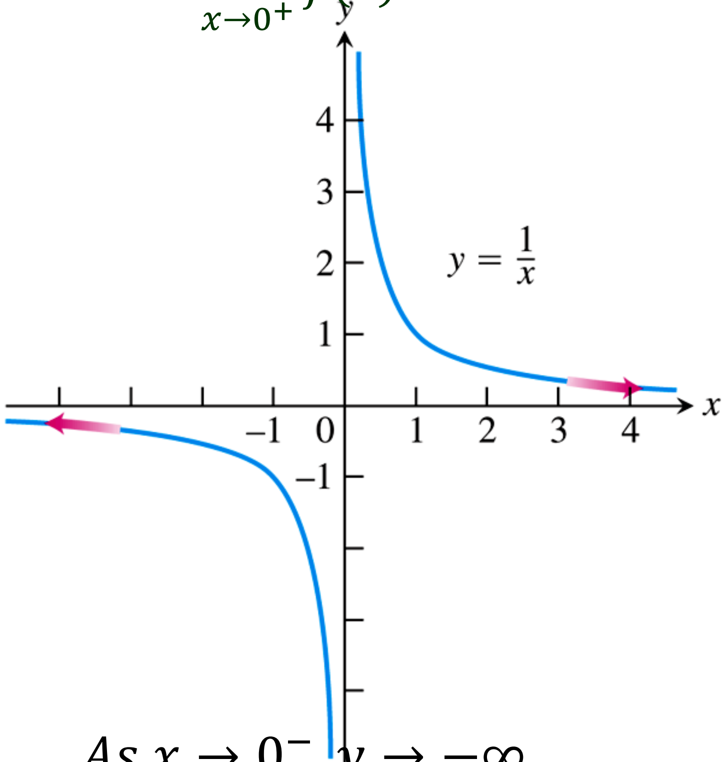
Limits Involving Infinity; Asymptotes of Graphs

$As\ x \to 0^+, y \to \infty$

$$\lim_{x \to 0^+} f(x) = \infty$$

$As\ x \to -\infty, y \to 0$

$$\lim_{x \to -\infty} f(x) = 0$$



$As\ x \to \infty, y \to 0$

$$\lim_{x \to \infty} f(x) = 0$$

$As\ x \to 0^-, y \to -\infty$

$$\lim_{x \to 0^-} f(x) = -\infty$$

Asymptotes:

DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

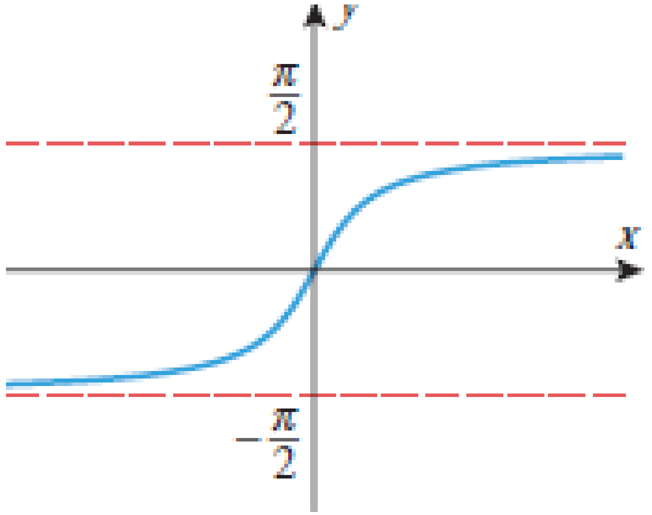
DEFINITION Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

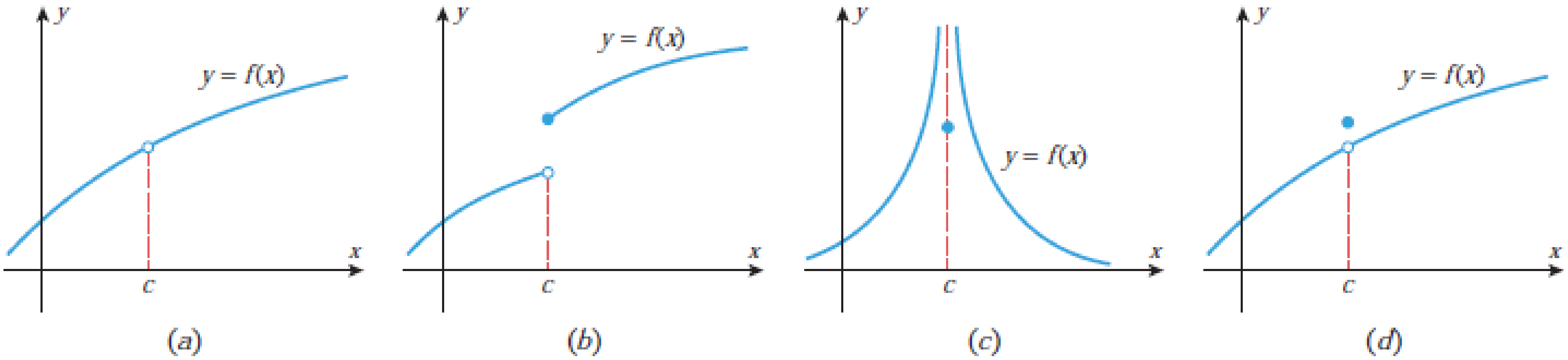
▶ **Example 2** Figure 1.3.3 is the graph of $f(x) = \tan^{-1} x$. As suggested by

$$\lim_{x\to+\infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$



Continuity:

- **Continuity** And Discontinuity. A function is **continuous** if it can be drawn without picking up the pencil; otherwise, it is discontinuous.



undefined

Limit does not exist

Limit of f(x) not equal to value of function

1.5.1 DEFINITION A function f is said to be ***continuous at*** $x = c$ provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

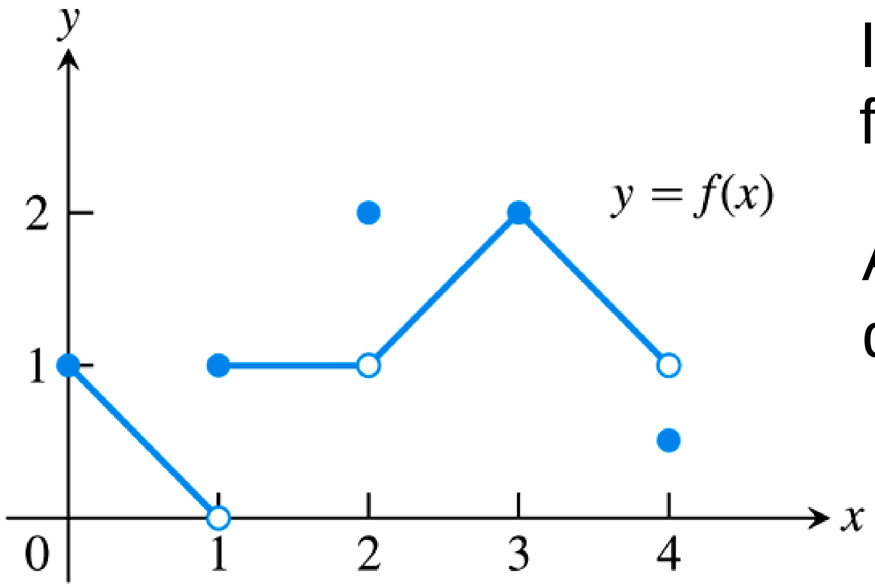
1.5.4 THEOREM

(*a*) *A polynomial is continuous everywhere.*

(*b*) *A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.*

1.5– Algebraic Limits and Continuity

A continuous function is one that can be plotted without the plot being broken.



Is the graph of f(x) a continuous function on the interval [0, 4]? No

At what values of x is the function discontinuous and why?

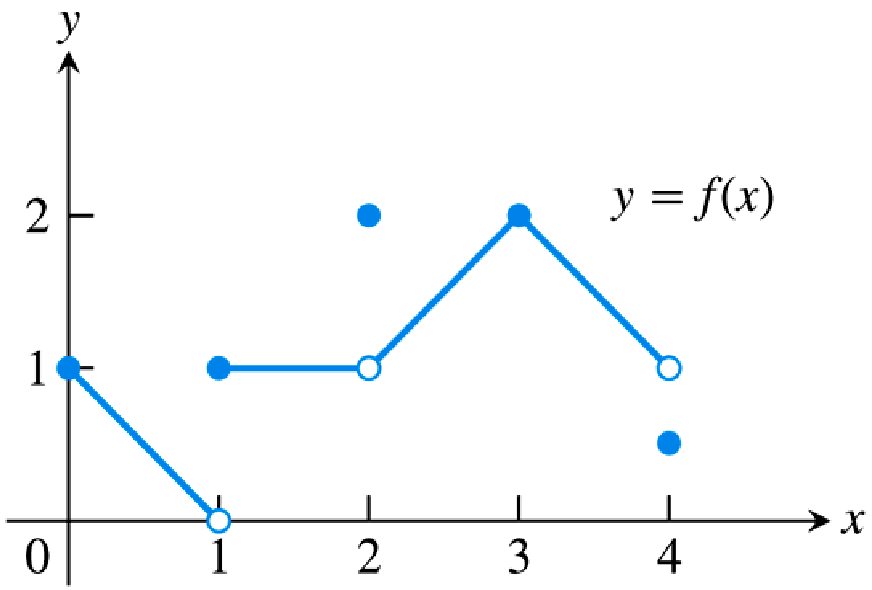
$x = 1 \quad There\ is\ a\ jump.$

$x = 2 \quad There\ is\ a\ hole.$

$x = 4 \quad There\ is\ a\ jump.$

Is the graph of f(x) continuous at $x = 3$? Yes

1.5 – Algebraic Limits and Continuity



$\lim_{x\to 3^-} f(x) = 2$ $\lim_{x\to 3^+} f(x) = 2$

$\lim_{x\to 3} f(x) = 2$ $f(3) = 2$

$\lim_{x\to 1^-} f(x) = 0$ $\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1} f(x) = DNE$ $f(1) = 1$

$\lim_{x\to 2^-} f(x) = 1$ $\lim_{x\to 2^+} f(x) = 1$

$\lim_{x\to 2} f(x) = 1$ $f(2) = 2$

$\lim_{x\to 4^-} f(x) = 1$ $\lim_{x\to 4^+} f(x) = not\ defined$

$\lim_{x\to 4} f(x) = none$ $f(4) = 0.5$

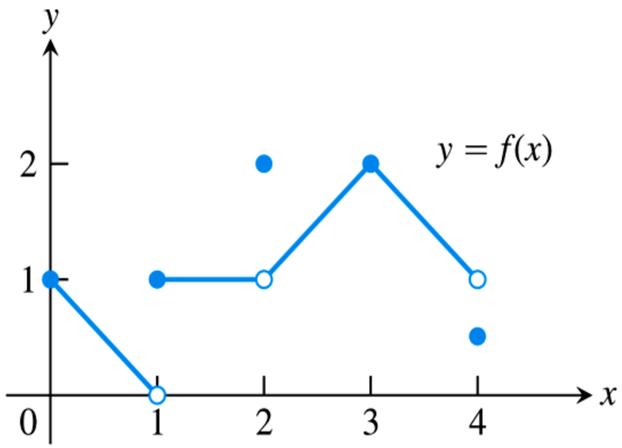
What are the rules for continuity at a point?

1.5 – Algebraic Limits and Continuity

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).



$x = 1$

✓ $f(c) exists$ $f(1) = 1$

✗ $\lim_{x \to c} f(x)$ exists $\lim_{x \to 1} f(x) = DNE$

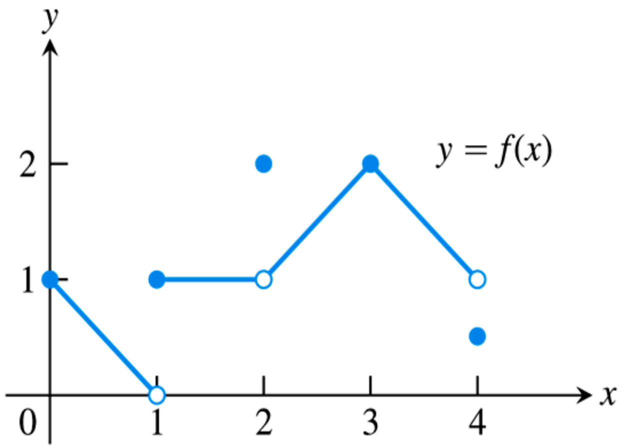
$\therefore$ $f(x)$ is not continuous at $x = 1$.

1.5 – Algebraic Limits and Continuity

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (*c* lies in the domain of *f*).
2. $\lim_{x \to c} f(x)$ exists (*f* has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).



$x = 2$

✓ $f(c)$ *exists* $\quad f(2) = 2$

✓ $\lim\limits_{x \to c} f(x)$ *exists* $\quad \lim\limits_{x \to 2} f(x) = 1$

✗ $\lim\limits_{x \to c} f(x) = f(c) \quad 2 \neq 1$

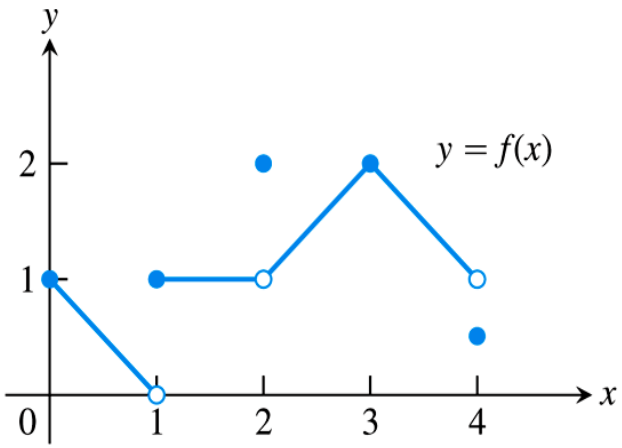
$\therefore \quad f(x)$ *is not continuous at* $x = 2.$

1.5 – Algebraic Limits and Continuity

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).



$x = 3$

✓ $f(c) exists$ $\qquad f(2) = 2$

✓ $\lim_{x\to c} f(x)\ exists$ $\qquad \lim_{x\to 3} f(x) = 2$

✓ $\lim_{x\to c} f(x) = f(c)$ $\qquad 2 = 2$

$\therefore \quad f(x)\ is\ continuous\ at\ x = 3.$

1.5 – Algebraic Limits and Continuity

Is the following function continuous at $x = 3$?

$$f(x) = x^2 - 5$$

✓ $f(c) \text{ exists}$ $\quad f(3) = \quad (3)^2 - 5 = 4$

✓ $\lim_{x \to c} f(x) \text{ exists}$ $\quad \lim_{x \to 3} f(x) = (3)^2 - 5 = 4$

✓ $\lim_{x \to c} f(x) = f(c)$ $\quad 4 = 4$

$\therefore$

$f(x) \text{ is continuous at}$
$x = 3.$

Is the following function continuous at $x = -2$?

$$g(x) = \begin{cases} \frac{1}{2}x + 3, & \text{for} \quad x < -2 \\ x - 1, & \text{for} \quad x \geq -2 \end{cases}$$

✓ $f(c) exists$ $\quad g(-2) = (-2) - 1 = -3$

✗ $\lim\limits_{x \to c} f(x) \ exists$ $\quad \lim\limits_{x \to -2} g(x)$

$$\lim_{x \to -2^-} g(x) = \frac{1}{2}(-2) + 3 = 2$$

$$\lim_{x \to -2^+} g(x) = (-2) - 1 = -3$$

$$2 \neq -3 \quad \therefore \quad \lim_{x \to -2} g(x) = DNE$$

$\therefore$

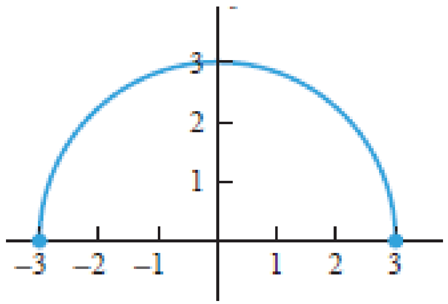
$g(x)\ is\ not\ continuous\ at\ x = -2.$

1.5.2 DEFINITION A function f is said to be ***continuous on a closed interval*** $[a, b]$ if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

Example 2 What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

If c is any point in the interval $(-3, 3)$.



$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to 3^-} (9 - x^2)} = 0 = f(3)$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = 0 = f(-3)$$

Thus, f is continuous on the closed interval $[-3, 3]$

1.5.3 THEOREM *If the functions f and g are continuous at c, then*

(a) $f + g$ *is continuous at* c.

(b) $f - g$ *is continuous at* c.

(c) fg *is continuous at* c.

(d) f/g *is continuous at* c *if* $g(c) \neq 0$ *and has a discontinuity at* c *if* $g(c) = 0$.

1.5.6 THEOREM

(a) *If the function g is continuous at c, and the function f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.*

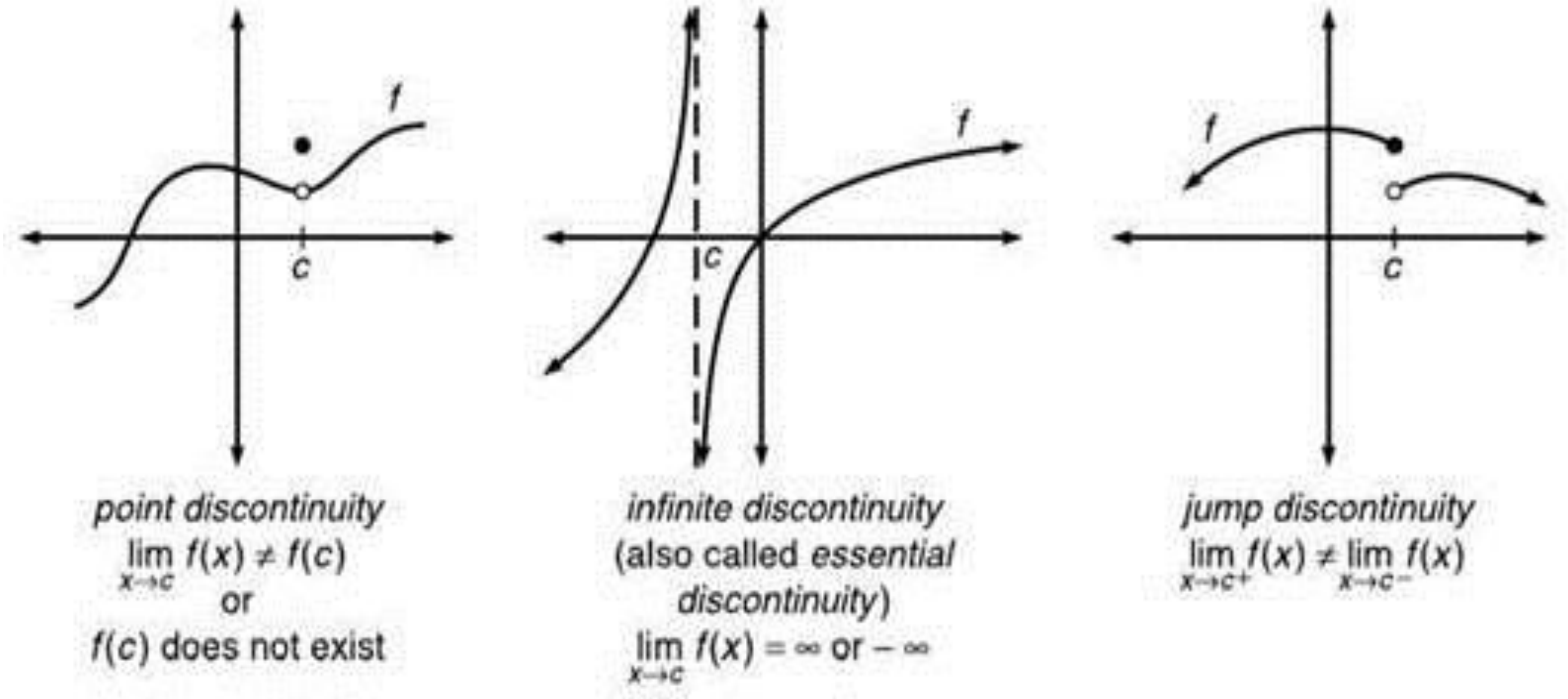
(b) *If the function g is continuous everywhere and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.*

Class Activity

$$1.\ f(x)=\begin{cases}2x+3\ ,\ x<5\\ -x+12\ ,\ x>5\end{cases}$$

$$2.\ f(x)=\begin{cases}x+2,\ x\neq 2\\ 3 \qquad ,x=2\end{cases} \qquad is\ f(x)\,Cont.at\ x=2$$

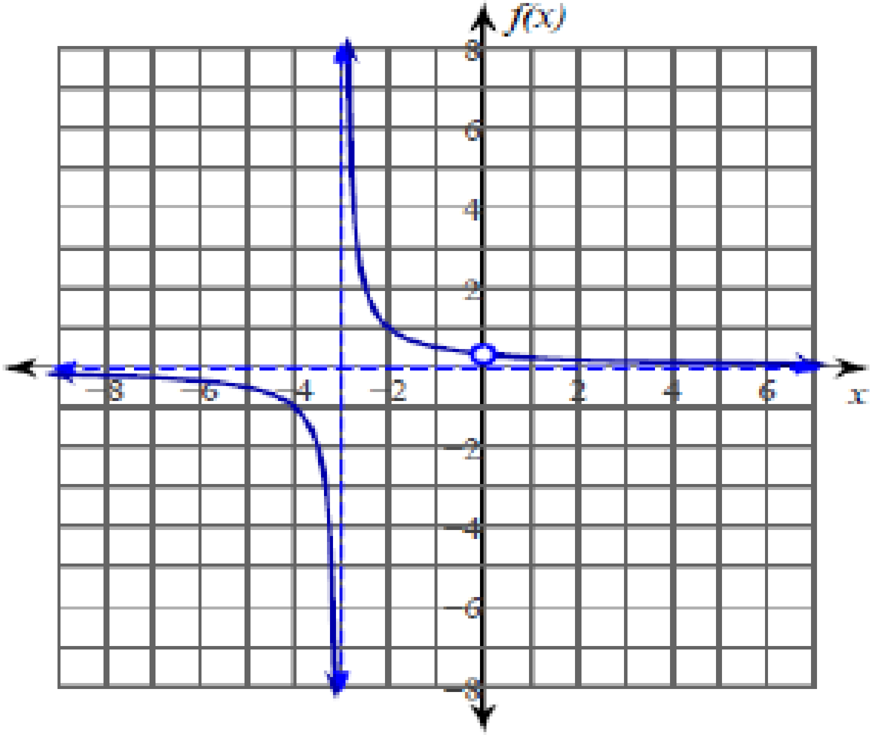
Types of discontinuity



Point or
Removable

Determine if each function is continuous at the given x-values. If not continuous, classify each discontinuity.

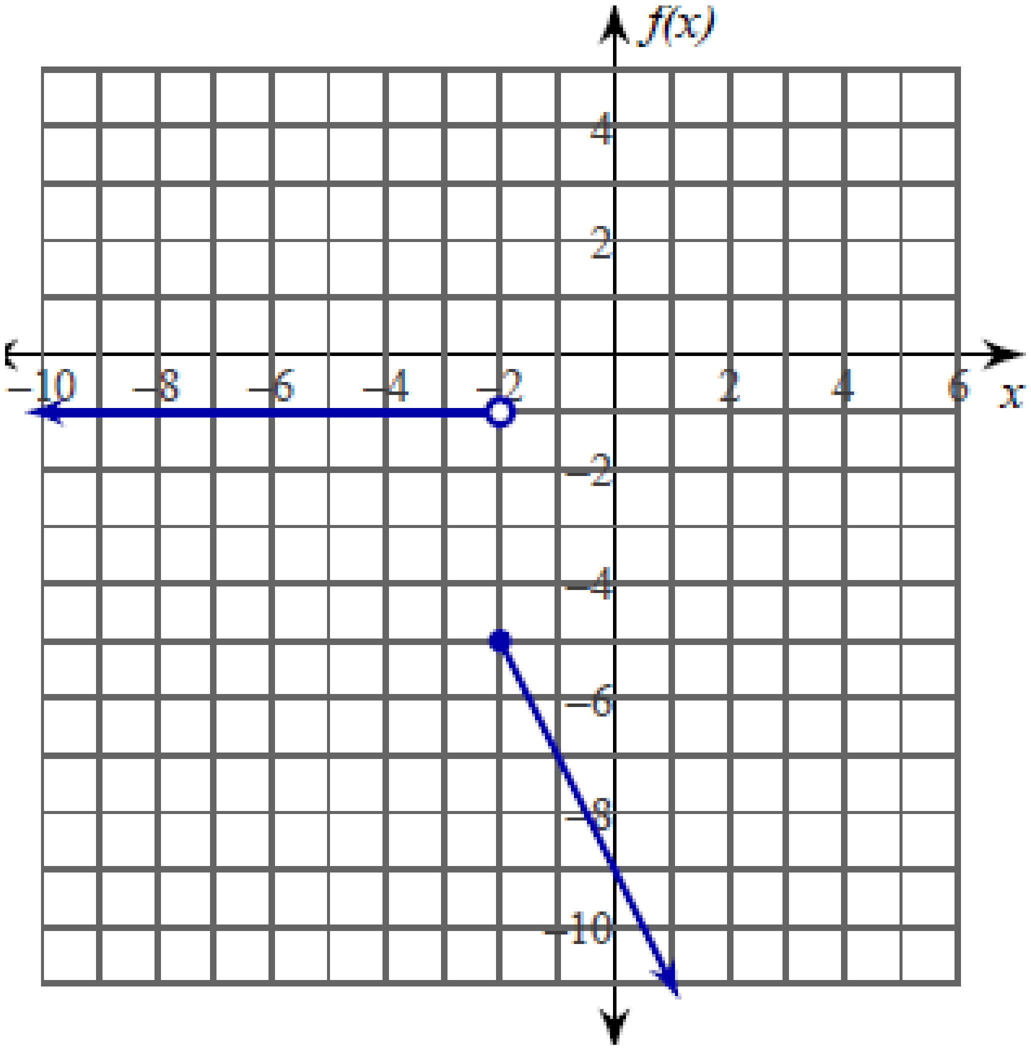
1) $f(x) = \dfrac{x}{x^2 + 3x}$; at $x = -3$ and $x = 0$



Infinite discontinuity

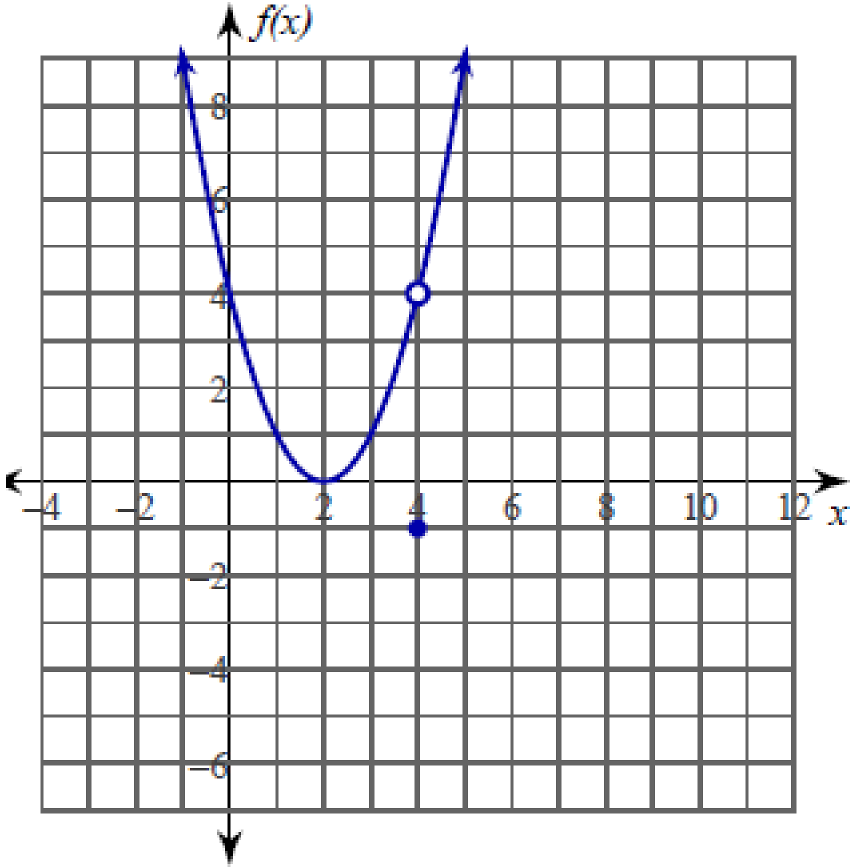
Jump discontinuity:

$$f(x) = \begin{cases} -1, & x < -2 \\ -2x - 9, & x \geq -2 \end{cases}$$



Removable discontinuity:

$$f(x)=\begin{cases} x^2-4x+4, & x\neq 4 \\ -1, & x=4 \end{cases}$$



Find the intervals on which each function is continuous.

13) $f(x) = \begin{cases} x, & x \neq 4 \\ 2, & x = 4 \end{cases}$

14) $f(x) = \begin{cases} -2, & x < 3 \\ -2x + 6, & x \geq 3 \end{cases}$

15) $f(x) = \dfrac{x - 1}{x^2 - 4x + 3}$

16) $f(x) = \dfrac{x^2}{2} + 4x + 10$

17) $f(x) = -x^2 - 4x + 2$

18) $f(x) = -\dfrac{x - 2}{x^2 - 3x + 2}$

Find values of x, if any, at which f is not continuous.

19) $f(x) = -\dfrac{x - 1}{x^2 - x}$

20) $f(x) = \dfrac{x}{x^2 - 6x + 9}$

Find the values of the parameters a and b such that the function

$$f(x) = \begin{cases} (2x+a)^3, & \text{if } x < 0 \\ 5bx + 8, & \text{if } 0 \le x < 1 \\ x^2 + 12, & \text{if } x \ge 1 \end{cases}$$

is continuous at all the points in its domain.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x+a)^3 = (2 \cdot 0 + a)^3 = a^3$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (5bx+8) = 5b \cdot 0 + 8 = 8$$
$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5bx+8) = 5b \cdot 1 + 8 = 5b + 8$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2+12) = 1^2 + 12 = 13$$

29–30 Find a value of the constant k, if possible, that will make the function continuous everywhere.

29. (a) $f(x) = \begin{cases} 7x - 2, & x \leq 1 \\ kx^2, & x > 1 \end{cases}$

(b) $f(x) = \begin{cases} kx^2, & x \leq 2 \\ 2x + k, & x > 2 \end{cases}$

30. (a) $f(x) = \begin{cases} 9 - x^2, & x \geq -3 \\ k/x^2, & x < -3 \end{cases}$