Exercise 2.2

1. Find the cube roots of -1,8, -27,64.

(i) The three cube roots of -1

Solution:

Let $x^3 = -1$

$(x)^3 + (1)^3 = 0$

$(x+1)(x^2 - x + 1) = 0$

Either $x + 1 = 0$ *or* $x^2 - x + 1 = 0$

$x = -1$ *Here* $a = 1, b = -1, c = 1$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1-4}}{2}$$

$$x = \frac{1 \pm \sqrt{-3}}{2}$$

$$x = \frac{1 + \sqrt{-3}}{2} \quad or \quad x = \frac{1 - \sqrt{-3}}{2}$$

$$= -\left(\frac{-1 - \sqrt{-3}}{2}\right) \qquad = -\left(\frac{-1 + \sqrt{-3}}{2}\right)$$

$$= -\omega^2 \qquad = -\omega$$

Three cube roots of -1 are $-1, -\omega, -\omega^2$

(ii) The three cube roots of 8

Solution:

Let $x^3 = 8$

$x^3 - 8 = 0$

$(x)^3 - (2)^3 = 0$

$(x-2)(x^2 + 2x + 4) = 0$

$Either\ \ x-2=0 \qquad or \qquad x^2+2x+4=0$

$x=2 \qquad Here\ a=1, b=2, c=4$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-2\pm\sqrt{(2)^2-4(1)(4)}}{2(1)}$$

$$x=\frac{-2\pm\sqrt{4-16}}{2}$$

$$x=\frac{-2\pm\sqrt{-12}}{2}$$

$$x=\frac{-2\pm2\sqrt{-3}}{2}$$

$$x=\frac{2\left(-1\pm i\sqrt{3}\right)}{2} \because i=\sqrt{-1}$$

$$x=2\left(\frac{-1+i\sqrt{3}}{2}\right) \quad or \quad x=2\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$=2\omega \qquad\qquad =2\omega^2$$

Three cube roots of 8 are $2, 2\omega, 2\omega^2$

(iii) The three cube roots of -27

Solution:

Let $x^3=-27$

$x^3+27=0$

$(x)^3+(3)^3=0$

$(x+3)\left(x^2-3x+9\right)=0$

$Either\ \ x+3=0 \qquad or \qquad x^2-3x+9=0$

$x = -3$ $\quad\quad$ $Here\ a = 1, b = -3, c = 9$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9 - 36}}{2}$$

$$x = \frac{3 \pm \sqrt{-27}}{2}$$

$$x = \frac{3 \pm 3\sqrt{-3}}{2}$$

$$x = \frac{3\left(1 \pm i\sqrt{3}\right)}{2} \because i = \sqrt{-1}$$

$$x = 3\left(\frac{1 + i\sqrt{3}}{2}\right) \ or \ x = 3\left(\frac{1 - i\sqrt{3}}{2}\right)$$

$$x = -3\left(\frac{-1 - i\sqrt{3}}{2}\right) \ or \ x = -3\left(\frac{-1 + i\sqrt{3}}{2}\right)$$

$$= -3\omega^2 \quad\quad\quad = -3\omega$$

Three cube roots of -27 are $-3, -3\omega, -3\omega^2$

(iv) The three cube roots of 64

Solution:

Let $x^3 = 64$

$x^3 - 64 = 0$

$(x)^3 - (4)^3 = 0$

$(x-4)\left(x^2 + 4x + 16\right) = 0$

$Either\ \ x-4=0 \qquad or \qquad x^2+4x+16=0$

$x=4 \qquad\qquad Here\ a=1, b=4, c=16$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(4)\pm\sqrt{(4)^2-4(1)(16)}}{2(1)}$$

$$x=\frac{-4\pm\sqrt{16-64}}{2}$$

$$x=\frac{-4\pm\sqrt{-48}}{2}$$

$$x=\frac{-4\pm4\sqrt{-3}}{2}$$

$$x=\frac{4\left(-1\pm i\sqrt{3}\right)}{2}\because i=\sqrt{-1}$$

$$x=4\left(\frac{-1+i\sqrt{3}}{2}\right)\ or\ x=4\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$=4\omega \qquad\qquad =4\omega^2$$

Three cube roots of 64 are $4, 4\omega, 4\omega^2$

2. Evaluate

(i) $\left(1-\omega-\omega^2\right)^7$

Solution:

$$\left(1-\omega-\omega^2\right)^7=\left[1-\left(\omega+\omega^2\right)\right]^7$$

$$=\left[1-(-1)\right]^7 \qquad \because \omega+\omega^2=-1$$

$$=(1+1)^7$$

$$=2^7=128$$

(ii) $\left(1-3\omega-3\omega^2\right)^5$

Solution:

$$\begin{aligned}\left(1-3\omega-3\omega^2\right)^5 &= \left[1-3\left(\omega+\omega^2\right)\right]^5\\ &= \left[1-3(-1)\right]^5 \because \omega+\omega^2=-1\\ &= (1+3)^5\\ &= 4^5 = 1024\end{aligned}$$

(iii) $\left(9+4\omega+4\omega^2\right)^3$

Solution:

$$\begin{aligned}\left(9+4\omega+4\omega^2\right)^3 &= \left[9+4\left(\omega+\omega^2\right)\right]^3\\ &= \left[9+4(-1)\right]^3 \because \omega+\omega^2=-1\\ &= (9-4)^3\\ &= 5^3 = 125\end{aligned}$$

(iv) $\left(2+2\omega-2\omega^2\right)\left(3-3\omega+3\omega^2\right)$

Solution:

$$\begin{aligned}&\left(2+2\omega-2\omega^2\right)\left(3-3\omega+3\omega^2\right)\\ &= \left[2\left(1+\omega\right)-2\omega^2\right]\left[3\left(1+\omega^2\right)-3\omega\right]\\ &= \left[2\left(-\omega^2\right)-2\omega^2\right]\left[3\left(-\omega\right)-3\omega\right] \qquad \because 1+\omega=-\omega^2\\ &= \left(-2\omega^2-2\omega^2\right)\left(-3\omega-3\omega\right)\end{aligned}$$

$= \left(-4\omega^2\right)\left(-6\omega\right)$

$= (-4)(-6)\left(\omega^2.\omega\right)$

$= 24\omega^3$

$= 24(1) \qquad \because \omega^3 = 1$

$= 24$

(v) $\left(-1+\sqrt{-3}\right)^6 + \left(-1-\sqrt{-3}\right)^6$

Solution:

$\left(-1+\sqrt{-3}\right)^6 + \left(-1-\sqrt{-3}\right)^6$

$= \left(2\omega^6\right) + \left(2\omega^2\right)^6 \qquad \because \omega = \dfrac{-1+\sqrt{-3}}{2} \;\; and \;\; \omega^2 = \dfrac{-1-1\sqrt{-3}}{2}$

$= 2^6\left(\omega^6\right) + 2^6\left(\omega^{12}\right) \qquad 2\omega = -1+\sqrt{-3} \;\; and \;\; 2\omega^2 = -1-1\sqrt{-3}$

$= 2^6\left[\left(\omega^3\right)^2\right] + 2^6\left[\left(\omega^3\right)^4\right]$

$= 2^6\left[(1)^2\right] + 2^6\left[(1)^4\right] \qquad \because \omega^3 = 1$

$= 2^6[1+1]$

$= 2^6.2 = 2^{6+1} = 2^7$

$= 128$

(vi) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9 + \left(\dfrac{-1-\sqrt{-3}}{2}\right)^9$

Solution:

$\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9 + \left(\dfrac{-1-\sqrt{-3}}{2}\right)^9$

$= \omega^9 + (\omega^2)^9 \qquad \because \omega = \frac{-1+\sqrt{-3}}{2}$ *and* $\omega^2 = \frac{-1-1\sqrt{-3}}{2}$

$= \omega^9 + \omega^{18}$

$= (\omega^3)^3 + (\omega^3)^6$

$= (1)^3 + (1)^6 \qquad \because \omega^3 = 1$

$= 1 + 1 = 2$

(vii) $\omega^{37} + \omega^{38} - 5$

Solution:

$\omega^{37} + \omega^{38} - 5$

$= \omega^{37} + \omega^{38} - 5$

$= \omega^{36}.\omega + \omega^{36}.\omega^2 - 5$

$= (\omega^3)^{12}.\omega + (\omega^3)^{12}.\omega^2 - 5$

$= (1)^{12}.\omega + (1)^{12}.\omega^2 - 5 \qquad \because \omega^3 = 1$

$= \omega + \omega^2 - 5$

$= -1 - 5 \qquad \because \omega + \omega^2 = -1$

$= -6$

(viii) $\omega^{-13} + \omega^{-17}$

Solution:

$\omega^{-13} + \omega^{-17}$

$= \omega^{-13} + \omega^{-17}$

$= \omega^{-12-1} + \omega^{-15-2}$

$= \omega^{-12}.\omega^{-1} + \omega^{-15}.\omega^{-2}$

$= (\omega^3)^{-4}.\omega^{-1} + (\omega^3)^{-5}.\omega^{-2}$