Formulation of 1D Poisson FEM

Look @ $\frac{\partial^2 \psi}{\partial x^2} = s$ on $x \in [a, b]$ with periodic bcs

let $\varphi \in \{\text{space of continous functions}\}$ Then: (W written above "{")

$$\int_a^b \varphi \frac{\partial^2 \psi}{\partial x^2} dx = \int_a^b s \varphi dx \Rightarrow \varphi \frac{\partial \psi}{\partial x}\Big|_a^b - \int_a^b \frac{\partial \varphi}{\partial x}\frac{\partial \psi}{\partial x} dx = \int_a^b s \varphi dx$$

For periodicity $\varphi \frac{\partial \psi}{\partial x}\Big|_a^b = 0$

So we have weak-form $-\int_a^b \frac{\partial \varphi}{\partial x}\frac{\partial \psi}{\partial x} dx = \int_a^b s\varphi dx$

This needs to be satisfied for all $\varphi \in W$.

Discretize into cells & let $\varphi_j \in W_h$ st $\varphi_j\big|_{C} \in P^n$ ie restriction of φ_j on cell C is a polynomial of degree n.

So discrete weak-form on a single cell $c \equiv [x_{i-1/2}, x_{i+1/2}]$ is

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \varphi_j^{(c)} \frac{\partial^2 \psi}{\partial x} dx = \int_{x_{i-1/2}}^{x_{i+1/2}} s\varphi_j^{(c)} dx \Rightarrow \varphi_j^{(c)} \frac{\partial \psi}{\partial x}\Big|_{x_{i-1/2}}^{x_{i+1/2}} - \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial \varphi_j^{(c)}}{\partial x}\frac{\partial \psi}{\partial x} dx = \int_{x_{i-1/2}}^{x_{i+1/2}} s\varphi_j^{(c)} dx$$

Now express in each cell

$$\psi^{(c)}(x) = \sum_k \psi_k^{(c)} \varphi_k^{(c)}(x) \quad \& \quad s^{(c)}(x) = \sum_k s_k^{(c)} \varphi_k^{(c)}(x)$$

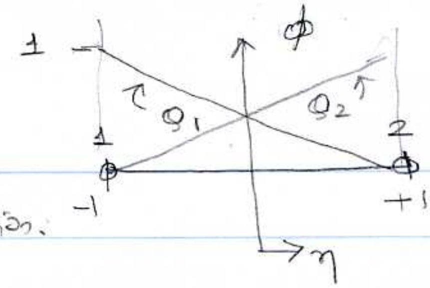
Then: $$\sum_k \varphi_j^c \frac{\partial \varphi_k^{(c)}}{\partial x}\Big|_{x_{i-1/2}}^{x_{i+1/2}} \psi_k^{(c)} - \sum_k \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial \varphi_j^{(c)}}{\partial x}\frac{\partial \varphi_k^{(c)}}{\partial x} dx\, \psi_k^{(c)} = \sum_k \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi_j^{(c)} \varphi_{(k)}^{(c)} dx\, s_k^{(c)}$$

$\forall c$ in grid

Let $$K_{jk}^c \equiv \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial \varphi_j^{(c)}}{\partial x}\frac{\partial \varphi_k^{(c)}}{\partial x} dx \qquad M_{jk}^c \equiv \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi_j^{(c)} \varphi_{(k)}^{(c)} dx.$$

these are local matrics. We need to compute these for every cell & assemble them to get global matrices.

[Note: Edge integrals dissapear on assembly. So ignore them]



Example: Consider a "standard" cell

Pick nodes 1 & 2 & define basis-function.

$$\phi_1(\eta) = \frac{1}{2}(1-\eta) \qquad \phi_2(\eta) = \frac{1}{2}(1+\eta).$$ The transform $\eta = \dfrac{x - x_i}{\Delta x/2}$

will take $\phi_j(\eta)$ & give $\phi_j^{(c)}(x)$ with $x_i \equiv (x_{i+1/2} + x_{i-1/2})/2$

& $\Delta x \equiv x_{i+1/2} - x_{i-1/2}$

Integrals like $\displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} f(x)\,dx \rightarrow \int_{-1}^{+1} f(x(\eta))\,d\eta\,\frac{\Delta x}{2}$

As $\dfrac{\partial \phi_j^{(c)}}{\partial x} = \dfrac{\partial \phi_j}{\partial \eta}\dfrac{\partial \eta}{\partial x} = \dfrac{2}{\Delta x}\dfrac{\partial \phi_j}{\partial \eta} \Rightarrow \dfrac{\partial \phi_1^{(c)}}{\partial x} = -\dfrac{1}{\Delta x} \qquad \dfrac{\partial \phi_2^{(c)}}{\partial x} = \dfrac{1}{\Delta x}$

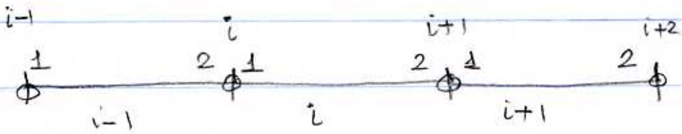
We hence get for these basis functions

$$\underset{\approx}{K}^c = \frac{1}{\Delta x}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \& \quad \underset{\approx}{M}^c = \Delta x \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

The local discrete weak-form is

$$-\frac{1}{\Delta x}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \psi_1^{(c)} \\ \psi_2^{(c)} \end{bmatrix} = \Delta x \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}\begin{bmatrix} s_1^{(c)} \\ s_2^{(c)} \end{bmatrix}$$ (Ignore edge terms for now)

How to do <u>global assembly</u>? Ans: Create connectivity matrix by identifying shared nodes.



global unknown

Eg: $\psi_1^{(i)} = \psi_2^{(i-1)} = \psi_i$

$\psi_2^{(i)} = \psi_1^{(i+1)} = \psi_{i+1}$

Let $\underset{\sim}{\psi}$ be vector of global unknowns. Then each cell needs a connectivity matrix $\underset{\approx}{G}^{(i)}$ which maps $\underset{\sim}{\psi}^{(i)}$ to $\underset{\sim}{\psi}$:

ie $\underset{\sim}{\psi} = \underset{\sim}{G}^{(i)}\underset{\sim}{\psi}^{(i)}$. A matrix like $\underset{\approx}{K}^{(c)}$ is mapped as

$$\underset{\approx}{K} = \underset{\approx}{G}^{(i)}\underset{\approx}{K}^{(i)}\underset{\approx}{G}^{(i)T}.$$ Hence to <u>assemble</u> do

$$\underset{\approx}{K} = \sum_i \underset{\approx}{G}^{(i)}\underset{\approx}{K}^{(i)}\underset{\approx}{G}^{(i)T} \qquad \underset{\approx}{M} \equiv \sum_i \underset{\approx}{G}^{(i)}\underset{\approx}{M}^{(i)}\underset{\approx}{G}^{(i)T}$$

For 1D Poisson it is easy to see that:

$$\underset{\sim}{K} = \frac{1}{\Delta x}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & & \cdots \end{bmatrix} \qquad S = \Delta x \begin{bmatrix} 2/3 & 1/6 & & \\ 1/6 & 2/3 & 1/6 & \\ & 1/6 & 2/3 & \cdots \end{bmatrix}$$

Hence the FEM looks like

$$\left[\frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{\Delta x^2} = \frac{1}{6} S_{i-1} + \frac{2}{3} S_i + \frac{1}{6} S_{i+1} \right]$$

which "looks" like a central difference on $\partial^2\psi/\partial x^2$ & some sort of averaging for the source term. Of course in 2D on general geometry this is no longer true, or even in 1D with order greater than 1 (linear basis)