N Equally Delta-Potential Barriers

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1 Stating The Problem

Consider the one dimensional scattering of a particle of mass m through a series of N equally spaced δ-potential barriers. All are defined by

$$V(x) = \sum_{n=0}^{N} V_0\, \delta(x - x_n) \qquad \text{where} \quad x_n = na$$

where $V_0 > 0$ is the potential strength, and a is the spacing between successive barriers. For this case the TISE given as

$$\frac{d^2}{dx^2}\,\psi = -\frac{2m}{\hbar^2}(E - V)\,\psi$$

In this setup we have only the *scattering state*, since $E > 0$.

2 Solve Single barrier

We start with simplest case, with a sigle δ-potential barrier at a general index n; $x_n = na$. The next barrier will be at x_{n+1}, and the previous one at x_{n-1}.

For the left region of the nth barrier, $x_{n-1} < x < x_n$ and $V(x) = 0$, the TISE becomes

$$\frac{d^2}{dx^2}\,\psi = -\frac{2m}{\hbar^2}E\,\psi \quad \Rightarrow \quad \frac{d^2}{dx^2}\,\psi = -k^2\,\psi \quad \text{where} \quad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

The general solution to this DE is

$$\psi_n(x) = A_n e^{ikx} + B_n e^{-ikx}$$

The same for the right region of the nth barrier, $x_n < x < x_{n+1}$, the general solution is

$$\psi_{n+a}(x) = A_{n+1} e^{ikx} + B_{n+1} e^{-ikx}$$

In summary, for both regions, the general solution is

$$\psi(x) = \begin{cases} A_n e^{ikx} + B_n e^{-ikx} & x_{n-1} < x < x_n \\ A_{n+1} e^{ikx} + B_{n+1} e^{-ikx} & x_n < x < x_{n+1} \end{cases}$$

3 Apply Boundary Conditions

3.1 The Continuity at The Barrier

We require that $\psi(x_n^-) = \psi(x_n^+)$, so

$$A_{n+1}e^{ikx_n} + B_{n+1}e^{-ikx_n} = A_n e^{ikx_n} + B_n e^{-ikx_n}$$

Factor out e^{-ikx_n},

$$A_{n+1}e^{2ikx_n} + B_{n+1} = A_n e^{2ikx_n} + B_n$$

3.2 The Continuity of Derivatives

First we calculate the derivatives,

$$\frac{d\psi}{dx} = \begin{cases} ikA_n e^{ikx} - ikB_n e^{-ikx} & x_{n-1} < x < x_n \\ ikA_{n+1}e^{ikx} - ikB_{n+1}e^{-ikx} & x_n < x < x_{n+1} \end{cases}$$

We require that

$$\left.\frac{d\psi}{dx}\right|_{x_n^+} - \left.\frac{d\psi}{dx}\right|_{x_n^-} = -\frac{2mV_0}{\hbar^2}\psi(x_n)$$

$$ik(A_{n+1}e^{ikx_n} - B_{n+1}e^{-ikx_n}) - ik(A_n e^{ikx_n} - B_n e^{-ikx_n}) = -\frac{2mV_0}{\hbar^2}\psi(x_n)$$

Substituting $\psi(x_n)$

$$ik(A_{n+1}e^{ikx_n} - B_{n+1}e^{-ikx_n}) - ik(A_n e^{ikx_n} - B_n e^{-ikx_n}) = -\frac{2mV_0}{\hbar^2}(A_n e^{ikx_n} + B_n e^{-ikx_n})$$

Divide through ike^{-ikx_n}

$$A_{n+1}e^{2ikx_n} - B_{n+1} - (A_n e^{2ikx_n} - B_n) = i\frac{2mV_0}{\hbar^2 k}(A_n e^{2ikx_n} + B_n)$$

In summary, from the boundary conditions we got

$$\begin{cases} A_{n+1}e^{2ikx_n} + B_{n+1} = A_n e^{2ikx_n} + B_n \\ A_{n+1}e^{2ikx_n} - B_{n+1} = A_n e^{2ikx_n} - B_n + i\frac{2mV_0}{\hbar^2 k}(A_n e^{2ikx_n} + B_n) \end{cases}$$

Add those two equations

$$2A_{n+1}e^{2ikx_n} = 2A_n e^{2ikx_n} + i\frac{2mV_0}{\hbar^2 k}(A_n e^{2ikx_n} + B_n)$$

Isolate A_{n+1}

$$A_{n+1} = A_n + i\frac{mV_0}{\hbar^2 k}(A_n + B_n e^{-2ikx_n})$$

Writing in terms of A_n and B_n

$$A_{n+1} = \left(1 + i\frac{mV_0}{\hbar^2 k}\right)A_n + \left(i\frac{mV_0}{\hbar^2 k}e^{-2ikx_n}\right)B_n$$

Now we substract those two equations (2nd from 1st), to wit

$$2B_{n+1} = 2B_n - i\frac{2mV_0}{\hbar^2 k}(A_n e^{2ikx_n} + B_n)$$

Isolate B_{n+1}

$$B_{n+1} = B_n - i\frac{mV_0}{\hbar^2 k}(A_n e^{2ikx_n} + B_n)$$

Writing in terms of A_n and B_n

$$B_{n+1} = \left(-i\frac{mV_0}{\hbar^2 k}e^{2ikx_n}\right) A_n + \left(1 - i\frac{mV_0}{\hbar^2 k}\right) B_n$$

Thus we have two equationd for B_{n+1} and B_{n+1} in terms of A_n and B_n

$$\begin{cases} A_{n+1} = (1 + i\beta) \quad A_n + \left(i\beta e^{-2ikx_n}\right) B_n \\ B_{n+1} = \left(-i\beta e^{2ikx_n}\right) \quad A_n + (1 - i\beta) B_n \end{cases} \qquad \beta \equiv \frac{mV_0}{\hbar^2 k}$$

Or in matrix form

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = M_n \begin{bmatrix} A_n \\ B_n \end{bmatrix} \qquad M_n = \begin{bmatrix} 1 + i\beta & i\beta e^{-2ikx_n} \\ -i\beta e^{2ikx_n} & 1 - i\beta \end{bmatrix}$$