Energy conservation

We have: $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0$

where $\phi = \int f\,dv$

We can write this as:

$$\frac{\partial f}{\partial t} + \nabla\cdot(\underset{\sim}{u} f) = 0 \qquad \underset{\sim}{u} = v\hat{e}_x - \frac{\partial\phi}{\partial x}\hat{e}_v$$

or in weak-form:

$$\langle \varphi \frac{\partial f}{\partial t}\rangle_{I_i} + \langle \varphi \underset{\sim}{u}\cdot n\hat{f}\rangle_{\partial C_i} - \langle \nabla\varphi\cdot u f\rangle_{C_i} = 0. \qquad (1)$$

$$\& \quad \langle w\phi\rangle_x = \langle w\int f\,dv\rangle_x \quad (2) \qquad (\langle\ \rangle_x \equiv \int dx$$

$$\langle\ \rangle_{C_i} \equiv \int dx\int dv \text{ etc})$$

In (1) use n as a test function.

Notice: $\nabla n = \frac{\partial n}{\partial x} e_x$ as n has no v dependence.

Then:

$$\langle n\frac{\partial f}{\partial t}\rangle_{C_i} + \langle n\underset{\sim}{u}\cdot n\hat{f}\rangle_{\partial C_i} - \langle \frac{\partial n}{\partial x} v f\rangle_{C_i} = 0.$$

Sum over all v direction cells: (actually this the linear operator that integrates in v-direction)

$$\langle n\frac{\partial}{\partial t}\sum_v f\rangle_{C_i} + \underbrace{\sum_v \langle n\underset{\sim}{u}\cdot n\hat{f}\rangle_{\partial C_i}}_{(A)} - \underbrace{\sum_v \langle\frac{\partial n}{\partial x} v f\rangle_{C_i}}_{(B)} = 0$$

$$(A) = \sum_v\left(\langle n v\hat{f}\rangle_R - \langle n v\hat{f}\rangle_L + \underbrace{\langle -n\frac{\partial\phi}{\partial x}\hat{f}\rangle_T - \langle -n\frac{\partial\phi}{\partial x}\hat{f}\rangle_B}\right)$$

$$= \sum_v\left(\langle n v\hat{f}\rangle_R - \langle n v\hat{f}\rangle_L\right)$$

vanishes on application of $\sum_v$ operator.

$$(B) = \langle\frac{\partial n}{\partial x}\sum_v v f\rangle_{C_i}$$

Hence:

$$\langle n \frac{\partial}{\partial t}\sum_v f\rangle_{C_i} = \langle \phi \frac{\partial}{\partial t}\sum_v f\rangle = -\sum_v \left(\langle n v \hat{f}\rangle_R - \langle n v \hat{f}\rangle_L\right) + \langle \frac{\partial n}{\partial x}\sum_v v f\rangle_{C_i}$$

Now pick $\frac{1}{2}v^2$ as test function. & sum over all v:

$$\langle \frac{\partial}{\partial t}\sum_v \frac{v^2}{2} f\rangle_{C_i} + \overset{\text{(I)}}{\sum_v \langle \frac{v^2}{2}\, \underline{u}\cdot n \hat{f}\rangle_{\partial C_i}} + \overset{\text{(II)}}{\sum_v \langle v \frac{\partial \phi}{\partial x} f\rangle_{C_i}} = 0$$

as $\nabla \frac{1}{2}v^2 = v e_v \Rightarrow \nabla \frac{1}{2}v^2 \cdot u = -v\frac{\partial\phi}{\partial x}$.

$$\text{(I)} = \sum_v \Big(\cancel{\langle n v f} \; \langle \frac{v^2}{2} v \hat{f}\rangle_R - \langle \frac{v^2}{2} v \hat{f}\rangle_L + \underbrace{\langle -\frac{v^2}{2}\frac{\partial\phi}{\partial x}\hat{f}\rangle_T - \langle -\frac{v^2}{2}\frac{\partial\phi}{\partial x}\hat{f}\rangle_B}_{\text{vanishes on application of } \sum_v} \Big)$$

$$= \sum_v \left(\langle \frac{v^3}{2}\hat{f}\rangle_R - \langle \frac{v^3}{2}\hat{f}\rangle_L \right)$$

$$\text{(II)} = \langle \frac{\partial\phi}{\partial x}\sum_v v f\rangle_{C_i}$$

Now: $\frac{\partial}{\partial t}\langle \frac{1}{2} n\phi\rangle_X = \frac{\partial}{\partial t}\langle \frac{n}{2}\sum_v f\rangle_X$

$$= \langle n \frac{\partial n}{\partial t}\rangle_X = -\sum_v \left(\langle n v \hat{f}\rangle_R - \langle n v \hat{f}\rangle_L \right) + \langle \underbrace{\frac{\partial n}{\partial x}}_{\frac{\partial \phi}{\partial x}} \sum_v v f\rangle_{C_i}$$

So:
$$\frac{\partial}{\partial t}\left(\sum_v \langle \frac{v^2}{2} f\rangle_{C_i} + \frac{1}{2}\langle n\phi\rangle_X \right)$$

$$= -\sum_v \left(\langle \frac{v^3}{2}\hat{f}\rangle_R - \langle \frac{v^3}{2}\hat{f}\rangle_L \right) - \cancel{\langle \frac{\partial\phi}{\partial x}\sum_v v f\rangle_{C_i}}$$

$$- \sum_v \left(\langle n v \hat{f}\rangle_R - \langle n v \hat{f}\rangle_L \right) + \cancel{\langle \frac{\partial\phi}{\partial x}\sum_v v f\rangle_{C_i}}$$

__Hence:__ Energy is conserved __iff__ $\frac{1}{2}v^2$ is a test __function__.