N Equally Delta-Potential Barriers

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1 Stating The Problem

Consider the one dimensional scattering of a particle of mass m through a series of N equally spaced δ-potential barriers. All are defined by

$$V(x) = \sum_{n=0}^{N} V_0\, \delta(x - x_n) \qquad \text{where} \quad x_n = na$$

where $V_0 > 0$ is the potential strength, and a is the spacing between successive barriers. For this case the TISE given as

$$\frac{d^2}{dx^2}\,\psi = -\frac{2m}{\hbar^2}(E - V)\,\psi$$

In this setup we have only the *scattering state*, since $E > 0$.

2 Solve Single barrier

We start with simplest case, with a sigle δ-potential barrier at a general index n; $x_n = na$. The next barrier will be at x_{n+1}, and the previous one at x_{n-1}.

For the left region of the nth barrier, $x_{n-1} < x < x_n$ and $V(x) = 0$, the TISE becomes

$$\frac{d^2}{dx^2}\,\psi = -\frac{2m}{\hbar^2}E\,\psi \quad \Rightarrow \quad \frac{d^2}{dx^2}\,\psi = -k^2\,\psi \quad \text{where} \quad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

The general solution to this DE is

$$\psi_n(x) = A_n e^{ikx} + B_n e^{-ikx}$$

The same for the right region of the nth barrier, $x_n < x < x_{n+1}$, the general solution is

$$\psi_{n+1}(x) = A_{n+1} e^{ikx} + B_{n+1} e^{-ikx}$$

In summary, for both regions, the general solution is

$$\psi(x) = \begin{cases} A_n e^{ikx} + B_n e^{-ikx} & x_{n-1} < x < x_n \\ A_{n+1} e^{ikx} + B_{n+1} e^{-ikx} & x_n < x < x_{n+1} \end{cases}$$

3 Apply Boundary Conditions

3.1 The Continuity at The Barrier

We require that $\psi(x_n^-) = \psi(x_n^+)$, so

$$A_{n+1}e^{ikx_n} + B_{n+1}e^{-ikx_n} = A_n e^{ikx_n} + B_n e^{-ikx_n}$$

Factor out e^{-ikx_n},

$$A_{n+1}e^{2ikx_n} + B_{n+1} = A_n e^{2ikx_n} + B_n$$

3.2 The Discontinuity of Derivatives

First we calculate the derivatives,

$$\frac{d\psi}{dx} = \begin{cases} ikA_n e^{ikx} - ikB_n e^{-ikx} & x_{n-1} < x < x_n \\ ikA_{n+1}e^{ikx} - ikB_{n+1}e^{-ikx} & x_n < x < x_{n+1} \end{cases}$$

To find the discontinuity we integrate the TISE around x_n, and taking the limit

$$\lim_{\epsilon \to 0} \int_{x_n-\epsilon}^{x_n+\epsilon} \frac{d^2\psi}{dx^2} dx = -\frac{2m}{\hbar^2} \lim_{\epsilon \to 0} \int_{x_n-\epsilon}^{x_n+\epsilon} (E - V)\psi \, dx$$

We require that

$$\left.\frac{d\psi}{dx}\right|_{x_n^+} - \left.\frac{d\psi}{dx}\right|_{x_n^-} = \frac{2mV_0}{\hbar^2}\psi(x_n)$$

$$ik(A_{n+1}e^{ikx_n} - B_{n+1}e^{-ikx_n}) - ik(A_n e^{ikx_n} - B_n e^{-ikx_n}) = \frac{2mV_0}{\hbar^2}\psi(x_n)$$

Substituting $\psi(x_n)$

$$ik(A_{n+1}e^{ikx_n} - B_{n+1}e^{-ikx_n}) - ik(A_n e^{ikx_n} - B_n e^{-ikx_n}) = \frac{2mV_0}{\hbar^2}(A_n e^{ikx_n} + B_n e^{-ikx_n})$$

Divide through ike^{-ikx_n}

$$A_{n+1}e^{2ikx_n} - B_{n+1} - (A_n e^{2ikx_n} - B_n) = -i\frac{2mV_0}{\hbar^2 k}(A_n e^{2ikx_n} + B_n)$$

In summary, from the boundary conditions we got

$$\begin{cases} A_{n+1}e^{2ikx_n} + B_{n+1} = A_n e^{2ikx_n} + B_n \\ A_{n+1}e^{2ikx_n} - B_{n+1} = A_n e^{2ikx_n} - B_n - i\frac{2mV_0}{\hbar^2 k}(A_n e^{2ikx_n} + B_n) \end{cases}$$

Adding those two equations

$$2A_{n+1}e^{2ikx_n} = 2A_n e^{2ikx_n} - i\frac{2mV_0}{\hbar^2 k}(A_n e^{2ikx_n} + B_n)$$

Isolate A_{n+1}

$$A_{n+1} = A_n - i\frac{mV_0}{\hbar^2 k}(A_n + B_n e^{-2ikx_n})$$

Writing in terms of A_n and B_n

$$A_{n+1} = \left(1 - i\frac{mV_0}{\hbar^2 k}\right) A_n - \left(i\frac{mV_0}{\hbar^2 k}e^{-2ikx_n}\right) B_n$$

Now we substract those two equations (2nd from 1st), to wit

$$2B_{n+1} = 2B_n + i\frac{2mV_0}{\hbar^2 k}(A_n e^{2ikx_n} + B_n)$$

Isolate B_{n+1}

$$B_{n+1} = B_n + i\frac{mV_0}{\hbar^2 k}(A_n e^{2ikx_n} + B_n)$$

Writing in terms of A_n and B_n

$$B_{n+1} = \left(i\frac{mV_0}{\hbar^2 k}e^{2ikx_n}\right) A_n + \left(1 + i\frac{mV_0}{\hbar^2 k}\right) B_n$$

Thus we have two equationd for B_{n+1} and B_{n+1} in terms of A_n and B_n

$$\begin{cases} A_{n+1} = (1 - i\beta) & A_n - (i\beta e^{-2ikx_n}) B_n \\ B_{n+1} = (i\beta e^{2ikx_n}) & A_n + (1 + i\beta) B_n \end{cases} \qquad \beta \equiv \frac{mV_0}{\hbar^2 k}$$

Or in matrix form

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = M_n \begin{bmatrix} A_n \\ B_n \end{bmatrix} \qquad M_n = \begin{bmatrix} 1 - i\beta & -i\beta e^{-2ikx_n} \\ i\beta e^{2ikx_n} & 1 + i\beta \end{bmatrix}$$

4 As a Black Box

The setup of the experiment is that there is an incident wave on the leftmost barrier, and we want to find the final resultant wave after transimtting through the rightmost barrier. Thus, the problem is to find the transmitted-wave amplitudes, $n = N + 1$, in terms of the incident-wave amplitudes, $n = 0$. For the transformation matrix we can take the product of all M_n elements. Hence, we have

$$\begin{bmatrix} A_{N+1} \\ B_{N+1} \end{bmatrix} = M \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \qquad M = \prod_{n=0}^{N} M_n \qquad |M| = 1$$

For this physically setup, there should be not traveling wave to the left at the rightmost barrier, since it will travell to infinity. Thus, we set $B_{N+1} = 0$.

5 Transmission Coefficient

The relative pobability of the incident wave to the transitted wave is called the *transmission coefficient*. It is defined as

$$T = \left|\frac{A_{N+1}}{A_0}\right|^2 = \left|\frac{\det(M)}{M_{22}}\right|^2 = \frac{1}{|M_{22}|^2}$$

6 Compute $T(\theta)$

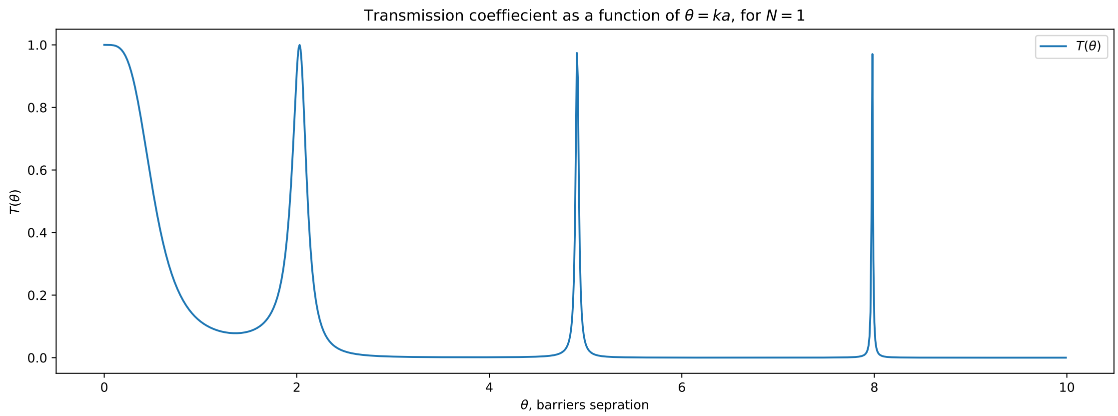
We want now to compute numerically the $T(\theta)$ as a fumction of the angle $\theta = ka$, for different values of n.

```
[ ]: import numpy as np
     import scipy as sp
     from matplotlib import pyplot as plt
     from matplotlib_inline import backend_inline
     plt.rcParams['figure.figsize'] = [15, 5]
     backend_inline.set_matplotlib_formats('png', 'pdf')
```

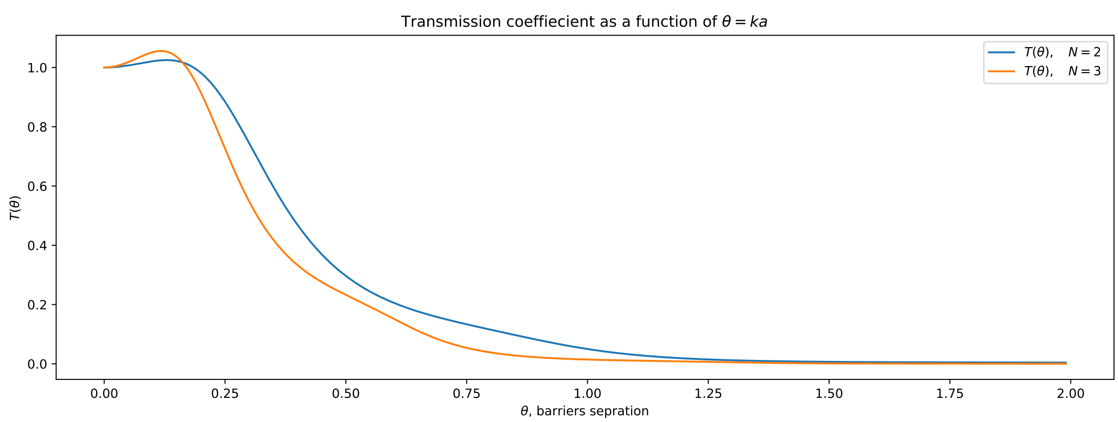
6.1 Transmission as Function of

We need to say the effect on the transmission coefficient when the wave number changes. We observe that there apppears a kind of resonant. for some values of θ the transmission is almost one, due to the constructive behavior of waves.Vice versa, some values porduves low probabilty for the particle to trunnel through.

```
[ ]: def Mn(n, theta):
         beta = theta / 1     # beta = theta / a
         return np.array([
             [ 1-1j*beta                , -beta*np.exp(-2j*n*theta) ],
             [ beta*np.exp(2j*n*theta), 1+1j*beta                  ],
         ])

     def M(N, theta):
         result = Mn(0, theta)
         for n in range(1, N+1, 1):
             result = result @ Mn(n, theta)
         return result

     def T(N, theta):
         M_ = M(N, theta)
         return abs(1 / M_[1][1]) **2

     thetas = np.arange(0, 10, 0.01)

     N = 1
     plt.plot(thetas, [T(N, x) for x in thetas], label=r'$T(\theta)$')
     plt.xlabel(r'$\theta$, barriers sepration')
     plt.ylabel(r'$T(\theta)$')
     plt.title(fr'Transmission coeffiecient as a function of $\theta=ka$, for␣
       ↪$N={N}$')
     plt.legend()
     plt.show()
```




```
[ ]: thetas = np.arange(0, 2, 0.01)

     N = 2
     plt.plot(thetas, [T(N, x) for x in thetas], label=rf'$T(\theta),\quad N={N}$')
     N = 3
     plt.plot(thetas, [T(N, x) for x in thetas], label=rf'$T(\theta),\quad N={N}$')
     plt.xlabel(r'$\theta$, barriers sepration')
     plt.ylabel(r'$T(\theta)$')
     plt.title(fr'Transmission coeffiecient as a function of $\theta=ka$')
     plt.legend()
     plt.show()
```



6.2 Number of Barriers

we check the effect of changing the number of barriers on transmission coeffiecient. We apply the matrix for 50 barriers. We notice that as the number of barriers increases the the tunneling probability decreases accordingly.

[ ]:

```
theta = 0.2
Ns = np.arange(50)
Ts = [T(N, theta) for N in Ns]

plt.plot(Ns, Ts, label=r'$T(N, \theta)$')
plt.xlabel('$N$, barriers number')
plt.ylabel(r'$T(N, \theta)$')
plt.title(fr'Transmission coeffiecient as a function of number of barriers, for␣
↪$\theta={theta}$')
plt.legend()
plt.show()
```