Random Number Generator

Pseudo Random Numbers:-

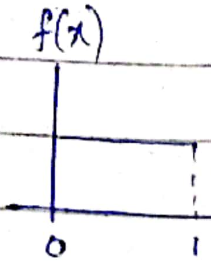
1. Long Cycle
2. Fast
3. Replicable
4. Maximum Density
5. Uniformity
6. Independence

Two Statistical Properties:-

1. Uniform
2. Independent

1. ⇒ Random Number R_i must be independently drawn from a uniform distribution with pdf.

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & e.w \end{cases}$$



$$F(x) = \int_0^1 x \, dx = \frac{1}{2}$$

Generators:-

① Linear Congruential Generators

② Combined Congruential Generators

① ⇒ $X_{i+1} = (a X_i + c) \bmod m$ ← modulus

$i = 0, 1, 2, \ldots\ldots m-1$

$a \rightarrow$ multiplier $\quad c \rightarrow$ increment $\quad X_0 =$ seed value

$m > 0 \quad a < m \quad c < m \quad X_0 < m$

• Convert the integer X_i to random Numbers

$$R_i = \frac{X_i}{m} \qquad i = 1, 2, \ldots\ldots$$

$X_i \in \{0, 1, \ldots, m-1\}$

$R_i \in [0, (m-1)/m]$

Use $X_0 = 27$ $a = 17$ $c = 43$ & $m = 100$

Find X_i & R_i

$X_{i+1} = (a X_i + c) \bmod m$

$X_1 = (17 X_0 + 43) \bmod 100$

$= (17(27) + 43) \bmod 100 = 502 \bmod 100 = 2$

$X_2 = (17(2) + 43) \bmod 100 = 77$

$$R_1 = \frac{X_1}{100} = \frac{2}{100} = 0.02$$

$$R_2 = \frac{X_2}{100} = \frac{77}{100} = 0.77$$

- Use $a = 13$, $c = 0$ $m = 64$

① $X_0 = 1$ $X_0 = 2$ $X_0 = 3$ $X_0 = 4$.

ⓐ $X_{i+1} = (a X_i + c) \bmod m$

$X_1 = (13(1) + 0) \bmod 64$

$= 13 \bmod 64 = 13$

$X_2 = ((13)(13) + 0) \bmod 64$

$X_2 = 169 \bmod 64$

$\boxed{C=0}$ multiplicative congruential generator

Changes in a, c, m create cycle.

$m = 2^{31} - 1$, $m = 2^{48}$ long cycles

Combined Linear Congruential Generator

Random No. ~~Variable~~ $x_{1}, x_{2} \leftarrow$ generator

- Let $X_{i,1}, X_{i,2} \ldots\ldots X_{i,k}$ be the i^{th} output from k different multiplicative congruential generators

* The i^{th} generator X_j

$$X_{i+1,j} = (a_j X_i + C_j) \bmod m_j$$

$$X_i = \left(\sum_{j=1}^{k} (-1)^{j-1} X_{i,j}\right) \bmod m_1 - 1$$

k = no. of generators

m_1 = modulus of first generator

for k = 2

$$X_i = (-1) * X_{i,1} + (-1)^{2-1} X_{i,2}$$

$$R_i = \begin{cases} \dfrac{X_i}{m_i} & X_i > 0 \\ \dfrac{m_1 - 1}{m_1} & X_i = 0 \end{cases}$$

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Use Combined Congruential Generators to generate Random Integer

$a_1 = 2 \quad m_1 = 10 \quad X_{0,1} = 5$

$a_2 = 3 \quad m_2 = 2^5 \quad X_{0,2} = 6$

$X_{i+1,1} = a_1 X_{0,1} \bmod m_1$

$X_{1,1} = (2)(5) \bmod 10 = 0$

$X_{1,2} = a_2 X_{0,2} \bmod m_2$

$= (3)(6) \bmod 2^5$

$X_{1,2} = 18$

Combined Generator:-

$X_1 = (-1)^0 (0) + (-1)^{2-1} (18)$

$= -18$

Find number that is divisible by 6

Negative Mod: $-13 \bmod 6$

$-13 - (-18) = 5$

Find one R_i by Combined Congruential Method

$m_1 = 483563$ $a_1 = 147$ $X_{0,1} = 798$

$m_2 = 483399$ $a_2 = 92$ $X_{0,2} = 96$

$$X_i = \sum_{j=1}^{k} (-1)^{j-1} X_{i,j} \bmod m-1$$

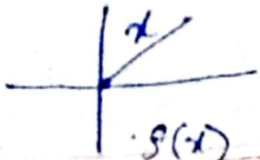
$X_{1,1} = 147 \times 798 \bmod 483563 = 117306$

$X_{1,2} = 92 \times 96 \bmod 483399 = 8832$

$X_1 = (117306 - 8832) \bmod (483563 - 1)$

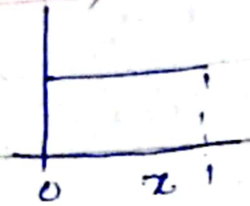
$X_1 = 108474$

cdf:



↓

Uniform Dist:



$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & ew \end{cases}$$

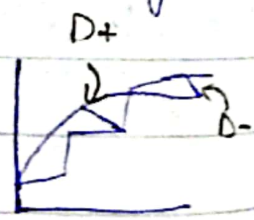
Random Variable Quantity is Same.

Sample Size = 1 KS Test else ~~KS~~ Chi Square Test
we compare Cdf

Kolmogorov - Smirnov (KS) Test:-
Compare Cdf of Random Variable.
Check if random numbers produce uniform dist.

① ⇒ Rank the data from smallest to largest.

② ⇒ $D^+ = \max\limits_{1 \le i \le N} \left\{ \dfrac{i}{N} - R_i \right\}$



$D^- = \max\limits_{1 \le i \le N} \left\{ R_i - \dfrac{i-1}{N} \right\}$

③ ⇒ $D = \max(D^+, D^-)$

Degree of freedom: No. of Samples.

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(4) Get D_α for significance level of α

(5) If $D < D_\alpha$ accept otherwise reject H_0

Q. 0.05, 0.14, 0.44, 0.81, 0.93 ← Random No.

R_i	0.05	0.14	0.44	0.81	0.93	$\alpha = 0.05$
(1) i	1	2	3	4	5	
R_i i/N	0.2	0.4	0.6	0.8	1.0	
$(i/N - R_i)$	0.15	0.26	0.16	Negative	0.07	
$(R_i - (i-1)/N)$	0.05	Neg.	0.04	~~Neg~~ 0.21	0.13	

$D^+ = 0.26$

$D^- = 0.21$

$D = \max(D^+, D^-) = 0.26$

$D_{0.05} = 0.565$

$D_\alpha > D$ Result: Accept H_0 - Uniform Distribution

Degree of freedom:-

$(n-1)$

↓ no. of interval

$E_i = \frac{N}{n} \geq 5$

Chi Square Test :-

N = 100 (no. of Random No.) n = 10 (interval) $E_i = 100/10 = 10$

Interval		E_i	O_i	$O_i - E_i$	$(O_i - E_i)^2 / E_i$
1	0 — 0.1	10	8	−2	0.4
2	0.1 — 0.2	10	8	−2	0.4
3	0.2 — 0.3	10	10	0	0
4	0.3 — 0.4	10	9	−1	0.1
5	0.4 — 0.5	10	12	2	0.4
6	0.5 — 0.6	10	8	−2	0.4
7	0.6 — 0.7	10	10	0	0
8	0.7 — 0.8	10	14	4	1.6
9	0.8 — 0.9	10	10	0	0
10	0.9 — 1.0	10	11	1	0.1
					3.4

less than equal to 0.1

Chi Square formula :-

$$X_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

E_i = Expected ≥ 5

O_i = Original

For this test degree of freedom = 9

$\alpha = 0.05$

from Table: $X_{0.05, 9} = 16.9$

Test statistics $X_0^2 < X_{0.05,9} \Rightarrow$ Accept Ho (follow uniform distribution)

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Validation, Verification, Credibility & accreditation of Simulation Models:-

Verification: It is concerned with determining whether the assumptions documents is correctly translated in to a computer program.

Validation:- It is the process of determining whether the simulation model is accurate with the presentation for the particular object of study.

Credibility: Acceptence of model by Institute

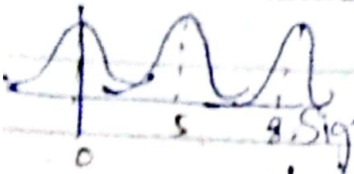
Accreditation: Models associated with authorize body.

Techniques:-

Verification: (1) Technique → Step by step debugging (2) Multiple Reviewer (3) Trace (Manual Test) (4) Different Parameter Setting (5) Animation (6) Verifying Given data from its output. (7) Use parameter from historic data (8) ... software

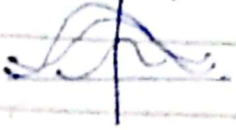
Parameter for histogram: location, shape & scale parameter.

Location Parameter: Define μ value. Change of x-axis

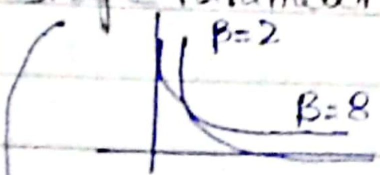


scale

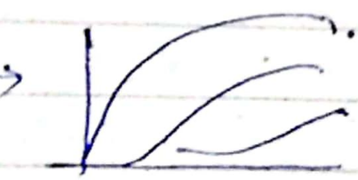
Sigma value change expand/contract

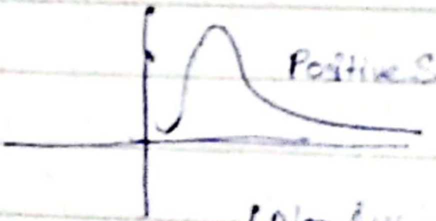


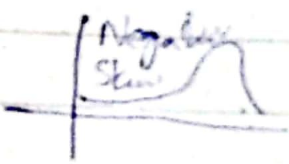
Shape Parameter: Change Skewness. β→ Scaling Parameter



Scaling







Normal Dist: Location, Scaling

Exponential Dist: Scaling

Gamma Dist: Scaling & Shape

Look for discrete / continuous distribution of your system

(i) Scenario
(ii) Histogram & check shape
Find what your data follow dist.

Random Variables: Random Number then convert into Uniform Dist.

Maximum Likelyhood Estimation: Used ~~to~~ to find parameter of dist.

Gamma Dist: Gamma function: $\Gamma\alpha$

$$f(x) = \begin{cases} \dfrac{\beta^{-\alpha} x^{\alpha-1} e^{-x/\beta}}{\Gamma\alpha} & x > 0 \\ 0 & \text{o.w} \end{cases}$$

Exponential Dist is special form of Gama dist
α & β are Parameter. MLE is used to find / estimate their value.
$\alpha \rightarrow$ shape Parameter $\beta \rightarrow$ Scale Parameter

Generate $[0 - \infty)$ Random No. Gama Dist.
Can't handle -ve scale of data

Exponential Dist: $f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x \geq 0 \end{cases}$

MLE = $\hat{\beta} = \bar{X}(n)$

12

5 0 7

— Continuous
Discrete

Normal Dist: <u>Note</u>: If data is -ve, we apply normal / poission dist.

Distribution: No closed form (No boundary)

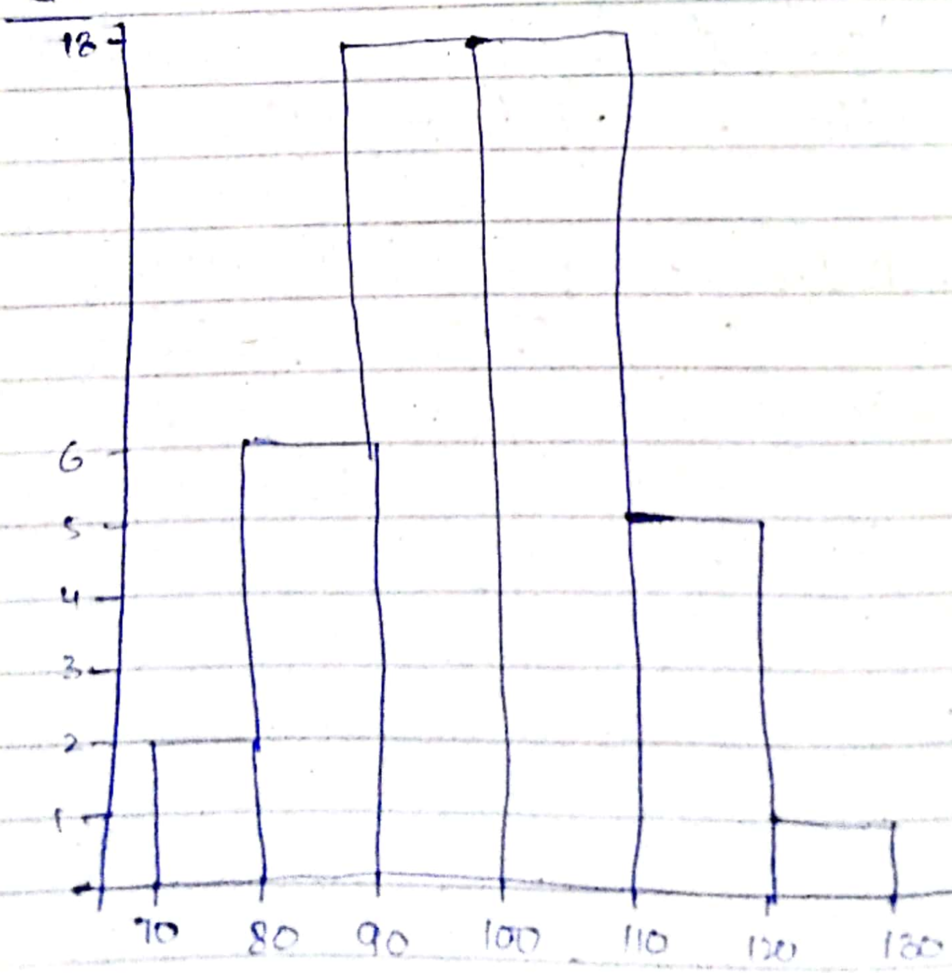
($\mu, \pm 3\sigma$)

Poission: Discrete Values. <u>MLE</u>: $\lambda = \bar{x}(n)$

No Parameter

$$p(x) = \begin{cases} \dfrac{e^{-\lambda} \lambda^{x}}{x!} & \lambda > 0 \\ 0 \end{cases}$$

Q.



Suggested MLE (Estimators)

Uniform Dist: $a, b \Rightarrow \hat{a} =$ min of data $\hat{b} =$ max of

Exponential Dist: $\hat{\lambda} = \frac{1}{\bar{x}}$

Gamma: $\cancel{(\alpha, \beta)}$ (β, α)

$\hat{\cancel{\beta}}^{\alpha} = \frac{1}{M}$ ← Table

$$M = \ln \bar{X} - \frac{1}{n} \sum_{i=1}^{n} \ln X_i$$

$$\hat{\beta} = \frac{1}{X}$$

Normal Dist: $\hat{\mu} = \bar{X}$ $\quad \sigma^2 = S^2$

$$S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}$$

Poission Dist: $\hat{\lambda} = \bar{X}$

Q. 99.79 99.56 100.17 100.33

99.79	99.56	100.17	100.33
100.26	100.41	99.98	99.83
100.23	100.27	100.02	100.47
99.55	99.62	99.65	99.82
99.96	99.90	100.06	99.85

It follows Normal Dist. Find Parameters.

$\hat{\mu} = \bar{X} = 1999.73 \div 20 = 99.9865$

$\sigma^2 = S^2 = \cancel{10418}\ 0.078$

$$S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}$$

$= \cancel{10418}\ 0.078$

Q.

70.292	25.292	30.215	16.314
10.107	14.413	17.137	28.073
48.386	39.166	44.024	39.019
20.480	17.421	10.552	32.330
13.053	13.905	37.298	36.547

It follows Gamma Dist. Find Parameters.

$\hat{\beta} = \frac{1}{\bar{X}} = \cancel{28.22}\ 0.035$

$\hat{\alpha} = 1/M = \cancel{7.142}\ 3.728$

$M = \ln \bar{X} - \frac{1}{n} \sum_{i=1}^{n} \ln X_i = 0.014$

$1/M = 7.142$ watch in table

Recommadation for numbers of class intervals

n	K (class Intervals)
20	do not use chi sequare
50	5 to 10
100	10 to 20
>100	$\sqrt{n}$ to $n/5$

degree of freedom = $K - S - 1$ $S \Rightarrow$ no. of parameter

K = no. of interval

Chi Sequare for Poission Dist:- $P(x) = \sum_0 \frac{e^{-\lambda}\lambda^x}{x!}$

≥ 5 $\frac{e^{-3.64}\lambda^0}{x!} = 0.026$

x_i	O_i	Total Sample points $E_i = nP(x)$	
0	12	$100 \times 0.026 = 2.6$ — not acceptable <5	$12+10 = 22$; $2.6 + 9.6 = 12.2$; $\frac{(22-12.2)^2}{12.2} = 7.87$
1	10	$P(1)$: $100 \times 0.096 = 9.6$	
2	19	$100 \times 0.174 = 17.4$	
3	17	$100 \times 0.211 = 21.1$	
4	10	$100 \times 0.192 = 19.2$	
5	8	$100 \times 0.140 = 14$	χ_0^2 27.68
6	7	$100 \times 0.085 = 8.5$	
7	5	$100 \times 0.044 = 4.4$	$<5 \Rightarrow 6$; 7.6
8	5	$100 \times 0.020 = 2.0$	
9	3	$100 \times 0.008 = 0.8$ +	can't make ≥ 5 we add previous with them
10	3	$100 \times 0.003 = 0.3$	
≥ 11	1	$100 \times 0.001 = 0.1$	add all the prob above it and then sub with 1.
	100		

$5 + 5 + 3 + 3 + 1 = 17$

last value 17

Poission = $\lambda = \bar{x} = 3.64$

$$\chi_0^2 = \sum_{i=1} \frac{(O_i - E_i)^2}{E_i}$$

degree of freedom $= 7 - 1 - 1 \Rightarrow 5$

$\chi^2_{0.05,5} = 11.1$

$\boxed{\text{Rejected}} \rightarrow \chi_0^2 > \chi_{0.05,5}$