

PHYS213

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Chapter 1

The Quantum Theory of Light

1.1 Hertz's Experiments; Light as EM Wave

- Between 1865 and 1873, James Clerk Maxwell, and according to his theory, an alternating current would set up fluctuating electric and magnetic fields in the region surrounding the original disturbance.
- These waves were predicted to have a frequency equal to the frequency of the current oscillations.
- And more importantly, they would behave in every way like *light*.
- Naturally this led to the unifying and simplifying postulate that light was also a type of Maxwell wave or electromagnetic disturbance, created by extremely high frequency electric oscillators in matter.
- it was apparent that this model of light emission was incapable of direct experimental verification, because the highest electrical frequencies then attainable were about $10^9 Hz$.
- Heinrich Hertz showed that Maxwell's theory was correct and that an oscillating electric current does indeed radiate electromagnetic waves that possess every characteristic of light except the same wavelength as light.
- By using a simple spark gap oscillator in Fig.(1.1).
- Hertz next proceeded to show that these electromagnetic waves could be reflected, refracted, focused, polarized, and made to interfere— in short, he convinced physicists of the period that Hertzian waves and light waves were one and the same.

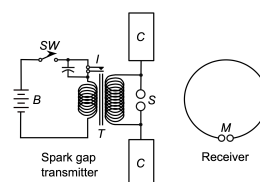


Figure 1.1: Hertz's Transmitter.

1.2 Blackbody Radiation

The problem is to predict the radiation intensity at a given wavelength emitted by a hot glowing solid at a specific temperature.

- In 1792, Thomas Wedgwood observed that all the objects in his ovens, regardless of their chemical nature, size, or shape, became red at the same temperature.
- In 1859, Gustav Kirchhoff proved a theorem when he showed by arguments based on thermodynamics that for any body in thermal equilibrium with radiation the emitted power is proportional to the power absorbed. More specifically,

$$e_f = J(f, T)A_f \quad (1.1)$$

- e_f is the power emitted per unit area per unit frequency by a heated object.
- A_f is the absorption power (fraction of the incident power absorbed per unit area per unit frequency by the heated object).
- $J(f, T)$ is a universal function (the same for all bodies) that depends only on f .

- A **blackbody** is defined as an object that absorbs all the electromagnetic radiation falling on it and consequently appears black. It has $a_f = 1$ for all frequencies and so Kirchhoff's theorem for a blackbody becomes,

$$e_f = J(f, T) \quad (1.2)$$

- Eq.(1.1) shows that the power emitted only per unit area per unit frequency by a blackbody depends only on the light frequency and the temperature.

- Because absorption and emission are connected by Kirchhoff's theorem, we see that a *blackbody* or perfect absorber is also an ideal *radiator*.

In practice, a small opening in any heated *cavity*, such as a port in an oven, behaves like a blackbody because such an opening traps all incident radiation.

- In 1879, Josef Stefan found experimentally that the total power per unit area emitted at all frequencies by a hot solid, was proportional to the fourth power of its absolute temperature. Therefore, *Stefan's law* may be written as

$$e_{total} = \int_0^\infty e_f df = \sigma T^4 \quad (1.3)$$

- e_{total} is the power per unit area emitted at the surface of the blackbody at all frequencies.
- e_f is the power per unit area per unit frequency emitted by the blackbody.
- T is the absolute temperature of the body.
- σ is the Stefan – Boltzmann constant $\sigma = 3 \times 10^5 \text{ W}/(\text{m}^2 \text{ K}^4)$.

A body that is not an ideal radiator will obey the same general law but with a coefficient, a , less than 1:

$$e_{total} = a\sigma T^4 \quad (1.4)$$

- Notice that in Fig.(1.2) λ_{max} shifts to shorter wavelengths as the blackbody gets hotter.

In 1893, Wilhelm Wien proposed a general form for the blackbody distribution law $J(f, T)$ that gave the correct experimental behavior of λ_{max} with temperature. This law is called **Wien's displacement law** and may be written

$$\lambda_{max} T = 2.898 \times 10^{-3} \text{ m K} \quad (1.5)$$

- λ_{max} is the the blackbody's maximum intensity.
- T is the absolute temperature of the surface.

- Let's consider the energy per unit volume per unit frequency of the radiation within the blackbody cavity, $u(f, T)$. For light in equilibrium with the walls, the power emitted per square centimeter of opening is simply proportional to the energy density of the light in the cavity.

- An important guess as to the form of the universal function $u(f, T)$ was made in 1893 by Wien and had the form

$$u(f, T) = A f^3 e^{-\beta f/T} \quad (1.6)$$

- A and β are constants

- Within a year Friedrich Paschen, had confirmed Wien's guess by working in the then difficult infrared range of 1 to $4 \mu\text{m}$ and at temperatures of 400 to 1600 K. Paschen had made most of his measurements in the maximum energy region of a body heated to 1500K and had found good agreement with Wien's exponential law. In 1900, however, Lummer and Pringsheim extended the measurements to $18 \mu\text{m}$, and Rubens and Kurlbaum went even farther— to $60 \mu\text{m}$. Both teams concluded that Wien's law failed in this region.

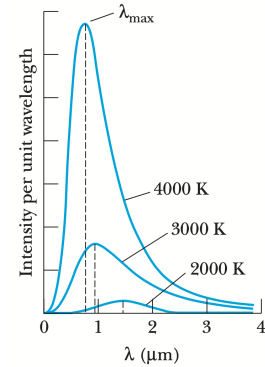


Figure 1.2: Emission from a glowing solid.

Enter Planck On a Sunday evening early in October of 1900, Max Planck discovered the famous blackbody formula.

Planck knew that Wien's law agreed well with the data at high frequency and indeed had been working hard for several years to derive Wien's exponential law from the principles of statistical mechanics and Maxwell's laws. See Eq.(1.17)

The Quantum Energy Planck was convinced that blackbody radiation was produced by vibrating submicroscopic electric charges, which he called *resonators*. He assumed that the walls of a glowing cavity were composed of literally billions of these *resonators* (whose exact nature was unknown at the time), all vibrating at different frequencies. Hence, according to Maxwell, each *oscillator* should emit radiation with a frequency corresponding to its vibration frequency. **Also according to classical Maxwellian theory, an oscillator of frequency f could have any value of energy and could change its amplitude continuously as it radiated any fraction of its energy.**

- This is where Planck made his revolutionary proposal. To secure agreement with experiment, Planck had to assume that the total energy of a resonator with mechanical frequency f could only be an integral multiple of hf or

$$E_{\text{resonator}} = nhf \quad n = 1, 2, 3, \dots \quad (1.7)$$

he concluded that emission of radiation of frequency f occurred when a resonator dropped to the next lowest energy state.

Thus the resonator can change its energy only by the difference ΔE according to

$$\Delta E = hf \quad (1.8)$$

That is, it cannot lose just any amount of its total energy, but only a finite amount, hf , the so-called quantum of energy.

- Planck explained the continuous spectrum of the blackbody by assuming that *the heated walls contained resonators vibrating at many different frequencies*, each emitting light at the same frequency as its vibration frequency.
- considering the conditions leading to equilibrium between the *wall resonators* and the *radiation* in the blackbody cavity, he found that the spectral energy density $u(f, T)$ could be expressed as

$$u(f, T) df = \bar{E} N(f) df \quad (1.9)$$

- $N(f)$ number of oscillators having frequency between f and $f + df$.
- \bar{E} average energy emitted per oscillator.

Furthermore, Planck showed that the number of oscillators with frequency between f and $f + df$ was proportional to f^2 or

$$N(f) df = \frac{8\pi f^2}{c^3} df \quad (1.10)$$

1.3 The Rayleigh-Jeans Law and Planck's Law

Both Planck's law and the Rayleigh-Jeans law¹ may be derived using the idea that the blackbody radiation energy per unit volume can be expressed as in Eq.(1.9). We will consider **Rayleigh-Jeans** and **Planck** calculations to see the effect on $u(f, T)$ of calculating \bar{E} from a *continuous* distribution of classical oscillator energies (Rayleigh-Jeans) as opposed to a *discrete* set of quantum oscillator energies (Planck).

¹the classical theory of blackbody radiation formulated by Lord Rayleigh, John William Strutt, 1842–1919, English physicist, and James Jeans, 1887–1946, English astronomer and physicist

Rayleigh-Jeans Law While Planck concentrated on the thermal equilibrium of cavity radiation with oscillating electric charges in the cavity walls, Rayleigh concentrated directly on the electromagnetic waves in the cavity.

- Rayleigh and Jeans reasoned that the *standing electromagnetic waves* in the cavity could be considered to have a **temperature** T , because they constantly exchanged energy with the walls and caused a thermometer within the cavity to reach the same temperature as the walls. Further, they considered a *standing polarized electromagnetic wave* to be equivalent to a one-dimensional oscillator (1.3).

- Using the same general idea as Planck, they expressed the energy density as a product of the number of standing waves (oscillators) and the average energy per oscillator. They found the average oscillator energy \bar{E} to be independent of frequency and equal to $k_B T$ from the *Maxwell-Boltzmann distribution law* (see Chapter 10). According to this distribution law,

$$P(E) = P_0 e^{-(E-E_0)/k_B T} \quad (1.11)$$

- the probability P of finding an individual system (such as a molecule or an atomic oscillator)
- with energy E above some
- minimum energy (E_0) in a large group of systems at
- temperature T
- where P_0 is the probability that a system has the minimum energy.

- In the case of a discrete set of allowed energies, the average energy (\bar{E}) is given by

$$\bar{E} = \frac{\sum E P(E)}{\sum P(E)} \quad (1.12)$$

where division by the sum in the denominator serves to normalize the total probability to 1.

- In the classical case considered by Rayleigh, an oscillator can have any energy E in a continuous range from 0 to ∞ . Thus the Eq.(1.12) must be replaced with integrals, so

$$\bar{E} = \frac{\int_0^\infty E e^{-E/k_B T} dE}{\int_0^\infty e^{-E/k_B T} dE} = k_B T \quad (1.13)$$

The derivation of the density of modes, $N(f) df$, gives

$$N(f) df = \frac{8\pi f^3}{c^3} df \quad \text{or in wavelength} \quad N(\lambda) d\lambda = \frac{8\pi}{\lambda^4} d\lambda \quad (1.14)$$

The spectral energy density is simply the *density of modes* multiplied by $k_B T$, or

$$u(f, T) df = \frac{8\pi f^2}{c^3} k_B T df \quad \text{or in wavelength} \quad u(\lambda, T) d\lambda = \frac{8\pi}{\lambda^4} k_B T d\lambda \quad (1.15)$$

- However, as in Figure 1.4, this classical expression, known as the *Rayleigh-Jeans law*, does not agree with the experimental results in the short wavelength region. Eq.(1.15) diverges as $\lambda \rightarrow 0$, predicting unlimited energy emission in the ultraviolet region, which was dubbed the “ultraviolet catastrophe”. To conclude that the classical theory fails to explain blackbody radiation.

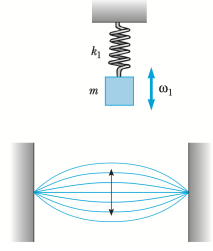


Figure 1.3: A one-dimensional harmonic oscillator.

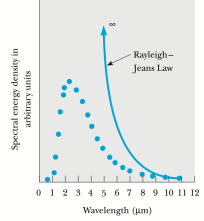


Figure 1.4: The failure of the classical Rayleigh-Jeans law Eq.(1.15) to fit the observed spectrum of a blackbody heated to 1000 K.

Planck Law Planck concentrated on the energy *states of resonators* in the cavity walls and used the condition that the **resonators and cavity radiation were inequilibrium** to determine the spectral quality of the radiation. By thermodynamic reasoning (and apparently unaware of Rayleigh’s derivation), he arrived at the same expression for $N(f)$ as Rayleigh. However, Planck arrived at a different form for E by allowing only *discrete* values of energy for his resonators. He found, using the Maxwell-Boltzmann distribution law,

$$\bar{E} = \frac{hf}{e^{hf/k_B T} - 1} \quad (1.16)$$

Multiplying \bar{E} by $N(f)$ gives the Planck distribution law,

$$\begin{aligned} u(f, T) df &= \frac{8\pi f^2}{c^3} \left(\frac{hf}{e^{hf/k_B T} - 1} \right) df \\ \text{or in wavelength} \quad u(\lambda, T) d\lambda &= \frac{8\pi hc}{\lambda^5 (e^{hc/\lambda k_B T} - 1)} d\lambda \end{aligned} \quad (1.17)$$

• h Planck’s constant $6.626 \times 10^{-34} \text{ J s}$

• k_B Boltzmann’s constant $1.380 \times 10^{-23} \text{ J K}^{-1}$

In summary, Planck arrived at his blackbody formula by making two startling assumptions:

1. The energy of a charged oscillator of frequency f is limited to discrete values nhf .
2. During emission or absorption of light, the change in energy of an oscillator is hf .

★ By integrating Eq.(1.17) for $0 \rightarrow \infty$ we will get Stefan’s law in Eq.(1.3).

1.4 Light Quantization and The Photoelectric Effect

Einstein recognized an inconsistency between Planck’s quantization of oscillators in the walls of the blackbody and Planck’s insistence that the cavity radiation consisted of classical electromagnetic waves.

★ By showing that the change in entropy of blackbody radiation was like the change in entropy of an ideal gas consisting of independent particles, Einstein reached the conclusion that light itself is composed of “grains,” irreducible finite amounts, or quanta of energy.

• Hertz first established that clean metal surfaces emit charges when exposed to ultraviolet light. In 1888 Hallwachs discovered that the emitted charges were negative, and in 1899 J. J. Thomson showed that the emitted charges were electrons, now called photoelectrons.

• In 1902, Philip Lenard was studying the photoelectric effect with intense carbon arc light sources. He found that electrons are emitted from the metal with a range of velocities and that the maximum kinetic energy of photoelectrons, K_{max} , does not depend on the *intensity* of the exciting light. He also indicated that K_{max} increases with light *frequency*.

• In the photoelectric setup, K_{max} is easily measured by applying a retarding voltage and gradually increasing it until the most energetic electrons are stopped and the photocurrent becomes zero. At this point,

$$K_{max} = \frac{1}{2} m_e v_{max}^2 = eV_s \quad (1.18)$$

As shown in Fig.(1.5) Lenard found that K_{max} or V_s is independent of light intensity I .

The classical electromagnetic theory has major difficulties explaining:

1. The linear dependence of K_{max} on light frequency, shown in Fig.(1.6).
2. There is no *time lag* between the start of illumination and the start of the photocurrent.

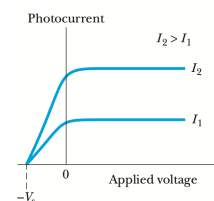


Figure 1.5: Photocurrent versus applied voltage

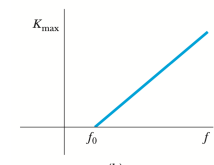


Figure 1.6: K_{max} versus frequency

★ Einstein's theory of the photoelectric effect: **He maintained that the energy of light is not distributed evenly over the classical wavefront, but is concentrated in discrete regions (or in "bundles"), called quanta, each containing energy, hf .**

- Therefore, according to Einstein, the maximum kinetic energy for emitted electrons is

$$K_{max} = hf - \phi \quad (1.19)$$

where ϕ is the **work function** of the metal, which corresponds to the minimum energy with which an electron is bound in the metal. Eq.(1.19) explained the independence of K_{max} and intensity found by Lenard. For a fixed light frequency f , an increase in light intensity means more photons and more photoelectrons per second, although K_{max} remains unchanged according to this equation.

- By setting $K_{max} = 0$ we get the characteristic threshold frequency,

$$f_0 = \frac{\phi}{h} \quad (1.20)$$

1.5 X-Rays

X-rays were discovered in 1895 by the German physicist Wilhelm Roentgen. He found that a beam of high-speed electrons striking a metal target produced a new and extremely penetrating type of radiation. Max von Laue in Germany and William Henry Bragg and William Lawrence Bragg (a father and son team) in England suggested using single crystals such as calcite as natural three-dimensional gratings, the periodic atomic arrangement in the crystals constituting the *grating rulings*.

- A particularly simple method of analyzing the scattering of x-rays from parallel crystal planes was proposed by W. L. Bragg in 1912. Consider two successive planes of atoms as shown in Fig.(1.7).

- The adjacent atoms in a single plane, A , will scatter constructively if the angle of incidence, θ_i , equals angle of reflection, θ_r .

Atoms in *successive planes* (A & B), will scatter constructively if the the path length difference for rays (1) & (2) is a whole number of wavelength, thus $n\theta$, so

$$\overline{AB} + \overline{BC} = n\theta, \quad n = 1, 2, 3, \dots \quad (1.21)$$

Thus Bragg equation is

$$\Delta L = 2d \sin \theta = n\lambda \quad (1.22)$$

- n is the order of the intensity maximum,
- λ is the x-ray wavelength,
- d is the spacing between planes,
- θ is the angle of the intensity maximum measured from plane A .

If λ is known d of crystal can be found. If d is also known, the x-ray intensity vs. wavelength may be determined as in Fig.(1.8). This continuous x-ray spectrum results from glancing or indirect scattering of electrons from metal atoms. In such collisions only part of the electron's energy is converted to electromagnetic radiation. This radiation is called **bremsstrahlung** (German for braking radiation), which refers to the radiation given off by any charged particle when it is *decelerated*.

- The λ_{min} is independent of the target composition and only depends on the tube voltage, V . This might be explained by assuming it to the case of a *head-on electron-atom collision* in which all of the incident electron's kinetic energy is converted to electromagnetic energy in the form of a single x-ray photon. For this case we have

$$eV = hf = \frac{hc}{\lambda_{min}} \quad \text{or} \quad \lambda_{min} = \frac{hc}{eV} \quad (1.23)$$

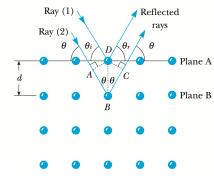


Figure 1.7: Bragg scattering of x-rays from successive planes of atoms.

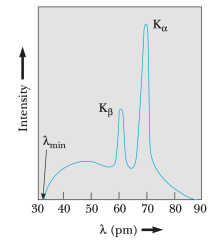


Figure 1.8: The x-ray spectrum of a metal target.

The Compton Effect In 1922, Arthur Holly Compton confirmed that x-ray photons behave like particles with momentum hf/c .

- classical theory predicted that incident radiation of frequency f_0 should accelerate an electron in the direction of propagation of the incident radiation, and it should cause *forced oscillations* of the electron and reradiation at frequency f' , where $f' \leq f$.² The frequency or wavelength of the scattered radiation should depend on the length of time the electron was exposed to the incident radiation as well as on the intensity of the incident radiation.

- Compton's result was that the wavelength of the x-ray is independent of the intensity of radiation and exposure duration, and depends only on the scattering angle!.

★ Quantum model easily explained the lower scattered frequency f' , where the incident photon gives some its energy hf and momentum to the recoiling electron.

- From Fig.(1.9) the shifted peak at λ' is caused by the scattering of x-rays from nearly free electrons. Assuming the photon exhibits particle-like behavior and collides elastically with a free electron initially at rest, Fig.(1.10). Because electron recoils at high speed, we treat the collision relativistically. Thus from the energy conservation,

$$E + m_e c^2 = E' + E_e \quad (1.24)$$

where E is incident photon energy, $m_e c^2$ is the rest mass-energy of the electron, E' is the energy of scattered photon, E_e is the total relativistic energy of the electron after the collision. Likewise, by the momentum conservation,

$$p = p' \cos \theta + p_e \cos \phi \quad p' \sin \theta = p_e \sin \phi \quad (1.25)$$

where p is the momentum of the incident photon, p' is the momentum of the scattered photon, and p_e is the recoiling electron momentum. We observed the angle of the scattered photons θ , so we shall solve Eq.(1.24) and Eq.(1.25) simultaneously to eliminate ϕ ,

$$p_e^2 = p'^2 + p^2 - 2pp' \cos \theta \quad (1.26)$$

Paradoxically, we need to use the wave nature of light to explain the particle-like behavior of photons. Recall that the energy of a photon and the frequency of the associated light wave are related by $E = hf$. If we assume that a photon obeys the relativistic express $E^2 = p^2 c^2 + m^2 c^4$, and the photon mass is zero,

$$p_{\text{photon}} = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda} \quad (1.27)$$

A paradoxical situation; a particle property, the photon momentum, is given in terms of a wave property, λ , of an associated light wave. If the relations $E = hf$ and $p = hf/c$ are substituted into Eq.(1.24) and Eq.(1.26), these become respectively

$$E_e = hf - hf' + m_e c^2, \quad p_e^2 = \left(\frac{hf'}{c}\right)^2 + \left(\frac{hf}{c}\right)^2 - \frac{2h^2 f f'}{c^2} \cos \theta \quad (1.28)$$

Because the Compton measurements do not concern the total energy and momentum of the electron, we eliminate E_e and p_e by substituting Eq.(1.27) into the following expression for the electron's relativistic energy,

$$E_e^2 = p_e^2 c^2 + m_e^2 c^4 \quad (1.29)$$

²This decrease in frequency of the reradiated wave is caused by a double Doppler shift, first because the electron is receding from the incident radiation, and second because the electron is a moving radiator as viewed from the fixed lab frame

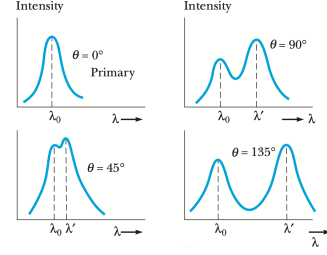


Figure 1.9: Scattered x-ray intensity versus wavelength of Compton scattering.

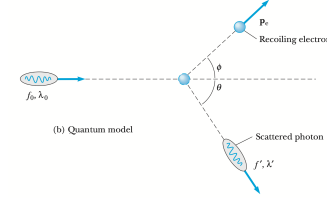


Figure 1.10: The quantum view

And we get the Compton's equation,

$$\lambda - \lambda' = \frac{h}{m_e c} (1 - \cos \theta) \quad (1.30)$$

- $\lambda_C = \frac{h}{m_e c}$ is Compton wavelength = 0.0243 Å

Particle-Wave Complementarity Compton effect offers evidence that when light *interacts* with matter it behaves as if it were composed of particles with energy hf and momentum h/λ .

Chapter 2

The Particle Nature of Matter

2.1 The Atomic Nature of Matter

French chemist Lavoisier and his wife, who established the conservation of matter in many careful chemical experiments;

Dalton, who perceived the atomicity of nature in the law of multiple proportions of compounds;

Avogadro, who in a most obscure and little-appreciated paper, postulated that all pure gases at the same temperature and pressure have the same number of molecules per unit volume;

and Maxwell, who showed with his molecular-kinetic theory of gases how macroscopic quantities, such as pressure and temperature, could be derived from averages over distributions of molecular properties. Jean Perrin and the ubiquitous Albert Einstein, who carried on very important theoretical and experimental work concerning Brownian motion, the zigzag movement of small suspended particles caused by molecular impacts. Their work produced additional confirmation of the atomic-molecular hypothesis and resulted in improved values of Avogadro's number as late as the early 1900s.

2.2 The Composition of Atoms

There were four major successful works:

- The discovery of the law of electrolysis in 1833 by Michael Faraday.
- The identification of cathode rays as electrons and the measurement of the charge-to-mass ratio (e/m_e) of these particles by Joseph John (J. J.) Thomson in 1897.
- The precise measurement of the electronic charge (e) by Robert Millikan in 1909.
- The establishment of the nuclear model of the atom by Ernest Rutherford and coworkers Hans Geiger and Ernest Marsden in 1913.

Faraday's Experiment of the electrolysis of molten common salt (NaCl). Faraday found that if 96 500 C of charge (1 faraday) is passed through such a molten solution, 23.0 g of Na will deposit on the cathode and 35.5 g of chlorine gas will bubble off the anode, see Fig.(2.1). In this case, exactly 1 gram atomic weight or mole of each element is released because both are monovalent. For divalent and trivalent elements, exactly 21 and 13 of a mole, respectively, would be released.

- Doubling the quantity of charge passed doubles the mass of neutral element freed. Thus **Faraday's law of electrolysis**,

$$m = \frac{qM}{(96\,500)(valence)} \quad (2.1)$$

- m the mass of the released substance in grams
- M the molar mass in grams
- q the total charge passed in coulombs

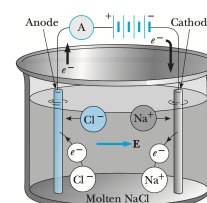


Figure 2.1: Electrolysis of molten NaCl.

- Faraday's law of electrolysis is explained in terms of atoms picture, see Fig.(2.1). Charges pass through the molten solution in form of ions, which they left an *excess* or *deficiency* of one or more electrons. By the electric field of the battery, these ions move to the anode (where they lose e^-) or the cathode (where they gain e^-) and are released as neutral elements.

★ Although it was not clear in 1833, Faraday's law of electrolysis confirmed three important ideas:

1. Matter consists of molecules and that molecules consist of atoms.
2. Charge is quantized, because only integral numbers of charges are transferred at the electrodes.
3. The subatomic parts of atoms are positive and negative charges, although they remained unknown.

Joseph John (J. J.) In 1897, Thomson discovered that the “rays” seen in low-pressure gas discharges were actually caused by negative particles (electrons) ended a debate dating back nearly 30 years: Were cathode rays material particles or waves?

- In Thomson's e/m_e experiment, electrons are accelerated from the cathode to the anode, collimated by slits in the anodes, and then allowed to drift into a region of perpendicular electric and magnetic fields. The \vec{E} and \vec{B} first are applied to produce an undeflected beam. If the \vec{B} field is then turned off, the beam deflected on the phosphorescent screen. From the deflection and measured values of \vec{E} and \vec{B} the charge-to-mass ratio, e/m_e can be determined.

- In Thomson experiment, see Fig.(2.2), consider when only \vec{E} exists between the plates. v_x remains the constant. y_x is constant except between the plates, where the e^- undergoes a constant upward acceleration due to electric force following a parabolic path.

To solve for the deflection angle, λ , we must solve for v_x and v_y . Because v_y initially is zero, e^- leaves the plates with y component velocity given by

$$v_y = a_y t \quad (2.2)$$

We know that $a = F_E/m_e = Ee/m_e = Ve/m_e d$, and $t = \ell/v_x$, where V is the applied voltage. Thus

$$v_y = \frac{V\ell}{m_e d v_x} \quad (2.3)$$

From Fig.(2.2), $\tan \theta = v_y/v_x$ and assuming $\tan \theta \approx \theta$,

$$\theta \approx \frac{V\ell}{dv_x^2} \left(\frac{e}{m_e} \right) \quad (2.4)$$

To find v_x , Thomson apply a \vec{B} field with existing \vec{E} field to balance the deflection, then their forces must be equal like

$$F_E = F_B \Leftrightarrow qE = qv_x B \quad \text{or} \quad v_x = \frac{E}{B} = \frac{V}{Bd} \quad (2.5)$$

Substituting Eq.(2.5) into Eq.(2.4),

$$\boxed{\frac{e}{m_e} = \frac{V\theta}{B^2 \ell d}} \quad (2.6)$$

★ Thomson's original value was only about 1×10^{11} C/kg (accepted value 1.758803×10^{11} C/kg), prior experiments on the electrolysis of hydrogen ions had given q/m values for hydrogen of about 10^8 C/kg. Thus it was clear that Thomson had discovered a particle with a mass about 1000 times smaller than the smallest atom.

★ Thomson noted that the e/m_e ratio was independent of the discharge gas and the cathode metal. Also, the particles emitted when electrical discharges were passed through different gases were found to be the same as those observed in the photoelectric effect. Based on these observations, Thomson concluded that these particles must be a universal constituent of all matter.

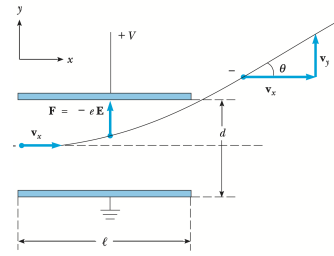


Figure 2.2: Deflection of negative particle

Milikan's Value of the Elementary Charge In 1897, Thomson had been unable to determine e or m_e separately. About two years later, he was close to the value of e (1.602×10^{-19} C) with values of 2.3×10^{-19} C for charges emitted from zinc illuminated by ultraviolet light and 1.1×10^{-19} C for charges produced by ionizing x rays and radium emissions.

- He concluded that “ e is the same in magnitude as the charge carried by the hydrogen atom in the electrolysis of solutions.” The technique used by Thomson and his students to measure e is especially interesting because it represents the first use of the **cloud chamber technique** in physics and also formed the starting point for the famous Millikan oil-drop experiment.

- Charles Wilson, one of Thomson's students, had discovered that ions act as condensation centers for water droplets when damp air is cooled by a rapid expansion. Thomson used this idea to form charged clouds by using the apparatus.

Q is the measured total charge of the cloud, W is the measured weight of the cloud, and v is the rate of fall or terminal speed. Thomson assumed that the cloud was composed of spherical droplets having a constant mass (no evaporation) and that the magnitude of the drag force D on a single falling droplet was given by **Stokes's law**,

$$D = 6\pi a\eta v \quad (2.7)$$

- a the radius of the droplet
- η the viscosity of the air
- v the terminal speed of the droplet

The following procedure was used to find a and w , the weight of a single drop. Because v is constant, the droplet is in equilibrium under its weight, w , and the drag force, D . Hence, we require that $w = D$, or

$$w = D \quad \equiv \quad \frac{4}{3}\pi a^3 \rho g = 6\pi a\eta v \quad \text{so} \quad a = \sqrt{\frac{9\eta v}{2\rho g}} \quad (2.8)$$

where ρ is the mass density of the droplet. We can find w , the weight. Then the number of drops n (or number of ions) is given by W/w and the electronic charge e is equal to Q/n , assuming that each droplet carries only *one* electronic charge. This method is inaccurate because the theory applies only to a *single* particle and the particles are all assumed to be *identical* in order to compare the theory to experiments performed on a cloud.

- By observing single droplets Millikan eliminated the problems of assuming all particles to be identical and of making uncertain measurements on a cloud. Millikan's basic idea was to measure the rate of fall of a single drop acted on by gravity and drag forces, apply Stokes's law to determine the drop radius and mass, then to measure its upward velocity in an opposing electric field, and hence determine the total charge on an individual drop¹.

- Millikan's experiment is to pass oil droplet charged by an atomizer into a small hole in the upper plate of parallel-capacitor plates. These droplets can be seen against the darkness when they are illuminated. If an \vec{E} field is applied to the capacitor plates, the drop may move slowly upward. A single droplet with constant mass and radius may be followed for hours, alternately *rising* and *falling*, by simply turning the \vec{E} field on and off. The charge is found after a long series of measurements of constant upward velocities one observes a discontinuous change or jump to a different upward velocity (higher or lower). *This discontinuous change is caused by the attraction of an ion to the charged droplet and a consequent change in droplet charge.* Such changes become more frequent when a source of ionizing radiation is placed between the plates.

¹This idea was first applied to charged clouds of water vapor by H. A. Wilson in 1903. Millikan switched from water to oil to avoid problems of a changing droplet mass and radius caused by water evaporation.

- The quantitative analysis of the Millikan experiment starts with Newton's second law applied to the oil drop, $\sigma F_y = ma_y$. Because the drag force D is large, a constant velocity of fall is quickly achieved, and all measurements are made for the case $a_y = 0$, or $\sigma F_y = 0$. If we assume that the magnitude of the drag force is proportional to the speed ($D = Cv$), and refer to Figure 4.9, we find

$$\begin{aligned} Cv - mg &= 0 & \vec{E} \text{ off} \\ q_1 E - mg - Cv'_1 &= 0 & \vec{E} \text{ on} \end{aligned}$$

Eliminating C ,

$$q_1 = \frac{mg}{E} \left(\frac{v + v'_1}{v} \right) \quad (2.9)$$

When the droplet undergoes change in its *upward velocity* from v'_1 to v'_2 (other remain constant),

$$q_2 = \frac{mg}{E} \left(\frac{v + v'_2}{v} \right) \quad (2.10)$$

Dividing Eq.(2.9) by Eq.(2.10)

$$\boxed{\frac{q_1}{q_2} = \frac{v + v'_1}{v + v'_2}} \quad (2.11)$$

★ In fact Eq.(2.10) is a proof of the quantization of charge, because if successive speed ratios are ratios of *whole* numbers, successive charges on the drop must be multiples of the same elementary charge.

- To determine the actual value of e , we need to find the mass. We can use the droplet radius a , from Stokes's law in Eq.(2.8). Thus the mass is

$$m = \rho \cdot \text{volume} = \rho \frac{4}{3} \pi a^3$$

- Stokes's law, as Millikan was aware, is only approximately correct for tiny spheres moving through a gas. The expression $D = 6\pi a\eta v$ holds quite accurately for a 0.1 cm radius sphere moving through a liquid or for any case where the moving-object radius, a , is large compared with the mean free path, L , of the surrounding molecules. (The mean free path is essentially the average distance between molecules.) In the Millikan experiment, however, a is of the same order of magnitude as the mean free path of air at STP. Consequently, Stokes's law overestimates the drag force, because the droplet actually moves for appreciable times through a frictionless "vacuum." Millikan corrected Stokes's law by using a drag force whose magnitude is

$$D \frac{6\pi a\eta v}{1 + \alpha(l/a)} \quad (2.12)$$

and found that $\alpha = 0.81$ gave the most consistent values of e for drops of different radii. Further corrections to Stokes's law were made by Perrin and Roux.

Rutherford's Model of the Atom Ernest Rutherford, and his students Ernest Marsden and Hans Geiger conducted a series of experiments 1909-1914. By noticing a beam of α particles broadened on a passing through a metal foil easily penetrated the thin film of metal. Rutherford's apparatus is about: a collimated beam of α particles emitted with speed 2×10^7 m/s hit a thin gold foil. Most of the α 's passed straight through the foil, but some were *scattered* at angle ϕ . The number of scattered α 's at each angle per unit detector area and per unit time was measured by aid of a microscope.

- Rutherford's insight was that, the mass and kinetic energy of the α 's are large, even for a head-on collision with a particle of mass of a hydrogen atom would only deflect the α particle with angle of 1° !
- However the repulsion force on a α particle would much greater the positive charge is assumed to be concentrated at a single point.

- Rutherford assumed that the large-angle scattering is produced by a single nuclear collision and the repulsion force is given by Coulomb's law

$$F = k \frac{(2e)(Ze)}{r^2} \quad (2.13)$$

where r is the distance between α & the nucleus, $+2e$ is the charge of α , and Ze is the nucleus charge. Rutherford found that the number of α particles entering the detector per unit time, Δn at an angle ϕ ,

$$\Delta n = \frac{k^2 Z^2 e^4 N n A}{4R^2 (\frac{1}{2} m_\alpha v_\alpha^2)^2 \sin^2 \phi} \quad (2.14)$$

- R distance of the detector and the foil
- ϕ scattering angle
- N number of nuclei per unit area of the foil
- n total number of α particles
- A area of the detector

Geiger confirmed the dependence of the scattering angle, the foil thickness, and the α speed.

- Rutherford realized that Eq.(2.13) holds iff the α particle does not have enough energy to deform or penetrate the scattering nucleus, he looked for the threshold where energy α should only just penetrate the nuclear radius at closest approach, so

$$\frac{1}{2} m_\alpha v_\alpha^2 = k \frac{(2e)(Ze)}{d_{min}} \quad (2.15)$$

At high kinetic energy this equation begins to fail. Rutherford confirmed the failure for Eq.(2.14) and Eq.(2.15) for heavy metals.

★ As a result Rutherford and his students had shown that all the mass and charge of Ze were concentrated in a minute nucleus, and that the electron must circle this nucleus.

However there were three questions raised:

1. If in the nucleus only the protons, where are the other half of the nuclear mass?
2. What is the force that holds protons in such small distance of 10×10^{-14} m?
3. How do the electrons move around the nucleus?

Rutherford's answer for the first one is that there are additional groupings of neutral particles of electron-proton pair. For the second one, that protons are held by an intense electrical force. However, James Chadwick discovered it is a very strong force as well as the neutron. The last question was done by Niels Bohr.

The Bohr Atom Recall that glowing solids and liquids (even gases at high densities) emit a continuous distribution of wavelengths. This exhibits the blackbody radiation curve of intensity vs. wavelength.

- In contrast, the discrete line spectrum emitted by low-pressure gas subject to an electric discharge. When light from such setup is examined through a spectroscope, it is found to consist of a few bright lines.
- Also these lines are a characteristic of a particular element. In 1859, Kirchhoff contributed to spectroscopy, where there were the foundation of **absorption spectroscopy**. Where this explained Joseph Fraunhofer's dark D-lines in the solar spectrum.
- Not all the emission lines exist in absorption lines. They depend on the temperature and of the absorbing vapor.

Lyman Series (uv)	$n_f = 1$
Balmer Series (vis-uv)	$n_f = 2$
Paschen Series (IR)	$n_f = 3$
Brackett Series (IR)	$n_f = 4$
Pfund Series (IR)	$n_f = 5$

Table 2.1: Special Series for the Hydrogen Atom

- Anders Ångström, measured the four visible lines of hydrogen atom, while in 1885, Johann J. Balmer found a formula that predict the wavelengths of these visible lines: H_α red, H_β green, H_γ blue, and H_σ violet. Balmer's formula is

$$\lambda(cm) = C_2 \left(\frac{n^2}{n^2 - 2^2} \right), \quad n = 3, 4, 5, \dots \quad (2.16)$$

where λ is the wavelength emitted in cm and $C_2 = 3645.6 \times 10^{-8} cm$, called the **convergence limit** because it gives the wavelength of the line with largest n . Today Balmer's formulas is in one single equation

$$\frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \quad (2.17)$$

where n_f and n_i are integers, and R Rydberg wavelength.

- Electron revolving idea around the nucleus has led to a disaster. According Maxwell's theory, accelerated charges revolving with orbital frequency f should radiate light with frequency f . As an electron radiate, its orbit radius decreases and its frequency increases, falling into the nucleus. Bohr postulated that classical radiation theory, which had been confirmed by Hertz's experiment, did not hold to atomic-sized systems. He used Plank and Einstein work, where he postulated that: electrons in atoms are generally confined to certain stable, nonradiating energy levels and orbits known as **stationary states**. He also used Einstein's concept of the photon.

Assumptions of the Bohr Theory For an atom of hydrogen:

- The electron moves in circular orbits around the protons under the influence of the Coulomb force.
- Only certain orbits are stable, where the electrons do not radiate, Hence the energy is fixed and classical mechanics is valid.
- Radiation is emitted by the atom when the electron 'jumps' from a higher energetic state to a lower one. This jump cannot be visualized or treated
- classically. **The frequency f of the emitted photon is independent of the electron's orbital motion.**

$$E_i - E_f = hf \quad (2.18)$$

- The size of allowed orbits is determined by an additional quantum condition imposed on the electron's orbital angular momentum; an integer multiple of \hbar .

$$m_e v r = n\hbar, \quad n = 1, 2, 3, \dots \quad (2.19)$$

Now the total energy of the electron-proton system is

$$E = K + U = \frac{1}{2} m_e v^2 - k \frac{e^2}{r} \quad (2.20)$$

Applying Newton's second law $F_r = m v^2 / r$,

$$\frac{k e^2}{r^2} = \frac{m_e v^2}{r}$$

Thus the kinetic energy

$$K = \frac{m_e v^2}{2} = \frac{ke^2}{2r} \quad (2.21)$$

Substituting Eq.(2.21) in Eq.(2.20),

$$E = -\frac{ke^2}{2r} \quad (2.22)$$

The negative indicate that this is a *bound* electron-proton system. We can find the orbits radius from Eq.(2.19) and Eq.(2.22),

$$r_n = \frac{\hbar^2}{m_e k e^2} n^2, \quad n = 1, 2, 3, \dots \quad (2.23)$$

The r_1 is called Bhor radius

$$a_0 = \frac{\hbar^2}{m_e k e^2} = 0.529 \text{ \AA}$$

The quantization of orbits resulted in the quantization of energy, substituting $r_n = n^2 a_0$ in Eq.(2.22),

$$E_n = -\frac{ke^2}{2a_0} \left(\frac{1}{n^2} \right) \quad (2.24)$$

$$E_n = -\frac{13.6 \text{ eV}}{n^2}$$

n corresponds to the discrete, or quantized vlues of the atom's energy, called **quantum numbers**. $n = 1$ is called the **ground state**, and $n = 2$ is called the **first excited state**. $n = \infty$ and $E = 0$, the electron is removed. The minimum required energy to remove an electron is called **ionization energy**.

By Eq.(2.24) and Bohr's third postulate, we can find the wavelength of emitted photon when it jumps,

$$f = \frac{E_i - E_f}{h}$$

or

$$\frac{1}{\lambda} = \frac{f}{c} = \frac{ke^2}{2a_0 h c} \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \quad (2.25)$$

This *theretical* result is identical to Balmer's *empirical* formula in Eq.(2.17).

- With this, Bohr explained the apectral series by the electrons trnsitions. He extended his model for hydrogen to other atoms; whose on electron is removed (ionized). Tn general, a single electron orbiting a nucleus of charge $+Ze$, Bohr's theory gives,

$$r_n = \frac{a_0}{Z} n^2, \quad \text{and} \quad E_n = -\frac{ke^2}{2a_0} \left(\frac{Z^2}{n^2} \right) \quad (2.26)$$

Beside all that Bohr explained more in his three-part papaer in 1913:

- He explained why fewer lines are seen unthe absorption hydrogen than in the emission spectrum.
- He explained the emmision of x rays from atoms.
- He explained the nuclear origin of β particles.
- He explained the chemical properties of atoms in terms of the electron shell model.
- He explained how atoms associate to form molecules.
- To go deeper, Bohr explained the absorption as the reverse process of emission. Ordinarily, hydrogen are in the ground state ($n = 1$), so only the Lyman series will take plce. Balmer series is not seen because the average thermal energy of each atoms is not sufficient to make the electron in the first excited state. That is, the number of electrons in $n = 2$ is insufficient in low temperatures.

- Bohr, in his paper part II, attempted to find stable electronic arrangements so that the total angular momentum of all the electrons is quantized and the total energy is minimum. Although this is so difficult when more electrons are introduced in the system, Bohr succeeded in showing that the hydrogen atom can take another electron to become an ion H^- , and the helium was stable by having a closed inner shell with two electrons. He also, considered the lithium ($Z=3$), and explained the tendency of lithium to lose the an electron.

★ Bohr's predictions about the multielectron atoms:

- Electrons of elements with higher atomic number form stable concentric *rings*, with definite numbers of electrons allowed for each ring.
- The number of electrons in the outer most ring determines the *valency*.

2.3 Bohr's Correspondence Principle; Quantization of Angular Momentum

Bohr's Correspondence Principle: **predictions of quantum theory must correspond to the predictions classical physics in the region of sizes where classical theory is known to hold.** For the classical sizes of length, mass, and time, if the quantum number becomes large because of increase size of these quantities, we can say that

$$\lim_{n \rightarrow \infty} [\text{quantumphysics}] = [\text{classicalphysics}]$$

This principle becomes a useful tool to test a new quantum results, As Bohr himself did to arrive to the concept of the quantized orbital angular momentum. Bohr argued that his correspondence principle, the quantum condition for emission ($\Delta E = hf$) and Maxwell's classical radiation theory (accelerated charges with orbital frequency f radiate light with the same frequency) *must simultaneously hold for the case of the extremely large electronic orbits.*

- Consider two large adjacent orbits r_1 and r_2 , separated with energy dE , and orbital angular frequency ω , where the last is approximately constant in transition between large orbits. To get know the allowed values of the angular momentum from the atom's change in energy, we need a connection between *total energy* of the atom Eq.(2.22), and the *total momentum* of the atom ($L = m_e v r = m_e \omega r^2$). If the electron is held by Coulomb force, we can show that

$$\frac{ke^2}{r^2} = \frac{m_e v^2}{r} \Rightarrow \frac{1}{r} = \frac{m_e k e^2}{L^2} \Rightarrow E = -\frac{m_e k^2 e^4}{2L^2} \quad (2.27)$$

Taking the derivative to get the connection between the change in energy and the change in momentum,

$$\frac{dE}{dL} = \frac{m_e k^2 e^4}{L^3} \quad (2.28)$$

Before let us find L in terms of ω (electron orbital angular frequency), from Eq.(2.27)

$$L^2 = m_e k e^2 v r \Rightarrow L^3 = (m_e v r) m_e k e^2 r \Rightarrow \quad (2.29)$$

Again from Eq.(2.27)

$$k e^2 = m_e v^2 r \Rightarrow m_e v r = \frac{k e^2}{v} \quad (2.30)$$

Substituting in Eq.(2.28)

$$\frac{dE}{dL} = \omega \quad (2.31)$$

Now consider the emission of a photon of energy $dE = \hbar \omega'$, transition from r_1 to r_2 , Eq.(2.31) becomes

$$dE = \omega dL \quad \text{or} \quad \hbar \omega' = \omega dL \quad (2.32)$$

where ω' is the photon angular frequency, and ω is the electron orbital frequency. Normally, they are not related, but according the correspondence principle for such large orbits, quantum theory must give the same frequency as the Maxwell's law of radiation. Thus, Maxwell's theory requires that the orbital frequency equals to the light frequency, $\omega' = \omega$, hence

$$\hbar\omega = \omega dL \quad \text{or} \quad dL = \hbar \quad (2.33)$$

Eq.(2.33) shows that the change in the electronic angular momentum for transition between adjacent, large orbits, is *always* \hbar . Thus, the magnitude of total angular momentum of an electron in a orbit is an integer multiple of \hbar

$$L = m_e v r = n\hbar \quad (2.34)$$

Bhor relized that although Eq.(2.34) was derived for large orbits case, it was a universal quantum principle.

2.4 Direct Confirmation of Atomic Energy Levels

We saw before the idea of the quantized energy levels in atoms, by observing the the optical line apectra. Howeverm in 1914, a directed experiment was conducted by James Franck and Gustav Hertz.

As in Fig.(2.4), Franck-Hertz apparatus is about electrons emitted by the filament and accelerated throught long regoin ($\approx 1cm$)by a positive potential on the grid, V . The electrons can reach the clector and be detected by elctrometer (a sensitive ammeter), if they have enough energy to pass potential of 1.5V between the grid and the collector.

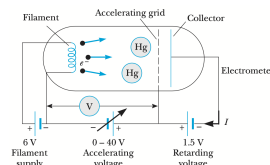


Figure 2.3: Franck-Hertz apparatus

- At low electron energies or accelerating voltages, perfectly elastic collisions occur between the electrons and Hg atoms, in which the *sum* of the kinetic energies of electron-atom system is conserved. Since Hg atom is much more massive than the electron, the electron transfers litle KE to the atom. Even with multiple collisions, the electron each the grid with KE of $e \cdot V$.
- As the accelerating voltage is increased, a threshold voltage where the inelastic collisions occur at the grid, with $KE = e \cdot$. In such collisions electrons can transfer almost all of thier KE to the atom, raising it to its first excited state. Thus these electrons are unable to overcome the retarding voltage, hence the current I is drop.
- For the second dip in the graph, where V is increased enough so that an electron have two successive inelastic collisions.
- Franck and Hertz with ammeter and voltmeter measurements showed that atoms only accept discrete amounts of energ from electron beam. Also, they showed that the enrgy levels obtained from electron bombardment agreed with the spectroscopic results.

Chapter 3

Matter Wave

3.1 The Pilot Wave of De Broglie

By 1920s scientists realized that Bohr theory had some issues

- It failed to predict of the observed intensities of the spectral lines.
- It had limited success in predicting emission and absorption wavelengths for multielectron atoms.
- It failed to provide an equation of motion governing the time development of atomic systems starting from some initial state.
- It overemphasized the particle nature of matter and could not explain the newly discovered wave-particle duality of light.
- It did not supply a general scheme for "quantizing" other systems, especially those without periodic motion.

In 1923, Louis Victor de Broglie postulated that *because photons have wave and particle characteristics, perhaps all forms of matter have wave as well as particle properties*. According to him, electrons had a dual wave-particle nature.

He found the the wavelength of a *matter wave*

$$\lambda = \frac{h}{p} \quad f = \frac{E}{h} \quad (3.1)$$

- h is Planck's constant,
- p is the relativistic momentum
- E is the total relativistic energy

De Broglie's Explanation of Quantization in Bohr Model De Broglie applied the ideas of familiar wave properties such as standing wave and interference. He postulated that the electron as an standing wave bent around the nucleus with radius of Bohr orbit r . Thus only discrete set of wavelengths is allowed for standing waves. Amazingly, Bohr theory for quantization of angular momentum can be easily explained by de Broglie's hypothesis: **the allowed Bohr orbits arise because the electron matter waves interfere constructively when an integral number of wavelengths exactly fits into the circumference of a circle orbit**. Thus

$$n\lambda = 2\pi r \quad (3.2)$$

where r is the radius of the orbit. From Eq.(3.1), we have $\lambda = h/m_e v$, substituting this into Eq.(3.2) and solving for L , $m_e v r$

$$m_e v r = n\hbar \quad (3.3)$$

which is exactly Bohr condition for quantization of angular momentum.

The Davisson-Germer Experiment In 1924, de Broglie had suggested that a stream of electrons traversing a small aperture should undergo diffraction phenomena. In 1925, Einstein was led to the need of postulating matter waves from an analysis of fluctuations of molecular gas. Also, he noted that a molecular beam should show a small diffraction effects. In 1927, the proof of the wave nature was obtained by the work of Davisson and Germer, and George P. Thomson.

- Davisson-Germer's experiment was to understand the arrangement of atoms on the surface of a nickel sample by elastically scattering a beam of low-speed electrons from a polycrystalline nickel target. They had three parameters that can be changed— *electron energy*; *nickel target orientation*, α ; and *scattering angle*, ϕ .
- Their experiment was going normally, with constant electron energy of 100 eV, the scattered intensity rapidly decreased as ϕ increased. Accidentally, someone dropped a flask of liquid air on the glass vacuum system, tearing the vacuum and oxidizing the nickel target, that had been in high temperature. To remove the oxide, they reduced the sample by heating it carefully in a flowing stream of hydrogen. After this reassembling, they found very different results: *Strong variations in the intensity of scattered electrons with angle*.
- This heating annealed the nickel target, causing large single-crystal regions to develop in the polycrystalline sample. These crystalline regions provided the extended regular lattice needed to observe electron diffraction. Davisson and Germer realized that it was the elastic scattering from *single crystals* that produced such unusual results.
- The idea is that they calculated the wavelength of the electrons from simple diffraction formula and compared it with de Broglie's formula. Thus, for an electron we have

$$\frac{1}{2}m_e v^2 = eV$$

Substitute v into de Broglie relation

$$\lambda = \frac{h}{m_e v} = \frac{h}{\sqrt{2Vem_e}}$$

Also, the wavelength can be obtained by considering the nickel atoms to be a reflection diffraction grating. Only the atoms at the surface are considered, since x-rays cannot penetrate deeply into crystal. Thus to find the constructive interference we have

$$d \sin \phi = n\lambda \quad (3.4)$$

The diffraction lines from high-energy electrons are so sharp, and these electrons can penetrate deeper in the crystal. As a result, we need to use Bragg's law instead.

- Davisson and Germer concluded that this wavelength agrees very well with that from de Broglie's formula, confirming the electron matter wave. If de Broglie's postulate is true for all matter, then any objects with mass m has wavelike property.

The Electron Microscope The ability of controlling electron wavelength, even shorter than the visible light, provides us with a great tool to investigate small objects. First **transmission electron microscope** (TEM), was built by Max Knoll and Ernst Ruska in 1931; it focuses electron beams accelerated by 100 kV with magnetic lenses and creates a flat-looking two-dimensional shadow pattern on its screen, the result of varying degrees of electron transmission through the object.

- **magnification** An optical microscope using ultraviolet has a magnification of 2000 and resolving power of 100 nm. TEM can reach a magnification of 1 000 000 and resolving power of 0.2 nm
- A second type of electron microscope with less resolution and magnification than TEM, but capable of producing three-dimensional images, is called **scanning electron microscope** (SEM).

3.2 Wave Groups and Dispersion

The matter wave of a moving particle has a large probability to be found in a small region of space only at a specific time. Hence, the traveling sinusoidal matter wave of finite extent and constant amplitude cannot represent a localized moving particle.

- Thus we need to introduce a pulse or a **wave group** of limited spatial extent. This pulse can be found by adding up sinusoidal wave with different wavelengths. The resulting wave group will have a **group speed** v_g identical to the classical particle speed.

- In fact, all observed waves are limited to definite regions of space and are called *pulses*, *wave groups*, or *wave packets* in matter waves. A plane wave with an exact wavelength and an infinite extension is an abstraction. Water waves, light waves, waves along a string, and sound wave all must be modeled by a group wave.

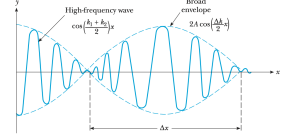


Figure 3.1: A wave group

★ A **wave group** consists of a superposition of waves with *different wavelengths*, with the amplitude and phase of each component wave adjusted to make a constructive interference over a small region of space. A good example is the beats in sound waves.

- Consider a one-dimensional wave propagating in the positive x direction with a **phase speed** v_p . This traveling wave with wavelength λ and frequency f , and amplitude A is written like

$$y = A \cos\left(\frac{2\pi}{\lambda}x - 2\pi ft\right) \quad \text{and} \quad v_p = \lambda f \quad (3.5)$$

take $\omega = 2\pi f$ (*angular frequency*) and $k = 2\pi/\lambda$ (*wavenumber*), thus

$$y = A \cos(kx - \omega t) \quad (3.6)$$

and the *phase velocity*

$$v_p = \frac{\omega}{k} \quad (3.7)$$

- Let us form a superposition of two waves having the same amplitude and traveling in the positive x , but different in wavelength, frequency, and phase velocities. Hence,

$$y = y_1 + y_2 = A \cos(k_1x - \omega_1t) + \cos(k_2x - \omega_2t) \quad (3.8)$$

Using the trigonometric identity: $\cos a + \cos b = 2 \cos \frac{1}{2}(a - b) \cos \frac{1}{2}(a + b)$, we get

$$y = 2A \cos \frac{1}{2}[(k_2 - k_1)x - (\omega_2 - \omega_1)t] \cos \frac{1}{2}[(k_1 + k_2)x - (\omega_1 + \omega_2)t] \quad (3.9)$$

For case of two waves with slightly different values of k and ω , we see that $\Delta k = k_2 - k_1$ $\Delta \omega = \omega_2 - \omega_1$ represent small wave, but $(k_1 + k_2)$ and $(\omega_1 + \omega_2)$ represent larger wave. So we can say that Eq.(3.9) has a broad sinusoidal envelope,

$$2A \cos\left(\frac{\Delta k}{2}x - \frac{\Delta \omega}{2}t\right) \quad (3.10)$$

modulating a high-frequency wave within the envelope

$$\cos\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right) \quad (3.11)$$

This superposition is shown in Fig.(3.2). We can find the speed of the high-frequency wave and the envelope wave by dividing the coefficient of t by the one of x ,

$$v_p = \frac{(\omega_1 + \omega_2)/2}{(k_1 + k_2)/2} \approx \frac{\omega_1}{k_1} = v_1 \quad (3.12)$$

and for the envelope velocity

$$v_g = \frac{\Delta \omega}{\Delta k} \quad (3.13)$$

- A general characteristic of wave groups is that both is a limited duration in time, Δt , and a limited extent in space, Δx . Thus the smaller the spatial width of the pulse, Δx , the larger the range of wavelengths or wavenumbers, k ,

$$\Delta x \Delta k \approx 1 \quad (3.14)$$

Likewise, the smaller the temporal duration, Δt , the larger the frequencies, ω

$$\Delta t \Delta \omega \approx 1 \quad (3.15)$$

In electronics, this condition is called *response time-bandwidth formula*.

★ **Both Δx and Δk cannot become arbitrarily small, but as one decreases the other must increase.** Our simple two-wave model can show the general principle given by Eq.(3.14) and Eq.(3.15). Let the spatial extent of our group be the distance between two adjacent minima ($y = 0$), labeled Δx in Fig.(3.2) we find from the envelope term: $2A \cos(\frac{1}{2}\Delta kx)$, then $\frac{1}{2}\Delta k \Delta x = \pi$

$$\Delta k \Delta x = 2\pi \quad (3.16)$$

where $\Delta k = k_2 - k_1$ is the present wavenumbers. Similarly, if we hold x constant and changing t , we have $\frac{1}{2}(\omega_2 - \omega_1)\Delta t = \pi$,

$$\Delta \omega \Delta t = 2\pi \quad (3.17)$$

as result we see that Eq.(3.16) and Eq.(3.17) agree with the *general principles*.

- The addition of two waves with discrete frequencies is informative, but produces an infinite wave instead of a true pulse. In general, many waves having a continuous distribution of wavelengths must be added to form a packet, finite over a limited range. In this case, the Eq.(3.13), for **group velocity** becomes,

$$v_g = \left. \frac{d\omega}{dk} \right|_{k_0} \quad (3.18)$$

where k_0 is the central wavenumber of the many waves present. We can find a connection between the group velocity and the phase velocity, if $\omega = kv_p$ then,

$$v_g = \left. \frac{d\omega}{dk} \right|_{k_0} = v_p \Big|_{k_0} + k \left. \frac{dv_p}{dk} \right|_{k_0} \quad (3.19)$$

- **Dispersion** happens in materials in which the *phase velocity* changes with wavelength.