Constraint Satisfaction Problems

Standard Search Problems

- Previous search problems:
  - Problems can be solved by searching in a space of states
  - state is a “black box” – any data structure that supports successor function, heuristic function, and goal test – problem-specific
- Constraint satisfaction problem
  - states and goal test conform to a standard, structured and simple representation
  - general-purpose heuristic

Constraint Satisfaction Problems (CSP)

- CSP is defined by 3 components (X, D, C):
  - state: a set of variables X, each X_i , with values from domain D_i
  - goal test: a set of constraints C, each C_i involves some subset of the variables and specifies the allowable combinations of values for that subset
  - Each constraint C_i consists of a pair <scope, rel>, where scope is a tuple of variables and rel is the relation, either represented explicitly or abstractly
- X1 and X2 both have the domain {A, B}
  - Constraints:
    - <(X1, X2), [(A, B), (B, A)]>, or
    - <(X1, X2), X1 ≠ X2>

Solution

- Each state in a CSP is defined by an assignment of values to some or all of the variables
- An assignment that does not violate any constraints is called a consistent or legal assignment
- A complete assignment is one in which every variable is assigned
- A solution to a CSP is consistent and complete assignment
- Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map Coloring

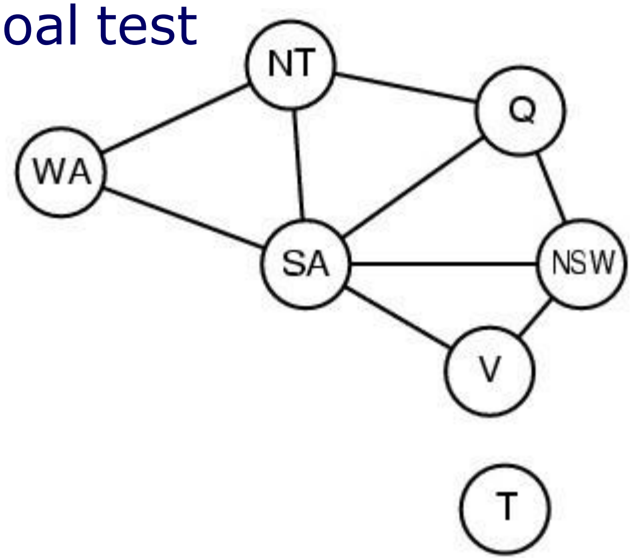
- Variables: X = {*WA, NT, Q, NSW, V, SA, T* }
- Domains: D_i = {red, green, blue}
- Constraints: adjacent regions must have different colors
- Solution?

Solution: Complete and Consistent Assignment

- Variables: X = {*WA, NT, Q, NSW, V, SA, T* }
- Domains: D_i = {red, green, blue}
- Constraints: adjacent regions must have different colors
- Solution? {WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = red}.

Constraint Graph

- Constraint graph: nodes are variables, arcs are constraints
- Binary CSP: each constraint relates two variables
- CSP conforms to a standard pattern
  - a set of variables with assigned values
  - generic successor function and goal test
  - generic heuristics
  - reduce complexity

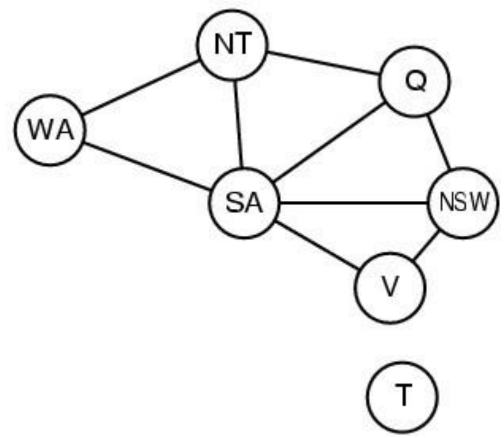


CSP as a Search Problem

- Initial state:
  - {} – all variables are unassigned
- Successor function:
  - a value is assigned to one of the unassigned variables with no conflict
- Goal test:
  - a complete assignment
- Path cost:
  - a constant cost for each step
- Solution appears at depth n if there are n variables
- Depth-first or local search methods work well

CSP Solvers Can be Faster

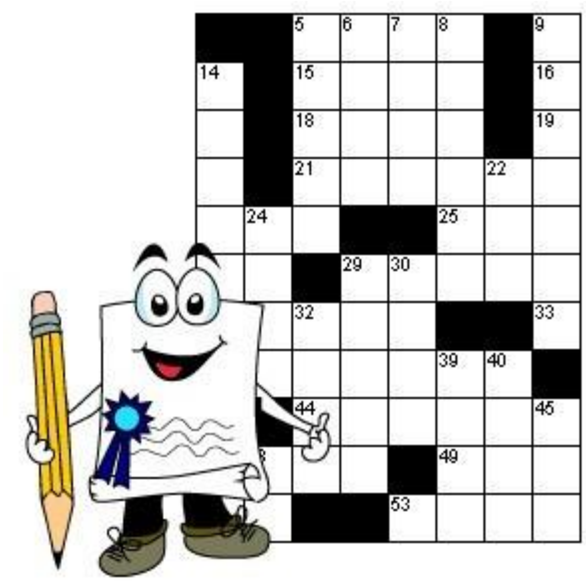
- CSP solver can quickly eliminate large part of search space
- If {SA = blue}
- Then 3^5 assignments can be reduced to 2^5 assignments, a reduction of 87%



- In a CSP, if a partial assignment is not a solution, we can immediately discard further refinements of it

Exercise (1)

- Consider the problem of crossword puzzle: fitting words into a rectangular grid. Assume that a list of words is provided and that the task is to fill in the blank squares using any subset of the list. Formulate this problem precisely in two ways:
  - As a general search problem. Choose an appropriate search algorithm.
  - As a constraint satisfaction problem.
  - Word vs. letters



Exercise (2)

- Problem formulation as CSP:
  - Class scheduling: There is a fixed number of professors and classrooms, a list of classes to be offered, and a list of possible time slots for classes. Each professor has a set of classes that he or she can teach.

Types of Variables

- Discrete variables
- finite domains:
    - n variables, domain size d $O(d^n)$ complete assignments
    - e.g., Boolean CSPs, such as 3-SAT (NP-complete)
    - Worst case, can't solve finite-domain CSPs in less than exponential time
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
- Continuous variables
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

Types of Constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables
  - e.g., cryptarithmetic column constraints

Real-World CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

What Search Algorithm to Use?

- Since we can formulate CSP problems as standard search problems, we can apply search algorithms from chapter 3 & 4
- If breadth-first search were applied
  - branching factor? nd
  - tree size? $nd * (n-1)d * \ldots * d = n! * d^n$ leaves
  - complete assignments? d^n

- A crucial property to all CSPs: commutativity
  - the order of application of any given set of actions has no effect on the outcome
  - Variable assignments are commutative, i.e., [WA = red then NT = green] same as [NT = green then WA = red]

Backtracking Search

- Only need to consider assignments to a single variable at each node ⇒ $b = d$ and there are d^n leaves

- Backtracking search is used for a depth-first search that chooses values for one variable at a time and backtracks when a variable has no legal values left to assign

- Backtracking search is the basic uninformed algorithm for CSPs

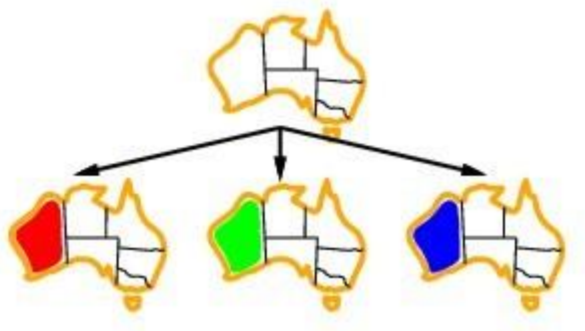
Backtracking Search Algorithm

```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
   return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment according to Constraints[csp] then
         add { var = value } to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failue then return result
         remove { var = value } from assignment
   return failure
```

Backtracking Example

Backtracking Example



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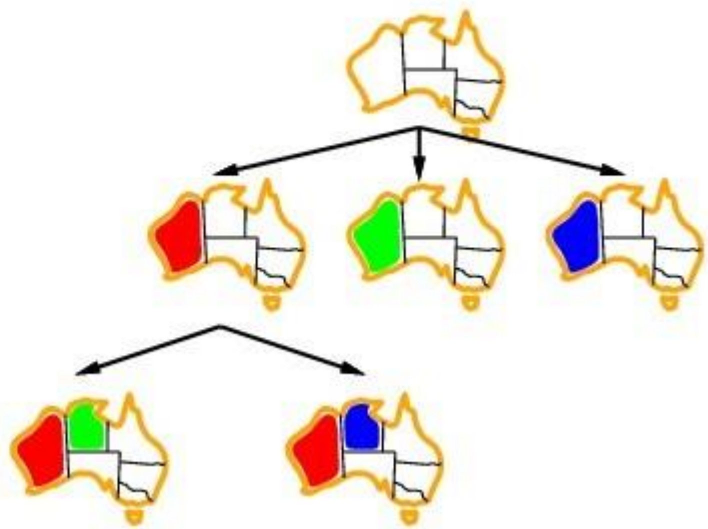

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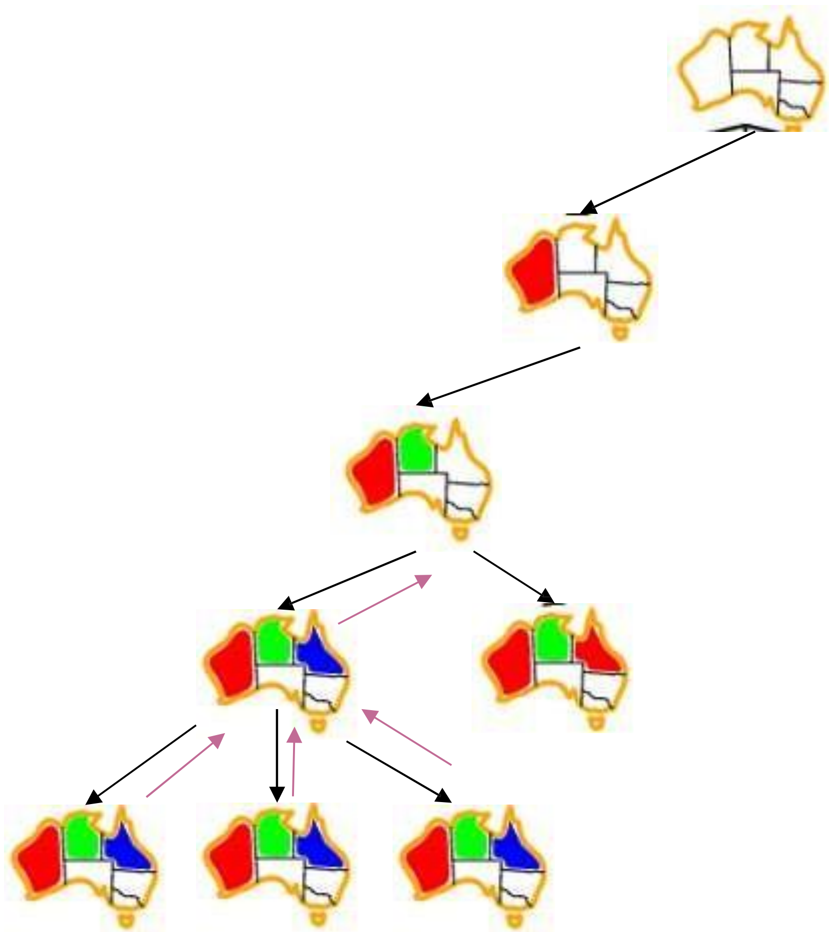
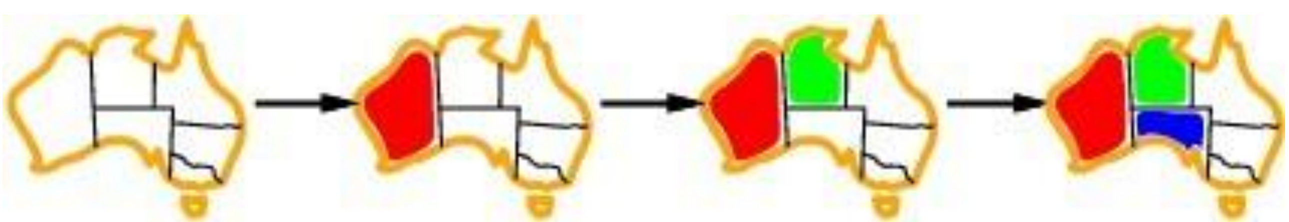


Improving Backtracking Efficiency

- We can solve CSPs efficiently without domain-specific knowledge, addressing the following questions:
  - in what order should variables be assigned, values be tried?
  - what are the implications of the current variable assignments for the other unassigned variables?
  - when a path fails, can the search avoid repeating this failure?

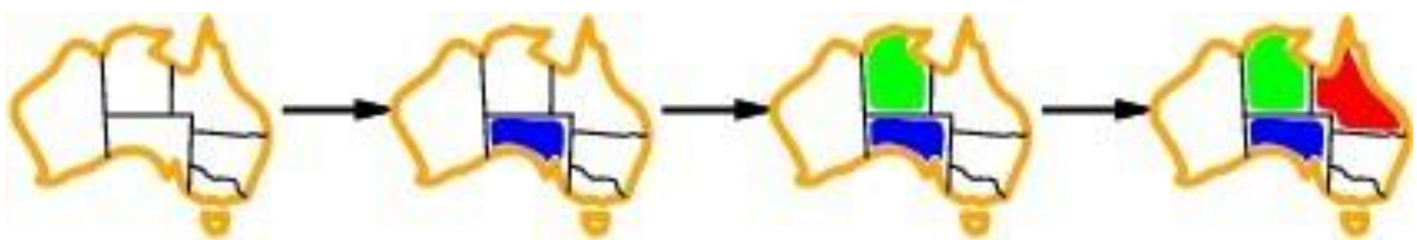
Variable and Value Ordering 1/3

- Minimum remaining values (MRV)
  - choose the variable with the fewest "legal" values
  - also called most constrained variable or fail-first heuristic
  - does it help in choosing the first variable?



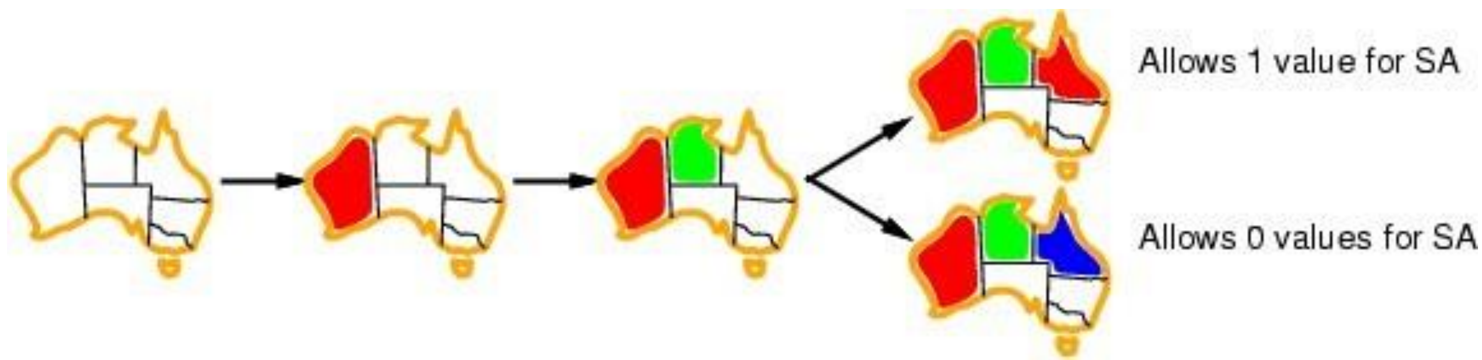
Variable and Value Ordering 2/3

- Most constraining variable:
  - selecting the variable that has the largest number of constraints on other unassigned variables
  - also called degree heuristics
- Tie-breaker among MRV



Variable and Value Ordering 3/3

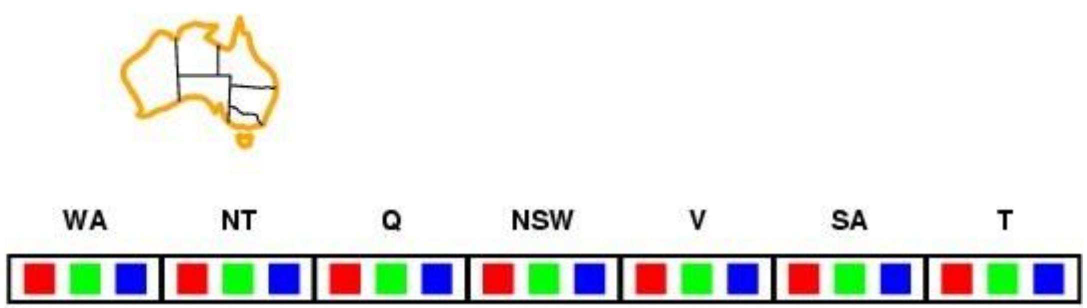
- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables



- Combining these heuristics makes 1000 queens feasible

Propagating Information through Constraints

- Forward checking:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



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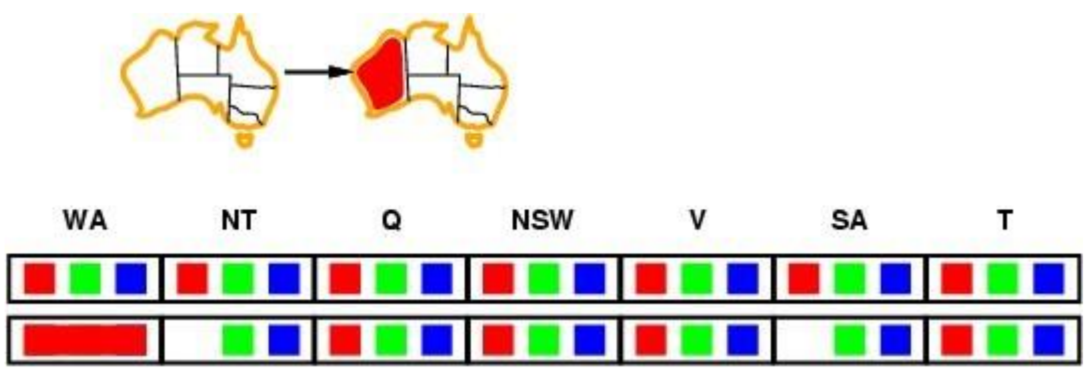

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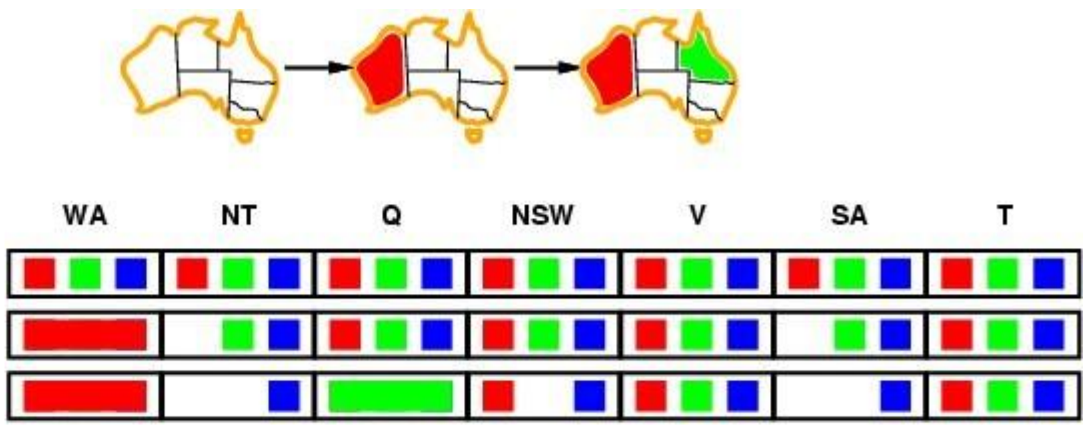

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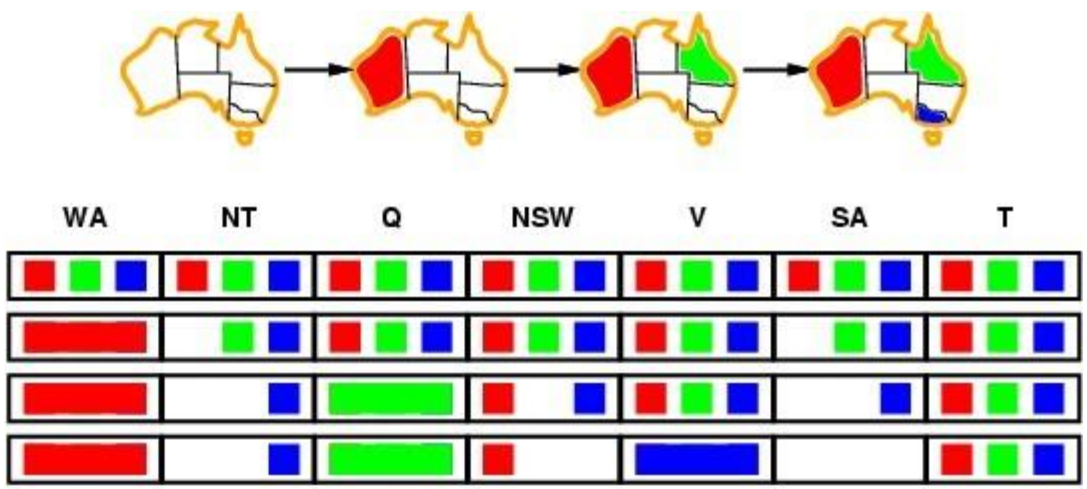
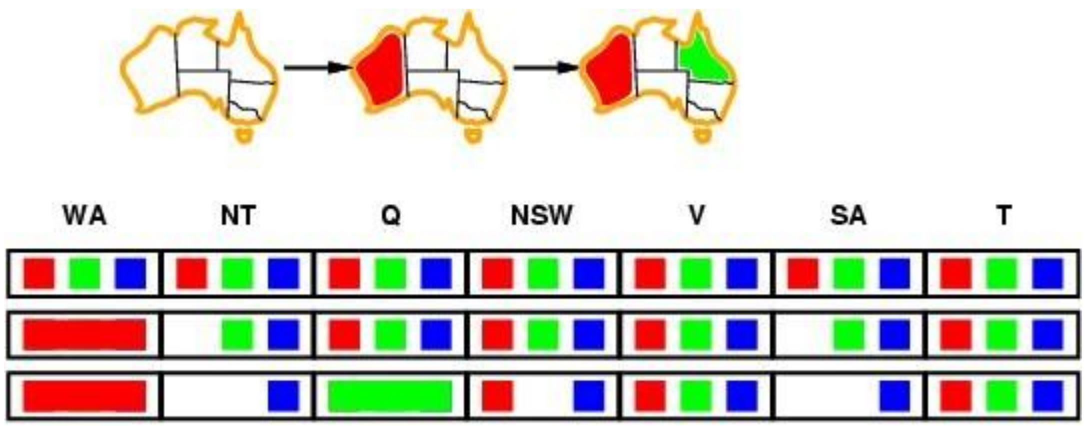


Propagating Information through Constraints

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally by propagating implications of a constraint of one variable onto other variables

Node Consistency

- A single variable is node-consistent if all the values in the variable’s domain satisfy the variable’s unary constraints

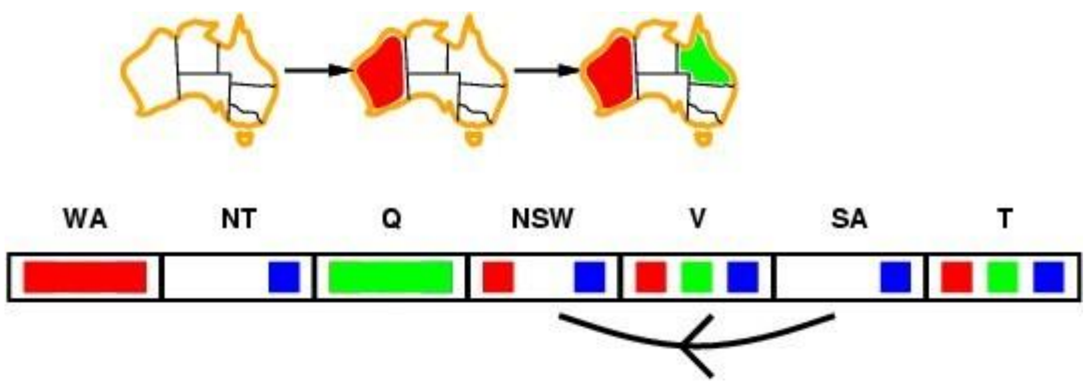
- For example, SA dislikes green

- A network is node-consistent if every variable in the network is node-consistent

Arc Consistency

- Simplest form of propagation makes each arc consistent
- *X* → *Y* is consistent iff

  for every value *x* of *X* there is some allowed *y*



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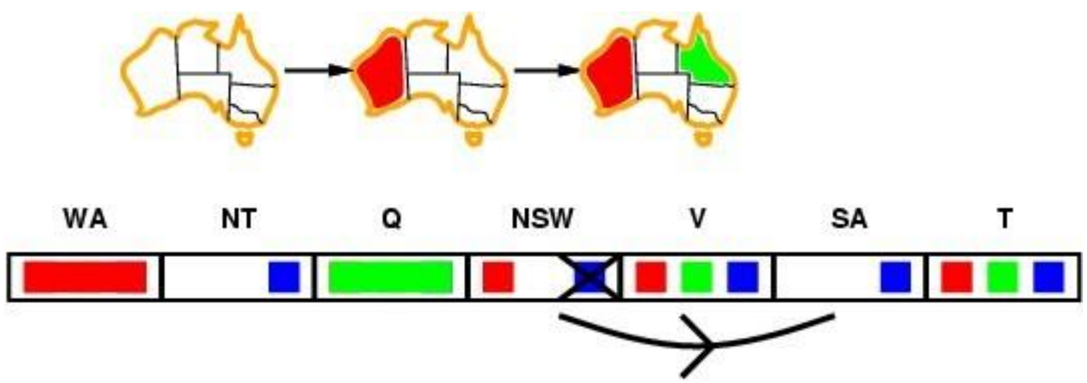

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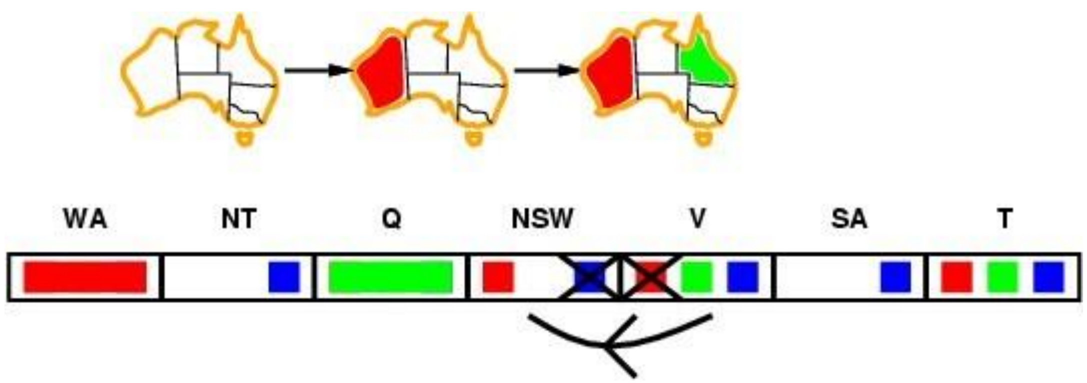

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Arc Consistency

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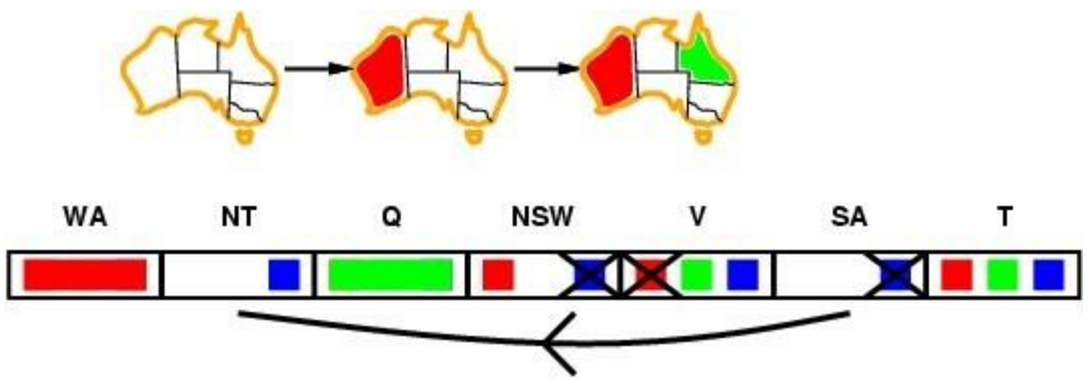


- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

AC-3 Algorithm

- Time complexity $O(n^2 d^3)$

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

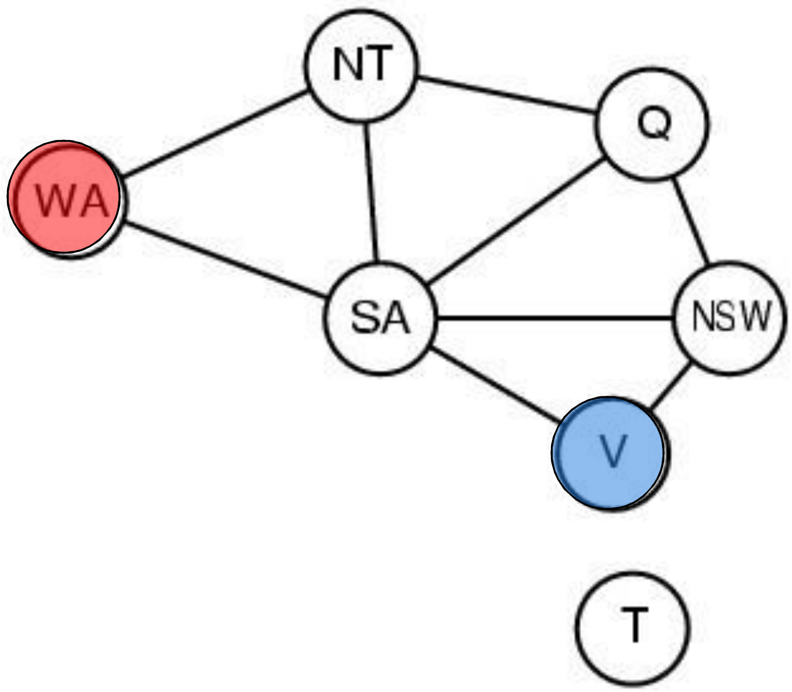
  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if RM-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS[X_i] **do**
        add (X_k, X_i) to *queue*

---

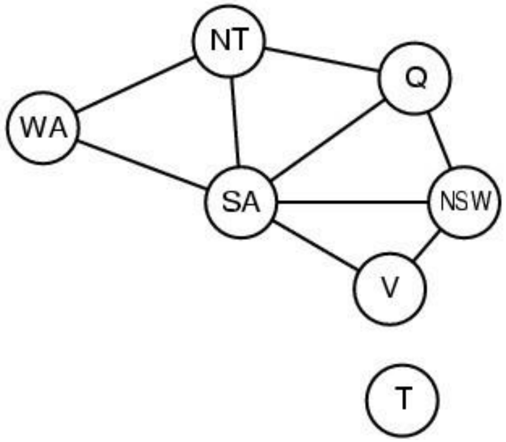
function RM-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff remove a value
  removed $\leftarrow$ *false*
  for each x **in** DOMAIN[X_i] **do**
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy constraint(X_i, X_j)
      then delete x from DOMAIN[X_i]; *removed* $\leftarrow$ *true*
  return *removed*

Example

- Use the AC-3 algorithm to show that arc consistency is able to detect the inconsistency of the partial assignment {WA = red, V = blue} in the following map coloring problem.



One Possible Trace



- One possible trace of the algorithm:
  - remove SA-WA, delete R from SA
  - remove SA-V, delete B from SA, leaving only G
  - remove NT-WA, delete R from NT
  - remove NT-SA, delete G from NT, leaving only B
  - remove NSW-SA, delete G from NSW
  - remove NSW-V, delete B from NSW, leaving only R
  - remove Q-NT, delete B from Q
  - remove Q-SA, delete G from Q
  - remove Q-NSW, delete R from Q, leaving no proper assignment for Q

Path Consistency

- Consider map-coloring with only two colors
- Every arc is consistent initially
- Check the set {WA, SA} path consistenty with respect to NT

- More generally, k-consistency
  - 1-consistency = node consistency
  - 2-consistency = arc consistency
  - 3-consistency = path consistency

Local Search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned

- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values

- Variable selection: randomly select any conflicted variable

- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with $h(n)$ = total number of violated constraints

Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice