Modal DG | Ammar Hakim

Consider
$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = s \qquad \text{let } q \in \mathbb{R} \quad \& \quad f: \mathbb{R} \mapsto \mathbb{R} \qquad f = f(q)$$

Let $C_i \equiv [x_{i-1/2}, x_{i+1/2}]$ be a cell. Multiply by $v_r(x)$ & integrate over cell:
$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial q}{\partial t} v_r(x)\,dx + \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial f}{\partial x} v_r(x)\,dx = \int_{x_{i-1/2}}^{x_{i+1/2}} s\, v_r(x)\,dx$$

$$= \frac{d}{dt}\int q(x,t)\,v_r(x)\,dx + f(q(x,t))\,v_r(x)\Big|_{x_{i-1/2}}^{x_{i+1/2}} - \int_{x_{i-1/2}}^{x_{i+1/2}} f(q(x,t))\frac{dv_r}{dx}\,dx = \int_{x_{i-1/2}}^{x_{i+1/2}} s\, v_r\,dx$$

Now:
$$q(x,t) = \sum_k q_k(t)\, v_k(x). \quad \text{Also assume} \quad \int_{x_{i-1/2}}^{x_{i+1/2}} v_r(x)\, v_k(x)\,dx = \Delta x\, C_r\, \delta_{rk}$$
$$\Delta x \equiv x_{i+1/2} - x_{i-1/2}$$

Then:
$$\left[\Delta x\, C_r \frac{dq_r}{dt} + f(q(x,t))\,v_r(x)\Big|_{x_{i-1/2}}^{x_{i+1/2}} - \int_{x_{i-1/2}}^{x_{i+1/2}} f(q(x,t))\frac{dv_r}{dx}\,dx = \int_{x_{i-1/2}}^{x_{i+1/2}} s\, v_r\,dx\right]$$

Let
$$v_r(x) = P_r(\eta(x)) \qquad \eta(x) \equiv \frac{x - x_i}{\Delta x/2} \qquad \eta \in [-1,1]$$

Then
$$\frac{dv_r}{dx} = \frac{dP_r}{d\eta}\frac{d\eta}{dx} = \frac{dP_r}{d\eta}\frac{2}{\Delta x}$$

Also
$$\int_{x_{i-1/2}}^{x_{i+1/2}} v_r(x)\, v_m(x)\,dx = \int_{-1}^{+1} P_r(\eta)\, P_m(x)\,d\eta\,\frac{\Delta x}{2} = \frac{2\delta_{rm}}{2r+1}\frac{\Delta x}{2}$$

$$\Rightarrow \boxed{C_r \equiv \frac{1}{2r+1}}$$

&
$$\int_{x_{i-1/2}}^{x_{i+1/2}} f(q(x,t))\frac{dv_r}{dx}\,dx = \int_{-1}^{+1} f(q(x(\eta),t))\frac{dP_r}{d\eta}\frac{\cancel{2}}{\cancel{\Delta x}}\,d\eta\,\frac{\cancel{\Delta x}}{\cancel{2}}$$

And:
$$v_r(x_{i+1/2}) = P_r(1) = 1 \qquad v_r(x_{i-1/2}) = P_r(-1) = (-1)^r$$

So finally:

modal DG ←
$$\left[\frac{dq_r}{dt} + \frac{f_{i+1/2} - (-1)^r f_{i-1/2}}{C_r \Delta x} - \frac{1}{C_r\Delta x}\int_{-1}^{+1} f(q(x(\eta)),t)\frac{dP_r}{d\eta}\,d\eta = \frac{1}{2C_r}\int_{-1}^{+1} s(q(\eta,t),x(\eta))\,P_r(\eta)\,d\eta\right]$$

is the DG update formula. $[\, f_{i\pm1/2} \equiv f(q(x_{i\pm1/2},t))\,]$

To compute $\int_{-1}^{+1} g(\eta)\,d\eta$ use <u>Gaussian</u> quadrature:
$$\int_{-1}^{+1} g(\eta)\,d\eta = \sum_{i=1}^{N} g(\eta_i)\, w_i$$
where η_i ← ordinates
w_i ← weights.