Chapter 5

THE INDEFINITE INTEGRAL

The process of finding antiderivatives is called ***antidifferentiation*** or ***integration***.

$$\frac{d}{dx}[F(x)] = f(x) \qquad \int f(x)\,dx = F(x) + C$$

5.2.3 THEOREM

The statements in Theorem 5.2.3 can be summarized by the following formulas:

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

Notation and Existence of the Definite Integral

Upper limit of integration

The function is the integrand.

x is the variable of integration.

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

INTEGRATION FORMULAS
TABLE 5.2.1

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\, dx = \sin x + C$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\, dx = -\cos x + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\, dx = -\cot x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\, dx = \sec x + C$

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
9. $\frac{d}{dx}[e^x] = e^x$	$\int e^x \, dx = e^x + C$
10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right] = b^x \quad (0 < b, b \neq 1)$	$\int b^x \, dx = \frac{b^x}{\ln b} + C \quad (0 < b, b \neq 1)$
11. $\frac{d}{dx}[\ln \lvert x \rvert] = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \ln \lvert x \rvert + C$
12. $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1 + x^2}$	$\int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + C$
13. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}$	$\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C$
14. $\frac{d}{dx}[\sec^{-1} \lvert x \rvert] = \frac{1}{x\sqrt{x^2 - 1}}$	$\int \frac{1}{x\sqrt{x^2 - 1}} \, dx = \sec^{-1} \lvert x \rvert + C$

Exercise 5.3

- **Example 2**

$$\int \sin(x+9)\, dx :$$

- **Example 3** Evaluate $\int \cos 5x\, dx$.

- **Example 4**

$$\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5} :$$

- **Example 5** Evaluate $\int \frac{dx}{1+3x^2}$.

- **Example 7** Evaluate $\int \sin^2 x \cos x\, dx$

- **Example 8** Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$.

- **Example 9** Evaluate $\int t^4 \sqrt[3]{3-5t^5}\, dt$.

- **Example 10** Evaluate $\int \frac{e^x}{\sqrt{1-e^{2x}}}\, dx$.

- **Example 11** Evaluate $\int x^2 \sqrt{x-1}\, dx$.

33. $\int \left[\frac{1}{2\sqrt{1-x^2}} - \frac{3}{1+x^2}\right] dx$

35. Evaluate the integral

$$\int \frac{1}{1+\sin x}\, dx$$

by multiplying the numerator and denominator by an appropriate expression.

36. Use the double-angle formula $\cos 2x = 2\cos^2 x - 1$ to evaluate the integral

$$\int \frac{1}{1+\cos 2x}\, dx$$

43–46 Solve the initial-value problems.

43. (a) $\frac{dy}{dx} = \sqrt[3]{x},\ y(1) = 2$

(b) $\frac{dy}{dt} = \sin t + 1,\ y\left(\frac{\pi}{3}\right) = \frac{1}{2}$

61–62 Evaluate the integrals with the aid of Formulas (5), (6), and (7).

61. (a) $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$ (b) $\displaystyle\int \frac{dx}{5+x^2}$ (c) $\displaystyle\int \frac{dx}{x\sqrt{x^2-\pi}}$

62. (a) $\displaystyle\int \frac{e^x}{4+e^{2x}}\,dx$ (b) $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}}$ (c) $\displaystyle\int \frac{dy}{y\sqrt{5y^2-3}}$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

Formula 5 , 6 and 7

5.4 THE DEFINITION OF AREA AS A LIMIT; SIGMA NOTATION

The index k ends at $k = n$.

The summation symbol (Greek letter sigma) — $\displaystyle\sum_{k=1}^{n} a_k$ — a_k is a formula for the kth term.

The index k starts at $k = 1$.

▸ **Example 1**

The sum in sigma notation

$$\sum_{k=1}^{5} k$$

$$\sum_{k=1}^{3} (-1)^k k$$

$$\sum_{k=1}^{2} \frac{k}{k+1}$$

$$\sum_{k=4}^{5} \frac{k^2}{k-1}$$

$$\sum_{k=4}^{8} k^3 :$$

$$\sum_{k=1}^{5} 2k :$$

EXERCISE SET 5.4 C CAS

1. Evaluate.

(a) $\sum_{k=1}^{3} k^3$ (b) $\sum_{j=2}^{6} (3j-1)$ (c) $\sum_{i=-4}^{1} (i^2-i)$

(d) $\sum_{n=0}^{5} 1$ (e) $\sum_{k=0}^{4} (-2)^k$ (f) $\sum_{n=1}^{6} \sin n\pi$

10. Express in sigma notation.

(a) $a_1 - a_2 + a_3 - a_4 + a_5$

(b) $-b_0 + b_1 - b_2 + b_3 - b_4 + b_5$

(c) $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

(d) $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$

3–8 Write each expression in sigma notation but do not evaluate.

3. $1 + 2 + 3 + \cdots + 10$

4. $3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + \cdots + 3 \cdot 20$

5. $2 + 4 + 6 + 8 + \cdots + 20$

6. $1 + 3 + 5 + 7 + \cdots + 15$

7. $1 - 3 + 5 - 7 + 9 - 11$

8. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$

MACLAURIN SERIES

(uses of sigma in series)

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

5.4.1 THEOREM

(a) $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k \qquad (\textit{if } c \textit{ does not depend on } k)$$

(b) $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

(c) $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

5.4.2 THEOREM

(a) $$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(b) $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(c) $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

5.4.4 THEOREM

(a) $\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} 1 = 1$

(b) $\lim_{n \to +\infty} \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{2}$

(c) $\lim_{n \to +\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{3}$

(d) $\lim_{n \to +\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \frac{1}{4}$

► **Example 2** Evaluate $\sum_{k=1}^{30} k(k+1)$.

14. $\sum_{k=4}^{20} k^2$

15. $\sum_{k=1}^{30} k(k-2)(k+2)$

16. $\sum_{k=1}^{6} (k - k^3)$

Telescoping sum:

Consider the sum

$$\sum_{k=1}^{4}[(k+1)^3 - k^3]$$

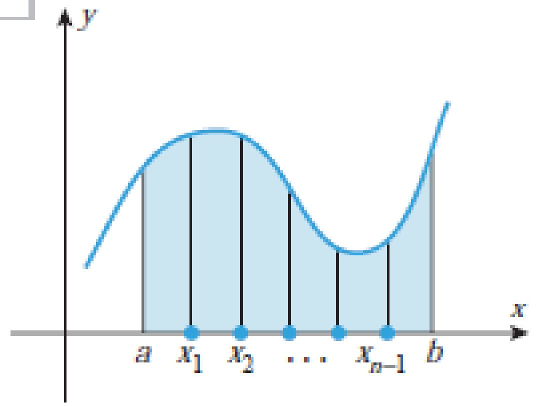
A sum is said to ***telescope*** when part of each term cancels part of an adjacent term, leaving only portions of the first and last terms uncanceled. Evaluate the telescoping sums in these exercises.

57. $\sum_{k=5}^{17}(3^k - 3^{k-1})$

58. $\sum_{k=1}^{50}\left(\frac{1}{k} - \frac{1}{k+1}\right)$

5.4.3 DEFINITION ***(Area Under a Curve)*** If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the ***area*** A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

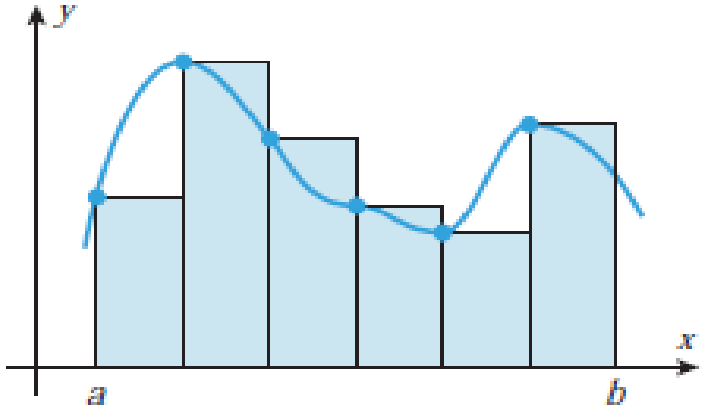
$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \qquad (2)$$



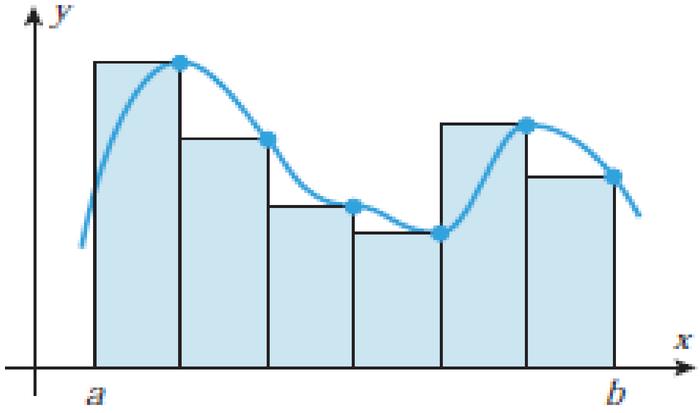
$$x_k^* = x_{k-1} = a + (k-1)\Delta x$$ Left endpoint

$$x_k^* = x_k = a + k\Delta x$$ Right endpoint

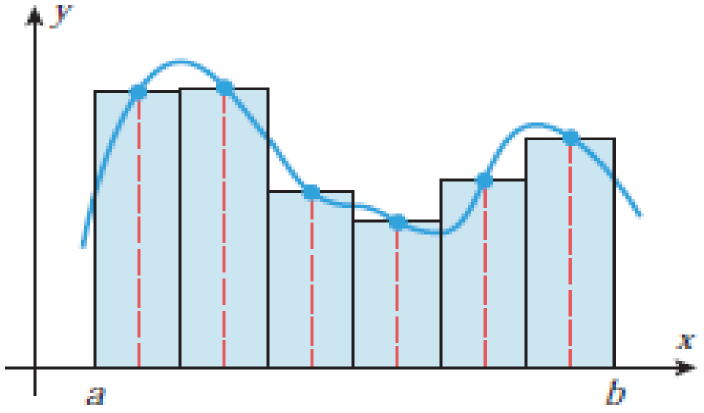
$$x_k^* = \tfrac{1}{2}(x_{k-1} + x_k) = a + \left(k - \tfrac{1}{2}\right)\Delta x$$ Midpoint



Left endpoint approximation



Right endpoint approximation



Midpoint approximation

► **Example 4** Use Definition 5.4.3 with x_k^* as the right endpoint of each subinterval to find the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$.

► **Example 7** Confirm that the net signed area between the graph of $f(x) = x - 1$ and the interval $[0, 2]$ is zero by using Definition 5.4.5 with x_k^* chosen to be the left endpoint of each subinterval.

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \qquad \text{Left endpoint}$$

$$x_k^* = x_k = a + k\Delta x \qquad \text{Right endpoint}$$

$$x_k^* = \tfrac{1}{2}(x_{k-1} + x_k) = a + \left(k - \tfrac{1}{2}\right)\Delta x \qquad \text{Midpoint}$$

27–30 Divide the specified interval into $n = 4$ subintervals of equal length and then compute

$$\sum_{k=1}^{4} f(x_k^*)\Delta x$$

with x_k^* as (a) the left endpoint of each subinterval, (b) the mid-point of each subinterval, and (c) the right endpoint of each

27. $f(x) = 3x + 1$; $[2, 6]$ **28.** $f(x) = 1/x$; $[1, 9]$

Exercise 5.4

35–40 Use Definition 5.4.3 with x_k^* as the *right* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

35. $f(x) = x/2;\ [1, 4]$

36. $f(x) = 5 - x;\ [0, 5]$

37. $f(x) = 9 - x^2;\ [0, 3]$

38. $f(x) = 4 - \frac{1}{4}x^2;\ [0, 3]$

39. $f(x) = x^3;\ [2, 6]$

40. $f(x) = 1 - x^3;\ [-3, -1]$

41–44 Use Definition 5.4.3 with x_k^* as the *left* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

41. $f(x) = x/2;\ [1, 4]$

42. $f(x) = 5 - x;\ [0, 5]$

43. $f(x) = 9 - x^2;\ [0, 3]$

44. $f(x) = 4 - \frac{1}{4}x^2;\ [0, 3]$

45–48 Use Definition 5.4.3 with x_k^* as the *midpoint* of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

45. $f(x) = 2x;\ [0, 4]$

46. $f(x) = 6 - x;\ [1, 5]$

47. $f(x) = x^2;\ [0, 1]$

48. $f(x) = x^2;\ [-1, 1]$

Class Activity:

Q-37 Find Area with $x_k = a + k\Delta x$ as the right end point of subinterval for the curve $f(x) = 9 - x^2 \; over \; [0,3]$

- Solution:

RIEMANN Sum:

5.5.1 DEFINITION A function f is said to be *integrable* on a finite closed interval $[a, b]$ if the limi

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

5.5.2 THEOREM *If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is*

$$A = \int_a^b f(x)\,dx \qquad (1)$$

PROPERTIES OF THE DEFINITE INTEGRAL

5.5.3 DEFINITION

(a) If a is in the domain of f, we define

$$\int_a^a f(x)\,dx = 0$$

(b) If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

5.5.5 THEOREM *If f is integrable on a closed interval containing the three points a, b, and c, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

no matter how the points are ordered.

PROPERTIES OF THE DEFINITE INTEGRAL

Using Symmetry to Evaluate Integrals

- An EVEN function is symmetric about the y-axis
- An ODD function is symmetric about the origin

If f is EVEN, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

If f is ODD, then

$$\int_{-a}^{a} f(x)dx = 0$$

EXERCISE SET 5.5

1–4 Find the value of

(a) $\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ (b) $\max \Delta x_k$. ■

1. $f(x) = x + 1;\ a = 0, b = 4;\ n = 3;$

$\Delta x_1 = 1, \Delta x_2 = 1, \Delta x_3 = 2;$

$x_1^* = \frac{1}{3}, x_2^* = \frac{3}{2}, x_3^* = 3$

2. $f(x) = \cos x;\ a = 0, b = 2\pi;\ n = 4;$

$\Delta x_1 = \pi/2, \Delta x_2 = 3\pi/4, \Delta x_3 = \pi/2, \Delta x_4 = \pi/4;$

$x_1^* = \pi/4, x_2^* = \pi, x_3^* = 3\pi/2, x_4^* = 7\pi/4$

5–8 Use the given values of a and b to express the following limits as integrals. (Do not evaluate the integrals.)

5. $\lim\limits_{\max \Delta x_k \to 0} \sum\limits_{k=1}^{n} (x_k^*)^2 \Delta x_k$; $a = -1, b = 2$

9–10 Use Definition 5.5.1 to express the integrals as limits of Riemann sums. (Do not evaluate the integrals.)

9. (a) $\int_1^2 2x\,dx$ (b) $\int_0^1 \frac{x}{x+1}\,dx$

10. (a) $\int_1^2 \sqrt{x}\,dx$ (b) $\int_{-\pi/2}^{\pi/2} (1 + \cos x)\,dx$

Exercise 5.5

18. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} 2x, & x \le 1 \\ 2, & x > 1 \end{cases}$$

(a) $\int_0^1 f(x)\,dx$ (b) $\int_{-1}^1 f(x)\,dx$

(c) $\int_1^{10} f(x)\,dx$ (d) $\int_{1/2}^5 f(x)\,dx$

Evaluate $\int_0^3 f(x)\,dx$ if

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \ge 2 \end{cases}$$

21. Find $\int_{-1}^2 [f(x) + 2g(x)]\,dx$ if

$$\int_{-1}^2 f(x)\,dx = 5 \quad \text{and} \quad \int_{-1}^2 g(x)\,dx = -3$$

22. Find $\int_1^4 [3f(x) - g(x)]\,dx$ if

$$\int_1^4 f(x)\,dx = 2 \quad \text{and} \quad \int_1^4 g(x)\,dx = 10$$

23. Find $\int_1^5 f(x)\,dx$ if

$$\int_0^1 f(x)\,dx = -2 \quad \text{and} \quad \int_0^5 f(x)\,dx = 1$$

24. Find $\int_3^{-2} f(x)\,dx$ if

$$\int_{-2}^1 f(x)\,dx = 2 \quad \text{and} \quad \int_1^3 f(x)\,dx = -6$$

Exercise 5.5

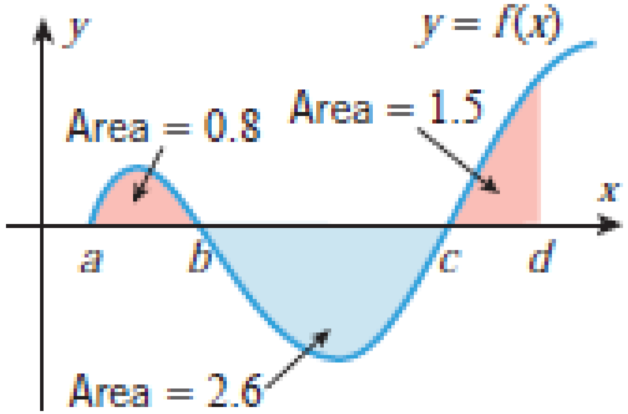
FOCUS ON CONCEPTS

19–20 Use the areas shown in the figure to find

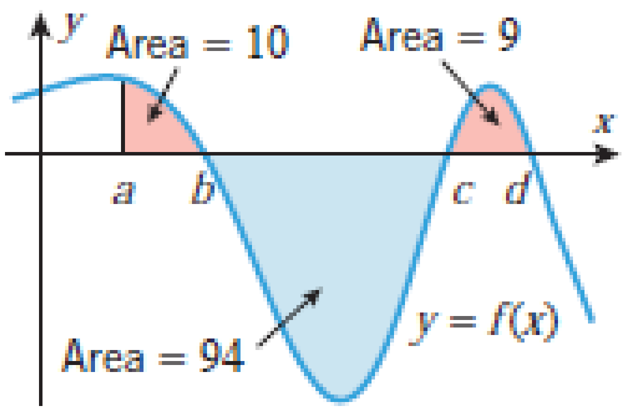
(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$

(c) $\int_a^c f(x)\,dx$ (d) $\int_a^d f(x)\,dx.$

19.



20.



Chapter 7 An Overview of Integration Methods

- 7.1 Substitution and basic integration
- 7.2 By parts and tabular integration by parts
- 7.2 Reduction formula
- 7.4 Trigonometric substitution
- 7.4 Completing square
- 7.5 Partial fraction
- 7.6 U = tan(x/2) substitution
- 7.8 Improper integral
- 6.9 Hyperbolic derivative and Integral

A REVIEW OF FAMILIAR INTEGRATION FORMULAS

1. $\int du = u + C$
2. $\int k\,du = ku + C$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C$ $(n \neq -1)$
5. $\int \frac{du}{u} = \ln |u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln |\cos u| + C$
13. $\int \cot u\,du = \ln |\sin u| + C$
    $= -\ln |\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C$ $(a > 0, a \neq$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C$

EXERCISE SET 7.1

1–30 Evaluate the integrals by making appropriate u-substitutions and applying the formulas reviewed in this section.

1. $\int (4-2x)^3\,dx$

2. $\int 3\sqrt{4+2x}\,dx$

3. $\int x\sec^2(x^2)\,dx$

4. $\int 4x\tan(x^2)\,dx$

5. $\int \frac{\sin 3x}{2+\cos 3x}\,dx$

6. $\int \frac{1}{9+4x^2}\,dx$

7. $\int e^x\sinh(e^x)\,dx$

8. $\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx$

9. $\int e^{\tan x}\sec^2 x\,dx$

10. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

11. $\int \cos^5 5x\sin 5x\,dx$

12. $\int \frac{\cos x}{\sin x\sqrt{\sin^2 x+1}}\,dx$

13. $\int \frac{e^x}{\sqrt{4+e^{2x}}}\,dx$

14. $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

15. $\int \frac{e^{\sqrt{x-1}}}{\sqrt{x-1}}\,dx$

16. $\int (x+1)\cot(x^2+2x)\,dx$

17. $\int \frac{\cosh\sqrt{x}}{\sqrt{x}}\,dx$

18. $\int \frac{dx}{x(\ln x)^2}$

19. $\int \frac{dx}{\sqrt{x}\,3^{\sqrt{x}}}$

7.2 INTEGRATION BY PARTS

$$\int u\,dv = uv - \int v\,du$$

strategy for choosing u

LIATE Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential

▶ **Example 2** Evaluate $\int xe^x\,dx$.

▶ **Example 3** Evaluate $\int \ln x\,dx$.

▶ **Example 4** Evaluate $\int x^2e^{-x}\,dx$.

▶ **Example 7** Evaluate $\int_0^1 \tan^{-1}x\,dx$.

Try now

Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Example: $\int x^2\sqrt{x-1}\,dx$

Solution.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/2}$
0		$\frac{8}{105}(x-1)^{7/2}$

$$\int x^2\sqrt{x-1}\,dx = \tfrac{2}{3}x^2(x-1)^{3/2} - \tfrac{8}{15}x(x-1)^{5/2} + \tfrac{16}{105}(x-1)^{7/2} + C$$

$$\int x^2\sqrt{x-1}\,dx$$

9. $\int x \ln x \, dx$ **10.** $\int \sqrt{x} \ln x \, dx$

11. $\int (\ln x)^2 \, dx$ **12.** $\int \frac{\ln x}{\sqrt{x}} \, dx$

13. $\int \ln(3x-2) \, dx$ **14.** $\int \ln(x^2+4) \, dx$

23. $\int x \sec^2 x \, dx$ **24.** $\int x \tan^2 x \, dx$

25. $\int x^3 e^{x^2} \, dx$ **26.** $\int \frac{xe^x}{(x+1)^2} \, dx$

43–44 Evaluate the integral by making a u-substitution and then integrating by parts.

43. $\int e^{\sqrt{x}} \, dx$ **44.** $\int \cos \sqrt{x} \, dx$

REDUCTION FORMULAS

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$$(1) \qquad \int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

$$(2) \qquad \int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

$$(3) \qquad \int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$

REDUCTION FORMULAS

$$(4) \qquad \int \cot^n(x)\,dx = -\frac{1}{n-1}\cot^{n-1}(x) - \int \cot^{n-2}(x)\,dx$$

$$(5) \qquad \int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

$$(6) \qquad \int \csc^n(x)\,dx = -\frac{1}{n-1}\csc^{n-2}(x)\cot(x) + \frac{n-2}{n-1}\int \csc^{n-2}(x)\,dx$$

Apply a reduction formula to evaluate $\int \sec^3 x\, dx$.

Solution

By applying the first reduction formula, we obtain

$$\begin{aligned}\int \sec^3 x\, dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\, dx \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

Evaluate $\int \tan^4 x\, dx$..

$$\begin{aligned}\int \tan^4 x\, dx &= \frac{1}{3}\tan^3 x - \int \tan^2 x\, dx \\ &= \frac{1}{3}\tan^3 x - (\tan x - \int \tan^0 x\, dx) \\ &= \frac{1}{3}\tan^3 x - \tan x + \int 1\, dx \\ &= \frac{1}{3}\tan^3 x - \tan x + x + C.\end{aligned}$$

Proof-1

$$\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

$$u = \cos^{n-1}x \qquad \text{and} \qquad dv = \cos x\,dx,$$

$$du = (n-1)\cos^{n-2}x\,(-\sin x\,dx) \qquad \text{and} \qquad v = \sin x.$$

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1}x\sin x + (n-1)\int \sin^2 x\cos^{n-2}x\,dx\\ &= \cos^{n-1}x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,dx,\\ &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

$$n\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx.$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

Proof-6

- $$\int \operatorname{cosec}^n x \, dx = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$$

Integrating by parts,

$$\int \operatorname{cosec}^n x \, dx = \operatorname{cosec}^{n-2} x\,(-\cot x) - \int (n-2)\operatorname{cosec}^{n-3} x\,(-\operatorname{cosec} x \cot x)(-\cot x)\, dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x\,(\operatorname{cosec}^2 x - 1)\, dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\left(\int \operatorname{cosec}^n x - \int \operatorname{cosec}^{n-2} x \, dx\right)$$

$$[1+(n-2)]\int \operatorname{cosec}^n x \, dx = -\cot x \operatorname{cosec}^{n-2} x + (n-2)\int \operatorname{cosec}^{n-2} x \, dx$$

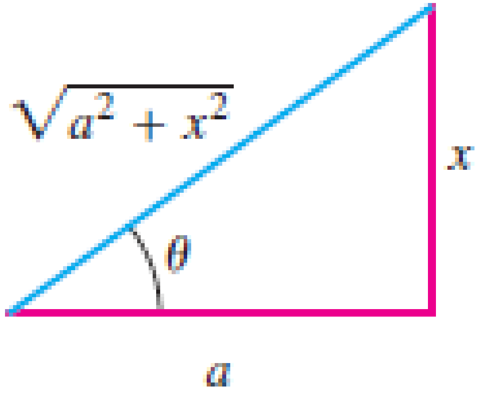
$$\int \operatorname{cosec}^n x \, dx = \frac{-\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x \, dx$$

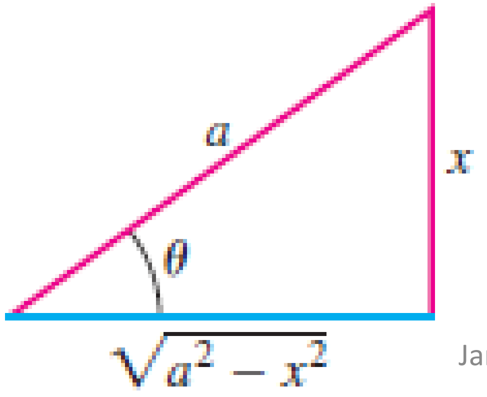
7.4 TRIGONOMETRIC SUBSTITUTIONS

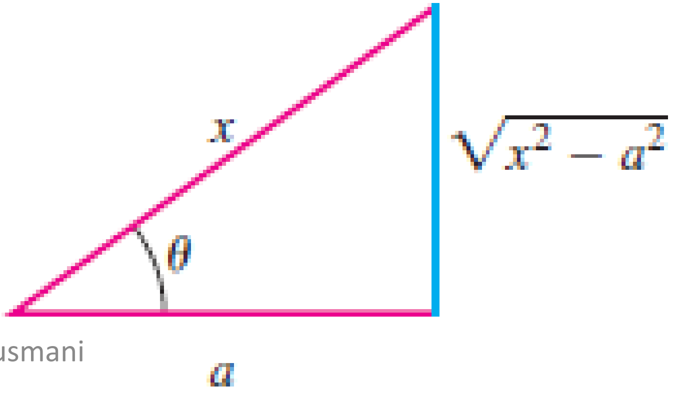
EXPRESSION IN THE INTEGRAND	SUBSTITUTION
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$

▶ **Example 1** Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{4 - x^2}}$.

▶ **Example 5** Evaluate $\displaystyle\int \frac{\sqrt{x^2 - 25}}{x}\,dx$,







Using the Substitution $x = a\tan\theta$

$$\int \frac{dx}{\sqrt{4+x^2}}.$$

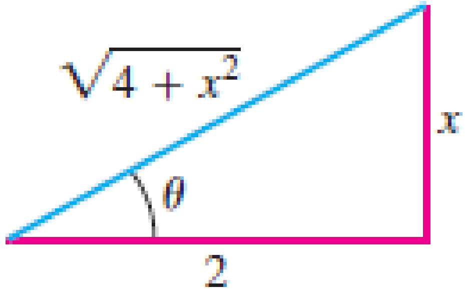
Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta,$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta.$$

Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \\
&= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C \\
&= \ln\left|\sqrt{4+x^2} + x\right| + C'.
\end{aligned}$$



EXERCISE SET 7.4 $\boxed{C}$ CAS

1–26 Evaluate the integral. ■

1. $\displaystyle\int \sqrt{4-x^2}\,dx$

2. $\displaystyle\int \sqrt{1-4x^2}\,dx$

3. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

4. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$

5. $\displaystyle\int \frac{dx}{(4+x^2)^2}$

6. $\displaystyle\int \frac{x^2}{\sqrt{5+x^2}}\,dx$

7. $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$

8. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-16}}$

9. $\displaystyle\int \frac{3x^3}{\sqrt{1-x^2}}\,dx$

10. $\displaystyle\int x^3\sqrt{5-x^2}\,dx$

11. $\displaystyle\int \frac{dx}{x^2\sqrt{9x^2-4}}$

12. $\displaystyle\int \frac{\sqrt{1+t^2}}{t}\,dt$

13. $\displaystyle\int \frac{dx}{(1-x^2)^{3/2}}$

14. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+25}}$

INTEGRALS INVOLVING $ax^2 + bx + c$ [Completing the square]

37–48 Evaluate the integral.

37. $\displaystyle\int \frac{dx}{x^2 - 4x + 5}$

38. $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$

39. $\displaystyle\int \frac{dx}{\sqrt{3 + 2x - x^2}}$

40. $\displaystyle\int \frac{dx}{16x^2 + 16x + 5}$

41. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 6x + 10}}$

42. $\displaystyle\int \frac{x}{x^2 + 2x + 2}\,dx$

43. $\displaystyle\int \sqrt{3 - 2x - x^2}\,dx$

44. $\displaystyle\int \frac{e^x}{\sqrt{1 + e^x + e^{2x}}}\,dx$

45. $\displaystyle\int \frac{dx}{2x^2 + 4x + 7}$

46. $\displaystyle\int \frac{2x + 3}{4x^2 + 4x + 5}\,dx$

47. $\displaystyle\int_1^2 \frac{dx}{\sqrt{4x - x^2}}$

48. $\displaystyle\int_0^4 \sqrt{x(4 - x)}\,dx$

Exercise 7.6

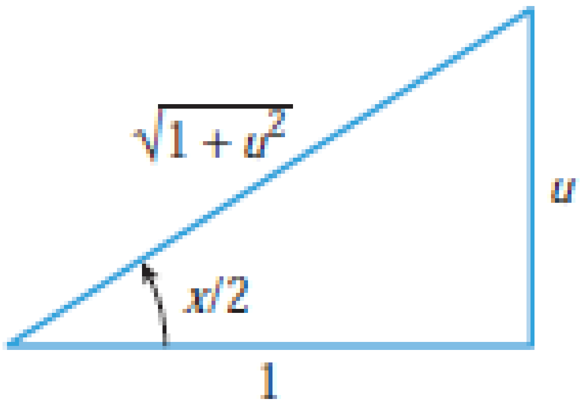
$$u = \tan(x/2),$$

$$x = 2\tan^{-1} u, \quad dx = \frac{2}{1+u^2}\,du$$

$$\sin x = 2\sin(x/2)\cos(x/2)$$
$$\cos x = \cos^2(x/2) - \sin^2(x/2)$$

$$\sin x = 2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right) = \frac{2u}{1+u^2}$$

$$\cos x = \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2 = \frac{1-u^2}{1+u^2}$$



$$\sin(x/2) = \frac{u}{\sqrt{1+u^2}} \quad \text{and} \quad \cos(x/2) = \frac{1}{\sqrt{1+u^2}}$$

Examples

a. $$\int \frac{1}{1+\cos x}\,dx = \int \frac{1+z^2}{2}\,\frac{2\,dz}{1+z^2}$$

$$= \int dz = z + C$$

$$= \tan\left(\frac{x}{2}\right) + C$$

b. $$\int \frac{1}{2+\sin x}\,dx = \int \frac{1+z^2}{2+2z+2z^2}\,\frac{2\,dz}{1+z^2}$$

$$= \int \frac{dz}{z^2+z+1} = \int \frac{dz}{(z+(1/2))^2+3/4}$$

$$= \int \frac{du}{u^2+a^2}$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1+2\tan(x/2)}{\sqrt{3}} + C$$

Exercise 7.6

65. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

66. $\displaystyle\int \frac{dx}{2+\sin x}$

67. $\displaystyle\int \frac{d\theta}{1-\cos\theta}$

68. $\displaystyle\int \frac{dx}{4\sin x-3\cos x}$

69. $\displaystyle\int \frac{dx}{\sin x+\tan x}$

70. $\displaystyle\int \frac{\sin x}{\sin x+\tan x}\,dx$

Determine: $\displaystyle\int \frac{dx}{\sin x+\cos x}$

Determine: $\displaystyle\int \frac{d\theta}{5+4\cos\theta}$

Determine: $\displaystyle\int \frac{dx}{7-3\sin x+6\cos x}$

$$\int \frac{\cos x\, dx}{1+\cos x}$$

7.5 Partial Fraction:

Form of the Rational Function	Form of the Partial Fraction
$\dfrac{px+q}{(x-a)(x-b)}, a \neq b$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}$
$\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}$
$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$
Where x^2+bx+c cannot be factorised further	

Basic structure for decomposition

Example-1

Integrating with an Irreducible Quadratic Factor in the Denominator

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\, dx$$

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}.$$

$$\begin{aligned} -2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D). \end{aligned}$$

Equating coefficients of like terms gives

Coefficients of x^3:	$0 = A + C$	$-4 = -2A, \quad A = 2$
Coefficients of x^2:	$0 = -2A + B - C + D$	$C = -A = -2$
Coefficients of x^1:	$-2 = A - 2B + C$	$B = 1$
Coefficients of x^0:	$4 = B - C + D$	$D = 4 - B + C = 1.$

We solve these equations simultaneously to find the values of A, B, C, and D:

Integrating with an Irreducible Quadratic Factor in the Denominator

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}.$$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

$$\begin{aligned}
\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\
&= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\
&= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C.
\end{aligned}$$

Example-2

Evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned} 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

$$A=1, \quad B=-1, \quad C=0, \quad D=-1, \text{ and } E=0.$$

$$\int \frac{dx}{x(x^2+1)^2} = \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] = \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K$$

Exercise7.5

11. $\displaystyle\int \frac{11x+17}{2x^2+7x-4}\,dx$

12. $\displaystyle\int \frac{5x-5}{3x^2-8x-3}\,dx$

13. $\displaystyle\int \frac{2x^2-9x-9}{x^3-9x}\,dx$

14. $\displaystyle\int \frac{dx}{x(x^2-1)}$

15. $\displaystyle\int \frac{x^2-8}{x+3}\,dx$

16. $\displaystyle\int \frac{x^2+1}{x-1}\,dx$

17. $\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx$

18. $\displaystyle\int \frac{x^2}{x^2-3x+2}\,dx$

19. $\displaystyle\int \frac{2x-3}{x^2-3x-10}\,dx$

20. $\displaystyle\int \frac{3x+1}{3x^2+2x-1}\,dx$

21. $\displaystyle\int \frac{x^5+x^2+2}{x^3-x}\,dx$

22. $\displaystyle\int \frac{x^5-4x^3+1}{x^3-4x}\,dx$

23. $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}\,dx$

24. $\displaystyle\int \frac{3x^2-x+1}{x^3-x^2}\,dx$

6.9 HYPERBOLIC FUNCTIONS

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{Even}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{Odd}}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

6.9.2 THEOREM

$$\cosh x + \sinh x = e^x$$

$$\cosh x - \sinh x = e^{-x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\coth^2 x - 1 = \operatorname{csch}^2 x$$

6.9.3 THEOREM

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx} \qquad \int \cosh u \, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx} \qquad \int \sinh u \, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx} \qquad \int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx} \qquad \int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx} \qquad \int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx} \qquad \int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

6.9.6 THEOREM *If $a > 0$, then*

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C$$

6.9.4 THEOREM *The following relationships hold for all x in the domains of the stated inverse hyperbolic functions:*

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

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▶ **Example 6** Evaluate $\int \frac{dx}{\sqrt{4x^2-9}}$, $x > \frac{3}{2}$.

Prove that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

Proof:

Let $y = sinh^{-1}x$

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$e^y - 2x - e^{-y} = 0$$

Multiplying this equation through by e^y

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Since $e^y > 0$, the solution involving the minus sign is extraneous and must be discarded. Thus,

$$e^y = x + \sqrt{x^2 + 1}$$

Taking natural logarithms yields

$$y = \ln(x + \sqrt{x^2 + 1}) \quad \text{or} \quad \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

Derivative of Inverse hyperbolic fn:

6.9.5 THEOREM

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$$

$$\frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d}{dx}(\operatorname{csch}^{-1} u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$\begin{aligned} y &= \sinh^{-1} x \\ \sinh y &= x \\ \frac{d}{dx}\sinh y &= \frac{d}{dx}x \\ \cosh y\frac{dy}{dx} &= 1. \end{aligned}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1 + \sinh^2 y}$. Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.$$

9–28 Find dy/dx.

Exercise 6.9

9. $y = \sinh(4x - 8)$

10. $y = \cosh(x^4)$

11. $y = \coth(\ln x)$

12. $y = \ln(\tanh 2x)$

13. $y = \operatorname{csch}(1/x)$

14. $y = \operatorname{sech}(e^{2x})$

15. $y = \sqrt{4x + \cosh^2(5x)}$

16. $y = \sinh^3(2x)$

17. $y = x^3 \tanh^2(\sqrt{x})$

18. $y = \sinh(\cos 3x)$

19. $y = \sinh^{-1}\left(\frac{1}{3}x\right)$

20. $y = \sinh^{-1}(1/x)$

21. $y = \ln(\cosh^{-1} x)$

22. $y = \cosh^{-1}(\sinh^{-1} x)$

23. $y = \dfrac{1}{\tanh^{-1} x}$

24. $y = (\coth^{-1} x)^2$

25. $y = \cosh^{-1}(\cosh x)$

26. $y = \sinh^{-1}(\tanh x)$

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{4x^2 - 1}} dx$

b. $\int \frac{1}{2x\sqrt{1 - 9x^2}} dx$

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{x^2 - 4}} dx, \quad x > 2$

b. $\int \frac{1}{\sqrt{1 - e^{2x}}} dx$

Solution

We can use u-substitution in both cases.

a. Let $u = 2x$. Then, $du = 2dx$ and we have

$$\int \frac{1}{\sqrt{4x^2 - 1}} dx = \int \frac{1}{2\sqrt{u^2 - 1}} du = \frac{1}{2}\cosh^{-1} u + C = \frac{1}{2}\cosh^{-1}(2x) + C.$$

b. Let $u = 3x$. Then, $du = 3dx$ and we obtain

$$\int \frac{1}{2x\sqrt{1 - 9x^2}} dx = \frac{1}{2}\int \frac{1}{u\sqrt{1 - u^2}} du = -\frac{1}{2}\text{sech}^{-1}|u| + C = -\frac{1}{2}\text{sech}^{-1}|3x| + C.$$

29–44 Evaluate the integrals.

29. $\int \sinh^6 x \cosh x \, dx$

30. $\int \cosh(2x - 3) \, dx$

31. $\int \sqrt{\tanh x} \operatorname{sech}^2 x \, dx$

32. $\int \operatorname{csch}^2(3x) \, dx$

33. $\int \tanh x \, dx$

34. $\int \coth^2 x \operatorname{csch}^2 x \, dx$

35. $\int_{\ln 2}^{\ln 3} \tanh x \operatorname{sech}^3 x \, dx$

36. $\int_0^{\ln 3} \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

37. $\int \frac{dx}{\sqrt{1 + 9x^2}}$

38. $\int \frac{dx}{\sqrt{x^2 - 2}} \quad (x > \sqrt{2})$

39. $\int \frac{dx}{\sqrt{1 - e^{2x}}} \quad (x < 0)$

40. $\int \frac{\sin \theta \, d\theta}{\sqrt{1 + \cos^2 \theta}}$

Home Activity:

1. $\int e^{-x} \sinh 2x dx$
2. $\int_0^{ln2} 4\, e^x \sinh x dx$
3. $\int \frac{dx}{\sinh x - \cosh x}$
4. $\int \frac{dx}{\sinh x + 2\cosh x}$
5. $\int \frac{dx}{3\sinh x - 5\cosh x}$
6. $\int \frac{dx}{1 + \cosh x}$

Practice

60. Prove:

(a) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$

(b) $\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \quad -1 < x < 1.$

62. Prove:

$$\operatorname{sech}^{-1} x = \cosh^{-1}(1/x), \quad 0 < x \leq 1$$

$$\coth^{-1} x = \tanh^{-1}(1/x), \quad |x| > 1$$

$$\operatorname{csch}^{-1} x = \sinh^{-1}(1/x), \quad x \neq 0$$

64. Show that

(a) $\frac{d}{dx}[\operatorname{sech}^{-1}|x|] = -\frac{1}{x\sqrt{1-x^2}}$

(b) $\frac{d}{dx}[\operatorname{csch}^{-1}|x|] = -\frac{1}{x\sqrt{1+x^2}}.$

55. In parts (a)–(f) find the limits, and confirm that they are consistent with the graphs in Figures 6.9.1 and 6.9.6.

(a) $\lim_{x \to +\infty} \sinh x$ (b) $\lim_{x \to -\infty} \sinh x$

(c) $\lim_{x \to +\infty} \tanh x$ (d) $\lim_{x \to -\infty} \tanh x$

(e) $\lim_{x \to +\infty} \sinh^{-1} x$ (f) $\lim_{x \to 1^-} \tanh^{-1} x$

Improper Integrals

There are two types of improper integrals:

- The limit ’a’ or ’b’ (or both the limits) are infinite;
- The function f(x) has one or more points of discontinuity in the given interval [a,b].

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx .$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx .$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx ,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Class Activity:

Example 1 Evaluate

$$\text{(a)} \int_1^{+\infty} \frac{dx}{x^3} \qquad \text{(b)} \int_1^{+\infty} \frac{dx}{x}$$

Example 3 Evaluate $\displaystyle\int_0^{+\infty} (1-x)e^{-x}\,dx$.

Example 4 Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

Example 4 Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \tan^{-1} x \Big]_a^0$$

$$= \lim_{a \to -\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{b \to \infty} \tan^{-1} x \Big]_0^b$$

$$= \lim_{b \to \infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

An Incorrect Calculation

$$\int_0^3 \frac{dx}{x-1}.$$

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big]_0^3 = \ln 2 - \ln 1 = \ln 2.$$ convergent

This result is *wrong* because the integral is improper. The correct evaluation uses limits:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

Divergent

Type II

Example

$$\int_0^1 \frac{dx}{x-1} = \lim_{b \to 1^-} \int_0^b \frac{dx}{x-1}$$

$$= \lim_{b \to 1^-} \ln|x-1| \Big]_0^b$$

$$= \lim_{b \to 1^-} (\ln|b-1| - \ln|-1|)$$

$1 - b \to 0^+$ as $b \to 1^-$

$$= \lim_{b \to 1^-} \ln(1-b) = -\infty.$$

Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}. \quad = 6.78$$

Solution:

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b$$

$$= \lim_{b\to 1^-}\left[3(b-1)^{1/3} + 3\right]$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3$$

$$= \lim_{c\to 1^+}\left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}$$

Properties of log:

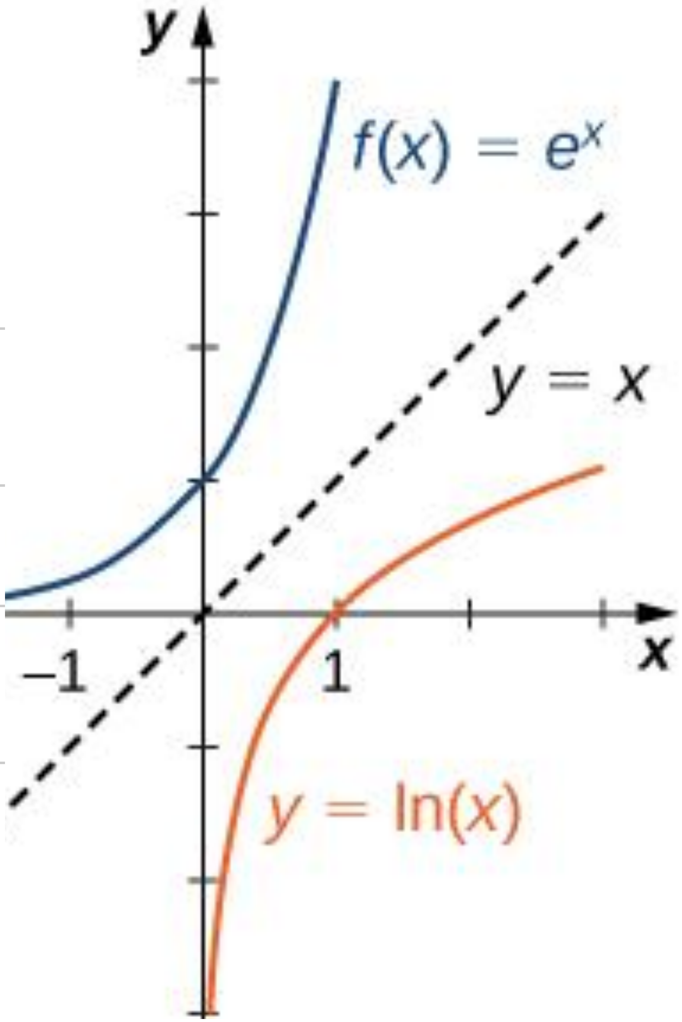
$\ln(x)$ is undefined when $x \leq 0$

$\ln(0)$ is undefined

$\lim_{x \to 0^+} \ln(x) = -\infty$

$\ln(1) = 0$

$\lim \ln(x) = \infty$, when $x \longrightarrow \infty$



x	$\ln x$
0	undefined
0^+	$-\infty$
0.0001	-9.210340
0.001	-6.907755
0.01	-4.605170
0.1	-2.302585
1	0
2	0.693147
e ≈ 2.7183	1
3	1.098612
4	1.386294
5	1.609438
6	1.791759

Self check :

Evaluate

$$\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}\,dx. \approx 1.1458$$

Check for convergence $\displaystyle\int_0^2 \frac{dx}{1-x}$?

EXERCISE SET 7.8 Graphing Utility C CAS

1. In each part, determine whether the integral is improper, and if so, explain why.

(a) $\int_1^5 \frac{dx}{x-3}$ (b) $\int_1^5 \frac{dx}{x+3}$ (c) $\int_0^1 \ln x\, dx$

(d) $\int_1^{+\infty} e^{-x}\, dx$ (e) $\int_{-\infty}^{+\infty} \frac{dx}{\sqrt[3]{x-1}}$ (f) $\int_0^{\pi/4} \tan x\, dx$

3–32 Evaluate the integrals that converge.

3. $\displaystyle\int_0^{+\infty} e^{-2x}\,dx$

4. $\displaystyle\int_{-1}^{+\infty} \frac{x}{1+x^2}\,dx$

5. $\displaystyle\int_3^{+\infty} \frac{2}{x^2-1}\,dx$

6. $\displaystyle\int_0^{+\infty} xe^{-x^2}\,dx$

7. $\displaystyle\int_e^{+\infty} \frac{1}{x\ln^3 x}\,dx$

8. $\displaystyle\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}}\,dx$

9. $\displaystyle\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$

10. $\displaystyle\int_{-\infty}^{3} \frac{dx}{x^2+9}$

11. $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx$

12. $\displaystyle\int_{-\infty}^{0} \frac{e^x\,dx}{3-2e^x}$

13. $\displaystyle\int_{-\infty}^{+\infty} x\,dx$

14. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\,dx$

15. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{(x^2+3)^2}\,dx$

16. $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{-t}}{1+e^{-2t}}\,dt$

17. $\displaystyle\int_0^4 \frac{dx}{(x-4)^2}$

18. $\displaystyle\int_0^8 \frac{dx}{\sqrt[3]{x}}$

19. $\displaystyle\int_0^{\pi/2} \tan x\,dx$

20. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

Name: Std ID: Section:

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QUIZ-2 25-10-2019

Evaluate (any three)

a) Derive reduction formula for $\int sec^n x dx$ and evaluate $\int sec^4 x dx$

b) Show that $cosh^{-1}x = ln\left(x + \sqrt{x^2 - 1}\right)$

c) Evaluate $\int \frac{dx}{sinx+cosx}$ OR $\int \frac{dx}{1+coshx}$

d) Compute $A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$ as a right end-point for curve $f(x) = \frac{x}{2}$, $over\ [1,4]$

Evaluate (any three)

a) Use $A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$ as a left end-point for curve $f(x) = \frac{x}{2}$, $over$ $[1,4]$

b) Derive reduction formula for $\int sin^n x dx$ and evaluate $\int sin^4 x dx$

c) Show that $cosh^{-1} x = ln\left(x + \sqrt{x^2 - 1}\right)$

d) Evaluate $\int \frac{dx}{sinhx+2coshx}$ OR $\int \frac{sinx\, dx}{sinx+tanx}$

Evaluate (any three)

a) Derive reduction formula for $\int cosec^n x dx$ and evaluate $\int cosec^4 x dx$

b) Show that $\frac{d}{dx} tanh^{-1} x = \frac{1}{1-x^2}$

c) Evaluate $\int \frac{dx}{5+4cosx}$ OR $\int \frac{dx}{\sqrt{1-e^{2x}}}$

d) Compute $A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$ as a right end-point for curve $f(x) = 5 - x$, over $[0,5]$