PHYS306 - Summary Homework

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Abstract

So far we have dealt with infinite-extent plane waves; now we need to consider electromaganetic waves confined to the interior of a hollow pipe, called **wave guide**

Boundary conditoins We assume the wave guide is perfect conductor, so that $\vec{E} = \vec{0}$ and $\vec{B} = \vec{0}$ inside the the material itself, thus the boundary conditoins at the inner wall are

$$\begin{cases} \vec{E}^{||} = \vec{0} \\ \vec{B}^{\perp} = 0 \end{cases} \tag{1}$$

Wave Equations Notice that free charges and currents will be induced at the surface to enforce these constraints. We are intreseted in monochromatic waves that propagate through the tube. Thus, $\vec{E}$ and $\vec{B}$ are in form

$$\begin{cases} \vec{\tilde{E}}(x, y, z, t) = \vec{\tilde{E}}_0(x, y)e^{i(kz-\omega t)}, \\ \vec{\tilde{B}}(x, y, z, t) = \vec{\tilde{B}}_0(x, y)e^{i(kz-\omega t)}, \end{cases} \tag{2}$$

Also the electric magnetic field must satisfy Maxwell's equations in the interior of the wave guide:

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = 0 & \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} = 0 & \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \end{cases} \tag{3}$$

Now we need just to find those amplitudes, $\vec{\tilde{E}}_0$ and $\vec{\tilde{B}}_0$, such that the fields Eq.(2) obey the differential equations Eq.(3), subject to boundary conditions Eq.(1).

Longitudinla Component In general, *confined* eaves are not transverse; in order to fit the boundary conditoins we shall have to include longitudinal components E_z and B_z:

$$\vec{\tilde{E}}_0 = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} \qquad \vec{\tilde{B}}_0 = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} \tag{4}$$

where each componen is a function of x and y.

Pluging Eq.9.180 into Maxwell-divergence equations yields:

$$\begin{cases} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 + k^2\right] E_z = 0 \\ \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 + k^2\right] B_z = 0 \end{cases} \tag{5}$$

If $E_z = 0$, we call these **TE waves**; if $B_z = 0$, they are called **TM waves**; if both vanishes, we call it **TEM waves**. But, TEM situatoin can never happen in a hollow wave guide.

TE Waves in a Recatngular Guide Now we consider a wave guide of recatangular shape, with height a and width b, we want to study the propagation of TE waves. Thus, the problem is to solve B_z in Eq.(5), subject to the boundary condition of B in Eq.(1). By separation of varaiables,

$$B_z(x, y) = X(x)Y(y) \tag{6}$$

so that

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \left[(\omega/c)^2 - k^2)\right] = 0 \tag{7}$$

We notice that the x- and y-dependent terms must be constant, So

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2 \tag{8}$$

The general solution is

$$X(x) = A\sin(k_x x) + B\cos(k_x x) \tag{9}$$

But the boundary condition for B_x must be zero, that is, $dX/dx = 0$. This gives

$$k_x = m\pi/a, \quad k_y = n\pi/b \qquad m, n = 0, 1, 2, \ldots \tag{10}$$

Finaly, the solution of TE_{mn} mode is

$$B_z = B_0 \cos(m\pi/ax)\cos(n\pi/bx) \tag{11}$$

The wave number is given by

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]} \tag{12}$$

If the vlaue under the root is negative, there will be no propagating wave, but an exponentialy attenuated fileds. Hence we degine the **cutoff frequency** ω_{mn} as,

$$\text{lowest possible} < \omega_{mn} \equiv c\pi\sqrt{(m/a)^2 + (n/b)^2} \le \omega \tag{13}$$

The velocities are

$$v = \frac{\omega}{k}, \qquad v_g = \left(\frac{dk}{d\omega}\right)^{-1} \tag{14}$$