Chapter 5: Relation

5.1 Relations and Their Properties

5.2 n-ary Relations and Their Applications

5.3 Representing Relations

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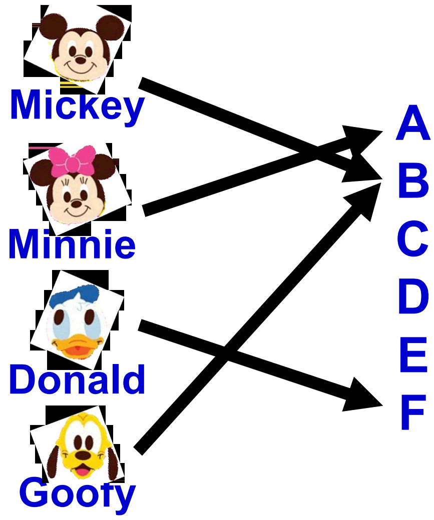
Agenda

- What is Relation?
- Representation of Relation
  - Graph
  - Matrix
- Operators of Relation
- Properties of Relation

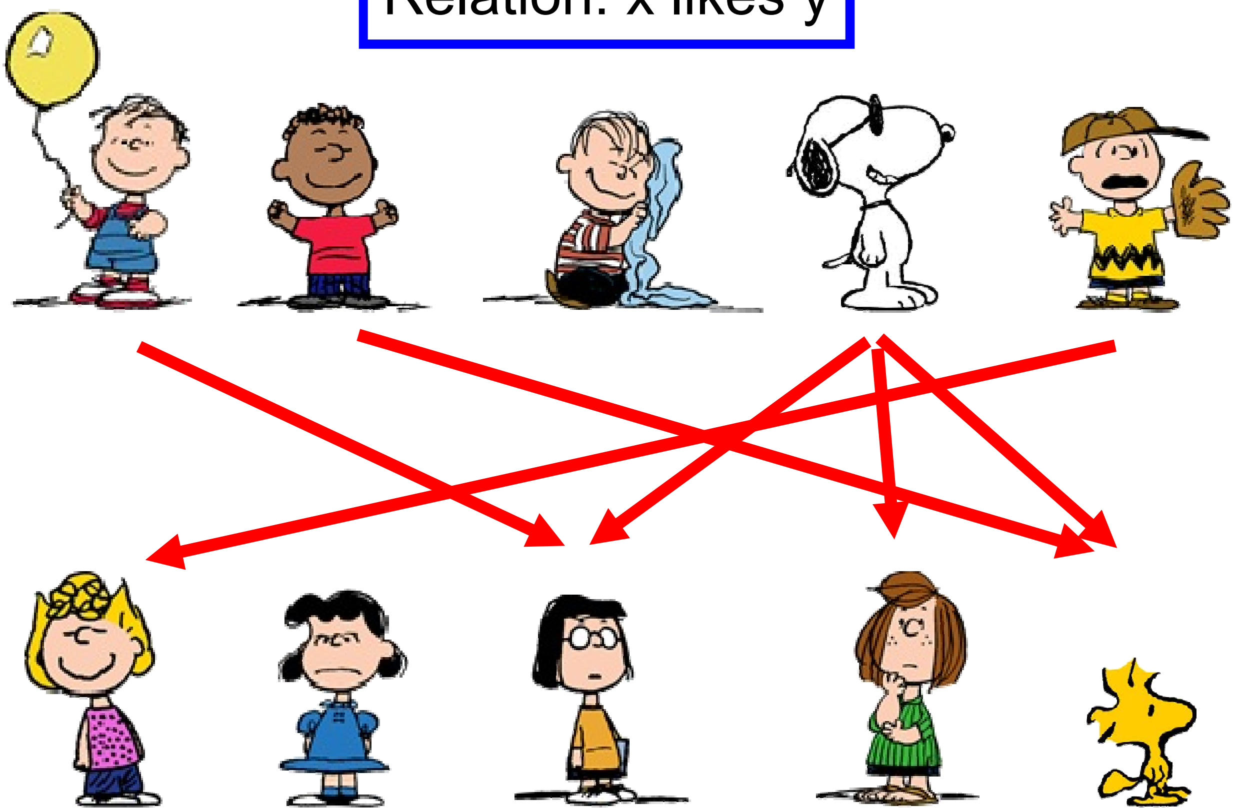
Recall, Function is...

- Let **A** and **B** be nonempty sets
  Function *f* from **A** to **B** is an assignment of
  exactly one element of **B** to each element of **A**
- By **defining** using a **relation**, a **function** from A to B contains **unique** ordered pair (a, b) for **every** element a $\in$ A



What is Relation?

Relation: x likes y



Relation

- Let A and B be sets
  A binary relation from A to B is a subset of A x B

- Recall, for example:
  - $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$
  - $A \times B = \{ (a_1, b_1), (a_1, b_2), (a_1, b_3),$
    $(a_2, b_2), (a_1, b_2), (a_1, b_3)\}$

Relation

- **R** is defined as
  - A binary relation from A to B
  - Ordered pairs, which
    - First element comes from A
    - Second element comes from B
- **aRb**: $(a, b) \in R$
- **a$\cancel{R}$b**: $(a, b) \notin R$
- Moreover, when (a, b) belongs to R,
  a is said to be related to b by R

Relation: Example

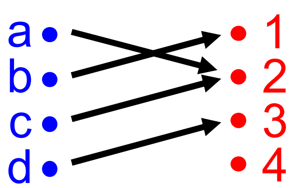
- S = {Peter, Paul, Mary}
- C = {C++, DisMath}

- Given
  - Peter takes C++ — Peter R C++ — Peter ~~R~~ DisMath
  - Paul takes DisMath — Paul ~~R~~ C++ — Paul R DisMath
  - Mary takes none of them — Mary ~~R~~ C++ — Mary ~~R~~ DisMath

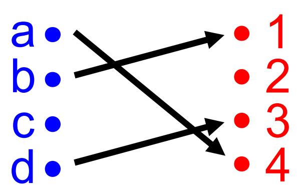
- R = {(Peter, C++), (Paul, DisMath)}
- (S x C) – R = ~~R~~

Relation VS Function

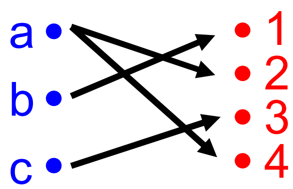
- **Function**
  from a set A to a set B
  - All elements of A are assigned to B
  - Exactly one element of B to each element of A
- **Relation**
  from a set A to a set B
  - Some elements of A are assigned to B
  - Zero, One or more elements of B to an element of A
- Function is a **special case** of Relation



Function
Relation



Not a Function
Relation

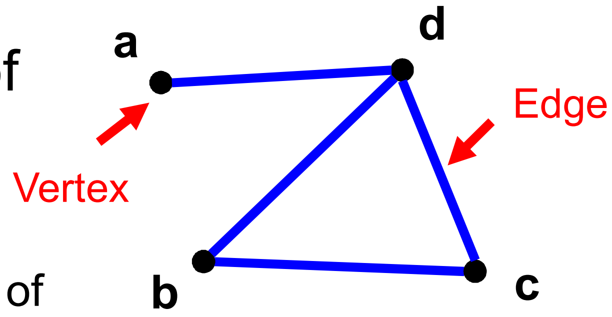


Not a Function
Relation

Relation Representation
Graph

- Relations can be represented by **Directed Graph**
  - You will learn the directed graph in detail in <Discrete Math Part 2>

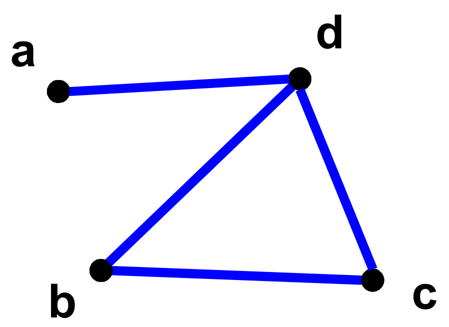
- Graph **G** = (**V**, **E**) consists of
  - a set of vertices **V**
  - a set of edges **E**,
    - a connection between a pair of vertices

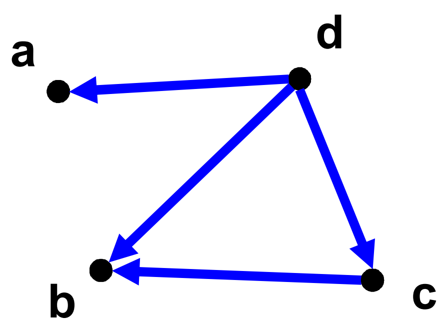


V = { a, b, c, d}

E = { (a,b), (b,c), (b,d), (c,d) }

Relation Representation
Graph



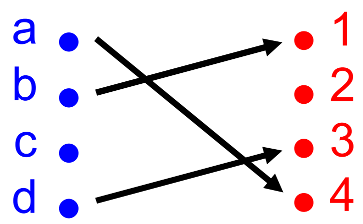


- **Undirected Graph**
  - Edges are not directed
  - E.g. (a, d) = (d, a)

- **Directed Graph** (**Digraph**)
  - Edges are directed
  - E.g. (a, d) ≠ (d, a)

Relation Representation
Graph

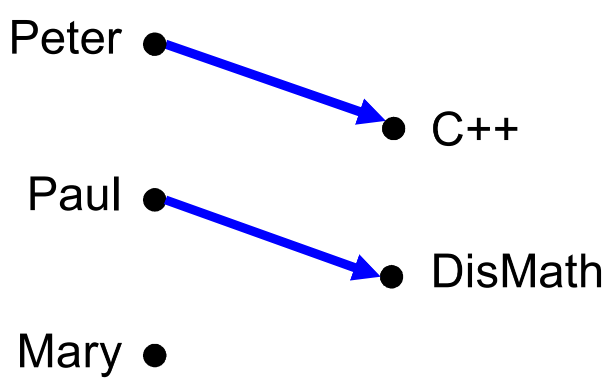
- **G** to present a relation from A to B is
  - vertices $V \subseteq A \cup B$
  - edges $E \subseteq A \times B$



- For example
  - If there is an ordered pair (x, y) in R, then there is an edge from x to y in D

Relation Representation
Graph: Example

- Peter R C++, Peter $\not R$ DisMath
  Paul $\not R$ C++, Paul R DisMath
  Mary $\not R$ C++, Mary $\not R$ DisMath

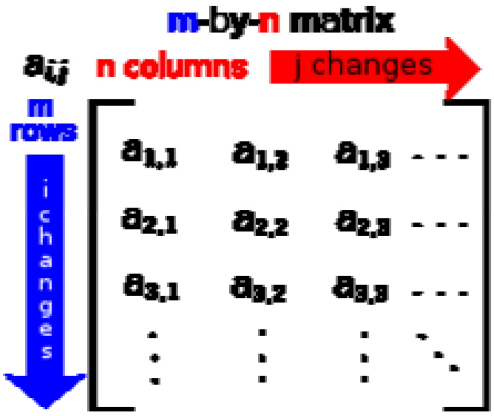


Directed Graph

Relation Representation
Matrix

- Let **R** be a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$
- An $m \times n$ connection **matrix M** for R is defined by

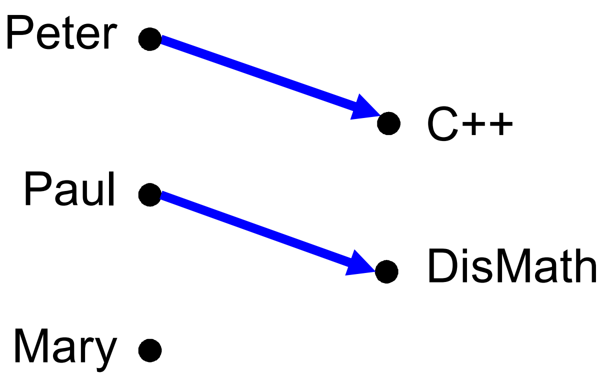
$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$$



$$\begin{array}{c|c} & \begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix} \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Relation Representation
Matrix: Example

- Peter R C++, Peter ~~R~~ DisMath
  Paul ~~R~~ C++, Paul R DisMath
  Mary ~~R~~ C++, Mary ~~R~~ DisMath



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Directed Graph **Matrix**

Relation on One Set

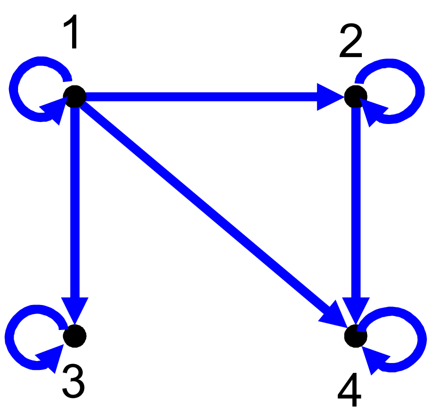
- Relation on the set A is a relation from A to A
  - Special case in relations

- Example:
  - A = {1, 2, 3, 4}
  - R = {(1,1), (1,4), (2,1), (2,3), (2,4), (3,1), (4,1), (4,2)}

Relation on One Set
Example 1

- Let A be the set {1, 2, 3, 4}, which ordered pairs are in the relation R = {(a, b) | a divides b}?

- **R** = { (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) }



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2

- How many different relations are there on a set with n elements?

- Suppose A has n elements
- Recall, a relation on a set A is a subset of A x A
- A x A has n^2 elements
- If a set has m element, its has 2^m subsets
- Therefore, the answer is 2^{n^2}

Relation Properties

- **Reflexive**

  $\forall a \, ((a, a) \in R)$

- **Irreflexive**

  $\forall a \, ((a \in A) \to ((a, a) \notin R))$

- **Transitive**

  $\forall a \forall b \forall c \, (((a,b) \in R \wedge (b,c) \in R) \to ((a,c) \in R))$

Relation on One Set

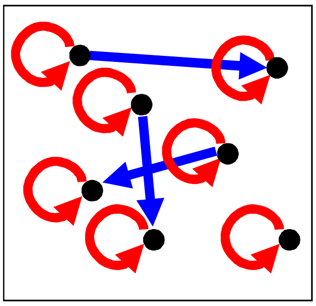
Relation Properties

- **Symmetric**
  $\forall a\ \forall b\ (\ ((a,\ b) \in R) \rightarrow ((b,\ a) \in R)\ )$
- **Asymmetric** ((a,a) cannot be an element in R)
  $\forall a\ \forall b\ (\ ((a,\ b) \in R) \rightarrow ((b,\ a) \notin R)\ )$
- **Antisymmetric** ((a,a) may be an element in R)
  $\forall a\ \forall b\ (\ ((a,\ b) \in R \wedge (b,\ a) \in R) \rightarrow (a = b)\ )$

- Asymmetry = Antisymmetry + Irreflexivity

Relation on One Set

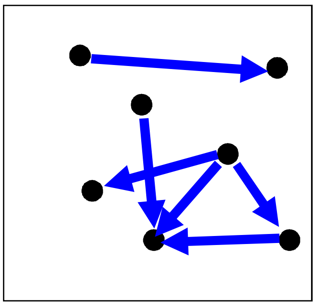
Relation Properties: Graph



Reflexive

$\forall a\ (\ (a,\ a) \in R\ )$

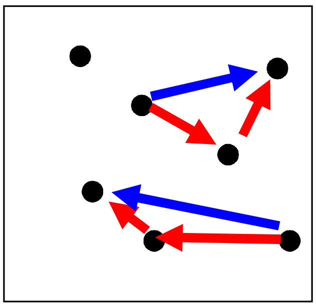
Every node has a self-loop



Irreflexive

$\forall a\ (\ (a \in A) \rightarrow ((a,\ a) \notin R)\ )$

No node links to itself

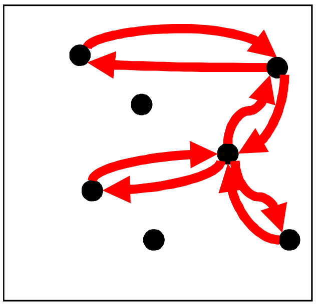


Transitive

$\forall a \forall b \forall c\ (\ ((a,b) \in R \wedge (b,c) \in R) \rightarrow ((a,c) \in R))$

Every two adjacent forms a triangle
(Not easy to observe in Graph)

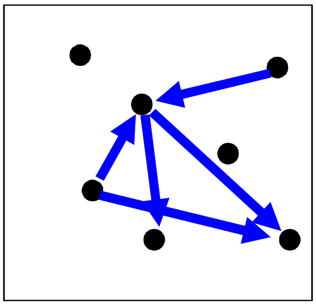
Relation Properties: Graph



Symmetric

∀a ∀b (((a, b)∈R) → ((b, a)∈R))

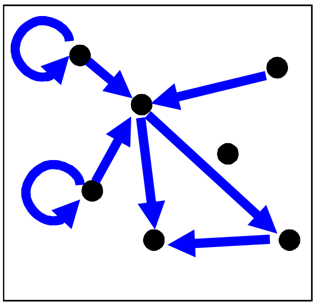
Every link is bidirectional



Asymmetric

∀a ∀b (((a, b)∈R) → ((b, a)∉R))

No link is bidirectional (Antisymmetric)
No node links to itself (Irreflexive)



Antisymmetric

∀a ∀b (((a, b)∈R ∧ (b, a)∈R) → (a = b))

No link is bidirectional

Relation Properties: Matrix

$$\begin{bmatrix} 1 & & ? & \\ & 1 & & \\ ? & & 1 & \\ & & & 1 \end{bmatrix}$$

Reflexive

∀a ((a, a) ∈ R)

All 1’s on diagonal

$$\begin{bmatrix} 0 & & ? & \\ & 0 & & \\ ? & & 0 & \\ & & & 0 \end{bmatrix}$$

Irreflexive

∀a ((a ∈ A) → ((a, a) ∉ R))

All 0’s on diagonal

?

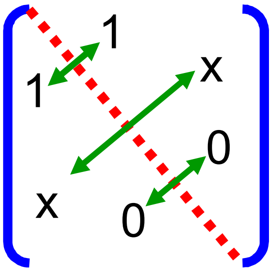
Transitive

∀a∀b∀c (((a,b)∈R∧(b,c)∈R) → ((a,c)∈R))

Not easy to observe in Matrix

Relation on One Set

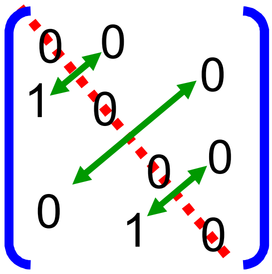
Relation Properties: Matrix



Symmetric

$\forall a\ \forall b\ (\ ((a, b) \in R) \rightarrow ((b, a) \in R)\ )$

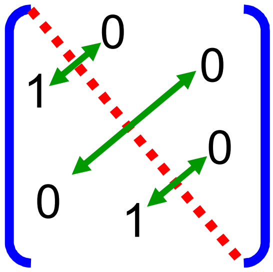
All identical across diagonal



Asymmetric

$\forall a\ \forall b\ (\ ((a, b) \in R) \rightarrow ((b, a) \notin R)\ )$

All 1's are across from 0's (Antisymmetric)
All 0's on diagonal (Irreflexive)



Antisymmetric

$\forall a\ \forall b\ (\ ((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b)\ )$

All 1's are across from 0's

Relation on One Set: Properties of Relation

Example 1

- Consider the following relations on {1, 2, 3, 4}, Which properties these relations have?
  - $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

~~Reflexive~~ ~~Irreflexive~~ ~~Transitive~~ ~~Symmetric~~ ~~Asymmetric~~ ~~Antisymmetric~~

  - $R_2 = \{(1,1), (1,2), (2,1)\}$

~~Reflexive~~ ~~Irreflexive~~ ~~Transitive~~ Symmetric ~~Asymmetric~~ ~~Antisymmetric~~

  - $R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

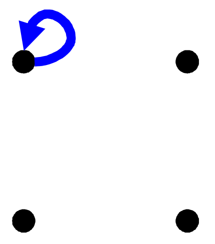
Reflexive ~~Irreflexive~~ ~~Transitive~~ Symmetric ~~Asymmetric~~ ~~Antisymmetric~~

  - $R_6 = \{(3,4)\}$

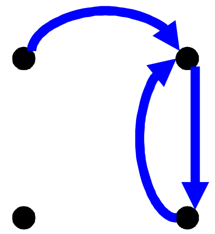
~~Reflexive~~ Irreflexive Transitive ~~Symmetric~~ Asymmetric Antisymmetric

Relation on One Set: Properties of Relation

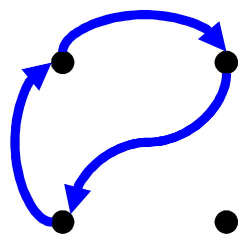
Example 2



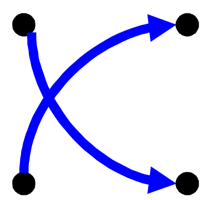
~~Reflexive~~
~~Irreflexive~~
Transitive
Symmetric
~~Asymmetric~~
Antisymmetric



~~Reflexive~~
Irreflexive
~~Transitive~~
~~Symmetric~~
~~Asymmetric~~
~~Antisymmetric~~



~~Reflexive~~
Irreflexive
~~Transitive~~
~~Symmetric~~
Asymmetric
Antisymmetric



~~Reflexive~~
Irreflexive
Transitive
~~Symmetric~~
Asymmetric
Antisymmetric

Relation on One Set: Properties of Relation

Example 3

- Let $A = Z^+$, $R = \{ (a,b) \in A \times A \mid a \text{ divides } b \}$
  Is R symmetric, asymmetric, or antisymmetric?

- **Symmetric** $(\forall a \forall b (((a, b) \in R) \rightarrow ((b, a) \in R)))$ ✗
  - If aRb, it does not follow that bRa
- **Asymmetric** $(\forall a \forall b (((a, b) \in R) \rightarrow ((b, a) \notin R)))$ ✗
  - If a=b, then aRb and bRa
- **Antisymmetric** $(\forall a \forall b (((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b)))$
  - If aRb and bRa, then a=b ✓

Combining Relations

- As R is a subsets of A x B, the set operations can be applied
  - Complement (‾)
  - Union (U)
  - Intersection (∩)
  - Difference (-)
  - Symmetric Complement (⊕)

Combining Relations

Example

- Given, A = {1,2,3}, B = {1,2,3,4}
- $R_1 = \{(1,1),(2,2),(3,3)\}$,
  $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$
- $R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$
- $R_1 \cap R_2 = \{(1,1)\}$
- $R_1 - R_2 = \{(2,2),(3,3)\}$
- $R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$
- $R_1 \oplus R_2 = \{(1,2),(1,3),(1,4),(2,2),(3,3)\}$

Combining Relations

- Let R be relation from a set A to a set B
- Inverse Relation ($\mathbf{R^{-1}}$) = $\{(b,a) \mid (a,b) \in R\}$
- Complementary Relation ($\overline{\mathbf{R}}$) = $\{(a,b) \mid (a,b) \notin R\}$
- Example
  - $X = \{a, b, c\}$ $Y=\{1, 2\}$
  - $R = \{(a, 1), (b, 2), (c, 1)\}$
  - $R^{-1} = \{(1, a), (2, b), (1, c)\}$
  - $E = X \times Y = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$
  - $\overline{R} = \{(a, 2), (b, 1), (c, 2)\} = E - R$

Combining Relations

Theorems

- Let R_1 and R_2 be relations from A to B. Then
  - $(R^{-1})^{-1} = R$
  - $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$
  - $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$
  - $(A \times B)^{-1} = B \times A$
  - $\varnothing^{-1} = \varnothing$
  - $(\overline{R})^{-1} = \overline{(R^{-1})}$
  - $(R_1 - R_2)^{-1} = R_1^{-1} - R_2^{-1}$
  - If $R_1 \subseteq R_2$ then $R_1^{-1} \subseteq R_2^{-1}$

Example for the Proof

- Proof $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$

Recall...
- $A \cup B = \{ x \mid x \in A \vee x \in B \}$
- $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

- Assume
  $(a,b) \in R_1$ & $(a,b) \in R_2$
- L.H.S.
  - $(R_1 \cup R_2) = \{(a,b) \mid (a,b) \in R_1 \vee (a,b) \in R_2 \}$
  - $(R_1 \cup R_2)^{-1} =$ **$\{(b,a) \mid (a,b) \in R_2 \vee (a,b) \in R2 \}$**
- R.H.S.
  - $R_1^{-1} = \{(b,a) \mid (a,b) \in R_1\}$
  - $R_2^{-1} = \{(b,a) \mid (a,b) \in R_2\}$
  - $R_1^{-1} \cup R_2^{-1} =$ **$\{(b,a) \mid (a,b) \in R_2 \vee (a,b) \in R2 \}$**

Example 1

- Given
  - R_1 is symmetric
  - R_2 is antisymmetric
- Does it $R_1 \cup R_2$ is transitive?
- Not transitive by giving a counterexample
  - $R_1 = \{(1,2),(2,1)\}$ which is symmetric
  - $R_2 = \{(1,2),(1,3)\}$ which is antisymmetric
  - $R_1 \cup R_2 = \{(1,2),(2,1),(1,3)\}$, not transitive

Example 2

- Given R_1 and R_2 are transitive on A
- Does R_1 U R_2 is transitive?

- Not transitive by giving a counterexample
  - A = {1, 2}
  - $R_1 = \{(1,2)\}$, which is transitive
  - $R_2 = \{(2,1)\}$, which is transitive
  - $R_1 \cup R_2 = \{(1,2), (2,1)\}$, not transitive

Combining Relations: Matrix

- Suppose that R_1 and R_2 are relations on a set A represented by the matrices M_{R_1} and M_{R_2}, respectively
- **Join operator (OR)**

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

- **Meet operator (AND)**

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Combining Relations: Matrix

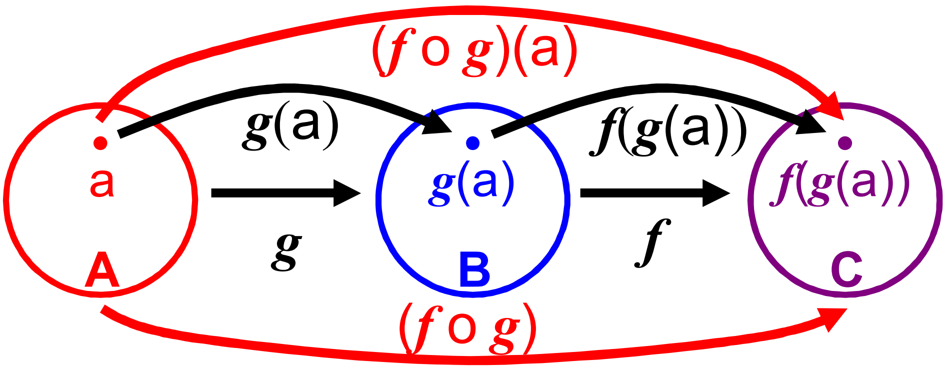
- Example

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Combining Relations
Composite

- Recall, the composition in functions…
- Let
  - g be a function from the set A to the set B
  - f be a function from the set B to the set C
- The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(\, g(a)\,)$

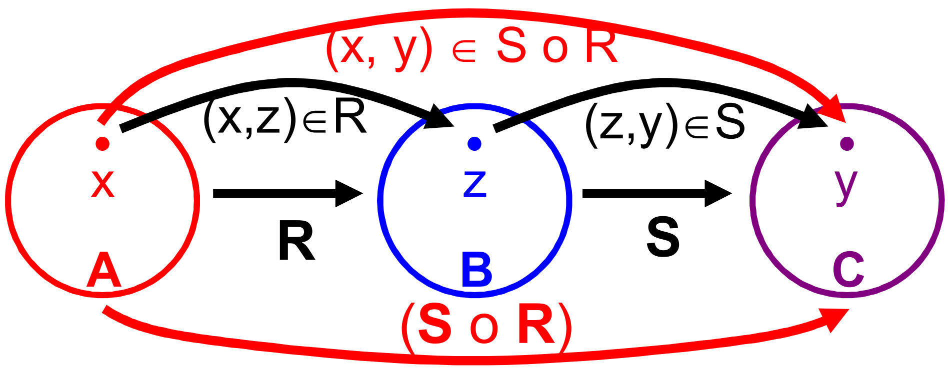


Composite

- Let
  - R be a relation from a set A to a set B
  - S be a relation from a set B to a set C
- The **composite** of R and S is the relation consisting of ordered pairs (a, c), where
  - $a \in A$, $c \in C$, and
  - There exists an element $b \in B$, such that $(a, b) \in R$ and $(b, c) \in S$
- The composite of R and S is denoted by **S o R**

Composite

- Suppose
  - R be a relation from a set A to a set B
  - S be a relation from a set B to a set C
- $\boxed{(x, y) \in S \circ R \text{ implies } \exists z\, ((x, z) \in R \land (z, y) \in S)}$

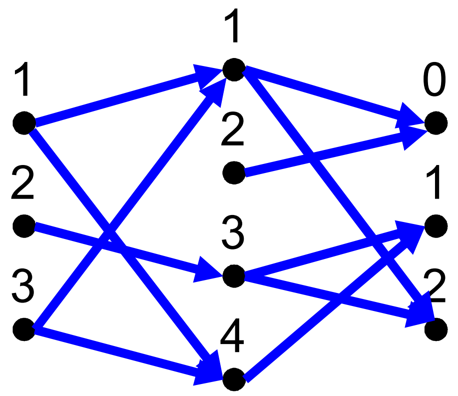


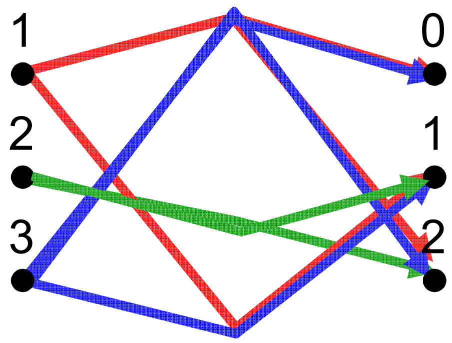
  - Remark: May be more than one element z, where $(x, z) \in R$ and $(z, y) \in S$

Composite: Example

- What is the composite of the relations R and S, where
  - R is the relation from **{1,2,3}** to **{1,2,3,4}** with $R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$
  - S is the relation from **{1,2,3,4}** to **{0,1,2}** with $S = \{(1,0),(1,2),(2,0),(3,1),(3,2),(4,1)\}$?

- $S \text{ o } R = \{(1,0),(1,2),(1,1),(2,1),(2,2),(3,0),(3,2),(3,1)\}$





Composite: Properties

- Let R_1 and R_2 be relations on the set A.
- Show $(R_1 \text{ o } R_2)^{-1} = R_2^{-1} \text{ o } R_1^{-1}$
- Proof:

Let $(x, y) \in (R_1 \text{ o } R_2)^{-1}$

(x, y) ∈ S o R implies
∃z ((x, z)∈R ∧ (z, y)∈S)

$(x, y) \in (R_1 \text{ o } R_2)^{-1}$
$\Leftrightarrow (y, x) \in R_1 \text{ o } R_2$
$\Leftrightarrow \exists z \, ((y, z) \in R_2 \wedge (z, x) \in R_1)$
$\Leftrightarrow \exists z \, ((z, y) \in R_2^{-1} \wedge (x, z) \in R_1^{-1})$
$\Leftrightarrow (x, y) \in R_2^{-1} \text{ o } R_1^{-1}$

Composite: Properties

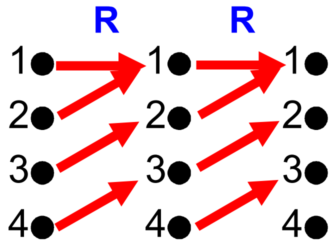
- Let F,G and H be relations on the set A, then
  - $F \circ (G \cup H) = (F \circ G) \cup (F \circ H)$
  - $F \circ (G \cap H) \subseteq (F \circ G) \cap (F \circ H)$
  - $(G \cup H) \circ F = (G \circ F) \cup (H \circ F)$
  - $(G \cap H) \circ F \subseteq (G \circ F) \cap (H \circ F)$

Composite

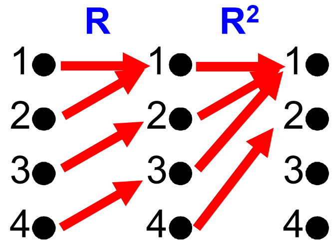
- Let R be a relation on the set A. The powers R^n , n = 1, 2, 3, ..., are defined recursively by
  - $R^1 = R$
  - $R^2 = R \circ R$
  - $R^3 = R^2 \circ R = (R \circ R) \circ R$

  ...
  - $R^{n+1} = R^n \circ R$

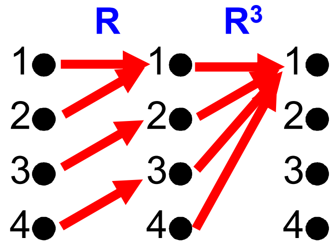
Composite: Example



- Let R = {(1,1), (2,1), (3,2), (4,3)}
- Find the powers R^n, n = 2,3,4,…



- R^2 = R o R = {(1,1),(2,1),(3,1),(4,2)}
- R^3 = R^2 o R = {(1,1),(2,1),(3,1),(4,1)}
- R^4 = R^3 o R = {(1,1),(2,1),(3,1),(4,1)}
- $R^n = R^3$ for n = 5, 6, 7, ….



Composite: Matrix

- Suppose
  - R_1 be relation from set A to set B represented by M_{R_1}
  - R_2 be relation from set B to set C represented by M_{R_2}
- The matrix for the composite of R_1 and R_2 is:

$$M_{R_2 \circ R_1}$$

  - Size of M_{R_1} and M_{R_2} is |A| x |B| and |B| x |C|
  - Size of $M_{R_2 \circ R_1}$ is |A| x |C|

Composite: Matrix

- Define:

n : the number of row of R_1
the number of column of R_2

$$M_{R_2 \circ R_1} = M_{R_2} \odot M_{R_1}$$

where $(M_{R_2} \odot M_{R_1})_{ij} = \bigvee_{k=1}^{n} [(M_{R_1})_{ik} \wedge (M_{R_2})_{kj}]$

An element in the matrix

- Such that
  $(M_{R_2} \odot M_{R_1})_{ij} = 1$
  if and only if
  $(M_{R_1})_{ik} = (M_{R_2})_{kj} = 1$ for some k

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & \cdots \\ x_{21} & & \vdots & \\ \vdots & \cdots & x_{ij} & \cdots \\ \vdots & & \vdots & \end{bmatrix}$$

Composite: Matrix: Example

n : the number of column of R_1
the number of row of R_2

$$M_R = \begin{array}{c} \overbrace{\hphantom{0\quad 1\quad 0\quad 0}}^{4} \\ \left.\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\} 3 \end{array}$$

3 x 4

$$M_S = \begin{array}{c} \overbrace{\hphantom{0\quad 0\quad 1\quad 0\quad 0}}^{5} \\ \left.\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\right\} 4 \end{array}$$

4 x 5

$$M_{SoR} = \begin{array}{c} \overbrace{\hphantom{0\quad 0\quad 0\quad 0\quad 1}}^{5} \\ \left.\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\right\} 3 \end{array}$$

$i = 1, j =$ [illegible]

$k =$ [illegible]

n=4

$$(M_{R_2} \odot M_{R_1})_{ij} = \bigvee_{k=1}^{n} [(M_{R_1})_{ik} \wedge (M_{R_2})_{kj}]$$

Composite: Matrix

- The powers R^n can defined using matrix as:

$$M_{R^n} = (M_R)^n$$

- Example
  - Find the matrix representing the relation R^2

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = (M_R)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Composite: Property 1

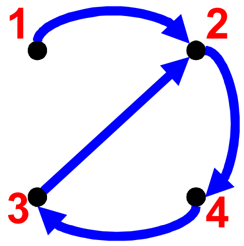
- **Theorem**
  If $R \subset S$, then $S \circ R \subset S \circ S$

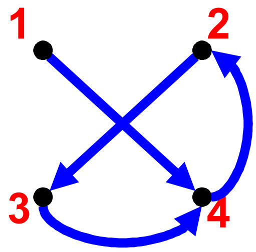
- Assume $(x,y) \in S \circ R$, there exists a element z, which $(x,z) \in R$ and $(z,y) \in S$
- As $R \subset S$ and $(x,z) \in R$, $(x,z) \in S$
- Therefore, as $(x,z) \in S$ and $(z,y) \in S$, $(x,y) \in S \circ S$
- $S \circ R \subset S \circ S$

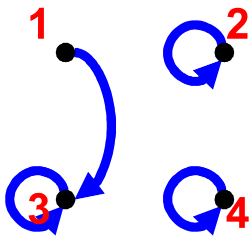
- It implies:
  If $R \subset S$ and $T \subset U$, then $R \circ T \subset S \circ U$

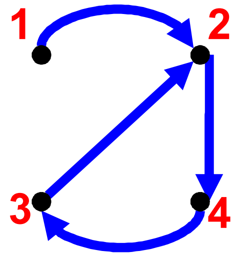
Combining Relations: Relation on One Set
Composite: Property 2

- An ordered pairs (x, y) is in R^n iff there is a path of length n from x to y in R







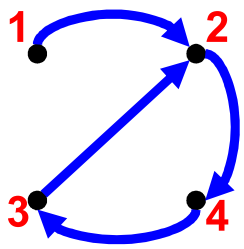


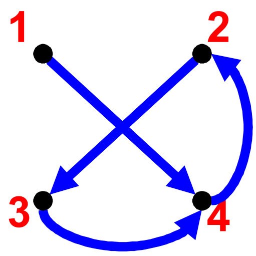
R $\qquad R^2$ $\qquad R^3$ $\qquad R^4$

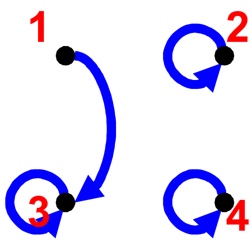
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

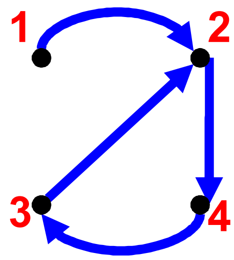
Combining Relations: Relation on One Set
Composite: Property 2

- An ordered pairs (x, y) is in R^n iff there is a path of length n from x to y in R









R $\qquad R^2$ $\qquad R^3$ $\qquad R^4$

- Example
  - In R, 1 > 2 > 4, length = 2 $\Leftrightarrow$ $(1,4) \in R^2$
  - In R, 3 > 2 > 4 > 3, length = 3 $\Leftrightarrow$ $(3,3) \in R^3$
  - $(1,2) \in R^4$ $\Leftrightarrow$ In R, 1 > 2 > 4 > 3 > 2, length = 4

Composite: Property 2

- **Theorem**

  Let R be a relation on A. There is a path of length n from a to b in R iff $(a, b) \in R^n$

Composite: Property 2

a path of length n from a to b iff $(a, b) \in R^n$

- Proof by Induction
  - **Show n=1 is true**
    - An arc from a to b is a path of length 1, which is in $R^1 = R$
    - Hence the assertion is true for n = 1
  - **Assume it is true for k. Show it is true for k+1**
    - As it is true for n = 1,
      suppose (a, x) is a path of length 1, then $(a, x) \in R$
    - As it is true for n = k,
      suppose (x, b) is a path of length k, then $(x, b) \in R^k$
    - Considering, $(a, x) \in R$ and $(x, b) \in R^k$,
      $(a, b) \in R^{k+1} = R^k \circ R$ as there exists an element x, such that $(a, x) \in R$ and $(x, b) \in R^k$
    - The length of (a,b) is k+1

Composite: Property 3

- R is transitive iff $R^n \subseteq R$ for n > 0.

- Proof

1. $(R^n \subseteq R) \rightarrow$ R is transitive

- Suppose $(a,b) \in R$ and $(b,c) \in R$
- (a,c) is an element of R^2 as $R^2 = R \circ R$
- As $R^2 \subseteq R$, $(a,c) \in R$
- Hence R is transitive

Composite: Property 3

2. R is transitive $\rightarrow (R^n \subseteq R)$

- Use a proof by induction:
  - **Basis:** Obviously true for n = 1.
  - **Induction:** Assume true for n, show it is true for n + 1
    - For any (x, y) is in R^{n+1}, there is a z such that $(x, z) \in R$ and $(z, y) \in R^n$
    - But since $R^n \subseteq R$, $(z, y) \in R$
    - As R is transitive, (x, z) and (z, y) are in R, so (x, y) is in R
    - Therefore, $R^{n+1} \subseteq R$

Composite: Property 4

- Proof: If R is transitive, R^n is also transitive

- When n = 1, R is transitive
- Assume R^k is transitive
- Show R^{k+1} is transitive

  Given $(a,b) \in R^{k+1}$ and $(b,c) \in R^{k+1}$, show $(a,c) \in R^{k+1}$
  - $R^{k+1} = R^k \circ R$
  - As $(a,b) \in R^{k+1}$, $(\mathbf{d},b) \in R^k$ and $(a,\mathbf{d}) \in R$
  - As $(b,c) \in R^{k+1}$, $(\mathbf{f},c) \in R^k$ and $(b,\mathbf{f}) \in R$
  - As $(a,c) \in R^{k+1}$, $(\mathbf{?},c) \in R^k$ and $(a,\mathbf{?}) \in R$

Composite: Property 4

- Given $(a,b) \in R^{k+1}$ and $(b,c) \in R^{k+1}$, show $(a,c) \in R^{k+1}$
  - $R^{k+1} = R^k \circ R$
  - As $(a,b) \in R^{k+1}$, $(\mathbf{d},b) \in R^k$ and $(a,\mathbf{d}) \in R$
  - As $(b,c) \in R^{k+1}$, $(\mathbf{f},c) \in R^k$ and $(b,\mathbf{f}) \in R$
  - As $(a,c) \in R^{k+1}$, $(\mathbf{?},c) \in R^k$ and $(a,\mathbf{?}) \in R$
- As “R is transitive iff $R^n \subseteq R$ for $n > 0$”
- $(d,b) \in R^k \subseteq R$
- As R is transitive, $(d,b) \in R$ and $(b,f) \in R$ imply $(d,f) \in R$
- As R is transitive, $(d,f) \in R$ and $(a,d) \in R$ imply $(a,f) \in R$
- Therefore, by considering, $(f,c) \in R^k$ and $(a,f) \in R$, $(a,c) \in R^{k+1}$

Composite: Property 4

- Proof: If R is transitive, R^n is also transitive
- When n = 1, R is transitive
- Assume R^k is transitive
- Show R^{k+1} is transitive

  Given $(a,b) \in R^{k+1}$ and $(b,c) \in R^{k+1}$, show $(a,c) \in R^{k+1}$
  - $R^{k+1} = R^k \circ R$
  - As $(a,b) \in R^{k+1}$, $(a,\mathbf{d}) \in R^k$ and $(\mathbf{d},b) \in R$
  - As $(b,c) \in R^{k+1}$, $(b,\mathbf{f}) \in R^k$ and $(\mathbf{f},c) \in R$
  - As $(a,c) \in R^{k+1}$, $(a,\mathbf{?}) \in R^k$ and $(\mathbf{?},c) \in R$

Composite: Property 4

- Given $(a,b) \in R^{k+1}$ and $(b,c) \in R^{k+1}$, show $(a,c) \in R^{k+1}$
  - $R^{k+1} = R^k \circ R$
  - As $(a,b) \in R^{k+1}$, $(a,\mathbf{d}) \in R^k$ and $(\mathbf{d},b) \in R$
  - As $(b,c) \in R^{k+1}$, $(b,\mathbf{f}) \in R^k$ and $(\mathbf{f},c) \in R$
  - As $(a,c) \in R^{k+1}$, $(a,\mathbf{?}) \in R^k$ and $(\mathbf{?},c) \in R$
- As "R is transitive iff $R^n \subseteq R$ for n > 0"
- $(b,f) \in R^k \subseteq R$
- As R is transitive, $(d,b) \in R$ and $(b,f) \in R$ imply $(d,f) \in R$
- As R is transitive, $(d,f) \in R$ and $(f,c) \in R$ imply $(d,c) \in R$
- Therefore, by considering, $(a,d) \in R^k$ and $(d,c) \in R$, $(a,c) \in R^{k+1}$

n-ary Relation

- Let A_1, A_2, ..., A_n be sets
  An **n-ary relation** on these sets is a subset of $A_1 \times A_2 \times \dots \times A_n$
- **Domains** of the relation:
  the sets A_1, A_2, ..., A_n
- **Degree** of the relation: n

n-ary Relation: Example

- Let R be the relation on $Z \times Z \times Z^+$ consisting of triples
  (a, b, m), where a, b, and m are integers with
  $m \geq 1$ and $a = b \pmod{m}$, (i.e. m divides a-b)

- Degree of the relation? **3**
- First domain is: **the set of all integers**
- Second domain is: **the set of all integers**
- Third domain: **the set of positive integers**
- Do they belong to R?
  - (8,2,3) **Y**
  - (7,2,3) **N**
  - (-1,9,5) **Y**
  - (-2,-8,5) **N**

Relational Database VS n-ary Relation

- A **database** consists of records made up of fields
- Each **record** is a n-tuple (n fields)
  - For example:

	ID num	Name	Major	GPA
▪	888323	Adams	Data Structure	85
▪	231455	Peter	C++	61

  - Domain: ID num, Name, Major, GPA
  - Relation: (888323, Adams, Data Structure, 85),
    (231455, Sam, C++, 61)
- Relations are displayed as tables

ID_number	Student_name	Major	Grade
888323	Adams	Data Structure	85
231455	Peter	C++	61
678543	Sam	Data Structure	98

Relational Database VS n-ary Relation

- n-ary relation can be:
  - Determining all n-tuples satisfy certain conditions
  - Joining the records in different tables

ID_number	Major	Grade
888323	Data Structure	85
231455	C++	61
678543	Data Structure	98
453876	Discrete Math	83

ID_number	Student_name
231455	Adams
888323	Peter
102147	Sam
453876	Goodfriend
678543	Rao
786576	Stevens