Symmetric functions of the roots of a quadratic equation:

Define symmetric functions of the roots of a quadratic equation:

Definition:

Symmetric functions are those functions in which the roots involved are such that the value of the expressions involving them remain unaltered, when roots are interchanged.

For example, if

$$f(\alpha,\beta)=\alpha^2+\beta^2, then$$
$$f(\beta,\alpha)=\beta^2+\alpha^2=\alpha^2+\beta^2 \qquad (\because \beta^2+\alpha^2=\alpha^2+\beta^2)$$
$$=f(\alpha,\beta)$$

Exercise 2.4

1. If α, β are the roots of the equation $x^2+px+q=0$, then evaluate

(i) $\alpha^2+\beta^2$

Solution:

$\alpha^2+\beta^2$

$x^2+px+q=0$

Here a=1, b=p, c=q

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$$=-\frac{p}{1} \qquad \alpha\beta=\frac{q}{1}$$

$$=-p \qquad =q$$

Now $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$=(-p)^2-2(q)$$

$$=p^2-2q$$

(ii) $\alpha^3\beta+\alpha\beta^3$

Solution:

$\alpha^3\beta+\alpha\beta^3$

$x^2+px+q=0$

Here a=1, b=p, c=q

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$=-\frac{p}{1}$ $\alpha\beta=\frac{q}{1}$

$=-p$ $=q$

Now $\alpha^3\beta+\alpha\beta^3=\alpha\beta(\alpha^2+\beta^2)$

$$=\alpha\beta\left[(\alpha+\beta)^2-2\alpha\beta\right]$$

$$=q\left[(-p)^2-2q\right]$$

$$=q(p^2-2q)$$

(iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

Solution:

$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

$x^2+px+q=0$

Here a=1, b=p, c=q

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$=-\frac{p}{1}$ $\alpha\beta=\frac{q}{1}$

$=-p$ $=q$

Now $$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{(-p)^2-2q}{q}=\frac{1}{q}\left(p^2-2q\right)$$

2. If α, β are the roots of the equation $4x^2-5x+6=0$, then find the values of

(i) $\frac{1}{\alpha}+\frac{1}{\beta}$

Solution:

$\frac{1}{\alpha}+\frac{1}{\beta}$

$4x^2-5x+6=0$

Here a=4, b=-5, c=6

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$=-\frac{(-5)}{4}$ $\alpha\beta=\frac{6}{4}$

$=\frac{5}{4}$ $=\frac{3}{2}$

Now $$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$$

$$=\frac{\frac{5}{4}}{\frac{3}{2}}=\frac{5}{4}\times\frac{2}{3}=\frac{5}{6}$$

(ii) $\alpha^2\beta^2$

Solution:

$\alpha^2\beta^2$

$4x^2-5x+6=0$

Here a=4, b=-5, c=6

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$=-\frac{(-5)}{4}$ $\alpha\beta=\frac{6}{4}$

$=\frac{5}{4}$ $=\frac{3}{2}$

Now $\alpha^2\beta^2=(\alpha\beta)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}$

(iii) $\frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}$

Solution:

$4x^2-5x+6=0$

Here a=4, b=-5, c=6

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$=-\frac{(-5)}{4}$ $\alpha\beta=\frac{6}{4}$

$=\frac{5}{4}$ $=\frac{3}{2}$

Now $\frac{1}{\alpha^2\beta}+\frac{1}{\alpha\beta^2}=\frac{\alpha+\beta}{\alpha^2\beta^2}=\frac{\alpha+\beta}{(\alpha\beta)^2}$

$$=\frac{\frac{5}{4}}{\left(\frac{3}{2}\right)^2}=\frac{\frac{5}{4}}{\frac{9}{4}}=\frac{5}{4}\times\frac{4}{9}=\frac{5}{9}$$

(iv) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}$

Solution:

$4x^2-5x+6=0$

Here a=4, b=-5, c=6

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\dfrac{b}{a}$ *and* $\alpha\beta=\dfrac{c}{a}$

$$=-\frac{(-5)}{4} \qquad \alpha\beta=\frac{6}{4}$$

$$=\frac{5}{4} \qquad =\frac{3}{2}$$

Now $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}=\dfrac{\alpha^3+\beta^3}{\alpha\beta}=\dfrac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$$=\frac{\left(\frac{5}{4}\right)^3-3\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)}{\frac{3}{2}}=\frac{\frac{125}{64}-\frac{45}{8}}{\frac{3}{2}}$$

$$=\frac{125-360}{64}\times\frac{2}{3}=-\frac{235}{64}\times\frac{2}{3}$$

$$=-\frac{235}{96}$$

3. If α, β are the roots of the equation lx²+mx+n=0 $(1\neq 0)$, then find the values of:

(i) $\alpha^3\beta^2+\alpha^2\beta^3$

Solution:

$lx^2+mx+n=0$

Here a=l, b=m, c=n

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$=-\frac{m}{l}$ $\alpha\beta=\frac{n}{l}$

Now $\alpha^3\beta^2+\alpha^2\beta^3=\alpha^2\beta^2(\alpha+\beta)=(\alpha\beta)^2(\alpha+\beta)$

$$=\left(\frac{n}{l}\right)^2\left(-\frac{m}{l}\right)=-\frac{mn^2}{l^3}$$

(ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

Solution:

$lx^2+mx+n=0$

Here a=l, b=m, c=n

Let α,β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$=-\frac{m}{l}$ $\alpha\beta=\frac{n}{l}$

Now $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$

$$=\frac{\left(-\frac{m}{l}\right)^2-2\left(\frac{n}{l}\right)}{\left(\frac{n}{l}\right)^2}=\frac{\frac{m^2}{l^2}-\frac{2n}{l}}{\frac{n^2}{l^2}}$$

$$=\frac{m^2-2nl}{l^2}\times\frac{l^2}{n^2}=\frac{m^2-2nl}{n^2}$$

$$=\frac{1}{n^2}\left[m^2-2nl\right]$$