a. $\lim\limits_{x \to \infty} \dfrac{\ln\left(x^{2020}\right)}{x}$

$= \lim\limits_{x \to \infty} \dfrac{2020 \ln(x)}{x}$

$= 2020 \lim\limits_{x \to \infty} \dfrac{\ln(x)}{x} = \dfrac{\infty}{\infty}$ (bertemu indeterminate form)

Menggunakan aturan L'Hopital

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

sehingga didapat:

$$2020 \lim_{x \to \infty} \frac{\ln(x)}{x} = 2020 \lim_{x \to \infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}x}$$

$$= 2020 \lim_{x \to \infty} \frac{1/x}{1} = 2020 \lim_{x \to \infty} \frac{1}{x} = 2020 \cdot 0 = 0$$

b. $\lim\limits_{x \to \infty} \dfrac{x^{2020}}{e^x} = \lim\limits_{x \to \infty} \left(x^{2020} e^{-x}\right) = \infty \cdot 0$ (bertemu indeterminate form)

Menggunakan aturan L'Hopital

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

sehingga didapat:

$$\lim_{x \to \infty} \frac{x^{2020}}{e^x} = \lim_{x \to \infty} \frac{\frac{d}{dx}x^{2020}}{\frac{d}{dx}e^x}$$

$= \lim\limits_{x \to \infty} \dfrac{2020x^{2019}}{e^x} = \dfrac{\infty}{\infty}$ (masih bertemu indeterminate form, maka diturunkan lagi)

$= \lim\limits_{x \to \infty} \dfrac{2020 \cdot 2019x^{2018}}{e^x} = \dfrac{\infty}{\infty}$ (masih bertemu indeterminate form, maka diturunkan lagi)

$= \lim\limits_{x \to \infty} \dfrac{2020 \cdot 2019 \cdot 2018x^{2017}}{e^x} = \dfrac{\infty}{\infty}$ (masih bertemu indeterminate form, maka diturunkan lagi)

...

diturunkan sebanyak 2020 kali, sehingga tidak menghasilkan indeterminate form lagi

...

... (proses disingkat, hasilnya membentuk pola yang bagus, mudah dibaca)

...

menjadi factorial berikut

$$= \lim_{x \to \infty} \frac{2020 \cdot 2019 \cdot 2018 \cdot 2017 \cdot \ldots \cdot 1}{e^x} = \lim_{x \to \infty} \frac{2020!}{e^x} = \frac{2020!}{\infty} = 0$$

c. $\lim_{x\to\pi/2} \dfrac{3\sec(x)+5}{\tan(x)}$

$$\boxed{\text{Catatan: } \tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \sec(x) = \frac{1}{\cos(x)}}$$

$$= \lim_{x\to\pi/2} \frac{3\sec(x)+5}{\sin(x)\,/\cos(x)}$$

$$= \lim_{x\to\pi/2} \left(\frac{3\sec(x)}{\sin(x)\,/\cos(x)} + \frac{5}{\sin(x)\,/\cos(x)}\right)$$

$$= \lim_{x\to\pi/2} \left(\frac{3(1/\cos(x))}{\sin(x)\,/\cos(x)} + \frac{5}{\sin(x)\,/\cos(x)}\right)$$

$$= \lim_{x\to\pi/2} \left(\frac{3\cancel{\cos(x)}}{\cancel{\cos(x)}\sin(x)} + \frac{5\cos(x)}{\sin(x)}\right)$$

$$= \lim_{x\to\pi/2} \left(\frac{3}{\sin(x)} + \frac{5\cos(x)}{\sin(x)}\right)$$

$$= \left(\frac{3}{\sin(\pi/2)} + \frac{5\cos(\pi/2)}{\sin(\pi/2)}\right)$$

$$= \frac{3}{1} + \frac{5\cdot 0}{1} = \frac{3}{1} + 0 = 3$$

d. $\lim_{x\to\infty} [x - \ln(x)]$

$$= \lim_{x\to\infty} x\left[1 - \frac{\ln(x)}{x}\right] \text{ (dipecah menjadi 2 limit)}$$

$$= \left(\lim_{x\to\infty} x\right)\cdot\left(\lim_{x\to\infty}\left[1 - \frac{\ln(x)}{x}\right]\right) \text{ (bertemu indeterminate form pada limit ke-2)}$$

$$\boxed{\begin{array}{l}\text{Menggunakan aturan L'Hopital} \\ \displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}\end{array}}$$

$$= \left(\lim_{x\to\infty} x\right)\cdot\left(\lim_{x\to\infty}\left[1 - \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}x}\right]\right)$$

$$= \left(\lim_{x\to\infty} x\right)\cdot\left(\lim_{x\to\infty}\left[1 - \frac{1/x}{1}\right]\right)$$

$$= \left(\lim_{x\to\infty} x\right)\cdot\left(\lim_{x\to\infty}\left[1 - \frac{1}{x}\right]\right)$$

$$= \left(\lim_{x\to\infty} x\right)\cdot(1-0)$$

$$= \infty\cdot 1 = \infty$$

e. $\lim\limits_{x\to 0}\left[\csc^2(x)-\frac{1}{x^2}\right]$

Catatan: $\csc(x)=\frac{1}{\sin(x)}, \quad \sin(2x)=2\sin(x)\cos(x)$

$=\lim\limits_{x\to 0}\left[\frac{1}{\sin^2(x)}-\frac{1}{x^2}\right]$

$=\lim\limits_{x\to 0}\left[\frac{x^2}{x^2\sin^2(x)}-\frac{\sin^2(x)}{x^2\sin^2(x)}\right]$ (disamakan penyebutnya)

$=\lim\limits_{x\to 0}\frac{x^2-\sin^2(x)}{x^2\sin^2(x)}=\frac{0^2-\sin^2(0)}{0^2\sin^2(0)}=\frac{0}{0}$ (bertemu indeterminate form)

Menggunakan aturan L'Hopital

$\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$

sehingga $\lim\limits_{x\to 0}\frac{x^2-\sin^2(x)}{x^2\sin^2(x)}=\lim\limits_{x\to 0}\frac{\frac{d}{dx}\left(x^2-\sin^2(x)\right)}{\frac{d}{dx}\left(x^2\sin^2(x)\right)}$

$=\lim\limits_{x\to 0}\frac{2x-\sin(2x)}{2x\sin^2(x)+\sin(2x)\,x^2}=\frac{0}{0}$ (masih bertemu indeterminate form, diturunkan lagi)

$=\lim\limits_{x\to 0}\frac{\frac{d}{dx}(2x-\sin(2x))}{\frac{d}{dx}(2x\sin^2(x)+\sin(2x)\,x^2)}$

$=\lim\limits_{x\to 0}\frac{2-\cos(2x)\cdot\ 2}{2x^2\cos(2x)+4x\sin(2x)+2\sin^2(x)}$

$=\lim\limits_{x\to 0}\frac{1-\cos(2x)}{x^2\cos(2x)+2x\sin(2x)+\sin^2(x)}=\frac{0}{0}$ (masih bertemu indeterminate form, diturunkan lagi)

$=\lim\limits_{x\to 0}\frac{\frac{d}{dx}(1-\cos(2x))}{\frac{d}{dx}(x^2\cos(2x)+2x\sin(2x)+\sin^2(x))}$

$=\lim\limits_{x\to 0}\frac{2\sin(2x)}{2x\cos(2x)-2x^2\sin(2x)+2\sin(2x)+4x\cos(2x)+\sin(2x)}$

$=\lim\limits_{x\to 0}\frac{2\sin(2x)}{6x\cos(2x)+3\sin(2x)-2x^2\sin(2x)}=\frac{0}{0}$ (masih bertemu indeterminate form, diturunkan lagi)

$=\lim\limits_{x\to 0}\frac{\frac{d}{dx}(2\sin(2x))}{\frac{d}{dx}(6x\cos(2x)+3\sin(2x)-2x^2\sin(2x))}$

$=\lim\limits_{x\to 0}\frac{4\cos(2x)}{6\cos(2x)-12x\sin(2x)+6\cos(2x)-2\left(2x\sin(2x)+\cos(2x)\cdot\ 2x^2\right)}$

$=\lim\limits_{x\to 0}\frac{4\cos(2x)}{12\cos(2x)-4x^2\cos(2x)-16x\sin(2x)}$

$=\lim\limits_{x\to 0}\frac{\cancel{4}\cos(2x)}{\cancel{4}(3\cos(2x)-x^2\cos(2x)-4x\sin(2x))}$

$=\lim\limits_{x\to 0}\frac{\cos(2x)}{3\cos(2x)-x^2\cos(2x)-4x\sin(2x)}$

$=\frac{\cos(2(0))}{3\cos(2(0))-0^2\cos(2(0))-4(0)\sin(2(0))}$

$=\frac{\cos(0)}{3\cos(0)-0-0}=\frac{1}{3}$