PHYS210

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Contents

Chapter 1

Series

1.1 Infinite Series

- A *sequence* is simply a set of quantities, one for each n, represented with a_n.
- A *series* is an indicated sum of such represented with $'\sum'$ symbol.
- In general **infinite series** can be written as $\Rightarrow a_1 + a_2 + a_3 + \cdots + a_n + \ldots$
- The geomtric series are written as $\Rightarrow a + ar + ar^2 + \cdots + ar^{n-1} + \ldots$
- or in short notation: $\Rightarrow \sum_{n=1}^{\infty} a_n$
- The sum first of first: n term is $S_n = \frac{a(1-r^n)}{1-r}$

  ★ The sum of an infinite series is the limit of the sum of n terms as $n \to \infty$. The *sum of series* is $S = \lim_{n\to\infty} S_n = \frac{a}{1-r}$

  ★ The geomtric series have a finite sum if and only if $|r| < 1$.
  Then the series is called *convergent*, Otherwise it is called *divergent*.

- the *remainder* (or the remainder after n terms) is $R_n = S - S_n$.
  Thus, $\lim_{n\to\infty} R_n = 0$.

1.2 Series Tests

- If the terms of an infinite series do not tend to zero (that is, if a $\lim_{n\to\infty}$), the series diverges. If $\lim_{n\to\infty} a_n = 0$, we must test further (Except the alternating series).
- An alternating series is an example of a *conditional series*.
- Its positive or negative terms (alone) *diverges*, thus we can control to which number (the sum, S) such series approaches.
- Physically, We cannot stop at some point and say that the rest of the series is negligible as we could in the bouncing ball problem in Section 1.
- But if we specify the order in which the charges are to be placed, then the sum S of the series is determined (S is probably different from F in (8.1) unless the charges are placed alternately).

★ The convergence or divergence of a series is not affected by multiplying every term of the series by the same nonzero constant. Neither is it affected by changing a finite number of terms (for example, omitting the first few terms).

★ Two *convergent* series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ may be added (or subtracted) term by term $(a_n + b_n)$. The resulting series is *convergent*, and its sum is obtained by adding (subtracting) the sums of the two given series.

★ The terms of an *absolutely convergent series* may be rearranged in any order without affecting either the *convergence* or the *sum*. This is not true of *conditionally convergent series* as we have seen in Section 8.

1.3 Power Series

It is distiguished by having a varible x^n in its terms multiply by a constant.
By definition:

$$\sum_{n=0}^{\infty} a_n(x-a) = a_0 + a_1(x-a)^1 + a_2(x-a)^2 + \dots \tag{1.1}$$

.

- The *radius of convergence* R of Eq:1.1 depends on x values .
- We find R by the *ratio test* so that $L < 1$.

★ We see then that a power series (within its interval of convergence) defines a function of x, namely $S(x)$.

Theorems

1. A power series may be differentiated or integrated term by term; the resulting series converges to the derivative or integral of the function represented by the original series *within* the same interval of convergence as the original series (that is, not necessarily at the endpoints of the interval).
2. Two power series may be added, subtracted, or multiplied; the resultant series converges at least in the common interval of convergence. You may divide two series if the denominator series is not zero at $x = 0$, or if it is and the zero is canceled by the numerator [as, for example, in $(\sin x)/x$; see (13.1)]. The resulting series will have *some* interval of convergence (which can be found by the ratio test or more simply by complex variable theory—see Chapter 2, Section 7).
3. One series may be substituted in another provided that the values of the substituted series are in the interval of convergence of the other series.
4. The power series of a function is unique, that is, there is just one power series of the form $\sum_{n=0}^{\infty} a_n x^n$ which converges to a given function.

★ Expanding functions can be done using *Taylor series*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1.2}$$

- *Maclaurin series* for $f(x)$ is a special case of Eq(1.2) where $a = 0$.

Chapter 2

Complex Numbers

2.1 Introduction

The *imaginary* number is written as $i = \sqrt{-1}$. We use the term *complex number* to mean any one of the whole set of numbers, *real*, *imaginary*, or combinations of the two

- The complex number can araise from the quadratic equation
$$az^2 + bz + c = 0 \tag{2.1}$$
- Where z is a unknown variable and its solution is
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.2}$$
if the *discriminant* $d = b^2 - 4ac < 0$, z will be a complex number.
- Notice the pattern if i to power of some numbers
$$i^2 = -1,\ i^3 = -i,\ i^{4n} = 1 \tag{2.3}$$
- A *complex* number such $3 + 5i$ has two parts a *real part* (here, 3) and an *imaginary part* (5).

2.2 The Complex Plane

- The rectangular coordinates representation for a complex number in the form $x + yi$ is (x, y).
- In the polar coordinates recall that
$$x = r\cos\theta, \tag{2.4}$$
$$y = r\sin\theta \tag{2.5}$$
- Then we have (by euler formula) assigned to z. All θ are in *radian*.
$$z = x + yi = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{2.6}$$
- The *modulus* or *absolute value* of z is
$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \tag{2.7}$$

2.3 Complex Algebra

- Note that in Eq:(2.7) ≥ 0, always real.
- The *conjugate* of z is found by substituting $\theta = -\theta$ in Eq:(2.6)
$$\bar{z} = r(\cos\theta - i\sin\theta) = re^{-i\theta} \tag{2.8}$$
- Inotherwords,anyequation involving complex numbers is really two equations involving real numbers.

Complex Equations In other words, any equation involving complex numbers is really two equations involving real numbers.

2.4 Complex Infinite Series

- The partial sums of a series of complex numbers will be complex numbers.
- It can be written as $S_n = X_n + Y_n i$ where X_n and Y_n are real.
- *Convergence* is defined just as real series: if S_n approaches a limit $(S = X + Yi)$ as $n \to \infty$, we call the series convergent and call S its sum.
- This means that $X_n \to X$ and $Y_n \to Y$.
- It can be proved that an *absolutely convergent series* converges, recall that Eq:(2.7) is positive term.
- $\sum_{n=0}^{\infty} z^n$ ia a geomtric series, with ratio $= z$ and convergent when $|z| < 1$.

★ Thus any of the tests given in Chapter:1 for convergence of series of positive terms may be used here to test a *complex series* for *absolute convergence*.

2.5 Complex Power Series

They are in form of

$$\sum a_n z^n \tag{2.9}$$

where $z = x + yi$ and a_n are complex numbers.

- Note that Eq:(2.9) includes the *real series* as a special case when $y = 0$.

2.6 Euler's Formula

To derive Euler's Formula, for real θ we have from Chapter:1

$$\sin\theta = \dots \tag{2.10}$$

$$\cos\theta = \dots \tag{2.11}$$

thus we have

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.12}$$

2.7 Powers and Roots of Complex Numbers

- The nth power (and nth root) of z is given by

$$z^n = (re^{i\theta})^n = r^n e^{ni\theta} \tag{2.13}$$

- When $r = 1$ Eq:(2.13) becomes **DeMoivre's theorem**

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{2.14}$$

2.8 The Exponential and The Trigonometric

- Although we have already defined e^z by a power series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, it is worth while to write it in another form.

$$e^z = e^{x+yi} = e^x e^{yi} = e^x(\cos y + i \sin y) \tag{2.15}$$

- We have already seen that there is a close relationship [Euler's formula (2.12)] between complex exponentials and trigonometric functions of real angles. It is useful to write this relation in another form.

- from Eq(2.12) we can rearrange it to

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{2.16}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{2.17}$$

- It can be shown that in Eq(2.16) θ can be z a complex number.

- The rest of the trigonometric functions of z are defined in the usual way in terms of these.

  ★ If z is a complex number, $\sin z$ and $\cos z$ can have *any* value we like.

2.9 Hyperbolic Functions

- Let us look at $\sin z$ and $\cos z$ for pure imaginary z in Eq(2.16), that is, $z = iy$:

$$\sin yi = \frac{e^{-y} - e^{y}}{2i} = i\frac{e^{y} - e^{-y}}{2} \tag{2.18}$$

$$\cos yi = \frac{e^{-y} + e^{y}}{2} = \frac{e^{y} + e^{-y}}{2} \tag{2.19}$$

- The real functions on the right have special names because these particular combinations of exponentials arise frequently in problems. They are called the hyperbolic. Their definitions for all z:

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \tag{2.20}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \tag{2.21}$$

- The other hyperbolic functions are definded as in normal trig. functions.

- Thus from Eq(2.18) we have:

$$\sin yi = i \sinh y \tag{2.22}$$

$$\cos yi = \cosh y \tag{2.23}$$

Chapter 3

Linear Algebra

3.1 Introduction

- problems in various fields of science and mathematics involve the solution of sets of linear equations. Suppose you have solved two simultaneous linear equations and have found $x = 2$ and $y = -3$. We can think of $x = 2$, $y = -3$ as the point $(2, -3)$ in the (x, y) plane. Since two linear equations represent two straight lines, the *solution* is then the *point* of intersection of the lines.
- The language of vectors is very useful in studying sets of simultaneous equations. *Quantities* such as the velocity of an object, which have both *magnitude* and *direction*. Such quantities are called **vectors**; contrast them with such quantities as mass, which have *magnitude* only and are called **scalars**.
- Vector formulas are independent of the choice of coordinate system.
- A vector equation in two dimensions is equivalent to two component equations.

3.2 Matrices: Row Reduction

- A matrix (plural: matrices) is just a rectangular array of quantities, such as

$$A = \begin{bmatrix} 1 & 5 & -2 \\ -3 & 0 & 6 \end{bmatrix} \tag{3.1}$$

- The A letter does not have a numerical value; it simply stands for the array.
- To indicate a number in the array, we will write A_{ij} where i is the *row* number and j is the *column* number.
- We will call a matrix with m rows and n columns as m *by* n *matrix*.
- **Transpose of a Matrix** We write:

$$A = \begin{bmatrix} 1 & -3 \\ 5 & 0 \\ -2 & 6 \end{bmatrix} \tag{3.2}$$

  and call A^T the transpose of the matrix A in Matrix(3.1).
- To transpose a matrix, we simply write the rows as columns.
- Not that, $(A^T)_{ij} = A_{ji}$
- Consider the set of equations of:

$$\begin{cases} 2x - z = 2 \\ 6x + 5y + 3z = 7 \\ 2x - y = 4 \end{cases} \tag{3.3}$$

- Let's agree in organizing them in *standard form*, each column be for a specific variable.
- There are three markable matricies to extract from (3.3); first one is the *matrix of the coefficients*

$$M = \begin{bmatrix} 2 & 0 & -1 \\ 6 & 5 & 3 \\ 2 & -1 & 0 \end{bmatrix} \tag{3.4}$$

Also, there are 3×1 matrices, r and k.

$$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad k = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \tag{3.5}$$

- The Eqs(3.3) can be written in matrix as $Mr = k$.
- Now we want to write Eqs(3.3) in an *augmented matrix*

$$A = \begin{bmatrix} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{bmatrix} \tag{3.6}$$

and then solve it by the *row reduction* with the following *elementary row operations*:

1. Interchange two rows.
2. Multiply (or divide) a row by a (nonzero) constant.
3. Add a multiple of one row to another; this includes subtracting, that is, using a negative multiple.

(3.7)

- If the last row in a rediced augmented matrix is like $0 \times z = 5$, which cannot be true for any finite number on z, then it is called *inconsistent.*
- **Rank of a Matrix** The number of nonzero rows remaining when a matrix has been row reduced is called the rank of the matrix.

1. If $(\text{rank}\, M) < (\text{rank}\, A)$, the equations are inconsistent and there is no solution.
2. If $(\text{rank}\, M) = (\text{rank}\, A) = n$(# of unknowns), there is one solution.
3. If $(\text{rank}\, M) = (\text{rank}\, A) = R < n$, then R unknowns can be found in terms of the remaining $n - R$ unknowns.

(3.8)

3.3 Determinants; Carmer's Rule

For a square matrix, however, there is a useful number called the determinant of the matrix.

- For a 2×2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad det\, A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{3.9}$$

- If we remove one row and one column from a determinant of order n, we have a determinant of order $n - 1$.
- When removing the *row* and *column* containing the element a_{ij} and call the remaining determinant M_{ij}, which is called the **minor** of a_{ij}.
- The **cofactor** of a_{ij} is:

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{3.10}$$

- In general, for $n \times n$ matrix we have,

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}, \quad \text{where } j \text{ is fixed} \tag{3.11}$$

- The signs goes like $\begin{vmatrix} + & - & \\ - & + & \\ & & \ddots \end{vmatrix}$.

★ The value of a *determinant*: Multiply each element of one row (or one column) by its cofactor and add the results.

Useful Fact About Determinants:

1. If each element of one row (or one column) of a determinant is multiplied by a number k, the value of the determinant is multiplied by k.
2. The value of a determinant is zero if
   (a) all elements of one row (or column) are zero; or if
   (b) two rows (or two columns) are identical; or if
   (c) two rows (or two columns) are proportional.
3. If two rows (or two columns) of a determinant are interchanged, the value of the determinant changes sign.
4. The value of a determinant is unchanged if
   (a) rows are written as columns and columns as rows; or if
   (b) we add to each element of one row, k times the corresponding element of another row, where k is any number (and a similar statement for columns).

(3.12)

Carmer's Rule This is a formula in terms of determinants for the solution of n linear equations in n unknowns when there is exactly one solution.
Let's start with the following equations:

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases} \tag{3.13}$$

If we multiply the first equation by b_2, the second by b_1, and then subtract the results and solve for x, we get if $(a_1 b_2 - a_2 b_1 \neq 0)$

$$x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1}, \quad y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} \tag{3.14}$$

Using the definition (3.9) we can write Eq.(3.14)

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{1}{D} \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{1}{D} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad D \neq 0 \tag{3.15}$$

★ To remember it, the denominator for both x and y is the *determinant of the coefficients*,for the numerator for x replace its column with the k matrix (constants column), and do so for y.

This method of solution of a set of linear equations is called Cramer's rule. It may be used to solve n equations in n unknowns if $D \neq 0$; the solution then consists of one value for each unknown. The denominator determinant D is the n by n determinant of the *coefficients* when the equations are arranged in standard form. The numerator determinant for each unknown is the determinant obtained by replacing the column of coefficients of that unknown in D by the constant terms from the right-hand sides of the equations. Then to find the unknowns, we must evaluate each of the determinants and divide. (3.16)

Rank of Matrix Here is another way to find the rank of a matrix. A submatrix means a matrix remaining if we remove some rows and/or remove some columns from the original matrix. To find the rank of a matrix, we look at all the square submatrices and find their determinants. The *order of the largest nonzero determinant* is the rank of the matrix.

3.4 Vectors

Notation We shall indicate a vector by a boldface letter (for example, **A**) and a component of a vector by a subscript (for example A_x is the 4 component of **A**). For handwriting you should write a vector with an arrow (like, $\vec{A}$).

Magnitude of a Vector The *length* of the arrow representing a vector **A** is called the *length* or the *magnitude* of **A** (written $|\mathbf{A}|$ or A) or the *norm* of **A** (written $||\mathbf{A}||$). Note the use of A to mean the magnitude of **A**.

- By the Pythagorean theorem, we find

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{3.17}$$

Addition of Vectors There two ways to add vectors: Addition of Vectors by the parallelogram law: To find $\vec{A} + \vec{B}$, place the tail of $\vec{B}$ at the head of $\vec{A}$ and draw the vector from the tail of $\vec{A}$ to the head of $\vec{B}$. Or by adding their components togather, like $A_x + B_x$ and $A_y + B_y$. They follow:

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}, \quad \text{(commutative law for addition);} \tag{3.18}$$

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}), \quad \text{(associative law for addition).} \tag{3.19}$$

- In other words, vectors may be added together by the usual laws of algebra. They can be multiplied by a number. Ane we can define vectors substraction so:

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

- The **zero vector** is a vector of zero magnitude; its components are all zero and it does not have a direction.

A vector of length or magnitude 1 is called a **unit vector**. Then for any $\vec{A} \neq 0$, the vector $\vec{A}/|\vec{A}|$ is a unit vector.

Vectors in Terms of Components We consider a set of rectangular axes as in Fig.(3.1) let $\hat{\imath}$ be a unit vector in the positive x direction; $\hat{\jmath}$ and $\hat{k}$ in y and z positive direction. For $\vec{A}$

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \tag{3.20}$$

The vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ are called *unit basis vectors*.

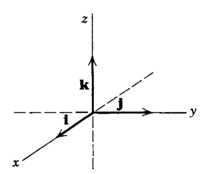

Figure 3.1: The unit basis vectors in a rectangular system.

Multiplication of Vectors There are two kinds of product of two vectors. One, called the scalar product (or **dot product** or **inner product**), gives a result which is a scalar; the other, called the *vector product* (or **cross product**), gives a vector answer.

Scalar Product For vectors $\vec{A}$, and $\vec{B}$, and the angle $\theta(\leq 180^\circ)$ between them, we have:

$$\vec{A}\cdot\vec{B} = (\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z)\cdot(\hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z)$$

$$\boxed{\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = A_xB_x + A_yB_y + A_zB_z} \tag{3.21}$$

The dot product in Eq.(3.21) holds the commutative law.
And the distributive law.

- A vector dot itself gives its magnitude:

$$\vec{A}\cdot\vec{A} = |\vec{A}|^2\cos 0 = |A|^2 = A^2$$

Perpendicular and Parallel Vectors If two vectors are perpendicular, then $\cos\theta = 0$; thus

$$\begin{aligned} &\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = 0, && \text{if } \vec{A} \text{ and } \vec{B} \text{ are perpendicular vectors.}\\ &\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}, && \text{if } \vec{A} \text{ and } \vec{B} \text{ are perpendicular vectors.}\end{aligned} \tag{3.22}$$

Vector Product The vector or cross product of $\vec{A}$ and $\vec{B}$ is written $\vec{A}\times\vec{B}$. By definition, $\vec{A}\times\vec{B}$ is a vector whose magnitude and direction are given as follows:

$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta, \tag{3.23}$$

The direction of $\vec{A}\times\vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$, using *right-hand rule*

Intresting properties:

$$\begin{aligned} &\vec{A}\times\vec{B} = -\vec{B}\times\vec{A} && \text{not commutative,}\\ &\vec{A}\times\vec{B} = 0 && \text{if } \vec{A} \text{ and } \vec{B} \text{ are parallel or antiparallel,}\\ &\vec{A}\times\vec{A} = 0 && \text{for any } \vec{A}\end{aligned} \tag{3.24}$$

★ A good way to remember the basis-unit vectors cross-product is to write them cyclically. Reading around the circle counterclockwise (positive θ direction), we get the positive products (for example, $\hat{\imath}\times\hat{\jmath} = \hat{k}$); reading the other way we get the negative products (for example, $\hat{\imath}\times\hat{k} = -\hat{\jmath}$). This works with the *right-haned systems.*

To write A $\times$ B in component form we need the distributive law, namely

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{c} \tag{3.25}$$

Thus we have:

$$\begin{aligned}\vec{A}\times\vec{B} &= (\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z)\times(\hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z)\\ &= \hat{\imath}(A_yB_z - A_zB_y) + \hat{\jmath}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)\\ &= \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\end{aligned} \tag{3.26}$$

3.5 Lines and Planes

In analytic geometry a point is a set of three coordinates (x, y, z); we shall think of this point as the *head* of a vector $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$ with *tail* at the `origin`. In two dimensions, we write the equation of a *straight line* through (x_0, y_0) with slope m as

$$\frac{y - y_0}{x - x_0} = m \tag{3.27}$$

• Suppose, instead of the slope, we are given a vector in the direction of the line, say $\vec{A} = \hat{\imath}a + \hat{\jmath}b$. Then the line through (x_0, y_0) and parallel to $\vec{A}$ we can write its equation. If we have two points on the line from (x_0, y_0) to any point (x, y), then the vector $\vec{r} - \vec{r_0}$ with components $x - x_0$ and $y - y_0$:

$$\vec{r} - \vec{r_0} = (x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} \tag{3.28}$$

Since this vector is parallel to $\vec{A}$, then their components are proportional, so (for $a, b \neq 0$):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \qquad \textit{or} \quad \frac{y - y_0}{x - x_0} = \frac{b}{a} \tag{3.29}$$

The equation is for a given **line**, see tht Eq.(3.27) and Eq.(3.29) are identical.

• Since $\vec{r} - \vec{r_0}$ and $\vec{A}$ are parallel, thus they are only differen by a factor of t, so

$$\vec{r} - \vec{r_0} = \vec{A}t \quad \textit{or} \quad \vec{r} = \vec{r_0} + \vec{A}t \tag{3.30}$$

Then theier *components form*:

$$\begin{matrix} x - x_0 = at \\ y - y_0 = bt \end{matrix} \quad \textit{or} \quad \begin{matrix} x = x_0 + at \\ y = y_0 + bt \end{matrix} \tag{3.31}$$

Eliminating t yields the equation of the line in Eq.(3.29)

• In three dimensions, we want the equations of a straight line through a given point (x_0, y_0, z_0) and parallel to a given vector $\vec{A} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$. If (x, y, z) is any point on the line, the vector joining (x_0, y_0, z_0) and (x, y, z) is parallel to $\vec{A}$. Then,

$$\begin{array}{ll} \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c} & a, b, c \neq 0 \\ \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b}, z = z_0 & \text{if say } c = 0 \end{array} \tag{3.32}$$

The paramtric equations for a line, see Fig.(3.5), from Eq.(3.32)

$$\vec{r} = \vec{r_0} + \vec{A}t \quad \textit{or} \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \tag{3.33}$$

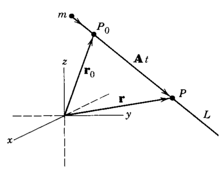

Figure 3.2:

• Suppose we want the equation of a straight line L through the point (x_0, y_0) and `perpendicular` to a given vector $\vec{N} = a\hat{\imath} + b\hat{\jmath}$. Let the vector in Eq.(3.28) lies along the line but to be perpendicular to $\vec{N}$. *Recall* from (3.22), we have $(\vec{r} - \vec{r_0}) \cdot \vec{N} = 0$, thus their components

$$a(x - x_0) + b(y - y_0) = 0 \quad \textit{or} \quad \frac{y - y_0}{x - x_0} = -\frac{a}{b} \tag{3.34}$$

This is the equation for a line that is perpendicular to $\vec{N}$

• In 3D, we can use this method to assign an equation for a plne. Suppose that two points on the plane, $(x_{0,0}, z_0)$ and (x, y, z) is any point and represented by the vector in (3.28).
If $\vec{N} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ is a normal vector to the plane, we have $(\vec{r} - \vec{r_0}) \cdot \vec{N} = 0$, so

$$\begin{aligned} &a(x-x_0) + b(y-y_0) + c(z-z_0) = 0, \\ &\qquad ax + by + cz = d \quad where \quad d = ax_0 + by_0 + cz_0 \end{aligned} \tag{3.35}$$

3.6 Matrix Operations

In Section 2 we used matrices simply as arrays of numbers. Now we want to go farther into the subject and discuss the meaning and use of multiplying a matrix by a number and of combining matrices by addition, subtraction, multiplication, and even (in a sense) division.

Matrix Equations Two matrices are only equal if they are identical, for example,

$$\begin{bmatrix} w & m \\ r & k \end{bmatrix} = \begin{bmatrix} 4 & 5i \\ 5 & 0 \end{bmatrix} \tag{3.36}$$

then $w = 4, \quad m = 5i, \quad r = 5, \quad k = 0.$
Remeber this consept in the equation $z = x + iy = 2 - 3i$ is equivalent to the two real equations $x = 2$, $y = -3$;
a vector equation in three dimensions is equivalent to three component equations.

Multiplication of a Matrix by a Number We can write a vector $\vec{A} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ in matrix-like way,

$$\begin{aligned} A &= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ a column matrix or column vector,} \\ A^T &= \begin{bmatrix} a & b & c \end{bmatrix} \text{ called a row matrix or row vector.} \end{aligned} \tag{3.37}$$

The row matrix A^T is the transpose of the column matrix A

• Suppose we multiply the vector $\vec{A}$ by a constant C, them $C\vec{A} = aC\hat{\imath} + bC\hat{\jmath} + cC\hat{k}$ and its matrix becomes

$$A = \begin{bmatrix} aC & bC & cC \end{bmatrix} \tag{3.38}$$

★ Thus when a matrix is multiplied by a number each element is multiplied by it.
Multiplying a matrix by a number k means multiplying every element by k,
But multiplying just one *row* of a determinant by k multiplies the determinant by k. Thus $\det kA = k^2 \det A$ for a 2 by 2 matrix.

Addition of Matrices When we add vectors algebraically, we add them by components. Matrices are added in the same way, but by adding *corresponding elements*. If the two matrices have different m by n, we say that the sum is undefined or meaningless.

Multiplication of Matrices Let us start by defining the product of two matrices,

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} = C \tag{3.39}$$

Each row for each column. :"row times column".

The element in row i and column j of the product matrix AB is equal to row i of A times column j of B. In index notation

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} \tag{3.40}$$

The product AB (in that order) can be found if and only if the number of elements in a *row* of A equals the number of elements in a *column* of B; the matrices A, B in that order are then called **conformable**.
(Observe that the number of rows in A and of columns in B have nothing to do with the question of whether we can find AB or not.)

$$[A, B] = AB - BA = \textit{commutator} \text{ of } A \text{ and } B. \tag{3.41}$$

Zero Matrix The **zero** or **null** matrix means one with all its elements equal to zero. It is often abbreviated by 0, but we must be careful about this

Identity Matrix or Unit Matrix This is a square matrix with every element of the main diagonal equal to 1 and all other elements equal to zero. For example

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.42}$$

In multiplication, a unit matrix acts like the number 1, that is, if A is any matrix and I is the *unit* matrix *conformable* with A in the order in which we multiply, then $IA = AI = A$.

Operations with Determinants we multiply determinants the same way we multiply matrices.

$$\det AB = \det BA = \det A \cdot \det B \tag{3.43}$$

Applications of Matrix Multiplication We can now write sets of simultaneous linear equations in a very simple form using matrices. Consider the matrix equation

$$\begin{bmatrix} 2 & 6 & -3 \\ 7 & 4 & 4 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \tag{3.44}$$

By matrices multiplication,

$$\begin{bmatrix} 2x + 6y - 3z \\ 7x + 4y + 4z \\ 2x + y + 8z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \tag{3.45}$$

Recall from (3.6), then we get

$$\begin{cases} 2x + 6y - 3z &= 5, \\ 7x + 4y + 4z &= 4, \\ 2x + y + 8z &= 2 \end{cases} \tag{3.46}$$

Consequently (3.44) is the matrix form for the set of equations in (3.46), if

$$M = \begin{bmatrix} 2x + 6y - 3z \\ 7x + 4y + 4z \\ 2x + y + 8z \end{bmatrix}, \quad r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad k = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \tag{3.47}$$

then we can write the (3.44) as $Mr = k$ or $\sum_j M_{ij} r_i = k_i$.

Inverse of a Matrix *Inverse of a Matrix* The reciprocal or inverse of a number x is x^{-1} such that the product $xx^{-1} = 1$. We define the inverse of a matrix M (if it has one) as the matrix M^{-1} such that MM^{-1} and $M^{-1}M$ are both equal to a unit matrix I.

- Note that only square matrices can have inverses (otherwise we could not multiply both MM^{-1} and $M^{-1}M$). Actually, some square matrices do not have inverses either.

You can see from (3.43) that if $M^{-1}M = I$, then $(\det M^{-1})(\det M) = \det I = 1$. If two numbers have product = 1, then neither of them is zero; thus $\det M \neq 0$ is a requirement for M to have an inverse.

- If a matrix has an inverse we say that it is **invertible**; if it doesn't have an inverse, it is called **singular**. We can find the inverse of a matrix as

$$M^{-1} = \frac{1}{\det M} C^T \tag{3.48}$$

3.7 Linear Combinations, Functions, and Opreators

- Given two vectors $\vec{A}$ and $\vec{B}$, the vector $3\vec{A} - 2\vec{B}$ is called a **linear combination** of $\vec{A}$ and $\vec{B}$. In general, a linear combination of $\vec{A}$ and $\vec{B}$ means $a\vec{A} + b\vec{B}$ where a and b are *scalars*. Geometrically, if $\vec{A}$ and $\vec{B}$ have the same *tail* and do not lie along a line, then they determine a plane. All linear combinations of $\vec{A}$ and $\vec{B}$ lie in the plane.

- It is also true that every vector in the plane can be written as a linear combination of $\vec{A}$ and $\vec{B}$. The vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ with tail at the origin (which we used in writing equations of lines and planes) is a linear combination of the unit basis vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

A function of a vector, $f(\vec{r})$ ia called linear if

$$f(\vec{r_1} + \vec{r_2}) = f(\vec{r_1}) + f(\vec{r_2}), \quad and \quad f(a\vec{r}) = af(\vec{r}) \tag{3.49}$$

where a is a scalar.

- Note, $f(\vec{r}) = |\vec{r}|$ is not a linear function, because the length of the sum of two vectors is not in general the sum of their lengths. That is,

$$f(\vec{r_1} + \vec{r_2}) = |\vec{r_1} + \vec{r_2}| \neq |\vec{r_1}| + |\vec{r_2}| = f(\vec{r_1}) + f(\vec{r_2})$$

as in Fig.(3.3). Also noticw that we call $y = ms + b$ a linear a equation, however the function $f(x) = mx + b$ is not linear by the test (3.49) (unless $b = 0$).

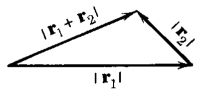

Figure 3.3:

- Now consider **vector function** of a vector $\vec{r}$

$\vec{F}(\vec{r})$ is a linear vector function if

$$\vec{F}(\vec{r_1} + \vec{r_2}) = \vec{F}(\vec{r_1}) + \vec{F}(\vec{r_2}) \quad and \quad \vec{F}(a\vec{r}) = a\vec{F}(\vec{r}) \tag{3.50}$$

where a is a scalar.

- Recall that from calculus:

$$\begin{aligned} \frac{d}{dx}\left[f(x) + g(x)\right] &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \quad and \\ \frac{s}{dx}\left[kf(x)\right] &= k\frac{d}{dx}f(x) \end{aligned} \tag{3.51}$$

where k is a constant

This is silimilar to (3.49), then we call d/dx a **linear opreator**. An **operator** or **operation** simply means a rule or some kind of instruction telling us what to do with whatever follows it. In other words, a *linear operator* is a *linear function*, so

O is a linear operator if

$$O(A + B) = O(A) + O(B) \qquad and \qquad O(kA) = kO(A) \tag{3.52}$$

where k is a number

Matrix Operators, Linear Transformations Consider the following equation set

$$\begin{cases} X = ax + by \\ Y = cx + dy \end{cases} \qquad or \qquad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad or \qquad R = Mr \tag{3.53}$$

where a, b, c, and d are constants. For every point (x, y), these equations results a point (X, Y).

If we think of each point of the (x, y) plane being moved to some other point (the origin not being moved), we can call this process a **mapping** or **transformation** of the plane into itself. All the information about this *transformation* is contained in the matrix M. We say that this matrix is an *operator* which maps the plane into itself.

- Any matrix can be thought of as an *operator* on (conformable) column matrices r. From (3.52), M is a linear operator.

- We can interpreted Eqs.(3.53) geometrically in two ways. In Fig.(3.4) The vector $\vec{r}$ has been changed to the vector $\vec{R}$ by the transformation (3.53).

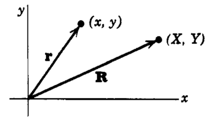

Figure 3.4: fixed coordinates axes

- However, in Fis.(3.5) `two` sets of coordinates axes (x, y) and (x', y'), and `one` vector $\vec{r} = \vec{r'}$. The transformation can ve written as

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \qquad or \qquad \begin{bmatrix} z' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad or \qquad r' = Mr \tag{3.54}$$

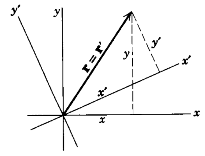

Figure 3.5: A fixed vector

Orthogonal Transformations This is a special case of a *linear transformation* which preserves the length of a vector. In (3.54) is an *orthogonal transformation* if

$$x'^2 + y'^2 = x^2 + y^2 \tag{3.55}$$

then either it is reflected or rotated. The matrix M of an orthogonal transformation is called an **orthogonal matrix**. The inverse of an orthogonal matrix equals its transpose.

$$M^{-1} = M^T, \quad M \text{ is orthogonal} \tag{3.56}$$

From Eq.(3.54) and (3.55)

$$\begin{aligned} x'^2 + y'^2 &= (ax + by)^2 + (cx + dy)^2 \\ &= (a^2 + c^2)x^2 + 2(ab + cd)xy + (b^2 + d^2)y^2 \equiv x^2 + y^2 \end{aligned}$$

Thus we must have $a^2 + c^2 = b^2 + d^2 = 1, \quad ab + cd = 0$

$$MM^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let Eq.(3.56) be $MM^T = I$ and from Eq.(3.43) we have

$$\begin{aligned} \det M^T M &= \det M^T \det M \\ and \quad det M^T &= \det M \\ so \quad \det M^2 &= \det M^T M = \det I = 1 \end{aligned}$$

$$\det M = \pm 1 \tag{3.57}$$

★ $\det M = 1$ corresponds geometrically to a *rotation*, and $\det M = -1$ means that a *reflection* is involved.

Rotations in 2 Dimensions In Fig.(3.6) we have two vectors $\vec{R}$ and $\vec{r}$ which is rotated by θ. we can write in matrix their transformation as

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3.58}$$

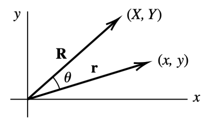

Figure 3.6: Vectors rotation with an angle θ

- However for axes rotation as in Fig.(3.7) we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3.59}$$

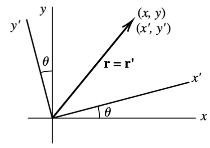

Figure 3.7: Axes rotation

- Both Eq.(3.58) and Eq.(3.59) are called **rotation equations** and the matrices that contain θ are called **rotation matrices**. Also notice that Eq.(3.58) and Eq.(3.59) are inverses of each other.

Rotations and Reflections in 3 Dimensions For a vector $\vec{r} = \langle x, y, z \rangle$. The the following matrix is a rotation about $z - axis$,

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.60}$$

And the following is a rotation and reflection about $xy - plane$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.61}$$

3.8 Linear Dependence and Independence

In general, a set of vectors is linearly dependent, if there are some linear combinations of them equals *zero* (other than all the coefficients equal zero). We can use row reduction to form linear combinations by *elementary row operations*. These operations are reversible, se we can combine the remaining vectors to form the original vectors. Thus the remaining vectors are *independent* and called as *basis vectors*.

Linear Independence of Functions Similar to vectors, the functiond $f_1(x)$, $f_2(x)$, $\ldots$, $f_n(x)$ are linearly dependent if there are some linear combinations of them is identically *zero*, meaning, if there are constants $k_1, k_2, \ldots, k_n$ (not all zero), such that,

$$k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) \equiv 0$$

For a given set of functiond we have the fillowing theorem

if $f_1(x)$, $f_2(x)$, $\ldots$, $f_n(x)$ have derivatives of order $n-1$ and if the determinant

$$W = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ f_1''(x) & f_2''(x) & \ldots & f_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{n-1}(x) & f_2^{n-1}(x) & \ldots & f_n^{n-1}(x) \end{vmatrix} \neq 0 \tag{3.62}$$

then the functions are linearly independent. the determinant of W is called *Wronskian* of the functions. If $W \equiv 0$, it is not necessarily imply "functions dependent".

Homogeneous Equations We consider a set of equations, but a special case raises when the all the right side of those equations are zero.

(3.63)

Homogeneous equations are never *inconsistent*; they always have the solution "all unkowns = 0" (called *trivial solution*). If the number of the independent *equations* (the rank of the matrix) is the same as the the number of the *unknowns*, this is the only solution. If the *rank* is less than the *unknowns*, there are infinitely many solutions.

- For n homogeneous equations in n unknowns, they have only a trivial solution, unless their rank is less than n. Meaning that, at least one row of the reduced n by n coefficients matrix is zero row. Then its determinant is zero. Thus we say

(3.64)

> A system of n homogeneous equations in n unknowns has solutions other than the trivial solution iff the determinant of the coefficients is zero

3.9 Special Matrices and Formulas

There are some importants special matrices. Consider a matrix A, so

Name of matrix	Notations	Procedure
Transponse of A.	A^T or $\tilde{A}$ or A'	Interchange rows and columns in A.
Complex conjugate of A.	$\bar{A}$ or A^*	The complex conjugate of each element in A.
Transponse conjugate, Hermitian conjugate, adjoint, Hermitian adjoint.	$A^\dagger$	The complex conjugate of each element in A^T.
Inverse of A	A^{-1}	See Par.(3.6)

Also the following are special properties,

A matrix is	if it satisfies
real	$A = \bar{A}$
symmetric	$A = A^T$, A real
skew-symmetric or antisymmetric	$A = -A^T$, A real
orthogonal	$A^{-1} = A^T$, A real
pure imaginary	$A = -\bar{A}$
Hermitian	$A = A^\dagger$
anti-Hermitian	$A = -A^\dagger$
unitary	$A^{-1} = A^\dagger$
normal	$AA^\dagger = A^\dagger A$

Kronecker σ is defined as

$$\sigma_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{3.65}$$

We can define the unit matrix as one whose elements $I = \sigma_{ij}$. We use Eq.(3.40) to prove matrix associative law for matrix, that is

$$A(BC) = (AB)C = ABC$$

First we write, $(BC)_{kj} = \sum_l B_{kl}C_{li}$ then we have

$$\begin{aligned}
[A(BC)]_{ij} &= \sum_k A_{ik}(BC)_{kj} = \sum_k A_{ik} \sum_l B_{kl}C_{lj} \\
&= \sum_k \sum_l A_{ik}B_{kl}C_{lj} = (ABC)_{ij}
\end{aligned}$$

which is the index notation of $A(BC) = ABC$.

Silimilarly, we can fins the transpose of the product ot two matrices. Recall that $A^T_{ik} = A_{ki}$.

$$\begin{aligned}(AB)^T_{ik} &= (AB)_{ki} = \sum_j A_{kj}B_{ji} = \sum_j A^T_{jk}B^T_{ij} \\ &= \sum_j B^T_{ij}A^T_{jk} = (B^T A^T)_{ik} \\ (AB)^T &= B^T A^T\end{aligned}$$

In general,

The transpose of a product of matrices is equal to the product of the transposes in reverse order. $$(ABCD)^T = D^T D^T B^T A^T \tag{3.66}$$

A similar theorem foe the inverse

The inverse of a product of matrices is equal to the product of the inverses in reverse order. $$(ABCD)^{-1} = D^{-1}D^{-1}B^{-1}A^{-1} \tag{3.67}$$

Trace of a Matrix The **trace** (or **spur**) of a matrix A is the sum of the elements in the main diagonal. The trace of a product of matrices is not changed by arrange them in cyclic order. Meaning that,

$$\text{Tr}\, ABC = \text{Tr}\, BCA = \text{Tr}\, CAB$$

proof,

$$\begin{aligned}\text{Tr}\, ABC = \sum_i (ABC)_{ii} &= \sum_i \sum_j \sum_k A_{ij}B_{jk}C_{ki} \\ &= \sum_i \sum_j \sum_k B_{jk}C_{ki}A_{ij} = \text{Tr}\, BCA \\ &= \sum_i \sum_j \sum_k C_{ki}A_{ij}B_{jk} = \text{Tr}\, CAB\end{aligned}$$

Heads up: In general $\text{Tr}\, ABC$ is not equal to $\text{Tr}\, ACB$. To memorize it, take the last matrix an insetrt it before the first one, and on.

3.10 Linear Vector Spaces

We dealt with vectors such that $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, that represents a vector from the origin to the point (x, y, z). The all possible sets of vectors make up a 3-dimensional space R_3 (R for real) or V_3 (V for vector) or E_3 (E for Euclidean). Generally, an ordered set of n points or vectors make up a n-dimensional space V_n. All the termonolgy are also aplies for n-D, such the distance and the orthogonality.

Subspace, Span, Basis, Dimension

- Two vectors in 3-dimensional space represent a plane, or a vector space V_2. All thier possible combinations lie on that plane. Thye are also a a part of V_3, we say that V_2 is a *subspace* of V_3.

- A set of vectors **spans** a space if all the vectors in the space be written as linear combinations of the spanning set. A *basis* vector is a set of *linearly independent* vectors which span a vector space.

- A *dimension* of a vector space is equal to the number of the basis vectors.

For example, the three unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ span the vector space V_3; they are linearly independent, thus a basis for the space V_3.

Inner Product, Norm, Orthogonality Let us generalize the the dot product in Eq(3.21) to n dimensions

$$\vec{A}\cdot\vec{B} = \sum_{i=1}^{n} A_i B_i \tag{3.68}$$

Similarly, to generalize the *norm* of $\vec{A}$

$$A = |\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{\sum_{i=1}^{n} A_i^2} \tag{3.69}$$

For orthogonality we have

$$\vec{A} \text{ and } \vec{B} \text{ are orthogonal if } \quad \sum_{i=0}^{n} A_i B_i = 0 \tag{3.70}$$

Schwarz Inequality We can also use the formula $\vec{A}\cdot\vec{B} = AB\cos\theta$ to find the angle between two vectors in n-dimensions. The result must satisfy $|\cos\theta| \leq 1$, thus

$$|\vec{A}\cdot\vec{B}| \leq AB, \qquad or \quad \left|\sum_{i=1}^{n} A_i B_i\right| \leq \sqrt{\sum_{i=1}^{n} A_i^2}\sqrt{\sum_{i=1}^{n} B_i^2} \tag{3.71}$$

Proof: if $\vec{B} = \vec{0}$, Eq.(3.71) is stisfied. If $\vec{B} \neq \vec{0}$, consider the vector $\vec{C} = B\vec{A} - (\vec{A}\cdot\vec{B})\vec{B}/B$. Then, $\vec{C}\cdot\vec{C} = \sum C_i^2 \geq 0$

$$\begin{aligned}
\vec{C}\cdot\vec{C} &= B^2(\vec{A}\cdot\vec{A}) - 2B(\vec{A}\cdot\vec{B})(\vec{A}\cdot\vec{B})/B + (\vec{A}\cdot\vec{B})^2(\vec{B}\cdot\vec{B})/B^2 \\
&= A^2B^2 - 2(\vec{A}\cdot\vec{B})^2 + (\vec{A}\cdot\vec{B})^2 \\
&= A^2B^2 - (\vec{A}\cdot\vec{B})^2 = C^2 \geq 0
\end{aligned}$$

Thus we can define the angle between two vectors as $\cos\theta = \vec{A}\cdot\vec{B}/AB$.

Orthonormal Basis; Gram-Schmidt Method We call a set of vectors **orthonormal** if they are all mutually *orthogonal* (perpendicular), and each vector is *normal*ized; $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ as an example.

- Consider we have three basis vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. First, normalize $\vec{A}$. Substract from $\vec{B}$ its projection along $\vec{A}$, then normalize it. And so on.

Complex Euclidean Space It is allowed to have a vector with components of comlex numbers. In Eq.(3.68) we need to make what under the square root positive. So we replace A_i^2 by $|A_i|^2 = A_i^* A_i$. likewise, in Eq.(3.69), Eq.(3.70), and Eq.(3.71).

$$\vec{A}\cdot\vec{B} = \sum_{i=1}^{n} A_i^* B_i$$

$$|\vec{A}| = \sqrt{\sum_{i=1}^{n} A_i^* A_i}$$

- In matrix form, suppose that A is a column matrix with elemnts A_i. Then its transpose conjugate $A^\dagger$ is a row matrix with elements A_i^*. We can write, $\sum A_i^* B_i = A^\dagger B$.

3.11 Eigenvalues and Eigenvectors; Diagonalizing Matrices

In some special matrix transformation, there are some specific vectors that do not move off of their span; such vectors are called **eigenvectors**. They satisfiy $\vec{R} = \lambda\vec{r}$, where λ is called the **eigenvalue**.

- **Eigenvalues** Consider the transformation of $\vec{R} = M\vec{r}$,

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3.72}$$

The eigenvector condition $\vec{R} = \lambda\vec{r}$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \tag{3.73}$$

Thus we get

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0 \qquad or \begin{cases} (a-\lambda)x + by = 0 \\ cx + (d-\lambda)y = 0 \end{cases} \tag{3.74}$$

These are homogeneous equations. Thus from (3.63), a set of homogeneous equations has a solution other than $x = y = 0$, iff the determinant of coefficients is zero.

$$\text{For the maatrix } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{the characteristic equation is} \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \tag{3.75}$$

- **Eigenvectors** By finding the the eigenvalue, λ, and substituting it back into Eq.(3.74), will result line equations. Any vector from the origin to a point along those lines is considered as an eigenvector, where they stay in thier span.

Diagonalizing a Matrix Again substitute the eigenvalues into Eq.(3.74), and distinguish the eigenvectors by subscripts,

$$\begin{aligned} ax_1 + by_1 &= \lambda_1 x_1 \quad & ax_2 + by_2 &= \lambda_2 x_2 \\ cx_1 + dy_1 &= \lambda_1 x_1 \quad & cx_2 + dy_2 &= \lambda_2 x_2 \end{aligned} \tag{3.76}$$

then by writing these equation as one matrix equation, $\boxed{MC = CD}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{3.77}$$

where (x_1, y_1) is a point along the found line obtained from λ_1; which is the eigenvector of the transformation. If $\det C \neq 0$, then C has an inverse C^{-1}, so

The matrix D has nonzero numbers only in its main diagonal, so it is called **diagonal matrix**. Also, the matrix D is called **similar** to M. When we obtain D given M, we *diagonalized* M by a *similarity tranformation*

$$C^{-1}MC = D \tag{3.78}$$

- The diagonalization process simplify problems by a better choice of variables. For example, it is simpler to to describe some of deformations by using axes along the eigenvectors. Thus geometrically,

- A complete example, consider two coordinate systems, (x, y) and (x', y'), and two vectors that are beign transform from $\vec{r} = (x, y)$ into $\vec{R} = (X, Y)$. The $\vec{r}$ and $\vec{r'}$ in the two systems are related to each other by Eq.(3.59), to find (z, y),

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned} \quad \Leftrightarrow \quad r = Cr', \quad C = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{3.79}$$

Similarly we have, $R = CR'$. Consider a transformation matrix M in (x, y) system, as

$$R = Mr \tag{3.80}$$

Now, we can get the transformation equation in (x', y') system by substituting in Eq.(3.80), so

$$R' = C^{-1}MCr' \Leftrightarrow R' = Dr' \tag{3.81}$$

Thus we conclude that

$D = C^{-1}MC$ is the transformation matrix described in the (x', y') system, where M describes the tranformation in the (x, y) system. (3.82)

★ If C is chosen to make D a diagonal matrix, then the new axes (x', y') are along the eigenvectors of M. If the eigenvectors are perpendicular, then the new axess (x', y') are a set of perpendicular axes rotated by θ from axes (x, y).

- However, if C is not an orthogonal matrix, then the axes (x', y') are not perpendicular and $|\vec{r}| \neq |\vec{r'}|$.

- Thus, if the unit eigenvectors in matrix C; are perpendicular, then C is *orthogonal.* This is only if the matrix M is *symmetric.*

Degeneracy means two (or more) independent eigenvectors correspond to the same eigenvalue. For a *symmetric* matrix, we saw that the eigenvectors corresponding to different eigenvalues are *orthogonal.*

Diagonalizing Hermitian Matrices We saw how to diagonalize symmetric matrices by *orthogonal similarity transformation.* Consider thid analogy

Real	Complex
Symmetric $S^T = S$	Hermitian $H^\dagger = H$
Orthogonal $O^T = O^{-1}$	Unitary $U^\dagger = U^{-1}$

★ The eigenvalues of a Hermitian matrix are always *real.*
Consider H to be a Hermitian matrix, and $\vec{r}$ the column matrix of a non-zero eigenvector of H corresponding to the eigenvalue λ. Thus the eigenvector condition is

$$Hr = \lambda r$$

Taking the transpose conjugate ($\dagger$) of this this equation, we get

$$(Hr)^\dagger = r^\dagger H^\dagger = r^\dagger H \qquad \text{and} \quad \lambda r \Rightarrow \lambda^* r^\dagger$$

For a Hermitian matrix we have $H^\dagger = H$, λ is just a number. Hence,

$$Hr = \lambda r \qquad \text{and} \qquad r^\dagger H = \lambda^* r^\dagger$$

Then multiply the left by the row matrix $r^\dagger$, and the right by the column matrix r,

$$r^\dagger H r = \lambda r^\dagger r \qquad \text{and} \qquad r^\dagger H r = \lambda^* r^\dagger r$$

Substracting the two equations we get $(\lambda - \lambda^*) r^\dagger r = 0$. But we assumed $r \neq 0$, so $\Lambda^* = \lambda$, hence the eigenvalue λ is always *real.*

- For a Hermitian matrix, the eigenvectors corresponding to two different eigenvalues are *orthogonal.* Consider this,

$$Hr_1 = \lambda_1 r_1 \qquad Hr_2 = \lambda_2 r_2$$

$$r_1^\dagger H r_2 = \lambda_1 r_1^\dagger r_2 = \lambda_2 r_1^\dagger r_2 \qquad \text{or} \quad (\lambda_1 - \lambda_2) r_1^\dagger r_2 = 0$$

If $\lambda_1 \neq \lambda_2$, then $r_1 \cdot r_2 = 0$, hence, they are *orthogonal.*

★ If a matrix M has a real eigenvalues and can be diagonalize by a **unitary similarity transformation**, then M is *Hermitian.* We state $\boxed{\text{MU = UD}}$

$$\begin{aligned} U^{-1} M U &= D \\ (U^{-1} M U)^\dagger &= U^{-1} M^\dagger U = D^\dagger = D \\ \text{thus} \quad U^{-1} M U &= U^{-1} M^\dagger U \end{aligned}$$

We get, $M = M^\dagger$, hence, M is Hermitian.

$$\text{A matrix has real eigenvalues and can be diagonalize by a } \textit{unitary similarity transformation} \text{ iff it is } \textbf{Hermitian}. \tag{3.83}$$

But a <u>real</u> Hermitian matrix is a symmetric matrix, and <u>real</u> unitary is a orthogonal matrix,

$$\text{A matrix has real eigenvalues and can be diagonalize by a } \textit{orthogonality similarity transformation} \text{ iff it is } \textbf{symmetric}. \tag{3.84}$$

The normal matrices include, symmetric, Hermitian, orthogonal, and unitary matrices, thus

$$\text{A matrix has real eigenvalues and can be diagonalize by a } \textit{unitary similarity transformation} \text{ iff it is } \textbf{normal}. \tag{3.85}$$

Powers and Functions of Matrices For the matrix M raised to the power n, we can use the diagonalization,

$$M^n = CD^nC^{-1} \qquad \text{where} \qquad D = C^{-1}MC$$

3.12 An Introduction to Groups

Consider $\pm 1, \pm i$. No matter what the products and powers of them we compute, we never get numbers other than these foure. This property of a set of elements with a law of combinations is called **closure**. We are interested in groups of matrices, that is, in matrix representaions os of groups.

A Group is a set $A, B, C, \ldots$ of elements –can be numbers, matrices, or operations– togather with a law of combinations of two elements, having four conditions

1. *Closure*: The combination of any two elements is an element of the group.
2. *Associative*: The of combination satisfies the associative law; $(AB)C = A(BC)$.
3. *Unit element*: There is a unit element I with the property that $AI = IA = A$ for every element of the group.
4. *Inverse*: Every element of the group has an inverse in the group; like, for A, there is an element B such that $AB = BAI$.

- Thus, the set $\pm 1, \pm i$ is valid under these conditions. The **order of a finite group** is the number of elements in the group. If the elements of a group of order n are of the form $A, A^2, A^3, \ldots, A^n = 1$, it is called a **cyclic group**.

- Hence $\pm 1, \pm i$ is a cyclic group of order 4. A **subgroup** is a subset which is itself a group. The whole group or the unit element, are called **trivial subgroup**, any other subgroup is called **proper subgroup**.Here $\pm 1, \pm i$, has the proper subgroup ± 1.

- A *group multiplication* refers to the law of combination for the group.

- Two gourps are called **isomorphic** if their multiplication tables are identical except for the names we attach to the elements.

- If evrey two group elements *commute*, the group is called **Abelian**.

- Two group elements A and B are called **conjugate** elements, if there is a a group element C such that $C^{-1}AC = B$. By making C be successively one *group element* after another, we can find all the *group conjugate* of A. This set is called a **class**.

Chapter 4

Vector Analusis

4.1 Theorems

Line Integral The line integral for $f(x,y)$; $x = x(t)$ & $y = y(t)$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))L\,dt$$

The integral of '$f(x,y)$' along 'C' with respect to 'x'. Or,

$$\int_C f(x,y)\,dx\,dy = \int_a^b f(x(t),y(t))x'(t)\,dt$$

where $x'(t)dt = dx$.

★ $\vec{r}(t) = (1-t)\vec{r_1} + \vec{r_2}$

Line integral of vector fields If $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, where Q, P, and R are scalra functions of (x,y,z).

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)dt = \int_C \vec{F}\cdot\vec{T}\,ds$$
$$\Rightarrow \int_C P\,dx + Q\,dy + R\,dz$$

where $\vec{T}$ is the unit tangent vector.

Fundamental theorem for line integral

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\int_C \nabla f\cdot d\vec{r} = \int_C df = \vec{r}(b) - \vec{r}(a)$$

which says that the line integral of ∇f is the net change in f. Note that $\oint \vec{F}\cdot d\vec{r} = 0$

Green's Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_R P\,dx + Q\,dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

let $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ to get the area A.

In vector form

$$\oint_R \vec{F}\cdot d\vec{r} = \int_D \nabla\vec{F}\cdot\hat{k}\,dA$$

As an special case of Stoke's theorem The integral of the tangential component of $\vec{F}$ <u>along</u> C is the the double integral of the vertical component of curl$\vec{F}$ <u>over</u> the region D enclosed by C. Also we have

$$\oint_R \vec{F} \cdot \hat{n}\, ds = \int_D \nabla \cdot \vec{F}\, dA$$

The line integral of the normal component of $\vec{F}$ <u>along</u> C is the double integral of the divergence of $\vec{F}$ <u>over</u> region D enclosed by C.

★ tangent $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$

★ normal $\hat{n}ds = dy\hat{\imath} - dx\hat{\jmath}$

Stoke's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \int_S \nabla \times \vec{F} \cdot d\vec{S}$$

The line integral <u>around</u> the boundry curve of S of the tangential component of $\vec{F}$ is the the surface integral <u>over</u> S of the normal component of curl$\vec{F}$.

Divergence Theoprem

$$\oint_A \vec{F} \cdot d\vec{s} = \int_E \nabla \cdot \vec{F} dV$$

The flux of F <u>across</u> the boundry surface of E is the triple integral of the divergence of F <u>over</u> E.