Ammar Hakim

Derivation of axisymmetric equilibria

Ideal MHD $\partial/\partial t$ eqns are

$$\nabla\cdot(\rho v) = 0 \quad (1)$$

$$\rho\, v\cdot\nabla v - J\times B = -\nabla p \quad (2)$$

$$v\cdot\nabla S = 0 \quad (3)$$

$$E + v\times B = 0 \quad (4)$$

$$\nabla\times B = \mu_0 J \quad (5)$$

$$\nabla\times E = 0 \quad (6)$$

as $\nabla\cdot B = 0$

scalar

$$\Rightarrow B = B_\phi e_\phi + \frac{1}{r}\nabla\psi\times e_\phi \quad (7)$$

As $\nabla\cdot J = 0$ (from 5)

we have

scalar

$$J = J_\phi e_\phi + \frac{1}{r}\nabla f\times e_\phi \quad (8)$$

Thus (5) is.

$$\mu_0 J = \nabla\times B = -\frac{\Delta^*\psi}{r} e_\phi + \frac{1}{r}\nabla(rB_\phi)\times e_\phi$$

$$\Rightarrow \mu_0 J_\phi = -\frac{\Delta^*\psi}{r} \quad (9)$$

$$\& \quad \mu_0 f = rB_\phi \quad (10)$$

Case I: No flow

$$J \times B = \nabla p \qquad (11)$$

$$\Rightarrow \quad B \cdot (J \times B) = B \cdot \nabla p$$

$$\text{or} \quad 0 = \frac{1}{r} (\nabla p \times \nabla \psi) \cdot e_\phi \qquad (\text{Identity } \underline{\underline{52}})$$

$$\Rightarrow \quad p = p(r, z) = P(\psi) \qquad \psi = \psi(r, z) \qquad (12)$$

also $\quad J \cdot (J \times B) = J \cdot \nabla p$

$$\Rightarrow \quad 0 = \frac{1}{r} \nabla p \times \nabla (r B_\phi)$$

$$\Rightarrow \quad 0 = \nabla \psi \times \nabla (r B_\phi)$$

$$\Rightarrow \quad r B_\phi = \mu_0 f(r, z) = F(\psi) \qquad (13)$$

Now (11) can be written as:

$$\frac{J_\phi}{r} \nabla \psi - \frac{B_\phi}{r \mu_0} \underbrace{\nabla (r B_\phi)}_{F' \nabla \psi} + \frac{1}{r \mu_0} \cancel{\nabla (r B_\phi) \times \nabla \psi}_{\to 0} = P' \nabla \psi$$

$$\Rightarrow \left(\frac{J_\phi}{r} - \frac{F F'}{\mu_0 r^2} \right) \nabla \psi = P' \nabla \psi$$

$$\Rightarrow \quad J_\phi = r P' + \frac{1}{\mu_0 r} F F' \qquad (14)$$

Thus using in (9) gives

$$\left. \begin{aligned} \Delta^* \psi &= - r \mu_0 J_\phi \\ &= - \mu_0 r^2 P' - F F' \end{aligned} \right\} \qquad (15)$$

No flow eqn

Case II : With Flow

Eqns (4) & (5) give

$$\nabla \times (v \times B) = 0$$

$$\Rightarrow \quad v \times B = \nabla \underset{\uparrow \text{scalar}}{g} \qquad (16)$$

Eq (1) gives

$$v = v_\phi e_\phi + \frac{1}{r\rho} \nabla \overset{\text{scalar}\downarrow}{\psi_v} \times e_\phi \qquad (17)$$

Thus (identity $\underline{\underline{53}}$)

$$\frac{v_\phi}{r} \nabla \psi - \frac{B_\phi}{r\rho} \nabla \psi_v + \frac{1}{r^2 \rho} \nabla \psi_v \times \nabla \psi = \nabla g$$

$$\Rightarrow \quad (\nabla \psi_v \times \nabla \psi) \cdot e_\phi = 0$$

Thus $\psi_v = \overline{\Psi}(\psi)$

$$\Rightarrow v_p = \frac{\overline{\Psi}'}{r\rho} \nabla \psi \times e_\phi = \frac{\overline{\Psi}'}{\rho} B_p$$

& also:

$$\frac{v_\phi}{r} \nabla \psi - \frac{B_\phi}{r\rho} \overline{\Psi}' \nabla \psi = \nabla g$$

$$\Rightarrow \left(\frac{v_\phi}{r} - \frac{B_\phi}{r\rho} \overline{\Psi}' \right) \nabla \psi = \nabla g .$$

$\nabla g = \overline{G}'(\psi)$

$$\Rightarrow \quad g = g(r, z) = \overline{G}(\psi) \qquad (17b)$$

$$\Rightarrow \quad \frac{v_\phi}{r} - \frac{B_\phi}{r\rho} \overline{\Psi}' = \overline{G}'(\psi) = G(\psi) \qquad (18)$$

$$\text{or} \quad v_\phi - \frac{B_\phi}{\rho} \overline{\Psi}' = r\, G(\psi) \qquad (19)$$

Use identity

$$v \cdot \nabla v + v \times \nabla \times v = \nabla\left(\frac{v^2}{2}\right)$$

in eq (2) to give

$$-v \times \omega - \frac{J \times B}{\rho} + \nabla\left(\frac{v^2}{2}\right) = -\frac{\nabla p}{\rho} \qquad (20)$$

$$\omega = \nabla \times v \qquad (21)$$

$$\Rightarrow \quad \omega = \omega_\phi e_\phi + \frac{1}{r} \nabla \lambda \times e_\phi$$

(identity 48)

$$= -\frac{\Delta^*_\rho \psi_v}{r} e_\phi + \frac{1}{r} \nabla(r v_\phi) \times e_\phi$$

where

$$\Delta^*_\rho \psi_v \equiv \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial \psi_v}{\partial z}\right) + r\frac{\partial}{\partial r}\left(\frac{1}{r\rho}\frac{\partial \psi_v}{\partial r}\right) \qquad (22)$$

Thus $\quad \omega_\phi = -\dfrac{\Delta^*_\rho \psi_v}{r} \qquad (23)$

$$\lambda = r v_\phi \qquad (24)$$

Take e_ϕ component of (20)

$$\Rightarrow \quad \left(v \times \omega + \frac{J \times B}{\rho}\right) \cdot e_\phi = 0$$

use identity 53 & 51 to get

$$\left[\frac{1}{r^2 \rho} \nabla \psi_v \times \nabla (r v_\phi) + \frac{1}{r^2 \rho} \nabla (r B_\phi) \times \nabla \psi\right] \cdot e_\phi = 0$$

$$\Rightarrow \quad \left[\nabla(r B_\phi) - \Psi' \nabla (r v_\phi)\right] \times \nabla\psi \cdot e_\phi = 0$$

$$\Rightarrow \quad \nabla(r B_\phi) - \Psi' \nabla(r v_\phi) \;/\!/\; \nabla\psi$$

$$\Rightarrow \quad r B_\phi - \Psi' r v_\phi = K(\psi) \qquad (25)$$

Now from thermodynamics (see thermo notes Eq 3)

$$\frac{dp}{\rho} = dh - T\,dS$$

Thus momentum Eq (20) becomes:

$$-v \times \omega - \frac{J \times B}{\rho} + \nabla\left(\frac{v^2}{2} + h\right) = T\nabla S \qquad (26)$$

Take dot product with v:

$$-v \cdot \frac{J \times B}{\rho} + v \cdot \nabla\left(\frac{v^2}{2} + h\right) = T\, \underbrace{v \cdot \nabla S}_{0 \text{ (eq 3)}}$$

Now

$$-v \cdot \frac{J \times B}{\rho} = v \cdot \frac{B \times J}{\rho} = v \times B \cdot \frac{J}{\rho}$$

$$= \nabla g \cdot \frac{J}{\rho} = G(\psi)\, \nabla\psi \cdot \frac{J}{\rho} \qquad \text{(see eq 17b)}$$

$$= G(\psi)\, \frac{\nabla\psi}{\rho} \cdot \left[e_\phi J_\phi + \frac{1}{r} \nabla(rB_\phi) \times e_\phi \right]$$

$$= G(\psi)\, \frac{\nabla\psi}{r\rho} \cdot \nabla(rB_\phi) \times e_\phi$$

$$= G(\psi)\, \frac{\nabla\psi \times \nabla(rB_\phi)}{r\rho} \cdot e_\phi = -G(\psi)\, \frac{\nabla(rB_\phi) \times \nabla\psi}{r\rho} \cdot e_\phi$$

Also

$$v \cdot \nabla\left(\frac{v^2}{2} + h\right) = \frac{1}{r\rho} \nabla\left(\frac{v^2}{2} + h\right) \times \nabla\psi_v \qquad \text{(identity } \underline{\underline{54}}\text{)}$$

$$= \frac{\Psi'}{r\rho} \nabla\left(\frac{v^2}{2} + h\right) \times \nabla\psi$$

Thus

$$\left[-G(\psi)\, \nabla(rB_\phi) + \Psi'\, \nabla\left(\frac{v^2}{2} + h\right) \right] \times \nabla\psi \cdot e_\phi = 0$$

or

$$\left[-\nabla(G\, rB_\phi) + \nabla\left(\Psi' \left\{ \frac{v^2}{2} + h \right\} \right) \right] \times \nabla\psi \cdot e_\phi$$

(verify by expansion)

$$\frac{r B_\phi}{\bar{\Psi}'} = r v_\phi + K$$

Thus $\nabla \left[-G r B_\phi + \bar{\Psi}' \left(\frac{v^2}{2} + h \right) \right] \times \nabla \psi \cdot e_\phi = 0$

or $$-G r B_\phi + \bar{\Psi}' \left(\frac{v^2}{2} + h \right) = \bar{H}(\psi) \qquad (27)$$

$$\Rightarrow \quad -G \frac{r B_\phi}{\bar{\Psi}'} + \frac{v^2}{2} + h = \frac{\bar{H}(\psi)}{\bar{\Psi}'}$$

$$\Rightarrow \quad -G \left(K(\psi) + r v_\phi \right) + \frac{v^2}{2} + h = \frac{\bar{H}(\psi)}{\bar{\Psi}'}$$

or finally:

$$h + \frac{v^2}{2} - r v_\phi G = H(\psi) \qquad (28)$$

↑ "total" enthalpy

Now take dot product with $\nabla\psi$

$$\underset{(A)}{-V\times\omega\cdot\nabla\psi} - \underset{(B)}{\frac{J\times B}{\rho}\cdot\nabla\psi} + \underset{(C)}{\nabla\left(\frac{V^2}{2}+h\right)\cdot\nabla\psi} = T S'|\nabla\psi|^2$$

$V\cdot\nabla S = 0$

$\Rightarrow \nabla S\times\nabla\psi\cdot e_\phi = 0$

$\Rightarrow S = S(\psi)$

Term (A)

$V\times\omega\cdot\nabla\psi$

$$= \left(\frac{V_\phi}{r}\nabla(rV_\phi) - \underbrace{\frac{\omega_\phi}{r\rho}\nabla\psi_V}_{\Psi'\nabla\psi} + \underbrace{\frac{1}{r^2\rho}\nabla\psi_V\times\nabla(rV_\phi)}_{\Psi'\nabla\psi\times\nabla(rV_\phi)}\right)\cdot\nabla\psi$$

$$= \frac{V_\phi}{r}\nabla(rV_\phi)\nabla\psi - \frac{\omega_\phi\Psi'|\nabla\psi|^2}{r\rho}$$

Term (B)

$$\frac{J\times B}{\rho}\cdot\nabla\psi$$

$$= \frac{1}{\rho}\left[\frac{J_\phi}{r}\nabla\psi - \frac{B_\phi}{r}\nabla(rB_\phi) + \frac{1}{r^2}\nabla(rB_\phi)\times\nabla\psi\right]\cdot\nabla\psi$$

$$= \frac{1}{\rho r}J_\phi|\nabla\psi|^2 - \frac{B_\phi}{r\rho}\nabla(rB_\phi)\cdot\nabla\psi$$

Term (C)

$$\nabla\left(\frac{V^2}{2}+h\right)\cdot\nabla\psi = \nabla\left[rV_\phi G + H\right]\cdot\nabla\psi$$

$$= G\nabla(rV_\phi)\nabla\psi + rV_\phi G'|\nabla\psi|^2 + H'|\nabla\psi|^2$$

Thus $\quad -A - B + C = TS'|\nabla\psi|^2$

$\Rightarrow$

$$\underset{(I)}{-\frac{V_\phi}{r}\nabla(rV_\phi)\nabla\psi} + \frac{\omega_\phi}{r\rho}\Psi'|\nabla\psi|^2$$

$$-\frac{J_\phi}{\rho r}|\nabla\psi|^2 + \underset{(II)}{\frac{B_\phi}{r\rho}\nabla(rB_\phi)\cdot\nabla\psi}$$

$$+\underset{(III)}{G\nabla(rV_\phi)\nabla\psi} + rV_\phi G'|\nabla\psi|^2 + H'|\nabla\psi|^2$$

$$= TS'|\nabla\psi|^2$$

Now $\quad -\mathrm{I} + \mathrm{II} + \mathrm{III}$

$$= \left[-\frac{V_\phi}{r} \nabla(r V_\phi) + \frac{B_\phi}{r\rho} \nabla(r B_\phi) + G \nabla(r V_\phi) \right] \nabla\psi$$

Now $\quad \nabla(r B_\phi) = \nabla\left[\overline{\Psi}' r V_\phi + K \right] \qquad$ (from Eq 25)

$$= \overline{\Psi}' \nabla(r V_\phi) + r V_\phi \overline{\Psi}'' \nabla\psi + K' \nabla\psi$$

thus

$$\rightsquigarrow \left[-\frac{V_\phi}{r} \nabla(r V_\phi) + \frac{B_\phi}{r\rho} \overline{\Psi}' \nabla(r V_\phi) + \frac{B_\phi}{r\rho} r V_\phi \overline{\Psi}'' \nabla\psi + \frac{B_\phi}{r\rho} K' \nabla\psi \right.$$

$$\left. + \; G \nabla(r V_\phi) \right] \nabla\psi$$

$$\Rightarrow \underbrace{\left(G - \frac{V_\phi}{r} + \frac{B_\phi}{r\rho} \overline{\Psi}' \right)}_{0 \text{ [Eq 18]}} \nabla(r V_\phi) \nabla\psi + \frac{B_\phi}{r\rho} r V_\phi \overline{\Psi}'' |\nabla\psi|^2 + \frac{B_\phi}{r\rho} K' |\nabla\psi|^2$$

Thus

$$\frac{B_\phi V_\phi}{\rho} \overline{\Psi}'' + \frac{B_\phi}{r\rho} K' + \frac{\omega_\phi}{r\rho} \overline{\Psi}' - \frac{J_\phi}{r\rho} + r V_\phi G' + H'$$

$$= T S'$$

or

$$\frac{\overline{\Psi}'}{r\rho} \omega_\phi - \frac{1}{r\rho} J_\phi = T S' - H' - r V_\phi G' - \frac{B_\phi}{r\rho} K' - \frac{B_\phi V_\phi}{\rho} \overline{\Psi}'' \qquad (29)$$

is the needed GS equation.

$$r^2 \nabla \cdot \left(\frac{\nabla\psi}{r^2} \right) \equiv \Delta^* \psi$$

A1

Appendix

axisymmetric system

Let $\nabla f = \kappa(r,z)\nabla\psi$

thus $$\frac{\partial f}{\partial r} = \kappa\frac{\partial\psi}{\partial r} \qquad (A_1)$$

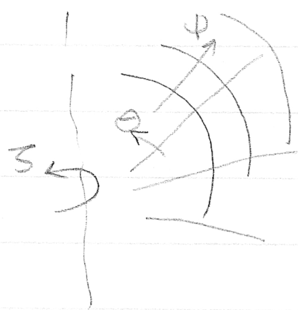
$$\frac{\partial f}{\partial z} = \kappa\frac{\partial\psi}{\partial z} \qquad (A_2)$$

Take z derivative of A_1:

$$\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial\kappa}{\partial z}\frac{\partial\psi}{\partial r} + \kappa\frac{\partial^2\psi}{\partial z\,\partial r}$$

$$\Rightarrow \frac{\partial}{\partial r}\left(\kappa\frac{\partial\psi}{\partial z}\right) = \frac{\partial\kappa}{\partial z}\frac{\partial\psi}{\partial r} + \kappa\frac{\partial^2\psi}{\partial z\,\partial r}$$

$$\Rightarrow \frac{\partial\kappa}{\partial r}\frac{\partial\psi}{\partial z} - \frac{\partial\kappa}{\partial z}\frac{\partial\psi}{\partial r} = 0 \qquad (A_3)$$



Now: $\kappa = \kappa(r,z) = \bar{\kappa}(\psi,\theta)$

$$\Rightarrow \frac{\partial\kappa}{\partial r} = \frac{\partial\bar{\kappa}}{\partial\psi}\frac{\partial\psi}{\partial r} + \frac{\partial\bar{\kappa}}{\partial\theta}\frac{\partial\theta}{\partial r}$$

$$\frac{\partial\kappa}{\partial z} = \frac{\partial\bar{\kappa}}{\partial\psi}\frac{\partial\psi}{\partial z} + \frac{\partial\bar{\kappa}}{\partial\theta}\frac{\partial\theta}{\partial z}$$

use in A_3:

$$\left(\frac{\partial\bar{\kappa}}{\partial\psi}\frac{\partial\psi}{\partial r} + \frac{\partial\bar{\kappa}}{\partial\theta}\frac{\partial\theta}{\partial r}\right)\frac{\partial\psi}{\partial z} - \left(\frac{\partial\bar{\kappa}}{\partial\psi}\frac{\partial\psi}{\partial z} + \frac{\partial\bar{\kappa}}{\partial\theta}\frac{\partial\theta}{\partial z}\right)\frac{\partial\psi}{\partial r} = 0$$

$$\Rightarrow \frac{\partial\bar{\kappa}}{\partial\psi}\cancel{\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial z}} - \cancel{\frac{\partial\bar{\kappa}}{\partial\psi}}\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial r}$$

$$+ \frac{\partial\bar{\kappa}}{\partial\theta}\frac{\partial\theta}{\partial r}\frac{\partial\psi}{\partial z} - \frac{\partial\bar{\kappa}}{\partial\theta}\frac{\partial\theta}{\partial z}\frac{\partial\psi}{\partial r} = 0$$

$$\Rightarrow \frac{\partial\bar{\kappa}}{\partial\theta}\left(\frac{\partial\theta}{\partial r}\frac{\partial\psi}{\partial z} - \frac{\partial\theta}{\partial z}\frac{\partial\psi}{\partial r}\right) = 0$$

$$\Rightarrow \frac{\partial\bar{\kappa}}{\partial\theta}(\nabla\theta\times\nabla\psi)\cdot\hat{z} = 0 \qquad \Rightarrow \frac{\partial\bar{\kappa}}{\partial\theta} = 0$$

(note $\nabla\theta$ is never $\parallel$ to $\nabla\psi$)

Thus $\boxed{\bar{\kappa} = \bar{\kappa}(\psi)}$

$\Rightarrow$ if $\nabla f = \kappa(r,z)\nabla\psi$ then $f = F(\psi)$