APPLICATION OF DEFINITE INTEGRALS

Chap-6

LENGTH(6.4) , AREA(6.1) , VOLUME(6.2)

Mid2 Exam Topics:

<table>
<tr><td>7</td><td>Relative Extrema(1^{st} and 2^{nd} derivative test) Absolute Maxima and Minima</td><td>4.2(Q#3-5,7-12,25-40)
4.4(Q#7-16,21-28)</td><td rowspan="2">Q2</td></tr>
<tr><td>8</td><td><u>Integral Calculus</u>:
Concept and idea of Integration
Indefinite Integrals. Sigma notation
Riemann sums</td><td>5.3(Q#1-50)
5.4(Q#1-20,35-48)
5.5(Q#1-24)</td></tr>
<tr><td>9</td><td>Techniques of integration
Basic Integration ,Integration by parts
Reduction formula ,Trigonometric substitution ,Hyperbolic function</td><td>7.1(Q#1-30)
7.2(Q#1-30,47-52,61,65)
7.4(Q#1-25,37-48)
6.9 (Q11-40,58-62)</td><td rowspan="2">A2</td></tr>
<tr><td>10</td><td>Integration of Rational function by Partial fraction , u= tan(x/2) substitution
Improper integrals.</td><td>7.5(Q#9-30)
7.6 (Q#65-70,87,88)
7.8(Q#3-32,37-40)</td></tr>
</table>

6.4 LENGTH OF A PLANE CURVE

6.4.2 DEFINITION If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \qquad (3)$$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (4)$$

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad (5)$$

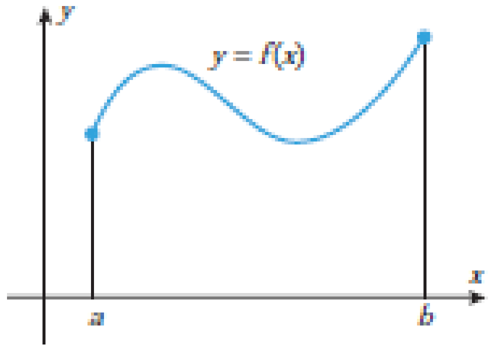


Figure 6.4.1

EXAMPLE Find the length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2} = 2\sqrt{2} x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2} x^{1/2}\right)^2 = 8x.$$

The length of the curve from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

$$= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big]_0^1 = \frac{13}{6}.$$

Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10\cosh(x/10)$ for $-15 \le x \le 15$, where x is measured in feet. Determine the length of the cable (in feet).

Solution

Recall from Section 6.4 that the formula for arc length is

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We have $f(x) = 10\cosh(x/10)$, so $f'(x) = \sinh(x/10)$. Then

$$\begin{aligned} \text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx. \end{aligned}$$

$$\text{Arc Length} = \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx$$

Now recall that $1 + \sinh^2 x = \cosh^2 x$.

$$= \int_{-15}^{15} \cosh\left(\frac{x}{10}\right) dx$$

$$= 10 \sinh\left(\frac{x}{10}\right)\Big|_{-15}^{15} = 10\left[\sinh\left(\frac{3}{2}\right) - \sinh\left(-\frac{3}{2}\right)\right] = 20 \sinh\left(\frac{3}{2}\right)$$

$$\approx 42.586 \text{ ft.}$$

Example: Assume a hanging cable has the shape $15\cosh(x/15)$ for $-20 \le x \le 20$. Determine the length

catenary a curve in the shape of the function $y = a\cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

▶ **Example 1** Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$ (Figure 6.4.4) in two ways: (a) using Formula (4) and (b) using Formula (5).

Exercise 6.4

3–8 Find the exact arc length of the curve over the interval. ■

3. $y = 3x^{3/2} - 1$ from $x = 0$ to $x = 1$

4. $x = \frac{1}{3}(y^2 + 2)^{3/2}$ from $y = 0$ to $y = 1$

5. $y = x^{2/3}$ from $x = 1$ to $x = 8$

6. $y = (x^6 + 8)/(16x^2)$ from $x = 2$ to $x = 3$

7. $24xy = y^4 + 48$ from $y = 2$ to $y = 4$

8. $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ from $y = 1$ to $y = 4$

Parametric equations. A **parametric equation** is where the x and y coordinates are both written in terms of another letter.
This is called a parameter and is usually given the letter t

If dx/dt and dy/dt are continuous functions for $a \leq t \leq b$, then it can be shown that as max $\Delta t_k \to 0$, this sum converges to

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

27–32 Use the arc length formula from Exercise 26 to find the arc length of the curve. ■

27. $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2 \quad (0 \leq t \leq 1)$

28. $x = (1+t)^2, \quad y = (1+t)^3 \quad (0 \leq t \leq 1)$

29. $x = \cos 2t, \quad y = \sin 2t \quad (0 \leq t \leq \pi/2)$

30. $x = \cos t + t \sin t, \quad y = \sin t - t \cos t \quad (0 \leq t \leq \pi)$

31. $x = e^t \cos t, \quad y = e^t \sin t \quad (0 \leq t \leq \pi/2)$

32. $x = e^t(\sin t + \cos t), \quad y = e^t(\cos t - \sin t) \quad (1 \leq t \leq 4)$

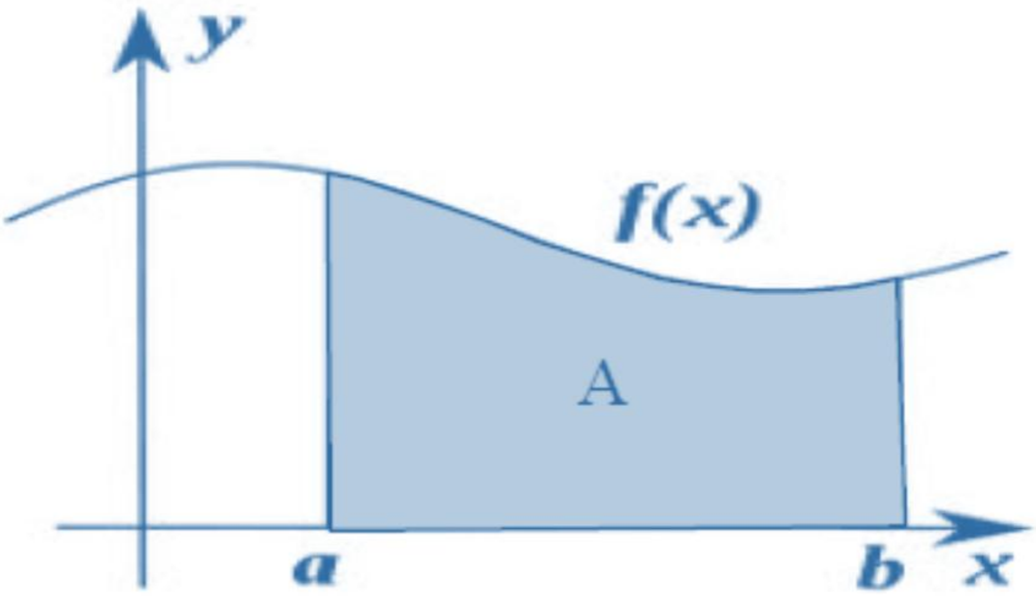
A REVIEW OF RIEMANN SUMS

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

$$\begin{array}{ccc}
\displaystyle\sum_{k=1}^{n} & f(x_k^*) & \Delta x_k \\
\downarrow & \downarrow & \downarrow \\
\displaystyle\int_a^b & f(x) & dx
\end{array}$$

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$$

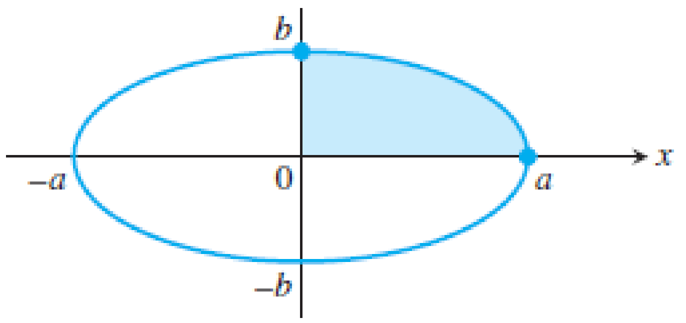
$$Area = \int_a^b f(x)dx$$



When calculating the area under a curve *f(x)*, follow the steps below:

1. Sketch the area.
2. Determine the boundaries a and b,
3. Set up the definite integral,
4. Integrate.

7.4 Finding the Area of an Ellipse



▸ **Example 3** Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution The area of the ellipse is

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2},$$

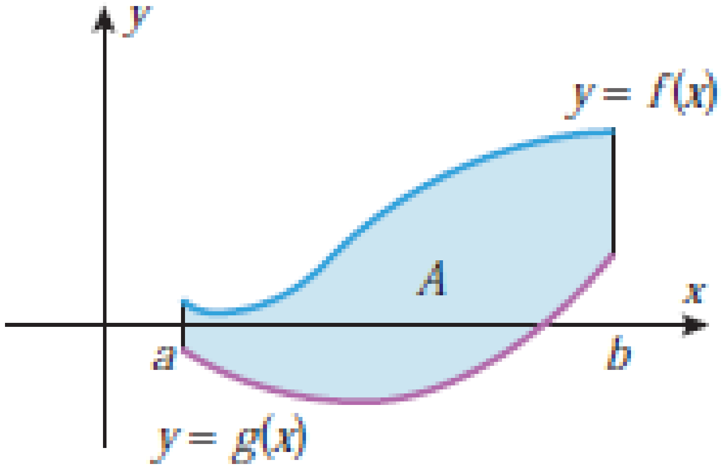
$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \le x \le a$$

$$\begin{aligned}
A &= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx \\
&= 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta \\
&= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta \\
&= 4ab\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\,d\theta \\
&= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\
&= 2ab\left[\frac{\pi}{2} + 0 - 0\right] = \pi ab.
\end{aligned}$$

AREA BETWEEN $y = f(x)$ AND $y = g(x)$

6.1.1 FIRST AREA PROBLEM Suppose that f and g are continuous functions on an interval $[a, b]$ and

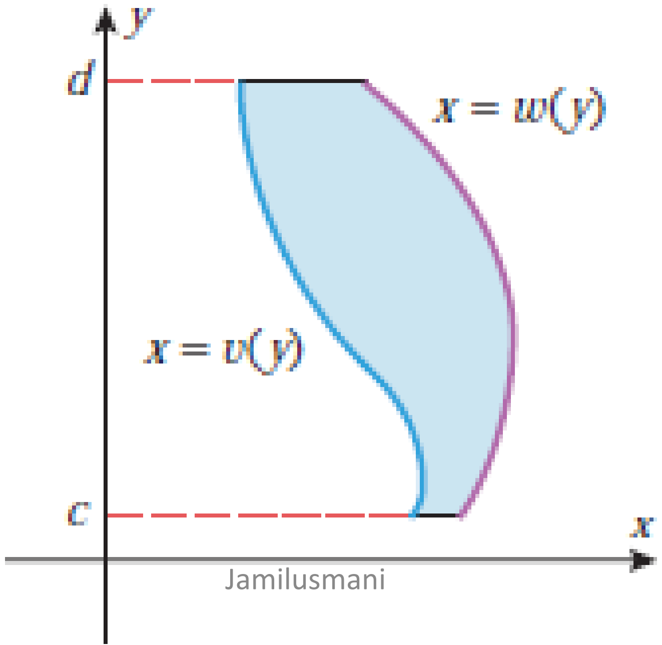
$$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$



$$A = \int_a^b [f(x) - g(x)]\, dx$$

6.1.4 AREA FORMULA

$$A = \int_c^d [w(y) - v(y)]\, dy$$

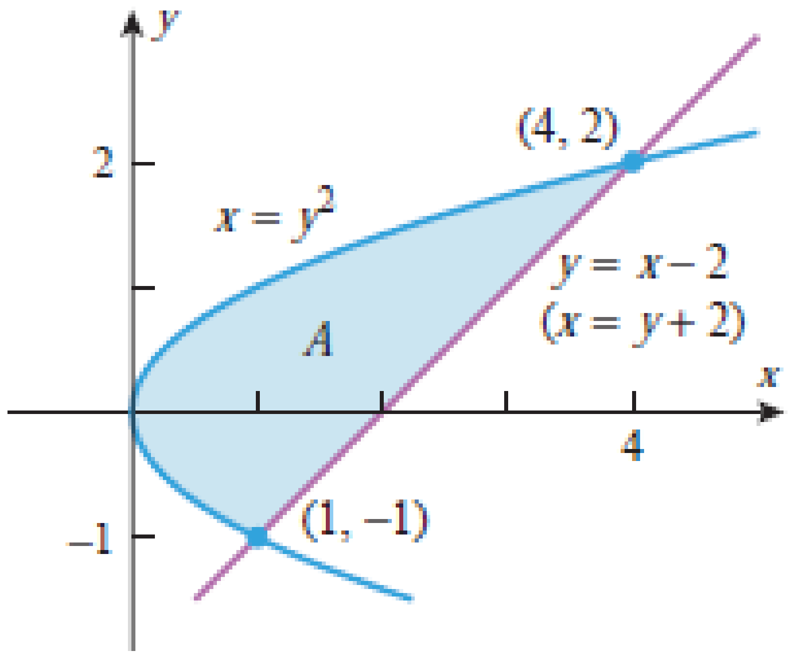


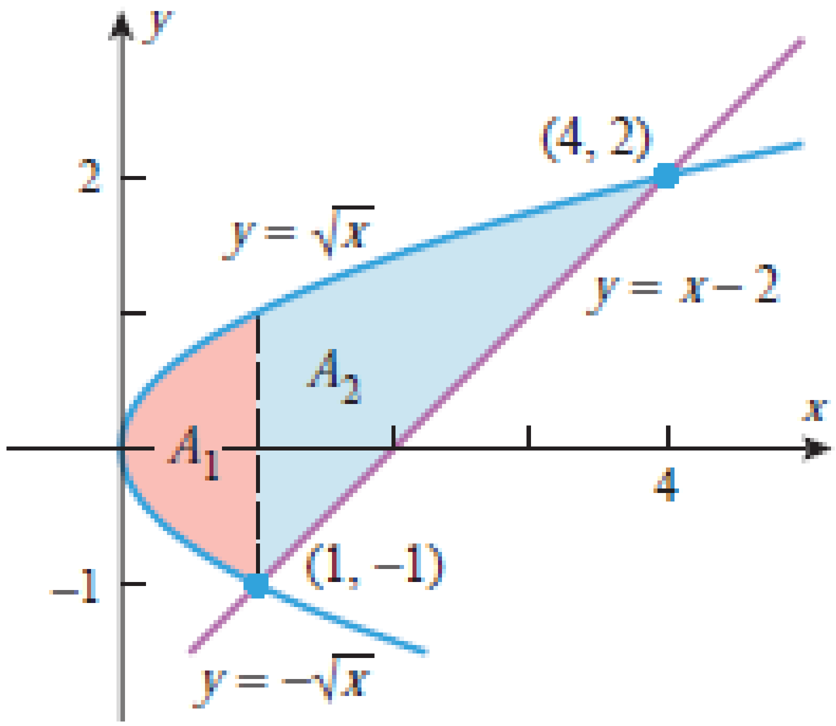
Practice with Theorem:

▶ **Example 1** Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

▶ **Example 2** Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

▶ **Example 4** Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.





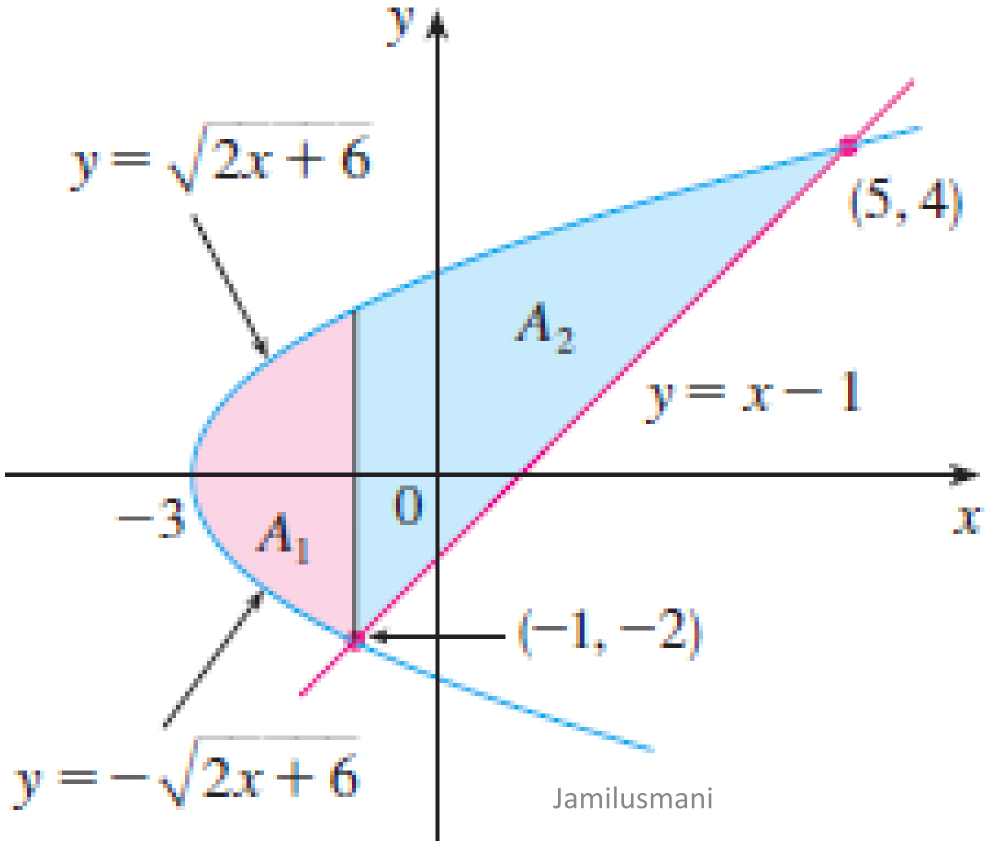
▶ **Example 5** Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y.

GROUP ACTIVITY

EXAMPLE 2 Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

$$\boxed{A = \int_0^1 (2x - 2x^2)\,dx}$$

EXAMPLE 6 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

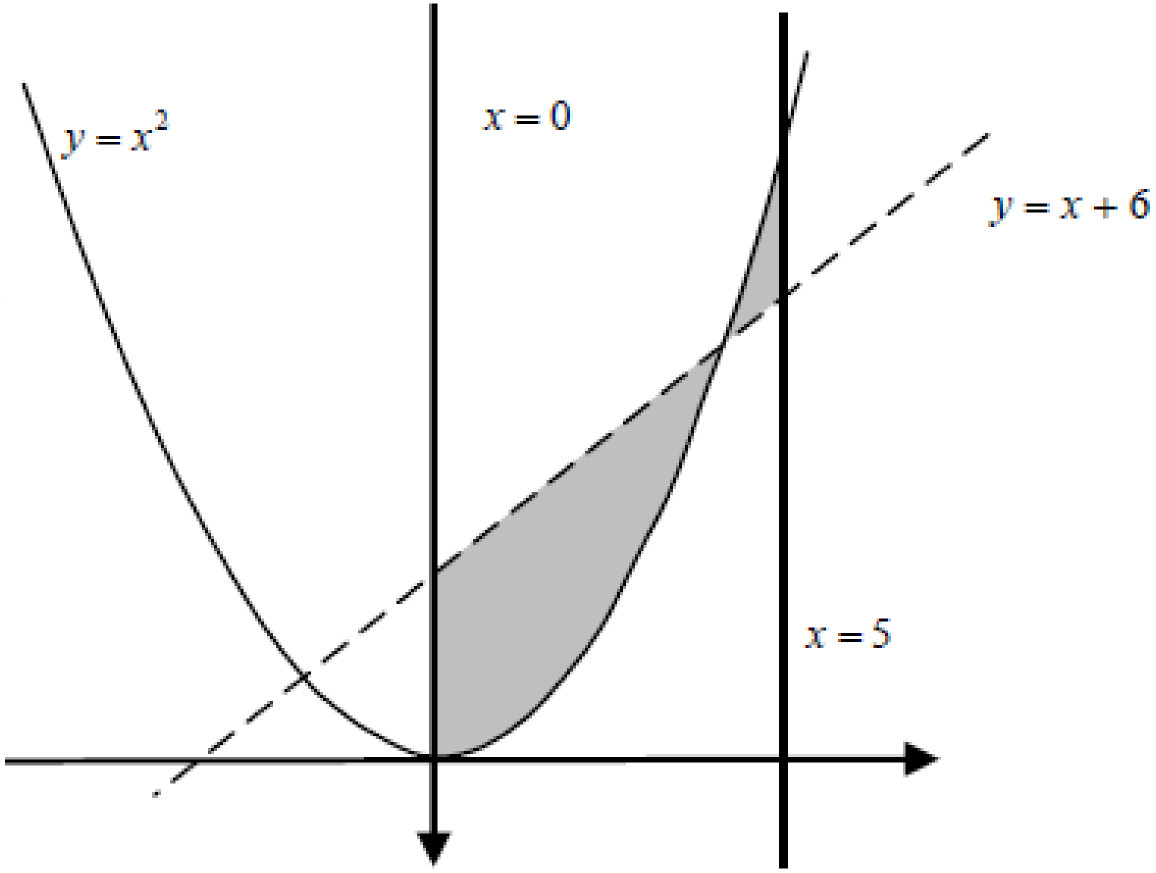


$$\boxed{= \int_{-2}^{4} \left[(y + 1) - \left(\tfrac{1}{2}y^2 - 3\right)\right] dy}$$

Find the area of the region enclosed by the following curves: $y = x^2$, $y = x + 6$, $x = 0$ and $x = 5$

$$A = \int_0^3 \left(x + 6 - x^2\right)dx + \int_3^5 \left(x^2 - (x+6)\right)dx$$

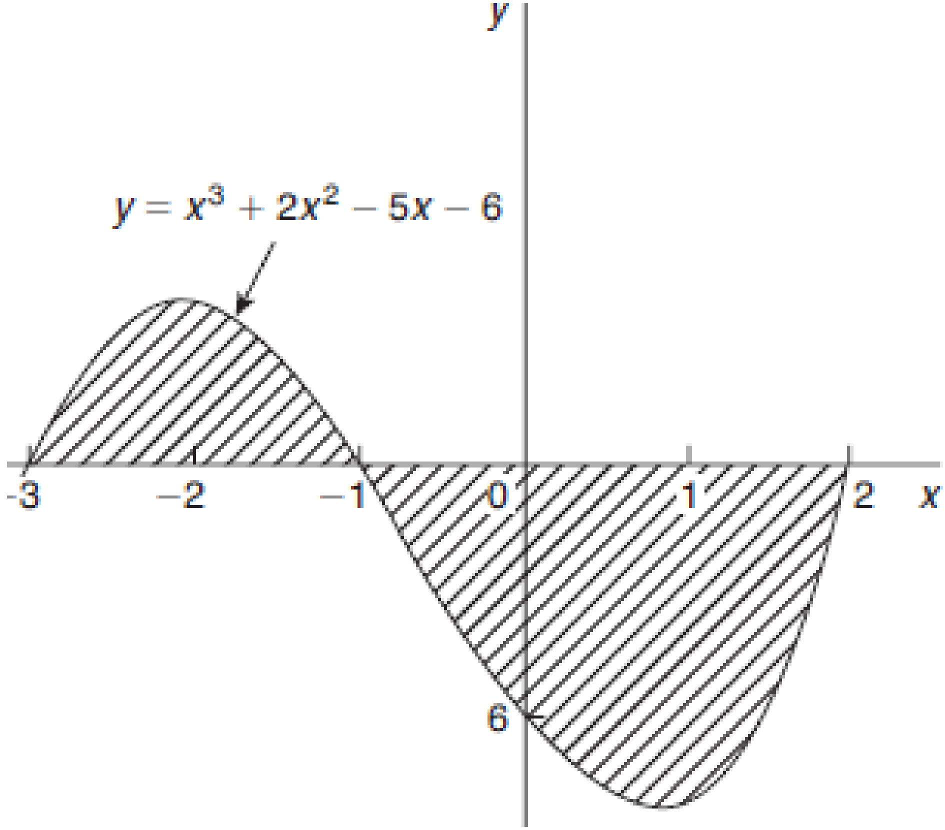
$$= \frac{157}{6} \text{ square units.}$$



Problem 3. Sketch the graph $y = x^3 + 2x^2 - 5x - 6$ between $x = -3$ and $x = 2$ and determine the area enclosed by the curve and the x-axis

- Table for Graph

x	-3	-2	-1	0	1	2
y	0	4	0	-6	-8	0



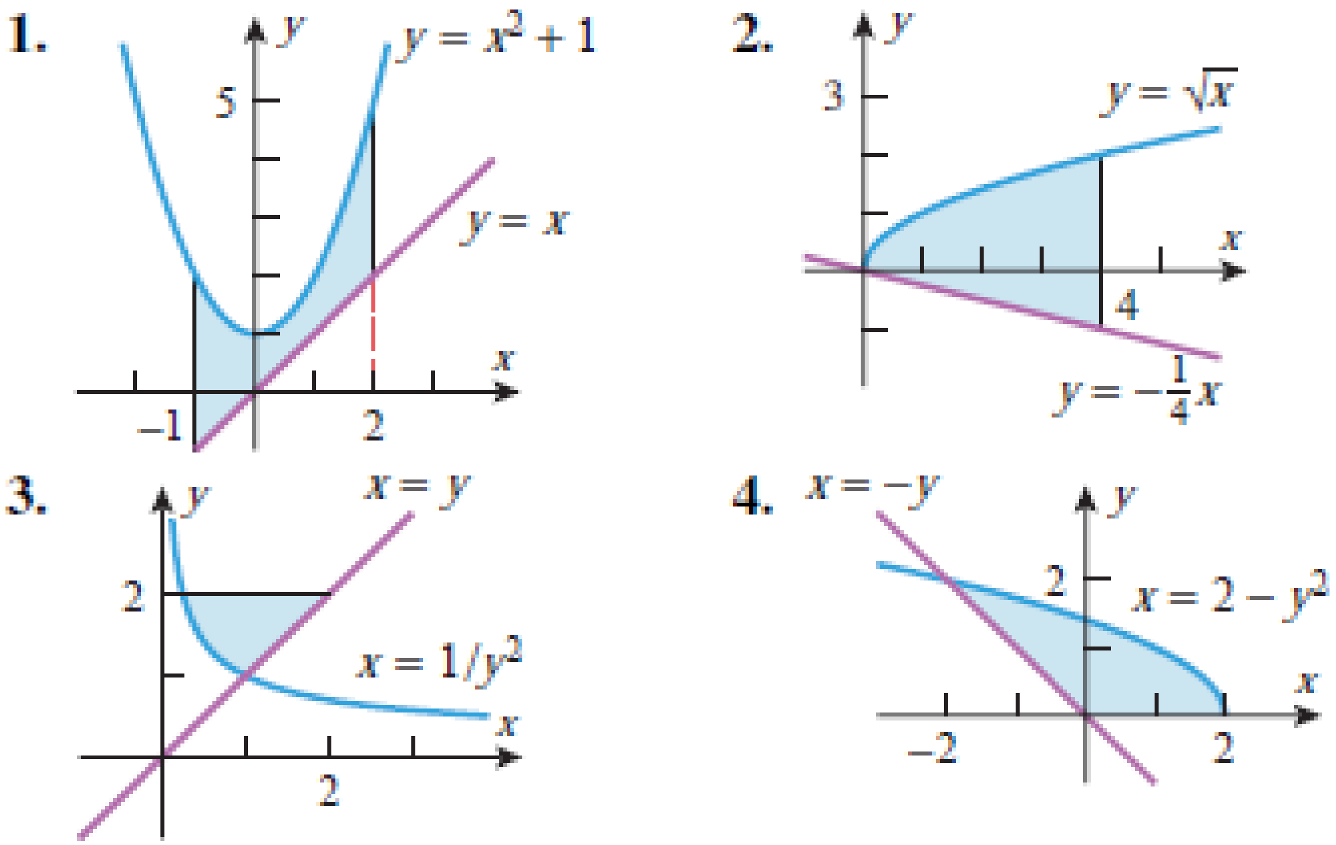
$$\text{Shaded area} = \int_{-3}^{-1} y\,dx - \int_{-1}^{2} y\,dx,$$

$$= \left[5\frac{1}{3}\right] - \left[-15\frac{3}{4}\right]$$

$$= 21\frac{1}{12} \quad \text{or} \quad 21.08 \text{ square units}$$

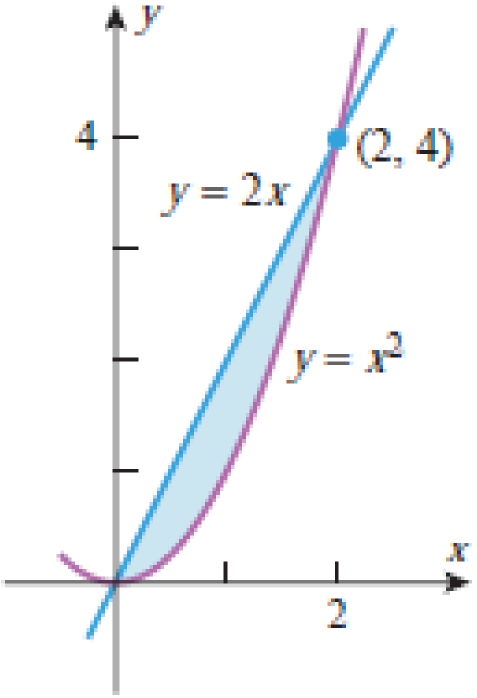
EXERCISE SET 6.1 Graphing Utility C CAS

1–4 Find the area of the shaded region.

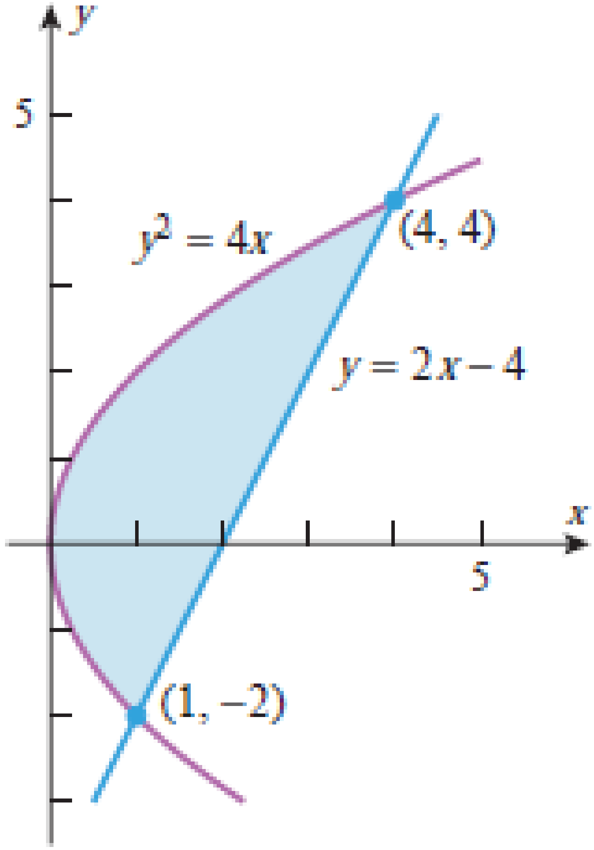


5–6 Find the area of the shaded region by (a) integrating with respect to x and (b) integrating with respect to y.

5.



6.



KEY EQUATIONS

- Disk Method along the x-axis

$$V = \int_a^b \pi[f(x)]^2\,dx$$

- Disk Method along the y-axis

$$V = \int_c^d \pi[g(y)]^2\,dy$$

- Washer Method

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$$

- Area between two curves, integrating on the *x*-axis

$$A = \int_a^b [f(x) - g(x)]dx$$

- Area between two curves, integrating on the *y*-axis

$$A = \int_c^d [u(y) - v(y)]dy$$

- Arc Length of a Function of *x*

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

- Arc Length of a Function of *y*

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy$$

Finding Volumes by Disk / Washer

Chapter 6(6.2)

Volume is the quantity of 3D space enclosed by a closed surface

VOLUME (DISK & WASHER)

$$V = A \cdot h = [\text{area of a cross section}] \times [\text{height}]$$

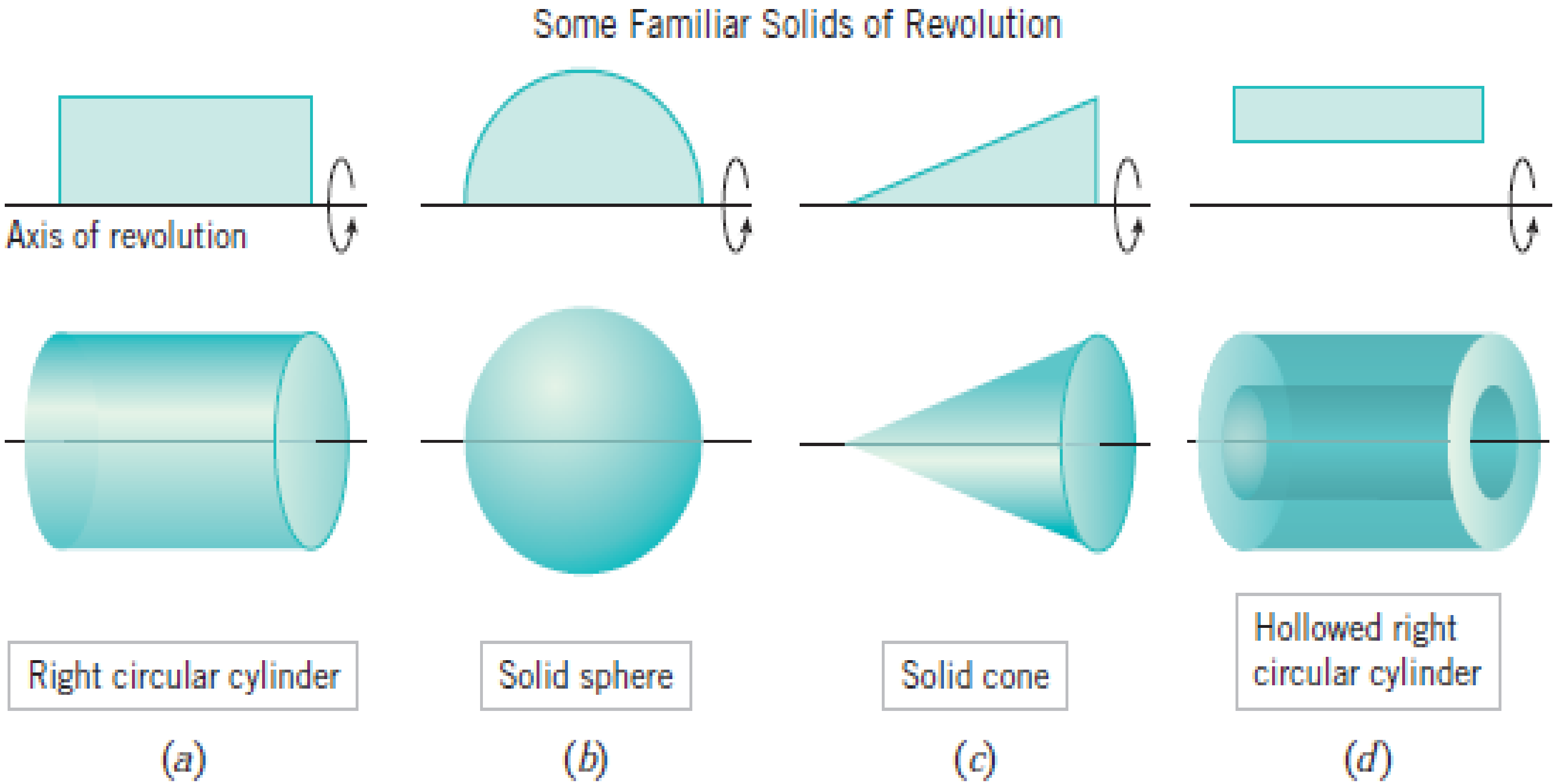
6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx \qquad (3)$$

provided $A(x)$ is integrable.

SOLIDS OF REVOLUTION

A *solid of revolution* is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the *axis of revolution*. Many familiar solids are of this type (Figure 6.2.8).



VOLUMES BY DISKS PERPENDICULAR TO THE x-AXIS

We will be interested in the following general problem.

6.2.4 PROBLEM Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.

$$V = \int_a^b \pi[f(x)]^2\, dx$$

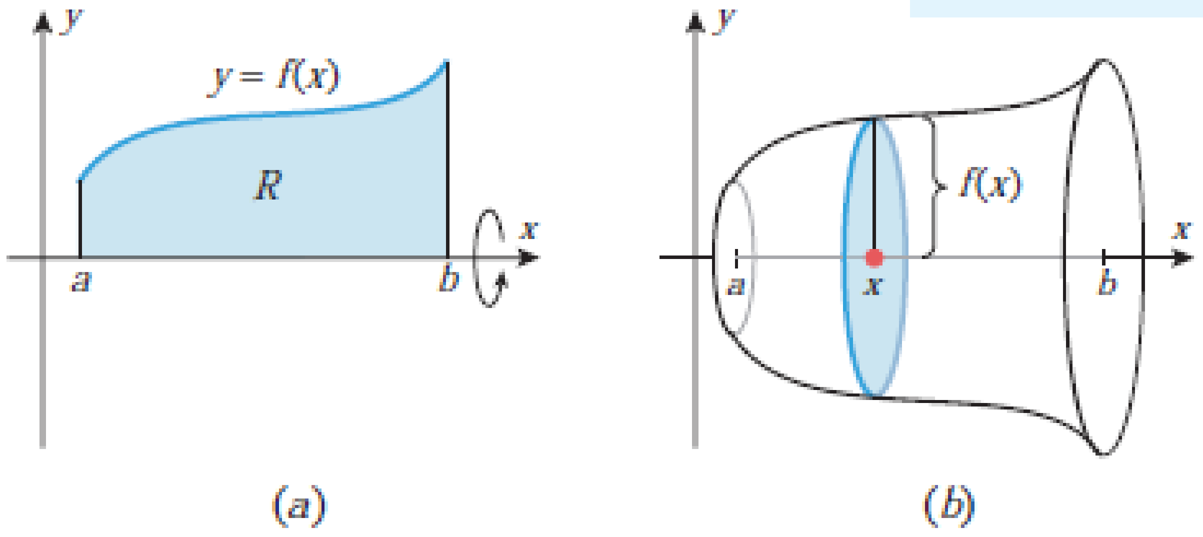
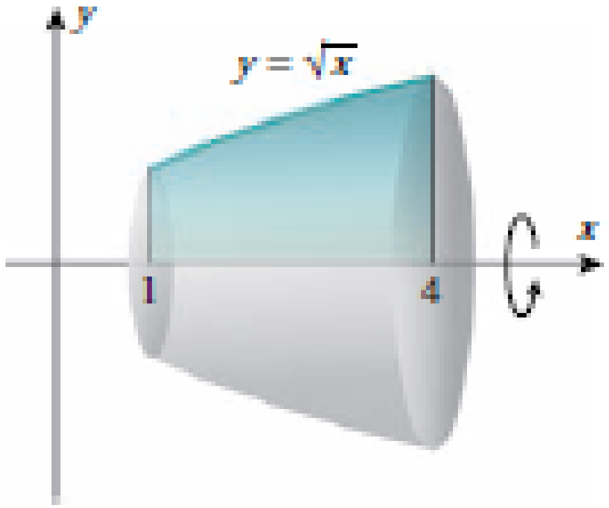


Figure 6.2.9

$$V = \int_a^b \pi[f(x)]^2\,dx$$

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

▶ **Example 2** Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis (Figure 6.2.10).

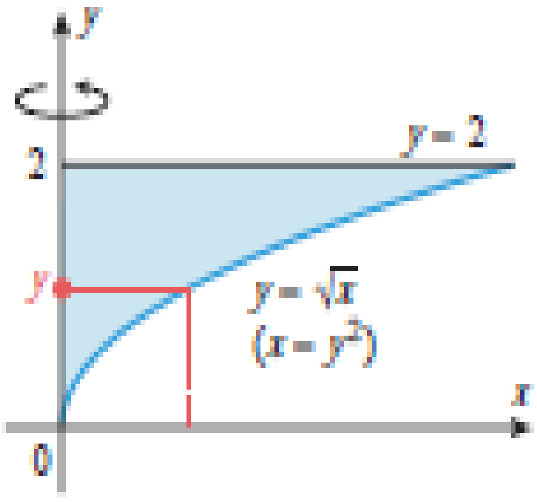


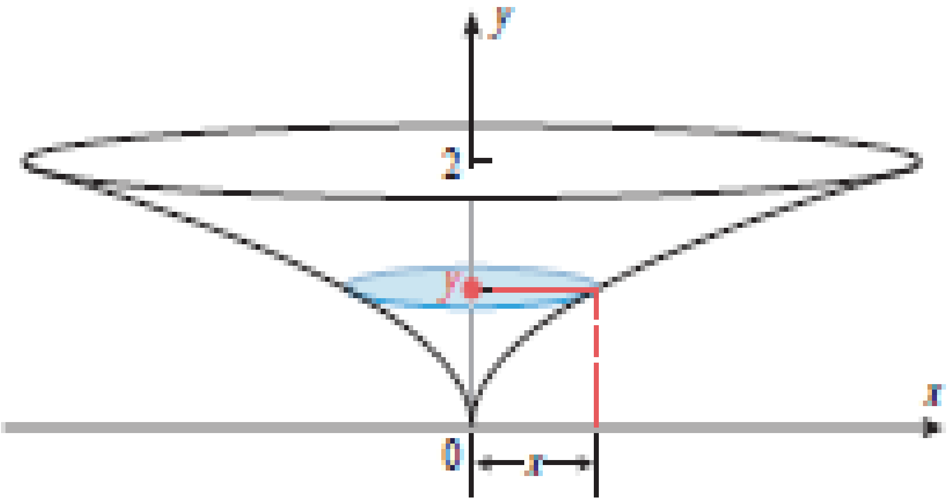
▲ Figure 6.2.10

EXAMPLE 7 Rotation About the y-Axis

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

► **Example 5** Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis.





washer

- A **washer** is a thin plate (typically disk-shaped) with a hole (typically in the middle)

VOLUMES BY WASHERS PERPENDICULAR TO THE x-AXIS

Not all solids of revolution have solid interiors; some have holes or channels that create interior surfaces, as in Figure 6.2.8*d*. So we will also be interested in problems of the following type.

> **6.2.5 PROBLEM** Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.12*a*). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis (Figure 6.2.12*b*).

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\, dx$$

Outer radius: $R(x)$

Inner radius: $r(x)$

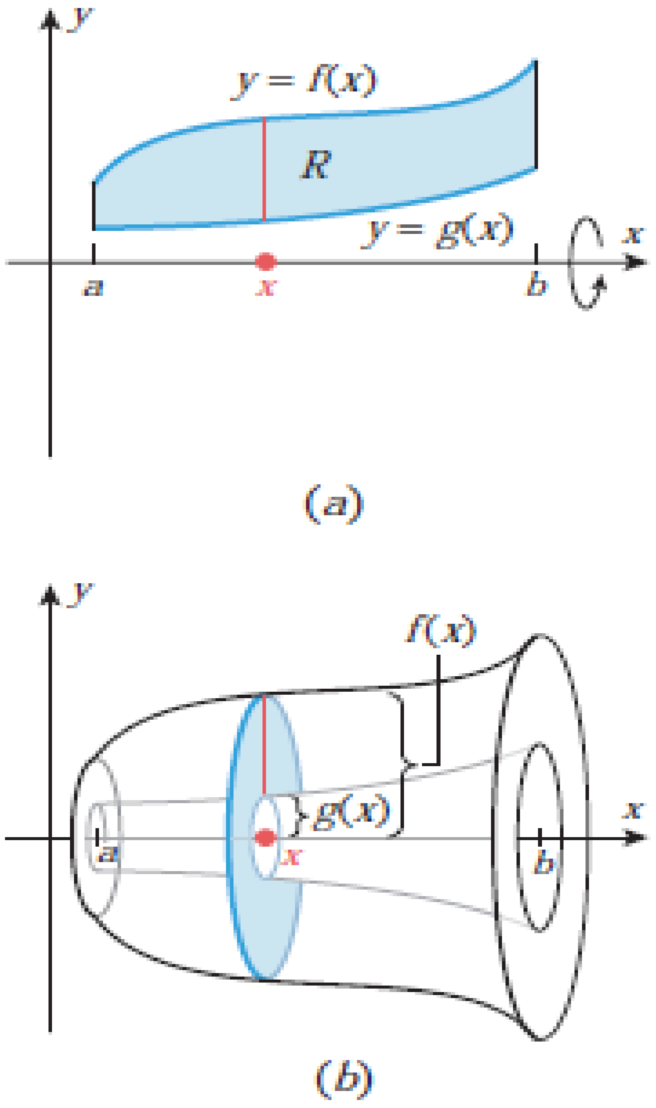
The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

$$\begin{pmatrix} Area \\ of \\ Outer \end{pmatrix} - \begin{pmatrix} Area \\ of \\ Inner \end{pmatrix}$$

Consequently, the definition of volume gives

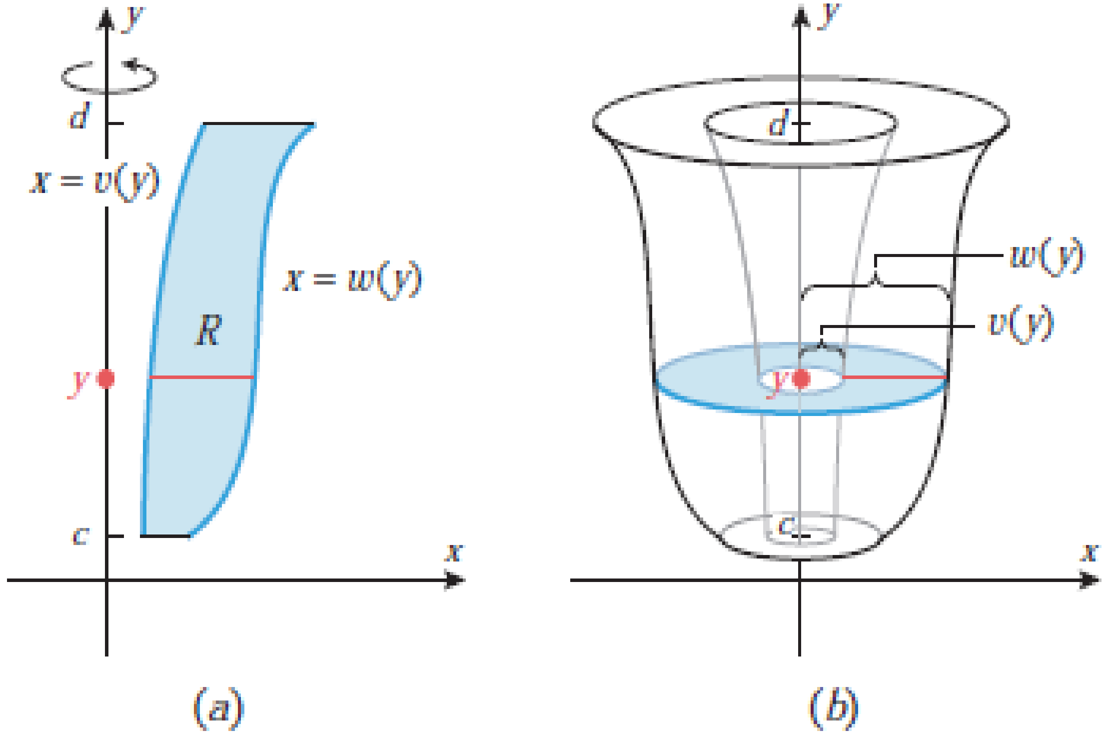
$$\boxed{V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.}$$



▲ Figure 6.2.12

$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy$$

Washers



Example:

The region R is bounded by the graphs of f(x) = √x and g(x) = x^2

between x = 0 and x = 1. What is the volume of the solid resulting when R is revolved about the x-axis?

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx \quad = \int_0^1 \pi\left(x - x^4\right)dx$$

$$= \pi\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{3\pi}{10}. \quad =0.942$$

Practice:

▶ **Example 4** Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.

Example:(washer method)

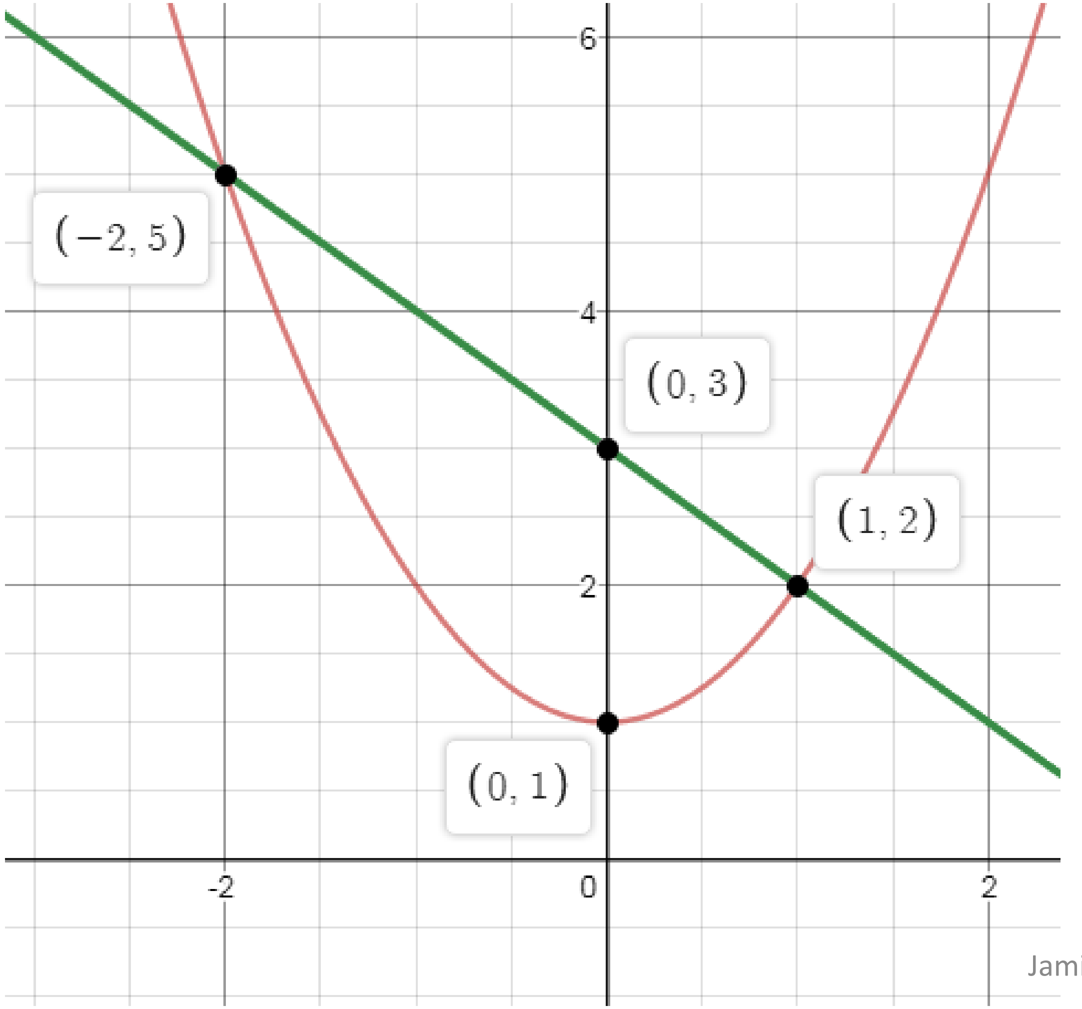
Self Practice

EXAMPLE 9 A Washer Cross-Section (Rotation About the *x*-Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution Ex-9

1. Draw the region



2. Find the outer and inner radii of the washer

Outer radius: $R(x) = -x + 3$

Inner radius: $r(x) = x^2 + 1$

3. Find the limits of integration

$$x^2 + 1 = -x + 3$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2, \quad x = 1$$

4-Established the formula and evaluate

$$V = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx$$

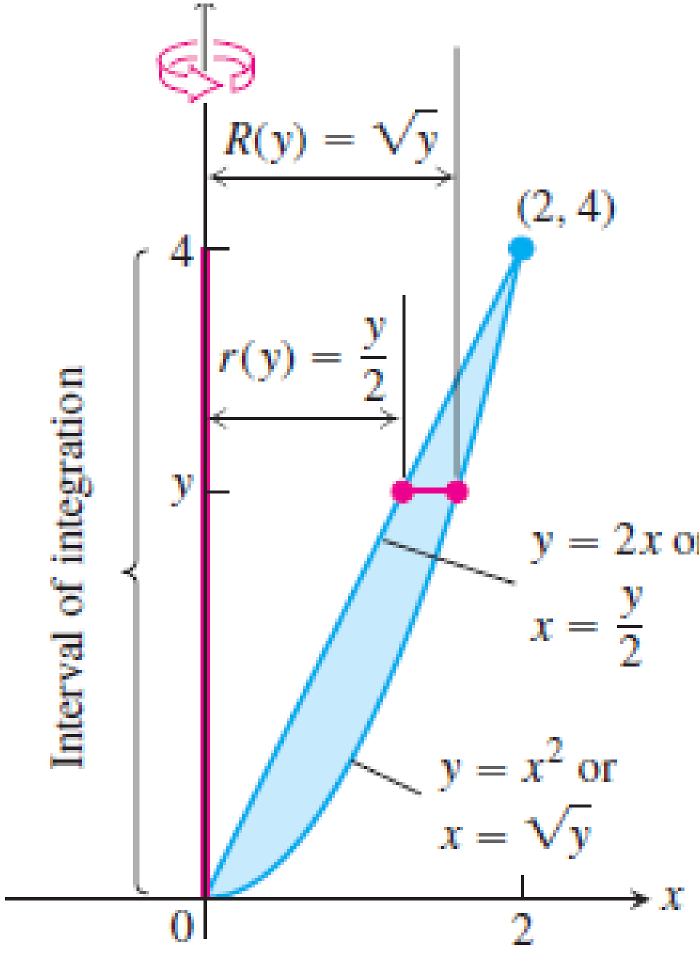
$$= \int_{-2}^{1} \pi((-x + 3)^2 - (x^2 + 1)^2)\,dx$$

$$= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\,dx$$

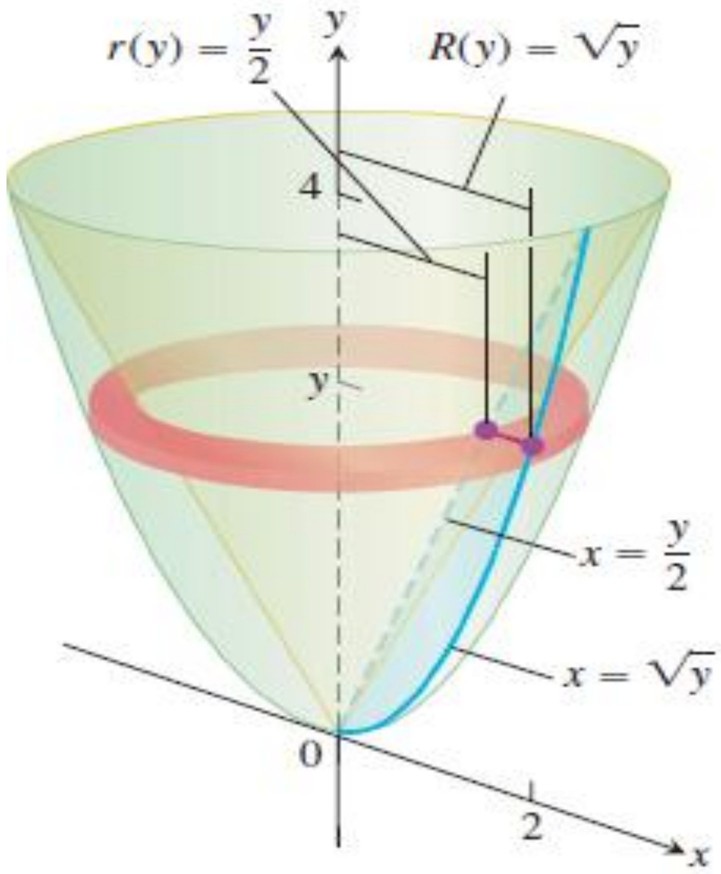
$$= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5} \quad =73.51$$

EXAMPLE 10 A Washer Cross-Section (Rotation About the y-Axis)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.



(a)

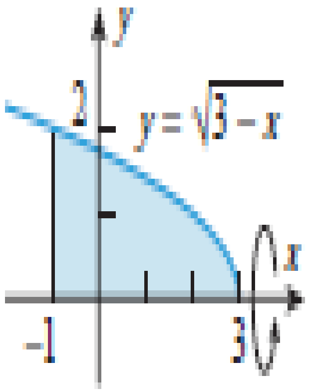


(b)

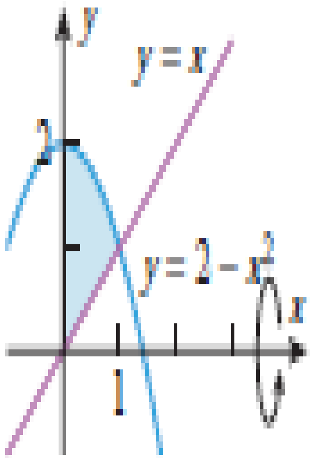
EXERCISE SET 6.2 $\boxed{C}$ CAS

1–8 Find the volume of the solid that results when the shaded region is revolved about the indicated axis.

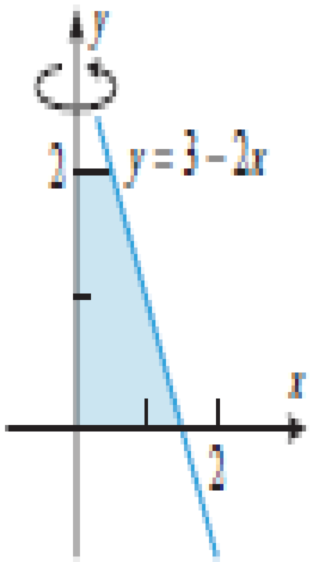
1.



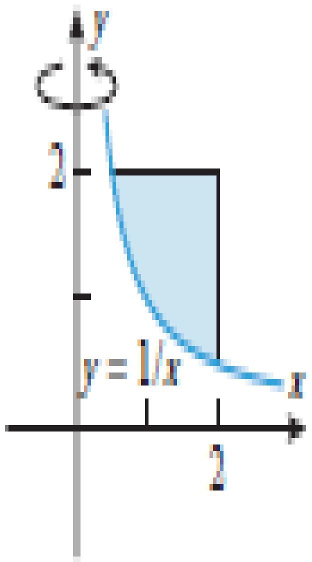
2.



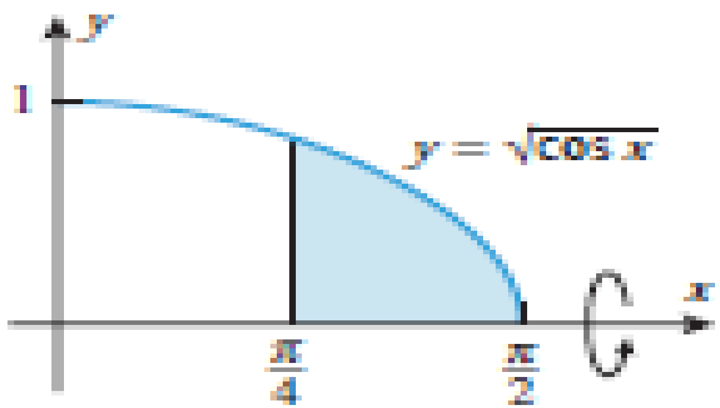
3.



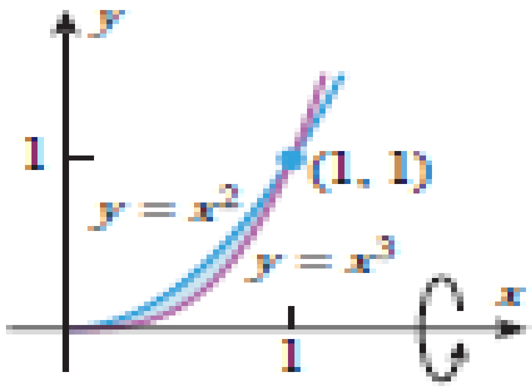
4.



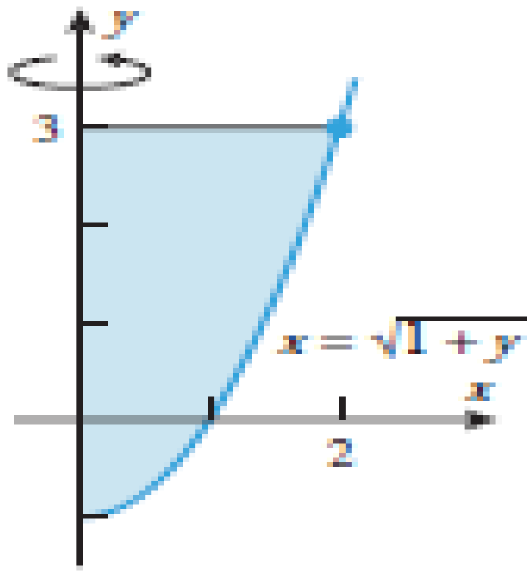
5.


y
1
$y = \sqrt{\cos x}$
x
$\frac{\pi}{4}$
$\frac{\pi}{2}$

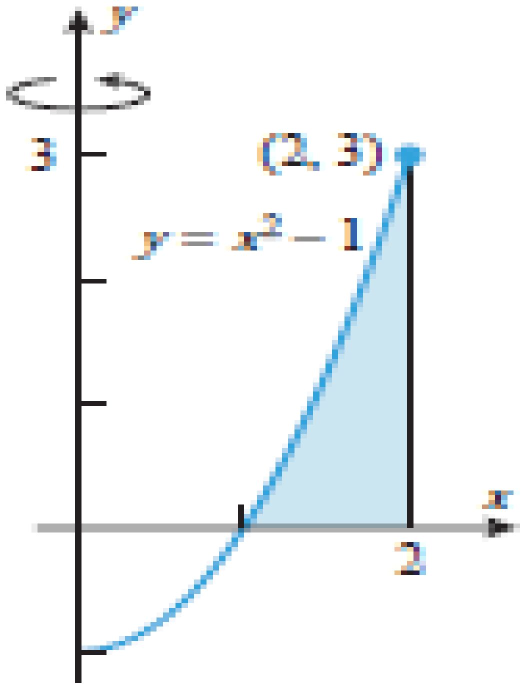
6.


y
1
(1, 1)
$y = x^2$
$y = x^3$
x
1

7.


y
3
$x = \sqrt{1 + y}$
x
2

8.


y
3
(2, 3)
$y = x^2 - 1$
x
2

11–18 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

11. $y = \sqrt{25 - x^2}, \ \ y = 3$

12. $y = 9 - x^2, \ \ y = 0$

13. $x = \sqrt{y}, \ \ x = y/4$

14. $y = \sin x, \ \ y = \cos x, \ \ x = 0, \ \ x = \pi/4$
[*Hint:* Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.]

15. $y = e^x, \ \ y = 0, \ \ x = 0, \ \ x = \ln 3$

16. $y = e^{-2x}, \ \ y = 0, \ \ x = 0, \ \ x = 1$

17. $y = \dfrac{1}{\sqrt{4 + x^2}}, \ \ x = -2, \ \ x = 2, \ \ y = 0$

18. $y = \dfrac{e^{3x}}{\sqrt{1 + e^{6x}}}, \ \ x = 0, \ \ x = 1, \ \ y = 0$

21–26 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the y-axis.

21. $x = \csc y,\ \ y = \pi/4,\ \ y = 3\pi/4,\ \ x = 0$

22. $y = x^2,\ \ x = y^2$

23. $x = y^2,\ \ x = y + 2$

24. $x = 1 - y^2,\ \ x = 2 + y^2,\ \ y = -1,\ \ y = 1$

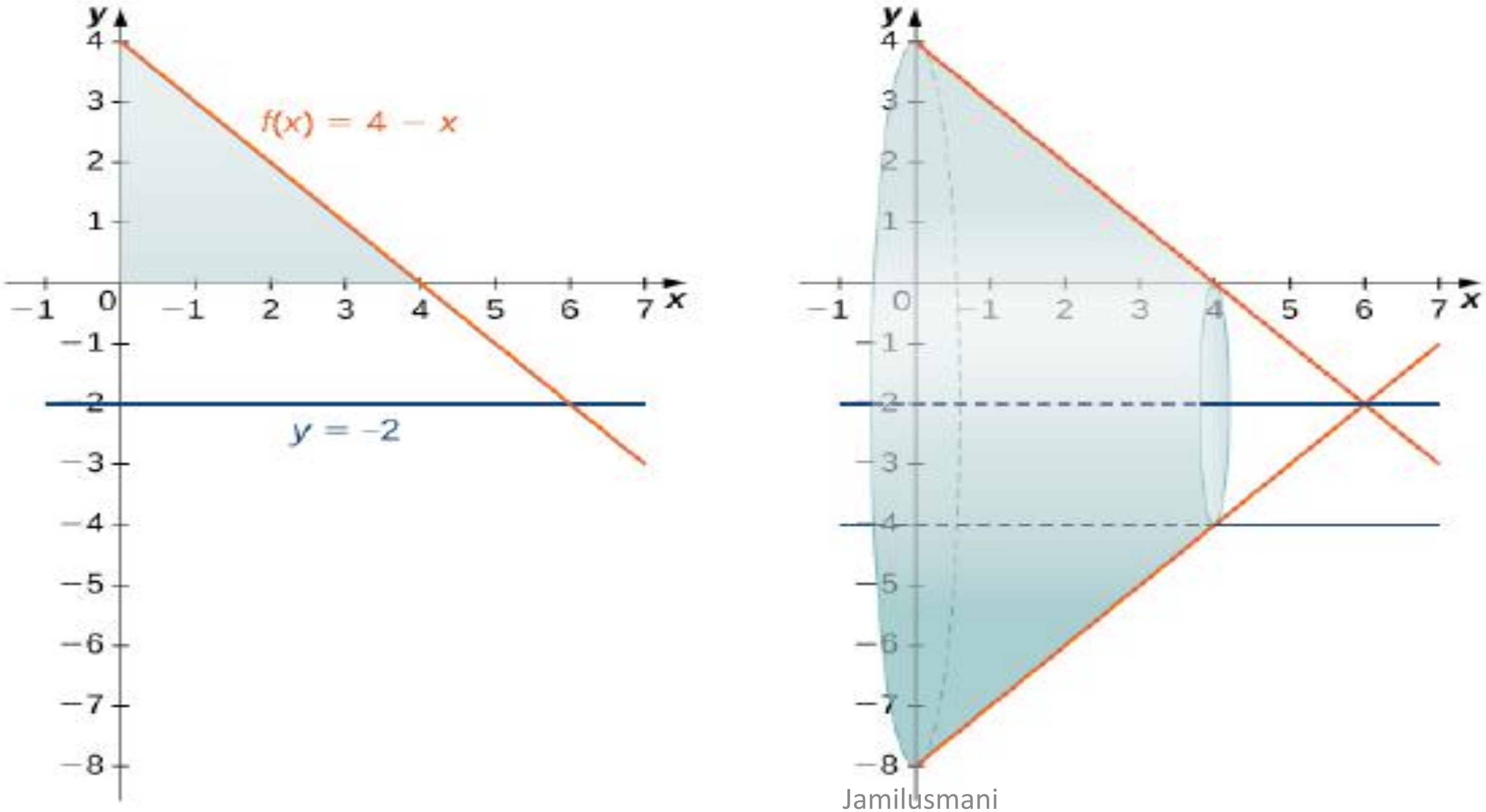
25. $y = \ln x,\ \ x = 0,\ \ y = 0,\ \ y = 1$

The Washer Method with a Different Axis of Revolution

Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the x-axis over the interval $[0, 4]$ around the line $y = -2$.

Solution

The graph of the region and the solid of revolution are shown in the following figure.



We can't apply the volume formula to this problem directly because the axis of revolution is not one of the

coordinate axes. However, we still know that the area of the cross-section is the area of the outer circle less the area of the inner circle. Looking at the graph of the function, we see the radius of the outer circle is given by $f(x)+2$, which simplifies to

$$f(x)+2=(4-x)+2=6-x.$$

The radius of the inner circle is $g(x)=2$. Therefore, we have

$$\begin{aligned} V &= \int_0^4 \pi\left[(6-x)^2-(2)^2\right]dx \\ &= \pi\int_0^4 \left(x^2-12x+32\right)dx = \pi\left[\frac{x^3}{3}-6x^2+32x\right]\Big|_0^4 = \frac{160\pi}{3}\,\text{units}^3. \end{aligned}$$