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Question No 2

a) Use decomposition result to express a_{ij} .... ?

Solution:-

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

So,

$a_{(ij)}$ and $a_{[ij]}$ Satisfy the appropriate conditions.

b)

Solution:-

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 6 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

So,

$a_{(ij)}$ and $a_{[ij]}$ Satisfy the appropriate conditions.

c)

Solution:-

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

So,

$a_{(ij)}$ and $a_{[ij]}$ Satisfy the appropriate conditions.

Question No 4

Explicitly verify the following kronecker delta

$$\delta_{ij} a_j = a_i$$
$$\delta_{ij} a_{jk} = a_{ik}$$

Solution:-

$$\delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3$$

$$= \begin{bmatrix} \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 \\ \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 \\ \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

$$\delta_{ij} a_{jk} = \begin{bmatrix} \delta_{11} a_{11} + \delta_{12} a_{21} + \delta_{13} a_{31} & \delta_{11} a_{12} + \delta_{12} a_{22} + \delta_{13} a_{32} & \delta_{11} a_{13} + \delta_{12} a_{23} + \delta_{13} a_{33} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

Question No 6

Determine components of vector b_i and matrix a_{ij} given in exercise in a new coordinate system.... ...... The rotation direction follows the tve sense presented in 1-2.

Solution:-

45° rotation about x_1-axis

$$\Rightarrow Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

Now, we know

$$b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

Also

$$b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{iq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 3.5 & 2.5 \\ -\sqrt{2}/2 & 1.5 & 0.5 \end{bmatrix}$$

Question No 8

Show that second order tensor $a\,\delta_{ij}$, where a is an arbitrary constant .... ?

Solution:-

$$a' \delta_{ij}' = Q_{ip} Q_{jp}\, a\, \delta_{pq}$$

$$= a\, Q_{ip} Q_{jp} = a\, \delta_{ij}$$

Question No 10

For the fourth order isotropic tensor ....

$C_{ijkl} = C_{klij}$.

Solution:-

$$C_{ijkl} = \alpha\, \delta_{ij} \delta_{kl} + \beta\, \delta_{ik} \delta_{jl} + \gamma\, \delta_{il} \delta_{jk}$$

$$= \alpha\, \delta_{ij} \delta_{kl} + \beta (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$= \alpha\, \delta_{kl} \delta_{ij} + \beta (\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li}) = C_{klij}$$

Question NO 12

Determine the invariants ....

a) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:-

$$a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I_a = -1, \ II_a = -2, \ III_a = 0$$

$\because$ Characteristic equation is $-\lambda^3 - \lambda^2 + 2\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 + \lambda - 2) = 0$$

$$\Rightarrow \lambda(\lambda+2)(\lambda-1) = 0$$

$$\Rightarrow \lambda_1 = 0, \ \lambda_2 = -2, \ \lambda_3 = 1$$

Now for $\lambda_1 = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{aligned} -n_1^{(1)} + n_2^{(1)} &= 0 \\ n_3^{(1)} &= 0 \\ n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} &= 1 \end{aligned} \quad \Rightarrow n_1^{(1)} = +n_2^{(1)} = \pm \sqrt{2}/2, \ n^{(1)} = \pm(\sqrt{2}/2)(1,1,0)$$

For $\lambda_2 = -2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{aligned} n_1^{(2)} + n_2^{(2)} &= 0 \\ n_3^{(2)} &= 0 \\ n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} &= 1 \end{aligned} \quad \Rightarrow n_1^{(2)} = -n_2^{(2)} = \pm \sqrt{2}/2, \ n^{(2)} = \pm(\sqrt{2}/2)(-1,1,0)$$

For $\lambda_3 = 1$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0.$$

$$\Rightarrow \begin{aligned} -2n_1^{(3)} + n_2^{(3)} &= 0 \\ n_1^{(3)} - 2n_2^{(3)} &= 0 \\ n_1^{(3)2} + n_2^{(3)2} + n_3^{(3)2} &= 1 \end{aligned} \quad \Rightarrow n_1 = n_2 = 0, \ n_3 = 1 \Rightarrow n^{(3)} = \pm(0,0,1)$$

The rotational matrix is given by

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

and

$$a'_{ij} = Q_{ip} Q_{ip} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution:-

$$a_{ij} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -4 \,,\; II_a = 3 \,,\; III_a = 0$$

Characteristic equation is

$$-\lambda^3 - 4\lambda^2 - 3\lambda = 0 \quad \Rightarrow \lambda(\lambda^2 + 4\lambda + 3) = 0 \Rightarrow \lambda(\lambda+3)(\lambda+1) = 0$$

So

$$\lambda_1 = -3 \,,\quad \lambda_2 = -1 \,,\quad \lambda_3 = 0$$

For $\lambda_1 = -3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \Rightarrow \begin{array}{l} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1 \end{array} \Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2$$

So $n_1^{(1)} = \pm(\sqrt{2}/2)(-1,1,0)$

For $\lambda_2 = -1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \Rightarrow \begin{array}{l} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0 \\ n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} = 1 \end{array} \Rightarrow n_1 = n_2 = \pm\sqrt{2}/2$$

So $n^{(2)} = \pm(\sqrt{2}/2)(1,1,0)$

For $\lambda_3 = 0$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \Rightarrow \begin{array}{l} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \\ n_1^{(3)2} + n_2^{(3)2} + n_3^{(3)2} = 1 \end{array} \Rightarrow n_1 = n_2 = 0 \,,\; n_3^{(3)} = 1$$

So

$n^{(3)} = \pm(0,0,1)$

The rotation matrix is

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

and

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution:-

$$a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -2, \; II_a = 0, \; III_a = 0$$

Characteristic equation is

$$-\lambda^3 - 2\lambda^2 = 0 \quad \text{or} \quad \lambda^2(\lambda+2) = 0$$

$$\lambda_1 = -2, \; \lambda_2 = \lambda_3 = 0.$$

for $\lambda_1 = -2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} n_1 + n_2 = 0 \\ n_3 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1 \end{matrix} \Rightarrow n_1 = -n_2 = \pm\sqrt{2}/2$$

So $n^{(1)} = \pm\sqrt{2}/2(-1, 1, 0)$

for $\lambda_2 = \lambda_3 = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -n_1 + n_2 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1 \end{matrix} \Rightarrow n_1 = n_2$$

So $n_3^2 = 1 - 2n_1^2$

$\Rightarrow n = \pm(k, k, \sqrt{1-2k^2})$

for arbitrary k, and thus direction are not uniquely determined, for convenience we choose $k = \sqrt{2}/2$ and 0 to get $n^{(2)} = \pm \sqrt{2}/2\,(1,1,0)$ and $n^{(3)} = \pm(0,0,1)$

The rotation matrix is given by

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

and

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Question NO 14

Calculate the quantities $\nabla \cdot u$, $\nabla \times u$, $\nabla^2 u$, ∇u, $tr(\nabla u)$....

a) $u = x_1 e_1 + x_1 x_2 e_2 + 2 x_1 x_2 x_3 e_3$

Solution:-

$$u = x_1 e_1 + x_1 x_2 e_2 + 2 x_1 x_2 x_3 e_3$$

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3} = 1 + x_1 + 2 x_1 x_2$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1 x_2 & 2 x_1 x_2 x_3 \end{vmatrix}$$

$$= 2 x_1 x_3 e_1 - 2 x_2 x_3 e_2 + x_2 e_3$$

$$\nabla^2 u = 0 e_1 + 0 e_2 + 0 e_3 = 0$$

$$\nabla u = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2 x_2 x_3 & 2 x_1 x_3 & 2 x_1 x_2 \end{bmatrix}, \quad tr(\nabla u) = 1 + x_1 + 2 x_1 x_2$$

b) $u = x_1^2 e_1 + 2x_1x_2e_2 + x_3^3 e_3$

Solution:-

$$u = x_1^2 e_1 + 2x_1x_2e_2 + x_3^3 e_3$$

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3} = 2x_1 + 2x_1 + 3x_3^2$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1x_2 & x_3^3 \end{vmatrix}$$

$$= 0e_1 - 0e_2 + 2x_2e_3$$

$$\nabla^2 u = 2e_1 + 0e_2 + 6x_3e_3 = 0.$$

$$\nabla u = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix}, \quad tr(\nabla u) = 4x_1 + 3x_3^2$$

c) $u = x_2^2 e_1 + 2x_2x_3e_2 + 4x_1^2 e_3$

Solution:-

$$u = x_2^2 e_1 + 2x_2x_3e_2 + 4x_1^2 e_3$$

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3} = 0 + 2x_3 + 0.$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{vmatrix}$$

$$= -2x_2e_1 - 8x_1e_2 - 2x_2e_3$$

$$\nabla^2 u = 2e_1 + 0e_2 + 8e_3 = 0$$

$$\nabla u = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix}, \quad tr(\nabla u) = 3x_3$$

Question No 16
Using Index Notation, verify vector identities

a) $(1.8.5)_{1,2,3}$

Solution:-

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\,\psi + \phi\nabla\psi$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k} = \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi$$

$$= \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot(\phi u) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k}u_k = \nabla\phi\cdot u + \phi(\nabla\cdot u)$$

b) $(1.8.5)_{4,5,6,7}$

Solution:-

$$\nabla\times(\phi u) = \varepsilon_{ijk}(\phi u_k)_{,j} = \varepsilon_{ijk}(\phi u_{k,j} + \phi_{,j}u_k)$$

$$= \varepsilon_{ijk}\phi_{,j}u_k + \phi\varepsilon_{ijk}u_k$$

$$= \nabla\phi\times u + \phi(\nabla\times u)$$

$$\nabla\cdot(u\times v) = (\varepsilon_{ijk}u_jv_k)_{,i} = \varepsilon_{ijk}(u_jv_{k,i} + u_{j,i}v_k)$$

$$= v_k\varepsilon_{ijk}u_{j,i} + u_j\varepsilon_{ijk}v_{k,i} = v\cdot(\nabla\times u) - u\cdot(\nabla\times v)$$

$$\nabla\times\nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj}$$

$$\nabla\cdot\nabla\phi = (\phi_{,k})_{,k} = \phi_{,kk} = \nabla^2\phi$$

c) $(1.8.5)_{8,9,10}$

Solution:-

$$\nabla\cdot(\nabla\times u) = (\varepsilon_{ijk}u_{k,j})_{,i} = \varepsilon_{ijk}u_{k,ji} = 0$$

$$\nabla\times(\nabla\times u) = \varepsilon_{mni}(\varepsilon_{ijk}u_{k,j}) = \varepsilon_{imn}\varepsilon_{ijk}u_{k,jn}$$

$$= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})u_{k,jn} = u_{n,nm} - u_{m,nn}$$

$$= \nabla(\nabla\cdot U) - \nabla^2 U$$

$$U\times(\nabla\times U) = \varepsilon_{ijk}\, U_j(\varepsilon_{kmn}\, U_{n,m}) = \varepsilon_{kij}\,\varepsilon_{kmn}\, U_j\, U_{n,m}$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\, U_j\, U_{n,m}$$

$$= U_n U_{n,i} - U_m U_{i,m}$$

$$= \frac{1}{2}\nabla(U\cdot U) - U\cdot\nabla U$$

Question No 18

For spherical coordinate system (R, ϕ, θ). show that

$$h_1 = 1,\quad h_2 = R,\quad h_3 = R\sin\phi$$

Solution:-

Spherical coordinates

$$\xi^1 = R,\quad \xi^2 = \phi,\quad \xi^3 = \theta$$

$$x^1 = \xi^1 \sin\xi^2 \cos\xi^3,\quad x^2 = \xi^1 \sin\xi^2 \sin\xi^3,\quad x^3 = \xi^1 \cos\xi^2$$

Scale factors

$$(h_1)^2 = \frac{\partial x^k}{\partial\xi^1}\frac{\partial x^k}{\partial\xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi$$

$$= 1 \quad\Rightarrow\quad h_1 = 1$$

$$(h_2)^2 = \frac{\partial x^k}{\partial\xi^2}\frac{\partial x^k}{\partial\xi^2} = R^2 \quad\Rightarrow\quad h_2 = R$$

$$(h_3)^2 = \frac{\partial x^k}{\partial\xi^3}\frac{\partial x^k}{\partial\xi^3} = R^2\sin^2\phi \quad\Rightarrow\quad h_3 = R\sin\phi$$

Unit vectors are

$$\hat{e}_R = \cos\theta\sin\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\phi\, e_3$$

$$\hat{e}_\phi = \cos\theta\cos\phi\, e_1 + \sin\theta\cos\phi\, e_2 - \sin\phi\, e_3$$

$$\hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2$$

$$\frac{\partial\hat{e}_R}{\partial R} = 0,\quad \frac{\partial\hat{e}_R}{\partial\phi} = \hat{e}_\phi,\quad \frac{\partial\hat{e}_R}{\partial\theta} = \sin\phi\,\hat{e}_\theta$$

$$\frac{\partial\hat{e}_\phi}{\partial R} = 0,\quad \frac{\partial\hat{e}_\phi}{\partial\phi} = 0,\quad \frac{\partial\hat{e}_\phi}{\partial\theta} = -\cos\phi\,\hat{e}_\theta.$$

$$\frac{\partial \hat{e}_\theta}{\partial R} = 0 \;, \quad \frac{\partial \hat{e}_\theta}{\partial \phi} = 0 \;, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\phi\, \hat{e}_\phi$$

Now

$$\bar{\nabla} = \hat{e}_R \frac{\partial}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{1}{R\sin\phi\,\partial\theta}$$

$$\bar{\nabla} f = \hat{e}_R \frac{\partial f}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial f}{\partial \phi} + \hat{e}_z \frac{1}{R\sin\phi}\frac{\partial f}{\partial \theta}$$

$$\bar{\nabla}\cdot U = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}(R^2\sin\phi\, U_R) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}(R\sin\phi\, U_\phi) + \frac{1}{R^2\sin\phi\,\partial\theta}(R U_\theta)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 U_R) + \frac{1}{R\sin\phi\,\partial\phi}(\sin\phi\, U_\phi) + \frac{1}{R\sin\phi}\frac{\partial}{\partial\theta}(U_\theta)$$

$$\bar{\nabla}^2 f = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left[R^2\sin\phi\frac{\partial f}{\partial R}\right] + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial\theta}\right)$$

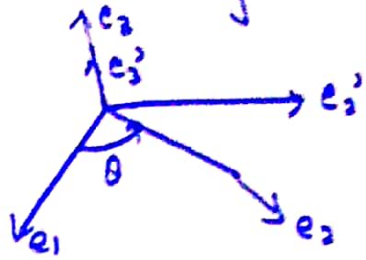
$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left[R^2\frac{\partial f}{\partial R}\right] + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}$$

$$\bar{\nabla}\times U = \left[\frac{1}{R^2\sin\phi}\left(\frac{\partial}{\partial\phi}(R\sin\phi\, U_\theta) - \frac{\partial}{\partial\theta}(R U_\phi)\right)\right]\hat{e}_R + \left[\frac{1}{R\sin\phi}\left[\frac{\partial}{\partial\theta}(U_R) - \frac{\partial}{\partial R}(R\sin\phi\, U_\theta)\right]\right]\hat{e}_\phi + \left[\frac{1}{R}\frac{\partial}{\partial R}\left[(R U_\phi) - \frac{\partial}{\partial\phi}(U_R)\right]\right]\hat{e}_\theta$$

$$= \left[\frac{1}{R\sin\phi}\left(\frac{\partial}{\partial\phi}(\sin\phi\, U_\theta)\right) - \frac{\partial U_\phi}{\partial\theta}\right]\hat{e}_R + \left[\frac{1}{R\sin\phi}\frac{\partial U_R}{\partial\theta} - \frac{1}{R}\frac{\partial}{\partial R}(R U_\theta)\right]\hat{e}_\phi + \left[\frac{1}{R}\left(\frac{\partial}{\partial R}(R U_\phi) - \frac{\partial U_R}{\partial\phi}\right)\right]\hat{e}_\theta$$

Example Proof:

Suppose the basis e_1', e_2', e_3' is obtain by rotating basis $\{e_1, e_2, e_3\}$ through angle θ about unit vector e_3. Write out rule for 2-tensor explicitly.



Solution:-

$$e_1' = \cos\theta e_1 + \sin\theta e_2$$
$$e_2' = -\sin\theta e_1 + \cos\theta e_2$$
$$e_3' = e_3$$

$$[A'] = [Q][A][Q^T]$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A_{11}' & A_{12}' & A_{13}' \\ A_{21}' & A_{22}' & A_{23}' \\ A_{31}' & A_{32}' & A_{33}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11}\cos\theta + A_{12}\sin\theta & -A_{11}\sin\theta + A_{12}\cos\theta & A_{13} \\ A_{21}\cos\theta + A_{22}\sin\theta & -A_{21}\sin\theta + A_{22}\cos\theta & A_{23} \\ A_{31}\cos\theta + A_{32}\sin\theta & -A_{31}\sin\theta + A_{32}\cos\theta & A_{33} \end{bmatrix}$$

$$R.H.S = \begin{bmatrix} A_{11}\cos^2\theta + A_{12}\sin\theta\cos\theta + A_{21}\cos\theta\sin\theta + A_{22}\sin^2\theta & -A_{11}\sin\theta\cos\theta + A_{12}\cos^2\theta \\ -A_{11}\sin\theta\cos\theta - A_{22}\cos^2\theta + A_{22}\sin\theta\cos\theta & -A_{22}\sin^2\theta + A_{22}\cos\theta\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta & \end{bmatrix}$$

R.H.S

$$= \begin{bmatrix} A_{11}\cos^2\theta + A_{12}\sin\theta\cos\theta + A_{21}\cos\theta\sin\theta + A_{22}\sin^2\theta & -A_{11}\sin\theta\cos\theta + A_{12}\cos^2\theta - A_{21}\sin^2\theta + A_{22}\cos\theta\sin\theta & A_{13}\cos\theta + A_{23}\sin\theta \\ -A_{11}\sin\theta\cos\theta - A_{22}\cos^2\theta + A_{22}\sin\theta\cos\theta & A_{11}\sin^2\theta - A_{12}\cos\theta\sin\theta - A_{21}\sin\theta\cos\theta + A_{22}\cos^2\theta & A_{23}\cos\theta - A_{13}\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta & A_{32}\cos\theta - A_{31}\sin\theta & A_{33} \end{bmatrix}$$

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$$= \begin{bmatrix} A_{11}\cos^2\theta + A_{22}\sin^2\theta + (A_{12}+A_{21})\sin\theta\cos\theta & A_{12}\cos^2\theta - A_{21}\sin^2\theta + (A_{22}-A_{11})\cos\theta\sin\theta & A_{13}\cos\theta + A_{23}\sin\theta \\ A_{21}\cos^2\theta - A_{22}\sin^2\theta + (A_{22}-A_{11})\sin\theta\cos\theta & A_{22}\cos^2\theta + A_{11}\sin^2\theta - (A_{12}+A_{21})\cos\theta\sin\theta & A_{23}\cos\theta - A_{13}\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta & A_{32}\cos\theta - A_{31}\sin\theta & A_{33} \end{bmatrix}$$

Using half angle identities

$$= \begin{bmatrix} \left(\frac{A_{11}+A_{22}}{2}\right) + \left(\frac{A_{11}-A_{22}}{2}\right)\cos2\theta + \left(\frac{A_{12}+A_{21}}{2}\right)\sin2\theta & \left(\frac{A_{12}-A_{21}}{2}\right) + \left(\frac{A_{12}+A_{21}}{2}\right)\cos2\theta + \left(\frac{A_{22}-A_{21}}{2}\right)\sin2\theta & A_{13}\cos\theta + A_{23}\sin\theta \\ \left(\frac{A_{21}-A_{12}}{2}\right) + \left(\frac{A_{21}+A_{12}}{2}\right)\cos2\theta + \left(\frac{A_{22}-A_{11}}{2}\right)\sin2\theta & \left(\frac{A_{22}+A_{11}}{2}\right) + \left(\frac{A_{23}-A_{11}}{2}\right)\cos2\theta - \left(\frac{A_{12}+A_{21}}{2}\right)\sin2\theta & A_{23}\cos\theta - A_{13}\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta & A_{32}\cos\theta - A_{31}\sin\theta & A_{33} \end{bmatrix}$$

Comparing both sides we get

$$A_{11}' = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos2\theta + \frac{A_{12}+A_{21}}{2}\sin2\theta$$

$$A_{12}' = \frac{A_{12}-A_{21}}{2} + \frac{A_{12}+A_{21}}{2}\cos2\theta + \frac{A_{22}-A_{11}}{2}\sin2\theta$$

$$A_{13}' = A_{13}\cos\theta + A_{23}\sin\theta$$

$$A_{21}' = \frac{A_{21}-A_{12}}{2} + \frac{A_{21}+A_{12}}{2}\cos2\theta + \frac{A_{22}-A_{11}}{2}\sin2\theta$$

$$A_{22}' = \frac{A_{22}+A_{11}}{2} + \frac{A_{22}-A_{11}}{2}\cos2\theta - \frac{A_{12}+A_{21}}{2}\sin2\theta$$

$$A_{23}' = A_{23}\cos\theta - A_{13}\sin\theta$$

$$A_{31}' = A_{31}\cos\theta + A_{32}\sin\theta$$

$$A_{32}' = A_{32}\cos\theta - A_{31}\sin\theta$$

$$A_{33}' = A_{33}$$

In special case (A) is symmetric in addition

$$A_{13} = A_{23} = 0$$

So nine equations simplify to

$$A_{11}' = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + A_{12}\sin 2\theta$$

$$A_{22}' = \frac{A_{11}+A_{22}}{2} - \frac{A_{11}-A_{22}}{2}\cos 2\theta - A_{12}\sin 2\theta$$

$$A_{12}' = -\frac{A_{11}-A_{22}}{2}\sin 2\theta,$$

together with $A_{13}' = A_{23}' = 0$ and $A_{33}' = A_{33}$.

They are well known equations underlying the Mohr's circle for transforming 2-tensors in 2D