Quantum Information Analysis of Majorana Fermion Braiding: Exploring Realization and Error Assessment in Series of Measurements

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December 10, 2023

1 Instroduction

The inception of the concept dates back to Majorana's proposal in 1937, wherein he postulated that electrically neutral spin-1/2 particles could be characterized by a real-valued wave equation known as the Majorana equation. According to this proposition, these particles would be indistinguishable from their antiparticles due to the relationship between their wave functions through complex conjugation, thereby leaving the Majorana wave equation unaltered.

This paper explores the realization of Majorana fermion braiding, shifting from the traditional approach to a series of measurements. Additionally, it employs the Jordan-Wigner transformation to express Majorana operators in the language of a spin (fermionic) system. By doing so, the study facilitates the utilization of techniques offered by the `qiskit` package to simulate and analyze this novel approach and its associated quantum system.

2 Majorana Definition

The distinction between Majorana fermions and Dirac fermions can be mathematically articulated through the creation and annihilation operators of second quantization. Specifically, the creation operator $\gamma_j^\dagger$ generates a fermion in quantum state j (described by a real wave function), while the annihilation operator γ_j eliminates it, or equivalently, creates the corresponding antiparticle. In the case of a Dirac fermion, the operators $\gamma_j^\dagger$ and γ_j are distinct, whereas for a Majorana fermion, they are identical. Expressing the ordinary fermionic annihilation and creation operators f and $f^\dagger$ in terms of two Majorana operators γ_1 and γ_2 can be achieved as follows:

$$f = \frac{1}{\sqrt{2}}\left(\gamma_1 + i\gamma_2\right),$$
$$f^\dagger = \frac{1}{\sqrt{2}}\left(\gamma_1 - i\gamma_2\right).$$

Having an even number of Majorana fermions, $2n$, they obey the anitcommutation relation

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \qquad i = 1, 2, \ldots, 2n. \tag{1}$$

Mathematically, the superconductor imposes electron hole "symmetry" on the quasiparticle excitations, relating the creation operator $\gamma(E)$ at energy E to the annihilation operator $\gamma^\dagger(-E)$ at energy $-E$. Majorana fermions can be bound to a defect at zero energy, and then the combined objects are called Majorana bound states or Majorana zero modes. This name is more appropriate than Majorana fermion (although the distinction is not always made in the literature), because the statistics of these objects is no longer fermionic. Instead, the Majorana bound states are an example of non-abelian anyons: interchanging them changes the state of the system in a way that depends only on the order in which the exchange was performed. The non-abelian statistics that Majorana bound states possess allows them to be used as a building block for a topological quantum computer.

3 Non-abelian Theroy

Any Hermitian operator A can be written as unitary operator in form $U = e^{i\beta A}$ to some angle β. For a Majorana fermions operator, we can start from their parity operator, which is Hermitian, $P_{nm} = i\gamma_n\gamma_m$ So we can have

$$U \equiv e^{\beta\gamma_n\gamma_m} \quad \text{or} \quad U = \cos\beta + \gamma_n\gamma_m \sin\beta \tag{2}$$

We'll try to find a unitary operator that evolve Majorana fermion as following, in Heisenberg picture,

$$\gamma_n \to U\gamma_n U^\dagger \tag{3}$$

$$\gamma_m \to U\gamma_m U^\dagger \tag{4}$$

Putting Equation 2 in Equation 4 we get

$$\gamma_n \to \cos 2\beta\gamma_n - \sin 2\beta\gamma_m, \tag{5}$$

$$\gamma_n \to \cos 2\beta\gamma_m - \sin 2\beta\gamma_n. \tag{6}$$

Then we should have $\beta = \pm\pi/4$. The we get braiding unitary operator of γ_n & γ_m

$$U = \exp\left(\pm\frac{\pi}{4}\gamma_n\gamma_m\right) = \frac{1}{\sqrt{2}}\left(1 \pm \gamma_n\gamma_m\right) \tag{7}$$

4 Majorana Zero Modes

Assume we have the configuration of Majorana fermions shown in Figure 1. The idea now is to do measurement only operations on the Majorana modes' configuration that will realize the same result as if we braid the two Majorana fermions.[1]

5 As Series of Measurement

As a demostration for the idea, we'll start with a system of 4 Majorana fermions, corresponding to two fermions. The configuration of Majorana fermions is shown in fig. The true braiding operator between γ_0 and γ_3 is given by

$$U = e^{\frac{\pi}{4}\gamma_0\gamma_3} \tag{8}$$

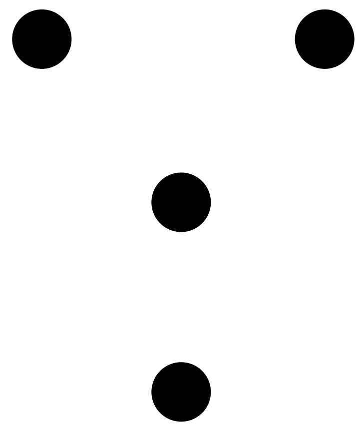

Figure 1: Four Majorana fermions configuration.

Then, to relize this braiding operator as just series of measurement we do this in four steps:

1. $(1 + i\gamma_1\gamma_2)$
2. $(1 + i\gamma_1\gamma_0)$
3. $(1 + i\gamma_3\gamma_1)$
4. $(1 + i\gamma_1\gamma_2)$

6 Jordan-Wigner Transformation

We shall redefine the our γ s in term of fermionic spin operators, giving us a way to model this system in much more fimiliar systems, such as qubits in quantum computing information. So we'll have:

- $\gamma_0 = Z_0$
- $\gamma_1 = X_0Z_1$
- $\gamma_2 = X_0Y_1$
- $\gamma_3 = Y_0$

Note: tensor product is understod, if there is one gate, tensor product with indentity of that subsystem is implicit. Then 4-step series of measurement on the system becomes

1. $(1 + iX_0Z_1X_0Y_1) = (1 + X_1)$
2. $(1 + iX_0Z_1Z_0) = (1 + Y_0Z_1)$
3. $(1 + iY_0X_0Z_1) = (1 + Z_0Z_1)$

4. $(1+X_1)$

Also for true braiding operator we get

$$e^{i\frac{\pi}{4}X_0} = \frac{1}{\sqrt{2}}(1+iX_0) \quad \text{or} \quad e^{-i\frac{\pi}{4}X_0} = \frac{1}{\sqrt{2}}(1-iX_0) \tag{9}$$

7 Applying all Measurements

Let's understand the possible outcomes from the general case of the measurement operator, that is,

$$(1+S_3X_1)(1+S_2Z_0Z_1)(1+S_1Y_0Z_1)(1+S_0X_1) \tag{10}$$

Expanding the middle two factors as

$$(1+S_3X_1)(1+S_2Z_0Z_1+S_1Y_0Z_1+S_2S_1Z_0Z_1Y_0Z_1)(1+S_0X_1) \tag{11}$$

Utilizing the Pauli gates anitcommutation relations, we move the LHS factor to RHS, as for first term we get

$$(1+S_3X_1)(1+S_0X_1) = \delta_{S_0,S_3}\,(1+S_0X_1)$$

For second term,

$$(1+S_3X_1)S_2Z_0Z_1(1+S_0X_1) = \delta_{S_0,-S_3}\,S_2Z_0Z_1(1+S_0X_1)$$

For the 3rd term,

$$(1+S_3X_1)S_1Y_0Z_1(1+S_0X_1) = \delta_{S_0,-S_3}\,S_1Y_0Z_1(1+S_0X_1)$$

For the 3rd term,

$$(1+S_3X_1)S_2S_1Z_0Z_1Y_0Z_1(1+S_0X_1) = \delta_{S_0,S_3}\, -iX_0S_2S_1(1+S_0X_1)$$

8 Constructing Protocol

Now, we'll investigate the protocol classic outcomes, then we shall decide based on it whether we did relize a braiding between γ_0 & γ_3, if not, what operators to apply to fix it. From Section 7, we simplify it to

$$[\delta_{S_0,S_3} + \delta_{S_0,-S_3}\,S_2Z_0Z_1 + \delta_{S_0,-S_3}\,S_1Y_0Z_1 + \delta_{S_0,S_3}\, -iX_0S_2S_1](1+S_0X_1)$$

Let's study different cases:

Case 1: $S_0 = S_3$ We get

$$[1-iX_0S_2S_1](1+S_0X_1)$$

Note, the right factor just acts on subsystem 1 that we don't care about it outcomes.

Case 1.1: $S_1 = -S_2$

$$[1+iX_0](1+S_0X_1)$$

relizing counterclockwise braiding operator in Equation 9.

Case 1.2: $S_1 = S_2$

$$[1 - iX_0](1 + S_0X_1)$$

relizing clockwise braiding operator in Equation 9.

Case 2: $S_0 \neq S_3$ We get

$$[S_2Z_0Z_1 + S_1Y_0Z_1]\,(1 + S_0X_1)$$

let's factor out Z_0Z_1

$$Z_0Z_1[S_2 - iS_1X_0]\,(1 + S_0X_1)$$

In this case we always want to multiply by Z_0, then we'll have

Case 2.1: $S_1 = S_2$

$$S_1Z_0Z_1[1 - iX_0](1 + S_0X_1)$$

relizing the inverse braiding operator

Case 2.2: $S_1 = -S_2$

$$S_1Z_0Z_1[S_1S_2 - iX_0](1 + S_0X_1)$$

But $S_1S_2 = -1$, then

$$-S_1Z_0Z_1[1 + iX_0](1 + S_0X_1)$$

relizing the braiding operator

9 Statevector and Density Matrix

Assume you have a state vector $|\psi\rangle$, which is a column vector with complex numbers:

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} \tag{12}$$

Here, n is the dimensionality of the quantum system. Take the conjugate transpose of the state vector, denoted by $\langle\psi|$. This is obtained by taking the complex conjugate of each element and then transposing the vector:

$$\langle\psi| = \begin{bmatrix} \psi_1^* & \psi_2^* & \dots & \psi_n^* \end{bmatrix}^T \tag{13}$$

Compute the outer product to obtain the density matrix ρ:

$$\rho = |\psi\rangle\langle\psi| = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} \begin{bmatrix} \psi_1^* & \psi_2^* & \dots & \psi_n^* \end{bmatrix} \tag{14}$$

The result is a square matrix of size $n \times n$, representing the density matrix. The density matrix ρ is Hermitian (equal to its conjugate transpose) and positive semi-definite. If the state vector $|\psi\rangle$ is normalized (has a magnitude of 1), then the trace of the density matrix is equal to 1, which is a requirement for a valid density matrix representing a physical state.

Pure State:

For a pure state density matrix ρ, which satisfies $\rho^2 = \rho$ (idempotent), you can find the state vector $|\psi\rangle$ by:

$$|\psi\rangle = \sqrt{\lambda} \cdot |\phi\rangle$$

Here, λ is the non-zero eigenvalue of ρ, and $|\phi\rangle$ is the corresponding eigenvector.

Mixed State: For a mixed state density matrix ρ with multiple non-zero eigenvalues, it is not possible to uniquely determine a single state vector. The system is in a statistical mixture of pure states. However, you can find a set of state vectors and their corresponding probabilities:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Here, $|\psi_i\rangle$ are the eigenvectors of ρ, and p_i are the corresponding eigenvalues. The state vectors represent the pure states in the mixture, and the probabilities p_i give the weight of each pure state in the mixture.

Implementation: To implement this in practice, you can use a numerical linear algebra library (such as NumPy for Python) to compute the eigenvalues and eigenvectors of the density matrix. Depending on the specific form of the density matrix, you might need to use different methods. Keep in mind that this process may not be unique, especially for mixed states, where different sets of pure states and probabilities can lead to the same density matrix.

References

[1] Martin Leijnse and Karsten Flensberg. Introduction to topological superconductivity and majorana fermions. *Semiconductor Science and Technology*, 27(12):124003, November 2012.