THREE-DIMENSIONAL SPACE

VECTORS

Chapter 11

Post Mid2 Topics

12	Applications of Integration, Definite Integrals ,Area bounded by the curves. Volume by Disk and washer method	6.1(Q#1-18) 6.2(Q#1-26)
13	Applications of Integration : Arc length	6.4(Q#3-8,27-32)
14	**Analytical Geometry:** Parametric equations of lines in 3D	11.5(Q#3-10,15-22, 29-34,49,50)
15	Plane in 3-space ,Distance Problems involving planes, Intersecting planes.	11.6(11-20,41-48)
16	Revision / Presentation	

Old & New Topics:

- Dot product 11.3
- Cross product 11.4

- Parametric equation of line 11.5
- Planes in 3-space 11.6
  - Intersecting planes
  - Distance problem involving planes

11.3 DOT PRODUCT;

11.3.1 DEFINITION If $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are vectors in 2-space, then the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is written as $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

Similarly, if $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ are vectors in 3-space, then their dot product is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

11.3.2 THEOREM *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 2- or 3-space and* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

(*d*) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

(*e*) $\mathbf{0} \cdot \mathbf{v} = 0$

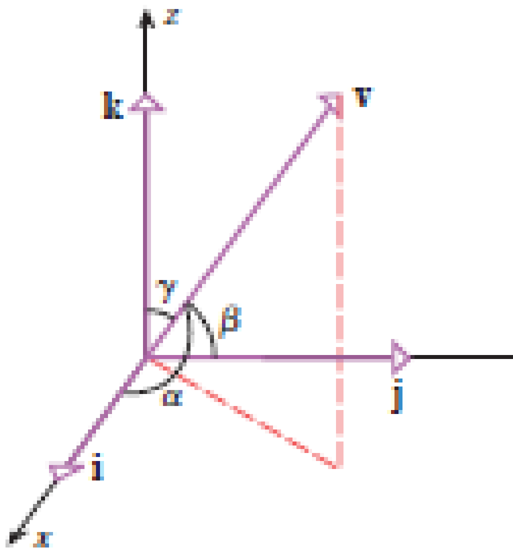
11.3.3 THEOREM *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are nonzero vectors in 2-space or 3-space, and if* θ *is the angle between them, then*

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \tag{2}$$

DIRECTION ANGLES

$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\,\|\mathbf{i}\|} = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\,\|\mathbf{j}\|} = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{\mathbf{v}\cdot\mathbf{k}}{\|\mathbf{v}\|\,\|\mathbf{k}\|} = \frac{v_3}{\|\mathbf{v}\|}$$

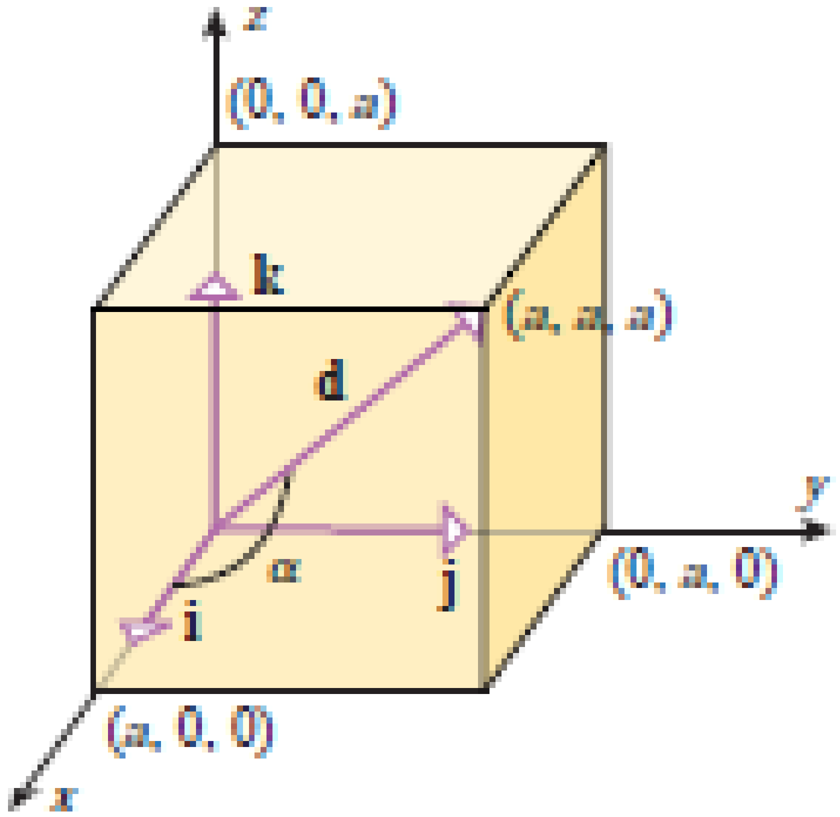


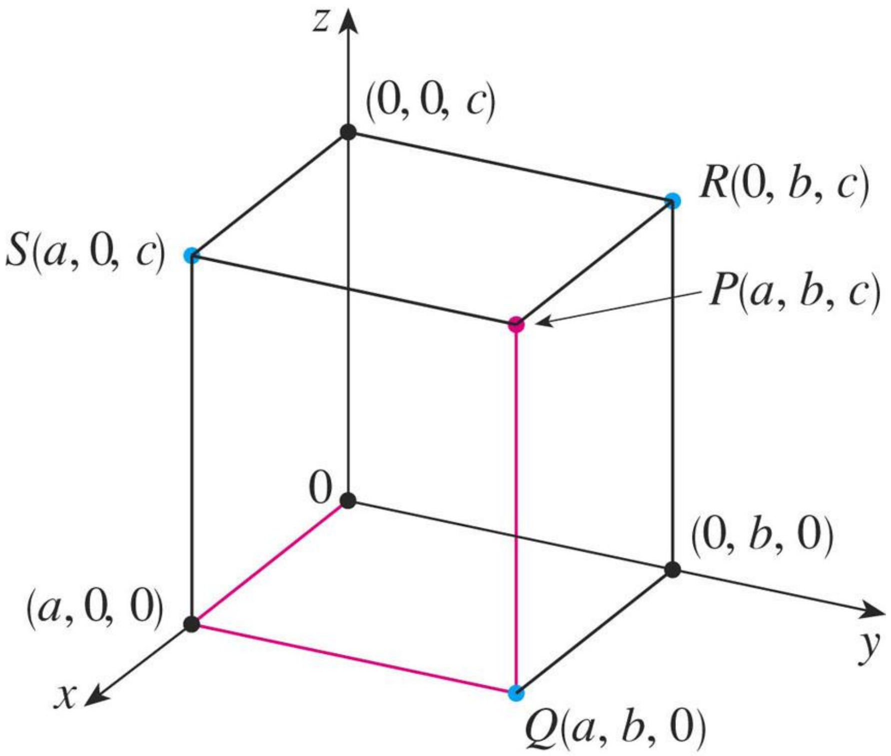
▲ Figure 11.3.5

11.3.4 THEOREM *The direction cosines of a nonzero vector* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ *are*

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

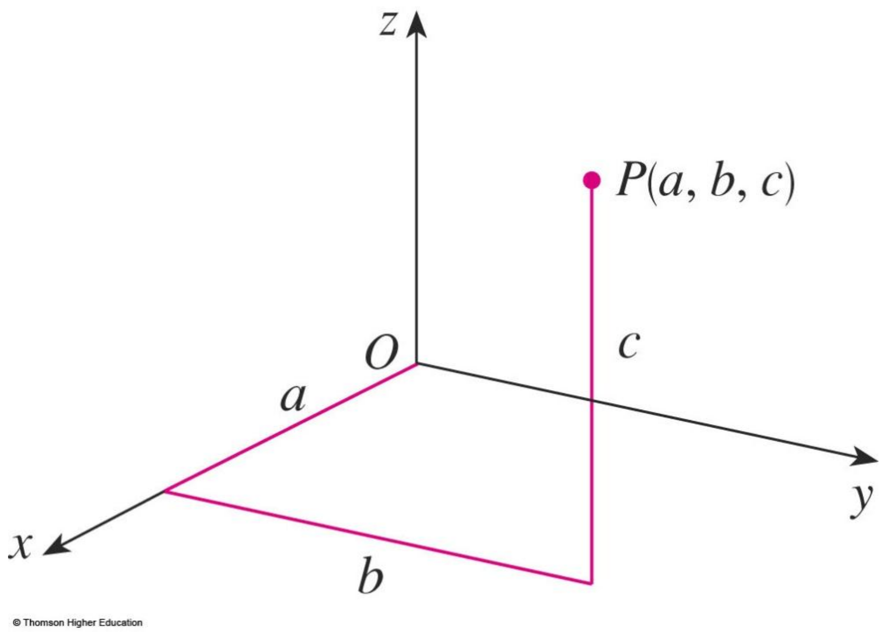
▶ **Example 3** Find the direction cosines of the vector $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, and approximate the direction angles to the nearest degree.

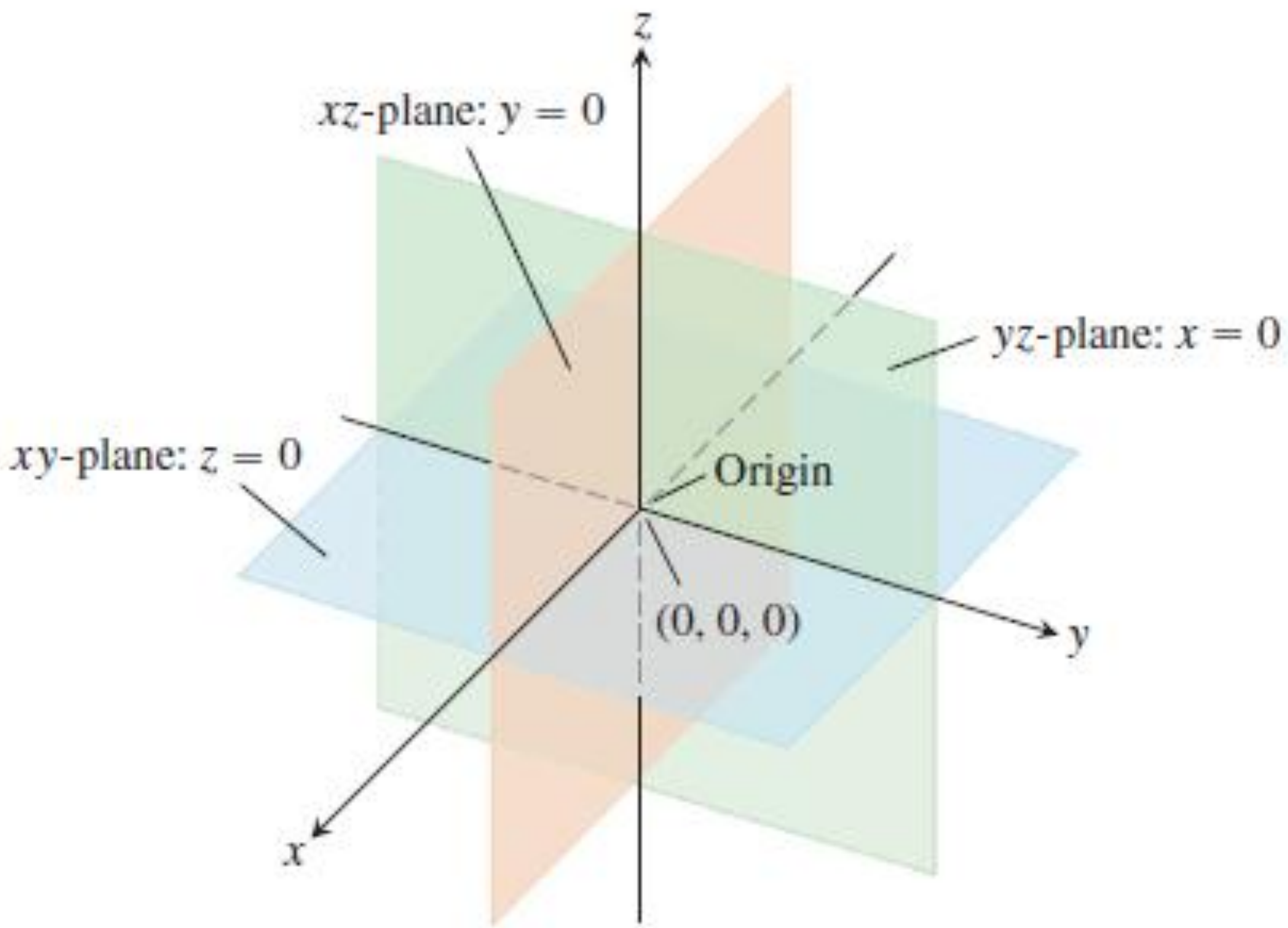




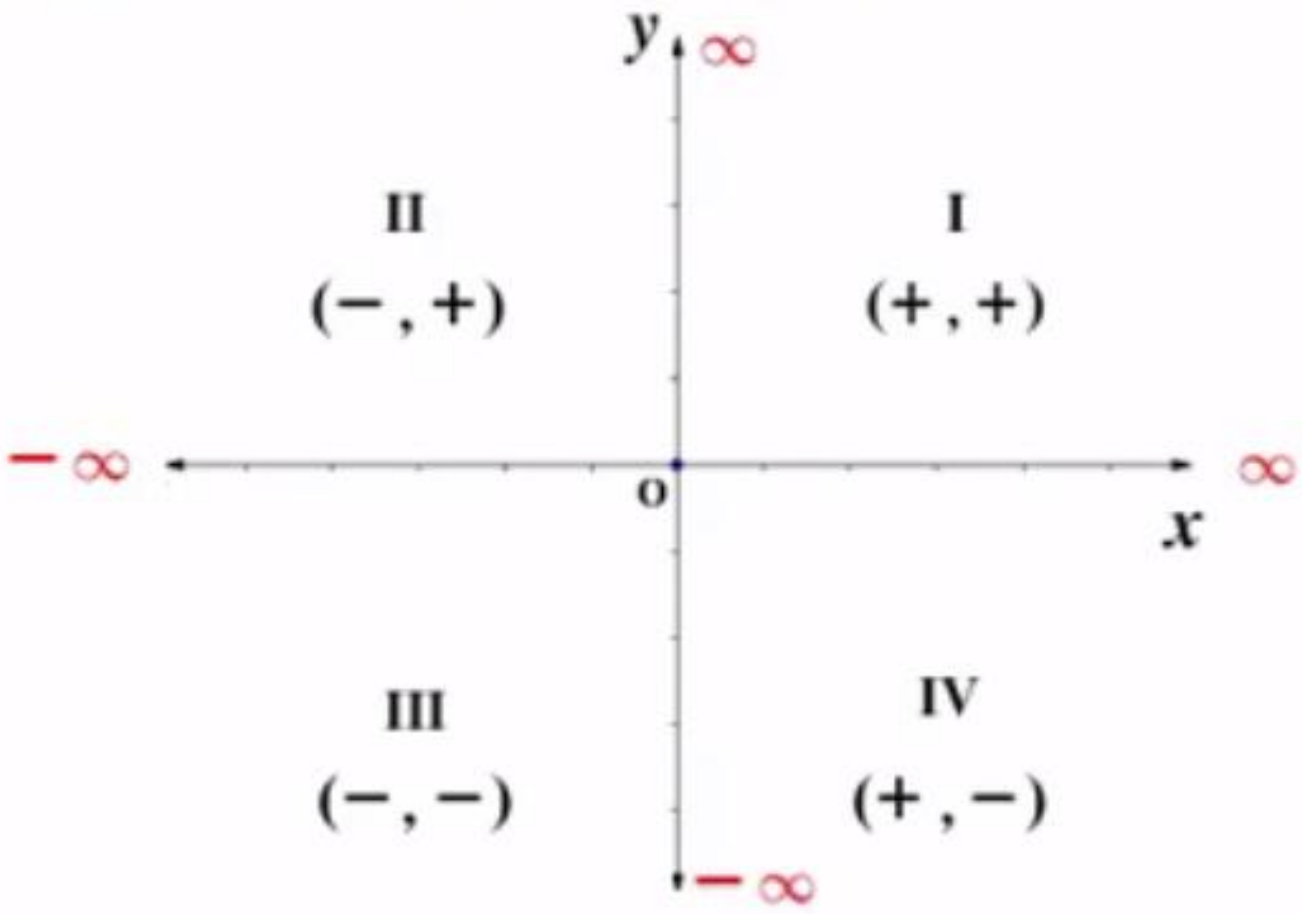
Coordinates system in 3D

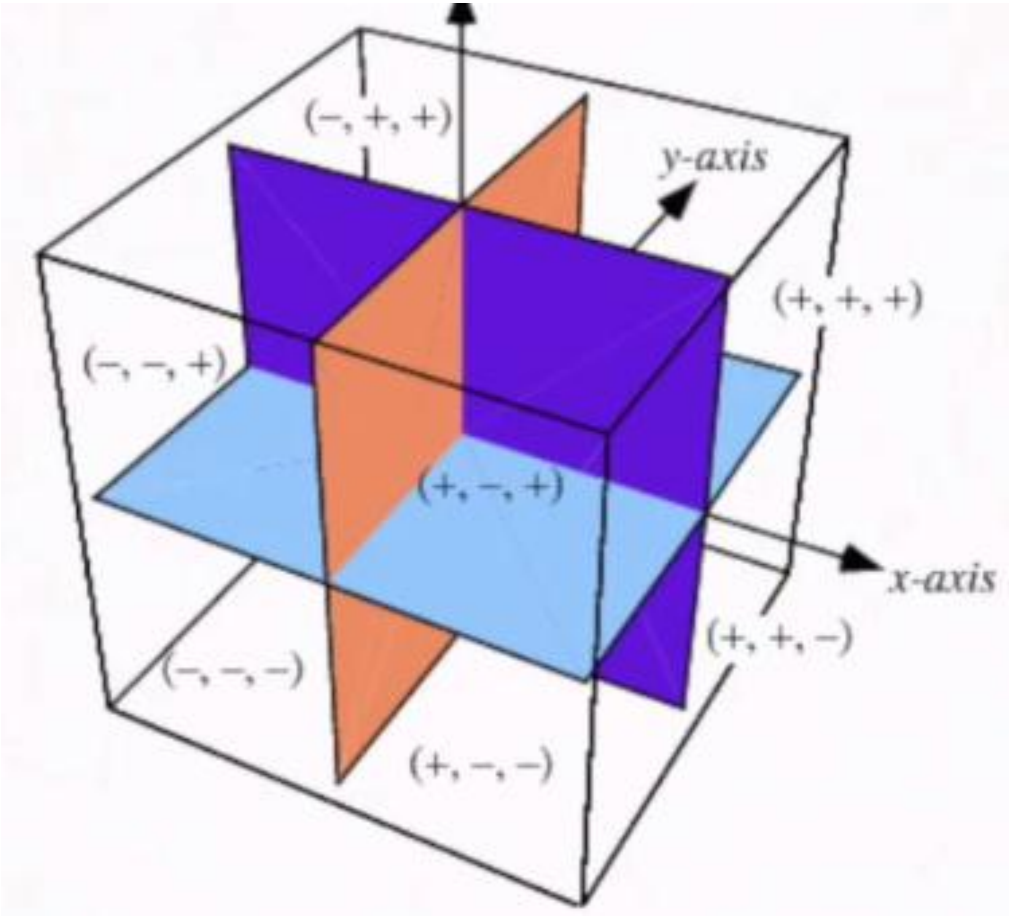
REGION	DESCRIPTION
xy-plane	Consists of all points of the form $(x, y, 0)$
xz-plane	Consists of all points of the form $(x, 0, z)$
yz-plane	Consists of all points of the form $(0, y, z)$
x-axis	Consists of all points of the form $(x, 0, 0)$
y-axis	Consists of all points of the form $(0, y, 0)$
z-axis	Consists of all points of the form $(0, 0, z)$

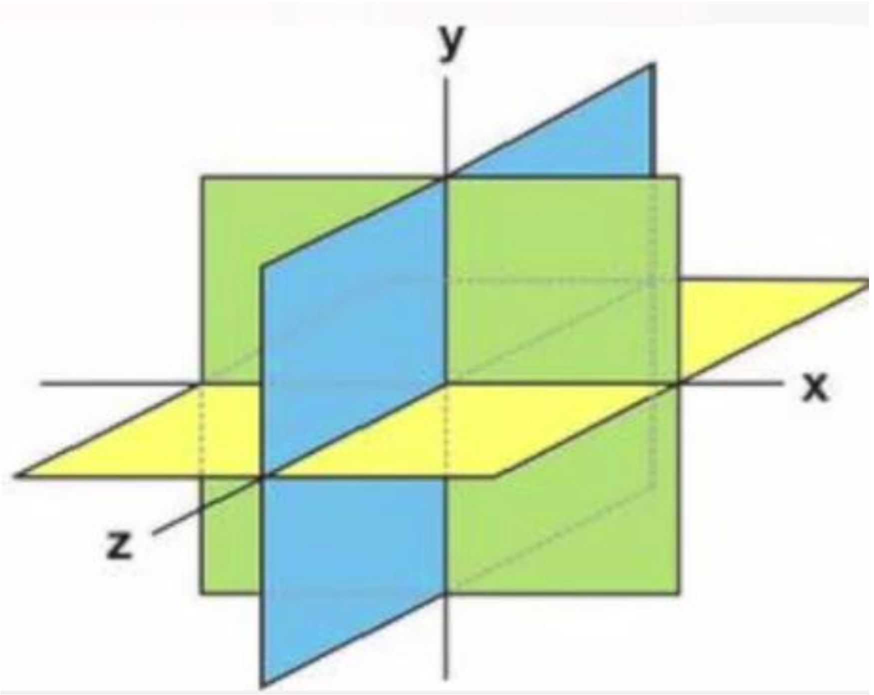



z
xz-plane: y = 0
yz-plane: x = 0
xy-plane: z = 0
Origin
(0, 0, 0)
y
x

Two Dimensional Coordinate System




(−, +, +)
y-axis
(+, +, +)
(−, −, +)
(+, −, +)
x-axis
(+, +, −)
(−, −, −)
(+, −, −)


y
x
z

Exact position of coordinates in 3-D

Octants / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

CROSS PRODUCT

We now turn to the main concept in this section.

11.4.2 DEFINITION If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the *cross product* $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \tag{3}$$

or, equivalently,

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k} \tag{4}$$

11.4.3 THEOREM *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$

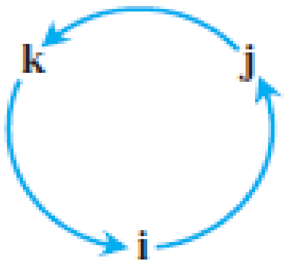


Diagram for recalling these products

11.4.4 THEOREM *If* **u** *and* **v** *are vectors in 3-space, then:*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* **u**)

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* **v**)

11.4.5 THEOREM *Let* **u** *and* **v** *be nonzero vectors in 3-space, and let* θ *be the angle between these vectors when they are positioned so their initial points coincide.*

(*a*) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$

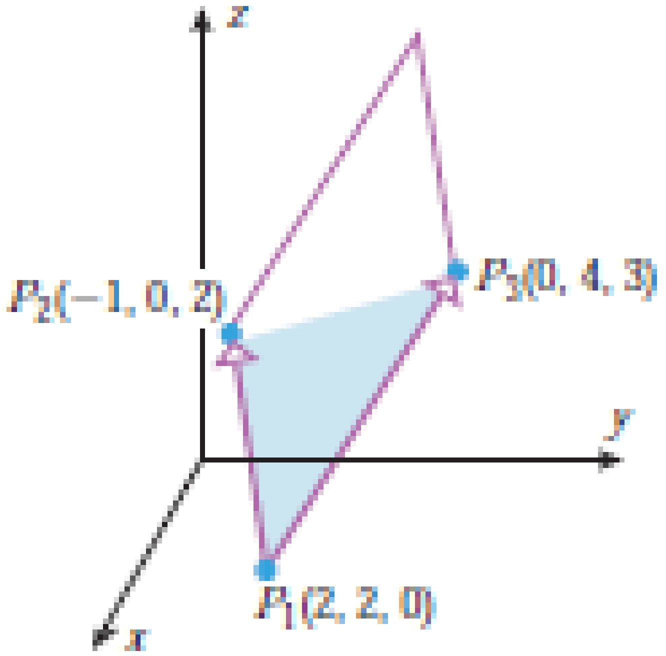
(*b*) *The area* A *of the parallelogram that has* **u** *and* **v** *as adjacent sides is*

$$A = \|\mathbf{u} \times \mathbf{v}\| \tag{8}$$

(*c*) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ *if and only if* **u** *and* **v** *are parallel vectors, that is, if and only if they are scalar multiples of one another.*

▸ Example 3 Find a vector that is orthogonal to both of the vectors $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (-7, 2, -1)$.

▸ Example 4 Find the area of the triangle that is determined by the points $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.



11.4.6 THEOREM *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be nonzero vectors in 3-space.*

(*a*) *The volume* V *of the parallelepiped that has* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *as adjacent edges is*

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \tag{10}$$

(*b*) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ *if and only if* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *lie in the same plane.*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$$

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$$

EXAMPLE 1 Component Form and Length of a Vector

Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

EXAMPLE 3 Performing Operations on Vectors

Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find

(a) $2\mathbf{u} + 3\mathbf{v}$ **(b)** $\mathbf{u} - \mathbf{v}$ **(c)** $\left|\frac{1}{2}\mathbf{u}\right|$.

EXAMPLE 5 Finding a Vector's Direction

Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3-1)\mathbf{i} + (2-0)\mathbf{j} + (0-1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

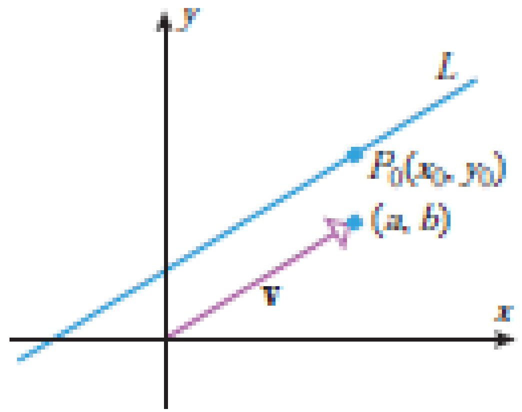
The unit vector $\mathbf{u}$ is the direction of $\overrightarrow{P_1P_2}$.

11.5 PARAMETRIC EQUATIONS OF LINES

11.5.1 THEOREM

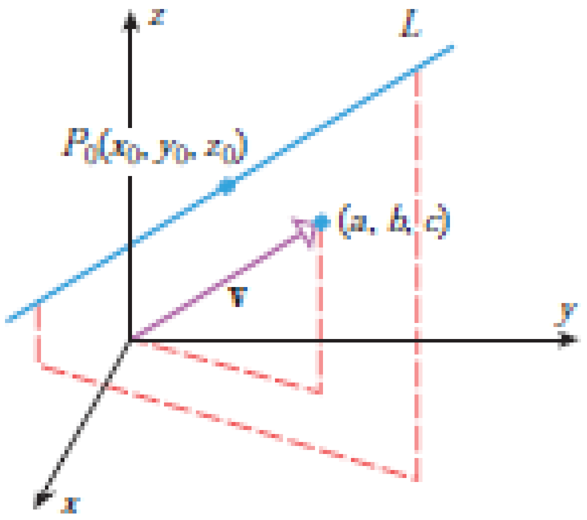
(a) *The line in 2-space that passes through the point $P_0(x_0, y_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$ has parametric equations*

$$x = x_0 + at, \quad y = y_0 + bt \tag{1}$$



(b) *The line in 3-space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has parametric equations*

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \tag{2}$$



$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

(symmetric equations)

► **Example 4** Find parametric equations describing the line segment joining the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

The line in 3-space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has parametric equations

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \tag{2}$$

VECTOR EQUATIONS OF LINES

$$(x, y) = (x_0 + at, y_0 + bt)$$

$$(x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct)$$

or, equivalently, as

$$(x, y) = (x_0, y_0) + t(a, b)$$

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$$

$$\mathbf{r} = (x, y), \quad \mathbf{r}_0 = (x_0, y_0), \quad \mathbf{v} = (a, b)$$

$$\mathbf{r} = (x, y, z), \quad \mathbf{r}_0 = (x_0, y_0, z_0),$$

In general

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty, \tag{2}$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

► **Example 6** Find an equation of the line in 3-space that passes through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

Parametric / Vector equation:

► **Example 4** Find parametric equations describing the line segment joining the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t \qquad (0 \le t \le 1)$$

► **Example 6** Find an equation of the line in 3-space that passes through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

Thus, a vector equation of the line through P_1 and P_2 is

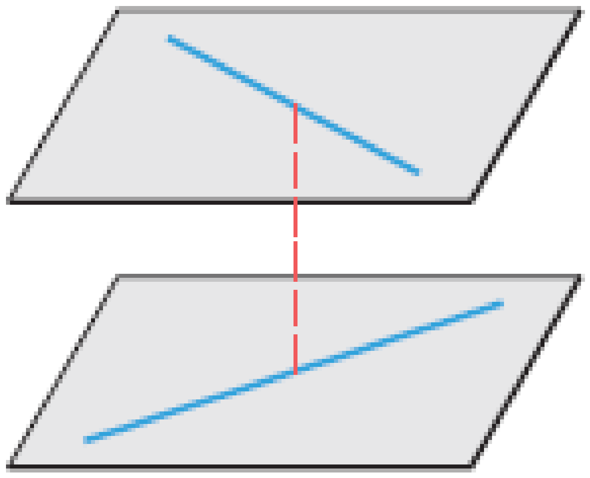
$$(x, y, z) = (2, 4, -1) + t(3, -4, 8)$$

▶ **Example 3** Let L_1 and L_2 be the lines

$$L_1: x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t$$
$$L_2: x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t$$

(a) Are the lines parallel?

(b) Do the lines intersect?



(b)

3–4 Find parametric equations for the line through P_1 and P_2 and also for the line segment joining those points. ■

Exercise : 11.5

3. (a) $P_1(3, -2)$, $P_2(5, 1)$ (b) $P_1(5, -2, 1)$, $P_2(2, 4, 2)$
4. (a) $P_1(0, 1)$, $P_2(-3, -4)$
   (b) $P_1(-1, 3, 5)$, $P_2(-1, 3, 2)$

5–6 Find parametric equations for the line whose vector equation is given. ■

5. (a) $\langle x, y \rangle = \langle 2, -3 \rangle + t\langle 1, -4 \rangle$
   (b) $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{k} + t(\mathbf{i} - \mathbf{j} + \mathbf{k})$

7–8 Find a point P on the line and a vector $\mathbf{v}$ parallel to the line by inspection. ■

7. (a) $x\mathbf{i} + y\mathbf{j} = (2\mathbf{i} - \mathbf{j}) + t(4\mathbf{i} - \mathbf{j})$
   (b) $\langle x, y, z \rangle = \langle -1, 2, 4 \rangle + t\langle 5, 7, -8 \rangle$

9–10 Express the given parametric equations of a line using bracket notation and also using $\mathbf{i}, \mathbf{j}, \mathbf{k}$ notation. ■

9. (a) $x = -3 + t$, $y = 4 + 5t$
   (b) $x = 2 - t$, $y = -3 + 5t$, $z = t$

15–22 Find parametric equations of the line that satisfies the stated conditions. ■

15. The line through $(-5, 2)$ that is parallel to $2\mathbf{i} - 3\mathbf{j}$.

16. The line through $(0, 3)$ that is parallel to the line $x = -5 + t$, $y = 1 - 2t$.

17. The line that is tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

18. The line that is tangent to the parabola $y = x^2$ at the point $(-2, 4)$.

19. The line through $(-1, 2, 4)$ that is parallel to $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.

20. The line through $(2, -1, 5)$ that is parallel to $\langle -1, 2, 7 \rangle$.

21. The line through $(-2, 0, 5)$ that is parallel to the line given by $x = 1 + 2t$, $y = 4 - t$, $z = 6 + 2t$.

22. The line through the origin that is parallel to the line given by $x = t$, $y = -1 + t$, $z = 2$.

23. Where does the line $x = 1 + 3t$, $y = 2 - t$ intersect
(a) the x-axis (b) the y-axis
(c) the parabola $y = x^2$?

29–30 Show that the lines L_1 and L_2 intersect, and find their point of intersection. ■

29. $L_1: x = 2 + t, \;\; y = 2 + 3t, \;\; z = 3 + t$
$L_2: x = 2 + t, \;\; y = 3 + 4t, \;\; z = 4 + 2t$

30. $L_1: x + 1 = 4t, \;\; y - 3 = t, \;\; z - 1 = 0$
$L_2: x + 13 = 12t, \;\; y - 1 = 6t, \;\; z - 2 = 3t$

31–32 Show that the lines L_1 and L_2 are skew. ■

31. $L_1: x = 1 + 7t, \;\; y = 3 + t, \;\; z = 5 - 3t$
$L_2: x = 4 - t, \;\; y = 6, \;\; z = 7 + 2t$

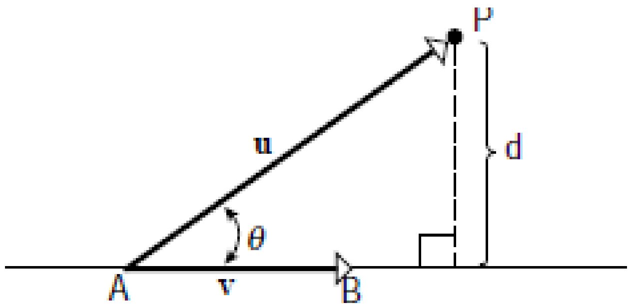
32. $L_1: x = 2 + 8t, \;\; y = 6 - 8t, \;\; z = 10t$
$L_2: x = 3 + 8t, \;\; y = 5 - 3t, \;\; z = 6 + t$

33–34 Determine whether the lines L_1 and L_2 are parallel. ■

33. $L_1: x = 3 - 2t, \;\; y = 4 + t, \;\; z = 6 - t$
$L_2: x = 5 - 4t, \;\; y = -2 + 2t, \;\; z = 7 - 2t$

34. $L_1: x = 5 + 3t, \;\; y = 4 - 2t, \;\; z = -2 + 3t$
$L_2: x = -1 + 9t, \;\; y = 5 - 6t, \;\; z = 3 + 8t$

49–50 Show that the lines L_1 and L_2 are parallel, and find the distance between them.



49. $L_1: x = 2 - t,\ \ y = 2t,\ \ z = 1 + t$
$L_2: x = 1 + 2t,\ \ y = 3 - 4t,\ \ z = 5 - 2t$

50. $L_1: x = 2t,\ \ y = 3 + 4t,\ \ z = 2 - 6t$
$L_2: x = 1 + 3t,\ \ y = 6t,\ \ z = -9t$

54. Consider the lines L_1 and L_2 whose symmetric equations are

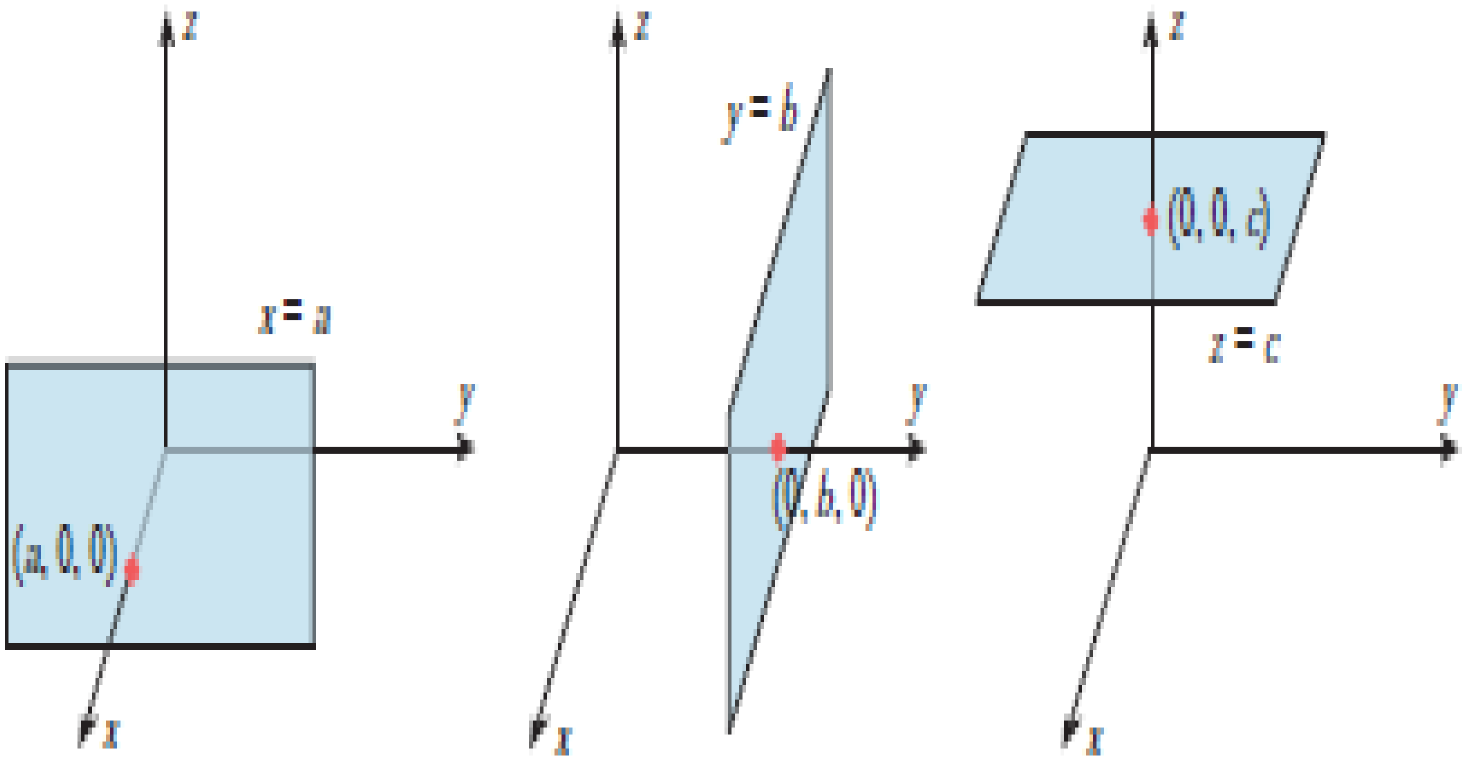
$$L_1: \frac{x-1}{2} = \frac{y+\frac{3}{2}}{1} = \frac{z+1}{2}$$

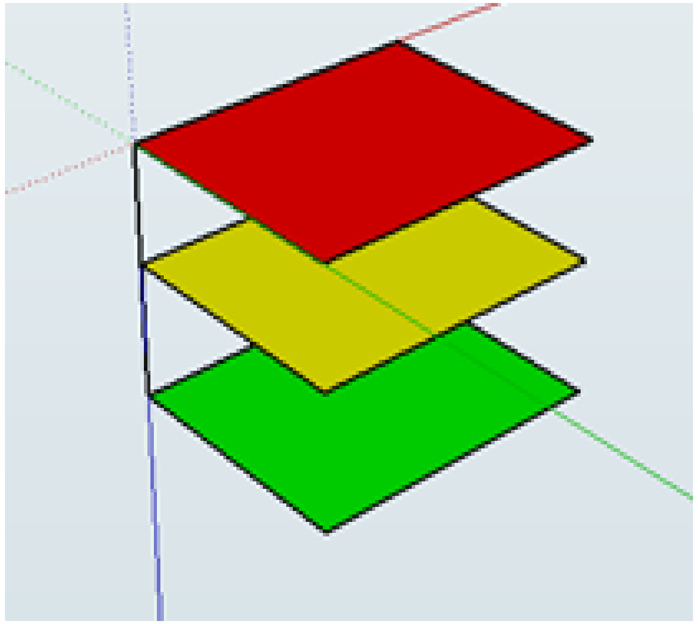
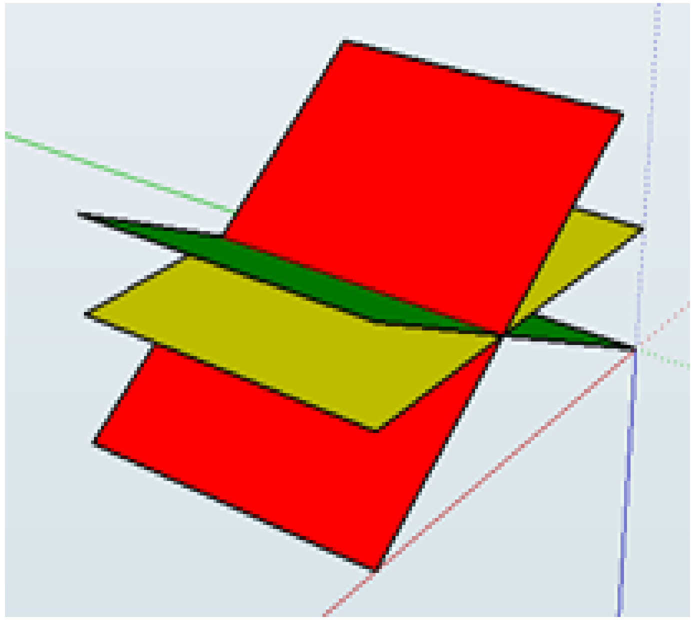
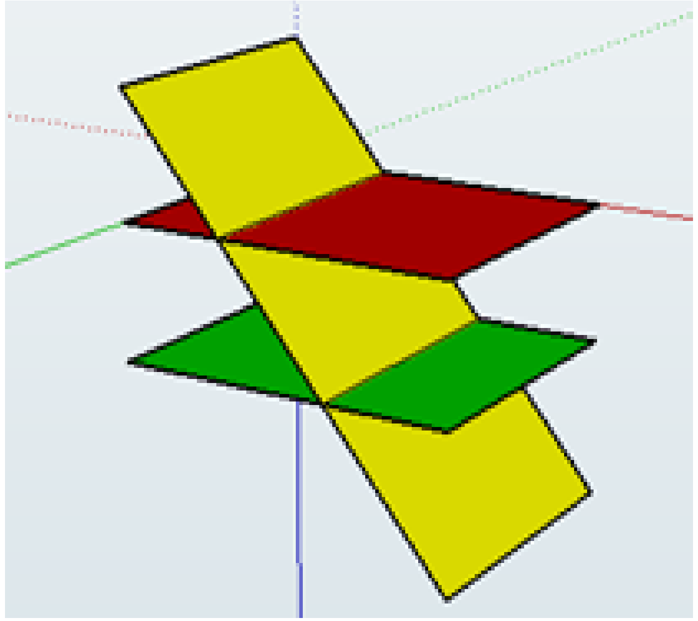
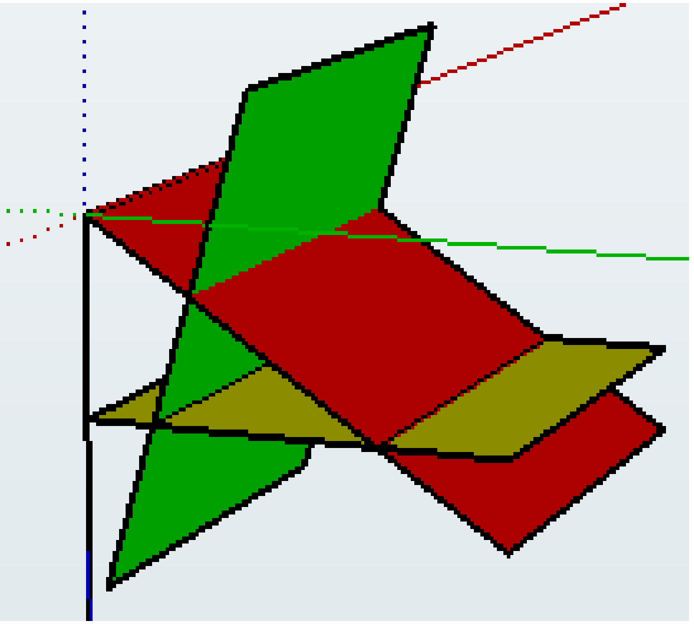
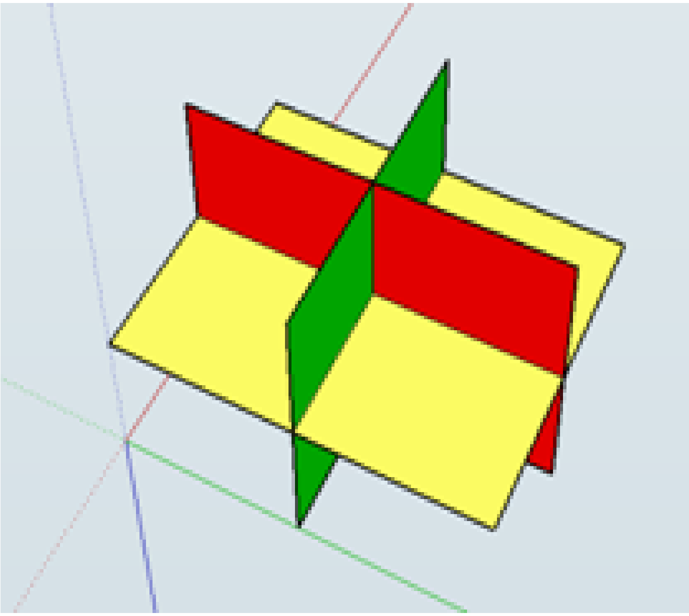
$$L_2: \frac{x-4}{-1} = \frac{y-3}{-2} = \frac{z+4}{2}$$

(see Exercise 52).

(a) Are L_1 and L_2 parallel? Perpendicular?
(b) Find parametric equations for L_1 and L_2.
(c) Do L_1 and L_2 intersect? If so, where?

A **plane** is a flat surface which extends without end in all directions.
Coplanar points are three or more points which lie in the same **plane**.
Two planes are parallel if their normal vectors are parallel.

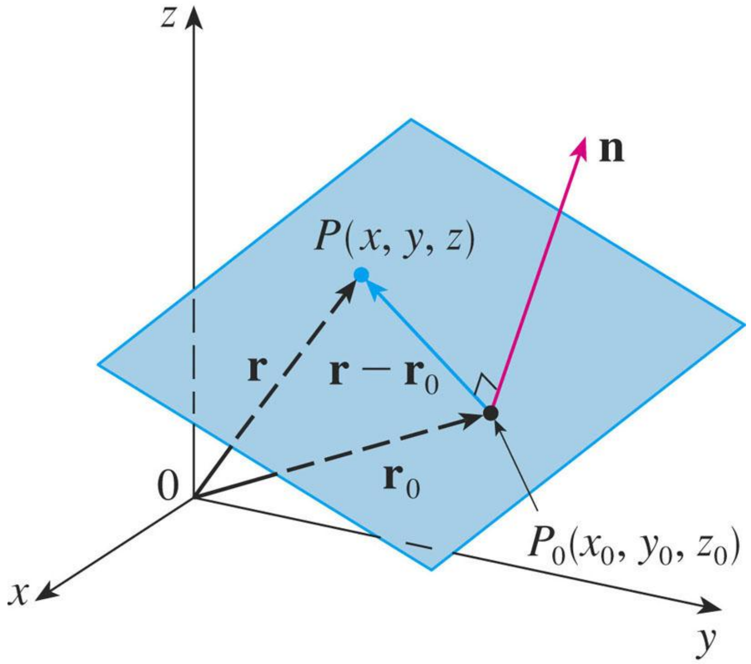


Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation:	$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$
Component equation:	$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
Component equation simplified:	$Ax + By + Cz = D$, where
	$D = Ax_0 + By_0 + Cz_0$



EXAMPLE 4 Finding a Unit Normal to a Plane

Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|}$$

11.6.1 THEOREM *If a, b, c, and d are constants, and a, b, and c are not all zero, then the graph of the equation*

$$ax + by + cz + d = 0 \tag{6}$$

is a plane that has the vector $\mathbf{n} = \langle a, b, c \rangle$ as a normal.

► **Example 1** Find an equation of the plane passing through the point $(3, -1, 7)$ and perpendicular to the vector $\mathbf{n} = \langle 4, 2, -5 \rangle$.

► **Example 2** Determine whether the planes

$$3x - 4y + 5z = 0 \quad \text{and} \quad -6x + 8y - 10z - 4 = 0$$

are parallel.

► **Example 3** Find an equation of the plane through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$.

► **Example 4** Determine whether the line

$$x = 3 + 8t, \quad y = 4 + 5t, \quad z = -3 - t$$

is parallel to the plane $x - 3y + 5z = 12$.

EXAMPLE 10 Finding the Intersection of a Line and a Plane

Find the point where the line

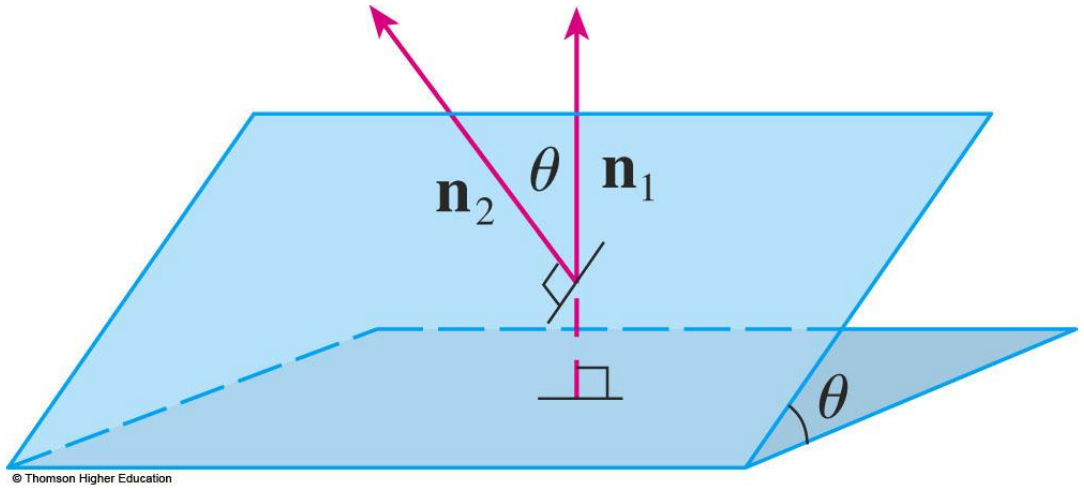
$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

The point of intersection is

$$(x, y, z)|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right).$$

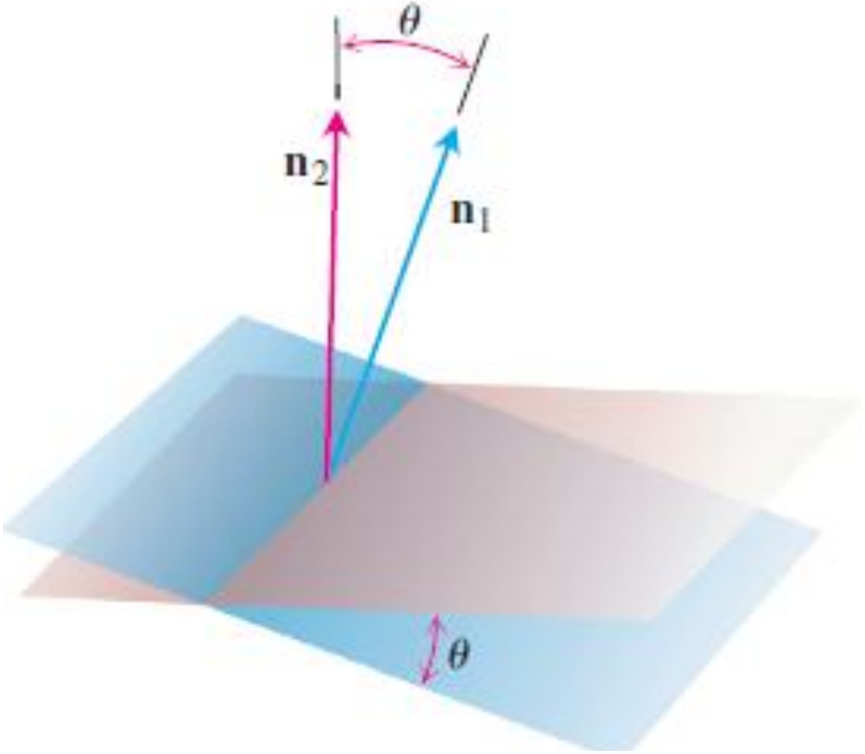
- If two planes are not parallel, then
  - They intersect in a straight line.
  - The angle between the two planes is defined as the acute angle between their normal vectors.



Angles Between Planes

The angle between them is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$$

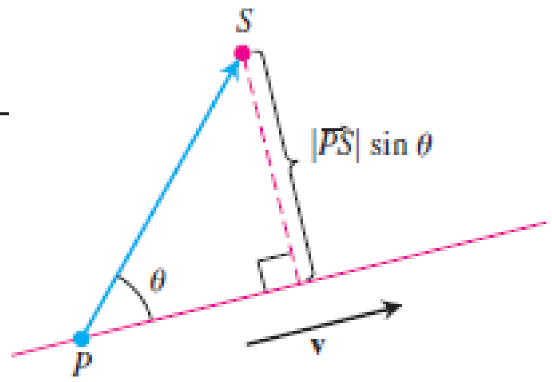


EXAMPLE 12 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

$$\approx 1.38 \text{ radians.}$$

The Distance from a Point to a Line in Space

To find the distance from a point S to a line that passes through a point P parallel to a vector $\mathbf{v}$, we find the absolute value of the scalar component of $\overrightarrow{PS}$ in the direction of a vector normal to the line (Figure 12.38). In the notation of the figure, the absolute value of the scalar component is, $|\overrightarrow{PS}| \sin\theta$, which is $\dfrac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$.



Distance from a Point S to a Line Through P Parallel to $\mathbf{v}$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

EXAMPLE 5 Finding Distance from a Point to a Line

Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

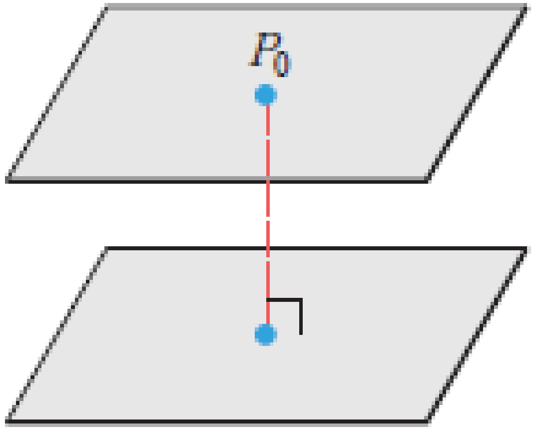
$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

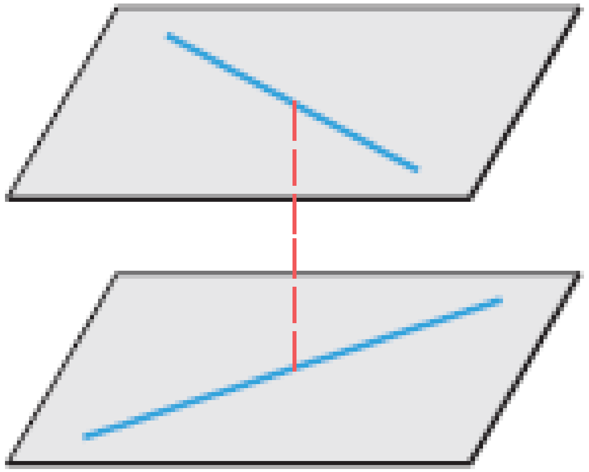
DISTANCE PROBLEMS INVOLVING PLANES

Next we will consider three basic distance problems in 3-space:

- Find the distance between a point and a plane.
- Find the distance between two parallel planes.
- Find the distance between two skew lines.



(a)



(b)

11.6.2 THEOREM *The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is*

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{10}$$

▶ **Example 8** Find the distance D between the point $(1, -4, -3)$ and the plane

$$2x - 3y + 6z = -1$$

▶ **Example 9** The planes

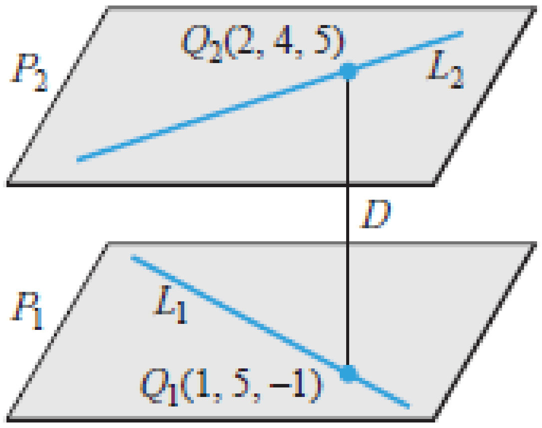
$$x + 2y - 2z = 3 \quad \text{and} \quad 2x + 4y - 4z = 7$$

are parallel since their normals, $\langle 1, 2, -2 \rangle$ and $\langle 2, 4, -4 \rangle$, are parallel vectors. Find the distance between these planes.

▶ Example 10 It was shown in Example 3 of Section 11.5 that the lines

$$L_1: x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t$$
$$L_2: x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t$$

are skew. Find the distance between them.



▲ **Figure 11.6.10**

Solution:

S-1 vector $\mathbf{u}_1 = \langle 4, -4, 5 \rangle$ is parallel to line L_1, and therefore also parallel to planes P_1 and P_2.

S-2 Similarly, $\mathbf{u}_2 = \langle 8, -3, 1 \rangle$ is parallel to L_2 and hence parallel to P_1 and P_2.

S-3 $$\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11\mathbf{i} + 36\mathbf{j} + 20\mathbf{k}$$ is normal to both P_1 and P_2.

S-4 an equation for P_2:

$$11(x-2)+36(y-4)+20(z-5)=0$$

S-5 The distance between $Q_1(1, 5, -1)$ and this plane is

$$D=\frac{|(11)(1)+(36)(5)+(20)(-1)-266|}{\sqrt{11^2+36^2+20^2}}=\frac{95}{\sqrt{1817}}$$

11–12 Find an equation of the plane that passes through the given points. ■

Exercise : 11.6

11. $(-2, 1, 1)$, $(0, 2, 3)$, and $(1, 0, -1)$

12. $(3, 2, 1)$, $(2, 1, -1)$, and $(-1, 3, 2)$

13–14 Determine whether the planes are parallel, perpendicular, or neither. ■

13. (a) $2x - 8y - 6z - 2 = 0$
$-x + 4y + 3z - 5 = 0$

(b) $3x - 2y + z = 1$
$4x + 5y - 2z = 4$

(c) $x - y + 3z - 2 = 0$
$2x + z = 1$

15–16 Determine whether the line and plane are parallel, perpendicular, or neither. ■

15. (a) $x = 4 + 2t,\ \ y = -t,\ \ z = -1 - 4t;$
$3x + 2y + z - 7 = 0$

(b) $x = t,\ \ y = 2t,\ \ z = 3t;$
$x - y + 2z = 5$

(c) $x = -1 + 2t,\ \ y = 4 + t,\ \ z = 1 - t;$
$4x + 2y - 2z = 7$

17–18 Determine whether the line and plane intersect; if so, find the coordinates of the intersection. ■

17. (a) $x = t,\ \ y = t,\ \ z = t;$
$3x - 2y + z - 5 = 0$

(b) $x = 2 - t,\ \ y = 3 + t,\ \ z = t;$
$2x + y + z = 1$

18. (a) $x = 3t,\ \ y = 5t,\ \ z = -t;$
$2x - y + z + 1 = 0$

(b) $x = 1 + t,\ \ y = -1 + 3t,\ \ z = 2 + 4t;$
$x - y + 4z = 7$

19–20 Find the acute angle of intersection of the planes to the nearest degree. ■

19. $x = 0$ and $2x - y + z - 4 = 0$

20. $x + 2y - 2z = 5$ and $6x - 3y + 2z = 8$

41–42 Find parametric equations of the line of intersection of the planes. ■

41. $-2x + 3y + 7z + 2 = 0$
$x + 2y - 3z + 5 = 0$

42. $3x - 5y + 2z = 0$
$z = 0$

43–44 Find the distance between the point and the plane. ■

43. $(1, -2, 3)$; $2x - 2y + z = 4$

44. $(0, 1, 5)$; $3x + 6y - 2z - 5 = 0$

45–46 Find the distance between the given parallel planes. ■

45. $-2x + y + z = 0$
$6x - 3y - 3z - 5 = 0$

46. $x + y + z = 1$
$x + y + z = -1$

47–48 Find the distance between the given skew lines. ■

47. $x = 1 + 7t,\ y = 3 + t,\ z = 5 - 3t$
$x = 4 - t,\ y = 6,\ z = 7 + 2t$

48. $x = 3 - t,\ y = 4 + 4t,\ z = 1 + 2t$
$x = t,\ y = 3,\ z = 2t$

49. Find an equation of the sphere with center $(2, 1, -3)$ that is tangent to the plane $x - 3y + 2z = 4$.