Quantum Information Analysis of Majorana Fermion Braiding

Exploring Realization and Error Assessment via a Series of Measurements

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Contents

Introduction

- In 1937, Majorana proposed the Majorana equation.
- This work explores Majorana fermion braiding using a sequence of measurements.
- Utilizes Jordan-Wigner transformation and qiskit package.

Braiding in Two Dimensions

- Two exchanges of particle positions mirror topological equivalence.
- Braiding encapsulates a historical record and alters wave functions.
- Introduction of braiding operator adds complexity to non-Abelian statistics.

Fault-Tolerant Quantum Computing

- Quantum systems are susceptible to errors and decoherence.
- Measurement-based quantum computers offer fault tolerance and scalability.
- Topological properties in quantum computation are crucial for stability.

Majorana Definition & Non-abelian Theory

Majorana Definition

Majorana Fermions and Non-Abelian Theory

- Majorana fermions are identical to their antiparticles.
- Majorana bound states exhibit non-Abelian statistics.

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\gamma_1 + i\gamma_2\right),$$
$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\gamma_1 - i\gamma_2\right).$$

Having an even number of Majorana fermions, $2n$, they obey the anitcommutation relation

$$\begin{aligned} \{\gamma_i, \gamma_j\} &= 2\delta_{ij}, \qquad i = 1, 2, \ldots, 2n, \\ \gamma_i^2 &= 1. \end{aligned} \tag{1}$$

Indistinguishable Particles

Consider the wavefunctions ψ_1 and ψ_2 associated with two indistinguishable particles. Indistinguishable particles give

$$|\langle\psi_1\psi_2|\psi_2\psi_1\rangle|^2 = 1.$$

Upon applying for an exchange operator, one anticipates the form

$$\hat{B}|\psi_1\,\psi_2\rangle = e^{i\phi}|\psi_2\,\psi_1\rangle, \tag{2}$$

where for fermions, $\phi = \pi \to -1$, and for bosons, $\phi = 0 \to +1$–both being constant. Neither, will be called non-Abelian particles, i.e., Majorana fermions.

Parity Operator for Majorana

In general, for $2N$ Majorana modes will have N fermionic modes. Each mode has the flexibility to be occupied or unoccupied by a fermion, resulting in two potentially degenerate quantum states $|0\rangle$ and $|1\rangle$ for every pair of Majoranas. Consider the the *fermion parity* operator

$$\hat{P} = 1 - 2\hat{N} = i\gamma_1\gamma_2, \tag{3}$$

where $N = \hat{a}^\dagger \hat{a}$ is the *fermion number* operator. Let's apply it to a pair of Majoranas states as

$$\hat{P}|0\rangle = (1 - 2\hat{N})\,|0\rangle = +|0\rangle,$$
$$\hat{P}|1\rangle = (1 - 2\hat{N})\,|1\rangle = -|1\rangle.$$

Majorana Fermions Braiding

- Braiding involves exchanging positions in space.
- Non-trivial matrix elements characterize the transformation.
- Unitary operators and the parity operator play crucial roles.

Unitary Operators

In general, any unitary operator U can be expressed in terms of a Hermitian operator A of the form $U = e^{i\beta A}$ with some angle β.

we begin with their parity operator, denoted as $P_{nm} = i\gamma_n\gamma_m$. This allows us to define

$$U \equiv e^{\beta\gamma_n\gamma_m} \quad \text{or} \quad U = \cos\beta + \gamma_n\gamma_m \sin\beta, \tag{4}$$

To find a unitary operator that evolves Majorana fermions according to the Heisenberg picture, i.e.,

$$\gamma_n \to U\gamma_n U^\dagger,$$
$$\gamma_m \to U\gamma_m U^\dagger,$$

while leaving other elements unaffected, we substitute Equation 4 into the transformation equation:

$$\gamma_n \to \gamma_n \cos 2\beta - \gamma_m \sin 2\beta,$$
$$\gamma_m \to \gamma_m \cos 2\beta + \gamma_n \sin 2\beta.$$

This leads to the condition $\beta = \pm\pi/4$-through the text, we will refer to $\pi/4$ as the braiding operator and $-\pi/4$ as the inverse braiding operator. Substituting this into the expression, we obtain the braiding unitary operator for γ_n and γ_m:

$$U = \exp\left(\pm\frac{\pi}{4}\gamma_n\gamma_m\right) = \frac{1}{\sqrt{2}}\left(1 \pm \gamma_n\gamma_m\right).$$

Realizations of Braiding Via a Series of Measurements

Realizations of Braiding Via a Series of Measurements

Braiding Realization Protocol

- Series of measurements designed for a system of 4 Majorana fermions.
- Measurement outcomes used to correct and realize braiding.
- Ancilla Majorana fermions contribute to robustness.

Simple Measurement Projector

In case of two-level fermionic systems, to make projective measurements along z-axis we apply the following operator

$$(1 + S\sigma_z),$$

where S can be ± 1 with some probability, corresponding to the eigenvalues of σ_z. The analogy operator for Majorana fermions to a projective measurement will be

$$(1 + Si\gamma_n\gamma_m),$$

where again S can be $+1$ or -1.

4 Majorana Fermions Braiding

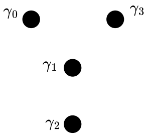

Figure 1: Configuration of four Majorana fermions.

To illustrate this idea, we begin with a system of 4 Majorana fermions, corresponding to two fermions. The configuration of Majorana fermions is depicted in Figure 1. The true braiding operator between γ_0 and γ_3 is given by

$$U = e^{\frac{\pm\pi}{4}\gamma_3\gamma_0} = \frac{1}{\sqrt{2}}(1 \pm \gamma_3\gamma_0). \tag{5}$$

4 Majorana Fermions Configuration

$$(1 + iS_3\gamma_1\gamma_2)(1 + iS_2\gamma_3\gamma_1)(1 + iS_1\gamma_1\gamma_0)(1 + iS_0\gamma_1\gamma_2)$$

as illustrated in Figure 2.

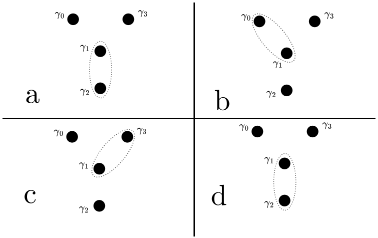

Figure 2: a-d shows measurements sequence that eventually realizes braiding of γ_0 and γ_3.

Analysis of Series of Measurements

$$(1 + iS_3\gamma_1\gamma_2)[1 + iS_2\gamma_3\gamma_1 + iS_1\gamma_1\gamma_0 - S_2S_1\gamma_3\gamma_0](1 + iS_0\gamma_1\gamma_2),$$

we used $\gamma_n^2 = 1$ on the last term. We employ the operators on the sides to encapsulate each term within the brackets. Starting with the first terms we get

$$(1 + iS_3\gamma_1\gamma_2)(1 + iS_0\gamma_1\gamma_2).$$

We have two cases; $S_3 = S_0$, then we get just $(1 + iS_0\gamma_1\gamma_2)$, since projectors are idempotent (i.e., $P^2 = P$). And if $S_3 \neq S_0$, it vanishes; since getting a measurement output S_0, then trying to get a measurement on another output will always result in zero. In compact, we write

$$\delta_{S_3,S_0}\,(1 + iS_0\gamma_1\gamma_2).$$

Analysis of Series of Measurements

Moving to the second term, we get

$$(1 + iS_3\gamma_1\gamma_2)iS_2\gamma_3\gamma_1(1 + iS_0\gamma_1\gamma_2).$$

As in Equation 7, we exploit the property that all Majorana fermions anticommute with each other, and commute with themselves. Then, by moving the enclosed factor to the right we introduce a minus sign,

$$(1 + iS_3\gamma_1\gamma_2)(1 - iS_0\gamma_1\gamma_2)iS_2\gamma_3\gamma_1.$$

In terms of the Kronecker Delta, we have

$$\delta_{S_3,-S_0}(1 - iS_0\gamma_1\gamma_2)iS_2\gamma_3\gamma_1.$$

Analysis of Series of Measurements

We continue the same steps for the remaining terms, adding them app we get

$$\delta_{S_3,S_0}\,(1 + iS_0\gamma_1\gamma_2) + \delta_{S_3,-S_0}\,(1 - iS_0\gamma_1\gamma_2)iS_2\gamma_3\gamma_1$$
$$+\,\delta_{S_3,-S_0}\,(1 - iS_0\gamma_1\gamma_2)iS_1\gamma_1\gamma_0 + \delta_{S_3,S_0}\,(1 + iS_0\gamma_1\gamma_2)S_2S_1\gamma_3\gamma_0.$$

Factoring out the same Kroneckers and projectors

$$\begin{aligned}&\delta_{S_3,S_0}\,(1 + iS_0\gamma_1\gamma_2)(1 + S_2S_1\gamma_3\gamma_0)\\&+\delta_{S_3,-S_0}\,(1 - iS_0\gamma_1\gamma_2)(iS_2\gamma_3\gamma_1 + iS_1\gamma_1\gamma_0).\end{aligned} \tag{6}$$

Analysis Results

If $S_3 = S_0$, we get

$$(1 + S_2 S_1 \gamma_3 \gamma_0);$$

but $S_2 S_1 = \pm 1$, thus

$$(1 \pm \gamma_3 \gamma_0).$$

Analysis Results

Now consider the case where $-S_3 = S_0$. We have

$$(iS_2\gamma_3\gamma_1 + iS_1\gamma_1\gamma_0) = -iS_2\gamma_1\,(\gamma_3 - S_1S_2\gamma_0),$$

where $S_1S_2 = \pm 1$. Since the global phase factor has no physically observable consequences, we can ignore the factor $-iS_2$.

$$-iS_2\gamma_1\,(1 \mp \gamma_3\gamma_0).$$

Larger System

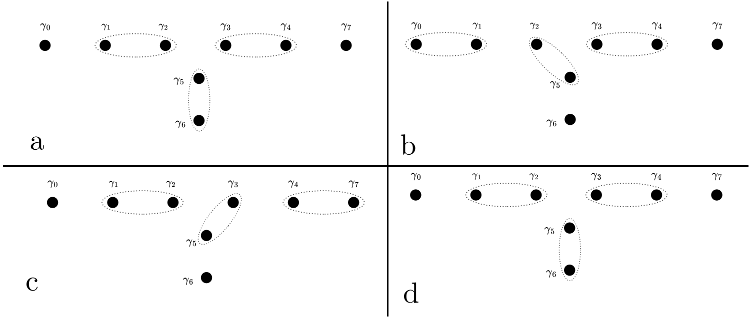

Figure 3: a-d shows measurements sequence that eventually realizes braiding of γ_0 and γ_7 for a system of eight Majorana fermions.

So the series of measurements will be, written from right to left,

$$\begin{aligned}&(1+iS_0\gamma_1\gamma_2)(1+iS_1\gamma_3\gamma_4)(1+iS_2\gamma_0\gamma_1)(1+iS_3\gamma_2\gamma_3)\\&\times(1+iS_4\gamma_1\gamma_3)(1+iS_5\gamma_2\gamma_5)(1+iS_6\gamma_1\gamma_2)(1+iS_7\gamma_3\gamma_4).\end{aligned} \tag{7}$$

Simulation with Ancilla Majorana Fermions

Simulation with Ancilla Majorana Fermions

- Explore the impact of ancilla Majorana fermions on fidelity.
- Ideal case vs. finite-time case.
- Increasing ancilla Majorana fermions may not always improve fidelity.

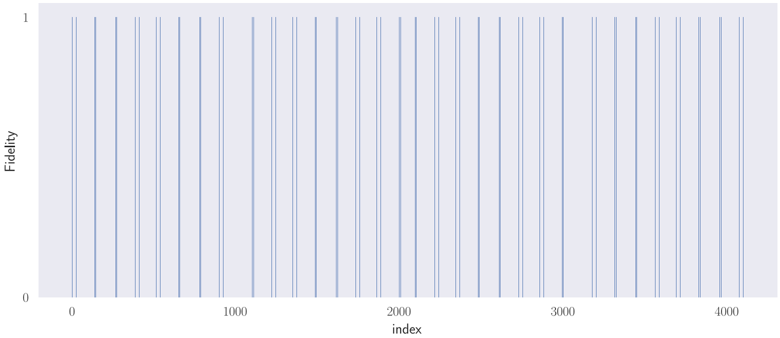

Figure 4: Shows the fidelity values, of all possible 4096 combinations of S_i for eight Majoranas, again their index. This implements the sequence in Equation 7.

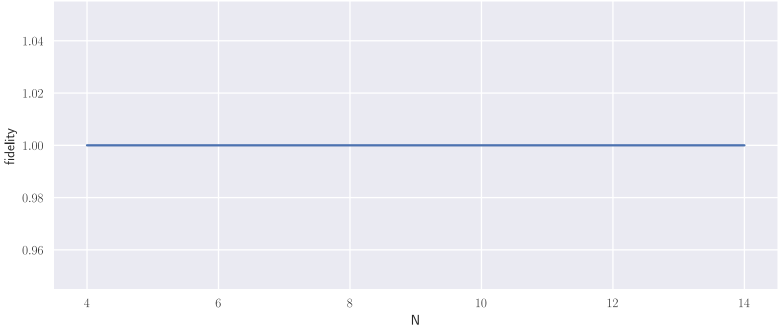

Figure 5: The fidelity of braiding operator realization plotted against the increasing number of employed ancilla Majoranas. The results are obtained using the sequence specified in Equation 7.

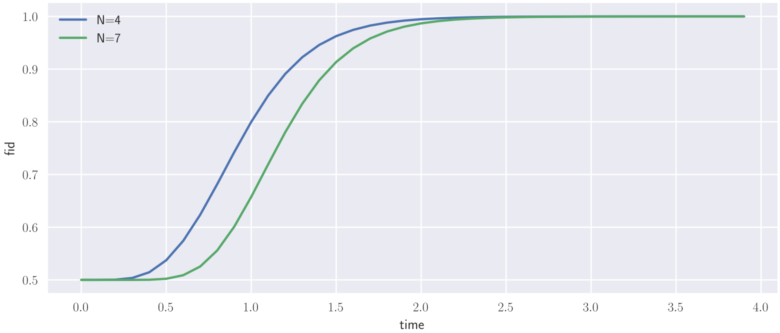

Figure 6: Applying the measurement sequence detailed in Equation 7, where the projectors take on a time-finite exponential form. The figure presents two plots corresponding to two different numbers of Majorana fermions.

Figure 7: Illustrating fidelity under the approximate measurement operator, this plot depicts how fidelity varies with the changing number of utilized Majorana fermions. The analysis maintains a constant time value set at 1.

Approximating Measurement Operators

$$e^{t(iS_i\gamma_n\gamma_m)} = \cosh t + (i\gamma_n\gamma_m)\sinh t,$$

which implies

$$\cosh t + (i\gamma_n\gamma_m)\sinh t \to (1 + iS_i\gamma_n\gamma_m) \quad \text{as} \quad t \to \infty$$

Constructing Protocol & Simulating on Quantum Circuit

Simulating Real-Life Conditions

- Quantum circuit construction for braiding realization.
- Utilizing qiskit for simulations.
- Fidelity comparisons and visualization on the Bloch sphere.

- $\gamma_0 = Z_0$,
- $\gamma_1 = X_0 Z_1$,
- $\gamma_2 = X_0 Y_1$,
- $\gamma_3 = Y_0$,

Here, X, Y, and Z represent the Pauli matrices associated with the two-level qubit,

1. $(1 + iX_0 Z_1 X_0 Y_1) = (1 + X_1)$
2. $(1 + iX_0 Z_1 Z_0) = (1 + Y_0 Z_1)$
3. $(1 + iY_0 X_0 Z_1) = (1 + Z_0 Z_1)$
4. $(1 + X_1)$

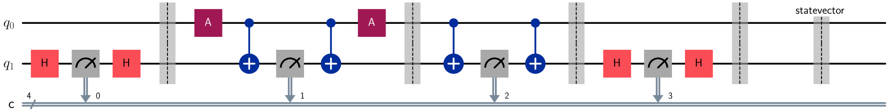

Figure 8: The figure illustrates the ultimate two-qubit circuit designed to implement the sequence of measurements outlined in. The drawing highlights the four distinct steps of applying each measurement operator, delineated by vertical dashed lines.

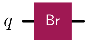

Figure 9: Illustration of the one-qubit circuit featuring a single unitary gate, denoted as Br.

Statevector after braiding

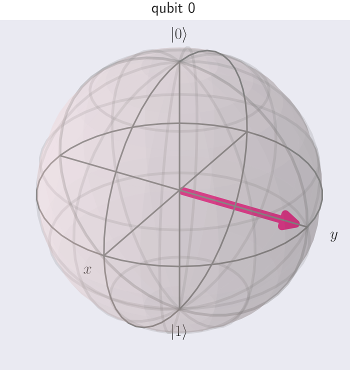

Figure 10: Visualization of the statevector for the single-qubit circuit subjected to the true braiding operator on the Bloch sphere.

Error Assessment

- Introducing Pauli X error and bit-flip error.
- Examining the impact on the realization of braiding.

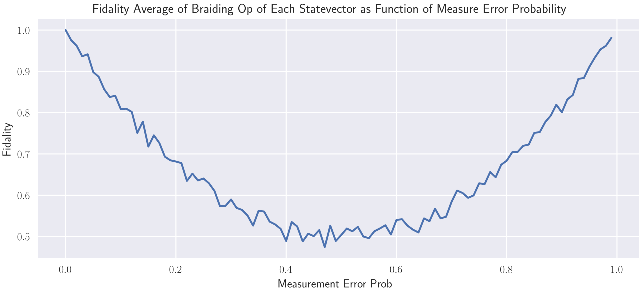

Figure 11: Illustration of the circuit in Figure 8 subjected to Pauli X error, with 1024 iterations for each probability setting. The resulting fidelities are averaged over the 1024 shots.

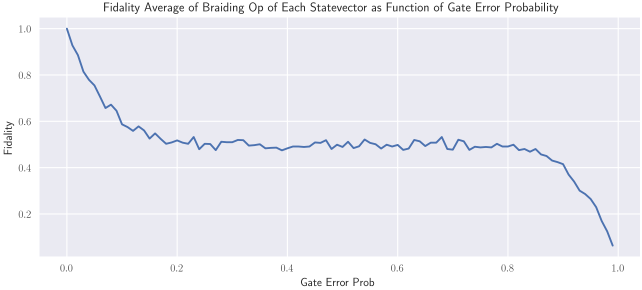

Figure 12: As in Figure 11, but with bit-flip applied to before each occurrence of `h` and `cx` operations.

References & Conclusion

Conclusion

- Majorana fermion braiding explored through a series of measurements.
- Ancilla Majorana fermions contribute to robustness.
- Error assessment provides insights into the fidelity of braiding.

References

Thank you for listening!