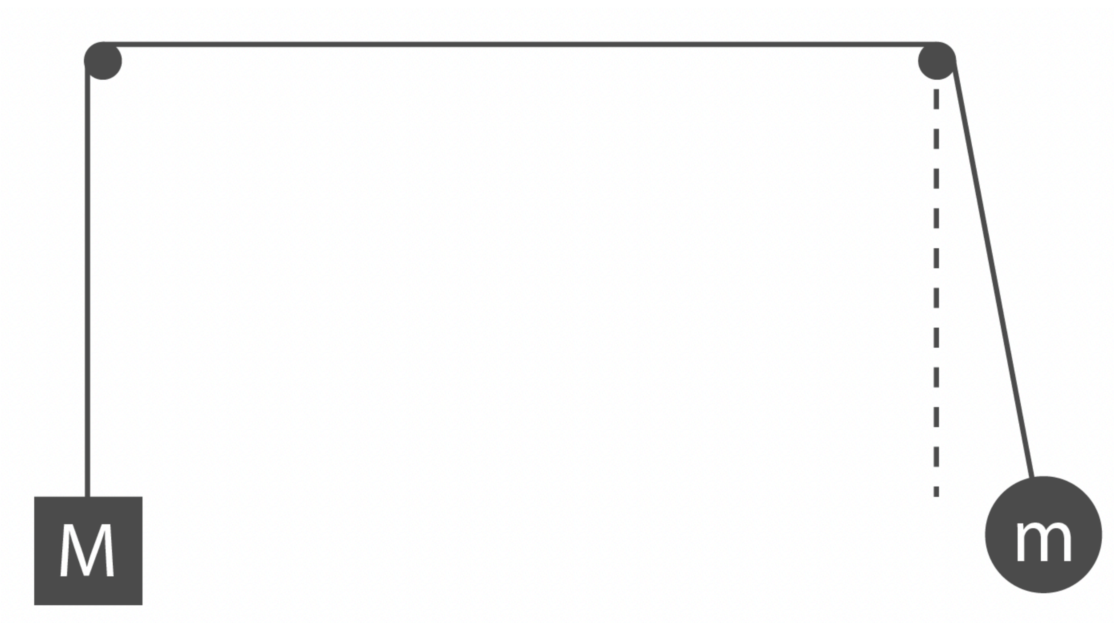
Swinging Atwood Machine

December 12, 2021

1 Abstract

A swinging Atwood machine consists of two non-colliding masses connected by an inextensible string over two frictionless support points (see figure below). The mass M moves only vertically, that is up and down, while the mass m oscillate in the vertical plane.

The lagrangian can be written as

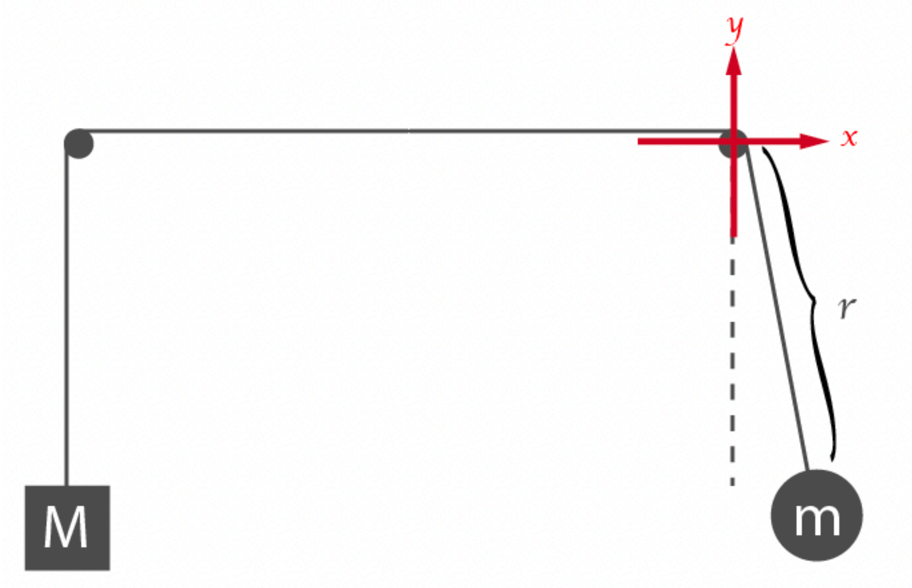


2 The Problem

We consider a system as the above picture. There are two masses m and M hangged from two supports by an inextensible string of length b. The spherical mass is allowed to swing with an angle of θ as a pendulum. Thus pendulum has is r long and it can change. The cubic mass is only allowed to move vertically.

3 The Lagrangian

3.1 The Mass m



For the spherical mass m and using a cartesian coordinate as in the above picture, we write the Lagrangian as,

$$L_m = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$$

But we have a constraint on the coordinates as

$$\begin{cases} x = r\sin(\theta) \\ y = -r\cos(\theta) \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x} = \dot{r}\sin(\theta) + r\cos(\theta) \\ \dot{y} = -\dot{r}\cos(\theta) + \sin(\theta) \end{cases}$$

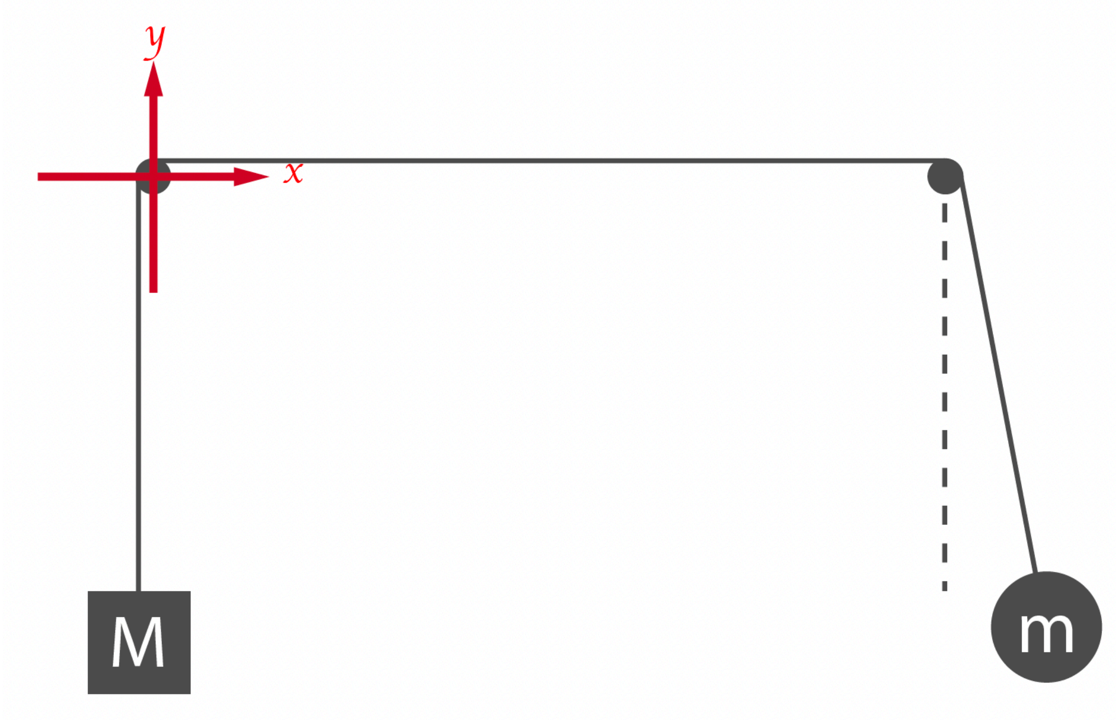
So

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

So L_m becomes

$$L_m = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos(\theta)$$

3.2 The Mass M



The lgransgian for the cubic mass M with coordiante system as above is

$$L_M = \frac{M}{2}(\dot{x}^2 + \dot{y}^2) - Mgy$$

The constraint is

$$\begin{cases} x = 0 \\ y = b - r \qquad \text{we ignore the horizontal length of the string} \end{cases}$$

Hence,

$$L_M = \frac{M}{2}\left[\frac{d}{dt}(b-r)\right]^2 - Mgy = \frac{M}{2}\dot{r}^2 - Mgr$$

3.3 The Entire System

We add up the two indivisual Lgrngian to get the one of the systme, then

$$L_s = L_m + L_M = \left[\frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos(\theta)\right] + \left[\frac{M}{2}\dot{r}^2 - Mgr\right]$$

Hence,

$$L_s = \frac{m+M}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 + gr(m\cos(\theta) - M)$$

3.4 The Lagrange Equation of Motions

Starting with the equtaion

$$\frac{\partial L_s}{\partial r} - \frac{d}{dt}\frac{\partial L_s}{\partial \dot{r}} = 0$$

Then,

$$\frac{\partial L_s}{\partial r} = mr\dot{\theta}^2 + g(m\cos(\theta) - M) \quad \text{and} \quad \frac{d}{dt}\frac{\partial L_s}{\partial \dot{r}} = (m+M)\ddot{r}$$

Hence,

$$(m+M)\ddot{r} - mr\dot{\theta}^2 - gm\cos(\theta) + gM = 0$$

Now for doing the same for θ

$$\frac{\partial L_s}{\partial \theta} - \frac{d}{dt}\frac{\partial L_s}{\partial \dot{\theta}} = 0$$

So we get

$$\frac{\partial L_s}{\partial \theta} = -mgr\sin(\theta) \quad \text{and} \quad \frac{d}{dt}\frac{\partial L_s}{\partial \dot{\theta}} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

Hence,

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin(\theta) = 0$$

To add up, we have these two EOMs,

$$(m+M)\ddot{r} - mr\dot{\theta}^2 - gm\cos(\theta) + gM = 0$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin(\theta) = 0$$

Let us divide those two by m, and define the mass ration $\mu = M/m$, so get the follwoing equations

$$(1+\mu)\ddot{r} - r\dot{\theta}^2 - g\cos(\theta) + g\mu = 0$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\sin(\theta) = 0$$

Finaly we isolate $\ddot{r}$, and $\ddot{\theta}$, like so

$$\ddot{r} = \frac{r\dot{\theta}^2 + g\cos(\theta) - g\mu}{1+\mu}, \qquad \ddot{\theta} = -\frac{2\dot{r}\dot{\theta} + g\sin\theta}{r}$$

We will use python to interate those equations for different initial values

4 Writing The Program

[ ]:
```python
from scipy.integrate import solve_ivp
import numpy as np
from matplotlib import pyplot as plt
from matplotlib.animation import FuncAnimation

# graph settings
import matplotlib.pyplot as plt
from matplotlib_inline import backend_inline
plt.rcParams['figure.figsize'] = [13, 5]
backend_inline.set_matplotlib_formats('svg', 'pdf')
```

4.1 Define the ODE of the radial coordinate θ

To start, we need to write the last two second-order ODEs in form of first-order ODEs, so we define

$$A \equiv \dot{r}, \qquad B \equiv \dot{\theta}$$

Also, we define a vector function $\vec{S}$ as

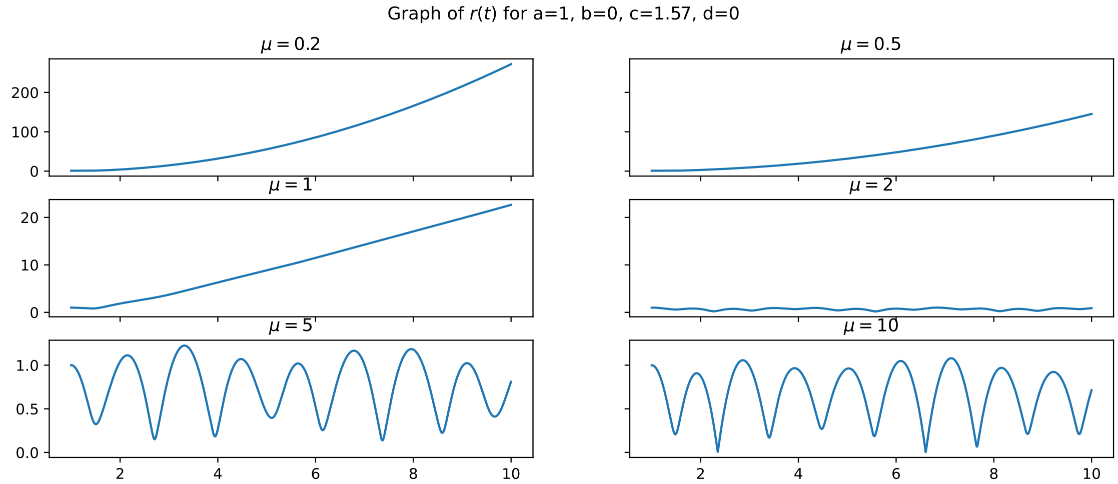
$$\vec{S} = \begin{bmatrix} r \\ A \\ \theta \\ B \end{bmatrix}, \qquad \frac{d\vec{S}}{dt} = \begin{bmatrix} \dot{r} \\ \dot{A} \\ \dot{\theta} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} A \\ \frac{rB^2 + g\cos(\theta) - g\mu}{1+\mu} \\ B \\ -\frac{2AB + g\sin(\theta)}{r} \end{bmatrix}$$

Now we start defining a python function that takes t and $\vec{S}$ as inputs, and return $\frac{d\vec{S}}{dt}$. Then, the latter will be solved numerically.

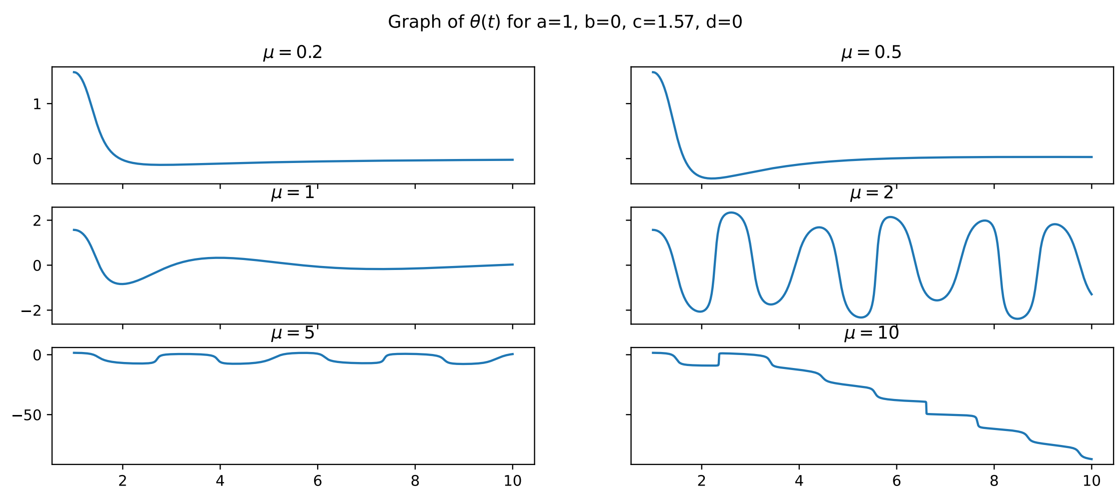
[ ]:
```
# define the dS/dt function
def dSdt(t, S, mu):
    g = 9.8 #earth acceleration
    r, A, th, B = S  # decompisite vecote S

    Adot = (r * B**2 + g * np.cos(th) - mu * g) / (1 + mu)
    Bdot = - (2 * A * B + g * np.sin(th)) / r

    return np.array([A, Adot, B, Bdot])

S0 = [1, 0, np.pi/2, 0]    # a, b, c, d
ts = np.arange(1, 10, 0.001)     # time values
mus = [0.2, 0.5, 1, 2, 5, 10]    # mu to test them

sol = solve_ivp(dSdt, y0=S0, t_span=[1, max(ts)], t_eval=ts, args=[1.4])
```

4.2 Graph $r(t)$ and $\theta(t)$

[ ]:
```
fig, axes = plt.subplots(3, 2, sharex=True, sharey='row')

for i in range(6):
    axes = axes.reshape(6)
    sol = solve_ivp(dSdt, y0=S0, t_span=[1, max(ts)], t_eval=ts, args=[mus[i]])
    axes[i].plot(ts, sol.y[0])
    axes[i].set_title(f'$\mu={mus[i]}$')

fig.suptitle(f'Graph of $r(t)$ for a={S0[0]}, b={S0[1]}, c={S0[2]:.2f},␣
↪d={S0[3]}')
plt.show()
```



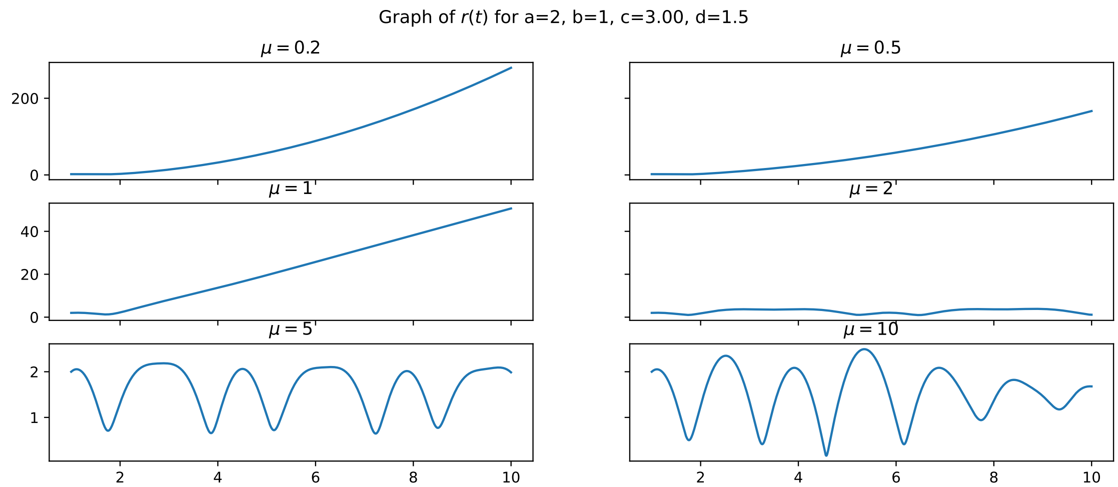

```
[ ]: fig, axes = plt.subplots(3, 2, sharex=True, sharey='row')

     for i in range(6):
         axes = axes.reshape(6)
         sol = solve_ivp(dSdt, y0=S0, t_span=[1, max(ts)], t_eval=ts, args=[mus[i]])
         axes[i].plot(ts, sol.y[2])
         axes[i].set_title(f'$\mu={mus[i]}$')

     fig.suptitle(rf'Graph of $\theta(t)$ for a={S0[0]}, b={S0[1]}, c={S0[2]:.2f},␣
      ↪d={S0[3]}')
     plt.show()
```



Now doing the same but with different initial set values.

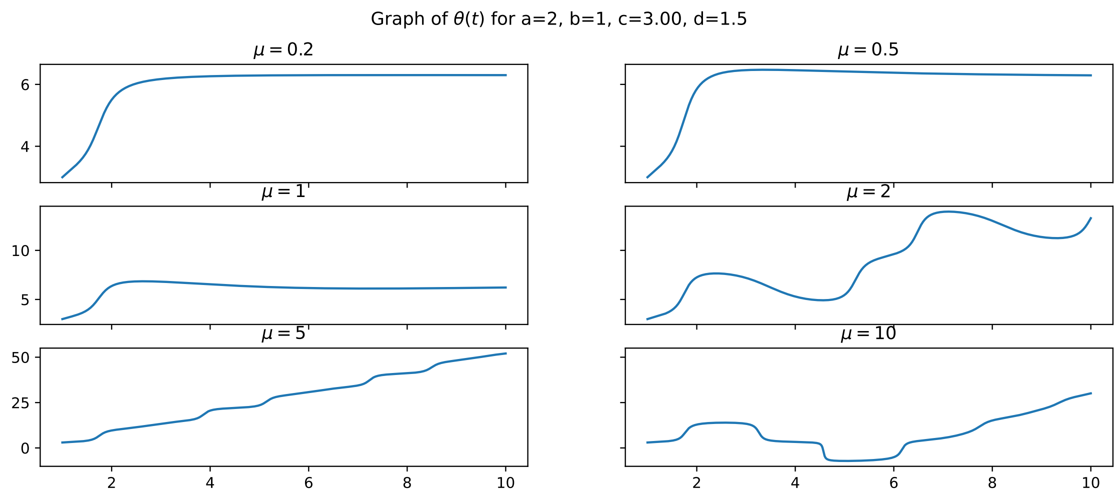
[ ]:

```python
S0 = [2, 1, 3, 1.5]    # new ones
fig, axes = plt.subplots(3, 2, sharex=True, sharey='row')

for i in range(6):
    axes = axes.reshape(6)
    sol = solve_ivp(dSdt, y0=S0, t_span=[1, max(ts)], t_eval=ts, args=[mus[i]])
    axes[i].plot(ts, sol.y[0])
    axes[i].set_title(f'$\mu={mus[i]}$')

fig.suptitle(f'Graph of $r(t)$ for a={S0[0]}, b={S0[1]}, c={S0[2]:.2f},␣
↪d={S0[3]}')
plt.show()
```



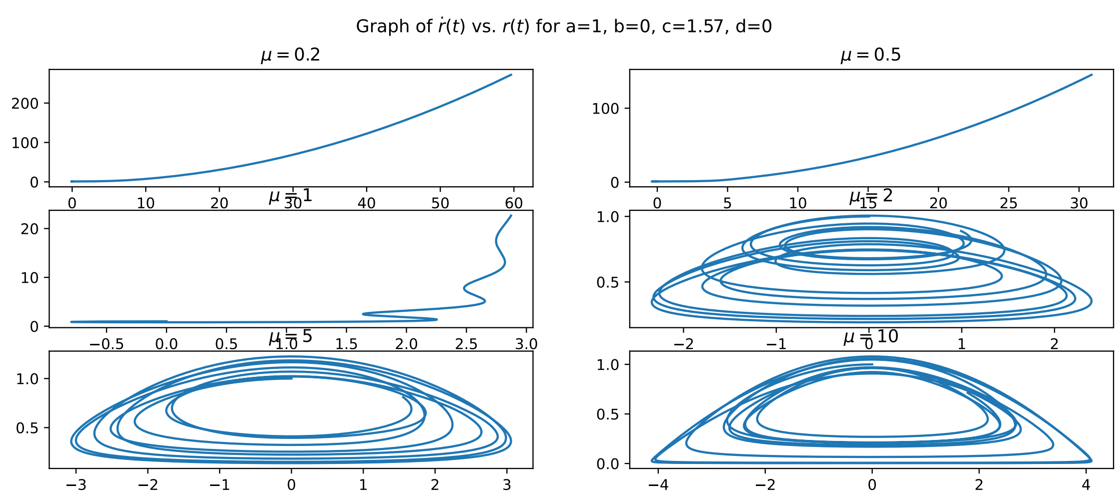
[ ]:

```python
fig, axes = plt.subplots(3, 2, sharex=True, sharey='row')

for i in range(6):
    axes = axes.reshape(6)
    sol = solve_ivp(dSdt, y0=S0, t_span=[1, max(ts)], t_eval=ts, args=[mus[i]])
    axes[i].plot(ts, sol.y[2])
    axes[i].set_title(f'$\mu={mus[i]}$')

fig.suptitle(rf'Graph of $\theta(t)$ for a={S0[0]}, b={S0[1]}, c={S0[2]:.2f},␣
↪d={S0[3]}')
plt.show()
```



4.3 Graph The Phase Diagram; $\dot{r}$ vs. r

[ ]:
```
S0 = [1, 0, np.pi/2, 0]    # initials
fig, axes = plt.subplots(3, 2)

for i in range(6):
    axes = axes.reshape(6)
    sol = solve_ivp(dSdt, y0=S0, t_span=[1, max(ts)], t_eval=ts, args=[mus[i]])
    axes[i].plot(sol.y[1], sol.y[0])
    axes[i].set_title(f'$\mu={mus[i]}$')

fig.suptitle(rf'Graph of $\dot{{r}}(t)$ vs. $r(t)$ for a={S0[0]}, b={S0[1]},␣
↪c={S0[2]:.2f}, d={S0[3]}')
plt.show()
```



4.4 Graph The Phase Diagram; $\dot{\theta}$ vs. r in Polar

[ ]:
```
S0 = [1, 0, np.pi/2, 0]    # initials
fig, axes = plt.subplots(3, 2, subplot_kw={'polar': True})

for i in range(6):
    axes = axes.reshape(6)
    sol = solve_ivp(dSdt, y0=S0, t_span=[1, max(ts)], t_eval=ts, args=[mus[i]])
    axes[i].plot(sol.y[2], sol.y[0])
    axes[i].set_title(f'$\mu={mus[i]}$')

fig.suptitle(rf'Graph of $\theta(t)$ for a={S0[0]}, b={S0[1]}, c={S0[2]:.2f},␣
 ↪d={S0[3]}')
fig.set_size_inches(8, 10)
plt.show()
```

Graph of $\theta(t)$ for a=1, b=0, c=1.57, d=0