HW3

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1 Homework: Lagranian Mechanics

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2 Stating the Equations

The lagrangian is defined as $L \equiv T - U$, where T is the kinetic energy for the considred mass, and U is the potential energy on due to forces on the mass. For the jth particle, we have T_j and U_j as

$$T_j = \frac{1}{2}m\dot{x_j}^2 \qquad U_j = \frac{1}{2}kx_j^2$$

We relize that we have N particles connected by $N+1$ springs. Start analayzing by assuming all the particles have shifted from their equilibrium a distance x_j to the right.

2.1 First Atom

Consider the free body diagram for the first atom we find its potential as

$$U_1 = \frac{k}{2}kx_1^2 + \frac{k}{2}(x_2^2 - x_1^2) = \frac{k}{2}(x_2^2 - 2x_1x_2 + 2x_1^2)$$

And the kinetic energy is

$$T_1 = \frac{m}{2}\dot{x_1}^2$$

Then the Lagrangian is

$$L_1 = T_1 - U_1 = \frac{m}{2}\dot{x_1}^2 - \frac{k}{2}(x_2^2 - 2x_1x_2 + 2x_1^2)$$

Since we take the partial derivatives with respect to x_1 it is enough to use L_1, then the Lagrange equation is

$$\frac{\partial L_1}{\partial x_1} - \frac{d}{dt}\frac{\partial L_1}{\partial \dot{x_1}} = 0$$

So,

$$-\frac{k}{2}(-2x_2 + 4x_1) - \frac{d}{dt}m\dot{x}_1 = 0$$
$$-k(-x_2 + 2x_1) - m\ddot{x}_1 = 0$$
$$\ddot{x}_1 + 2\omega_0^2 x_1 - \omega_0^2 x_2 = 0$$

where, $\omega_0^2 \equiv k/m$

2.2 Last Atom; N

Consider the last atom at $j = N$, its previous spring streached by a distance $x_N - x_{N-1}$, and its next spring compressed by a distance x_N. The Lagrangian is

$$L_N = \frac{m}{2}\dot{x}_N^2 - \left(\frac{k}{2}x_N^2 + \frac{k}{2}(x_N - x_{N-1})^2\right)$$

We can use the last equation and just replace x_N with x_N and x_{N+1} with x_{N+1}, that is,

$$\ddot{x}_N + 2\omega_0^2 x_N - \omega_0^2 x_{N-1} = 0$$

2.3 Middle Atoms

Again, we analyze the free body diagram for a given in between atom to its potential energy. We find that the Lagrangian is,

$$L_j = T_j - U_j = \frac{m}{2}\dot{x}_j^2 - \frac{k}{2}\left[(x_j - x_{j-1})^2 + (x_{j+1} - x_j)^2\right]$$

Or

$$L_j = \frac{m}{2}\dot{x}_j^2 - \frac{k}{2}\left[2x_j^2 - 2x_j x_{j-1} - 2x_j x_{j+1} + x_{j-1}^2 + x_{j+1}^2\right]$$

Then the equation of motion is,

$$\frac{\partial L}{\partial x_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} = 0$$

Then,

$$m\ddot{x} + k[2x_j - x_{j-1} - x_{j+1}] = 0$$
$$\ddot{x} + 2\omega_0^2 x_j - \omega_0^2 x_{j-1} - \omega_0^2 x_{j+1} = 0$$

2.4 Generalization of Lagrangian

$$L = \sum_{j=1}^{N}(T_j - U_j) = \sum_{j=1}^{N}\left(\frac{m}{2}\dot{x_j}^2 - \frac{k}{2}(x_{j+1}^2 - 2x_j x_{j+1} + 2x_j^2)\right)$$

The Lagrange equation is

$$\frac{\partial L}{\partial x_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} = 0, \qquad j = 1, 2, 3, \ldots, N$$

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3 The Eigenmodes

We write the equation of motion in matrix form as, $\ddot{X} + \Omega X = 0$

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# import needed packages
import numpy as np

# graph settings
import matplotlib.pyplot as plt
from matplotlib_inline import backend_inline
plt.rcParams['figure.figsize'] = [15, 4]
backend_inline.set_matplotlib_formats('png', 'pdf')
```