Lecture 2: Making Sequences of Good Decisions Given a Model of the World

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CS234 Reinforcement Learning

Refresh Your Knowledge 1. Piazza Poll

In a Markov decision process, a large discount factor γ means that short term rewards are much more influential than long term rewards. [Enter your answer in piazza]

- True
- False
- Don't know

Refresh Your Knowledge 1. Piazza Poll

In a Markov decision process, a large discount factor γ means that short term rewards are much more influential than long term rewards. [Enter your answer in piazza]

- True
- False
- Don't know

False. A large γ implies we weigh delayed / long term rewards more. $\gamma = 0$ only values immediate rewards

Today's Plan

- Last Time:
  - Introduction
  - Components of an agent: model, value, policy
- This Time:
  - Making good decisions given a Markov decision process
- Next Time:
  - Policy evaluation when don't have a model of how the world works

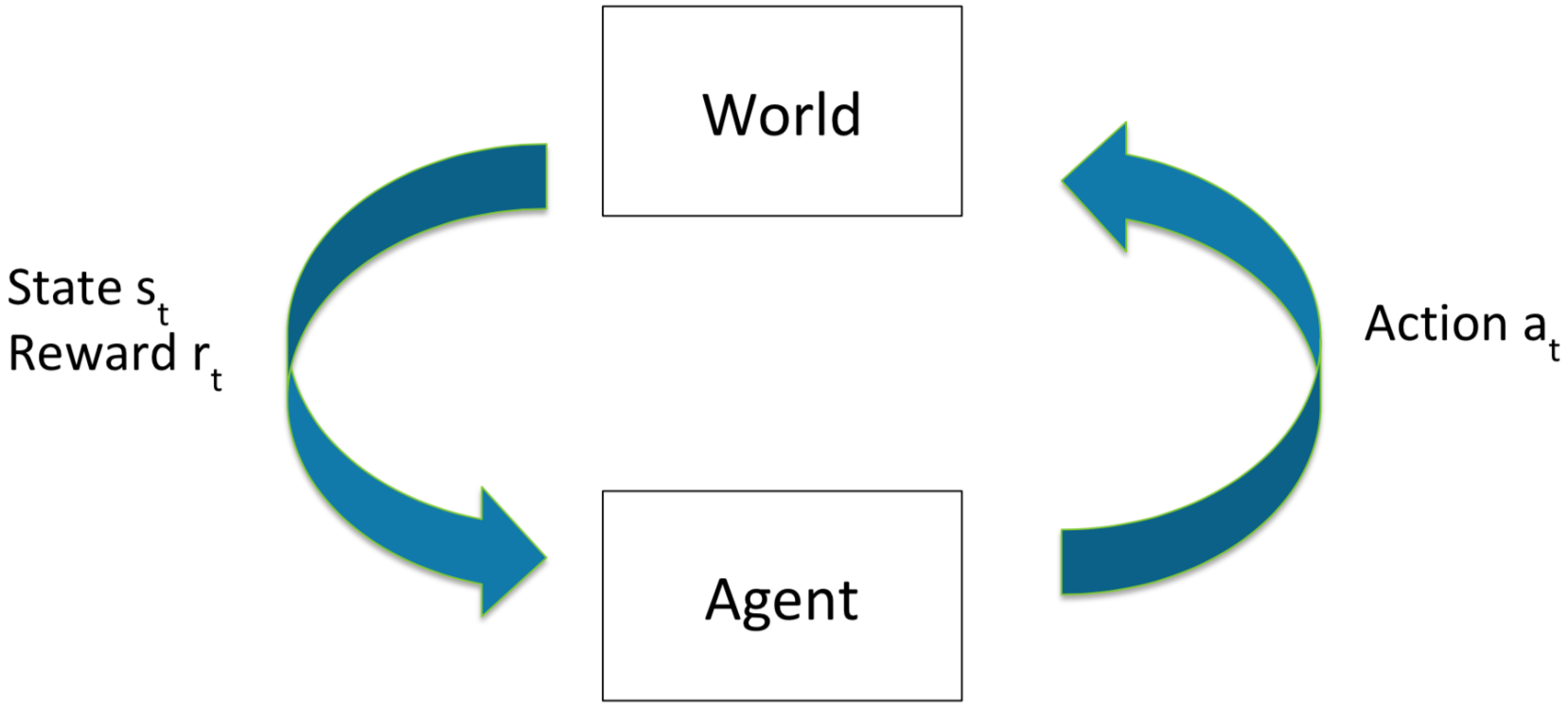
Models, Policies, Values

- **Model**: Mathematical models of dynamics and reward
- **Policy**: Function mapping agent's states to actions
- **Value function**: future rewards from being in a state and/or action when following a particular policy

Today: Given a model of the world

- Markov Processes
- Markov Reward Processes (MRPs)
- Markov Decision Processes (MDPs)
- Evaluation and Control in MDPs

Full Observability: Markov Decision Process (MDP)



- MDPs can model a huge number of interesting problems and settings
  - Bandits: single state MDP
  - Optimal control mostly about continuous-state MDPs
  - Partially observable MDPs = MDP where state is history

Recall: Markov Property

- Information state: sufficient statistic of history
- State s_t is Markov if and only if:

$$p(s_{t+1}|s_t, a_t) = p(s_{t+1}|h_t, a_t)$$

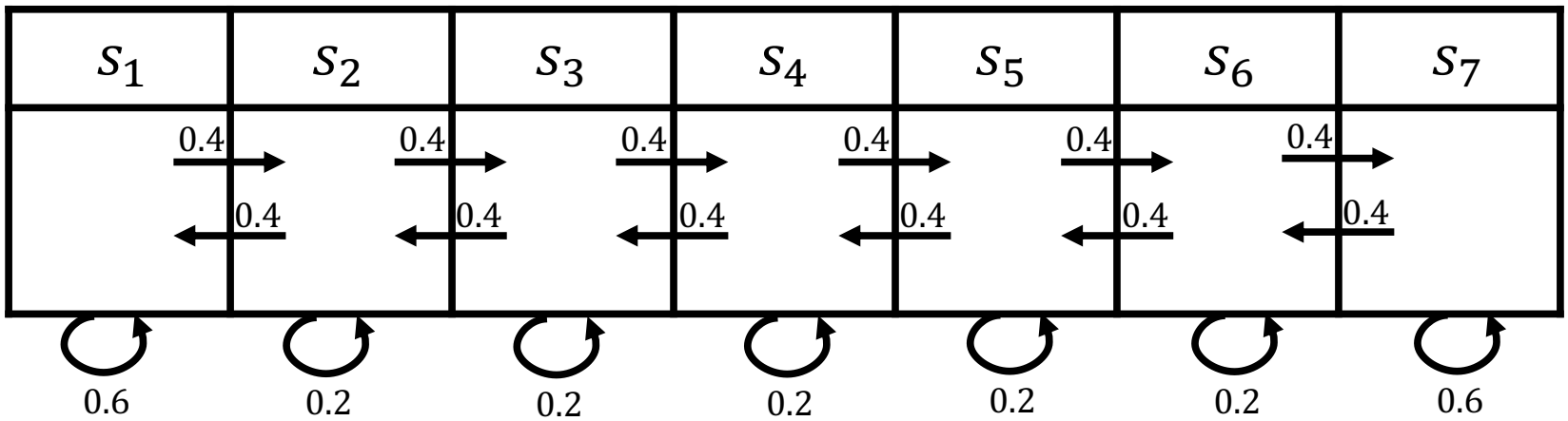
- Future is independent of past given present

Markov Process or Markov Chain

- Memoryless random process
  - Sequence of random states with Markov property
- Definition of Markov Process
  - S is a (finite) set of states ($s \in S$)
  - P is dynamics/transition model that specifices $p(s_{t+1} = s' | s_t = s)$
- Note: no rewards, no actions
- If finite number (N) of states, can express P as a matrix

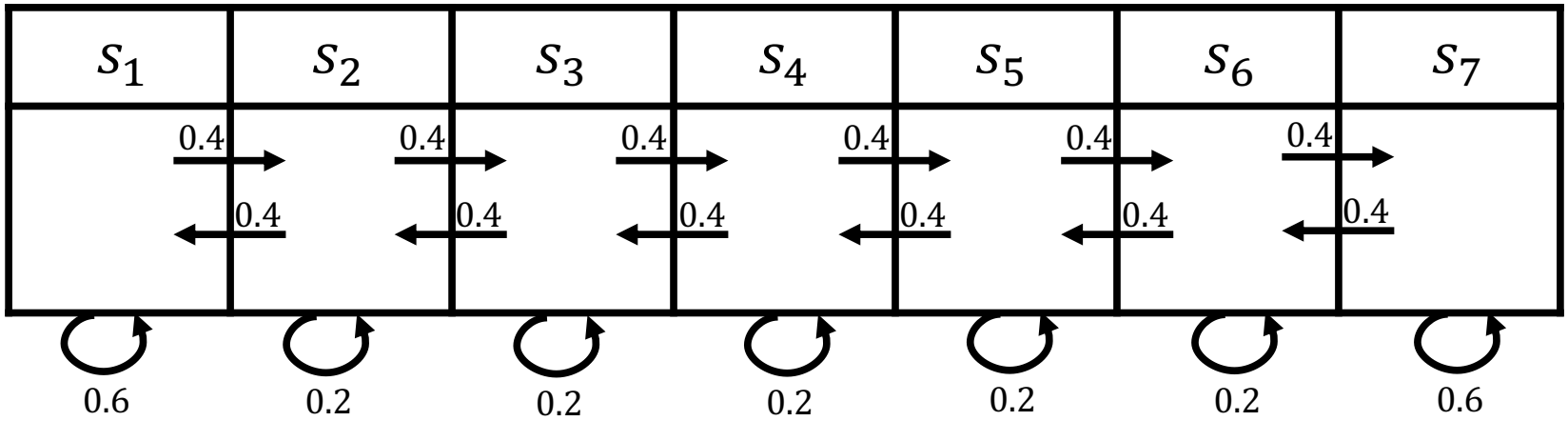
$$P = \begin{pmatrix} P(s_1|s_1) & P(s_2|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & P(s_2|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \vdots & \ddots & \vdots \\ P(s_1|s_N) & P(s_2|s_N) & \cdots & P(s_N|s_N) \end{pmatrix}$$

Example: Mars Rover Markov Chain Transition Matrix, P



$$P = \begin{pmatrix} 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.2 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.2 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \end{pmatrix}$$

Example: Mars Rover Markov Chain Episodes



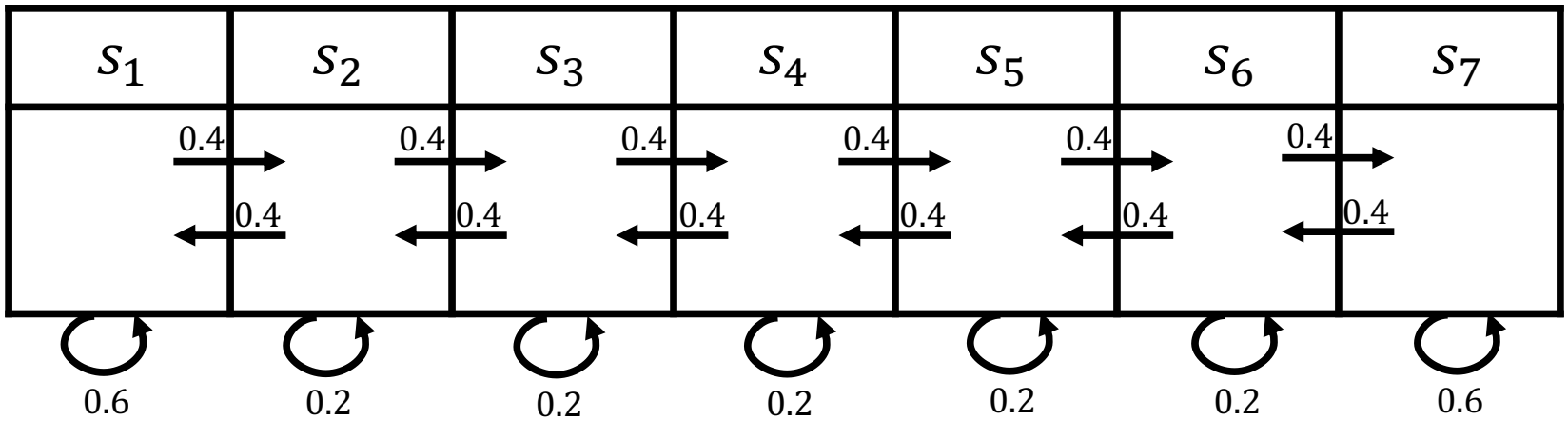
Example: Sample episodes starting from S4

- $s_4, s_5, s_6, s_7, s_7, s_7, \ldots$
- $s_4, s_4, s_5, s_4, s_5, s_6, \ldots$
- $s_4, s_3, s_2, s_1, \ldots$

Markov Reward Process (MRP)

- Markov Reward Process is a Markov Chain + rewards
- Definition of Markov Reward Process (MRP)
  - S is a (finite) set of states ($s \in S$)
  - P is dynamics/transition model that specifices $P(s_{t+1} = s' | s_t = s)$
  - R is a reward function $R(s_t = s) = \mathbb{E}[r_t | s_t = s]$
  - Discount factor $\gamma \in [0, 1]$
- Note: no actions
- If finite number (N) of states, can express R as a vector

Example: Mars Rover MRP



- Reward: $+1$ in s_1, $+10$ in s_7, 0 in all other states

Return & Value Function

- Definition of Horizon (H)
  - Number of time steps in each episode
  - Can be infinite
  - Otherwise called **finite** Markov reward process
- Definition of Return, G_t (for a MRP)
  - Discounted sum of rewards from time step t to horizon H

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots + \gamma^{H-1} r_{t+H-1}$$

- Definition of State Value Function, $V(s)$ (for a MRP)
  - Expected return from starting in state s

$$V(s) = \mathbb{E}[G_t | s_t = s] = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots + \gamma^{H-1} r_{t+H-1} | s_t = s]$$

Discount Factor

- Mathematically convenient (avoid infinite returns and values)
- Humans often act as if there's a discount factor < 1
- $\gamma = 0$: Only care about immediate reward
- $\gamma = 1$: Future reward is as beneficial as immediate reward
- If episode lengths are always finite ($H < \infty$), can use $\gamma = 1$

Computing the Value of a Markov Reward Process

- Markov property provides structure
- MRP value function satisfies

$$V(s) = \underbrace{R(s)}_{\text{Immediate reward}} + \underbrace{\gamma \sum_{s' \in S} P(s'|s)V(s')}_{\text{Discounted sum of future rewards}}$$

Matrix Form of Bellman Equation for MRP

- For finite state MRP, we can express $V(s)$ using a matrix equation

$$\begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix} = \begin{pmatrix} R(s_1) \\ \vdots \\ R(s_N) \end{pmatrix} + \gamma \begin{pmatrix} P(s_1|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \ddots & \vdots \\ P(s_1|s_N) & \cdots & P(s_N|s_N) \end{pmatrix} \begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix}$$

$$V = R + \gamma P V$$

Analytic Solution for Value of MRP

- For finite state MRP, we can express $V(s)$ using a matrix equation

$$\begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix} = \begin{pmatrix} R(s_1) \\ \vdots \\ R(s_N) \end{pmatrix} + \gamma \begin{pmatrix} P(s_1|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \ddots & \vdots \\ P(s_1|s_N) & \cdots & P(s_N|s_N) \end{pmatrix} \begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix}$$

$$\begin{aligned} V &= R + \gamma P V \\ V - \gamma P V &= R \\ (I - \gamma P) V &= R \\ V &= (I - \gamma P)^{-1} R \end{aligned}$$

- Solving directly requires taking a matrix inverse $\sim O(N^3)$
- Note that $(I - \gamma P)$ is invertible

Iterative Algorithm for Computing Value of a MRP

- Dynamic programming
- Initialize $V_0(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S

$$V_k(s) = R(s) + \gamma \sum_{s' \in S} P(s'|s) V_{k-1}(s')$$

- Computational complexity: $O(|S|^2)$ for each iteration ($|S| = N$)

Markov Decision Process (MDP)

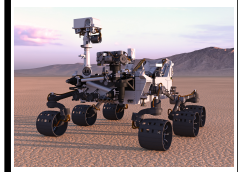
- Markov Decision Process is Markov Reward Process + actions
- Definition of MDP
  - S is a (finite) set of Markov states $s \in S$
  - A is a (finite) set of actions $a \in A$
  - P is dynamics/transition model for **each action**, that specifies $P(s_{t+1} = s' | s_t = s, a_t = a)$
  - R is a reward function[1]

$$R(s_t = s, a_t = a) = \mathbb{E}[r_t | s_t = s, a_t = a]$$

  - Discount factor $\gamma \in [0, 1]$
- MDP is a tuple: (S, A, P, R, γ)

[1] Reward is sometimes defined as a function of the current state, or as a function of the (state, action, next state) tuple. Most frequently in this class, we will assume reward is a function of state and action

Example: Mars Rover MDP

s_1	s_2	s_3	s_4	s_5	s_6	s_7
						

$$P(s'|s,a_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad P(s'|s,a_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- 2 deterministic actions

MDP Policies

- Policy specifies what action to take in each state
  - Can be deterministic or stochastic
- For generality, consider as a conditional distribution
  - Given a state, specifies a distribution over actions
- Policy: $\pi(a|s) = P(a_t = a|s_t = s)$

MDP + Policy

- MDP + $\pi(a|s)$ = Markov Reward Process
- Precisely, it is the MRP $(S, R^\pi, P^\pi, \gamma)$, where

$$R^\pi(s) = \sum_{a \in A} \pi(a|s) R(s,a)$$

$$P^\pi(s'|s) = \sum_{a \in A} \pi(a|s) P(s'|s,a)$$

- Implies we can use same techniques to evaluate the value of a policy for a MDP as we could to compute the value of a MRP, by defining a MRP with R^π and P^π

MDP Policy Evaluation, Iterative Algorithm

- Initialize $V_0(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S

$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

- This is a **Bellman backup** for a particular policy

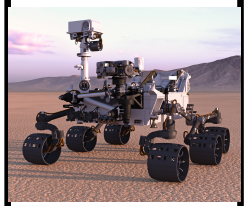
Check Your Understanding: MDP 1 Iteration of Policy Evaluation, Mars Rover Example

- Dynamics: $p(s_6|s_6, a_1) = 0.5$, $p(s_7|s_6, a_1) = 0.5$, ...
- Reward: for all actions, $+1$ in state s_1, $+10$ in state s_7, 0 otherwise
- Let $\pi(s) = a_1 \ \forall s$, assume $V_k = [1\ 0\ 0\ 0\ 0\ 0\ 10]$ and $k = 1$, $\gamma = 0.5$
- Compute $V_{k+1}(s_6)$

Select answer on piazza poll:

- 0
- 2.5
- 10
- I don't know

Check Your Understanding

s_1	s_2	s_3	s_4	s_5	s_6	s_7
						

- We will shortly be interested in not just evaluating the value of a single policy, but finding an optimal policy. Given this it is informative to think about properties of the potential policy space.
- First for the Mars rover example [7 discrete states (location of rover); 2 actions: Left or Right]
- How many deterministic policies are there?
- Select answer on piazza poll: 2 / 14 / 7^2 / 2^7 / Not sure
- Is the optimal policy (one with highest value) for a MDP unique?
- Select answer on piazza poll: Yes / No / Not sure

Check Your Understanding: MDP 1 Iteration of Policy Evaluation, Mars Rover Example

- Dynamics: $p(s_6|s_6, a_1) = 0.5$, $p(s_7|s_6, a_1) = 0.5$, ...
- Reward: for all actions, $+1$ in state s_1, $+10$ in state s_7, 0 otherwise
- Let $\pi(s) = a_1 \ \forall s$, assume V_k =[1 0 0 0 0 0 10] and $k = 1$, $\gamma = 0.5$
-

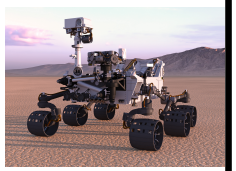
$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

$$V_{k+1}(s_6) = r(s_6, a_1) + \gamma * 0.5 * V_k(s_6) + \gamma * 0.5 * V_k(s_7)$$

$$V_{k+1}(s_6) = 0 + 0.5 * 0.5 * 0 + .5 * 0.5 * 10$$

$$V_{k+1}(s_6) = 2.5$$

Check Your Understanding

s_1	s_2	s_3	s_4	s_5	s_6	s_7
						

- 7 discrete states (location of rover)
- 2 actions: Left or Right
- How many deterministic policies are there?
  2^7

- Is the optimal policy for a MDP always unique?
  No, there may be two actions that have the same optimal value function

MDP Control

- Compute the optimal policy

$$\pi^*(s) = \arg\max_{\pi} V^{\pi}(s)$$

- There **exists a unique optimal value function**
- Optimal policy for a MDP in an infinite horizon problem is deterministic

MDP Control

- Compute the optimal policy

$$\pi^*(s) = \arg\max_{\pi} V^{\pi}(s)$$

- There exists a unique optimal value function
- Optimal policy for a MDP in an infinite horizon problem (agent acts forever is
  - Deterministic
  - Stationary (does not depend on time step)
  - Unique? Not necessarily, may have state-actions with identical optimal values

Policy Search

- One option is searching to compute best policy
- Number of deterministic policies is $|A|^{|S|}$
- Policy iteration is generally more efficient than enumeration

MDP Policy Iteration (PI)

- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$

New Definition: State-Action Value Q

- State-action value of a policy

$$Q^{\pi}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi}(s')$$

- Take action a, then follow the policy π

Policy Improvement

- Compute state-action value of a policy π_i
  - For s in S and a in A:

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

- Compute new policy π_{i+1}, for all $s \in S$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a) \ \forall s \in S$$

MDP Policy Iteration (PI)

- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
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  - $i = i + 1$

Delving Deeper Into Policy Improvement Step

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

Delving Deeper Into Policy Improvement Step

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

$$\max_a Q^{\pi_i}(s,a) \geq R(s,\pi_i(s)) + \gamma \sum_{s' \in S} P(s'|s,\pi_i(s)) V^{\pi_i}(s') = V^{\pi_i}(s)$$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a)$$

- Suppose we take $\pi_{i+1}(s)$ for one action, then follow π_i forever
  - Our expected sum of rewards is at least as good as if we had always followed π_i
- But new proposed policy is to always follow π_{i+1} ...

Monotonic Improvement in Policy

- Definition

$$V^{\pi_1} \geq V^{\pi_2} : V^{\pi_1}(s) \geq V^{\pi_2}(s), \forall s \in S$$

- Proposition: $V^{\pi_{i+1}} \geq V^{\pi_i}$ with strict inequality if π_i is suboptimal, where π_{i+1} is the new policy we get from policy improvement on π_i

Proof: Monotonic Improvement in Policy

$$V^{\pi_i}(s) \leq \max_a Q^{\pi_i}(s,a)$$
$$= \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

Proof: Monotonic Improvement in Policy

$$
\begin{aligned}
V^{\pi_i}(s) \leq & \max_a Q^{\pi_i}(s,a) \\
= & \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s') \\
= & R(s,\pi_{i+1}(s)) + \gamma \sum_{s' \in S} P(s'|s,\pi_{i+1}(s)) V^{\pi_i}(s') \text{ //by the definition of } \pi_{i+1} \\
\leq & R(s,\pi_{i+1}(s)) + \gamma \sum_{s' \in S} P(s'|s,\pi_{i+1}(s)) \left(\max_{a'} Q^{\pi_i}(s',a') \right) \\
= & R(s,\pi_{i+1}(s)) + \gamma \sum_{s' \in S} P(s'|s,\pi_{i+1}(s)) \\
& \left(R(s',\pi_{i+1}(s')) + \gamma \sum_{s'' \in S} P(s''|s',\pi_{i+1}(s')) V^{\pi_i}(s'') \right) \\
& \vdots \\
= & V^{\pi_{i+1}}(s)
\end{aligned}
$$

Check Your Understanding: Policy Iteration (PI)

- Note: all the below is for finite state-action spaces
- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$
- **If policy doesn't change, can it ever change again?**
- Select on piazza poll: Yes / No / Not sure
- **Is there a maximum number of iterations of policy iteration?**
- Select on piazza poll: Yes / No / Not sure

Results for Check Your Understanding Policy Iteration

- Note: all the below is for finite state-action spaces
- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$
- **If policy doesn't change, can it ever change again?**
  No
- **Is there a maximum number of iterations of policy iteration?**
  $|A|^{|S|}$ since that is the maximum number of policies, and as the policy improvement step is monotonically improving, each policy can only appear in one round of policy iteration unless it is an optimal policy.

Check Your Understanding Explanation of Policy Not Changing

- Suppose for all $s \in S$, $\pi_{i+1}(s) = \pi_i(s)$
- Then for all $s \in S$, $Q^{\pi_{i+1}}(s,a) = Q^{\pi_i}(s,a)$
- Recall policy improvement step

$$Q^{\pi_i}(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_i}(s')$$

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s,a)$$

$$\pi_{i+2}(s) = \arg\max_a Q^{\pi_{i+1}}(s,a) = \arg\max_a Q^{\pi_i}(s,a)$$

Therefore policy cannot ever change again

MDP: Computing Optimal Policy and Optimal Value

- Policy iteration computes optimal value and policy
- Value iteration is another technique
  - Idea: Maintain optimal value of starting in a state s if have a finite number of steps k left in the episode
  - Iterate to consider longer and longer episodes

Bellman Equation and Bellman Backup Operators

- Value function of a policy must satisfy the Bellman equation

$$V^{\pi}(s) = R^{\pi}(s) + \gamma \sum_{s' \in S} P^{\pi}(s'|s) V^{\pi}(s')$$

- Bellman backup operator
  - Applied to a value function
  - Returns a new value function
  - Improves the value if possible

$$BV(s) = \max_{a} R(s,a) + \gamma \sum_{s' \in S} p(s'|s,a) V(s')$$

  - BV yields a value function over all states s

Value Iteration (VI)

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states s
- Loop until [convergence]:
  - For each state s

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

  - View as Bellman backup on value function

$$V_{k+1} = BV_k$$

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

Policy Iteration as Bellman Operations

- Bellman backup operator B^π for a particular policy is defined as

$$B^\pi V(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s) V(s)$$

- Policy evaluation amounts to computing the fixed point of B^π
- To do policy evaluation, repeatedly apply operator until V stops changing

$$V^\pi = B^\pi B^\pi \cdots B^\pi V$$

Policy Iteration as Bellman Operations

- Bellman backup operator B^π for a particular policy is defined as

$$B^\pi V(s) = R^\pi(s) + \gamma \sum_{s' \in S} P^\pi(s'|s) V(s)$$

- To do policy improvement

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi_k}(s')$$

Going Back to Value Iteration (VI)

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states s
- Loop until [convergence]:
  - For each state s

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

  - Equivalently, in Bellman backup notation

$$V_{k+1} = BV_k$$

- To extract optimal policy if can act for $k+1$ more steps,

$$\pi(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_{k+1}(s')$$

Contraction Operator

- Let O be an operator,and $|x|$ denote (any) norm of x
- If $|OV - OV'| \leq |V - V'|$, then O is a contraction operator

Will Value Iteration Converge?

- Yes, if discount factor $\gamma < 1$, or end up in a terminal state with probability 1
- Bellman backup is a contraction if discount factor, $\gamma < 1$
- If apply it to two different value functions, distance between value functions shrinks after applying Bellman equation to each

Proof: Bellman Backup is a Contraction on V for $\gamma < 1$

- Let $\|V - V'\| = \max_s |V(s) - V'(s)|$ be the infinity norm

$$\|BV_k - BV_j\| = \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left(R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\|$$

Proof: Bellman Backup is a Contraction on V for $\gamma < 1$

- Let $\|V - V'\| = \max_s |V(s) - V'(s)|$ be the infinity norm

$$
\begin{aligned}
\|BV_k - BV_j\| &= \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left(R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\| \\
&\leq \max_a \left\| \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') - R(s,a) - \gamma \sum_{s' \in S} P(s'|s,a) V_j(s') \right) \right\| \\
&= \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a)(V_k(s') - V_j(s')) \right\| \\
&\leq \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a) \|V_k - V_j\|) \right\| \\
&= \max_a \left\| \gamma \|V_k - V_j\| \sum_{s' \in S} P(s'|s,a)) \right\| \\
&= \gamma \|V_k - V_j\|
\end{aligned}
$$

- Note: Even if all inequalities are equalities, this is still a contraction if $\gamma < 1$

Opportunities for Out-of-Class Practice

- Homework question: Prove value iteration converges to a unique solution for discrete state and action spaces with $\gamma < 1$
- Does the initialization of values in value iteration impact anything?

Value Iteration for Finite Horizon H

V_k = optimal value if making k more decisions
π_k = optimal policy if making k more decisions

- Initialize $V_0(s) = 0$ for all states s
- For $k = 1 : H$
  - For each state s

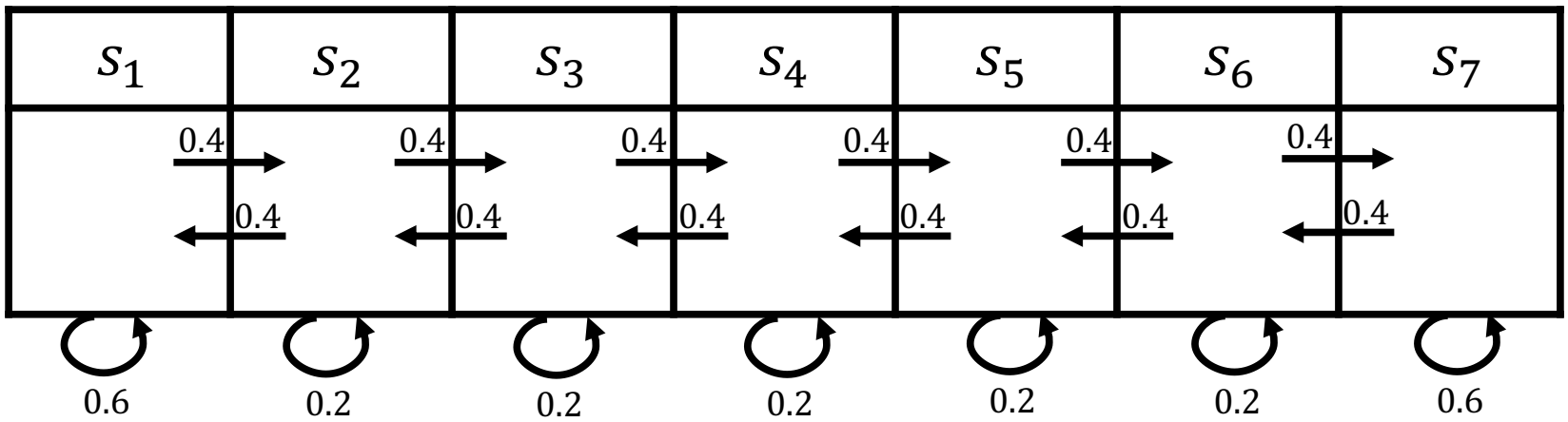
$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

Computing the Value of a Policy in a Finite Horizon

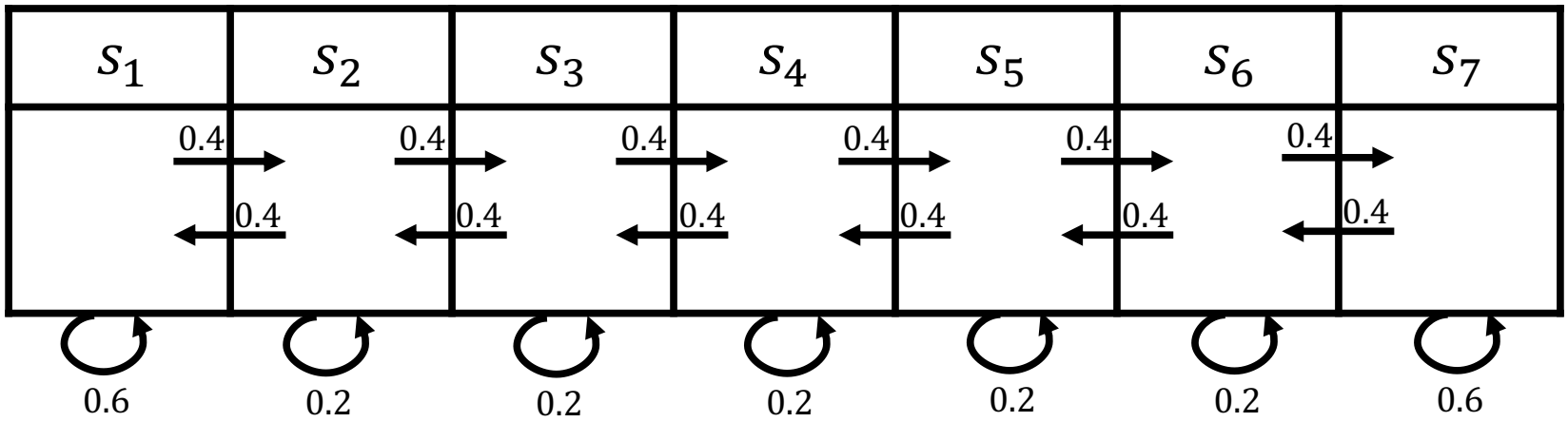
- Alternatively can estimate by simulation
  - Generate a large number of episodes
  - Average returns
  - Concentration inequalities bound how quickly average concentrates to expected value
  - Requires **no assumption** of Markov structure

Example: Mars Rover



- Reward: $+1$ in s_1, $+10$ in s_7, 0 in all other states
- Sample returns for sample 4-step (H=4) episodes, $\gamma = 1/2$
  - s_4, s_5, s_6, s_7: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 10 = 1.25$

Example: Mars Rover



- Reward: $+1$ in s_1, $+10$ in s_7, 0 in all other states
- Sample returns for sample 4-step (H=4) episodes, start state s_4 , $\gamma = 1/2$
  - s_4, s_5, s_6, s_7: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 10 = 1.25$
  - s_4, s_4, s_5, s_4: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 0 = 0$
  - s_4, s_3, s_2, s_1: $0 + \frac{1}{2} \times 0 + \frac{1}{4} \times 0 + \frac{1}{8} \times 1 = 0.125$

Check Your Understanding: Finite Horizon Policies

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states s
- Loop until $k == H$:
  - For each state s

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

Is optimal policy stationary (independent of time step) in finite horizon tasks? ? Piazza poll: Yes / No / Not sure

Check Your Understanding: Finite Horizon Policies

- Set $k = 1$
- Initialize $V_0(s) = 0$ for all states s
- Loop until $k == H$:
  - For each state s

$$V_{k+1}(s) = \max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

$$\pi_{k+1}(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s')$$

Is optimal policy stationary (independent of time step) in finite horizon tasks? In general no.

Value vs Policy Iteration

- Value iteration:
  - Compute optimal value for horizon $= k$
    - Note this can be used to compute optimal policy if horizon $= k$
  - Increment k
- Policy iteration
  - Compute infinite horizon value of a policy
  - Use to select another (better) policy
  - Closely related to a very popular method in RL: policy gradient

What You Should Know

- Define MP, MRP, MDP, Bellman operator, contraction, model, Q-value, policy
- Be able to implement
  - Value Iteration
  - Policy Iteration
- Give pros and cons of different policy evaluation approaches
- Be able to prove contraction properties
- Limitations of presented approaches and Markov assumptions
  - Which policy evaluation methods require the Markov assumption?

Where We Are

- Last Time:
  - Introduction
  - Components of an agent: model, value, policy
- This Time:
  - Making good decisions given a Markov decision process
- Next Time:
  - Policy evaluation when don't have a model of how the world works

Lecture 3: Model-Free Policy Evaluation: Policy Evaluation Without Knowing How the World Works[1]

Emma Brunskill

CS234 Reinforcement Learning

[1] Material builds on structure from David SIlver's Lecture 4: Model-Free Prediction. Other resources: Sutton and Barto Jan 1 2018 draft Chapter/Sections: 5.1; 5.5; 6.1-6.3

Refresh Your Knowledge 2 [Piazza Poll]

- What is the max number of iterations of policy iteration in a tabular MDP?
  1. $|A||S|$
  2. $|S|^{|A|}$
  3. $|A|^{|S|}$
  4. Unbounded
  5. Not sure
- In a tabular MDP asymptotically value iteration will always yield a policy with the same value as the policy returned by policy iteration
  1. True.
  2. False
  3. Not sure
- Can value iteration require more iterations than $|A|^{|S|}$ to compute the optimal value function? (Assume $|A|$ and $|S|$ are small enough that each round of value iteration can be done exactly).
  1. True.
  2. False
  3. Not sure

Refresh Your Knowledge 2

- What is the max number of iterations of policy iteration in a tabular MDP?
  Answer: $|A|^{|S|}$: There are only $|A|^{|S|}$ policies in a tabular MDP and each policy can only be considered at most once, since policy improvement either results in a policy with a higher value or returns the same policy if the optimal policy has been found.

- In a tabular MDP asymptotically value iteration will always yield a policy with the same value as the policy returned by policy iteration
  Answer. True. Both are guaranteed to converge to the optimal value function and a policy with an optimal value

- Can value iteration require more iterations than $|A|^{|S|}$ to compute the optimal value function? (Assume $|A|$ and $|S|$ are small enough that each round of value iteration can be done exactly).
  Answer: True. As an example, consider a single state, single action MDP where $r(s, a) = 1$, $\gamma = .9$ and initialize $V_0(s) = 0$. $V^*(s) = \frac{1}{1-\gamma}$ but after the first iteration of value iteration, $V_1(s) = 1$.

Today's Plan

- Last Time:
  - Markov reward / decision processes
  - Policy evaluation & control when have true model (of how the world works)
- **Today**
  - **Policy evaluation without known dynamics & reward models**
- Next Time:
  - Control when don't have a model of how the world works

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
- Temporal Difference (TD)
- Certainty Equivalence with dynamic programming
- Metrics to evaluate and compare algorithms

Recall

- Definition of Return, G_t (for a MRP)
  - Discounted sum of rewards from time step t to horizon

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$$

- Definition of State Value Function, $V^\pi(s)$
  - Expected return from starting in state s under policy π

$$V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s] = \mathbb{E}_\pi[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s]$$

- Definition of State-Action Value Function, $Q^\pi(s, a)$
  - Expected return from starting in state s, taking action a and then following policy π

$$\begin{aligned} Q^\pi(s, a) &= \mathbb{E}_\pi[G_t | s_t = s, a_t = a] \\ &= \mathbb{E}_\pi[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots | s_t = s, a_t = a] \end{aligned}$$

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
- Temporal Difference (TD)
- Certainty Equivalence with dynamic programming
- Metrics to evaluate and compare algorithms

Monte Carlo (MC) Policy Evaluation

- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_{T\sim\pi}[G_t | s_t = s]$
  - Expectation over trajectories T generated by following π
- Simple idea: Value = mean return
- If trajectories are all finite, sample set of trajectories & average returns

Monte Carlo (MC) Policy Evaluation

- If trajectories are all finite, sample set of trajectories & average returns
- Does not require MDP dynamics/rewards
- Does not assume state is Markov
- Can **only** be applied to episodic MDPs
  - Averaging over returns from a complete episode
  - Requires each episode to terminate

Monte Carlo (MC) On Policy Evaluation

- Aim: estimate $V^{\pi}(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^{\pi}(s) = \mathbb{E}_{\pi}[G_t | s_t = s]$
- MC computes empirical mean return
- Often do this in an incremental fashion
  - After each episode, update estimate of V^{π}

First-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0,\ G(s) = 0\ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each time step t till the end of the episode i
  - If this is the **first** time t that state s is visited in episode i
    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^{\pi}(s) = G(s)/N(s)$

Evaluation the Quality of a Policy Estimation Approach: Bias, Variance and MSE

- Consider a statistical model that is parameterized by θ and that determines a probability distribution over observed data $P(x|\theta)$
- Consider a statistic $\hat{\theta}$ that provides an estimate of θ and is a function of observed data x
  - E.g. for a Gaussian distribution with known variance, the average of a set of i.i.d data points is an estimate of the mean of the Gaussian
- Definition: the bias of an estimator $\hat{\theta}$ is:

$$Bias_\theta(\hat{\theta}) = \mathbb{E}_{x|\theta}[\hat{\theta}] - \theta$$

- Definition: the variance of an estimator $\hat{\theta}$ is:

$$Var(\hat{\theta}) = \mathbb{E}_{x|\theta}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2]$$

- Definition: mean squared error (MSE) of an estimator $\hat{\theta}$ is:

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias_\theta(\hat{\theta})^2$$

First-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0,\ G(s) = 0\ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
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    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^{\pi}(s) = G(s)/N(s)$

Properties:

- V^{π} estimator is an unbiased estimator of true $\mathbb{E}_{\pi}[G_t | s_t = s]$
- By law of large numbers, as $N(s) \to \infty,\ V^{\pi}(s) \to \mathbb{E}_{\pi}[G_t | s_t = s]$

Every-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0,\ G(s) = 0\ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each time step t till the end of the episode i
  - state s is the state visited at time step t in episodes i
  - Increment counter of total visits: $N(s) = N(s) + 1$
  - Increment total return $G(s) = G(s) + G_{i,t}$
  - Update estimate $V^{\pi}(s) = G(s)/N(s)$

Every-Visit Monte Carlo (MC) On Policy Evaluation

Initialize $N(s) = 0,\ G(s) = 0\ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
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  - Increment total return $G(s) = G(s) + G_{i,t}$
  - Update estimate $V^{\pi}(s) = G(s)/N(s)$

Properties:

- V^{π} every-visit MC estimator is a **biased** estimator of V^{π}
- But consistent estimator and often has better MSE

Worked Example First Visit MC On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0$ $\forall s \in S$

Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- For each time step t till the end of the episode i
  - If this is the **first** time t that state s is visited in episode i
    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^{\pi}(s) = G(s)/N(s)$
- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1$ $\forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)

Worked Example MC On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0 \; \forall s \in S$

Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
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    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^{\pi}(s) = G(s)/N(s)$
- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- Let $\gamma = 1$. First visit MC estimate of V of each state?
-
- Now let $\gamma = 0.9$. Compare the first visit & every visit MC estimates of s_2.
-

Worked Example MC On Policy Evaluation

Initialize $N(s) = 0$, $G(s) = 0 \; \forall s \in S$

Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
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    - Increment counter of total first visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^{\pi}(s) = G(s)/N(s)$
- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- Let $\gamma = 1$. First visit MC estimate of V of each state?
  V = [1 1 1 0 0 0 0]
- Now let $\gamma = 0.9$. Compare the first visit & every visit MC estimates of s_2.
  First visit: $V^{MC}(s_2) = \gamma^2$, Every visit: $V^{MC}(s_2) = \frac{\gamma^2 + \gamma}{2}$

Incremental Monte Carlo (MC) On Policy Evaluation

After each episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots$

- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots$ as return from time step t onwards in ith episode
- For state s visited at time step t in episode i
  - Increment counter of total visits: $N(s) = N(s) + 1$
  - Update estimate

$$V^{\pi}(s) = V^{\pi}(s)\frac{N(s)-1}{N(s)} + \frac{G_{i,t}}{N(s)} = V^{\pi}(s) + \frac{1}{N(s)}(G_{i,t} - V^{\pi}(s))$$

Incremental Monte Carlo (MC) On Policy Evaluation

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- for $i = 1 : T_i$ where T_i is the length of the i-th episode
  - $V^{\pi}(s_{it}) = V^{\pi}(s_{it}) + \alpha(G_{i,t} - V^{\pi}(s_{it}))$

Check Your Understanding: Piazza Poll Incremental MC

First or Every Visit MC

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
  - For all s, for **first or every** time t that state s is visited in episode i
    - $N(s) = N(s) + 1$, $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Incremental MC

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- for $t = 1 : T_i$ where T_i is the length of the i-th episode
  - $V^\pi(s_{it}) = V^\pi(s_{it}) + \alpha(G_{i,t} - V^\pi(s_{it}))$

1. Incremental MC with $\alpha = 1$ is the same as first visit MC
2. Incremental MC with $\alpha = \frac{1}{N(s_{it})}$ is the same as first visit MC
3. Incremental MC with $\alpha = \frac{1}{N(s_{it})}$ is the same as every visit MC
4. Incremental MC with $\alpha > \frac{1}{N(s_{it})}$ could be helpful in non-stationary domains

Check Your Understanding: Piazza Poll Incremental MC Answers

First or Every Visit MC

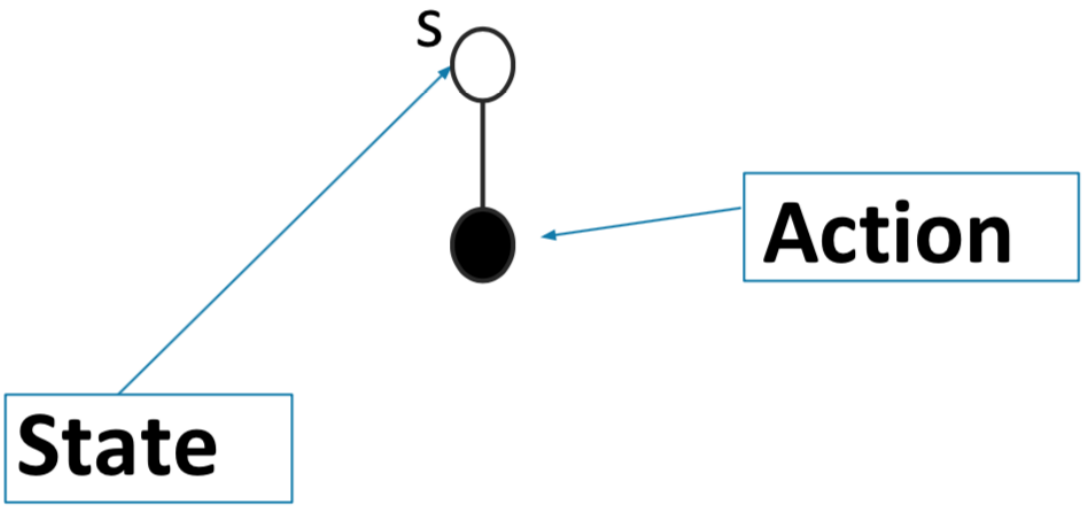
- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
  - For all s, for **first or every** time t that state s is visited in episode i
    - $N(s) = N(s) + 1$, $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

Incremental MC

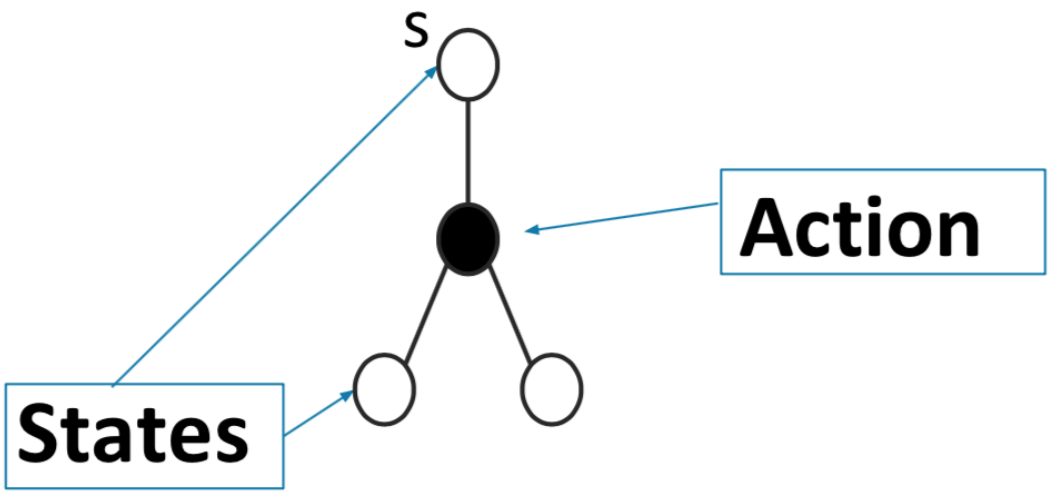
- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- for $t = 1 : T_i$ where T_i is the length of the i-th episode
  - $V^\pi(s_{it}) = V^\pi(s_{it}) + \alpha(G_{i,t} - V^\pi(s_{it}))$

1. Incremental MC with $\alpha = 1$ is the same as first visit MC (false)
2. Incremental MC with $\alpha = \frac{1}{N(s_{it})}$ is the same as first visit MC (false)
3. Incremental MC with $\alpha = \frac{1}{N(s_{it})}$ is the same as every visit MC (true)
4. Incremental MC with $\alpha > \frac{1}{N(s_{it})}$ could help in non-stationary domains (true)

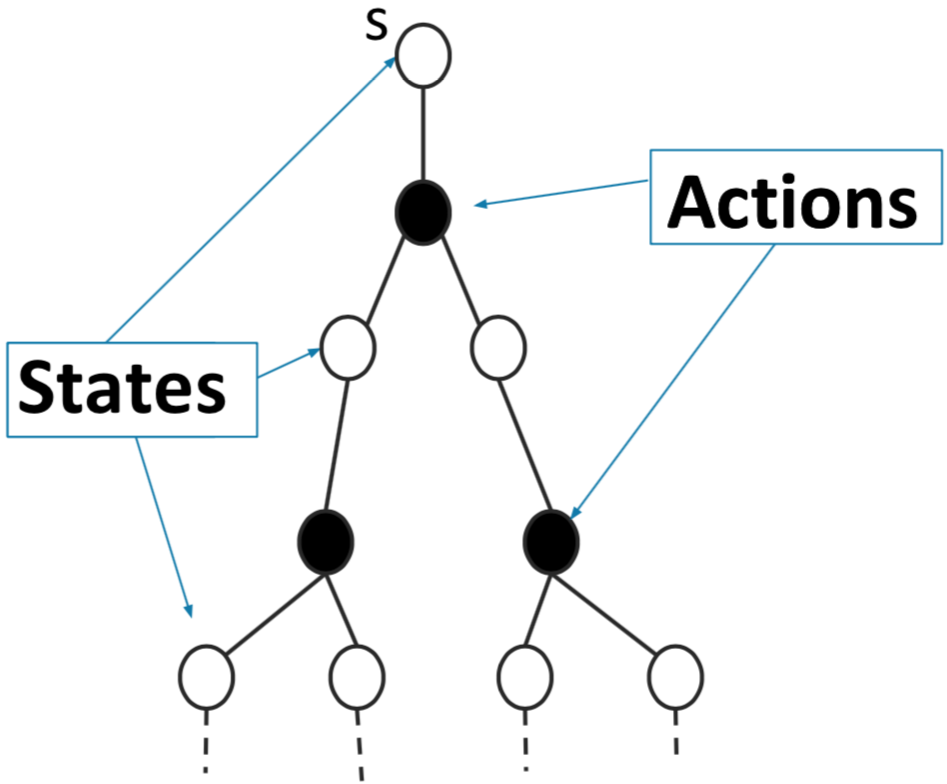
Policy Evaluation Diagram



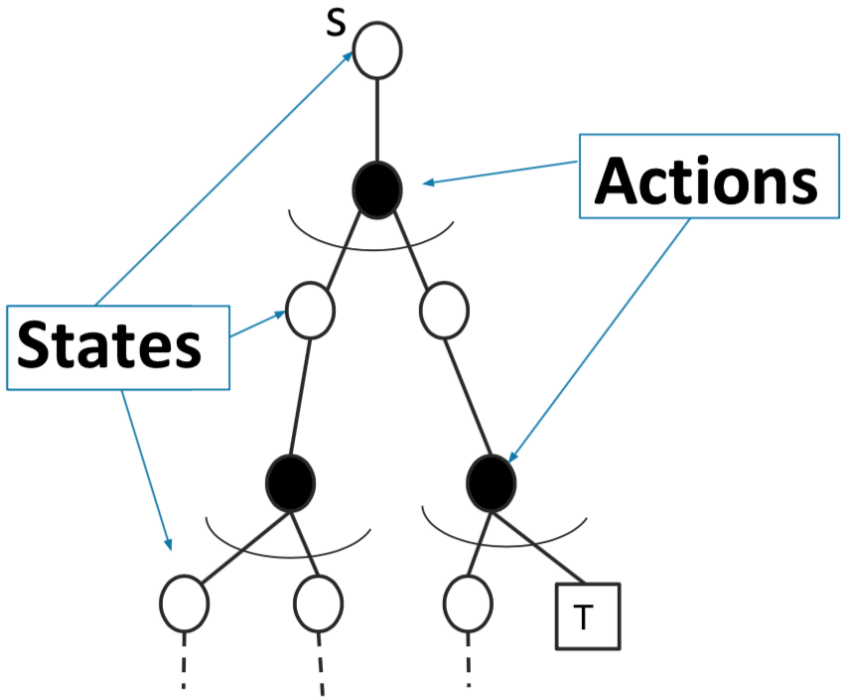
Policy Evaluation Diagram



Policy Evaluation Diagram



Policy Evaluation Diagram

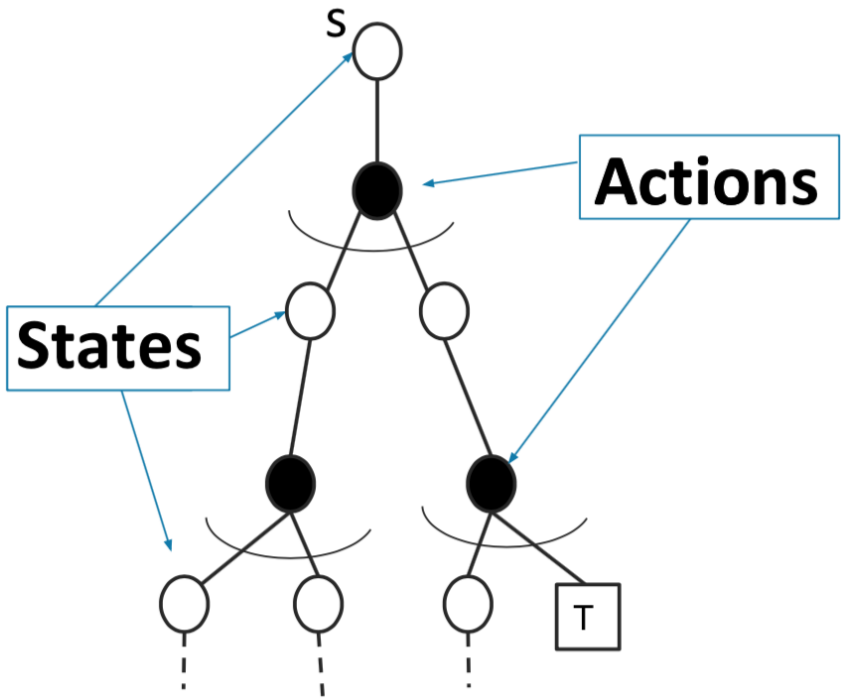


= Expectation

T = **Terminal state**

MC Policy Evaluation

$$V^{\pi}(s) = V^{\pi}(s) + \alpha(G_{i,t} - V^{\pi}(s))$$



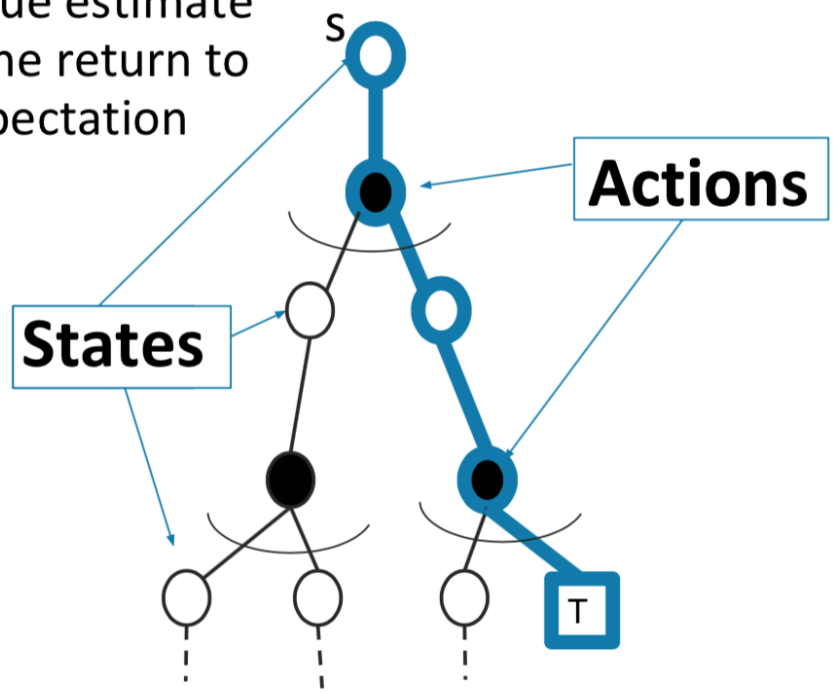
= Expectation

T = **Terminal state**

MC Policy Evaluation

$$V^{\pi}(s) = V^{\pi}(s) + \alpha(G_{i,t} - V^{\pi}(s))$$

MC updates the value estimate using a **sample** of the return to approximate an expectation



= Expectation

T = **Terminal state**

Monte Carlo (MC) Policy Evaluation Key Limitations

- Generally high variance estimator
  - Reducing variance can require a lot of data
  - In cases where data is very hard or expensive to acquire, or the stakes are high, MC may be impractical
- Requires episodic settings
  - Episode must end before data from episode can be used to update V

Monte Carlo (MC) Policy Evaluation Summary

- Aim: estimate $V^{\pi}(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ under policy π
- $V^{\pi}(s) = \mathbb{E}_{\pi}[G_t | s_t = s]$
- Simple: Estimates expectation by empirical average (given episodes sampled from policy of interest)
- Updates V estimate using **sample** of return to approximate the expectation
- Does not assume Markov process
- Converges to true value under some (generally mild) assumptions

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
- **Temporal Difference (TD)**
- Certainty Equivalence with dynamic programming
- Metrics to evaluate and compare algorithms

Temporal Difference Learning

- "If one had to identify one idea as central and novel to reinforcement learning, it would undoubtedly be temporal-difference (TD) learning." – Sutton and Barto 2017
- Combination of Monte Carlo & dynamic programming methods
- Model-free
- Can be used in episodic or infinite-horizon non-episodic settings
- Immediately updates estimate of V after each (s, a, r, s') tuple

Temporal Difference Learning for Estimating V

- Aim: estimate $V^\pi(s)$ given episodes generated under policy π
- $G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \gamma^3 r_{t+3} + \cdots$ in MDP M under policy π
- $V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s]$
- Recall Bellman operator (if know MDP models)

$$B^\pi V(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V(s')$$

- In incremental every-visit MC, update estimate using 1 sample of return (for the current ith episode)

$$V^\pi(s) = V^\pi(s) + \alpha(G_{i,t} - V^\pi(s))$$

- Insight: have an estimate of V^π, use to estimate expected return

$$V^\pi(s) = V^\pi(s) + \alpha([r_t + \gamma V^\pi(s_{t+1})] - V^\pi(s))$$

Temporal Difference [$TD(0)$] Learning

- Aim: estimate $V^{\pi}(s)$ given episodes generated under policy π
  - $s_1, a_1, r_1, s_2, a_2, r_2, \ldots$ where the actions are sampled from π
- Simplest TD learning: update value towards estimated value

$$V^{\pi}(s_t) = V^{\pi}(s_t) + \alpha(\underbrace{[r_t + \gamma V^{\pi}(s_{t+1})]}_{\text{TD target}} - V^{\pi}(s_t))$$

- TD error:

$$\delta_t = r_t + \gamma V^{\pi}(s_{t+1}) - V^{\pi}(s_t)$$

- Can immediately update value estimate after (s, a, r, s') tuple
- Don't need episodic setting

Temporal Difference $[TD(0)]$ Learning Algorithm

Input: α

Initialize $V^{\pi}(s) = 0, \forall s \in S$

Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^{\pi}(s_t) = V^{\pi}(s_t) + \alpha([\underbrace{r_t + \gamma V^{\pi}(s_{t+1})}_{\text{TD target}}] - V^{\pi}(s_t))$

Compute new V^π at the end of 1 trajectory

Input: α
Initialize $V^\pi(s) = 0, \forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha([\underbrace{r_t + \gamma V^\pi(s_{t+1})}_{\text{TD target}}] - V^\pi(s_t))$

Example Mars rover: R = [1 0 0 0 0 0 +10] for any action

- $\pi(s) = a_1 \ \forall s, \ \gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)

Worked Example TD Learning

Input: α
Initialize $V^\pi(s) = 0, \forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Example:

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1 \ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- TD estimate of all states (init at 0) with $\alpha = 1$?
  V = [1 0 0 0 0 0 0 0]

- First visit MC estimate of V of each state? [1 1 1 0 0 0 0]

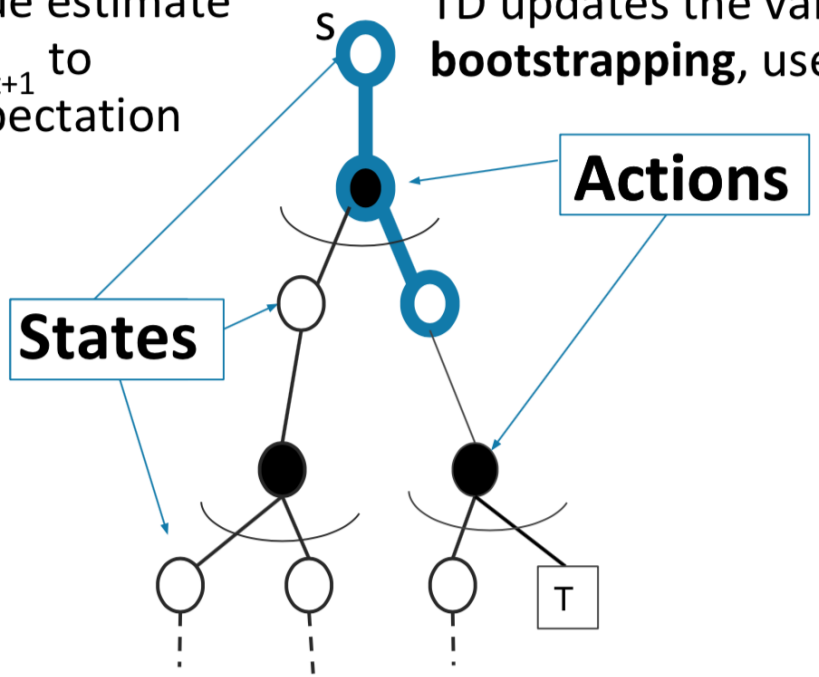
Temporal Difference (TD) Policy Evaluation

$$V^{\pi}(s_t) = r(s_t, \pi(s_t)) + \gamma \sum_{s_{t+1}} P(s_{t+1}|s_t, \pi(s_t)) V^{\pi}(s_{t+1})$$

$$V^{\pi}(s_t) = V^{\pi}(s_t) + \alpha([r_t + \gamma V^{\pi}(s_{t+1})] - V^{\pi}(s_t))$$

TD updates the value estimate using a **sample** of s_{t+1} to approximate an expectation

TD updates the value estimate by **bootstrapping**, uses estimate of $V(s_{t+1})$



= Expectation

Check Your Understanding: Piazza Poll Temporal Difference [$TD(0)$] Learning Algorithm

Input: α
Initialize $V^\pi(s) = 0, \forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha([\underbrace{r_t + \gamma V^\pi(s_{t+1})}_{\text{TD target}}] - V^\pi(s_t))$

Select all that are true

1. If $\alpha = 0$ TD will weigh the TD target more than the past V estimate
2. If $\alpha = 1$ TD will update the V estimate to the TD target
3. If $\alpha = 1$ TD in MDPs where the policy goes through states with multiple possible next states, V may oscillate forever
4. There exist deterministic MDPs where $\alpha = 1$ TD will converge

Check Your Understanding: Piazza Poll Temporal Difference [$TD(0)$] Learning Algorithm

Input: α
Initialize $V^\pi(s) = 0, \forall s \in S$
Loop

- Sample **tuple** (s_t, a_t, r_t, s_{t+1})
- $V^\pi(s_t) = V^\pi(s_t) + \alpha(\underbrace{[r_t + \gamma V^\pi(s_{t+1})]}_{\text{TD target}} - V^\pi(s_t))$

Answers. If $\alpha = 1$ TD will update to the TD target. If $\alpha = 1$ TD in MDPs where the policy goes through states with multiple possible next states, V may oscillate forever. There exist deterministic MDPs where $\alpha = 1$ TD will converge.

Summary: Temporal Difference Learning

- Combination of Monte Carlo & dynamic programming methods
- Model-free
- **Bootstraps and samples**
- Can be used in episodic or infinite-horizon non-episodic settings
- Immediately updates estimate of V after each (s, a, r, s') tuple

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
- Temporal Difference (TD)
- Certainty Equivalence with dynamic programming
- Metrics to evaluate and compare algorithms

Recall: Dynamic Programming for Policy Evaluation

- If we knew dynamics and reward model, we can do policy evaluation
- Initialize $V_0^\pi(s) = 0$ for all s
- For $k = 1$ until convergence
  - For all s in S

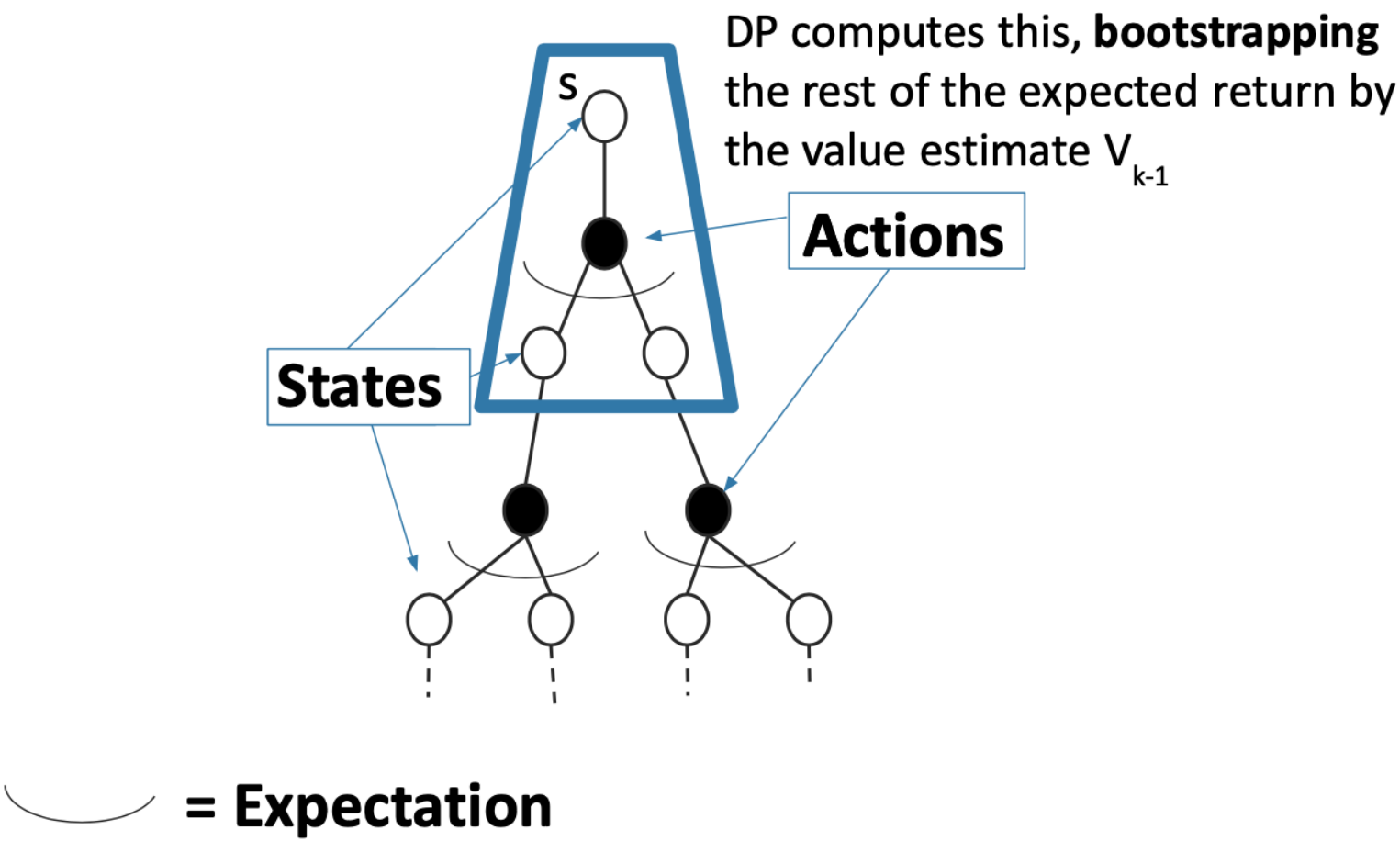
$$V_k^\pi(s) = r(s, \pi(s)) + \gamma \sum_{s' \in S} p(s'|s, \pi(s)) V_{k-1}^\pi(s')$$

- $V_k^\pi(s)$ is exactly the k-horizon value of state s under policy π
- $V_k^\pi(s)$ is an **estimate of the infinite horizon** value of state s under policy π

$$V^\pi(s) = \mathbb{E}_\pi[G_t | s_t = s] \approx \mathbb{E}_\pi[r_t + \gamma V_{k-1} | s_t = s]$$

Dynamic Programming Policy Evaluation

$$V^{\pi}(s) \leftarrow \mathbb{E}_{\pi}[r_t + \gamma V_{k-1} | s_t = s]$$



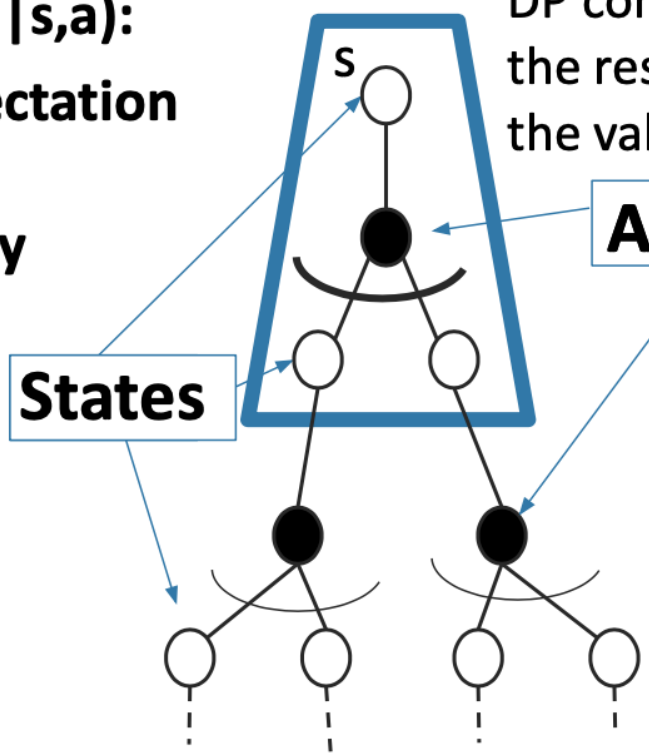
- Bootstrapping: Update for V uses an estimate

Dynamic Programming Policy Evaluation

$$V^{\pi}(s) \leftarrow \mathbb{E}_{\pi}[r_t + \gamma V_{k-1} | s_t = s]$$

Know model P(s'|s,a):

reward and expectation over next states computed exactly



DP computes this, bootstrapping the rest of the expected return by the value estimate V_{k-1}

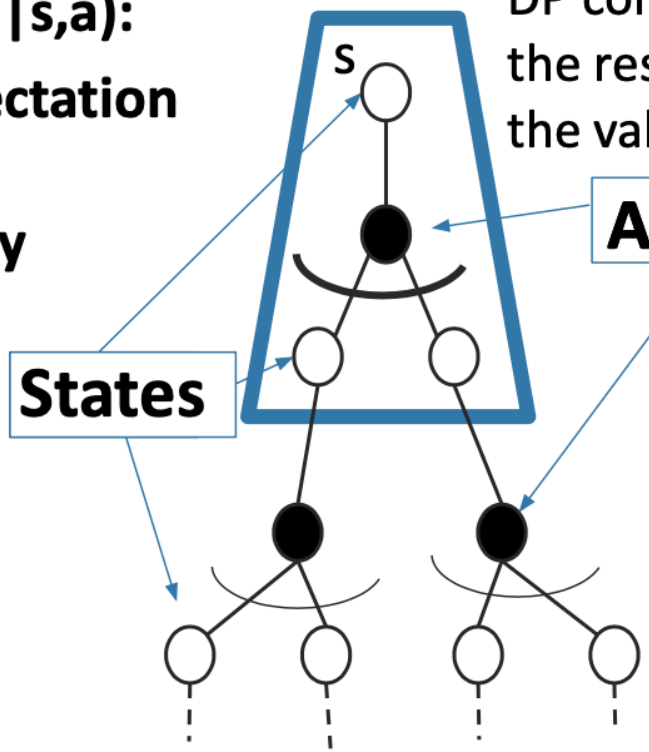
= Expectation

- Bootstrapping: Update for V uses an estimate

What about when we don't know the models?

Know model P(s'|s,a):

reward and expectation over next states computed exactly



DP computes this, bootstrapping the rest of the expected return by the value estimate V_{k-1}

= Expectation

Alternative: Certainty Equivalence V^π MLE MDP Model Estimates

- Model-based option for policy evaluation without true models
- After each (s, a, r, s') tuple
  - Recompute maximum likelihood MDP model for (s, a)

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute V^π using MLE MDP [1] (using any method from lecture 2))

[1] Requires initializing for all (s, a) pairs

s_1	s_2	s_3	s_4	s_5	s_6	s_7
$R(s_1) = +1$ *Okay Field Site*	$R(s_2) = 0$	$R(s_3) = 0$	$R(s_4) = 0$	$R(s_5) = 0$	$R(s_6) = 0$	$R(s_7) = +10$ *Fantastic Field Site*

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1 \ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of V of each state? [1 1 1 0 0 0 0]
- TD estimate of all states (init at 0) with $\alpha = 1$ is [1 0 0 0 0 0 0]
- What is the certainty equivalent estimate?
- $\hat{r} = [1\ 0\ 0\ 0\ 0\ 0\ 0]$, $\hat{p}(terminate|s_1, a_1) = \hat{p}(s_2|s_3, a_1) = 1$
- $\hat{p}(s_1|s_2, a_1) = .5$, $\hat{p}(s_2|s_2, a_1) = .5$, V = [1 1 1 0 0 0 0]

Alternative: Certainty Equivalence V^π MLE MDP Model Estimates

- Model-based option for policy evaluation without true models
- After each (s, a, r, s') tuple
  - Recompute maximum likelihood MDP model for (s, a)

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} 1(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} 1(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute V^π using MLE MDP
- Cost: Updating MLE model and MDP eval at each update ($O(|S|^3)$ for analytic matrix solution, $O(|S|^2|A|)$ for iterative methods)
- Very data efficient and very computationally expensive
- Consistent (will converge to right estimate for Markov models)
- Can also easily be used for off-policy evaluation

This Lecture: Policy Evaluation

- Estimating the expected return of a particular policy if don't have access to true MDP models
- Monte Carlo policy evaluation
  - Policy evaluation when don't have a model of how the world work
    - Given on-policy samples
- Temporal Difference (TD)
- Certainty Equivalence with dynamic programming
- **Metrics to evaluate and compare algorithms**

Check Your Understanding: Properties of Algorithms for Evaluation.

	DPCE	MC	TD
Can use w/out access to true MDP models			
Usable in continuing (non-episodic) setting			
Assumes Markov process			
Converges to true value in limit[2]			
Unbiased estimate of value			

- DPCE = Dynamic Programming w/certainty equivalence estimates, MC = Monte Carlo, TD = Temporal Difference

[2] For tabular representations of value function. More on this in later lectures

Check Your Understanding: Properties of Algorithms for Evaluation.

	DPCE	MC	TD
Can use w/out access to true MDP models	X	X	X
Usable in continuing (non-episodic) setting	X		X
Assumes Markov process	X		X
Converges to true value in limit[3]	X	X	X
Unbiased estimate of value		X	

- DPCE = Dynamic Programming w/certainty equivalence estimates, MC = Monte Carlo, TD = Temporal Difference

[3] For tabular representations of value function. More on this in later lectures

Some Important Properties to Evaluate Model-free Policy Evaluation Algorithms

- Bias/variance characteristics
- Data efficiency
- Computational efficiency
- Mostly focus on comparing MC and TD methods but we will connect back to dynamic programming with certainty equivalence methods later

Bias/Variance of Model-free Policy Evaluation Algorithms

- Return G_t is an unbiased estimate of $V^{\pi}(s_t)$
- TD target $[r_t + \gamma V^{\pi}(s_{t+1})]$ is a biased estimate of $V^{\pi}(s_t)$
- But often much lower variance than a single return G_t
- Return function of multi-step sequence of random actions, states & rewards
- TD target only has one random action, reward and next state
- MC
  - Unbiased (for first visit)
  - High variance
  - Consistent (converges to true) even with function approximation
- TD
  - Some bias
  - Lower variance
  - TD(0) converges to true value with tabular representation
  - TD(0) does not always converge with function approximation

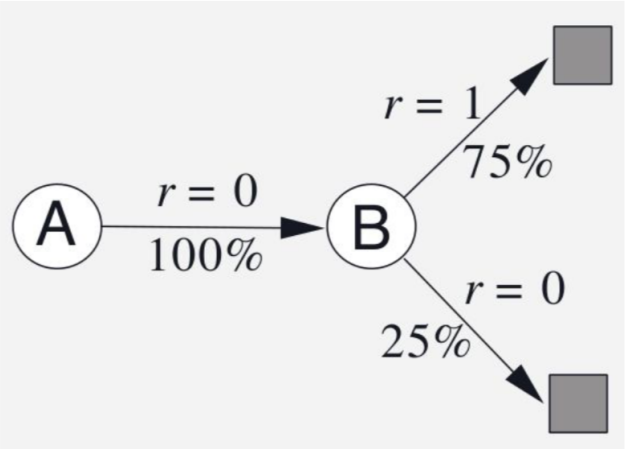
s_1	s_2	s_3	s_4	s_5	s_6	s_7
$R(s_1) = +1$ *Okay Field Site*	$R(s_2) = 0$	$R(s_3) = 0$	$R(s_4) = 0$	$R(s_5) = 0$	$R(s_6) = 0$	$R(s_7) = +10$ *Fantastic Field Site*

- Mars rover: R = [1 0 0 0 0 0 +10] for any action
- $\pi(s) = a_1 \ \forall s$, $\gamma = 1$. any action from s_1 and s_7 terminates episode
- Trajectory = (s_3, a_1, 0, s_2, a_1, 0, s_2, a_1, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of V of each state? [1 1 1 0 0 0 0]
- TD estimate of all states (init at 0) with $\alpha = 1$ is [1 0 0 0 0 0 0]
- TD(0) only uses a data point (s, a, r, s') once
- Monte Carlo takes entire return from s to end of episode

Batch MC and TD

- Batch (Offline) solution for finite dataset
  - Given set of K episodes
  - Repeatedly sample an episode from K
  - Apply MC or TD(0) to the sampled episode
- What do MC and TD(0) converge to?

AB Example: (Ex. 6.4, Sutton & Barto, 2018)



- Two states A, B with $\gamma = 1$
- Given 8 episodes of experience:
  - $A, 0, B, 0$
  - $B, 1$ (observed 6 times)
  - $B, 0$
- Imagine run TD updates over data infinite number of times
- $V(B) =$

AB Example: (Ex. 6.4, Sutton & Barto, 2018)

- TD Update: $V^{\pi}(s_t) = V^{\pi}(s_t) + \alpha([\underbrace{r_t + \gamma V^{\pi}(s_{t+1})}_{\text{TD target}}] - V^{\pi}(s_t))$
- Two states A, B with $\gamma = 1$
- Given 8 episodes of experience:
  - $A, 0, B, 0$
  - $B, 1$ (observed 6 times)
  - $B, 0$
- Imagine run TD updates over data infinite number of times
- $V(B) = 0.75$ by TD or MC
- What about $V(A)$?

AB Example: (Ex. 6.4, Sutton & Barto, 2018)

- TD Update: $V^{\pi}(s_t) = V^{\pi}(s_t) + \alpha([\underbrace{r_t + \gamma V^{\pi}(s_{t+1})}_{\text{TD target}}] - V^{\pi}(s_t))$
- Two states A, B with $\gamma = 1$
- Given 8 episodes of experience:
  - $A, 0, B, 0$
  - $B, 1$ (observed 6 times)
  - $B, 0$
- Imagine run TD updates over data infinite number of times
- $V(B) = 0.75$ by TD or MC
- What about $V(A)$?
  $V^{MC}(A) = 0$ $V^{TD}(A) = .75$

Batch MC and TD: Converges

- Monte Carlo in batch setting converges to min MSE (mean squared error)
  - Minimize loss with respect to observed returns
  - In AB example, $V(A) = 0$
- TD(0) converges to DP policy V^{π} for the MDP with the maximum likelihood model estimates
- Aka same as dynamic programming with certainty equivalence!
  - Maximum likelihood Markov decision process model

$$\hat{P}(s'|s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a, s_{k,t+1} = s')$$

$$\hat{r}(s,a) = \frac{1}{N(s,a)} \sum_{k=1}^{K} \sum_{t=1}^{L_k - 1} \mathbb{1}(s_{k,t} = s, a_{k,t} = a) r_{t,k}$$

  - Compute V^{π} using this model
  - In AB example, $V(A) = 0.75$

Some Important Properties to Evaluate Model-free Policy Evaluation Algorithms

- Data efficiency & Computational efficiency
- In simplest TD, use (s, a, r, s') once to update $V(s)$
  - $O(1)$ operation per update
  - In an episode of length L, $O(L)$
- In MC have to wait till episode finishes, then also $O(L)$
- MC can be more data efficient than simple TD
- But TD exploits Markov structure
  - If in Markov domain, leveraging this is helpful
- Dynamic programming with certainty equivalence also uses Markov structure

Summary: Policy Evaluation

Estimating the expected return of a particular policy if don't have access to true MDP models. Ex. evaluating average purchases per session of new product recommendation system

- Monte Carlo policy evaluation
  - Policy evaluation when we don't have a model of how the world works
    - Given on policy samples
    - Given off policy samples
- Temporal Difference (TD)
- Dynamic Programming with certainty equivalence
- Metrics to evaluate and compare algorithms
  - Robustness to Markov assumption
  - Bias/variance characteristics
  - Data efficiency
  - Computational efficiency

Today's Plan

- Last Time:
  - Markov reward / decision processes
  - Policy evaluation & control when have true model (of how the world works)
- Today
  - Policy evaluation without known dynamics & reward models
- Next Time:
  - Control when don't have a model of how the world works

Lecture 4: Model Free Control

Emma Brunskill

CS234 Reinforcement Learning.

- Structure closely follows much of David Silver's Lecture 5. For additional reading please see SB Sections 5.2-5.4, 6.4, 6.5, 6.7

Refresh Your Knowledge 3. Piazza Poll

- Which of the following equations express a TD update?
  1. $V(s_t) = r(s_t, a_t) + \gamma \sum_{s'} p(s'|s_t, a_t) V(s')$
  2. $V(s_t) = (1-\alpha) V(s_t) + \alpha (r(s_t, a_t) + \gamma V(s_{t+1}))$
  3. $V(s_t) = (1-\alpha) V(s_t) + \alpha \sum_{i=t}^{H} r(s_i, a_i)$
  4. $V(s_t) = (1-\alpha) V(s_t) + \alpha \max_a (r(s_t, a) + \gamma V(s_{t+1}))$
  5. Not sure
- Bootstrapping is when
  1. Samples of (s,a,s') transitions are used to approximate the true expectation over next states
  2. An estimate of the next state value is used instead of the true next state value
  3. Used in Monte-Carlo policy evaluation
  4. Not sure

Refresh Your Knowledge 3. Piazza Poll

- Which of the following equations express a TD update?
  True. $V(s_t) = (1 - \alpha)V(s_t) + \alpha(r(s_t, a_t) + \gamma V(s_{t+1}))$

- Bootstrapping is when
  An estimate of the next state value is used instead of the true next state value

Table of Contents

Class Structure

- Last time: Policy evaluation with no knowledge of how the world works (MDP model not given)
- This time: Control (making decisions) without a model of how the world works
- Next time: Generalization – Value function approximation

Evaluation to Control

- Last time: how good is a specific policy?
  - Given no access to the decision process model parameters
  - Instead have to estimate from data / experience
- Today: how can we learn a good policy?

Recall: Reinforcement Learning Involves

- Optimization
- Delayed consequences
- Exploration
- Generalization

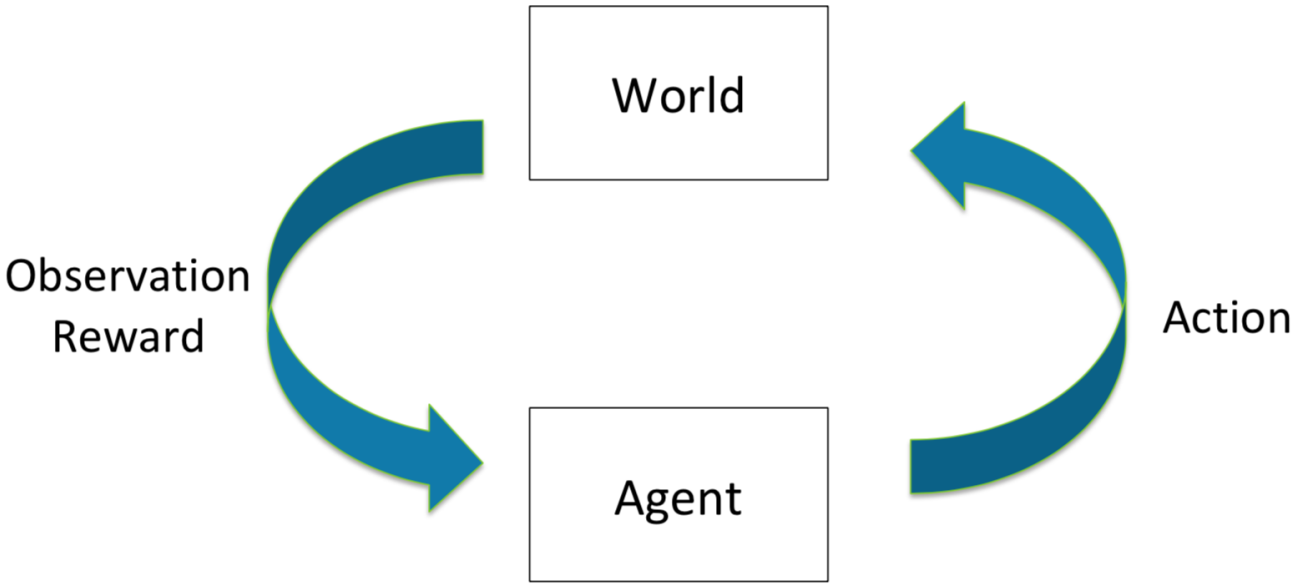
Today: Learning to Control Involves

- Optimization: Goal is to identify a policy with high expected rewards (similar to Lecture 2 on computing an optimal policy given decision process models)
- Delayed consequences: May take many time steps to evaluate whether an earlier decision was good or not
- Exploration: Necessary to try different actions to learn what actions can lead to high rewards

Today: Model-free Control

- Generalized policy improvement
- Importance of exploration
- Monte Carlo control
- Model-free control with temporal difference (SARSA, Q-learning)
- Maximization bias

Reinforcement Learning



- Goal: Learn to select actions to maximize total expected future reward

Model-free Control Examples

- Many applications can be modeled as a MDP: Backgammon, Go, Robot locomation, Helicopter flight, Robocup soccer, Autonomous driving, Customer ad selection, Invasive species management, Patient treatment
- For many of these and other problems either:
  - MDP model is unknown but can be sampled
  - MDP model is known but it is computationally infeasible to use directly, except through sampling

On and Off-Policy Learning

- On-policy learning
  - Direct experience
  - Learn to estimate and evaluate a policy from experience obtained from following that policy
- Off-policy learning
  - Learn to estimate and evaluate a policy using experience gathered from following a different policy

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Recall Policy Iteration

- Initialize policy π
- Repeat:
  - Policy evaluation: compute V^{π}
  - Policy improvement: update π

$$\pi'(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V^{\pi}(s') = \arg\max_a Q^{\pi}(s,a)$$

- Now want to do the above two steps without access to the true dynamics and reward models
- Last lecture introduced methods for model-free policy evaluation

Model Free Policy Iteration

- Initialize policy π
- Repeat:
  - Policy evaluation: compute Q^{π}
  - Policy improvement: update π

MC for On Policy Q Evaluation

Initialize $N(s,a) = 0,\ G(s,a) = 0,\ Q^{\pi}(s,a) = 0,\ \forall s \in S,\ \forall a \in A$
Loop

- Using policy π sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
- For each **state,action** (s,a) visited in episode i
  - For **first or every** time t that (s,a) is visited in episode i
    - $N(s,a) = N(s,a) + 1,\ G(s,a) = G(s,a) + G_{i,t}$
    - Update estimate $Q^{\pi}(s,a) = G(s,a)/N(s,a)$

Model-free Generalized Policy Improvement

- Given an estimate $Q^{\pi_i}(s, a)\ \forall s, a$
- Update new policy

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s, a)$$

Check Your Understanding: Model-free Generalized Policy Improvement

- Given an estimate $Q^{\pi_i}(s, a)\ \forall s, a$
- Update new policy

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s, a)$$

- Question: is this π_{i+1} deterministic or stochastic?
- Answer: Deterministic, Stochastic, Not Sure
- Recall in model-free policy evaluation, we estimated V^π by using π to generate new trajectories
- Question: Can we compute $Q^{\pi_{i+1}}(s, a)\ \forall s, a$ by using this π_{i+1} to generate new trajectories?
- Answer: True, False, Not Sure

Check Your Understanding: Model-free Generalized Policy Improvement

- Given an estimate $Q^{\pi_i}(s, a)\ \forall s, a$
- Update new policy

$$\pi_{i+1}(s) = \arg\max_a Q^{\pi_i}(s, a)$$

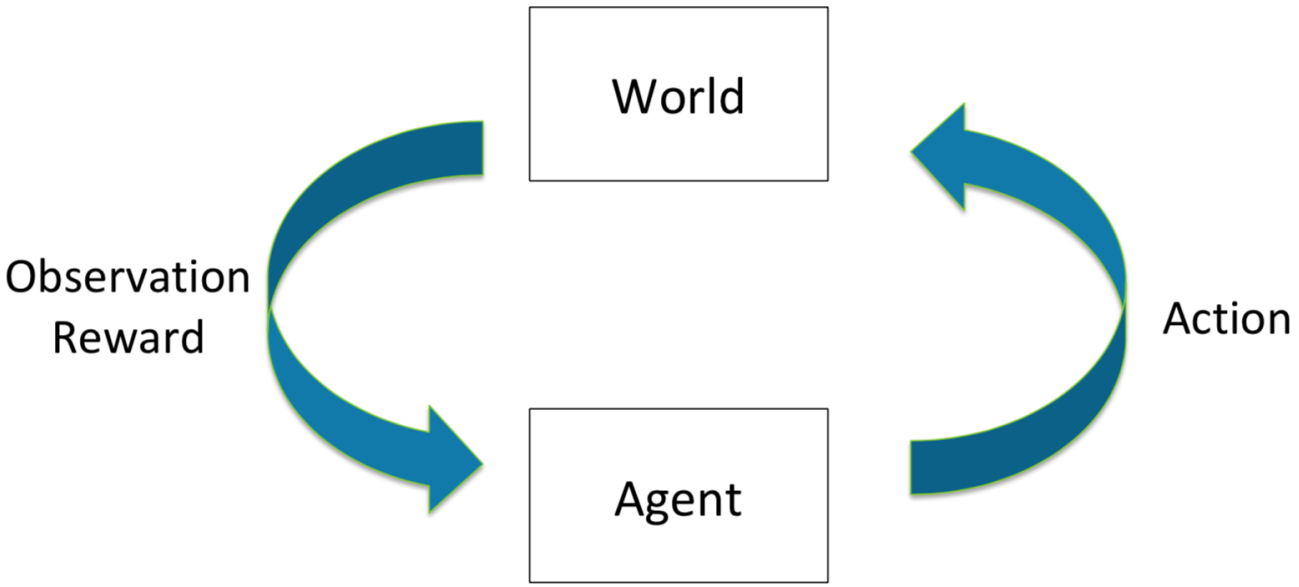
- Question: is this π_{i+1} deterministic or stochastic?
- :Answer: Deterministic
- Recall in model-free policy evaluation, we estimated V^π by using π to generate new trajectories
- Question: Can we compute $Q^{\pi_{i+1}}(s, a)\ \forall s, a$ by using this π_{i+1} to generate new trajectories?
- Answer: No.

Model-free Policy Iteration

- Initialize policy π
- Repeat:
  - Policy evaluation: compute Q^{π}
  - Policy improvement: update π given Q^{π}

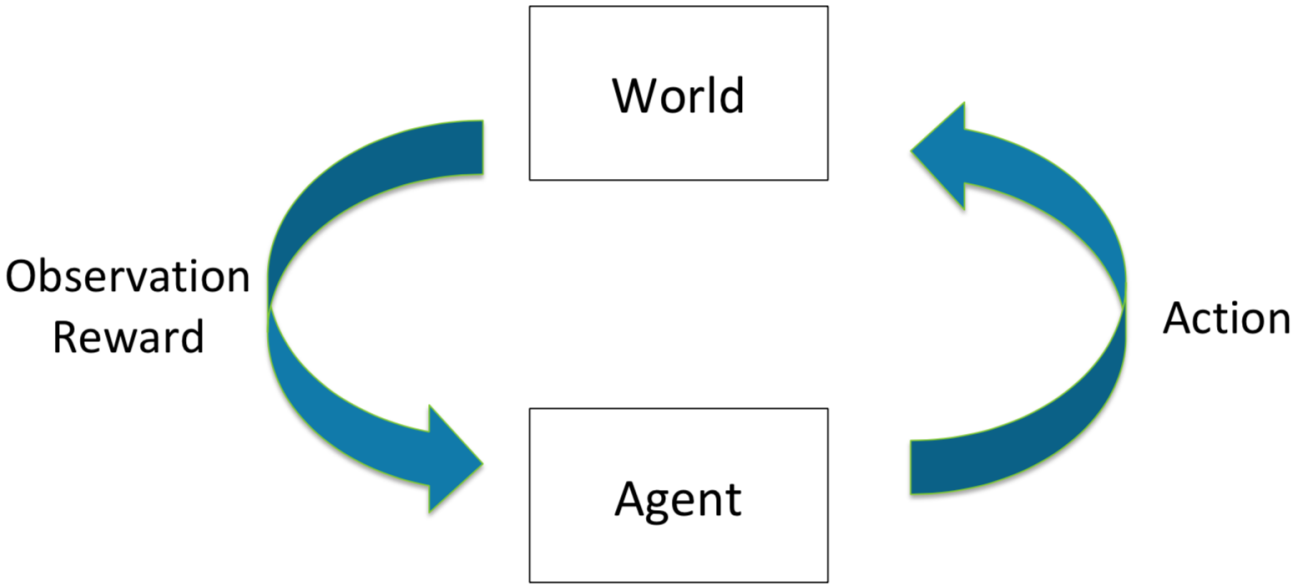
- May need to modify policy evaluation:
  - If π is deterministic, can't compute $Q(s, a)$ for any $a \neq \pi(s)$
- How to interleave policy evaluation and improvement?
  - Policy improvement is now using an estimated Q

The Problem of Exploration



- Goal: Learn to select actions to maximize total expected future reward
- Problem: Can't learn about actions without trying them
- Problem: But if we try new actions, spending less time taking actions that our past experience suggests will yield high reward

Explore vs Exploit



- Explore: take actions haven't tried much in a state
- Exploit: take actions estimate will yield high discounted expected reward
- We will discuss this much more later in the course

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Policy Evaluation with Exploration

- Want to compute a model-free estimate of Q^π
- In general seems subtle
  - Need to try all (s, a) pairs but then follow π
  - Want to ensure resulting estimate Q^π is good enough so that policy improvement is a monotonic operator
- For certain classes of policies can ensure all (s,a) pairs are tried such that asymptotically Q^π converges to the true value

ϵ-greedy Policies

- Simple idea to balance exploration and exploitation
- Let $|A|$ be the number of actions
- Then an ϵ-greedy policy w.r.t. a state-action value $Q(s, a)$ is $\pi(a|s) =$

ϵ-greedy Policies

- Simple idea to balance exploration and exploitation
- Let $|A|$ be the number of actions
- Then an ϵ-greedy policy w.r.t. a state-action value $Q(s, a)$ is $\pi(a|s) =$
  - $\arg\max_a Q(s, a)$, w. prob $1 - \epsilon + \frac{\epsilon}{|A|}$
  - $a' \neq \arg\max Q(s, a)$ w. prob $\frac{\epsilon}{|A|}$

For Later Practice: MC for On Policy Q Evaluation

Initialize $N(s,a) = 0,\ G(s,a) = 0,\ Q^{\pi}(s,a) = 0,\ \forall s \in S,\ \forall a \in A$
Loop

- Using policy π sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$
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  - For **first or every** time t that (s,a) is visited in episode i
    - $N(s,a) = N(s,a) + 1,\ G(s,a) = G(s,a) + G_{i,t}$
    - Update estimate $Q^{\pi}(s,a) = G(s,a)/N(s,a)$
- Mars rover with new actions:
  - $r(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10],\ r(-, a_2) = [\ 0\ 0\ 0\ 0\ 0\ 0\ +5],\ \gamma = 1.$
- Assume current greedy $\pi(s) = a_1\ \forall s,\ \epsilon$=.5
- Sample trajectory from ϵ-greedy policy
- Trajectory = (s_3, a_1, 0, s_2, a_2, 0, s_3, a_1, 0, s_2, a_2, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of Q of each (s,a) pair?
- $Q^{\epsilon-\pi}(-, a_1) = [1\ 0\ 1\ 0\ 0\ 0\ 0],\ Q^{\epsilon-\pi}(-, a_2) = [0\ 1\ 0\ 0\ 0\ 0\ 0]$

Monotonic ϵ-greedy Policy Improvement

Theorem

For any ϵ-greedy policy π_i, the ϵ-greedy policy w.r.t. Q^{π_i}, π_{i+1} is a monotonic improvement $V^{\pi_{i+1}} \geq V^{\pi_i}$

$$
\begin{aligned}
Q^{\pi_i}(s, \pi_{i+1}(s)) &= \sum_{a \in A} \pi_{i+1}(a|s) Q^{\pi_i}(s, a) \\
&= (\epsilon/|A|) \left[\sum_{a \in A} Q^{\pi_i}(s, a) \right] + (1 - \epsilon) \max_a Q^{\pi_i}(s, a)
\end{aligned}
$$

Monotonic ϵ-greedy Policy Improvement

Theorem

For any ϵ-greedy policy π_i, the ϵ-greedy policy w.r.t. Q^{π_i}, π_{i+1} is a monotonic improvement $V^{\pi_{i+1}} \geq V^{\pi_i}$

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&= (\epsilon/|A|) \left[\sum_{a \in A} Q^{\pi_i}(s, a) \right] + (1 - \epsilon) \max_a Q^{\pi_i}(s, a) \frac{1 - \epsilon}{1 - \epsilon} \\
&= (\epsilon/|A|) \left[\sum_{a \in A} Q^{\pi_i}(s, a) \right] + (1 - \epsilon) \max_a Q^{\pi_i}(s, a) \sum_{a \in A} \frac{\pi_i(a|s) - \frac{\epsilon}{|A|}}{1 - \epsilon} \\
&\geq \frac{\epsilon}{|A|} \left[\sum_{a \in A} Q^{\pi_i}(s, a) \right] + (1 - \epsilon) \sum_{a \in A} \frac{\pi_i(a|s) - \frac{\epsilon}{|A|}}{1 - \epsilon} Q^{\pi_i}(s, a) \\
&= \sum_{a \in A} \pi_i(a|s) Q^{\pi_i}(s, a) = V^{\pi_i}(s)
\end{aligned}
$$

- Therefore $V^{\pi_{i+1}} \geq V^{\pi}$ (from the policy improvement theorem)

ϵ-greedy

- Monotonic ϵ-greedy policy improvement theorem is a sanity check about using ϵ-greedy policies for policy iteration
- But note the assumption when we learned about policy iteration was that policy evaluation was done **exactly**
- In the last lecture we learned about doing policy evaluation without access to the true dynamics and reward models
- MC and TD yielded estimates of V^{π}
- Should not yet be clear that we can ensure monotonic improvement or convergence given such estimates...

Today: Model-free Control

- Generalized policy improvement
- Importance of exploration
- **Monte Carlo control**
- Model-free control with temporal difference (SARSA, Q-learning)
- Maximization bias

Recall Monte Carlo Policy Evaluation, **Now for Q**

1: Initialize $Q(s,a) = 0, N(s,a) = 0\ \forall (s,a),\ k = 1$, Input $\epsilon = 1,\ \pi$
2: **loop**
3: Sample k-th episode $(s_{k,1}, a_{k,1}, r_{k,1}, s_{k,2}, \ldots, s_{k,T})$ given π
3: Compute $G_{k,t} = r_{k,t} + \gamma r_{k,t+1} + \gamma^2 r_{k,t+2} + \cdots \gamma^{T_i - 1} r_{k,T_i}\ \forall t$
4: **for** $t = 1, \ldots, T$ **do**
5: **if** First visit to **(s,a)** in episode k **then**
6: $N(s,a) = N(s,a) + 1$
7: $Q(s_t, a_t) = Q(s_t, a_t) + \frac{1}{N(s,a)}(G_{k,t} - Q(s_t, a_t))$
8: **end if**
9: **end for**
10: $k = k + 1$
11: **end loop**

Monte Carlo Online Control / On Policy Improvement

1: Initialize $Q(s,a) = 0, N(s,a) = 0\ \forall (s,a)$, Set $\epsilon = 1$, $k = 1$
2: $\pi_k = \epsilon$-greedy(Q) // Create initial ϵ-greedy policy
3: **loop**
4: Sample k-th episode $(s_{k,1}, a_{k,1}, r_{k,1}, s_{k,2}, \ldots, s_{k,T})$ given π_k
4: $G_{k,t} = r_{k,t} + \gamma r_{k,t+1} + \gamma^2 r_{k,t+2} + \cdots \gamma^{T_i - 1} r_{k,T_i}$
5: **for** $t = 1, \ldots, T$ **do**
6: **if** First visit to (s,a) in episode k **then**
7: $N(s,a) = N(s,a) + 1$
8: $Q(s_t, a_t) = Q(s_t, a_t) + \frac{1}{N(s,a)}(G_{k,t} - Q(s_t, a_t))$
9: **end if**
10: **end for**
11: $k = k + 1$, $\epsilon = 1/k$
12: $\pi_k = \epsilon$-greedy(Q) // Policy improvement
13: **end loop**

Poll. Check Your Understanding: MC for On Policy Control

- Mars rover with new actions:
  - $r(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $r(-, a_2) = [\ 0\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$.
- Assume current greedy $\pi(s) = a_1\ \forall s$, ϵ=.5. $Q(s, a) = 0$ for all (s, a)
- Sample trajectory from ϵ-greedy policy
- Trajectory = (s_3, a_1, 0, s_2, a_2, 0, s_3, a_1, 0, s_2, a_2, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of Q of each (s, a) pair?
- $Q^{\epsilon-\pi}(-, a_1) = [1\ 0\ 1\ 0\ 0\ 0\ 0]$

After this trajectory (Select all)

- $Q^{\epsilon-\pi}(-, a_2) = [0\ 0\ 0\ 0\ 0\ 0\ 0]$
- The new **greedy** policy would be: $\pi = [1\ \text{tie}\ 1\ \text{tie tie tie tie}]$
- The new **greedy** policy would be: $\pi = [1\ 2\ 1\ \text{tie tie tie tie}]$
- If $\epsilon = 1/3$, prob of selecting a_1 in s_1 in the new ϵ-greedy policy is $1/9$.
- If $\epsilon = 1/3$, prob of selecting a_1 in s_1 in the new ϵ-greedy policy is $2/3$.
- If $\epsilon = 1/3$, prob of selecting a_1 in s_1 in the new ϵ-greedy policy is $5/6$.
- Not sure

Check Your Understanding: MC for On Policy Control

- Mars rover with new actions:
  - $r(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $r(-, a_2) = [\ 0\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$.
- Assume current greedy $\pi(s) = a_1\ \forall s$, ϵ=.5
- Sample trajectory from ϵ-greedy policy
- Trajectory = (s_3, a_1, 0, s_2, a_2, 0, s_3, a_1, 0, s_2, a_2, 0, s_1, a_1, 1, terminal)
- First visit MC estimate of Q of each (s, a) pair?
- $Q^{\epsilon-\pi}(-, a_1) = [1\ 0\ 1\ 0\ 0\ 0\ 0]$, $Q^{\epsilon-\pi}(-, a_2) = [0\ 1\ 0\ 0\ 0\ 0\ 0]$
- What is $\pi(s) = \arg\max_a Q^{\epsilon-\pi}(s, a)\ \forall s$?
  $\pi = [1\ 2\ 1$ tie tie tie tie$]$

- Under the new ϵ-greedy policy, if $k = 3$, $\epsilon = 1/k$
  With probability $2/3$ choose $\pi(s)$ else choose randomly. As an example, $\pi(s_1) = a_1$ with prob $(2/3)$ else randomly choose an action. So the prob of picking a_1 will be $2/3 + (1/3) * (1/2) = 5/6$

MC control with ϵ-greedy policies

- Is the prior algorithm a reasonable one?
- Will it eventually converge to the optimal Q^* function?
- Now see a set of conditions that is sufficient to ensure this desirable outcome

Greedy in the Limit of Infinite Exploration (GLIE)

Definition of GLIE

- All state-action pairs are visited an infinite number of times

$$\lim_{i \to \infty} N_i(s, a) \to \infty$$

- Behavior policy (policy used to act in the world) converges to greedy policy
  $\lim_{i \to \infty} \pi(a|s) \to \arg\max_a Q(s, a)$ with probability 1

Greedy in the Limit of Infinite Exploration (GLIE)

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- Behavior policy (policy used to act in the world) converges to greedy policy
  $\lim_{i \to \infty} \pi(a|s) \to \arg\max_a Q(s, a)$ with probability 1

- A simple GLIE strategy is ϵ-greedy where ϵ is reduced to 0 with the following rate: $\epsilon_i = 1/i$

GLIE Monte-Carlo Control

Theorem

GLIE Monte-Carlo control converges to the optimal state-action value function $Q(s, a) \rightarrow Q^*(s, a)$

Model-free Policy Iteration with TD Methods

- Use temporal difference methods for policy evaluation step
- Initialize policy π
- Repeat:
  - Policy evaluation: compute Q^π using temporal difference updating with ϵ-greedy policy
  - Policy improvement: Same as Monte carlo policy improvement, set π to ϵ-greedy (Q^π)
- Important issue: is it necessary for policy evaluation to converge to the true value of Q^π before updating the policy?
- Answer: Perhaps surprizingly, no. A single update of TD (policy evaluation updating of Q) can be sufficient!

Model-free Policy Iteration with TD Methods

- Use temporal difference methods for policy evaluation step
- Initialize policy π
- Repeat:
  - Policy evaluation: compute Q^{π} using temporal difference updating with ϵ-greedy policy
  - Policy improvement: Same as Monte carlo policy improvement, set π to ϵ-greedy (Q^{π})
- First consider SARSA, which is an on-policy algorithm.
- On policy: SARSA is trying to compute an estimate Q of the policy being followed.

General Form of SARSA Algorithm

1: Set initial ϵ-greedy policy π randomly, $t = 0$, initial state $s_t = s_0$
2: Take $a_t \sim \pi(s_t)$
3: Observe (r_t, s_{t+1})
4: **loop**
5: Take action $a_{t+1} \sim \pi(s_{t+1})$ // Sample action from policy
6: Observe (r_{t+1}, s_{t+2})
7: Update Q given $(s_t, a_t, r_t, s_{t+1}, a_{t+1})$:

8: Perform policy improvement:

9: $t = t + 1$
10: **end loop**

General Form of SARSA Algorithm

1: Set initial ϵ-greedy policy π, $t = 0$, initial state $s_t = s_0$
2: Take $a_t \sim \pi(s_t)$ // Sample action from policy
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4: **loop**
5: Take action $a_{t+1} \sim \pi(s_{t+1})$
6: Observe (r_{t+1}, s_{t+2})
7: $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$
8: $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
9: $t = t + 1$
10: **end loop**

Worked Example: SARSA for Mars Rover

1: Set initial ϵ-greedy policy π, $t = 0$, initial state $s_t = s_0$
2: Take $a_t \sim \pi(s_t)$ // Sample action from policy
3: Observe (r_t, s_{t+1})
4: **loop**
5: Take action $a_{t+1} \sim \pi(s_{t+1})$
6: Observe (r_{t+1}, s_{t+2})
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8: $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
9: $t = t + 1$
10: **end loop**

- Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$, $Q(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $Q(-, a_2) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$
- Assume starting state is s_6 and sample a_1

Worked Example: SARSA for Mars Rover

1: Set initial ϵ-greedy policy π, $t = 0$, initial state $s_t = s_0$
2: Take $a_t \sim \pi(s_t)$ // Sample action from policy
3: Observe (r_t, s_{t+1})
4: **loop**
5: Take action $a_{t+1} \sim \pi(s_{t+1})$
6: Observe (r_{t+1}, s_{t+2})
7: $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$
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10: **end loop**

- Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$, $Q(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $Q(-, a_2) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$
- Tuple: $(s_6, a_1, 0, s_7, a_2, 5, s_7)$.
- $Q(s_6, a_1) = .5 * 0 + .5 * (0 + \gamma Q(s_7, a_2)) = 2.5$

SARSA Initialization

- Mars rover with new actions:
  - $r(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $r(-, a_2) = [\ 0\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$.
- Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$, $Q(-, a_1) = r(-, a_1)$, $Q(-, a_2) = r(-, a_2)$
- SARSA: $(s_6, a_1, 0, s_7, a_2, 5, s_7)$.
- Does how Q is initialized matter (initially? asymptotically?)? Asymptotically no, under mild condiditions, but at the beginning, yes

Convergence Properties of SARSA

Theorem

SARSA for finite-state and finite-action MDPs converges to the optimal action-value, $Q(s,a) \rightarrow Q^*(s,a)$, under the following conditions:

1. The policy sequence $\pi_t(a|s)$ satisfies the condition of GLIE
2. The step-sizes α_t satisfy the Robbins-Munro sequence such that

$$\sum_{t=1}^{\infty} \alpha_t = \infty$$

$$\sum_{t=1}^{\infty} \alpha_t^2 < \infty$$

- For ex. $\alpha_t = \frac{1}{t}$ satisfies the above condition.

Q-Learning: Learning the Optimal State-Action Value

- SARSA is an **on-policy** learning algorithm
- SARSA estimates the value of the current **behavior** policy (policy using to take actions in the world)
- And then updates the policy trying to estimate
- Alternatively, can we directly estimate the value of π^* while acting with another behavior policy π_b?
- Yes! Q-learning, an **off-policy** RL algorithm

On and Off-Policy Learning

- On-policy learning
  - Direct experience
  - Learn to estimate and evaluate a policy from experience obtained from following that policy
- Off-policy learning
  - Learn to estimate and evaluate a policy using experience gathered from following a different policy

Q-Learning: Learning the Optimal State-Action Value

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- SARSA estimates the value of the current **behavior** policy (policy using to take actions in the world)
- And then updates the policy trying to estimate
- Alternatively, can we directly estimate the value of π^* while acting with another behavior policy π_b?
- Yes! Q-learning, an **off-policy** RL algorithm
- Key idea: Maintain state-action Q estimates and use to bootstrap– use the value of the best future action
- Recall SARSA

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha((r_t + \gamma Q(s_{t+1}, a_{t+1})) - Q(s_t, a_t))$$

- Q-learning:

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha((r_t + \gamma \max_{a'} Q(s_{t+1}, a')) - Q(s_t, a_t))$$

Q-Learning with ϵ-greedy Exploration

1: Initialize $Q(s, a), \forall s \in S, a \in A$ $t = 0$, initial state $s_t = s_0$
2: Set π_b to be ϵ-greedy w.r.t. Q
3: **loop**
4: Take $a_t \sim \pi_b(s_t)$ // Sample action from policy
5: Observe (r_t, s_{t+1})
6: $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t))$
7: $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
8: $t = t + 1$
9: **end loop**

Worked Example: ϵ-greedy Q-Learning Mars

1: Initialize $Q(s,a), \forall s \in S, a \in A$ $t = 0$, initial state $s_t = s_0$
2: Set π_b to be ϵ-greedy w.r.t. Q
3: **loop**
4: Take $a_t \sim \pi_b(s_t)$ // Sample action from policy
5: Observe (r_t, s_{t+1})
6: $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t))$
7: $\pi(s_t) = \arg\max_a Q(s_t, a)$ w.prob $1 - \epsilon$, else random
8: $t = t + 1$
9: **end loop**

- Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$, $Q(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $Q(-, a_2) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$
- Like in SARSA example, start in s_6 and take a_1.

Worked Example: ϵ-greedy Q-Learning Mars

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- Initialize $\epsilon = 1/k$, $k = 1$, and $\alpha = 0.5$, $Q(-, a_1) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +10]$, $Q(-, a_2) = [\ 1\ 0\ 0\ 0\ 0\ 0\ +5]$, $\gamma = 1$
- Tuple: $(s_6, a_1, 0, s_7)$.
- $Q(s_6, a_1) = 0 + .5 * (0 + \gamma \max_{a'} Q(s_7, a') - 0) = .5*10 = 5$
- Recall that in the SARSA update we saw $Q(s_6, a_1) = 2.5$ because we used the actual action taken at s_7 instead of the max
- Does how Q is initialized matter (initially? asymptotically?)? Asymptotically no, under mild condiditions, but at the beginning, yes

Check Your Understanding: SARSA and Q-Learning

- SARSA: $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t))$
- Q-Learning:
  $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r_t + \gamma \max_{a'} Q(s_{t+1}, a') - Q(s_t, a_t))$

Select all that are true

1. Both SARSA and Q-learning may update their policy after every step
2. If $\epsilon = 0$ for all time steps, and Q is initialized randomly, a SARSA Q state update will be the same as a Q-learning Q state update
3. Not sure

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2. If $\epsilon = 0$ for all time steps, and Q is initialized randomly, a SARSA Q state update will be the same as a Q-learning Q state update
3. Not sure

Both are true.

Q-Learning with ϵ-greedy Exploration

- What conditions are sufficient to ensure that Q-learning with ϵ-greedy exploration converges to optimal Q^*?
  Visit all (s, a) pairs infinitely often, and the step-sizes α_t satisfy the Robbins-Munro sequence. Note: the algorithm does not have to be greedy in the limit of infinite exploration (GLIE) to satisfy this (could keep ϵ large).

- What conditions are sufficient to ensure that Q-learning with ϵ-greedy exploration converges to optimal π^*?
  The algorithm is GLIE, along with the above requirement to ensure the Q value estimates converge to the optimal Q.

Maximization Bias[1]

- Consider single-state MDP ($|S| = 1$) with 2 actions, and both actions have 0-mean **random** rewards, ($\mathbb{E}(r|a = a_1) = \mathbb{E}(r|a = a_2) = 0$).
- Then $Q(s, a_1) = Q(s, a_2) = 0 = V(s)$
- Assume there are prior samples of taking action a_1 and a_2
- Let $\hat{Q}(s, a_1), \hat{Q}(s, a_2)$ be the finite sample estimate of Q
- Use an unbiased estimator for Q: e.g. $\hat{Q}(s, a_1) = \frac{1}{n(s,a_1)} \sum_{i=1}^{n(s,a_1)} r_i(s, a_1)$
- Let $\hat{\pi} = \arg\max_a \hat{Q}(s, a)$ be the greedy policy w.r.t. the estimated $\hat{Q}$

[1] Example from Mannor, Simester, Sun and Tsitsiklis. Bias and Variance Approximation in Value Function Estimates. Management Science 2007

Maximization Bias[2] Proof

- Consider single-state MDP ($|S| = 1$) with 2 actions, and both actions have 0-mean random rewards, ($\mathbb{E}(r|a = a_1) = \mathbb{E}(r|a = a_2) = 0$).
- Then $Q(s, a_1) = Q(s, a_2) = 0 = V(s)$
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- Let $\hat{Q}(s, a_1), \hat{Q}(s, a_2)$ be the finite sample estimate of Q
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- Let $\hat{\pi} = \arg\max_a \hat{Q}(s, a)$ be the greedy policy w.r.t. the estimated $\hat{Q}$
- *Even though each estimate of the state-action values is unbiased*, the estimate of $\hat{\pi}$'s value $\hat{V}^{\hat{\pi}}$ can be biased:
  $\hat{V}^{\hat{\pi}}(s) = \mathbb{E}[\max \hat{Q}(s, a_1), \hat{Q}(s, a_2)]$
  $\geq \max[\mathbb{E}[\hat{Q}(s, a_1)], [\hat{Q}(s, a_2)]]$
  $= max[0, 0] = V^{\pi}$,
  where the inequality comes from Jensen's inequality.

[2] Example from Mannor, Simester, Sun and Tsitsiklis. Bias and Variance Approximation in Value Function Estimates. Management Science 2007

Double Q-Learning

- The greedy policy w.r.t. estimated Q values can yield a maximization bias during finite-sample learning
- Avoid using max of estimates as estimate of max of true values
- Instead split samples and use to create two independent unbiased estimates of $Q_1(s_1, a_i)$ and $Q_2(s_1, a_i)$ $\forall a$.
  - Use one estimate to select max action: $a^* = \arg\max_a Q_1(s_1, a)$
  - Use other estimate to estimate value of a^*: $Q_2(s, a^*)$
  - Yields unbiased estimate: $\mathbb{E}(Q_2(s, a^*)) = Q(s, a^*)$

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  - Yields unbiased estimate: $\mathbb{E}(Q_2(s, a^*)) = Q(s, a^*)$
- Why does this yield an unbiased estimate of the max state-action value?
  Using independent samples to estimate the value
- If acting online, can alternate samples used to update Q_1 and Q_2, using the other to select the action chosen
- Next slides extend to full MDP case (with more than 1 state)

Double Q-Learning

1: Initialize $Q_1(s,a)$ and $Q_2(s,a), \forall s \in S, a \in A$ $t=0$, initial state $s_t = s_0$
2: **loop**
3: Select a_t using ϵ-greedy $\pi(s) = \arg\max_a Q_1(s_t,a) + Q_2(s_t,a)$
4: Observe (r_t, s_{t+1})
5: **if** (with 0.5 probability) **then**
6: $Q_1(s_t,a_t) \leftarrow Q_1(s_t,a_t) + \alpha(r_t + \gamma Q_2(s_{t+1}, \arg\max_a Q_1(s_{t+1},a)) - Q_1(s_t,a_t))$
7: **else**
8: $Q_2(s_t,a_t) \leftarrow Q_2(s_t,a_t) + \alpha(r_t + \gamma Q_1(s_{t+1}, \arg\max_a Q_2(s_{t+1},a)) - Q_2(s_t,a_t))$
9: **end if**
10: $t = t+1$
11: **end loop**

Compared to Q-learning, how does this change the: memory requirements, computation requirements per step, amount of data required?

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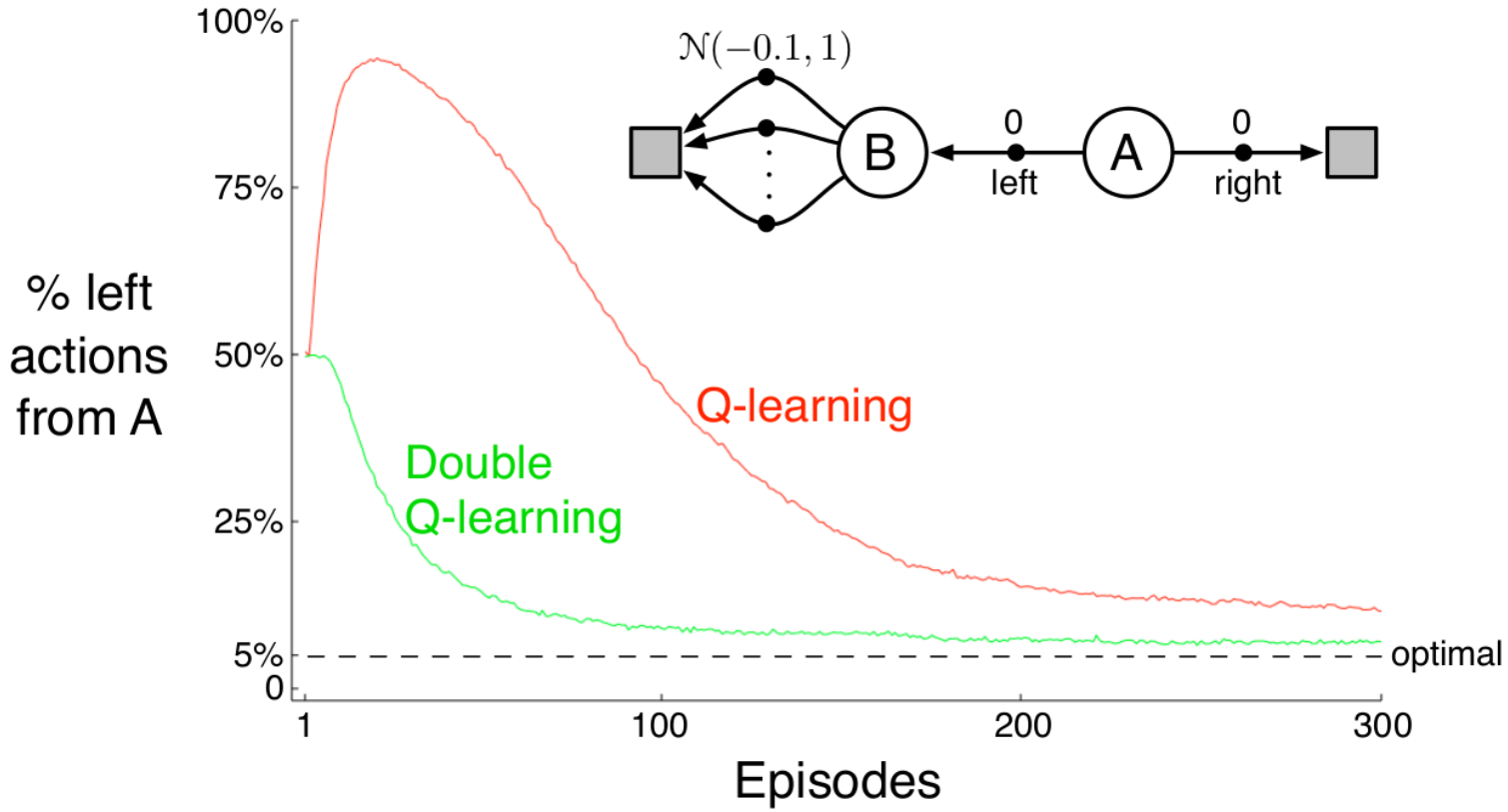
Compared to Q-learning, how does this change the: memory requirements, computation requirements per step, amount of data required?

Double Q-Learning (Figure 6.7 in Sutton and Barto 2018)



Due to the maximization bias, Q-learning spends much more time selecting suboptimal actions than double Q-learning.

What You Should Know

- Be able to implement MC on policy control and SARSA and Q-learning
- Compare them according to properties of how quickly they update, (informally) bias and variance, computational cost
- Define conditions for these algorithms to converge to the optimal Q and optimal π and give at least one way to guarantee such conditions are met.

Class Structure

- Last time: Policy evaluation with no knowledge of how the world works (MDP model not given)
- This time: Control (making decisions) without a model of how the world works
- **Next time: Generalization – Value function approximation**

Evaluation to Control

- Last time: how good is a specific policy?
  - Given no access to the decision process model parameters
  - Instead have to estimate from data / experience
- Today: how can we learn a good policy?

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