Ball and Beam System
EECE 460 Project

Project Report

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Abstract

The ball and beam system is a widely known nonlinear and unstable control problem that toggles the classical control theory principles. The system represents real-world problems making it a valuable tool for research and education. The ball and beam system can be modeled using Euler-Lagrange equations by considering its kinetic and potential energy[1] or Newton's Second Law of Motion by considering the sum of forces acting on the ball all of which capture the system behavior. Due to its inherently unstable nature, it can be modeled using classical controllers such as P, Lag, Lead, PD, Lag-lead, PI, or PID controllers [2]. Advanced controllers like Model Predictive Control (MPC) and Extended State Observer (ESO)-based and Linear Quadratic Regulators (LQR) also can be used [1] [3] [4] [2] [5].

I. Introduction

The ball and beam system is an important example of a control system in which the concepts of feedback, stability, and robustness are showcased. Its simplicity makes it an excellent problem to explore control concepts yet its nonlinear and unstable nature challenges advanced control theories. This Literature Review tackles the key aspects of the ball and beam system from an implementation, modeling, and control strategy perspective.

Modeling is the first step in understanding the system's dynamics to design an effective controller. The system can be modeled using the Euler-Lagrange method accounting for kinetic and potential energy (equations will be in the Mathematical Model section). Linearization simplifies the system for better analysis and design of classical controllers. However, advanced models can be used to consider nonlinearities and external disturbances to improve robustness in real-world applications [2] [5]. State-space representations and transfer functions are used to design controllers and do simulations (check the Mathematical Model section).

Classical control techniques are the go-to in terms of simplicity and effectiveness. It can be done using various controllers, not guaranteed all of them to work, such as P, Lag, Lead, PD, Lag-lead, PI, or PID controllers where every controller has its case to use if the changes need to be done to either transient or steady-state response [3] [2]. Modern control strategies solve the limitations of classical control techniques by incorporating robust and adaptive techniques.

II. Dynamic Model

A. Ball and Beam Model

Let KE_b be the Kinetic Energy of the ball

$$KE_b = \frac{1}{2}m_b v^2 + \frac{1}{2}J_b\omega^2 \tag{1}$$

If it is assumed there is no slipping, so:

$$\dot{r} = R_b\omega \tag{2}$$

Also, $v^2 = \dot{x}^2 + \dot{y}^2$, but $x = rcos(\alpha)$ and $y = rsin(\alpha)$.
Assuming α is small, then $x \approx r$ and $y \approx r\alpha$.
Thus, $\dot{x} = \dot{r}$ and $\dot{y} \approx \dot{r}\alpha$.
Now, substituting into 1 to get:

$$KE_b = \frac{1}{2}(\frac{J_b}{R_b^2} + m_b)\dot{r}^2 + \frac{1}{2}m_b r^2\dot{\alpha}^2 \tag{3}$$

Let U_b be the Potential Energy of the ball

$$U_b = m_b g sin(\alpha) r \tag{4}$$

Let KE_B be the Kinetic Energy of the beam

$$KE_B = \frac{1}{2}J_B\dot{\alpha}^2 \tag{5}$$

Let U_B be the Potential Energy of the beam

$$U_B = \frac{1}{2}L_B m_B g sin(\alpha) \tag{6}$$

Thus for the system of the beam and the ball:

$$KE = \frac{1}{2}(\frac{J_b}{R_b^2} + m_b)\dot{r}^2 + \frac{1}{2}(m_b r^2 + J_B)\dot{\alpha}^2 \tag{7}$$

and

$$U = g sin(\alpha)(m_b r + \frac{1}{2}L_B m_B) \tag{8}$$

Now, using the Lagrangian:

$$\mathcal{L} = KE - U \tag{9}$$

$$\frac{\partial}{\partial t}(\frac{\partial \mathcal{L}}{\partial \dot{r}}) - \frac{\partial \mathcal{L}}{\partial r} = F_{friction} \tag{10}$$

Hence:

$$(\frac{J_b}{R_b^2} + m_b)\ddot{r} + m_b g sin(\alpha) - m_b r \dot{\alpha}^2 = F_{friction} \tag{11}$$

Assuming α is small and no friction, we get:

$$(\frac{J_b}{R_b^2} + m_b)\ddot{r} + m_b g \alpha = 0 \tag{12}$$

When transforming into the Laplace domain, the resulting transfer function is:

$$\frac{r(s)}{\alpha(s)} = \frac{-m_b g}{(\frac{J_b}{R_b^2} + m_b)s^2} = 0 \tag{13}$$

1) Equations of Motion(including damping and stiffness): The governing equation for the ball and beam system, considering damping, is given by:

$$\left(\frac{J_b}{R_b^2} + m_b\right)\ddot{r} + b\dot{r} + cr + m_b g \alpha = 0. \tag{14}$$

Here: - $\frac{J_b}{R_b^2} + m_b$ represents the effective mass, - b is the damping coefficient, - c is due to non-linearities, - r is the radial position of the ball, - α is the angle of the beam, and - $m_b g \alpha$ represents the gravitational force acting on the ball.

2) Laplace Transform: Applying the Laplace transform to the equation of motion (with zero initial conditions):

$$\left(\frac{J_b}{R_b^2} + m_b\right)s^2 R(s) + bsR(s) + cR(s) + m_b g \alpha(s) = 0. \tag{15}$$

Here, $R(s)$ and $\alpha(s)$ are the Laplace transforms of the radial position $r(t)$ and the beam angle $\alpha(t)$, respectively.

3) Rearranging for Transfer Function: Rearrange the above equation to solve for the transfer function $\frac{R(s)}{\alpha(s)}$:

$$\frac{R(s)}{\alpha(s)} = \frac{m_b g}{\left(\frac{J_b}{R_b^2} + m_b\right)s^2 + bs + c}. \tag{16}$$

This is the transfer function relating the position of the ball $r(t)$ to the angle of the beam $\alpha(t)$.

B. Motor Model

Using Newton's Second law we can derive:

$$J\ddot{\theta} + b\dot{\theta} = K_m i \tag{17}$$

Using Kirchhoff's laws we derive:

$$L_m \frac{di}{dt} + R_m i = V_m - K_m \dot{\theta} \tag{18}$$

Transforming the equations to the Laplace domain and substituting to get the relation between V and θ we get:

$$\frac{\theta(s)}{V(s)} = \frac{K_m}{L_m J s^3 + (bL_m + JR_m)s^2 + (bR_m + K_m^2)s} \tag{19}$$

Since L_m is small, the equation simplifies to become:

$$\frac{\theta(s)}{V(s)} = \frac{K_m}{JR_m s^2 + (bR_m + K_m^2)s} \tag{20}$$

C. System Model

We know:

$$\alpha = \frac{d}{L_B}\theta \tag{21}$$

Using equations 13, 20 and 21 we get:

$$\frac{r(s)}{V(s)} = \frac{1}{s^3} \cdot \frac{-m_b g K_m d/(J_{eq} R_m L_B (\frac{J_b}{R_b^2} + m_b))}{s + (b + K_m^2/R_m)/J_{eq}} \tag{22}$$

D. System Identification

We did many trials using Matlab's system identification tool to get the best-fit transfer function. We started by testing the motor with the ball and beam on top of the rotor first when the disc was down perpendicular to the ground and deduced that it was not the region of operation we wanted to work in. Although it had 93.67% accuracy, we re-did the system identification process.

$$\frac{\theta(s)}{V(s)} = \frac{119.6}{s^2 + 17.91s + 490.8} \tag{23}$$

We started doing system identification at the desired region of operation when the disc is parallel to the ground, at 1.57 radians, by giving it a DC offset of around 5 and we got a transfer function that we will work with an accuracy of 95.12%.

$$\frac{\theta(s)}{V(s)} = \frac{40.42}{s^2 + 8.067s + 175.95} \tag{24}$$

We also made system identification for the ball and beam system of the overall system where we studied how the position of the theta to the position of the ball on the beam and we got this transfer function

$$\frac{r(s)}{\theta(s)} = \frac{0.292}{s^2 + 0.8837s + 1.275} \tag{25}$$

III. System Analysis

As shown from the equations of the motor, it is stable

IV. Control System Design

A. Introduction

The control system design aims to achieve the desired stability and performance criteria for the given dynamic system. This process involves determining the specifications, designing a suitable controller, and validating the system's behavior through simulation and analysis.

The control system must ensure robust performance under operating conditions and reject disturbances while achieving desired transient and steady-state characteristics. The following sections outline the design specifications, control strategy, and validation steps.

B. Derivation of Performance Requirements and Desired Pole Locations

The desired pole locations and time-domain performance requirements were derived based on the system's transient response and steady-state behavior specifications. The following steps outline the process:

1) Time-Domain Performance Requirements: The performance requirements, such as rise time, settling time, and overshoot, are linked to the damping ratio (ζ) and natural frequency (ω_n) of the system poles. Using standard second-order system equations, the system poles can be expressed as:

$$s = -\sigma \pm j\omega_d$$

where: - σ is the real part of the pole (also related to damping), - ω_d is the damped natural frequency (which relates to the oscillatory behavior).

The following relationships hold between the damping ratio (ζ), natural frequency (ω_n), and the real and imaginary parts of the pole:

$$\sigma = \zeta\omega_n$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Using the above formulas, we can derive the desired specifications.

- **Overshoot** (M_p): The percentage of overshoot is determined using the relationship:

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

For a maximum overshoot of 10%, solving for ζ gives:

$$\zeta = 0.591.$$

- **Settling Time** (t_s): The settling time for a 5% criterion is approximated by:

$$t_s \approx \frac{3}{\zeta\omega_n}.$$

For $t_s = 3\,\text{s}$, and using $\zeta = 0.591$, the required natural frequency is:

$$\omega_n = \frac{3}{0.591 \times 3} = 1.69\,\text{rad/s}.$$

2) Desired Pole Locations: Using the relations derived above, we can express the closed-loop poles as:

$$s = -\sigma \pm j\omega_d$$

Substituting $\zeta = 0.591$ and $\omega_n = 1.69\,\text{rad/s}$, we can calculate:

$$\sigma = \zeta\omega_n = 0.591 \times 1.69 = 1\,\text{rad/s}$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 1.69 \times \sqrt{1-0.591^2} \approx 1.36\,\text{rad/s}.$$

Therefore, the desired pole locations are:

$$s = -1 \pm j1.36.$$

These pole locations achieve the desired time-domain performance metrics while maintaining system stability.

C. Design Specifications

The design specifications are as follows:

1) **Stability**:
   - The system must remain stable under all operating conditions.
   - All poles must lie in the left-half s-plane (continuous systems) or inside the unit circle (discrete systems).
2) **Time-Domain Performance**:
   - **Settling Time** (t_s): $\leq 3.0\,\text{s}$. (5% criterion)
   - **Overshoot** (M_p): $\leq 10\%$.
   - **Steady-State Error** (ϵ_{ss}): $\leq 2\%$.
3) **Desired Pole Locations**:
   - For the closed-loop system, the poles are designed to be at $-1 \pm j1.36$, corresponding to a damping ratio of $\zeta = 0.591$ and a natural frequency of $\omega_n = 1.69\,\text{rad/s}$.

D. Choice of Control Strategy

Based on the open-loop transfer function, both transient and steady-state performance need to be improved, so we decided to use a PID controller to address both issues. We control the motor by changing the natural frequency ω_n to 40 rad/s and $\zeta = 0.7$ which change the closed loop poles to $\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ so the desired poles are $-28 \pm j28.5657$. Designing a PID for the motor based on these desired poles we get a PID controller in the form of $\frac{K(s+a)(s+b)}{s}$ where the angle deficiency of the motor was 86.5776 so based we design a PD compensator first in which we count for this angle and then the arbitrary PI controller and then get K from the magnitude criteria. for the overall ball and beam system we do the procedure where we substitute the desired poles at the open loop transfer function which is the multiplication of the CLTF of the compensated motor multiplied by the TF of the ball and beam.

E. Design Procedure

The overall system equation is:

$$\frac{r(s)}{V(s)} = \frac{11.803}{(s^2 + 0.8837s + 1.275) \times (s^2 + 8.067s + 175.95)} \tag{26}$$

The open-loop poles are:

$$s_{1,2} = -0.442 \pm 1.04j \ and \ s_{3,4} = -4.034 \pm 12.64j \tag{27}$$

Taking the system as 2 subsystems controlling the motor and then the overall system, we get a PID for the motor

$$PID_m = \frac{1.19(s + 29.708)(s + 0.2)}{s} \tag{28}$$

and a PID for the overall system is

$$PID_s = \frac{4.633(s + 1.39275)(s + 0.2)}{s} \tag{29}$$

1) Overall Lag-Lead Controller: It was decided to keep the PID controller for the motor because it has large steady state error, and it was decided to design a lag-lead compensator for the entire system because the system does not meet the transient and steady state requirements.

The lag-lead compensator to be designed, is of the form:

$$G_c(s) = K_c \frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}} \frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} \tag{30}$$

Designing the Lead Compensator: By plugging in the desired poles into the equation of the overall system, and subtracting the angle gotten from 180, the deficiency angle is:

$$\Phi = 73.871 \tag{31}$$

Then, using the graphical method:

$$zero\ location = z = -\frac{1}{T_1} = -0.7 \tag{32}$$

and

$$pole\ location = p = -\frac{\gamma}{T_1} = -3.5 \tag{33}$$

Then,

$$T_1 = \frac{10}{7}\ and\ \gamma = 5 \tag{34}$$

From the magnitude condition:

$$K_c = 12.6 \tag{35}$$

Designing he Lag Compensator: The desired steady state error is:

$$\epsilon_{ss} = \frac{R}{1+K_p} \tag{36}$$

Where R is the gain of the reference(in this case the gain of the step)

$$K_p = \frac{R}{\epsilon_{ss}} - 1 = \frac{410}{10} - 1 = 40 \tag{37}$$

$$K_p = lim_{s->0} G_c(s)G(s) = lim_{s->0} K_c \frac{\beta}{\gamma} G(s) = K_c \frac{\beta}{\gamma} \frac{0.292}{1.275} \tag{38}$$

So $beta$ is :

$$\beta = \frac{K_p \gamma}{K_c} \frac{1.275}{0.292} = 69.3 \tag{39}$$

T_2 was chosen to be 10 because it meets the limits(gain approximately 1 and angle of less that -5):

$$\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}|_{-1+1.36j} = 0.94\angle - 4.76 \tag{40}$$

Therefore:

$$G_c(s) = 12.6 \frac{s+0.7}{s+3.5} \frac{s+0.1}{0.001443} \tag{41}$$

However, the graphical method is not the best, so the location of the pole for the lead compensator was arbitrarily to chosen to be at -50, then, to meet the angle deficiency the zero was calculated to be at -1.35. And from the magnitude condition, $K_c = 215$. Then similar to the previous lag compensator β was calculated and $T_2 = 6$ was chosen and met the limits.

$$zero\ location = z = -\frac{1}{T_1} = -1.35 \tag{42}$$

and

$$pole\ location = p = -\frac{\gamma}{T_1} = -50 \tag{43}$$

Then,

$$T_1 = 0.74\ and\ \gamma = 36.96 \tag{44}$$

From the magnitude condition:

$$K_c = 215 \tag{45}$$

$beta$ is :

$$\beta = \frac{K_p \gamma}{K_c} \frac{1.275}{0.292} = 89.14 \tag{46}$$

T_2 was chosen to be 6 because it meets the limits(gain approximately 1 and angle of less that -5):

$$\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}|_{-1+1.36j} = 0.94\angle -4.8 \tag{47}$$

Therefore:

$$G_c(s) = 215\frac{s+0.1.35}{s+50}\frac{s+0.167}{0.0018697} \tag{48}$$

V. Simulation Results

A. Open-Loop analysis

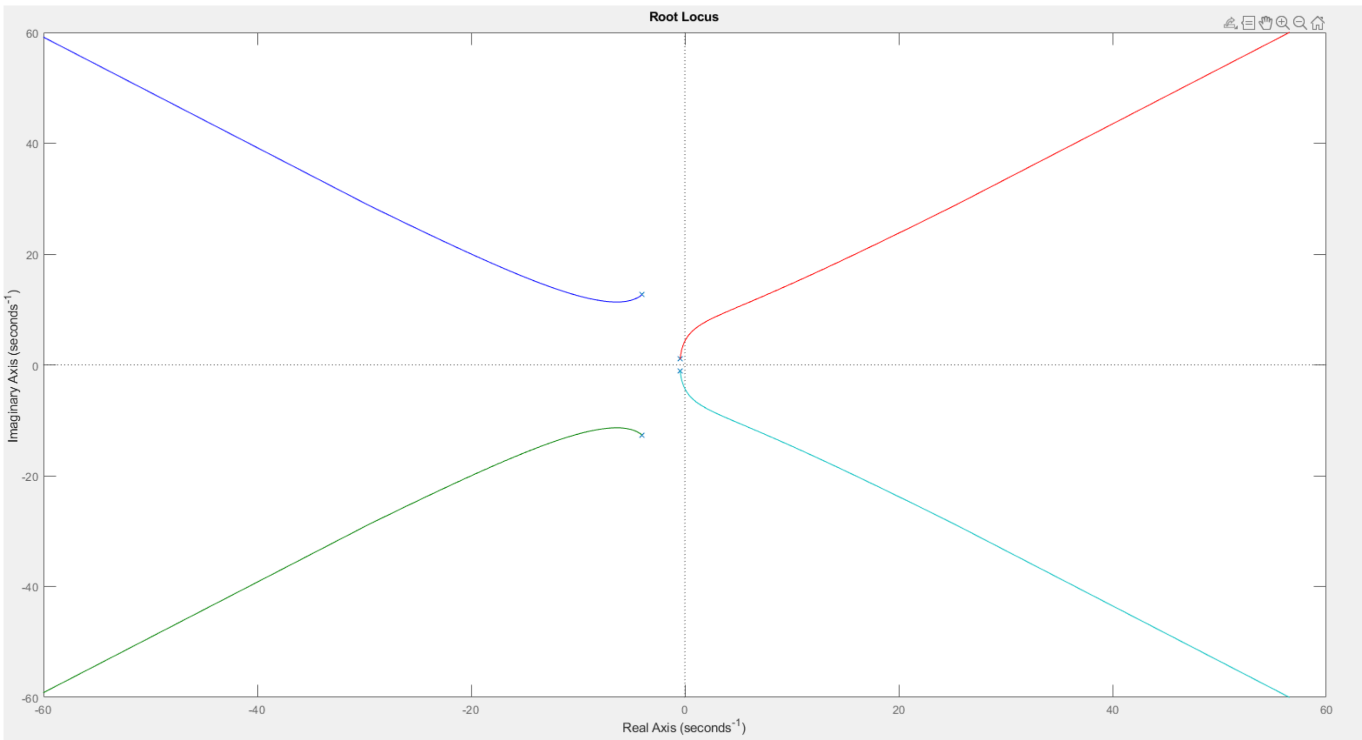

Fig. 1. Figure of the Open Loop root locus sketch

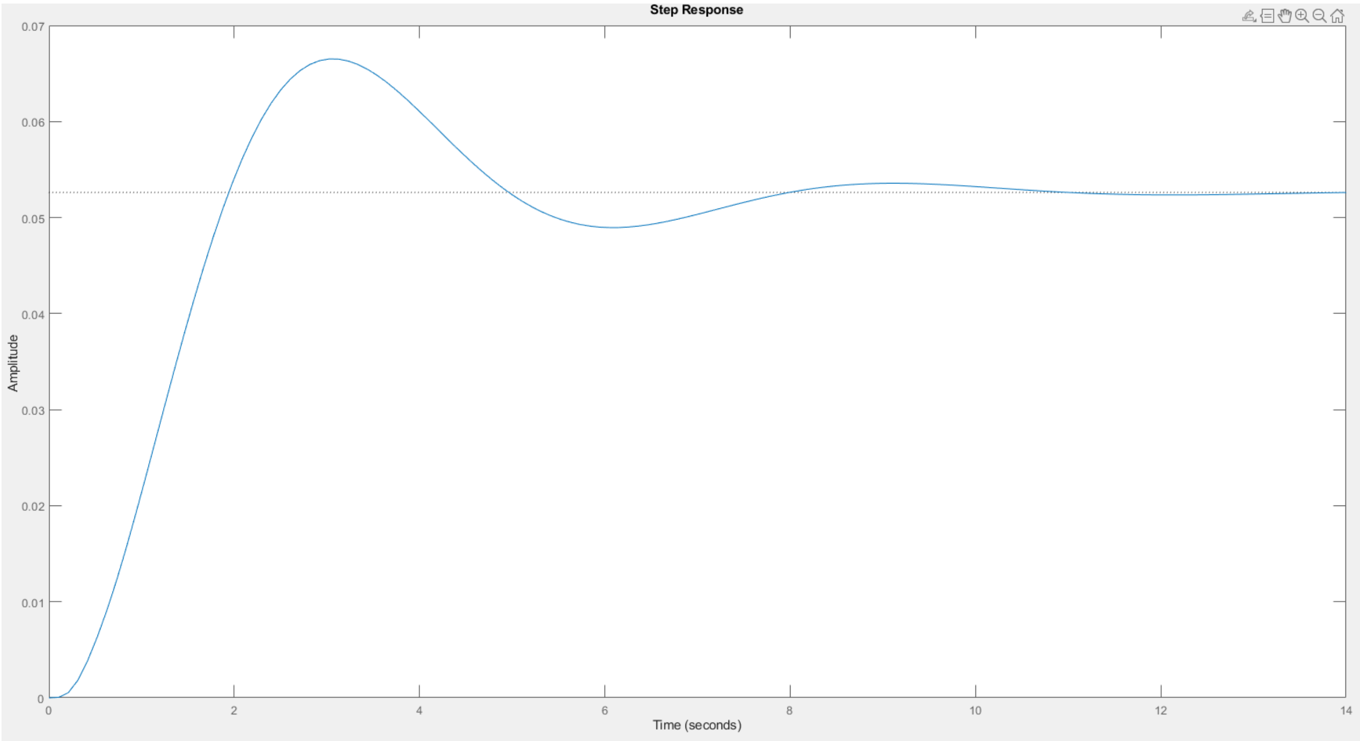

Fig. 2. Figure of the Open Loop step response

As shown in the open loop step response the system is dominated by a second-order step response but doesn't meet the requirements so it has to be controlled using compensators.

The root locus of the open loop transfer function above shows the instability of this system, although showing that for some gains K between 0 and ≈ 265 the system is "stable" or marginally stable. There can be two approaches: design a controller for the overall system or the motor, and then the ball and beam with the compensated motor. Our goal is to meet the requirements that are given 10 percent overshoot and 3 seconds settling time which leads to desired closed-loop poles at -1± j1.36.

Choosing the second approach (controlling the motor then controlling the compensated motor with the ball and beam) we start by designing a PID for the motor with the desired: $\omega_n = 40$ rad/s and $\zeta = 0.7$ so we get the desired complex poles of 28±j28.5657. Designing a PID for these dominant poles where the deficiency angle is 86.5776 we get:

$$PID_m = \frac{1.19(s + 29.708)(s + 0.2)}{s} \tag{49}$$

After controlling the motor we then multiply the the compensated motor equation by the ball and beam equation to get the open loop of the overall system.

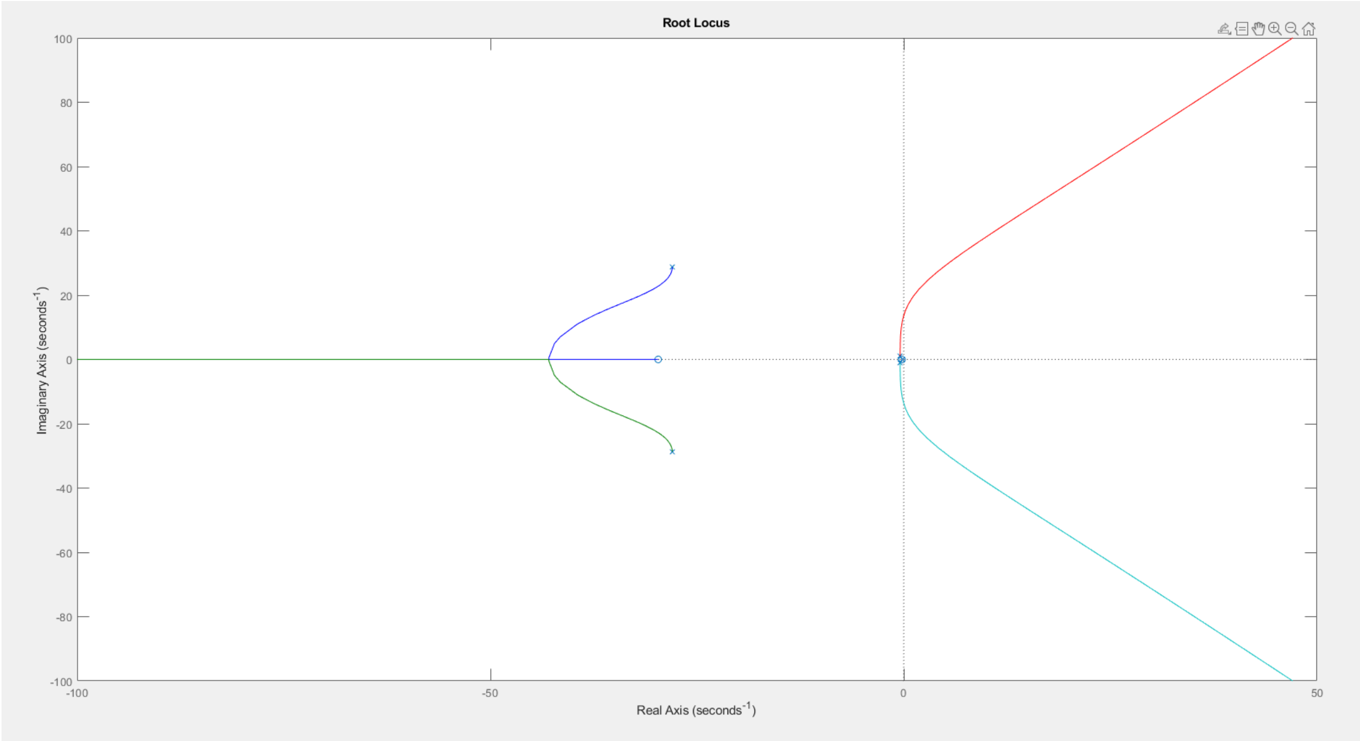

Fig. 3. Root locus of the open loop with the compensated motor.

Now we can start solving for the desired poles of the system from this root locus where we want the root locus to pass through -1±j1.36. Where we substitute the desired poles in the OLTF to get an angle deficiency consequently solving for the ”a”, ”b”, and ”K” in the overall PID controller.

So, what we worked at from the OLTF is getting to PID’s one for the motor and one for the overall system to control the system in our hands.

$$PID_m = \frac{1.19(s + 29.708)(s + 0.2)}{s} \tag{50}$$

and a PID for the overall system is

$$PID_s = \frac{4.633(s + 1.39275)(s + 0.2)}{s} \tag{51}$$

We get the overall closed-loop equation and see the results in the closed-loop analysis section.

Open Loop Frequency response

With the controllers being used on the motor and the whole system, between the motor and the beam and ball, the latter has more uncertainty to it due to friction and other disturbances. With the uncertainty of the beam and ball, it is our priority to get the margins of stability for it, and with having this topology being compensated motor and then full compensation of the system, the system stability margins are more of a priority.

Analyzing the Bode Plot, the margins of the system are very large for Gain and 82.7 for Phase. Furthermore, the phase change along the frequency sweep is relatively smooth and doesn’t have sharp dips which would have caused further instability.

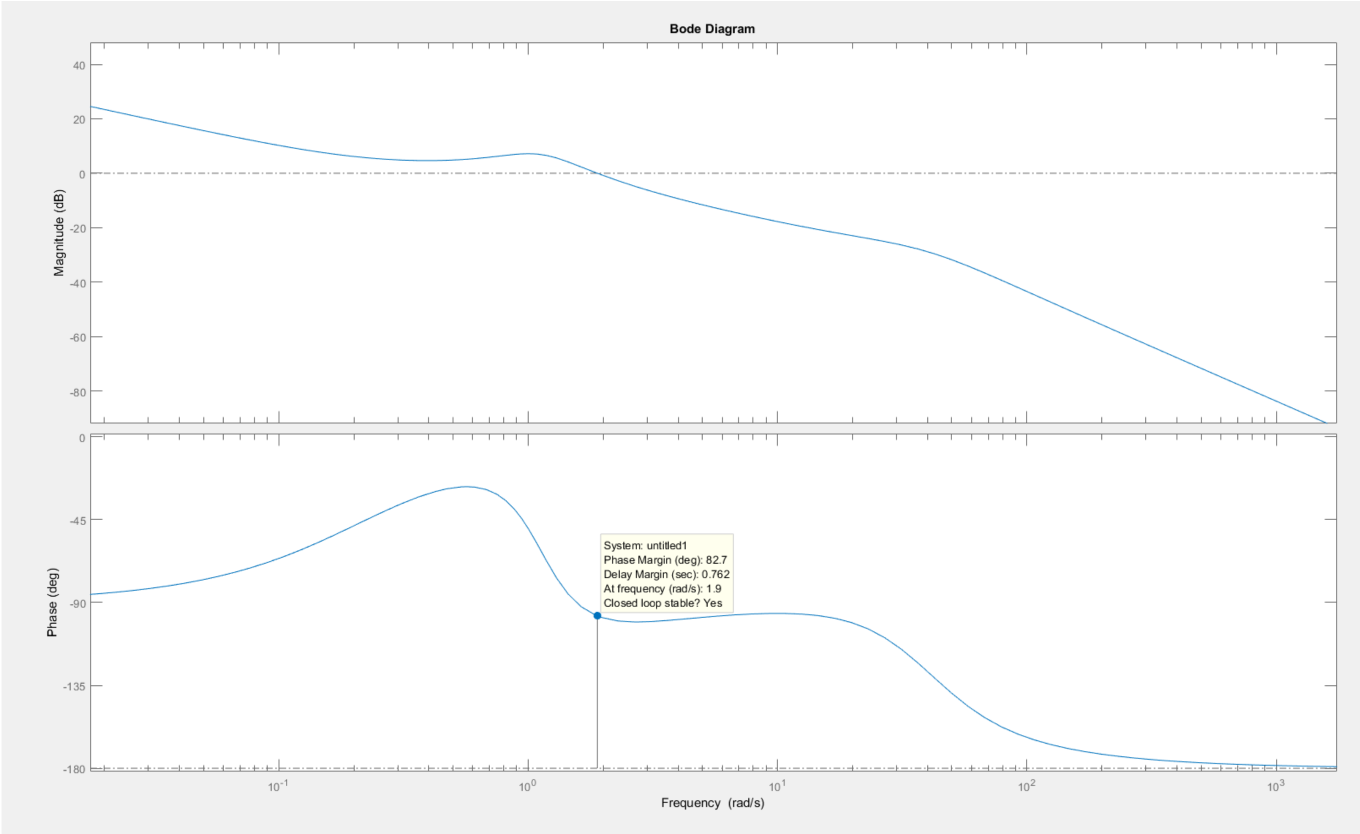

Fig. 4. Bode Plot

Moving to the Nyquist, the nearest our curve was to the -1+0j was around 0.9 meaning the Nyquist plot was far from being unstable having relatively sensitivity and more robustness.

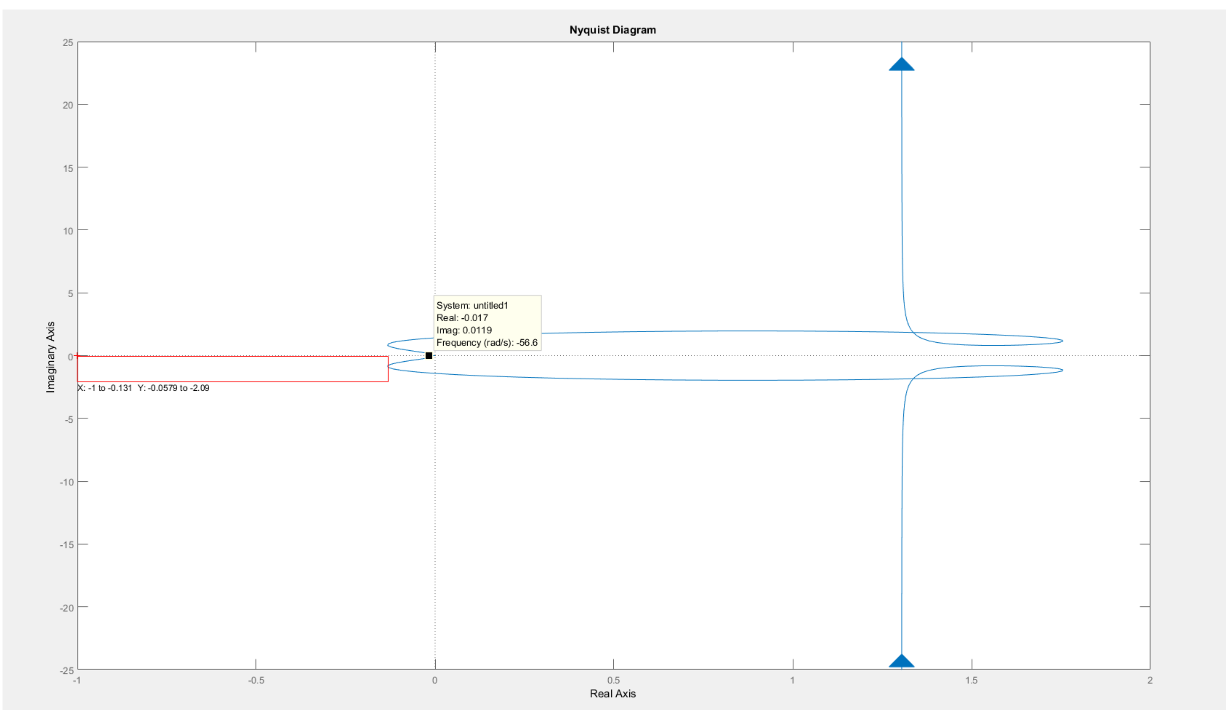

Fig. 5. Nyquist Plot

B. Closed-Loop analysis

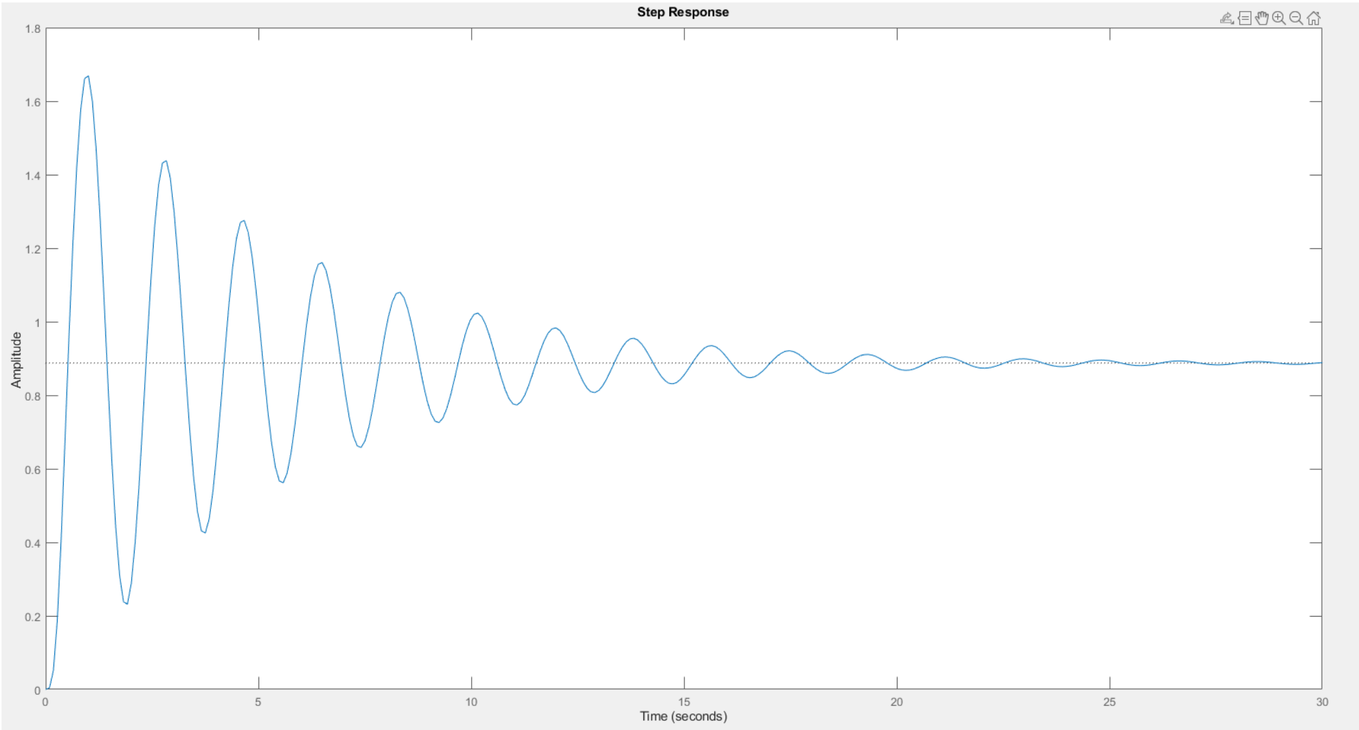

Fig. 6. Figure of the Closed-loop step response with only Proportional gain ”K” at 150.

In the figure above we can see the step response with only Kp, the system is stable but has a big overshoot in addition to huge steady state and transient response errors so it doesn’t meet the requirements therefore a proportional controller can’t and won’t work.

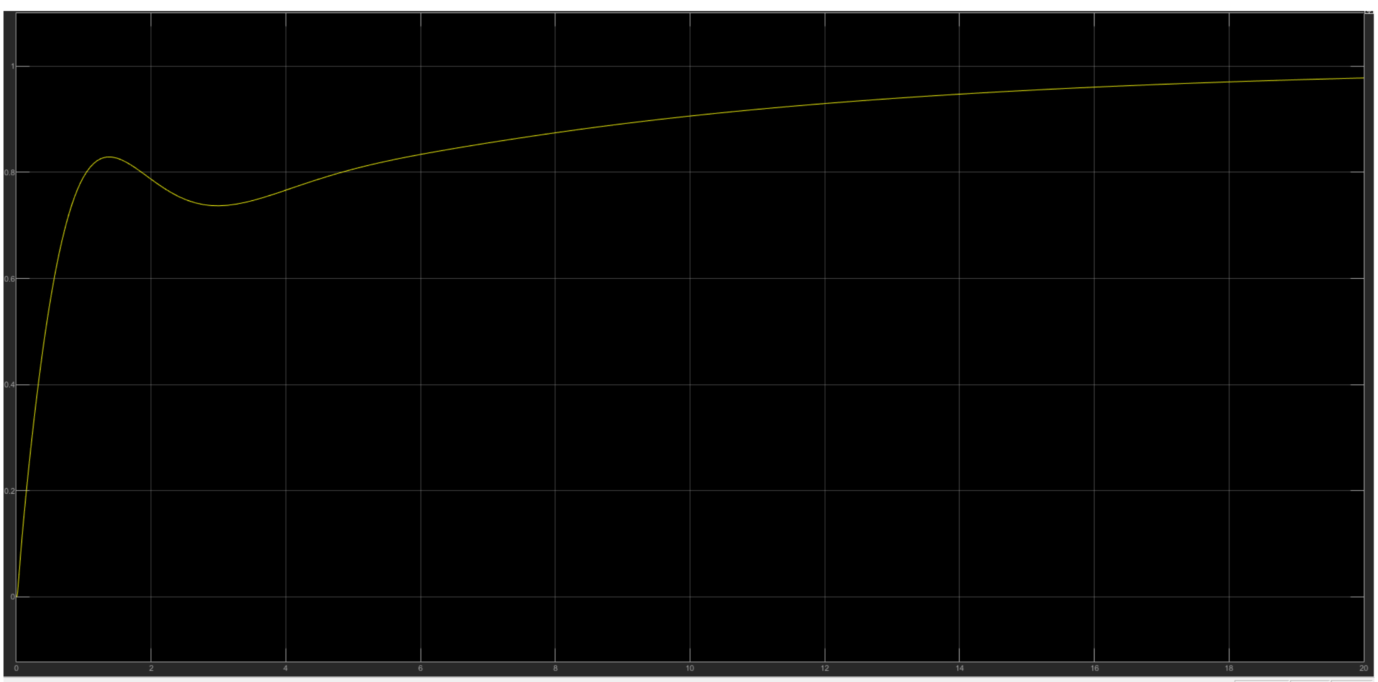

Fig. 7. Figure of the Closed-loop step response with the original PIDs we choose.

The system with the original PID controllers has an underdamped response and a huge settling time which we don’t want. So we try to fine-tune the parameters to get the requirements that we want.

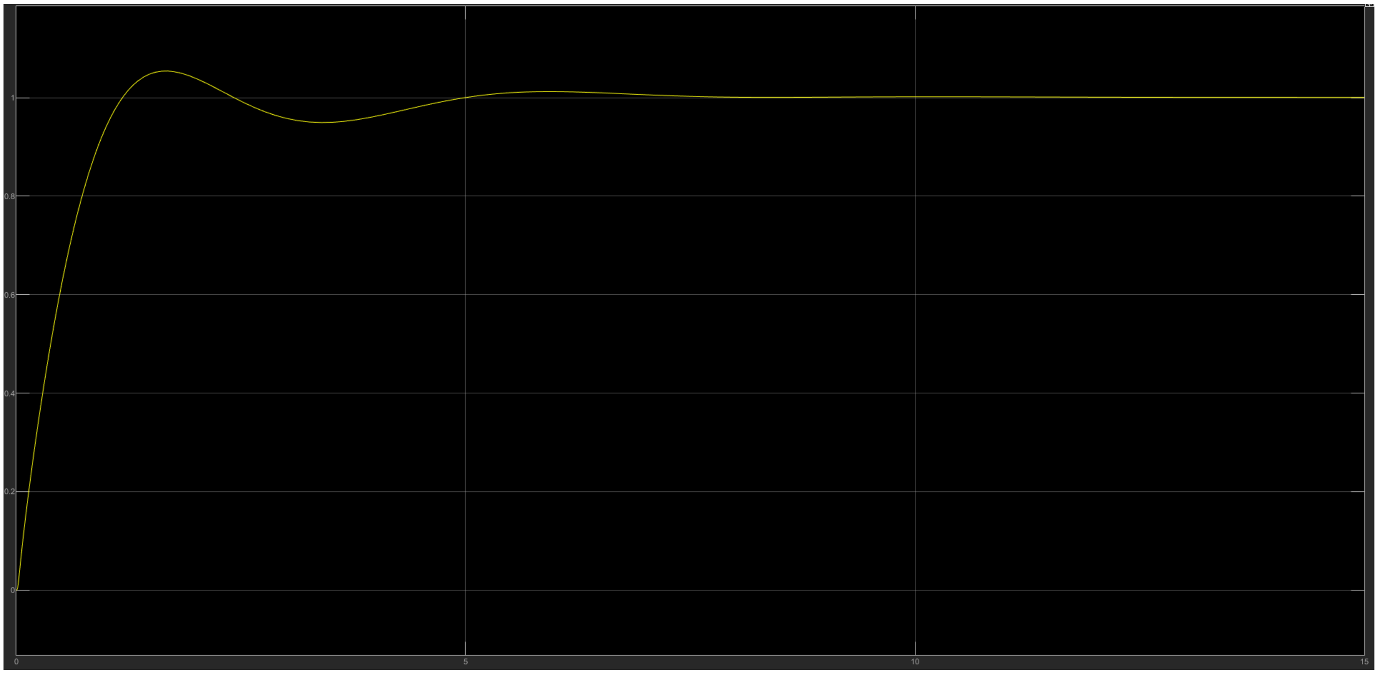

Fig. 8. step response of the compensated overall system.

When changing the Ki, Kp, Kd of the overall system from 1.2905, 7.379, and 4.633 to 8, 8, and 5 respectively it gave better simulation results notice the only difference maker is Ki that changed from 1.2905 to 8.

Now another approach for controlling the whole system is using a Lag-Lead compensator

VI. Experimental Results

A. PID experimental results

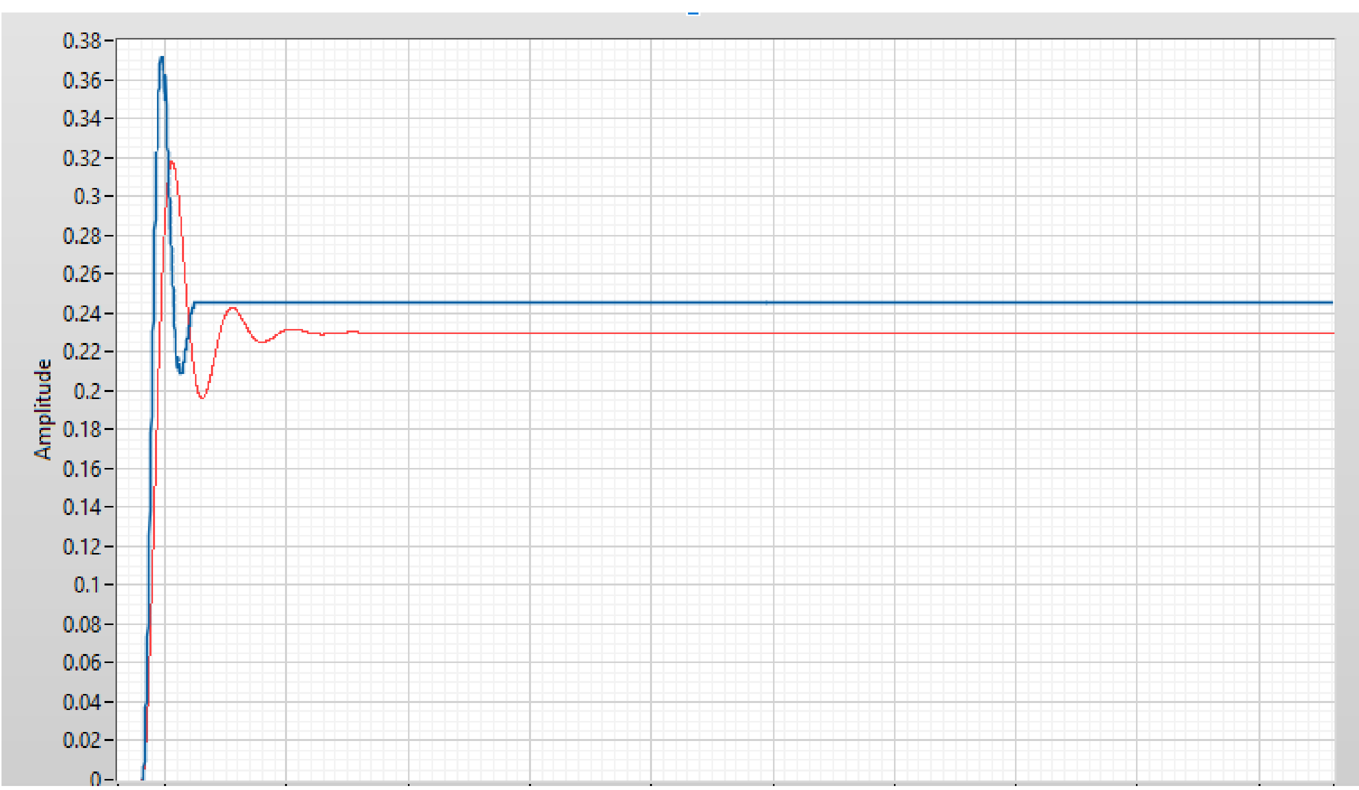

Fig. 9. Motor black box graph VS grey box graph

The graph above shows the motor transfer function that we have obtained using the system ID Matlab tool versus the real motor response(black box) which shows that our equation can be a good estimate for controlling the plant.

NOTE: scale is 500 ms.

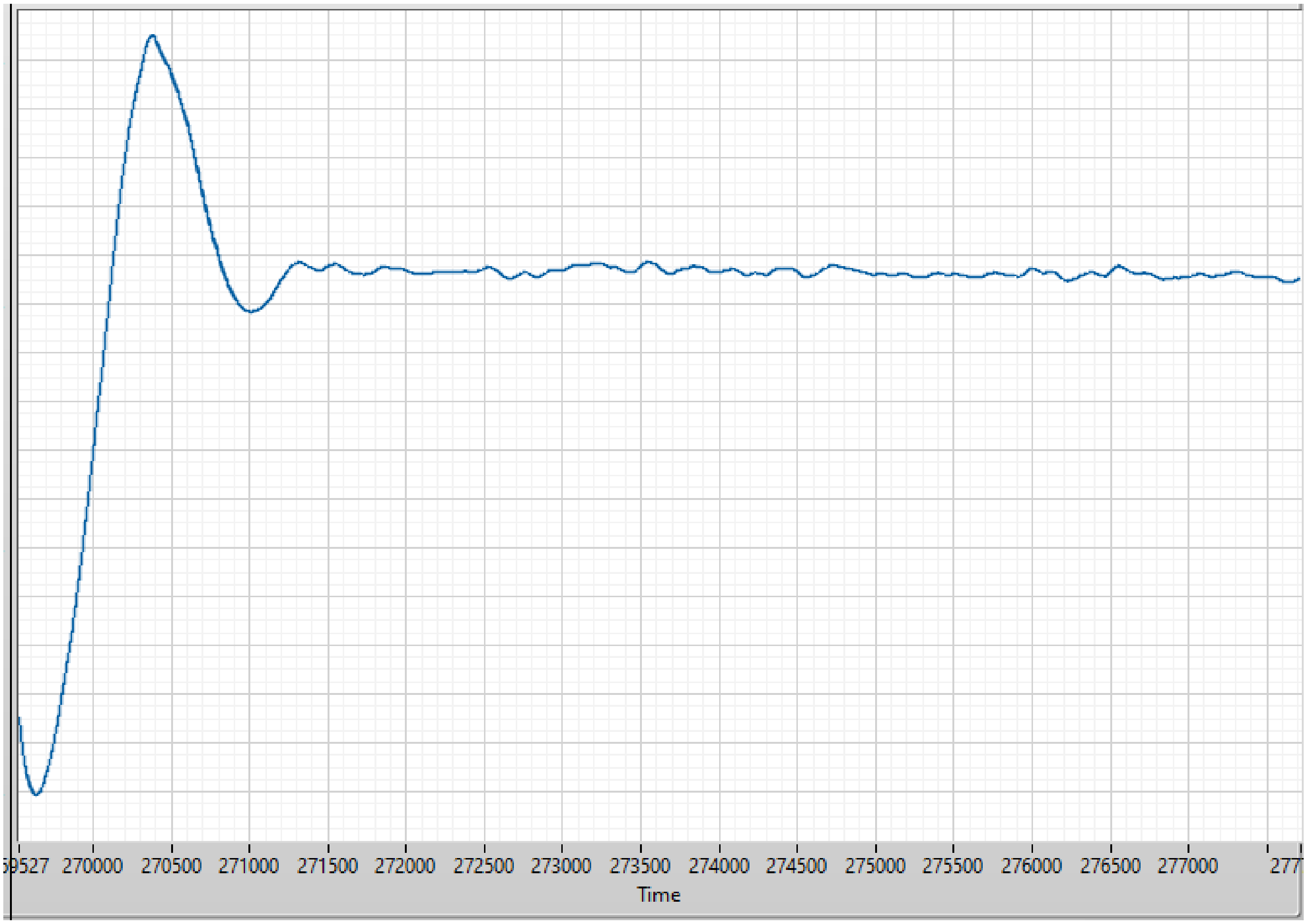

Fig. 10. Settling of the ball at 250 mm

The settling time at 250 mm is approximately 3-5 seconds after giving the ball a push which is equivalent to a step response. We can notice the transient response is very good and the steady state error will be discussed later.

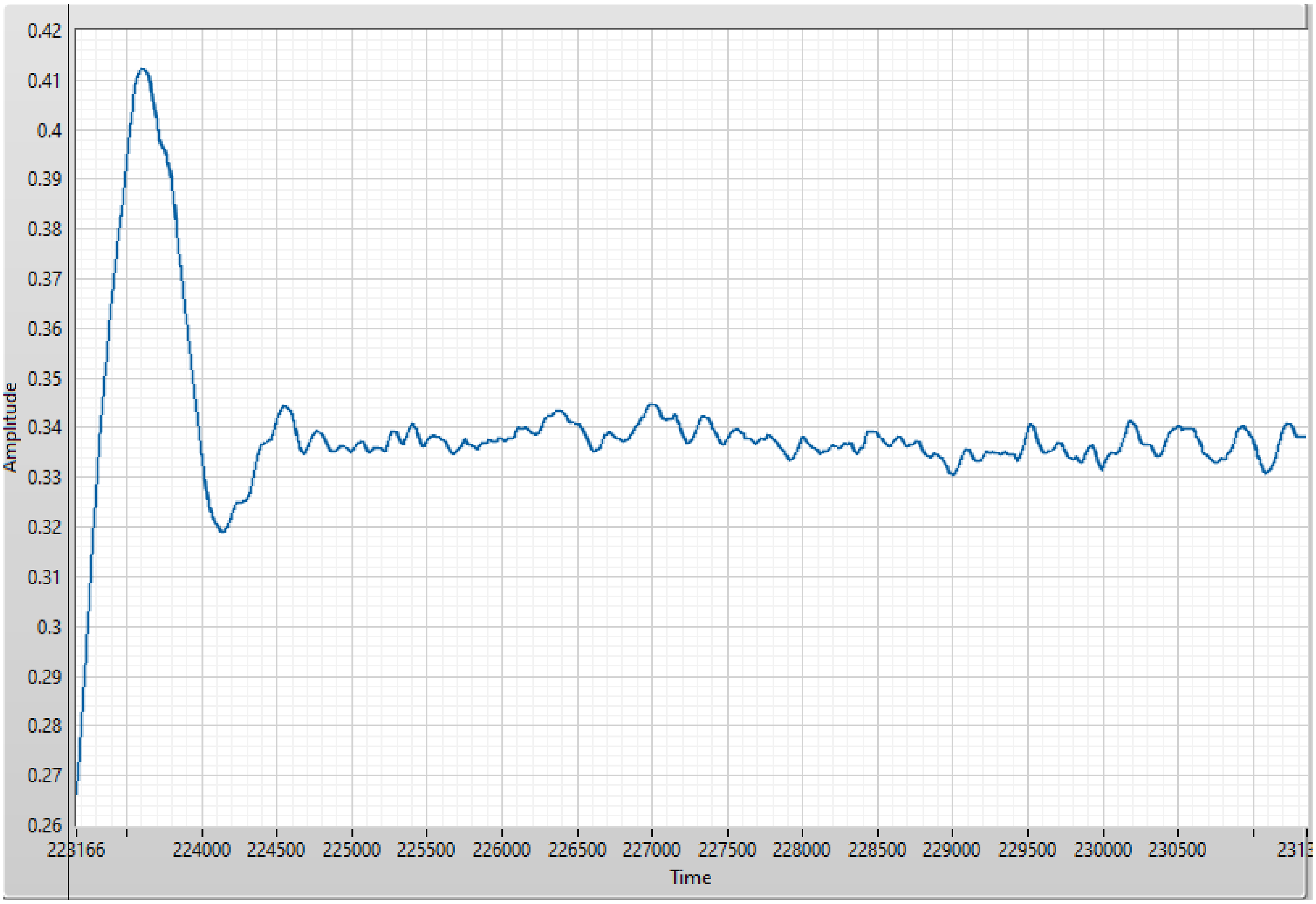

Fig. 11. Settling of the ball at 333 mm

We tried to balance the ball at 333 mm (2/3 of the length of the rod), as we can see the transient response is good and the steady state error is within the range of interest.

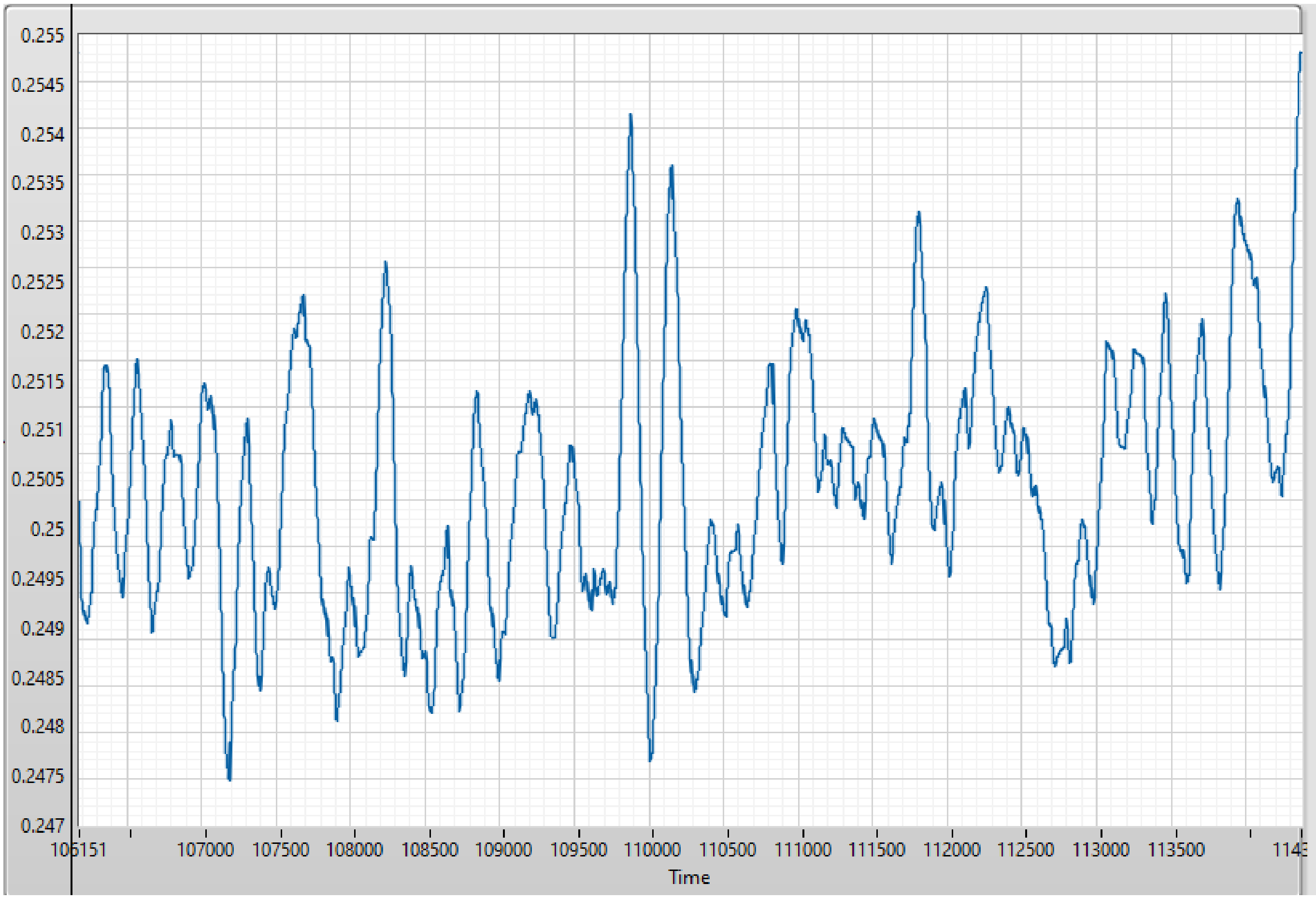

Fig. 12. Steady state error at 250 mm

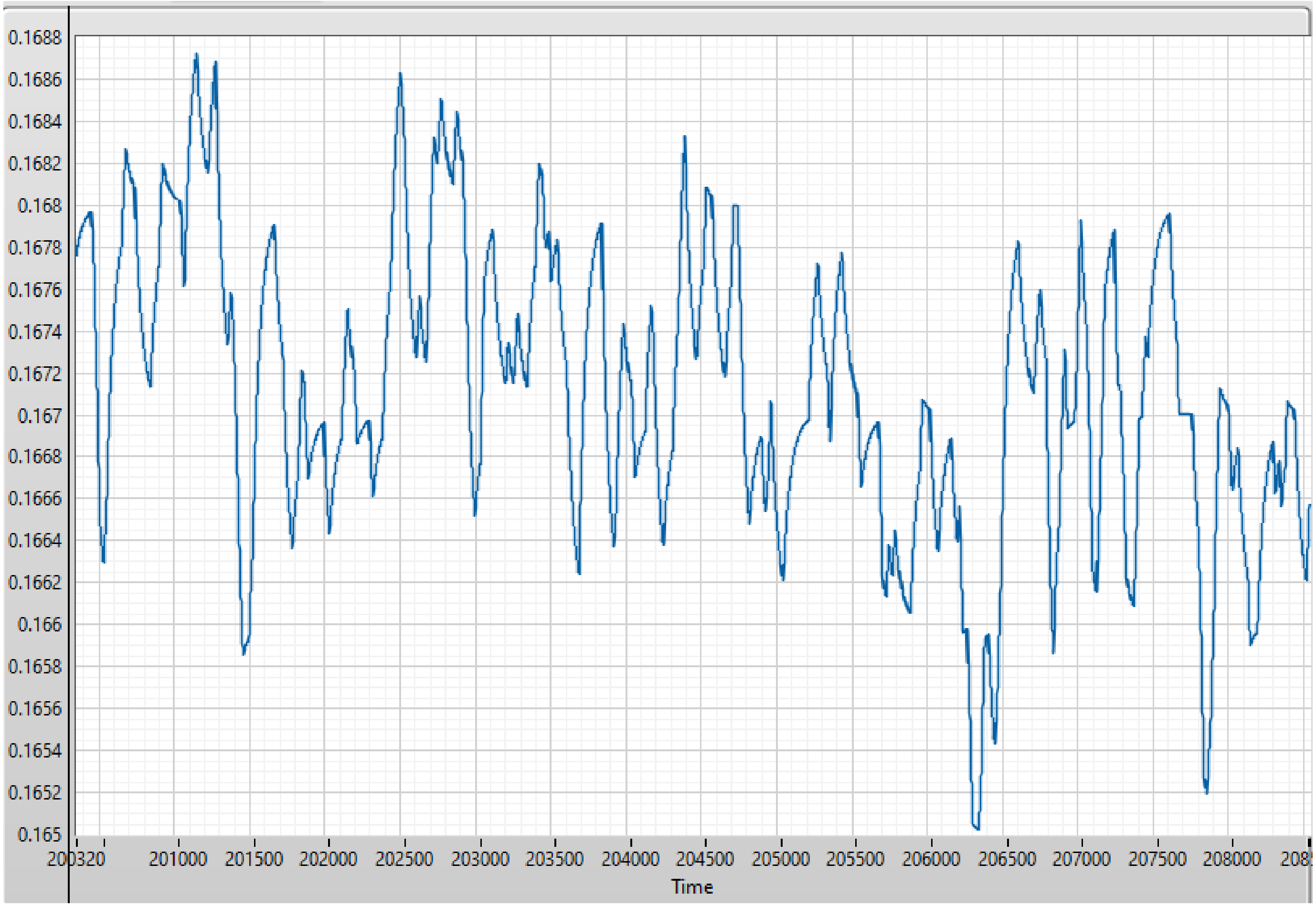

Fig. 13. Steady state error at 166 mm

As shown in the graph the steady-state error is within our range ±10mm so we have a good steady-state response.

VII. Conclusion

We tried to compensate the overall system together but it was challenging because of the difficult nature of the ball and beam and many uncertainties we don't know about.We conclude that compensators make systems stable since PID met all the requirements of the system with a good steady state and transient response. Since the motor was the bottleneck it was better to compensate for the motor before the overall system not the other way. To reiterate since the ball and beam had the most uncertainties it was better to compensate it with the overall system after compensating for the motor. Design is an iterative process we didn't get a success from the first try nor the second and guess not even the third.

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