Quantum Information Analysis of Majorana Fermion Braiding: Exploring Realization and Error Assessment via a Series of Measurements

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1 Instroduction

In 1937, Majorana proposed a groundbreaking idea suggesting that electrically neutral spin-1/2 particles could be accurately described by a wave equation with real-valued solutions, known as the Majorana equation [3]. This proposition introduced the concept that these particles and their antiparticles would be indistinguishable due to the unique relationship between their wave functions through complex conjugation, ultimately leaving the Majorana wave equation unaltered.

Building upon Majorana's pioneering work, this paper delves into the fascinating realm of Majorana fermion braiding, transitioning from conventional approaches to a novel method centered around a sequence of measurements. A key aspect of this exploration involves leveraging the Jordan-Wigner transformation to express Majorana operators within the framework of a spin (fermionic) system. This transformative step not only enhances the understanding of Majorana fermions but also opens up possibilities for utilizing advanced techniques provided by the `qiskit` package.

In the realm of two dimensions, executing two exchanges of particle positions mirrors the topological equivalence of one quasiparticle encircling another along a closed path, known as a braid. This action of exchanging identical particles or maneuvering one particle around another is formally termed "braiding" in mathematical terms. The act of "braiding" two anyons not only encapsulates a historical record of the event but also results in altered wave functions that effectively "count" the number of braids involved.[4]

Furthermore, the introduction of the braiding operator adds another layer of complexity and significance to the discussion. The braiding operator is a mathematical entity that encapsulates the non-Abelian statistics associated with the braiding of anyons, a fundamental concept in the study of topological quantum computation. Exploring the properties and applications of the braiding operator provides valuable insights into the behavior of Majorana fermions during braiding processes.

Fault-tolerant quantum computing is of particular interest due to the inherent challenges posed by the delicate nature of quantum information. Unlike classical computation, which can often rely on the reliability of classical gates, quantum systems are highly susceptible to errors and decoherence. In the classical realm, information storage on magnetic media, for instance, benefits from the collective behavior of spins in individual atoms. Although each spin is sensitive to thermal fluctuations, the interactions between spins tend to align them in the same direction. Consequently,

if a spin deviates, the interactions compel it to return to the alignment of other spins, resembling a self-correcting mechanism akin to the repetition code in error correction.[1]

Quantum computers that operate based on measurements employ a sophisticated quantum state referred to as a cluster state. This state comprises three (or more) qubits interconnected through the non-classical phenomenon of entanglement, where the properties of quantum particles remain correlated even across vast distances. The functionality of measurement-based quantum computers revolves around initiating a measurement on the initial qubit within the cluster state. The result of this measurement dictates the subsequent measurement on the second entangled qubit, a process known as feedforward. Subsequently, this determination influences how the third qubit is measured. Through this approach, any desired quantum gate or circuit can be realized by carefully selecting the sequence of measurements.[6] Motivation behind utilizing a measurement-based protocol for realizing the braiding operator deserves attention. By adopting a measurement-based approach, the study aims to exploit the inherent advantages offered by this methodology, such as fault tolerance and scalability.

Topological properties play a pivotal role in hosting quantum computation and Majorana braiding. The unique characteristics of topological systems, such as robustness against local perturbations, make them ideal candidates for quantum information processing. In the context of quantum gates simulation, the discussion extends to the utilization of measurement-only approaches for simulating quantum gates. Specifically, exploring how measurement-only strategies can effectively simulate quantum gates is crucial for comprehending the intricacies involved in the proposed Majorana fermion braiding scheme. Drawing parallels to existing quantum computing platforms, like the IBM Q computer, underscores the practicality and feasibility of implementing such schemes in real-world quantum hardware.

The exploration of accessing and running circuits on platforms such as the IBM Q computer is a crucial aspect. It is imperative to comprehend the processes and mechanisms involved in accessing and executing circuits on quantum computing hardware, as this knowledge forms the foundation for translating theoretical advancements into practical applications. In this report, we successfully employed the `qiskit` library and IBM online quantum computers to simulate our conceptual framework. This involved the creation of a two-qubit system that realizes our sequence of measurements through Pauli unitary gates. Through a comparison with a true braiding operator for a single qubit system, we assessed the results and error sensitivity of such circuits by calculating the average fidelity between the final state vector of the circuit after execution.

2 Majorana Definition & Non-abelian Theroy

The distinction between Majorana fermions and Dirac fermions can be mathematically articulated through the creation and annihilation operators of second quantization. Specifically, the creation operator $\gamma_j^\dagger$ generates a fermion in quantum state j (described by a real wave function), while the annihilation operator γ_j eliminates it, or equivalently, creates the corresponding antiparticle. In the case of a Dirac fermion, the operators $\gamma_j^\dagger$ and γ_j are distinct, whereas for a Majorana fermion, they are identical. Expressing the ordinary fermionic annihilation and creation operators f and $f^\dagger$ in terms of two Majorana operators γ_1 and γ_2 can be achieved as follows:

$$\hat{a} = \frac{1}{\sqrt{2}} \left(\gamma_1 + i\gamma_2 \right),$$
$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left(\gamma_1 - i\gamma_2 \right).$$

Having an even number of Majorana fermions, $2n$, they obey the anitcommutation relation

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \qquad i = 1, 2, \ldots, 2n. \tag{1}$$

Mathematically, the superconductor imposes electron hole "symmetry" on the quasiparticle excitations, relating the creation operator $\gamma(E)$ at energy E to the annihilation operator $\gamma^\dagger(-E)$ at energy $-E$.[7] Majorana fermions can be bound to a defect at zero energy, and then the combined objects are called Majorana bound states or Majorana zero modes. This name is more appropriate than Majorana fermion (although the distinction is not always made in the literature), because the statistics of these objects is no longer fermionic. Instead, the Majorana bound states are an example of non-abelian anyons: interchanging them changes the state of the system in a way that depends only on the order in which the exchange was performed. The non-abelian statistics that Majorana bound states possess allows them to be used as a building block for a topological quantum computer.

Consider the wavefunctions ψ_1 and ψ_2 associated with two particles. Upon applying an exchange operator, one anticipates the form

$$\hat{B}|\psi_1 \, \psi_2\rangle = \lambda|\psi_2 \, \psi_1\rangle, \tag{2}$$

where for fermions, $\lambda = -1$, and for bosons, $\lambda = +1$—both being constant. However, in the case of Majorana fermions, as we will elaborate on shortly, the constant takes the form of a matrix. This distinction categorizes the former as abelian particles and the latter as non-abelian particles [5]. The parity operator plays a crucial role in describing the non-abelian statistics associated with the braiding of these particles. The braiding of Majorana fermions involves exchanging their positions in space, and the resulting transformation is characterized by non-trivial matrix elements rather than simple constants.

To assign a state to a system of Majoranas, we need to consider them as fermionic system, i.e., as pairs of the form of Equation 2. In general, for $2N$ Majorana modes will have N fermionic modes. Each mode has the flexibility to be occupied or unoccupied by a fermion, resulting in two potentially degenerate quantum states $|0\rangle$ and $|1\rangle$ for every pair of Majoranas. Consdier the the *fermion parity* operator

$$\hat{P} = 1 - 2\hat{N} = i\gamma_1\gamma_2, \tag{3}$$

where $N = \hat{a}^\dagger\hat{a}$ is the *fermion number* operator. Let's apply it to a pair of Majoranas state as

$$\hat{P}|0\rangle = (1 - 2\hat{N})|0\rangle = +|0\rangle,$$
$$\hat{P}|1\rangle = (1 - 2\hat{N})|1\rangle = -|1\rangle.$$

The parity operator is characterized by eigenvalues of $+1$ and -1, hence its name. The Hilbert space associated with parity can be treated comparable to a qubit, functioning as a subspace within a qubit for information storage and quantum computation purposes. This non-local operator, endowed with robustness against errors in quantum computation, ensures stability over significant distances. Additionally, the preservation of the total parity within a system of Majorana particles becomes evident. Consequently, the application of unitary operators is constrained to those maintaining the

conservation of total parity. This insight leads to the realization that the braiding of Majorana particles adheres to this conservation principle.

In general, any unitary operator U can be expressed in terms of a Hermitian operator A of the form $U = e^{i\beta A}$ with some angle β. For Majorana fermion operators, we can begin with their parity operator, denoted as $P_{nm} = i\gamma_n\gamma_m$. This allows us to define

$$U \equiv e^{\beta\gamma_n\gamma_m} \quad \text{or} \quad U = \cos\beta + \gamma_n\gamma_m \sin\beta, \tag{4}$$

see Appendix A. To find a unitary operator that evolves Majorana fermions according to the Heisenberg picture, i.e.,

$$\gamma_n \to U\gamma_n U^\dagger,$$
$$\gamma_m \to U\gamma_m U^\dagger,$$

while leaving other elements unaffected, we substitute Equation 4 into the transformation equation:

$$\gamma_n \to \cos 2\beta\gamma_n - \sin 2\beta\gamma_m,$$
$$\gamma_m \to \cos 2\beta\gamma_m - \sin 2\beta\gamma_n.$$

This leads to the condition $\beta = \pm\pi/4$-through the text, we will refer to $\pi/4$ as the braiding operator and $-\pi/4$ the inverse braiding operator. Substituting this into the expression, we obtain the braiding unitary operator for γ_n and γ_m:

$$U = \exp\left(\pm\frac{\pi}{4}\gamma_n\gamma_m\right) = \frac{1}{\sqrt{2}}\left(1 \pm \gamma_n\gamma_m\right).$$

In case of two-level fermionic systems, to make projective measurement along z-axis we apply the following operator

$$(1 + S\sigma_z),$$

where S can be ± 1 with some probability, corresponding to the eigenvalues of σ_z. The analogy operator for Majorana fermions to a projective measurement will be

$$(1 + Si\gamma_n\gamma_m),$$

where again S can be $+1$ or -1.

3   Realizations of Braiding Via a Series of Measurements

Consider the configuration of Majorana fermions depicted in Figure 1. And, say, we want to braid the two Majorana fermions γ_0 & γ_3. The conceptual approach here involves executing measurement-only operations on the configuration of Majorana modes, effectively achieving the same outcome as physically braiding the two Majorana fermions [2]. To illustrate this idea, we begin with a system of 4 Majorana fermions, corresponding to two fermions. The configuration of Majorana fermions is depicted in Figure 1. The true braiding operator between γ_0 and γ_3 is given by

$$U = e^{\frac{\pm\pi}{4}\gamma_3\gamma_0} = \frac{1}{\sqrt{2}}(1 \pm \gamma_3\gamma_0). \tag{5}$$

We call γ_0 and γ_3 the main Majorana fermions, and the rest are called the ancilla Majorana fermions. By utilizing the ancilla Majorana fermions we want to realize braiding operator in Equation 5, using
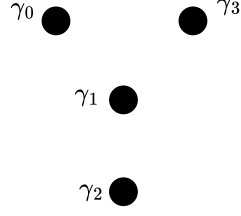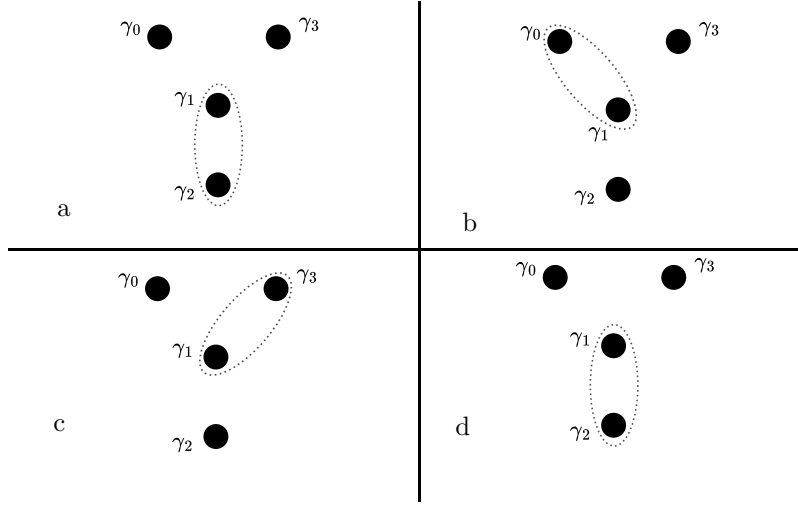
Figure 1: Configuration of four Majorana fermions.

Figure 2: a-b shows measurements sequence that evantually realizes braiding of γ_0 and γ_3.

only measurement operations. We do not want to do any measurement on both main Majorana fermions. Further, first step and last step should not invlove the main Majoranas.

To realize this braiding operator solely through a series of measurements, we follow these four steps:

1. Apply the operator $(1 + iS_0\gamma_1\gamma_2)$

2. Apply the operator $(1 + iS_1\gamma_1\gamma_0)$

3. Apply the operator $(1 + iS_2\gamma_3\gamma_1)$

4. Apply the operator $(1 + iS_3\gamma_1\gamma_2)$

as illustrated in Figure 2.

Now we need to verify this result wether it realizes the braiding operator. We start by writing all steps as one multiplication series from right to left, as

$$(1 + iS_3\gamma_1\gamma_2)(1 + iS_2\gamma_3\gamma_1)(1 + iS_1\gamma_1\gamma_0)(1 + iS_0\gamma_1\gamma_2)$$

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To simplify, we note the left most projector and the right most one are different only in S, if we could bring them togather we can use some helpful properties of projectors. First, expand the middle two projectors

$$(1 + iS_3\gamma_1\gamma_2)[1 + iS_2\gamma_3\gamma_1 + iS_1\gamma_1\gamma_0 - S_2S_1\gamma_3\gamma_0](1 + iS_0\gamma_1\gamma_2),$$

we used $\gamma_n^2 = 1$ on the last term. We employ the operators on the sides to encapsulate each term within the brackets. Starting with first terms we get

$$(1 + iS_3\gamma_1\gamma_2)(1 + iS_0\gamma_1\gamma_2).$$

We have two cases; $S_3 = S_0$, then we get just $(1 + iS_0\gamma_1\gamma_2)$, since projectors are idempotent (i.e., $P^2 = P$). And if $S_3 \neq S_0$, it vanishes; since getting a measurement output S_0, then trying to get a measurement on other output will always result in zero. In compact, we write

$$\delta_{S_3,S_0}(1 + iS_0\gamma_1\gamma_2).$$

Moving to the second term, we get

$$(1 + iS_3\gamma_1\gamma_2)iS_2\gamma_3\gamma_1(1 + iS_0\gamma_1\gamma_2).$$

As in Equation 2, we exploit the property that all Majorana fermions anticommute with each other, and commute with themselves. Then, by moving enclosed factor to the right we introduce a minus sign,

$$(1 + iS_3\gamma_1\gamma_2)(1 - iS_0\gamma_1\gamma_2)iS_2\gamma_3\gamma_1.$$

In term of the Kronecker delta, we have

$$\delta_{S_3,-S_0}(1 - iS_0\gamma_1\gamma_2)iS_2\gamma_3\gamma_1.$$

We continue the same steps for the remaining terms, adding them app we get

$$\delta_{S_3,S_0}(1 + iS_0\gamma_1\gamma_2) + \delta_{S_3,-S_0}(1 - iS_0\gamma_1\gamma_2)iS_2\gamma_3\gamma_1$$
$$+ \delta_{S_3,-S_0}(1 - iS_0\gamma_1\gamma_2)iS_1\gamma_1\gamma_0 + \delta_{S_3,S_0}(1 + iS_0\gamma_1\gamma_2)S_2S_1\gamma_3\gamma_0.$$

Factoring out the same Kroneckers and projectors

$$\delta_{S_3,S_0}(1 + iS_0\gamma_1\gamma_2)(1 + S_2S_1\gamma_3\gamma_0)$$
$$+ \delta_{S_3,-S_0}(1 - iS_0\gamma_1\gamma_2)(iS_2\gamma_3\gamma_1 + iS_1\gamma_1\gamma_0).$$

Since we are concerned about the main Majorana fermions (γ_3 & γ_0), we ignore the left most projector; it just affects the the ancilla Majorana fermions. If $S_3 = S_0$, we get

$$(1 + S_2S_1\gamma_3\gamma_0);$$

but $S_2S_1 = \pm 1$, thus

$$(1 \pm \gamma_3\gamma_0).$$

That's it. We realize the braiding operator in Equation 5. Based on the measurement outcomes, values of S_0 through S_3, we can get knowledge of the resulting overall operation as well as how to tune it gives always the braiding operator. Now consider $-S_3 = S_0$. We have

$$(iS_2\gamma_3\gamma_1 + iS_1\gamma_1\gamma_0) = -iS_2\gamma_1(\gamma_3 \mp \gamma_0),$$

where $S_1 S_2 = \pm 1$. Since the global phase factor has no physically observable consequences, we can ignore the factor $-iS_2$. For γ_1, again its effect is on the ancilla Majorana fermion. However, to realize a braiding operator we have to multiply by γ_3, to get

$$i\gamma_1 \left(1 \mp \gamma_3 \gamma_0\right).$$

By implementing a protocol that reads the values of the measurement outcomes we can, in predictable way, realize the braiding of γ_3 & γ_0 shown in Equation 5

4 Constructing Protocol

We shall redefine the our γ s in term of fermionic spin operators, giving us a way to model this system in much more fimiliar systems, such as qubits in quantum computing information. So we'll have:

- $\gamma_0 = Z_0$

- $\gamma_1 = X_0 Z_1$

- $\gamma_2 = X_0 Y_1$

- $\gamma_3 = Y_0$

Note that tensor product is understod, if there is one gate, tensor product with indentity of that subsystem is implicit. Then 4-step series of measurement on the system becomes

1. $(1 + i X_0 Z_1 X_0 Y_1) = (1 + X_1)$

2. $(1 + i X_0 Z_1 Z_0) = (1 + Y_0 Z_1)$

3. $(1 + i Y_0 X_0 Z_1) = (1 + Z_0 Z_1)$

4. $(1 + X_1)$

Also for true braiding operator we get

$$e^{i \frac{\pi}{4} X_0} = \frac{1}{\sqrt{2}} \left(1 + i X_0\right) \quad \text{or} \quad e^{-i \frac{\pi}{4} X_0} = \frac{1}{\sqrt{2}} \left(1 - i X_0\right)$$

Let's understand the possible outcomes from the general case of the measurement operator, that is,

$$(1 + S_3 X_1)(1 + S_2 Z_0 Z_1)(1 + S_1 Y_0 Z_1)(1 + S_0 X_1)$$

Expanding the middle two factors as

$$(1 + S_3 X_1)(1 + S_2 Z_0 Z_1 + S_1 Y_0 Z_1 + S_2 S_1 Z_0 Z_1 Y_0 Z_1)(1 + S_0 X_1)$$

Utilizing the Pauli gates anitcommutation relations, we move the LHS factor to RHS, as for first term we get

$$(1 + S_3 X_1)(1 + S_0 X_1) = \delta_{S_0, S_3} \left(1 + S_0 X_1\right)$$

For second term,

$$(1 + S_3 X_1) S_2 Z_0 Z_1 (1 + S_0 X_1) = \delta_{S_0, -S_3} S_2 Z_0 Z_1 (1 + S_0 X_1)$$

For the 3rd term,

$$(1 + S_3 X_1) S_1 Y_0 Z_1 (1 + S_0 X_1) = \delta_{S_0, -S_3} S_1 Y_0 Z_1 (1 + S_0 X_1)$$

For the 3rd term,

$$(1 + S_3 X_1) S_2 S_1 Z_0 Z_1 Y_0 Z_1 (1 + S_0 X_1) = \delta_{S_0, S_3} - i X_0 S_2 S_1 (1 + S_0 X_1)$$

Now, we'll investigate the protocol classic outcomes, then we shall decide based on it whether we did relize a braiding between γ_0 & γ_3, if not, what operators to apply to fix it. From previous paragraph, we simplify it to

$$[\delta_{S_0, S_3} + \delta_{S_0, -S_3} S_2 Z_0 Z_1 + \delta_{S_0, -S_3} S_1 Y_0 Z_1 + \delta_{S_0, S_3} - i X_0 S_2 S_1](1 + S_0 X_1)$$

Let's study different cases:

Case 1: $S_0 = S_3$ We get

$$[1 - i X_0 S_2 S_1](1 + S_0 X_1)$$

Note, the right factor just acts on subsystem 1 that we don't care about it outcomes.

Case 1.1: $S_1 = -S_2$

$$[1 + i X_0](1 + S_0 X_1)$$

relizing counterclockwise braiding operator in Equation 4.

Case 1.2: $S_1 = S_2$

$$[1 - i X_0](1 + S_0 X_1)$$

relizing clockwise braiding operator in Equation 4.

Case 2: $S_0 \neq S_3$ We get

$$[S_2 Z_0 Z_1 + S_1 Y_0 Z_1] (1 + S_0 X_1)$$

let's factor out $Z_0 Z_1$

$$Z_0 Z_1 [S_2 - i S_1 X_0] (1 + S_0 X_1)$$

In this case we always want to multiply by Z_0, then we'll have

Case 2.1: $S_1 = S_2$

$$S_1 Z_0 Z_1 [1 - i X_0](1 + S_0 X_1)$$

relizing the inverse braiding operator

Case 2.2: $S_1 = -S_2$

$$S_1 Z_0 Z_1 [S_1 S_2 - i X_0](1 + S_0 X_1)$$

But $S_1 S_2 = -1$, then

$$-S_1 Z_0 Z_1 [1 + i X_0](1 + S_0 X_1)$$

relizing the braiding operator

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5 Conclusion & Future Work

A   Using Eauler's Formula to Rewrite Exponential Form

The Euler formula is commonly expressed as $e^{ix} = \cos(x) + i\sin(x)$. To rewrite $U = e^{\beta i \gamma_n \gamma_m}$ in the desired form, we can use this formula:

$$U = e^{\beta i \gamma_n \gamma_m} = \cos(\beta) + i\gamma_n \gamma_m \sin(\beta)$$

Here, β is the coefficient of the exponential, and γ_n and γ_m are matrices or operators. The rewritten expression utilizes the Euler formula to represent the exponential term as a combination of cosine and sine functions. Specifically, for $\beta = \pi/4$

$$U = \frac{1}{\sqrt{2}} \left(1 + i\gamma_1 \gamma_2 \right)$$

B   Statevector and Density Matrix

Assume you have a state vector $|\psi\rangle$, which is a column vector with complex numbers:

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix}$$

Here, n is the dimensionality of the quantum system. Take the conjugate transpose of the state vector, denoted by $\langle\psi|$. This is obtained by taking the complex conjugate of each element and then transposing the vector:

$$\langle\psi| = \begin{bmatrix} \psi_1^* & \psi_2^* & \ldots & \psi_n^* \end{bmatrix}^T$$

Compute the outer product to obtain the density matrix ρ:

$$\rho = |\psi\rangle\langle\psi| = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} \begin{bmatrix} \psi_1^* & \psi_2^* & \ldots & \psi_n^* \end{bmatrix}$$

The result is a square matrix of size $n \times n$, representing the density matrix. The density matrix ρ is Hermitian (equal to its conjugate transpose) and positive semi-definite. If the state vector $|\psi\rangle$ is normalized (has a magnitude of 1), then the trace of the density matrix is equal to 1, which is a requirement for a valid density matrix representing a physical state.

Pure State:

For a pure state density matrix ρ, which satisfies $\rho^2 = \rho$ (idempotent), you can find the state vector $|\psi\rangle$ by:

$$|\psi\rangle = \sqrt{\lambda} \cdot |\phi\rangle$$

Here, λ is the non-zero eigenvalue of ρ, and $|\phi\rangle$ is the corresponding eigenvector.

Mixed State: For a mixed state density matrix ρ with multiple non-zero eigenvalues, it is not possible to uniquely determine a single state vector. The system is in a statistical mixture of pure states. However, you can find a set of state vectors and their corresponding probabilities:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Here, $|\psi_i\rangle$ are the eigenvectors of ρ, and p_i are the corresponding eigenvalues. The state vectors represent the pure states in the mixture, and the probabilities p_i give the weight of each pure state in the mixture.

Implementation: To implement this in practice, you can use a numerical linear algebra library (such as NumPy for Python) to compute the eigenvalues and eigenvectors of the density matrix. Depending on the specific form of the density matrix, you might need to use different methods. Keep in mind that this process may not be unique, especially for mixed states, where different sets of pure states and probabilities can lead to the same density matrix.

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