HW

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1 Homework: numerical methods

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[ ]:
```
# import needed packages
import numpy as np
from scipy.constants import g
from scipy.optimize import root
import pandas as pd

# graph settings
import matplotlib.pyplot as plt
from matplotlib_inline import backend_inline
plt.rcParams['figure.figsize'] = [10, 5]
plt.style.use('seaborn')
backend_inline.set_matplotlib_formats('png', 'pdf')
```

2 Range of a Projectile Under Air Drag

For a projectile with resistance, we start with these initial conditions

$$x(t=0) = y(t=0) = 0$$

$$\dot{x}(t=0) = v_0 cos\theta \equiv U$$

$$\dot{y}(t=0) = v_0 sin\theta \equiv V$$

Then using $\vec{F} = m\vec{a}$ for each dimension we get

$$m\ddot{x} = -km\dot{x}$$

$$m\ddot{y} = -km\dot{y}$$

Putting $y = 0$ we get the following equation, after defining $T \equiv kt$

$$T = (K+2)(1-e^{-T})$$

Then the range will be $R \equiv x(T)$, with $K \equiv kV/g$, and $R_0 = 2UV/g$

$$\frac{R}{R_0} = \frac{T}{2K(K+1)}$$

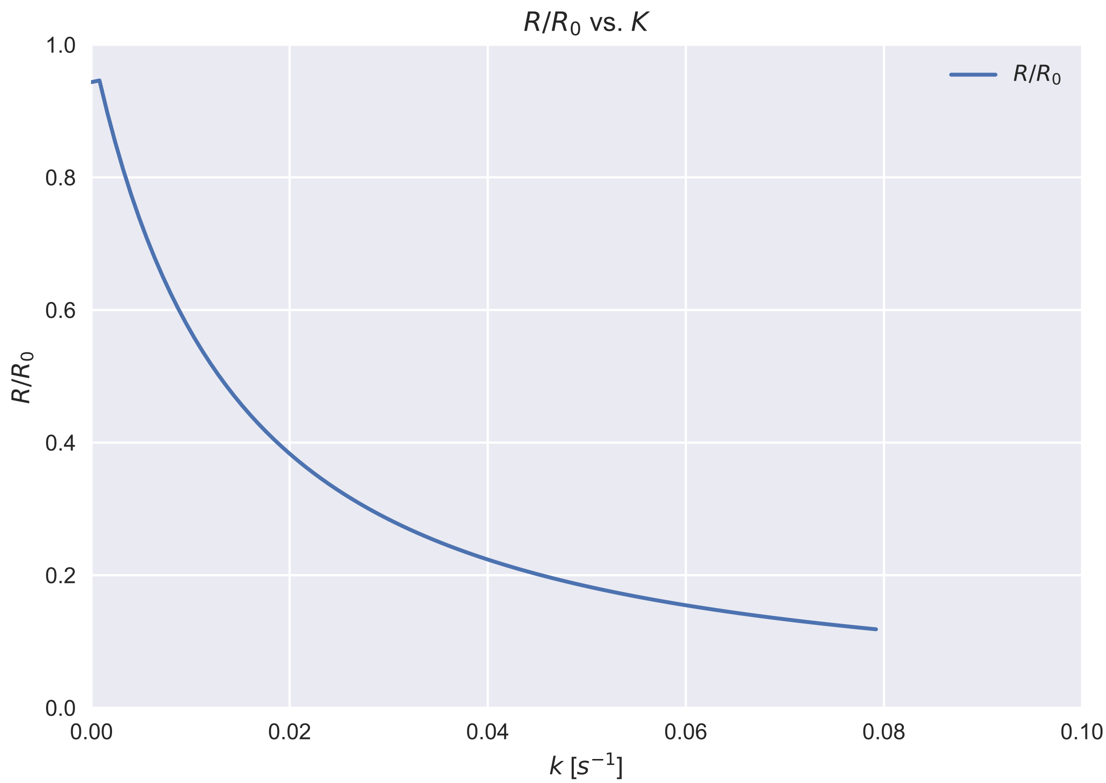
We need to find the value of T, and then substitute it back in the range equation

2.1 Check that $k \to 0$ leads to $R \to R_0$

Rewrite

2.2 Plot R/R_0 vs. K

[ ]:
```
# define initial constants
angle = np.radians(60)
v0 = 600.0 # speed
U = v0 * np.cos(angle) # x-axis speed
V = v0 * np.sin(angle) # y-axis speed
R0 = 2* U * V / g # range

eps = 1e-9 # cannot be zero
k = np.arange(eps, 0.08, 0.08/100)

def T(k, t):
    return k * t

def K(k):
    return k * V / g

# time equation = 0
def time(t, k):
    return (K(k)+1) * (1 - np.exp(-T(k, t))) - T(k, t)

# range equation
def range_(k, t):
    return T(k, t) / (2 * K(k) * (K(k)+1))

# apprximate T value for each k value
time_res = [root(time, args=(k_), x0=100)['x'][0] for k_ in k]
# find range R values
range_res = [range_(k_, t_) for k_, t_ in zip(k, time_res)]

plt.plot(k, range_res, label='$R/R_0$')
plt.xlim(0, 0.1)
plt.ylim(0, 1)
plt.legend()
plt.xlabel('$k$ [$s^{-1}$]')
plt.ylabel('$R/R_0$')
plt.title('$R/R_0$ vs. $K$')
plt.grid(True)
plt.show()
```



2.3 Check that when the limit $k \to 0$, $R/R_0 \to 1$

We use here the same calculation as before but with very small values of k.

[ ]:
```python
k = np.arange(1e-10, 0.01, 1/1000)
t = np.arange(0, 150)

def f(t, k):
    return (K(k)+1) * (1 - np.exp(-T(k, t)))

# apprximate T value for each k value
time_res = []
for k_ in k:
    res  = root(time, args=(k_), x0=105.8)
    if res['success']:
        time_res.append(res['x'][0])
    else: print(res)
# find range R values
range_res = [range_(k_, t_) for k_, t_ in zip(k, time_res)]

pd.DataFrame({'k': k, 'Range': range_res})
```

[ ]:

```
              k     Range
0  1.000000e-10  0.998377
1  1.000000e-03  0.933477
2  2.000000e-03  0.874337
3  3.000000e-03  0.821468
4  4.000000e-03  0.773966
5  5.000000e-03  0.731089
6  6.000000e-03  0.692224
7  7.000000e-03  0.656857
8  8.000000e-03  0.624557
9  9.000000e-03  0.594962
```

From the previous table we can deduce that as the drag constant $k \to 0$, then $R/R_0 \to 1$ or $R \to R_0$, which as expected.

3 Quadratic Drag Force

For a particle of mass m vertically thrown upward with initial speed v_0. The air resistance experienced by the particle is propotional to v^2; $f_R = -\alpha v^2$. We have the DE

$$m\dot{v} = -mg - \alpha v^2$$

The initial conditions are $t_0 = 0$, $y(t_0) = 0$, and $v(t_0) \equiv v_0$. ## Velocity

Factor $-mg$ from the RHS of the eq, we get

$$m\dot{v} = -mg(1 + \frac{\alpha}{mg}v^2)$$

We define the *terminal velocity* as $v_t \equiv \sqrt{mg/\alpha}$ then we get

$$\frac{dv}{dt} = -g(1 + \frac{v^2}{v_t^2})$$

rearranging and setting integrals,

$$\int_{v_0}^{v} \frac{dv}{1 + \frac{v^2}{v_t^2}} = -g\int_{t_0}^{t} dt$$

$$v_t(\tan^{-1}\frac{v}{v_t} - \tan^{-1}\frac{v_0}{v_t}) = -gt$$

Taking the *tangent* of both sides

$$v = v_t \tan\left(\tan^{-1}\frac{v_0}{v_t} - gt/v_t\right)$$

We define the *characteristic time* $\tau \equiv v_t/g$

$$v = v_t \tan\left(\tan^{-1}\frac{v_0}{v_t} - \frac{t}{\tau}\right)$$

3.1 Peak Time

To find the time needed to reach the trajectory peak t_p, we set $v = 0$, and take the arctangent of both sides

$$t_p = \tau \tan^{-1} \frac{v_0}{v_t}$$

3.2 Height

Now we find the $H(t) = y$

$$y = \int_{y_0}^{y} dy = v_t \int_{t_0}^{t} \tan\left(\tan^{-1} \frac{v_0}{v_t} - \frac{t}{\tau}\right) dt$$

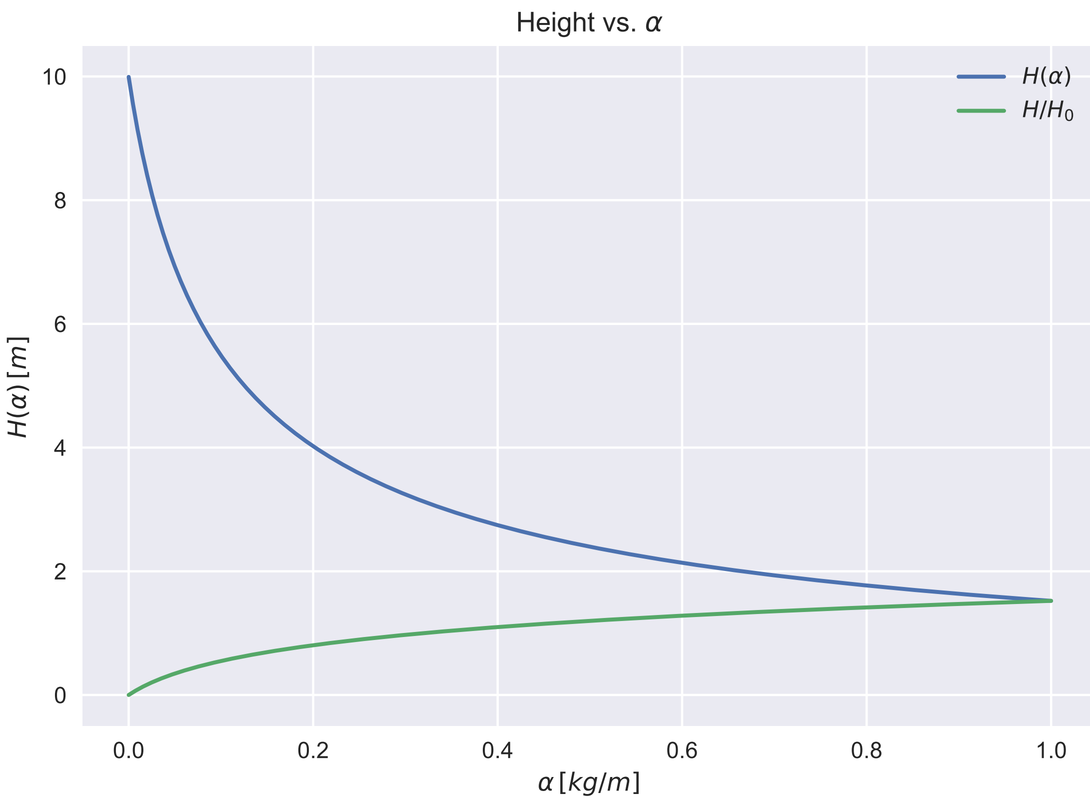
$$y = -v_t \tau \ln(cos(\tan^{-1} \frac{v_0}{v_t}))$$

3.3 Plot $H(\alpha)$

Let us set up the code to graph the function y vs α

[ ]:

```python
# define constants
mass = 1
v0 = 14
alphas = np.arange(1e-10, 1, 1e-5)

# define terminal velocity
def vt(alpha, m=mass):
    return np.sqrt(m * g / alpha)
# define characteristic time
def tau(vt):
    # return np.sqrt(m / (alpha*g))
    return vt / g
# define the height
def y(alpha):
    return - mass / alpha * np.log(np.cos(np.arctan(v0 / vt(alpha))))
def y2(alpha):
    return -  np.log(np.cos(np.arctan(v0 / vt(alpha))))

plt.plot(alphas, y(alphas), label=r'$H(\alpha)$')
plt.plot(alphas, y2(alphas), label=r'$H/H_0$')
plt.legend()
plt.xlabel(r'$\alpha\, [kg/m]$')
plt.ylabel(r'$H(\alpha)\, [m]$')
plt.title(r'Height vs. $\alpha$')
plt.show()
```




```
[ ]: alphas = np.arange(1e-16, 1e-5, 1e-6)
     height_res = [y(a) for a in alphas]

     pd.DataFrame({'alpha': alphas, 'Height': height_res})
```

[ ]:

	alpha	Height
0	1.000000e-16	9.992007
1	1.000000e-06	9.993119
2	2.000000e-06	9.993019
3	3.000000e-06	9.992919
4	4.000000e-06	9.992819
5	5.000000e-06	9.992720
6	6.000000e-06	9.992620
7	7.000000e-06	9.992520
8	8.000000e-06	9.992420
9	9.000000e-06	9.992320

3.4 Without resistance

Let us consider the case when $\alpha \to 0$. From the previous table we see theat the object do approaches the height $10m$, for no effect of air drag.