Formation of a quadratic equation:

If α and β are the roots of the required quadratic equation.

Let $\quad x=\alpha \quad$ and $\quad x=\beta$

i.e., $\quad x-\alpha=0 \quad , \quad x-\beta=0$

and $\quad (x-\alpha)(x-\beta)=0$

$\quad x^2-(\alpha+\beta)x+\alpha\beta=0$

which is the required quadratic equation in standard form.

Find a quadratic equation from given roots and establish the formula

$x^2-(\text{sum of the roots})x+\text{product of the roots}=0$

Let $\quad \alpha, \beta$ be the roots of the given equation

$\quad ax^2+bx+c=0 \quad , \quad (a \neq 0) \quad ....(i)$

Then $\quad \alpha+\beta=-\dfrac{b}{a} \quad$ *and* $\quad \alpha\beta=\dfrac{c}{a}$

Rewrite eq.(i) as $\quad x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$

or $\quad x^2-\left(-\dfrac{b}{a}\right)x+\dfrac{c}{a}=0$

$\quad x^2-(\alpha+\beta)x+\alpha\beta=0$

or $\quad x^2-(\text{sum of the roots})x+\text{product of the roots}=0$, that is

$x^2-Sx+P=0$ where $S=\alpha+\beta$ and $P=\alpha\beta$

Exercise 2.5

1. Write the quadratic equations having following roots.

(a) 1,5

Solution:

Since 1 and 5 are the roots of the required quadratic equation, therefore,

$$= -\frac{p}{1} \qquad \alpha\beta = \frac{q}{1}$$

$$= -p \qquad\qquad = q$$

Let roots of the new equation be $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

S = Sum of roots $\qquad$ P = Product of roots

$$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \qquad = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)$$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta} \qquad = 1$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(-p)^2 - 2q}{q}$$

$$= \frac{p^2 - 2q}{q}$$

Thus, the required quadratic equation is

$$x^2 - Sx + P = 0$$

$$x^2 - \left(\frac{p^2 - 2q}{q}\right)x + 1 = 0$$

$$qx^2 - \left(p^2 - 2q\right)x + q = 0$$

S = Sum of roots = 1 + 5 = 6

P = Product of roots = (1) (5) = 5

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 6x + 5 = 0$

(b) 4,9

Solution:

Since 4 and 9 are the roots of the required quadratic equation, therefore,

S = Sum of roots = 4 + 9 = 13

P = Product of roots = (4) (9) = 36

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 13x + 36 = 0$

(c) -2,3

Solution:

Since -2 and 3 are the roots of the required quadratic equation, therefore,

S = Sum of roots = -2 + 3 = 1

P = Product of roots = (-2) (3) = -6

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - (1)x + (-6) = 0$

$x^2 - x - 6 = 0$

(e) 2, – 6

Solution:

Since 2 and – 6 are the roots of the required quadratic equation, therefore,

S = Sum of roots = 2 + (– 6) = – 4

P = Product of roots = (2) (– 6) = – 12

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - (-4)x + (-12) = 0$

$x^2 + 4x - 12 = 0$

(f) – 1, – 7

Solution:

Since –1 and – 7 are the roots of the required quadratic equation, therefore,

S = Sum of roots = (–1) + (– 7) = – 1– 7 = – 8

P = Product of roots = (–1) (– 7) = 7

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - (-8)x + 7 = 0$

$x^2 + 8x + 7 = 0$

(g) 1+ i, 1 – i

Solution:

Since 1+ i and 1 – i are the roots of the required quadratic equation, therefore,

S = Sum of roots = (1+ i) + (1 – i) = 2

P = Product of roots = (1+ i) (1 – i) = $1 - i^2 = 1 - (-1) = 2$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 2x + 2 = 0$

(h) 3+ $\sqrt{2}$, 3 – $\sqrt{2}$

Solution:

Since 3+ $\sqrt{2}$ and 3 – $\sqrt{2}$ are the roots of the required quadratic equation, therefore,

S = Sum of roots = $3 + \sqrt{2} + 3 - \sqrt{2} = 6$

P = Product of roots = $(3 + \sqrt{2})\ (3 - \sqrt{2}) = (3)^2 - (\sqrt{2})^2 = 9 - 2 = 7$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 6x + 7 = 0$

2. If α, β are the roots of the equation $x^2 - 3x + 6 = 0$

Form equations whose roots are

a) $2\alpha + 1, 2\beta + 1$

$x^2 - 3x + 6 = 0$

Here a=1, b= − 3, c=6

Let α, β be the roots of the given equation

$$\text{Then} \quad \alpha + \beta = -\frac{b}{a} \quad \textit{and} \quad \alpha\beta = \frac{c}{a}$$

$$= -\frac{(-3)}{1} \qquad \alpha\beta = \frac{6}{1}$$

$$= 3 \qquad\qquad = 6$$

Let roots of the new equation be $2\alpha + 1, 2\beta + 1$

$$\begin{aligned} S &= \text{Sum of roots} & P &= \text{Product of roots} \\ &= (2\alpha+1)+(2\beta+1) & &= (2\alpha+1)(2\beta+1) \\ &= 2\alpha+1+2\beta+1 & &= (2\alpha+1)(2\beta+1) \\ &= 2\alpha+2\beta+2 & &= 4\alpha\beta+2\alpha+2\beta+1 \\ &= 2(\alpha+\beta)+2 & &= 4(6)+2(3)+1 \\ &= 2(3)+2 & &= 24+6+1 \\ &= 8 & &= 31 \end{aligned}$$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 8x + 31 = 0$

b) α^2, β^2

Solution:

$x^2 - 3x + 6 = 0$

Here a=1, b= − 3, c=6

Let α, β be the roots of the given equation

$$\text{Then} \quad \alpha + \beta = -\frac{b}{a} \quad \textit{and} \quad \alpha\beta = \frac{c}{a}$$

$$= -\frac{(-3)}{1} \qquad \alpha\beta = \frac{6}{1}$$

$$= 3 \qquad\qquad = 6$$

Let roots of the new equation be α^2, β^2

S = Sum of roots	P = Product of roots
$=\alpha^2+\beta^2$	$=(\alpha^2)(\beta^2)$
$=(\alpha+\beta)^2-2\alpha\beta$	$=(\alpha^2\beta^2)$
$=(3)^2-2(6)$	$=(\alpha\beta)^2$
$=9-12$	$=(6)^2$
$=-3$	$=36$

Thus, the required quadratic equation is

$x^2-Sx+P=0$

$x^2-(-3)x+36=0$

$x^2+3x+36=0$

c) $\dfrac{1}{\alpha},\dfrac{1}{\beta}$

Solution:

$x^2-3x+6=0$

Here a=1, b= −3, c=6

Let α, β be the roots of the given equation

Then $\quad \alpha+\beta=-\dfrac{b}{a} \quad$ *and* $\quad \alpha\beta=\dfrac{c}{a}$

$=-\dfrac{(-3)}{1} \qquad \alpha\beta=\dfrac{6}{1}$

$=3 \qquad\qquad =6$

Let roots of the new equation be $\dfrac{1}{\alpha},\dfrac{1}{\beta}$

S = Sum of roots	P = Product of roots
$=\dfrac{1}{\alpha}+\dfrac{1}{\beta}$	$=\left(\dfrac{1}{\alpha}\right)\left(\dfrac{1}{\beta}\right)$
$=\dfrac{\alpha+\beta}{\alpha\beta}$	$=\dfrac{1}{\alpha\beta}$
$=\dfrac{3}{6}=\dfrac{1}{2}$	$=\dfrac{1}{6}$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - \frac{1}{2}x + \frac{1}{6} = 0$

$\Rightarrow 6x^2 - 3x + 1 = 0$

d) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Solution:

$x^2 - 3x + 6 = 0$

Here a=1, b= -3, c=6

Let α, β be the roots of the given equation

Then $\alpha + \beta = -\frac{b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

$= -\frac{(-3)}{1}$ $\alpha\beta = \frac{6}{1}$

$= 3$ $= 6$

Let roots of the new equation be $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

S = Sum of roots

$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$= \frac{\alpha^2 + \beta^2}{\alpha\beta}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{(3)^2 - 2(6)}{6}$

$= \frac{9-12}{6}$

$= -\frac{3}{6}$

$= -\frac{1}{2}$

P = Product of roots

$= \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)$

$= 1$

e) $\alpha+\beta, \frac{1}{\alpha}+\frac{1}{\beta}$

Solution:

$x^2-3x+6=0$

Here a=1, b= −3, c=6

Let α, β be the roots of the given equation

Then $\alpha+\beta=-\frac{b}{a}$ *and* $\alpha\beta=\frac{c}{a}$

$=-\frac{(-3)}{1}$ $\alpha\beta=\frac{6}{1}$

$=3$ $=6$

Let roots of the new equation be $\alpha+\beta, \frac{1}{\alpha}+\frac{1}{\beta}$

$S=$ Sum of roots and $P=$ Product of roots

$=\alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}$ $=(\alpha+\beta)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$

$=(\alpha+\beta)+\frac{\alpha+\beta}{\alpha\beta}$ $=(\alpha+\beta)\left(\frac{\alpha+\beta}{\alpha\beta}\right)$

$=3+\frac{3}{6}$ $=(3)\left(\frac{3}{6}\right)$

$=3+\frac{1}{2}$ $=\frac{3}{2}$

$=\frac{7}{2}$

Thus, the required quadratic equation is

$x^2-Sx+P=0$

$x^2-\frac{7}{2}x+\frac{3}{2}=0$

$\Rightarrow 2x^2-7x+3=0$

3. If α, β are the roots of the equation $x^2 + px + q = 0$

Form equations whose roots are

a) α^2, β^2

Solution:

$x^2 + px + q = 0$

Here $a = 1,\ b = p,\ c = q$

Let α, β be the roots of the given equation

Then $\alpha + \beta = -\frac{b}{a}$ *and* $\alpha\beta = \frac{c}{a}$

$= -\frac{p}{1}$ $\quad \alpha\beta = \frac{q}{1}$

$= -p$ $\quad = q$

Let roots of the new equation be α^2, β^2

$S =$ Sum of roots and $P =$ Product of roots

$= \alpha^2 + \beta^2$ $\quad = (\alpha^2)(\beta^2)$

$= (\alpha + \beta)^2 - 2\alpha\beta$ $\quad = (\alpha\beta)^2$

$= (-p)^2 - 2q$ $\quad = q^2$

$= p^2 - 2q$

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - (p^2 - 2q)x + q^2 = 0$

b) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Solution:

$x^2 + px + q = 0$

Here $a = 1,\ b = p,\ c = q$

Let α, β be the roots of the given equation

Then $\alpha + \beta = -\frac{b}{a}$ *and* $\alpha\beta = \frac{c}{a}$