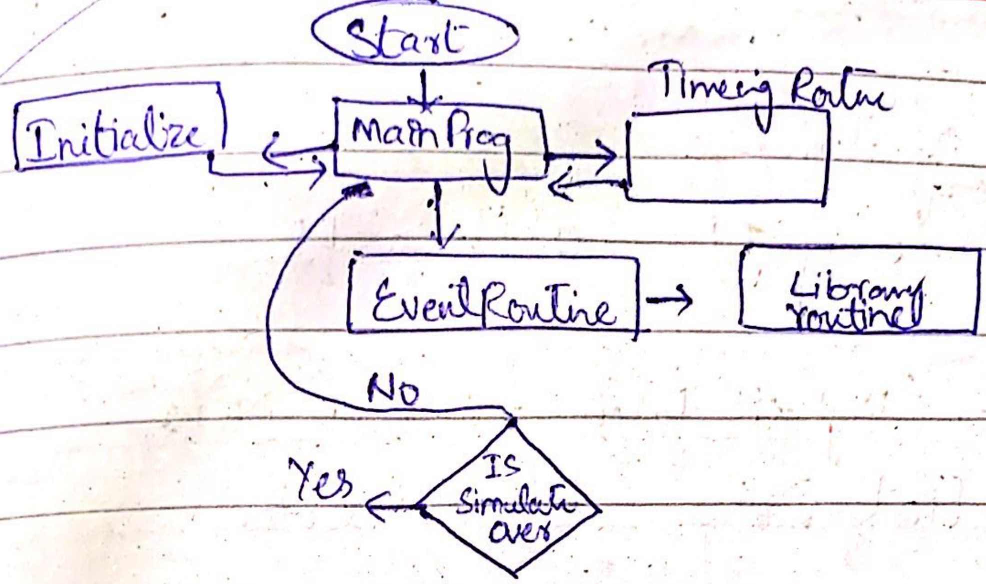


Differential Equations:-

Derivative (Change Happening) of equation

dx ← Dependent Variable ہے تبدیل ہونے سے تبدیلی ہو رہی ہے

dy ← Independent Variable ← ہے تبدیل ہو رہا ہے

An equation containing the derivative of one or more dependent variables with respect to one or more independent variables is said to be differentiated equations

① $\frac{dy}{dx} + y^2x + \sin x = 58\cos x$ ← One independent ⇒ Ordinary

② $y'' + 3y' + 6y = \sin x$

③ $\frac{\partial^2 y}{\partial x^2} + \frac{3\partial y}{\partial t} + \sin t \cos t = 0$

two, more independent Variable

Partial

Classification :-

i) Type :-

① Ordinary DE ② Partial DE

ii) Order (Highest taken derivative in DE) :-

- First Order
- Highest Order.

e.g :-

(i) $\frac{d^3y}{dx^3} + \frac{d^4y}{d^4x} + \sin x \cos y = 0$ (Fourth Order)

(ii) $\frac{\partial^3 y}{\partial^2 x \partial t} + \frac{\partial^2 y}{\partial^2 x} + \frac{\partial^2 y}{\partial x \partial t} = 0.$ (Third Order) Partial

Linearity :-

① Linear DE : ⓐ Dependent Variable and its derivatives have power one. d^2y/dx^2 (Power is 1)

ⓑ There shouldn't be any product of dependent variable and its derivatives

$y \cdot dy/dx$ ← should not

ⓒ No transcadental function should be apply on dependent variable & its derivatives

Transcadental : Trig, Inverse Trig, logarithmic, e

$y = x^5$

$\frac{dy}{dx} = 5x^4$

$\frac{d^2y}{dx^2} = 20x^3$

$y^{(5)} = \frac{d^5y}{dx^5}$

Ex 1.1 1–10

$\left(\frac{dy}{dx}\right)^2 = 25x^8$ — change

Not Linear due to Transcadental function on dependent variable

e.g: ① $\frac{d^3y}{dx^3} + \frac{d^4y}{dx^4} + \sin x \cos y = 0$

Not linear

② $y\frac{\partial^3 y}{\partial^2 x \partial t} + \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial x \partial t} = 0$

③ $y''' + y'' + 2y^5 = 0$ ← Non Linear Power is not 1.

④ $y\frac{\partial^3 x}{\partial^2 x \partial t} + \frac{\partial^2 x}{\partial y} + \frac{\partial x}{\partial y \partial t} = 0$ Linear

Initial Value Problem: Already know first value.

e.g $y'' + y'' + 3y' + 6y = 0$ $y(0) = \pi$

Boundary Value Problem: Two interval points.

eg $y'' + y'' + 3y' + 6y = 0$ $y(0) = \pi$ $y(2\pi) = 0$

Solutions of Differential Equations:-

① Interval of Solution: See the interested interval

② General Solution & Particular Sol:-

e.g: $\frac{dy}{dx} = -\frac{x}{y}$ (Non Linear)

Sol: $x + y = C$ (General Sol)

$\frac{dy}{dx} = -\frac{x}{y}$ — $y(0) = 5$

$x^2 + y^2 = 25$ (Particular Sol)

$(u.v)' = uv' + vu'$

$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

$y = f(x) \leftarrow$ explicit sol $\quad f(x,y) = 0 \leftarrow$ implicit sol

$\frac{d}{dx}\left\{x^2 + y^2 = c\right\} \Rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x}{y}$

ex: $y'' - 2y' + y = 0$

$y = c_1 e^x + c_2 x e^x$ general sol

$c_1 = 0 \quad c_2 = 1$

$y = xe^x$ partialar

$c_1 = 5 \quad c_2 = -2$

$y = 5e^x - 2xe^x$

$y = c_1 e^x + c_2 e^x$

$y' = c_1 e^x + c_2 (xe^x + e^x)$

$y'' = c_1 e^x + c_2 (xe^x + e^x + e^x)$

$= c_1 e^x + c_2 (xe^x + 2e^x)$

$c_1 e^x + c_2 (xe^x + 2e^x) - 2(c_1 e^x + c_2 (xe^x + e^x)) +$

$c_1 e^x + c_2 x e^x$

$= 0$

$xy' - y = x^2 \sin x$

$y = cx - x\cos x$

Particular Solution from General Solution:-

Ex 1.2:

$y' = y - y^2 \qquad y(0) = -\frac{1}{3}$

$$y = \frac{1}{1 + c_1 e^{-x}}$$

$$-\frac{1}{3} = \frac{1}{1 + c_1 e^{-0}} = \frac{1}{1 + c_1}$$

$-(1 + c_1) = 3$

$c_1 = -4$

$$\Rightarrow \quad y = \frac{1}{1 - 4e^{-x}}$$

$$y' = \frac{1}{1 - 4e^{-x}} - \left\{ \frac{1}{1 - 4e^{-x}} \right\}^2$$

7: $x'' + x = 0 \qquad x(0) = -1, \; x'(0) = 8$

$x = c_1 \cos t + c_2 \sin t$

$-1 = c_1 \cos 0 + c_2 \sin 0$

$\boxed{c_1 = -1}$

$x' = -c_1 \sin t + c_2 \cos t$

$8 = -c_1 \sin 0 + c_2 \cos 0$

$\boxed{c_2 = 8}$

$\boxed{0=0}$ Infinit Many Sol. $\boxed{0=5}$ No Soluti

$$x = -\cos t + 8\sin t \qquad \text{Particular solution}$$

Verify:

$$x'' = -c_1\cos t - c_2\sin t \qquad \text{General}$$
$$= \cos t - 8\sin t \qquad \text{Particular}$$

$x'' + x = 0$

$$\cos t - 8\sin t - \cos t + 8\sin t = 0$$
$$0 = 0.$$

Growth and Decay Model } First Order
Newton's Law of Cooling } Linear
Logistic Equation → First Order Non Linear

Ex 1: Bacterial Growth

A culture initially has P_0 number of bacteria. At $t = 1h$ the number of bacterias is measured to be $3/2\ P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t. Determine Time necessary for the number of bacteria to triple.

$$dP/dt \propto P(t)$$

$$\frac{dP}{dt} = \underset{\substack{\downarrow \\ \text{Constant}}}{K}P \xrightarrow{\text{Sol}} P = Ce^{kt}$$

Sol: $P(0) = P_0$
$P(1) = 3/2\, P_0$

$\ln\frac{3}{2} \cdot t = \ln e$

For $P(0) = P_0$:

$$P_0 = ce^0$$
$$\boxed{P_0 = c}$$

For $P(1) = 3/2 P_0$

& $c = P_0$

$$3/2 \cancel{P_0} = \cancel{P_0} e^{k(1)}$$
$$e^k = \frac{3}{2}$$
$$\ln e^k = \ln 3/2$$
$$k \ln e = \ln 3/2$$
$$k = \ln 3/2$$

$$\boxed{P(t) = P_0 e^{\ln 3/2 * t}}$$

we want $t = ?$ Population $= 3P_0$

$$3\cancel{P_0} = \cancel{P_0} e^{\ln 3/2\, t}$$
$$\ln 3 = \ln e^{\ln 3/2\, t}$$
$$t = \frac{\ln 3}{\ln 3/2} = 2.7$$

Ex2. Half life of Plotonium

A breeder reactor convertes stable uranim 238 into the isotope plutonium 239.

After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half life of the isotope if the rate of disintegration is directly prop. to the amount.

$$\frac{dA}{dt} \propto A(t)$$

$$\frac{dA}{dt} = KA$$

$$A = ce^{kt} = A_0 e^{kt}$$

$A(0) = A_0$

$A(15) = (100 - 0.043) A_0$

$= 99.957\% \, A_0$

For $A(0) = A_0$

$$A_0 = ce^{0}$$

$$\boxed{A_0 = c}$$

For $A(15) = 99.957\, A_0$

$$99.957\%\, A_0 = ce^{kt}$$

$$0.99957\, A_0 = A_0\, e^{k(15)}$$

$$\ln(0.99957) = \ln e^{15k}$$

$$\ln(0.99957) = 15k \cdot \ln e$$

$$k = \frac{\ln 0.99957}{15} = 24. \text{ V.}$$

Want $= t = ?$ $A = 1/2\, A_0$

$$1/2\, A_0 = A_0\, e^{vt}.$$

Newton's Law of Cooling:

$$\frac{dT}{dt} = K(T - T_m)$$ ambient Temperature → atmosphere temp.

Ex 4: When cake is removed from an oven, its temperature is measured at 300°F. Three minutes late its temperature is 200°F. How long will it take to cool off to a room Temp. to 70°F.

$$T(0) = 300°F$$

$$T(3) = 200°F$$

$$\frac{dT}{dt} = k(T - T_m)$$

$$T = T_m + ce^{kt}$$

For $T = 300°F$:-

$$300 = 70 + ce^{0}$$

$$c = 230°F$$

For $T = 200°F$:-

$$T = T_m + ce^{kt}$$

$$200 = 70 + (230)e^{3k}$$

$$\frac{130}{230} = e^{3k}$$

$$3k = \ln\left(\frac{130}{230}\right)$$

$$k = -0.19018$$

Want $t = ?$ ~~T~~ $70°F$

$$T(t) = T_m + ce^{kt}$$

$$70 = 70 + 230\, e^{-0.19018\,t}$$

$$0 = 230\, e^{-0.19018t}$$

$$0 = e^{-0.19018t}$$

$\because \ln(0) = \infty$

$\therefore$ No Solution

(mon) $1, 5, 7, 9, 11$

abcl * ky00 ⇒ len+2 $9-1=8/2=4$
for (i=0; i<4

$c = 60$
$k = -0.8109$ $T(1) = 36.667°F$; $t = 3.064$

Q13. A thermometer is removed from a room where the temperature is 70°F & is taken outside where the air temp is 10°F. After one-half minute the temp. reads 50°F. What is the reading of thermometer at $t = 1$ min? How long will it take for the thermometer to reach 15°F?

$t = ?$ $T = 15°F$

$T(0) = 70°F$
$T(1/2) = 50°F$ $T_m = 10°F$ $T(1) = ?$

$$\frac{dT}{dt} = k(T - T_m)$$

$$T = T_m + ce^{kt}$$

$T(0)$: $70°F$

$$70 = 10 + ce^0$$
$$60 = ce^0$$
$$\boxed{c = 60°F}$$

$T(1/2)$: $50°F$

$$50 = 10 + 60e^{1/2k}$$
$$\frac{4\cancel{0}}{6\cancel{0}} = e^{1/2k}$$
$$\ln 2/3 = (e^{1/2k})\ln \Rightarrow 1/2k \ln e$$
$$k = (\ln 2/3) * 2 = -0.8109$$

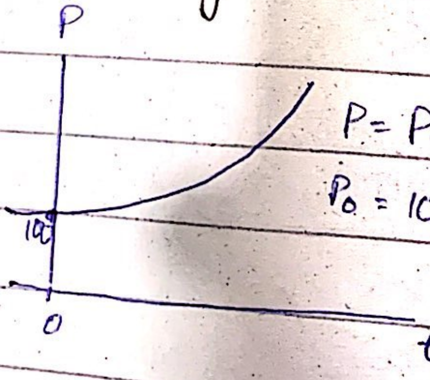
T(1):-

$$T = 10 + 60e^{-0.8109(1)}$$
$$= 10 + 26.667$$
$$= 36.667\ ^{\circ}F$$

Want $t = ?$ $T(t) = 15^{\circ}$

$$15 = 10 + 60e^{-0.8109t}$$
$$\frac{\cancel{5}\,1}{\cancel{60}\,12} = e^{-0.8109t}$$
$$\ln(0.083) = \ln e^{-0.8109t}$$
$$\ln(0.083) = -0.8109\,t$$
$$t = 3.0693 \text{ min.}$$

Growth & decay:-



$P = P_0 e^{kt}$

$P_0 = 100$ $k = 0.031$

It will grow exponential

Logistics Equation :- Population will not be grow exponentially, it will stop at certain spot.

$$\frac{dP}{dt} = r P \left(1 - \frac{P}{K}\right)$$

r = rate of population change
K = Carrying Capacity of Population of Area.

Solutions :-

General Sol: $P = \dfrac{c_1 K e^{rt}}{1 + c_1 e^{rt}}$

Initial Value: $P(0) = P_0$

$$c_1 = \frac{P_0}{K - P_0}$$

Specific / particular solution :-

$$P = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$

Ex: Consider the population of white tailed deer in state of Kentucky. Estimated Population at the state of observation is found to be 900,000 deers. Consider the growth rate of the ~~decay~~ deer $r = 0.2311$. Find the number of deers in the kentucky state after 3 years.

(a) Using growth and decay model

(b) Using Logistic Equation by considing the carrying capacity $K = 1,072,764$ deers.

(c) Draw graph of both solutions & compare it.

(a) $$\frac{dP}{dt} = rP$$

$$\frac{dP}{dt} = 0.231\,P$$

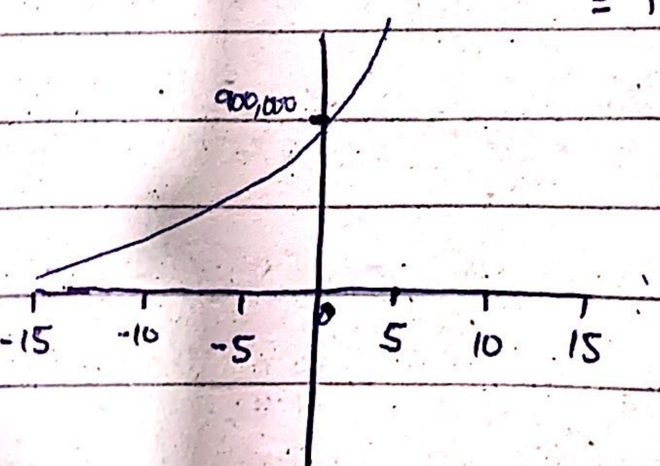
$$P = P_0 e^{rt}$$

$$= 900,000\, e^{0.2311\, t}$$

After 3 years

$$= 900,000\, e^{0.2311 \times 3}$$

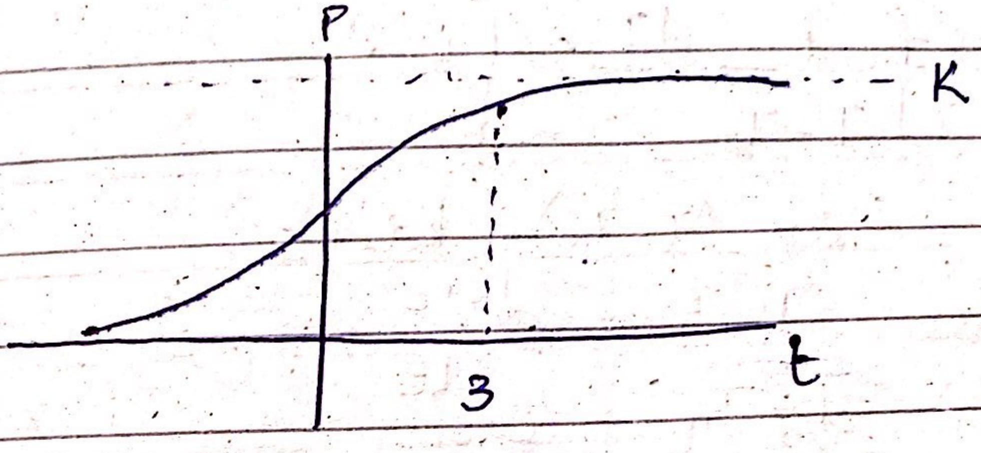
$$= 1800275.096$$



0.2311×3

(b) $P = \dfrac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}} = \dfrac{900{,}000 \times 1072764 \times e^{0.2311 \times 3}}{(1072764 - 900{,}000) + 900{,}000 \times e^{0.2311 \times 3}}$

After 3 years $P = 978830.230$



Growth & Decay Model:-

$$\frac{dP}{dt} = KP$$

$$\frac{1}{P}\,dP = K\,dt$$

$$\Rightarrow \int \frac{1}{P}\,dP = \int K\,dt$$

$$\ln P = Kt + c$$

$$P = e^{kt+c}$$

$$P = e^{kt}\,e^{c}$$

$$P = c\,e^{kt}$$

Newton's Law of Cooling :-

$$\frac{dT}{dt} = K(T - T_m)$$

$$\Rightarrow \int \frac{1}{T - T_m}\, dT = \int K\, dt$$

$$\ln(T - T_m) = Kt + c$$

$$T - T_m = e^{kt+c} = e^{kt} e^{c}$$

$$T - T_m = ce^{kt}$$

$$T = T_m + ce^{kt}$$

Logistic Equation :-

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\frac{dP}{dt} = rP\left(\frac{K - P}{K}\right)$$

$$\frac{dP}{P(K-P)} = \frac{r}{K}\, dt$$

$$\frac{K}{P(K-P)}\, dP = r\, dt$$

$$\int \left(\frac{1}{P} - \frac{1}{KP}\right) dP = \int r\, dt$$

$$\ln P - \ln KP = rt + c$$

$$\ln\left(\frac{P}{KP}\right) = rt + c$$

$$\frac{P}{K-P} = e^{rt+c} = e^{rt}e^{c} = Ce^{rt}$$

$$P = ce^{rt}K - Pce^{rt}$$

$$P + ce^{rt}P = ce^{rt}K$$

$$P = \frac{ce^{rt}K}{(1+ce^{rt})}$$

$\Rightarrow P(0) = P_0$

$$P_0 = \frac{cK}{1+c}$$

$$P_0 + P_0 c = cK$$

$$cK - P_0 c = P_0$$

$$c = \frac{P_0}{K-P_0} \quad \text{Put in } P.$$

$$P = \frac{\frac{P_0}{K-P_0} e^{rt} K}{1 + \frac{P_0}{K-P_0} e^{rt}}$$

$$P = \frac{P_0 e^{rt} K}{K - P_0 + P_0 e^{rt}}$$

Taylor Series:- $f(x)$ about c

$$F(c) + F'(c)(x-c) + \frac{F''(c)(x-c)^2}{2!} + \dots$$

Maclearin's Series:- $f(x)$ about 0

$$F(0) + F'(0)(x) + \frac{F''(0)(x)^2}{2!} + \dots$$

e.g: $e^x = F(0) + F'(0)x + \frac{F''(0)x^2}{2!}$

$$= 1 + 1 \cdot x + \frac{1 \cdot x^2}{2!}$$

$F(x) = e^x$

$\Rightarrow$ $F(0) = e^1 = 1$

$F'(0) = e^x$

$F'(0) = e^0 = 1$

e.g: $F(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$F''(x) = -\sin x \Rightarrow F''(0) = 0$

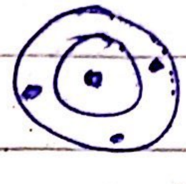
$F'''(x) = -\cos x \Rightarrow F'''(0) = -1$

$$\sin x = F(0) + F'(0)x + \frac{F''(0)(x)^2}{2!} + \frac{F'''(0)(x)^3}{3!}$$

$$= 0 + 1x + 0 - \frac{1x^3}{3!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

Accuracy: Close To the Answer 

Precision: Towards the region but apart from Answer 

$$E_T = \text{True Error} = \left| \frac{\text{True} - \text{Approx}}{\text{True}} \right| \times 100$$

True Value → Comes from Analytical method

Approx Value → Comes from Numerical Method

$$E_{aT} = \left| \frac{\text{Present Value} - \text{Previous Value}}{\text{Present Value}} \right| \times 100$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{0.5} = 1.64872$$

Term	Approx Value	E_T	E_{aT}
1	1	39.34	- ← first Present V. can't Calculate
2	1.5	9%	$\frac{1.5-1}{1.5} \times 100 = 33.3\%$
3	1.625	1.43	7.69%

Numerical Methods to Solve Differential Eq.

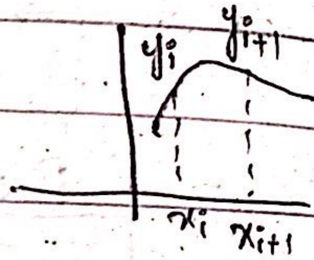
① Euler's method ② Heun's Method

① Euler's Method:-

Called as One Step Method

Single Step:- $y_{i+1} = y_i + \phi h$ → increment function

↳ Used to calculate points



$$\frac{dy}{dt} = f(t,y)$$

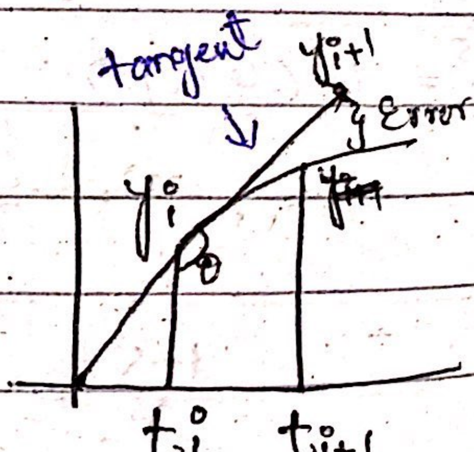
Euler formula:-

$$y_{i+1} = y_i + f(t_i, y)h$$

$\Rightarrow h = t_{i+1} - t_i$

↓

stepness



Problem Statement:- Use Euler's Method to solve $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to 4 with a step size of 1. The initial condition at $t = 0$ is $y = 2$. Exact solution of the given differential equation is $y = \frac{4}{1.3}(e^{0.8t} - e^{-0.5t}) + 2e^{-0.5t}$

t	Y_{true}	Y_{Euler}	ε_t
0	2		
1	6.19463	5	$\left\lvert \frac{6.19463-5}{6.19463} \right\rvert \times 100 \Rightarrow 19.28$
2	14.84392	11.40216	23.19%
3	33.67717	25.51321	24.24%
4	75.33896	56.84391	24.54%

Data:-

$h = 1$

$y(0) = 2$, $y = \frac{4}{1.3}(e^{0.8t} - e^{-0.5t}) + 2e^{-0.5t}$ ← Y_{true}

Sol:-

$$y_1 = y_0 + \underbrace{f(t_0, y_0)}_{\rightarrow \frac{dy}{dt}} h$$

$\because f(x_0, y_0) = 4e^{0.8t} - 0.5y$

$y_1 = 2 + 4e^{0.8(0)} - 0.5(2) 1 = 5$

$$y_2 = y_1 + f(t_1, y_1)\, h$$

$$= 5 + 4e^{0.8(1)} - 0.5(2) \cdot (1) = 11.40216$$

Heun's Method:- Multistep method

Predictor Correction Method:-

Predictor : $y_{i+1}^{0} = y_i^{m} + f(t_i, y_i) h$

Corrector: $y_{i+1}^{j} = y_i^{m} + \frac{f(t_i, y_i^{m}) + f(t_m, y_{i+1}^{j-1})}{2} h$

for $j = 1 \ldots m$

Example:- Use Heun's Method to solve $y' = 4e^{0.8t} - 0.5y$ from $t = 0$ to 4. with a step size of 1. The initial condition at $t=0$ is $y=2$. Employ stopping criterion of 0.00001% to terminate the corrector iteration

$\left(\frac{y_t - y_E}{y_t}\right)100$ $\quad$ 1 corrector $\quad \left(\frac{y_t - y_h}{y_t}\right)\times 100$ $\quad$ 2 corrector $\quad \left(\frac{y_t - y_{heun}^{2}}{y_t}\right)100$

t	y_{true}	y_{Euler}	ε_t	y_{heun}	ε_t	y_{heun}	ε_t
0	2	2.000	19.28%	6.701082	8.18%	6.275811	1.31%
1	6.19463	5.000	23.19%				
2	14.84392	11.40216					
3	33.67717	25.513121					
4	75.338916						

First Iteration:-

$h = 1$

$y(0) = 2$, $y_1 = y^{0}(1) = ?$ ← No. of Correct

$y_1^2 = ?$

Predictor:- $y_{0+1}^{0} = y_0 + f(t_0, y_0)(1)$

$y_1^{0} = 2 + f(0,2)(1) = 5$

Corrector: $y_1' = y_0 + \frac{f(t_0, y_0) + f(t_0, y_1^{0})}{2} (h)$

$y_1' = 2 + \frac{f(0,2) + f(1,5)}{2} (1) = 6.701082$

$y_1^2 = 2 + \frac{f(0,2) + f(1, 6.701082)}{2} (1) = 6.275811$

y(2):-

$y_2^{0} = y_1 + f(t_1, y_1)(1)$

$=$