Diffusion with Discontinuous Galerkin Schemes

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Outline

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2. Symmetric LDG
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4. Symmetric Direct DG
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Outline

- Looked at the following equation:

$$g = \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

- Explicit update formulas derived for approximating solution using piecewise linear basis functions:

$$u_h(x,t) = u_0 + \frac{x - x_j}{\Delta x/2} u_1$$

- Von Neumann analysis to find eigenvalues

Notation

$$
\begin{aligned}
[u] &= u^+ - u^- \\
\overline{u} &= \frac{u^+ + u^-}{2} \\
u^\pm &= \lim_{\epsilon \to 0^\pm} u(x + \epsilon, t)
\end{aligned}
$$

To simplify analysis we use the shift operator and its inverse

$$
\begin{aligned}
Tu(j) &= u(j+1) \\
T^{-1}u(j) &= u(j-1)
\end{aligned}
$$

Asymmetric Local DG (AS LDG)

$$\frac{\partial w}{\partial x} + f = 0, \ \frac{\partial g}{\partial x} + w = 0$$

- Two choices of fluxes:
  - $\hat{f} = f^+, \hat{w} = w^-$

    $$\frac{\partial}{\partial t}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \frac{1}{\Delta x^2}\begin{pmatrix} 4T^{-1} - 8 + 4T & 2T^{-1} + 2 - 4T \\ -12T^{-1} + 6 + 6T & -6T^{-1} - 24 - 6T \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

  - $\hat{f} = f^-, \hat{w} = w^+$

    $$\frac{1}{\Delta x^2}\begin{pmatrix} 4T^{-1} - 8 + 4T & 4T^{-1} - 2 - 2T \\ -6T^{-1} - 6 + 12T & -6T^{-1} - 24 - 6T \end{pmatrix}$$

AS LDG Eigenvalues

- Both choices of fluxes result in the same eigenvalues for the Von Neumann analysis
- Defining $x = k\Delta x$,

$$\lambda_1 = \frac{1}{\Delta x^2}\left(-16 - 2\cos x - \sqrt{186 + 136\cos x + 2\cos 2x}\right)$$
$$\lambda_2 = \frac{1}{\Delta x^2}\left(-16 - 2\cos x + \sqrt{186 + 136\cos x + 2\cos 2x}\right)$$

- $k << 1$ limit:

$$\lambda_1 = -\frac{36}{\Delta x^2} + 3k^2 - \frac{k^4 \Delta x^2}{6} + \frac{k^6 \Delta x^4}{270} + O[k^7 \Delta x^5]$$
$$\lambda_2 = -k^2 + \frac{k^6 \Delta x^4}{540} + O[k^7 \Delta x^5]$$

Symmetric Local DG (S LDG)

- By averaging the results from the two AS LDG schemes, we get a symmetric scheme for LDG

$$\frac{1}{\Delta x^2}\begin{pmatrix} 4T^{-1} - 8 + 4T & 3T^{-1} - 3T \\ -9T^{-1} + 9T & -6T^{-1} - 24 - 6T \end{pmatrix}$$

- Defining $x = k\Delta x$,

$$\lambda_1 = \frac{1}{\Delta x^2}\left(-16 - 2\cos x - 2\sqrt{42 + 40\cos x - \cos 2x}\right)$$
$$\lambda_2 = \frac{1}{\Delta x^2}\left(-16 - 2\cos x - 2\sqrt{42 + 40\cos x - \cos 2x}\right)$$

- $k << 1$ limit:

$$\lambda_1 = -\frac{36}{\Delta x^2} + 3k^2 - \frac{k^4\Delta x^2}{12} + O[k^6\Delta x^4]$$
$$\lambda_2 = -k^2 - \frac{k^4\Delta x^2}{12} + O[k^6\Delta x^4]$$

Direct DG (DDG)

- We looked at two versions of Direct DG, the standard asymmetric version with interface corrections (DDG) and a symmetric version (SDDG)
- DDG solves the following weak form:

$$\int_{I_j} u_t v \mathrm{d}x - \widehat{(u_x)} v \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + \int_{I_j} u_x v_x \mathrm{d}x + \frac{1}{2}[u](v_x)^-_{j+\frac{1}{2}} + \frac{1}{2}[u](v_x)^+_{j-\frac{1}{2}} = 0$$

- The flux is defined as:

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}]$$

- For $k = 1$, took $\beta_0 = 2$ and $\beta_1 = 0$

DDG Scheme

- Obtained the following update formulas:

$$\frac{1}{\Delta x^2}\begin{pmatrix} 2T^{-1} - 4 + 2T & T^{-1} - T \\ -3T^{-1} + 3T & -12 \end{pmatrix}$$

- Defining $x = k\Delta x$,

$$\lambda_1 = \frac{1}{\Delta x^2}\left(-8 + 2\cos x - 2\sqrt{6 + 4\cos x - \cos 2x}\right)$$
$$\lambda_2 = \frac{1}{\Delta x^2}\left(-8 + 2\cos x + 2\sqrt{6 + 4\cos x - \cos 2x}\right)$$

- $k << 1$ limit:

$$\lambda_1 = -\frac{12}{\Delta x^2} - k^2 + \frac{k^4 \Delta x^2}{4} + O[k^6 \Delta x^4]$$
$$\lambda_2 = -k^2 - \frac{k^4 \Delta x^2}{12} + \frac{k^6 \Delta x^4}{40} + O[k^7 \Delta x^5]$$

Symmetric DDG (SDDG)

- SDDG solves the following weak form:

$$\int_{I_j} u_t v \mathrm{d}x - \widehat{(u_x)} v \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + \int_{I_j} u_x v_x \mathrm{d}x + ([u]\widehat{v_x})^-_{j+\frac{1}{2}} + ([u]\widehat{v_x})_{j-\frac{1}{2}} = 0$$

- The flux is defined as:

$$\begin{aligned} \widehat{u_x} &= \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] \\ \widehat{v_x} &= \beta_0 \frac{[v]}{\Delta x} + \overline{v_x} + \beta_1 \Delta x [v_{xx}] \end{aligned}$$

- v is nonzero only inside the cell I_j, so only half of the terms contribute to the computation of $\widehat{v_x}$

Symmetric DDG Results

- Used same values of $\beta_0 = 2$ and $\beta_1 = 0$ from DDG
- Get same update equations for piecewise linear case as the symmetric LDG method

$$\frac{1}{\Delta x^2}\begin{pmatrix} 4T^{-1} - 8 + 4T & 3T^{-1} - 3T \\ -9T^{-1} + 9T & -6T^{-1} - 24 - 6T \end{pmatrix}$$

- The two methods might be different when higher-order basis functions are used ($k > 1$)

Interface Recovery Schemes

A second integration by parts is performed to give weak form

$$\int_{I_j} u_t v \mathrm{d}x = (v f_x - v_x f)\Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + \int_{I_j} v_{xx} u \mathrm{d}x$$

where $f(\zeta)$, $\zeta = x_{j-1/2} - x \in [-\Delta x, \Delta x]$, is a *reconstructed* poynomial that extends across two cells and defined as

$$f(\zeta) = f_0 + \zeta f' + \frac{1}{2}\zeta^2 f'' + \dots$$

and

$$\int_{I_{j-1}} v f \mathrm{d}x = \int_{I_{j-1}} v u \mathrm{d}x$$

$$\int_{I_j} v f \mathrm{d}x = \int_{I_j} v u \mathrm{d}x$$

For piecewise linear basis function this leads to

$$\frac{1}{4\Delta x^2}\begin{pmatrix} 9T - 18 + 9T^{-1} & -5T + 5T^{-1} \\ 15T - 15T^{-1} & -7T - 46 - 7T^{-1} \end{pmatrix}$$

The eigenvalues for this scheme are

$$\lambda_1 = \frac{1}{2}\left(\sqrt{75\sin^2 x + 64\cos^2 x + 112\cos x + 49} + \cos x - 16\right)$$
$$\lambda_2 = -\frac{1}{2}\left(\sqrt{75\sin^2 x + 64\cos^2 x + 112\cos x + 49} - \cos x + 16\right)$$

with expansion

$$\lambda_1 = -k^2 + \frac{k^6\Delta x^4}{360} + O(k^8\Delta x^6)$$

Comparison of LDG, DDG and Recovery Schemes