Solved Exercise 2.1

1. Find the discriminant of the following given quadratic equation:

$(i)\ 2x^2+3x-1=0$

Solution :

$2x^2+3x-1=0$

Compare it with

$ax^2+bx+c=0$

Here $a=2, b=3, c=-1$

$$\begin{aligned} Disc. &= b^2-4ac \\ &= (3)^2-4(2)(-1) \\ &= 9+8 \\ &= 17 \end{aligned}$$

$(ii)\ 6x^2-8x+3=0$

Solution :

$6x^2-8x+3=0$

Compare it with

$ax^2+bx+c=0$

Here $a=6, b=-8, c=3$

$$\begin{aligned} Disc. &= b^2-4ac \\ &= (-8)^2-4(6)(3) \\ &= 64-72 \\ &= -8 \end{aligned}$$

$(iii)\ 9x^2-30x+25=0$

$\Rightarrow 4a^2 - 4amc = 0$
$4a(a - mc) = 0$
$a - mc = 0$
$a = mc$
which is the required condition.

7. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, then $a = 0 \ or \ a^3 + b^3 + c^3 = 3abc$

Solution :

$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$

Here $\quad a = c^2 - ab, b = -2(a^2 - bc), c = b^2 - ac$

As the roots are equal, So

Disc. $= 0$

$b^2 - 4ac = 0$

$\left[-2(a^2 - bc)\right]^2 - 4(c^2 - ab)(b^2 - ac) = 0$

$4(a^2 - bc)^2 - 4(c^2 - ab)(b^2 - ac) = 0$

$4\left[(a^4 - 2a^2bc + b^2c^2) - (b^2c^2 - ac^3 - ab^3 + a^2bc)\right] = 0$

$a^4 - 2a^2bc + b^2c^2 - b^2c^2 + ac^3 + ab^3 - a^2bc = 0$

$\Rightarrow a^4 + ab^3 + ac^3 - 3a^2bc = 0$

$a(a^3 + b^3 + c^3 - 3abc) = 0$

Either a=0 $\quad$ or $\quad a^3 + b^3 + c^3 - 3abc = 0$

$\qquad a^3 + b^3 + c^3 = 3abc$

Hence proved.

8. Show that the roots of the following equations are rational.

$(i) \ a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

Solution :

$a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

Here $\quad a = a(b - c), b = b(c - a), c = c(a - b)$

Disc.
$= b^2 - 4ac$

$= [b(c-a)]^2 - 4[a(b-c)][c(a-b)]$
$= b^2(c-a)^2 - 4ac(b-c)(a-b)$
$= b^2(c-a)^2 - 4ac(ab - b^2 - ac + bc)$
$= b^2(c^2 + a^2 - 2ac) - 4ac(ab - b^2 - ac + bc)$
$= b^2c^2 + a^2b^2 - 2ab^2c - 4a^2bc + 4ab^2c + 4a^2c^2 - 4abc^2$
$= a^2b^2 + b^2c^2 + 4a^2c^2 + 2ab^2c - 4a^2bc - 4abc^2$
$= (ab)^2 + (bc)^2 + (-2ac)^2 + 2(ab)(bc) + 2(bc)(-2ac) + 2(-2ac)(ab)$
$= (ab + bc - 2ac)^2$
Hence the roots are rational.

$$(ii)\ (a+2b)x^2 + 2(a+b+c)x + (a+2c) = 0$$

Solution :
$(a+2b)x^2 + 2(a+b+c)x + (a+2c) = 0$
Here $a = a+2b,\ b = 2(a+b+c),\ c = a+2c$
Disc.
$= b^2 - 4ac$
$= [2(a+b+c)]^2 - 4(a+2b)(a+2c)$
$= 4(a+b+c)^2 - 4(a^2 + 2ac + 2ab + 4bc)$
$= 4(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) - 4(a^2 + 2ac + 2ab + 4bc)$
$= 4[a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - a^2 - 2ac - 2ab - 4bc]$
$= 4[b^2 + c^2 - 2bc]$
$= 4(b-c)^2$
$= [2(b-c)]^2$
Hence the roots are rational.

9. For all values of k, prove that the roots of the equation

$x^2 - 2\left(k + \frac{1}{k}\right)x + 3 = 0, k \neq 0$ are real.

Solution :

$x^2 - 2\left(k + \frac{1}{k}\right)x + 3 = 0$

Here $a = 1, b = -2\left(k + \frac{1}{k}\right), c = 3$

Disc.

$= b^2 - 4ac$

$= \left[-2\left(k + \frac{1}{k}\right)\right]^2 - 4(1)(3)$

$= 4\left(k + \frac{1}{k}\right)^2 - 12$

$= 4\left[\left(k + \frac{1}{k}\right)^2 - 3\right]$

$= 4\left[k^2 + \frac{1}{k^2} + 2 - 3\right]$

$= 4\left[k^2 + \frac{1}{k^2} - 1\right] > 0$

Hence, the roots are real.

10. Show that the roots of the equation

$(b-c)x^2 + (c-a)x + (a-b) = 0$ **are real.**

Solution :

$(b-c)x^2 + (c-a)x + (a-b) = 0$

Here $a = (b-c), b = (c-a), c = (a-b)$

Disc.

$= b^2 - 4ac$

$= (c-a)^2 - 4(b-c)(a-b)$

$= c^2 + a^2 - 2ac - 4(ab - b^2 - ac + bc)$

$= c^2 + a^2 - 2ac - 4ab + 4b^2 + 4ac - 4bc$

Solution :

$9x^2 - 30x + 25 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $\quad a = 9, b = -30, c = 25$

Disc. $\quad = b^2 - 4ac$

$$\begin{aligned} &= (-30)^2 - 4(4)(25) \\ &= 900 - 900 \\ &= 0 \end{aligned}$$

$(iv)\ 4x^2 - 7x - 2 = 0$

Solution :

$4x^2 - 7x - 2 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $\quad a = 4, b = -7, c = -2$

Disc. $\quad = b^2 - 4ac$

$$\begin{aligned} &= (-7)^2 - 4(4)(-2) \\ &= 49 + 32 \\ &= 81 \end{aligned}$$

2. Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations:

$(i)\ x^2 - 23x + 120 = 0$

Solution :

$x^2 - 23x + 120 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $\quad a = 1,\ b = -23,\ c = 120$

Disc. $\quad = b^2 - 4ac$

$$= (-23)^2 - 4(1)(120)$$
$$= 529 - 480$$
$$= 49$$
$$= (7)^2 > 0$$

As the disc. is positive and is a perect square. Therefore the roots are rational (real) and unequal. Verification by solving the equation.

$$x^2 - 23x + 120 = 0$$
$$x^2 - 15x - 8x + 120 = 0$$
$$x(x-15) - 8(x-15) = 0$$
$$(x-8)(x-15) = 0$$

Either $x - 8 = 0$ *or* $x - 15 = 0$

$x = 8$ $\quad$ $x = 15$

Thus, the roots are rational (real) and unequal.

(ii) $2x^2 + 3x + 7 = 0$

Solution :

$$2x^2 + 3x + 7 = 0$$

Compare it with

$$ax^2 + bx + c = 0$$

Here $a = 2, b = 3, c = 7$

Disc. $= b^2 - 4ac$

$$= (3)^2 - 4(2)(7)$$
$$= 9 - 56$$
$$= -47 < 0$$

As the disc. is negative

Therefore the roots are imaginary and unequal.

Verification by solving the equation.

$$2x^2 + 3x + 7 = 0$$

using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{(3)^2 - 4(2)(7)}}{2(2)}$$

$$= \frac{-3 \pm \sqrt{9-56}}{4}$$

$$= \frac{-3 \pm \sqrt{-47}}{4}$$

Thus, the roots are imaginary and unequal.

$(iii)\ 16x^2 - 24x + 9 = 0$

Solution :

$16x^2 - 24x + 9 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 16,\ b = -24,\ c = 9$

Disc. $= b^2 - 4ac$

$= (-24)^2 - 4(16)(9)$

$= 576 - 576$

$= 0$

As the disc. is zero.

Therefore the roots are real and equal.

Verification by solving the equation.

$16x^2 - 24x + 9 = 0$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-24) \pm \sqrt{(-24)^2 - 4(16)(9)}}{2(16)}$$

$$= \frac{24 \pm \sqrt{576-576}}{32}$$

$$= \frac{24 \pm \sqrt{0}}{32}$$

$= \frac{24}{32} = \frac{3}{4}$

Thus, the roots are real and equal.

$$(iv)\ 3x^2 + 7x - 13 = 0$$

Solution ;

$3x^2 + 7x - 13 = 0$

Compare it with

$ax^2 + bx + c = 0$

Here $a = 3, b = 7, c = -13$

Disc. $= b^2 - 4ac$

$= (7)^2 - 4(3)(-13)$

$= 49 + 156$

$= 205 > 0$

As the disc. is positive and not a perfect square.

Therefore the roots are irrational $(real)$ and unequal.

Verification by solving the equation.

$3x^2 + 7x - 13 = 0$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-13)}}{2(3)}$$

$$= \frac{-7 \pm \sqrt{49 + 156}}{6}$$

$$= \frac{-7 \pm \sqrt{205}}{6}$$

Thus, the roots are irrational (real) and unequal.

3. For what value of A, the expression $k^2x^2 + 2(k+1)x + 4$ **is perfect square.**

Solution :

$k^2x^2+2(k+1)x+4=0$

Compare it with

$ax^2+bx+c=0$

Here $a=k^2, b=2(k+1), c=4$

Disc. $=b^2-4ac$

$=[2(k+1)]^2-4(k^2)(4)$

$=4(k+1)^2-16k^2$

$=4(k^2+2k+1)-16k^2$

$=4k^2+8k+4-16k^2$

$=-12k^2+8k+4=0$

As the disc. of the given expression is a perfect square.

Therefore the roots are rational and unequal.

So, Disc=0

$-12k^2+8k+4=0$

$-(12k^2-8k-4)=0$

$\Rightarrow$ $12k^2-8k-4=0$

$12k^2-12k+4k-4=0$

$12k(k-1)+4(k-1)=0$

$(12k+4)(k-1)=0$

Either $12k+4=0$ *or* $k-1=0$

$12k=-4$ $\quad k=1$

$k=\frac{-4}{12}$

$k=\frac{-1}{3}$

4. Find the value of k, if the roots of the following equations are equal.

(i) $(2k-1)x^2+3kx+3=0$

Solution :

$(2k-1)x^2+3kx+3=0$

Compare it with

$ax^2+bx+c=0$

Here $a=2k-1,\ b=3k,\ c=3$

As the roots are equal, So

$Disc.=0$

$b^2-4ac=0$

$(3k)^2-4(2k-1)(3)=0$

$9k^2-12(2k-1)=0$

$9k^2-24k+12=0$

$3(3k^2-8k+4)=0$

$\Rightarrow 3k^2-8k+4=0$

$3k^2-6k-2k+4=0$

$3k(k-2)-2(k-2)=0$

$(3k-2)(k-2)=0$

Either $3k-2=0$ *or* $k-2=0$

$3k=2$ $\quad k=2$

$k=\frac{2}{3}$

ClassNotes

$(ii)\ x^2+2(k+2)x+(3k+4)=0$

Solution:

$x^2+2(k+2)x+(3k+4)=0$

Compare it with

$ax^2+bx+c=0$

Here $a=1,\ b=2(k+2),\ c=3k+4$

As the roots are equal, So

$Disc.=0$

$b^2-4ac=0$

$[2(k+2)]^2-4(1)(3k+4)=0$

$4(k+2)^2-4(3k+4)=0$

$4(k^2+4k+4)-12k-16=0$

$4k^2+16k+16-12k-16=0$

$4k^2+4k=0$

$4k(k+1)=0$

Either $\quad 4k=0 \quad$ *or* $\quad k+1=0$

$\quad\quad k=0 \quad\quad\quad k=-1$

(iii) $(3k+2)x^2-5(k+1)x+(2k+3)=0$

Solution :

$(3k+2)x^2-5(k+1)x+(2k+3)=0$

Compare it with

$ax^2+bx+c=0$

Here $\quad a=3k+2,\ b=-5(k+1),\ c=2k+3$

As the roots are equal, So

Disc. $=0$

$b^2-4ac=0$

$[-5(k+1)]^2-4(3k+2)(2k+3)=0$

$25(k^2+2k+1)-4(6k^2+13k+6)=0$

$25k^2+50k+25-24k^2-52k-24=0$

$k^2-2k+1=0$

$(k-1)^2=0$

$k-1=0$

$k=1$

5. Show that the equation $x^2+(mx+c)^2=a^2$ has equal roots, if $c^2=a^2(1+m^2)$

Solution :

$x^2+(mx+c)^2=a^2$

$x^2+m^2x^2+2mcx+c^2=a^2$

$(1+m^2)x^2+2mcx+c^2=a^2$

$\left(1+m^2\right)x^2+2mcx+c^2-a^2=0$

Here $\quad a=1+m^2,\ b=2mc,\ c=c^2-a^2$

As the roots are equal, So

$Disc.=0$

$b^2-4ac=0$

$(2mc)^2-4\left(1+m^2\right)\left(c^2-a^2\right)=0$

$4m^2c^2-4\left(c^2-a^2+m^2c^2-a^2m^2\right)=0$

$4m^2c^2-4c^2+4a^2-4m^2c^2+4a^2m^2=0$

$-4c^2+4a^2+4a^2m^2=0$

$-4\left(c^2-a^2-a^2m^2\right)=0$

$c^2-a^2-a^2m^2=0$

$c^2=a^2+a^2m^2$

$c^2=a^2\left(1+m^2\right)$

Hence proved.

6. Find the condition that the roots of the equation $(mx+c)^2-4ax=0$ are equal.

Solution :

$(mx+c)^2-4ax=0$

$m^2x^2+2mcx+c^2-4ax=0$

$m^2x^2+2mcx-4ax+c^2=0$

$m^2x^2+2(mc-2a)x+c^2=0$

Here $\quad a=m^2,\ b=2(mc-2a),\ c=c^2$

As the roots are equal, So

$Disc.=0$

$b^2-4ac=0$

$\left[2(mc-2a)\right]^2-4\left(m^2\right)\left(c^2\right)=0$

$4\left(m^2c^2-4amc+4a^2\right)-4\left(m^2c^2\right)=0$

$4\left(m^2c^2-4amc+4a^2-m^2c^2\right)=0$

$4\left(4a^2-4amc\right)=0$