We wish to solve

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\underset{\sim}{u} \psi) = 0 \quad \text{with} \quad \underset{\sim}{u} = e_z \times \nabla \psi \quad \& \text{ where}$$

$$\nabla^2 \psi = \omega \quad \text{on} \quad \underset{\sim}{x} \in \Omega \quad \text{with boundary } \partial\Omega$$

Let K be the element (cell) index & ∂K be cell boundary.

Then W is the space of continous functions s.t. for $\forall\, \varphi \in W$ the restriction of φ on K is of degree k : ie $\varphi|_K \in P^k$.

Let V be the space of piece-wise polynomials of degree k in each cell.

Let $\varphi_j \in K$ & $v_j \in V$. Then we need to find ω_h & ψ_h st

1 $\rightarrow$ $$\left\langle v_j \frac{\partial \omega_h}{\partial t} \right\rangle_K + \left\langle v_j\, \underset{\sim}{u} \cdot \hat{n} \nabla \psi_h \right\rangle_{\partial K} - \left\langle \nabla v \cdot \underset{\sim}{u}\, \hat{\omega} \right\rangle_K = 0 \quad \forall\, v_j \in V$$

$\boxed{\hat{\omega} \text{ is some average (limited)}}$

2 $\rightarrow$ & $$-\langle \nabla \varphi_j \cdot \nabla \psi \rangle_\Omega = \langle \varphi_j\, \omega \rangle_\Omega \quad \forall\, \varphi_j \in W$$

Here we have assumed $\langle \varphi_j\, \nabla \psi \cdot \hat{n} \rangle_{\partial\Omega} = 0$ (essentially net circulation is zero)

Energy conservation: take time derivative of 2 :

3 $\rightarrow$ $$-\left\langle \nabla \varphi_j \cdot \nabla \frac{\partial \psi}{\partial t} \right\rangle_\Omega = \langle \varphi_j\, \omega \rangle_\Omega$$ Now assume that:

$\boxed{\psi(x) = \sum_m \psi_m \varphi_m(x)}$ Then multiply 3 by ψ_j & sum over all j:

$$\left(E \equiv \frac{1}{2} |\nabla \psi|^2\right)$$

$$-\sum_j \psi_j \left\langle \nabla \varphi_j \cdot \nabla \frac{\partial \psi}{\partial t} \right\rangle_\Omega = -\left\langle \nabla \psi \cdot \nabla \frac{\partial \psi}{\partial t} \right\rangle_\Omega = -\frac{dE}{dt}$$

$$= \sum_j \psi_j \left\langle \varphi_j \frac{\partial \omega}{\partial t} \right\rangle_\Omega$$

Now if the set $\{\varphi_1, \varphi_2, \ldots, \}|_K \subset V$ then from 1

$$-\left\langle \varphi_j \frac{\partial \omega_h}{\partial t} \right\rangle_K = -\left\langle \varphi_j\, \underset{\sim}{u} \cdot \hat{n} \nabla \psi_h \right\rangle_{\partial K} + \left\langle \nabla \varphi_j \cdot \underset{\sim}{u}\, \hat{\omega} \right\rangle_K$$

multiply by ψ_j & sum over all m:

$$-\sum_j \psi_j \left\langle \varphi_j \frac{\partial \omega_h}{\partial t} \right\rangle_K = -\sum_j \psi_j \left\langle \varphi_j\, \underset{\sim}{u} \cdot \hat{n} \nabla \psi_h \right\rangle_{\partial K} + \sum_j \psi_j \left\langle \nabla \varphi_j \cdot \underset{\sim}{u}\, \hat{\omega} \right\rangle_K$$

Next sum over all cells:

$$-\sum_j \psi_j \sum_K \left\langle \varphi_j \frac{\partial \omega_h}{\partial t} \right\rangle_K = -\sum_j \psi_j \left\langle \varphi_j \frac{\partial \omega_h}{\partial t} \right\rangle_\Omega = -\sum_j \psi_j \sum_K \cancelto{0}{\left\langle \varphi_j\, \underset{\sim}{u} \cdot \hat{n} \nabla \psi_h \right\rangle_{\partial\Omega}}$$

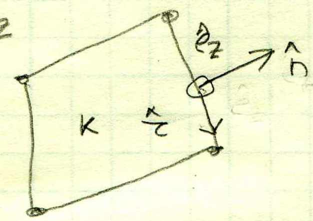
$$+ \left\langle \cancelto{0}{\nabla \psi \cdot \underset{\sim}{u}}\, \hat{\omega} \right\rangle_K = 0$$

Hence we have

$$\boxed{\frac{dE}{dt} = 0} \quad \text{if} \quad \{Q_1, Q_2, \ldots, \}\big|_K \subset V$$

Note $\underset{\sim}{u}\cdot\hat{n} = \hat{e}_z \times \nabla\psi \cdot \hat{n} = (\hat{n}\times\hat{e}_z)\cdot\nabla\psi = \hat{\tau}\cdot\nabla\psi$

where $\hat{\tau} = \hat{n}\times\hat{e}_z$



Hence: $\underset{\sim}{u}\cdot\hat{n} = \dfrac{d\psi}{d\tau}$ which is continous as ψ is continous along the edge.

However $\nabla\psi\cdot\hat{n}$ need not be continous across an edge, and in general will not be unless we select suitable basis function.

Also notice that $\underset{\sim}{u}\cdot\hat{\tau} = \hat{e}_z\times\nabla\psi\cdot\hat{\tau} = (\tau\times\hat{e}_z)\cdot\nabla\psi = -\hat{n}\cdot\nabla\psi$

Hence a term like $\oint_{\partial K} \hat{n}\cdot\nabla\psi \, dK$ represents the circulation in a cell K & need not vanish for individual cells. Also when summed over all cells the discrete net circulation if only computed from solution in individual cells will not vanish.

Hence implicitly the FEM has assumed that we have picked some average edge circulation when we got the global minimization problem. (Like the $\hat{\omega}$). It does not matter what this average is as it simply cancels on summing over all K to get Eq ②.