Chapter 14

Probabilistic Reasoning

Motivations

- Full joint probability distribution can answer any question but can become intractably large as number of variable increases

- Specifying probabilities for atomic events can be difficult, e.g., large set of data, statistical estimates, etc.

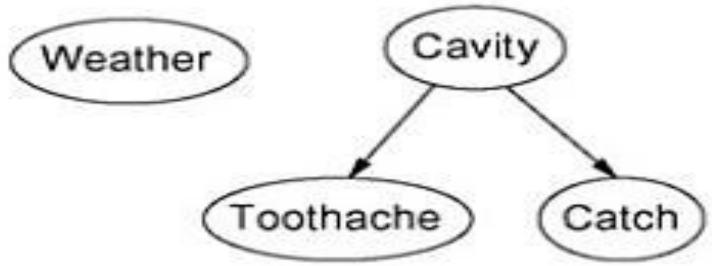
- Independence and conditional independence reduce the probabilities needed for full joint probability distribution.

Bayesian networks

- A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions
- A directed, acyclic graph (DAG)
- A set of nodes, one per variable (discrete or continuous)
- A set of directed links (arrows) connects pairs of nodes. X is a parent of Y if there is an arrow (direct influence) from node X to node Y.
- Each node X_i has a conditional probability distribution $P(X_i \mid Parents(X_i))$ that quantifies the effect of the parents on the node.
- Combinations of the topology and the conditional distributions specify (implicitly) the full joint distribution for all the variables.

✓ Causes should be parents that of effects

Topology of network encodes conditional independence assertions:



Weather is independent of the other variables

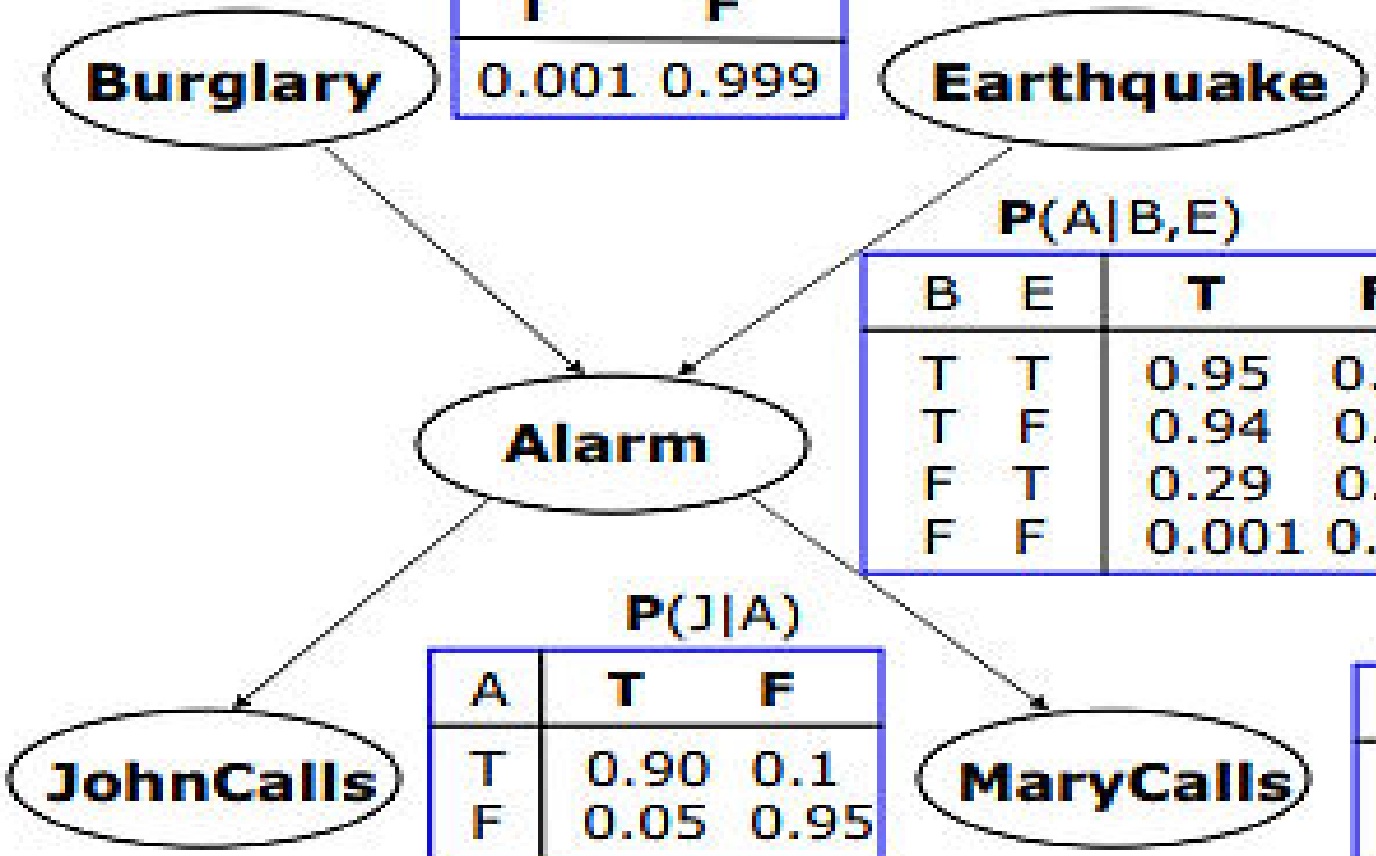
Toothache and *Catch* are conditionally independent given *Cavity*

Example: Burglar alarm system

- I have a burglar alarm installed at home
  - It is fairly reliable at detecting a burglary, but also responds on occasion to minor earth quakes.
- I also have two neighbors, John and Mary
  - They have promised to call me at work when they hear the alarm
  - John always calls when he hears the alarm, but sometimes confuses the telephone ringing with the alarm and calls then, too.
  - Mary likes rather loud music and sometimes misses the alarm altogether.
- Bayesian networks variables:
  - *Burglar*, *Earthquake*, *Alarm*, *JohnCalls*, *MaryCalls*

Bayesian belief network.

Variable Order= [John, Mary, Alarm, Burglar, Earthquake]



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

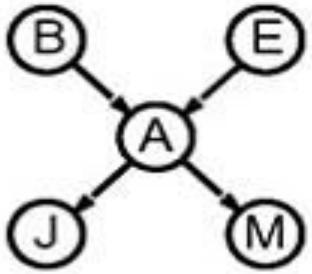
A	T	F
T	0.90	0.1
F	0.05	0.95

P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

Compactness of Bayesian networks

A CPT for Boolean X_i with k Boolean parents has
2^k rows for the combinations of parent values



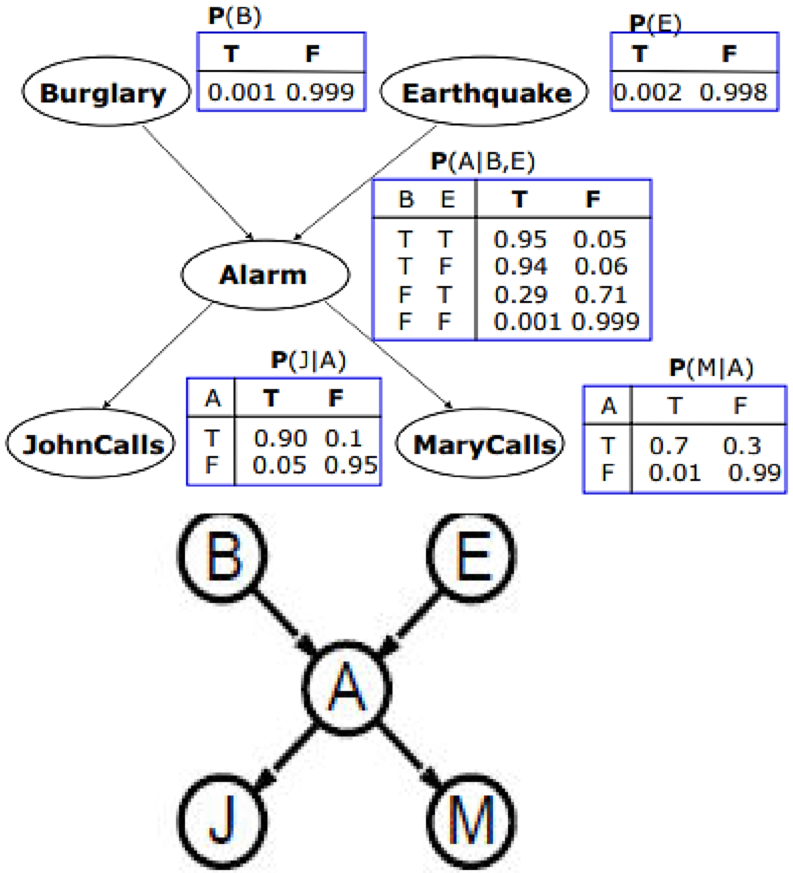
Each row requires one number p for $X_i = true$
(the number for $X_i = false$ is just $1 - p$)

If each variable has no more than k parents,
the complete network requires $O(n \cdot 2^k)$ numbers

I.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution

For burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$)

Global Semantics



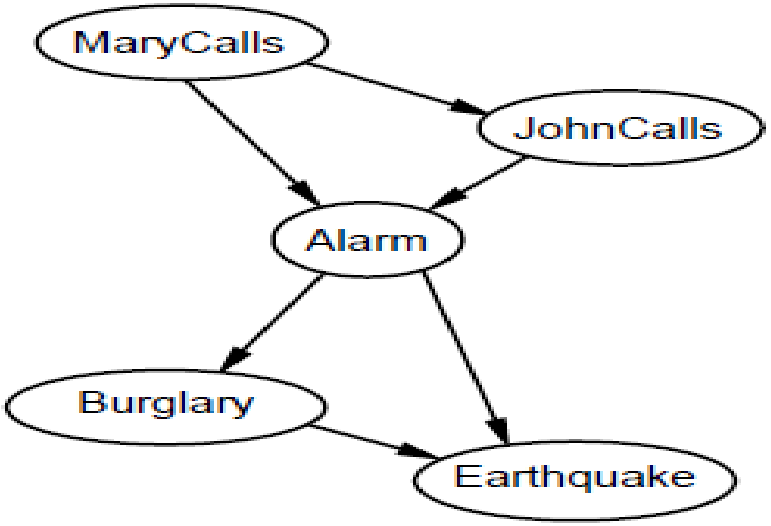
"Global" semantics defines the full joint distribution as the product of the local conditional distributions:

$$P(x_1, \ldots, x_n) = \Pi_{i=1}^{n} P(x_i | parents(X_i))$$

e.g., $P(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$$\begin{aligned} &= P(j|a)P(m|a)P(a|\neg b, \neg e)P(\neg b)P(\neg e) \\ &= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 \\ &\approx 0.00063 \end{aligned}$$

Example



Deciding conditional independence is hard in noncausal directions

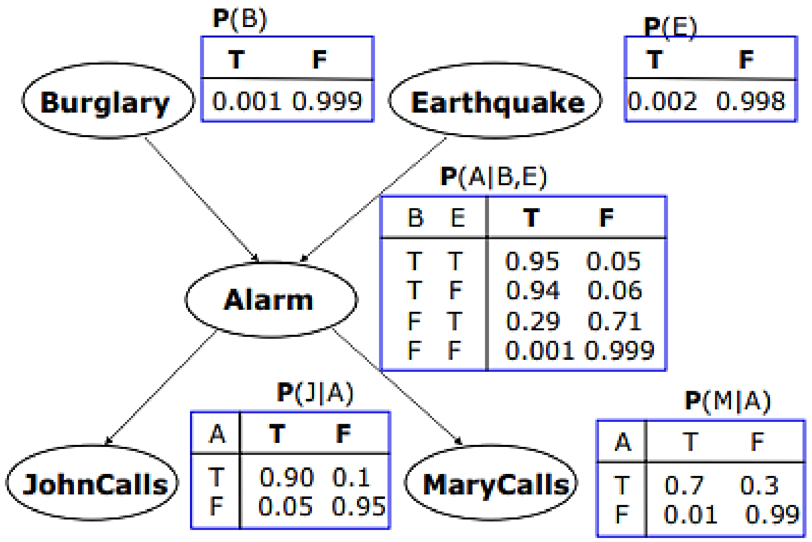
(Causal models and conditional independence seem hardwired for humans!)

Assessing conditional probabilities is hard in noncausal directions

Network is less compact: $1 + 2 + 4 + 2 + 4 = 13$ numbers needed

Example

$$P(A) = \sum_{b\in\{B,!B\}} \sum_{e\in\{E,!E\}} P(A,b,e)$$



Recall from conditional probability: P(A,B)=P(A|B)·P(B)

So P(A,B,E)=P(A|B,E)·P(B,E),
for example.
Which means:
P(A)=P(A|B,E)P(B,E)+P(A|B,!E)P(B,!E)+P(A|!B,E)P(!B,E)+P(A|!B,!E)P(!B,!E)

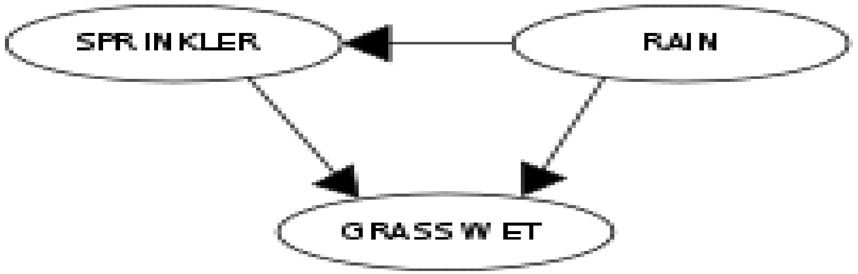
This is because there are 4 ways in which A can happen.
A can happen if events B and E happen, if B happens and not E, if E happens and not B, or if neither of them happen.
Working that out (and assuming B and E are independent), we have:

P(A)==(0.95*0.001*0.002)+(0.94*0.001*0.998)+(0.29*0.999*0.002)+(0.001*0.999*0.998)=0.002516442

Example:2

Two events can cause grass to be wet: an active sprinkler or rain. Rain has a direct effect on the use of the sprinkler (namely that when it rains, the sprinkler usually is not active). This situation can be modeled with a Bayesian network. Each variable has two possible values, T (for true) and F (for false).

	SPRINKLER	
RAIN	T	F
F	0.4	0.6
T	0.01	0.99



RAIN	
T	F
0.2	0.8

		GRASS WET	
SPRINKLER	RAIN	T	F
F	F	0.0	1.0
F	T	0.8	0.2
T	F	0.9	0.1
T	T	0.99	0.01

What is the probability that it is raining, given the grass is wet?

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- We can then calculate, for example:

$$P(\text{it is raining} \mid \text{grass is wet}) = \frac{P(\text{it is raining AND grass is wet})}{P(\text{grass is wet})}$$

$$= \frac{\sum_{\text{sprinkler} \in \{T,F\}} P(\text{grass is wet} = \text{T AND sprinkler AND raining=T})}{\sum_{\text{sprinkler} \in \{T,F\},\ \text{raining} \in \{T,F\}} P(\text{grass is wet} = \text{T AND sprinkler AND raining})}$$

The joint probability Distribution formula is

$$\Pr(G,S,R) = \Pr(G|S,R)\Pr(S|R)\Pr(R)$$

where *G* = *Grass wet (true/false)*, *S* = *Sprinkler turned on (true/false)*, and *R* = *Raining (true/false)*.

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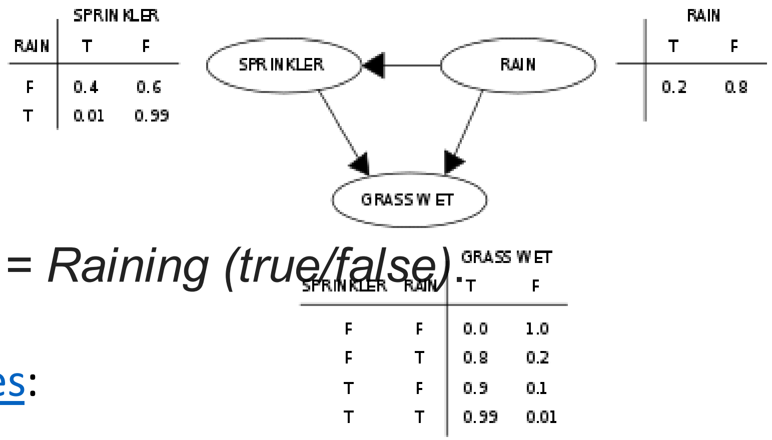

$$\Pr(G,S,R)=\Pr(G|S,R)\Pr(S|R)\Pr(R)$$

where G = *Grass wet (true/false)*, S = *Sprinkler turned on (true/false)*, and R = *Raining (true/false)*.

by using the conditional probability formula and summing over all random variables:

$$\Pr(R=T|G=T)=\frac{\Pr(G=T,R=T)}{\Pr(G=T)}=\frac{\sum_{S\in\{T,F\}}\Pr(G=T,S,R=T)}{\sum_{S,R\in\{T,F\}}\Pr(G=T,S,R)}$$

Using the expansion for the joint probability function $\Pr(G,S,R)$ and the conditional probabilities from CPT

$$\begin{aligned}\Pr(G=T,S=T,R=T)&=\Pr(G=T|S=T,R=T)\Pr(S=T|R=T)\Pr(R=T)\\&=0.99\times0.01\times0.2\\&=0.00198.\end{aligned}$$

$$\Pr(R=T|G=T)=\frac{0.00198_{TTT}+0.1584_{TFT}}{0.00198_{TTT}+0.288_{TTF}+0.1584_{TFT}+0.0_{TFF}}=\frac{891}{2491}\approx 35.77\%.$$

Conditional Probability

Q1: You, your Father and Mother lineup randomly in a queue to take a memorable picture at your Convocation. Find the P(A/B) such that

A= Daughter on one end, B= Father in Middle

Q2: Consider the following contingency table

	RIGHT-HANDED	LEFT-HANDED	TOTAL
MALE	0.41	0.08	0.49
FEMALE	0.45	0.06	0.51
TOTAL	0.86	0.14	1

Find the Probability that a randomly selected person is

A: A Male given that he is right handed.

B: Right handed given that he is a male.

C: A Female given that she is left handed.

D: Are the events being a female and being left handed Independent? Justify

Naïve Bays

Bayesian Classifiers

- Approach:
  - compute the posterior probability $P(C \mid A_1, A_2, \ldots, A_n)$ for all values of C using the Bayes theorem

$$\boxed{P(C \mid A_1 A_2 \ldots A_n) = \frac{P(A_1 A_2 \ldots A_n \mid C) P(C)}{P(A_1 A_2 \ldots A_n)}}$$

  - Choose value of C that maximizes
    $P(C \mid A_1, A_2, \ldots, A_n)$
  - Equivalent to choosing value of C that maximizes
    $P(A_1, A_2, \ldots, A_n \mid C)\, P(C)$
- How to estimate $P(A_1, A_2, \ldots, A_n \mid C)$?

Naïve Bayes Classifier

- Assume independence among attributes A_i when class is given:
  - $P(A_1, A_2, \ldots, A_n | C) = P(A_1 | C_j) P(A_2 | C_j) \ldots P(A_n | C_j)$
  - Can estimate $P(A_i | C_j)$ for all A_i and C_j.
  - New point is classified to C_j if $P(C_j) \Pi P(A_i | C_j)$ is maximum.

How to Estimate Probabilities from Data?

Tid	**Refund**	**Marital Status**	**Taxable Income**	**Evade**
1	Yes	Single	125K	**No**
2	No	Married	100K	**No**
3	No	Single	70K	**No**
4	Yes	Married	120K	**No**
5	No	Divorced	95K	**Yes**
6	No	Married	60K	**No**
7	Yes	Divorced	220K	**No**
8	No	Single	85K	**Yes**
9	No	Married	75K	**No**
10	No	Single	90K	**Yes**

- Class: $P(C) = N_c/N$
  - e.g., P(No) = 7/10,
    P(Yes) = 3/10
- For discrete attributes: $P(A_i \mid C_k) = |A_{ik}|/N_{kc}$
  - where $|A_{ik}|$ is number of instances having attribute A_i and belongs to class C_k
  - Examples:

    P(Status=Married|No) = 4/7
    P(Refund=Yes|Yes)=0