THE MUSCL-HANCOCK SCHEME FOR SOLUTION OF HYPERBOLIC EQUATIONS

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In this document I outline the MUSCl-Hancock scheme for the solution of 1D hyperbolic partial differential equations. This scheme is a predictor-corrector scheme and is second order accurate in both space and time. We start from the sysem of hyperbolic equations

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{1}$$

where $Q(x,t)$ is a vector of m conserved quantities and $F = F(Q)$ are fluxes. We will also assume (though this is not required) that the system can be put into the non-conservative ("primitive") quasi-linear form

$$\frac{\partial V}{\partial t} + A\frac{\partial V}{\partial x} = 0 \tag{2}$$

where $V(x,t)$ are a vector of m primitive quantities and $A(V)$ is a $m \times m$ matrix. Note that any transform $Q = \varphi(V)$ will transform Eq. (1) into Eq. (2).

In this note I will use the 1D Euler equations as an example. The conservative form of these equations are

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho u \\ E\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u \\ \rho u^2 + p \\ (E+p)u\end{bmatrix} = 0 \tag{3}$$

where

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \tag{4}$$

and γ is the gas adiabatic index. The non-conservative form of these equations are

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0 \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{6}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = 0 \tag{7}$$

$$\tag{8}$$

The essential idea of the MUSCL-Hancock scheme is to use cell averages to predict the values of the conserved (or primitive) quantities at cell edges and then use these predicted values to update the solution at the next time-step. The steps in the algorithm are as follows.

(1) Given cell averages reconstruct a linear representation of the variables inside each cell. This can be done for either the conserved variables or the primitive variables. Hence, in each cell we represent the solution as

$$W(x,t) = W_i + \frac{x - x_i}{\Delta x}\delta W_i \tag{9}$$

for $x_{i-1/2} < x < x_{i+1/2}$ and where $x_i \equiv (x_{i+1/2} + x_{i-1/2})/2$, $\Delta x \equiv x_{i+1/2} - x_{i-1/2}$ and δW_i are the reconstructed *slopes*. In Eq. (9) $W(x,t)$ stands for the variables we are reconstructing (either primitive or conserved variables). To determine the slopes we can use an averaging procedure

$$\delta W_i = \mathrm{ave}(W_i - W_{i-1}, W_{i+1} - W_i) \tag{10}$$

where $\mathrm{ave}(a, b)$ is a suitable "averaging" function. Note that using the standard average $\mathrm{ave}(a, b) = (a + b)/2$ leads to a central-difference computed slope, while $\mathrm{ave}(a, b) = 0$ leads to a zero slope or a first-order representation in each cell. See the next section for more details on the reconstruction and averaging steps.

(2) Use the slopes to predict the solution at half time-step, $\Delta t/2$. If the primitive variable slope have been determined then use the update formula

$$\tilde{V}_j = V_j - \frac{\Delta t}{2\Delta x} A(V_i) \delta V_i \tag{11}$$