Finite difference / volume scheme.

$$\frac{\partial f}{\partial t} + \nabla\cdot(\underset{\sim}{u} f) = 0 \qquad \phi = \int f\, dv \qquad u_x = v \qquad u_v = -\frac{\partial \phi}{\partial x}$$

In FV conservation form this is:

$$\frac{\partial F_{i,j}}{\partial t} + \underbrace{v_j \frac{\hat{f}_{i+1/2,j} - \hat{f}_{i-1/2,j}}{\Delta x}}_{A} - \left(\frac{\partial \phi}{\partial x}\right)_i \frac{\hat{f}_{i,j+1/2} - \hat{f}_{i,j-1/2}}{\Delta v} = 0$$

As there is no upwind direction for v, we pick:

$$\hat{f}_{i,j+1/2} = \frac{1}{2}(f_{i,j+1} + f_{i,j})$$

Pick the flux

$$\hat{f}_{i+1/2,j} = \frac{1}{2}(f_{i+1,j} + f_{i,j}) - \frac{[\![v_j]\!]}{2}(f_{i+1,j} - f_{i,j})$$

where: $[\![v_j]\!]$ is a function:

$[\![v_j]\!] = 0$ for central-scheme

$$[\![v_j]\!] \begin{cases} = 0 & v_j = 0 \\ = 1 & v_j > 0 \\ = -1 & v_j < 0 \end{cases} \qquad \text{upwind-scheme}$$

So we have

$$A = \frac{v_j}{\Delta x}\Big(\frac{1}{2}(f_{i+1,j} + f_{i,j}) - \frac{[\![v_j]\!]}{2}(f_{i+1,j} - f_{i,j})$$
$$- \frac{1}{2}(f_{i,j} + f_{i-1,j}) + \frac{[\![v_j]\!]}{2}(f_{i,j} - f_{i-1,j}) \Big)$$

$$= \underbrace{v_j \frac{(f_{i+1,j} - f_{i-1,j})}{2\Delta x}}_{\text{central flux}} - \underbrace{v_j \frac{[\![v_j]\!]}{2\Delta x}(f_{i+1,j} - 2f_{i,j} + f_{i-1,j})}_{\text{diffusion}}$$

Hence:

$$\frac{\partial F_{i,j}}{\partial t} + v_j \frac{(f_{i+1,j} - f_{i-1,j})}{2\Delta x} - v_j \frac{[\![v_j]\!]}{2\Delta x}(f_{i+1,j} - 2f_{i,j} + f_{i-1,j})$$
$$- \left(\frac{\partial \phi}{\partial x}\right)_i \frac{(f_{i,j+1} - f_{i,j-1})}{2\Delta v} = 0$$

← discrete system.

Sum over all v:

$$\sum_v \Delta v \frac{\partial F_{i,j}}{\partial t} + \sum_v \Delta v\, v_j \frac{(f_{i+1,j} - f_{i-1,j})}{2\Delta x} - \sum_v \Delta v\, v_j \frac{\llbracket v_j \rrbracket}{2\Delta x}\left(f_{i+1} - 2f_i + f_{i-1}\right) = 0$$

or $\frac{dn_i}{dt} = \dots\dots$

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compute kinetic energy:

$$\sum_v \Delta v \frac{1}{2} v_j^2 \frac{\partial F_{i,j}}{\partial t} + \sum_v \frac{\Delta v}{2} v_j^3 \frac{(f_{i+1} - f_{i-1})}{2\Delta x} - \sum_v \frac{v_j^3}{2} \frac{\llbracket v_j \rrbracket}{2\Delta x}\left(f_{i+1} - 2f_i + f_{i-1}\right)\Delta v$$

$$- \left(\frac{\partial \phi}{\partial x}\right)_i \underbrace{\sum_j \frac{1}{2} v_j^2 \frac{(f_{i,j+1} - f_{i,j-1})}{2}}_{B} = 0$$

$$B = \frac{1}{4}\left(\frac{\partial \phi}{\partial x}\right)_i \Big[v_1^2 (f_{i,2} - f_{i,0}) + v_2^2 (f_{i,3} - f_{i,1}) + v_3^2 (f_{i,4} - f_{i,2}) + \cdots \Big]$$

$$= \frac{1}{4}\left(\frac{\partial \phi}{\partial x}\right)_i \sum_j (v_{j-1} - v_{j+1})(v_{j-1} + v_{j+1}) f_{i,j} = -\left(\frac{\partial \phi}{\partial x}\right)_i \sum_j \Delta v \frac{(v_{j-1} + v_{j+1})}{2} f_{i,j}$$

$$= -\left(\frac{\partial \phi}{\partial x}\right)_i \sum_j \Delta v\, v_j f_{i,j}$$

Potential energy: $\frac{\partial}{\partial t}\left(\frac{1}{2}\phi n\right) = n \frac{\partial n}{\partial t} = \phi_i \frac{\partial n_i}{\partial t}$

Total energy:

$$\frac{\partial}{\partial t}\left(\sum_v \Delta v \frac{1}{2} v_j F_{i,j} + \frac{1}{2}\phi_i n_i \right) + \sum_v \frac{\Delta v}{2} v_j^3 \frac{(f_{i+1} - f_{i-1})}{2\Delta x} - \sum_v \frac{v_j^3}{2} \frac{\llbracket v_j \rrbracket}{2\Delta x}\left(f_{i+1} - 2f_i + f_{i-1}\right)\Delta v$$

$$+ \left(\frac{\partial \phi}{\partial x}\right)_i \sum_j \Delta v\, v_j f_{i,j} + \Delta v\, \phi_i \left(\sum_v v_j \frac{(f_{i+1,j} - f_{i-1,j})}{2\Delta x} - \sum_v v_j \frac{\llbracket v_j \rrbracket}{2\Delta x}\left(f_{i+1} - 2f_i + f_{i-} \right.$$

Sum over all i:

$$\frac{1}{\Delta v}\frac{\partial \mathcal{E}}{\partial t} - \sum_i \sum_v \frac{v_j^3}{2} \frac{\llbracket v_j \rrbracket}{2\Delta x}\left(f_{i+1} - 2f_i + f_{i-1}\right) + \overbrace{\sum_i \left(\frac{\partial \phi}{\partial x}\right)_i \sum_j v_j f_{i,j} + \sum_i \phi_i \sum_v v_j \frac{f_{i+1} - f_{i-1}}{2\Delta x}}^{A}$$

$$- \sum_i \phi_i \frac{v_j \llbracket v_j \rrbracket}{2\Delta x}\left(f_{i+1} - 2f_i + f_{i-1}\right)$$

$$A = \sum_j v_j \sum_i \left(\frac{\phi_{i+1}-\phi_{i-1}}{2\Delta x} f_{i,j} + \phi_i \frac{f_{i+1}-f_{i-1}}{2\Delta x} \right)$$

$$= \sum_i v_j \left((\phi_2-\phi_0) f_{1,j} + \phi_1 (f_{2,j}-f_{0,j}) + (\phi_3-\phi_1) f_{2,j} + \phi_2 (f_{3,j}-f_{1,j}) + \cdots \right)$$

$$= 0$$

So we are left with:

$$\frac{1}{\Delta v}\frac{\partial \mathcal{E}}{\partial t} - \sum_i \sum_v \frac{(f_{i+1}-2f_i+f_{i-1})}{2\Delta x} \left(\frac{v_j^3}{2} [\![v_j]\!] + \phi_i \, v_j [\![v_j]\!] \right) = 0$$

$$A = \sum_v \frac{v_j^2}{2} \frac{[\![v_j]\!]}{2\Delta x} \left(f_2 - 2f_1 + f_0 + f_3 - 2f_2 + f_1 + f_4 - 2f_3 + f_2 + \cdots \right)$$

$$= 0$$

$$B = \sum_i \sum_v \phi_i \frac{(f_{i+1}-2f_i+f_{i-1})}{2\Delta x} v_j [\![v_j]\!]$$

Hence.

$$\frac{1}{\Delta v}\frac{\partial \mathcal{E}}{\partial t} = \sum_i \sum_v \phi_i \frac{(f_{i+1,j}-2f_{i,j}+f_{i-1,j})}{2\Delta x} \overbrace{v_j [\![v_j]\!]}^{|v_j|}$$

$v_j \, \mathrm{sgn} [\![v_j]\!]$

if $v_j > 0$ $\quad v_j$

if $v_j < 0$ $\quad +|v_j|$

or $$\left[\frac{\partial \mathcal{E}}{\partial t} = \int_0^L \int_{-\infty}^{+\infty} \phi \frac{\partial^2 f}{\partial x^2} |v| \, dv \right]$$