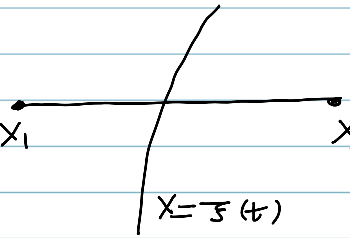
Proof of Lemma 1

Consider $\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0$. We are interested in **weak** solutions, so we should look @

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx + \left(F(u(x_2,t)) - F(u(x_1,t))\right) = 0$$

Integrate across the discontinuity:



$\cancel{\frac{d}{dt}\int_{x_1}^{x_2}}$

$$\lim_{\varepsilon \to 0}\left[\frac{d}{dt}\int_{x_1}^{\xi-\varepsilon} u(x,t)\,dx + \frac{d}{dt}\int_{\xi+\varepsilon}^{x_2} u(x,t)\,dx\right] = F(u_1) - F(u_2)$$

$u_1 \equiv u(x_1,t)$

$u_2 = u(x_2,t)$

$$\Rightarrow \lim_{\varepsilon \to 0}\left[u(\xi-\varepsilon,t)\dot{\xi} + \int_{x_1}^{\xi-\varepsilon}\frac{\partial u}{\partial t}\,dx - u(\xi+\varepsilon,t)\dot{\xi} + \int_{\xi+\varepsilon}^{x_2}\frac{\partial u}{\partial t}\,dx\right] = F(u_1) - F(u_2)$$

(as u is solution in each domain)

$$= (u^- - u^+)\dot{\xi} - \int_{x_1}^{\xi-\varepsilon}\frac{\partial F}{\partial x}\,dx - \int_{\xi+\varepsilon}^{x_2}\frac{\partial F}{\partial x}\,dx = F(u_1) - F(u_2)$$

$$\Rightarrow (u^- - u^+)\dot{\xi} - \left(F(u^-) - F(u_1)\right) - \cancel{\left(F(u_2) - F(u^+)\right)} = \cancel{F(u_1) - F(u_2)}$$

$$\Rightarrow \dot{\xi}(u^+ - u^-) = F(u^+) - F(u^-)$$

$\boxed{\text{QED}}$