New tools from applied topology for data analysis and signal processing

Physics & Astronomy Colloquium

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Mathematics Department

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Part I

Persistent homology

Points clouds and multi-scale approximation

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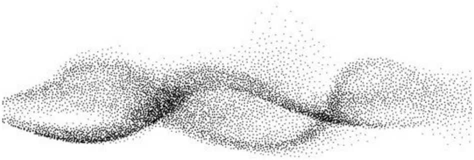

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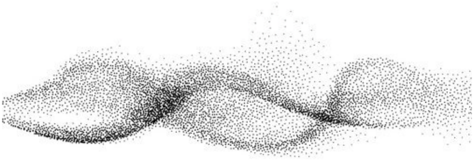

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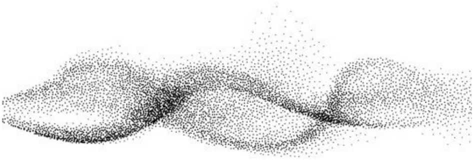

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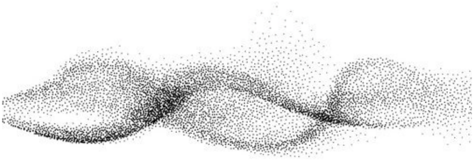

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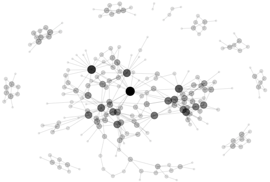

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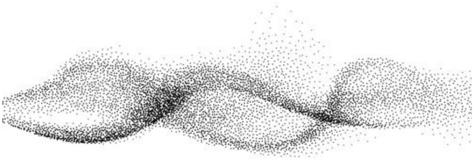

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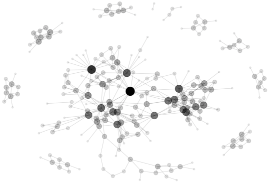

approximating it. What is the “right” scale?

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$$X_{t_0} \subset X_{t_1} \subset \cdots \subset X_{t_n}$$

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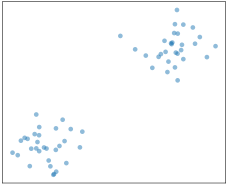

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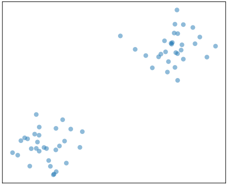

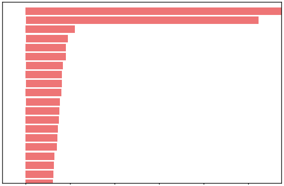

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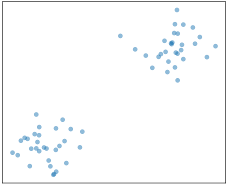

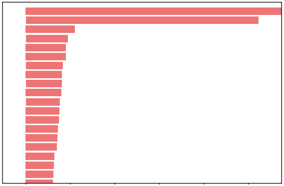

Longer bars correspond to features that persist.

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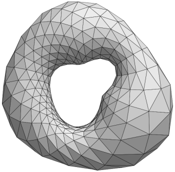

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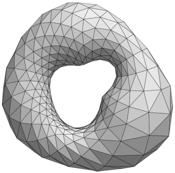

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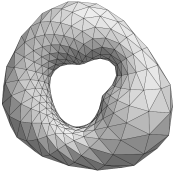

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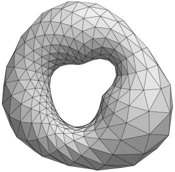

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Obtain filtered simplicial complex

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Basic example: (Euler characteristic)

$$\chi = \#vertices - \#edges + \cdots + (-1)^n \#(n\text{-}simplices).$$

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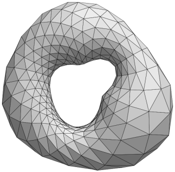
Given a simplicial complex X



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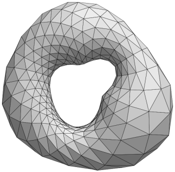

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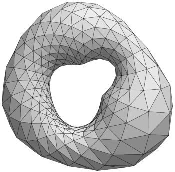

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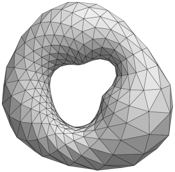

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Example if T is the above space

$$\beta_0(T) = 1, \quad \beta_1(T) = 2, \quad \beta_2(T) = 1, \quad \beta_i(T) = 0.$$

It is connected, has two independent loops,
and one 2-dimensional cavity.

Persistence homology

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Applying homology to the multi-scale approximation

$$X_0 \subset X_1 \subset \cdots \subset X_n$$

gives a family of vector spaces and linear maps

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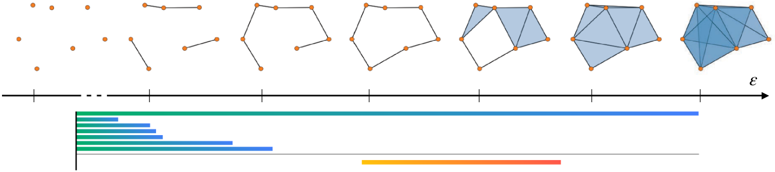

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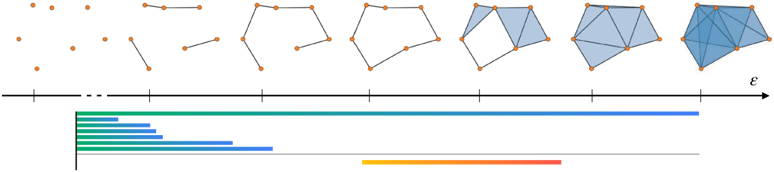
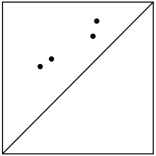


Abstractly, a barcode is a collection of pairs $(a, b) \in \mathbb{R}^2$ with multiplicity.



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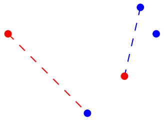

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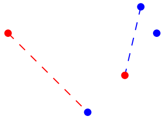

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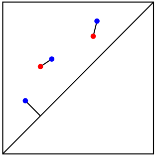

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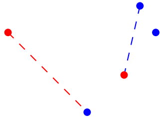

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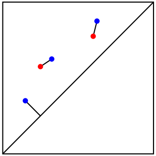

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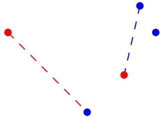

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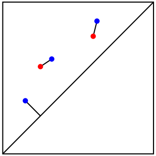

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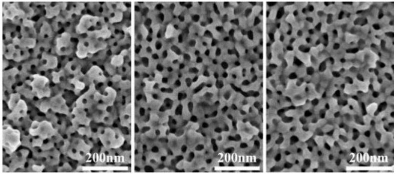
2. Computability: Based on matrix reduction algorithms, it complexity is

$$\sim O(\#\text{simplices}^3).$$

Western

Exemplar - Nanoporous materials (Lee et al.)



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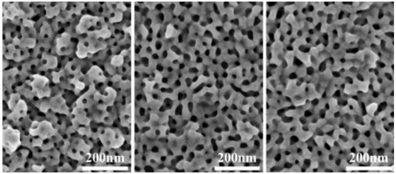

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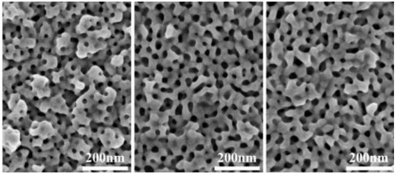
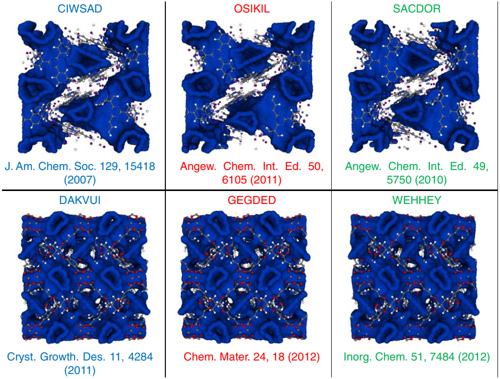


Comparing Barcodes

is done fast
and faithfully
by stability.



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Integration of persistence algorithms into `scikit-learn`.

The *giotto-tda* team

Giotto-tda

github.com/giotto-ai/giotto-tda

EPFL

ReDS

Reconfigurable & embedded Digital Systems

Sponsored by

Schweizerische Eidgenossenschaft
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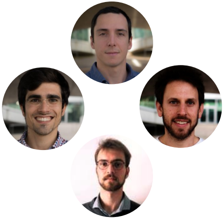

L2F

… and many others!

Part II

Steenrod barcodes

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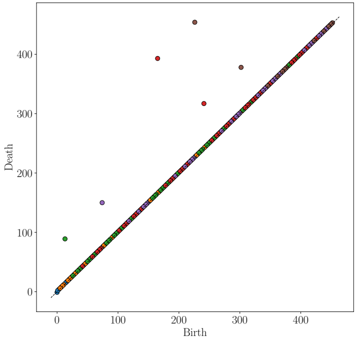

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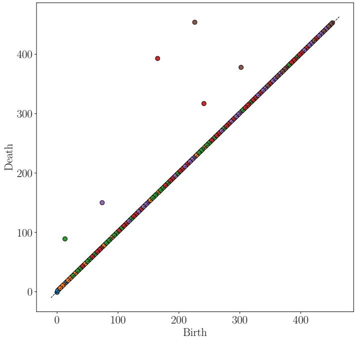

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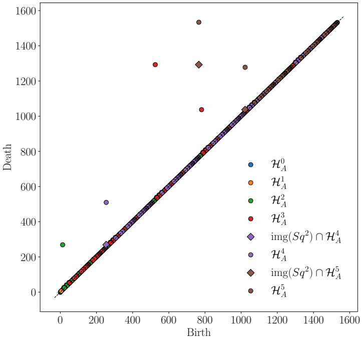

(b) $\mathrm{C}\,\Sigma\,\mathbb{C}\mathrm{P}^2$

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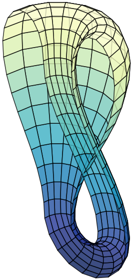
But these forget much information. For example, if K is the Klein bottle



over the field with two elements $\mathbb{F}_2$ we have

$$\beta_0(K) = 1, \quad \beta_1(K) = 2, \quad \beta_2(K) = 1, \quad \beta_i(K) = 0.$$

The same as the torus T.

Steenrod squares

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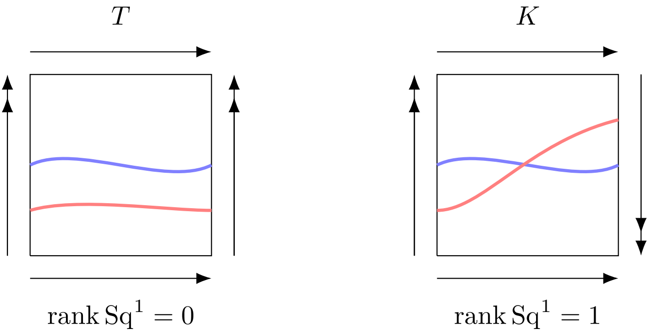

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Contribution (Med.): A faster way to compute Sq^k for simplicial complexes.

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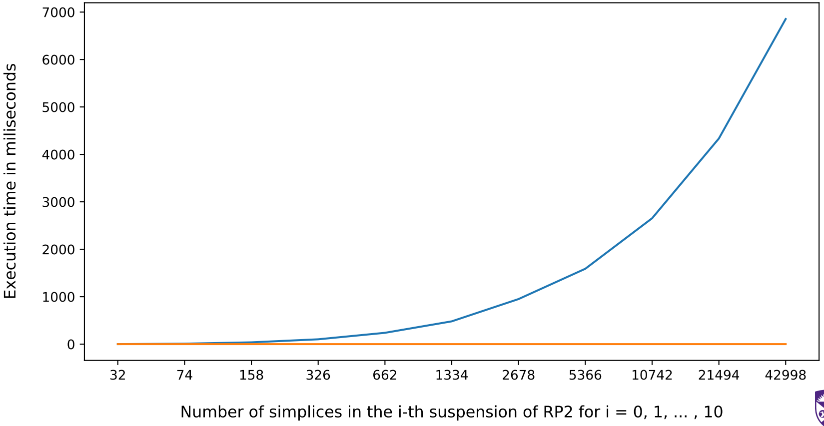
Example: Clocking the computation of Sq^1, the old and new ways.



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For each k and d, Sq^k induces a family of compatible linear maps

$$\begin{tikzcd}
H^d(X_n;\mathbb{F}_2) \arrow[r] \arrow[d, "\mathrm{Sq}^k"] & \cdots \arrow[r] & H^d(X_{n-1};\mathbb{F}_2) \arrow[r] \arrow[d, "\mathrm{Sq}^k"] & H^d(X_0;\mathbb{F}_2) \arrow[d, "\mathrm{Sq}^k"] \\
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Persistent Steenrod squares

Given a filtered simplicial complex

$$(1) \qquad X_0 \subset X_1 \subset \cdots \subset X_n,$$

For each k and d, Sq^k induces a family of compatible linear maps

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Computable?

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A ready-to-use package for computing Steenrod barcodes.

 

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Question: Are these finer invariants out there in the real world?

Space of conformations of C_8H_{16}

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Points in $\mathbb{R}^{24}$ (positions of 8 carbons in $\mathbb{R}^3$)

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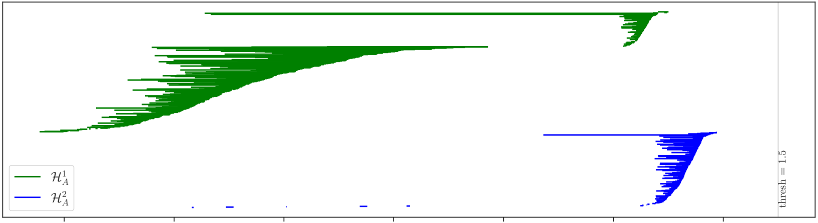
H^1 (green) and H^2 (blue) barcodes of (part of) this point cloud



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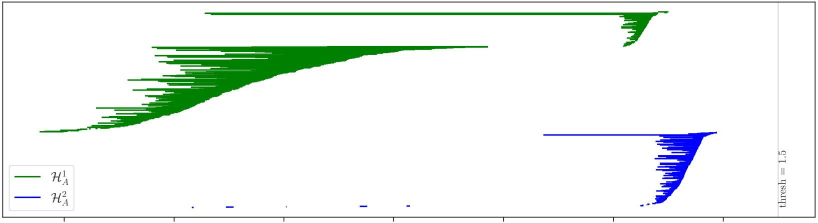
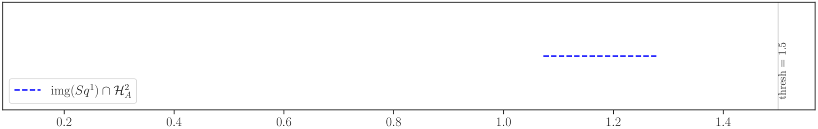


Sq^1 barcode



Consistent with a Klein bottle.

Summary

Persistent homology and their Steenrod generalization are principled tools for feature extraction.

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Persistent homology and their Steenrod generalization are principled tools for feature extraction.

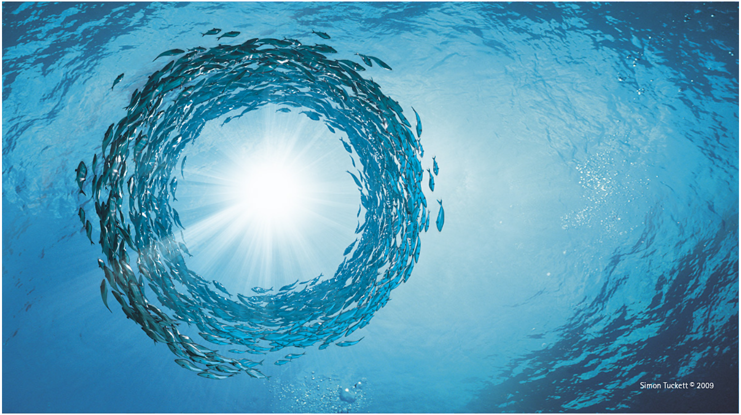
- They capture the primitive shape of data.
- They are relevant at multiple scales.
- Their output, barcodes, form a metric space.
- They are robust to noise.
- They are computable in practice: `giotto-tda` - `steenroder`.
- They are useful in the study real-world data.

Part III

(time-permitting)

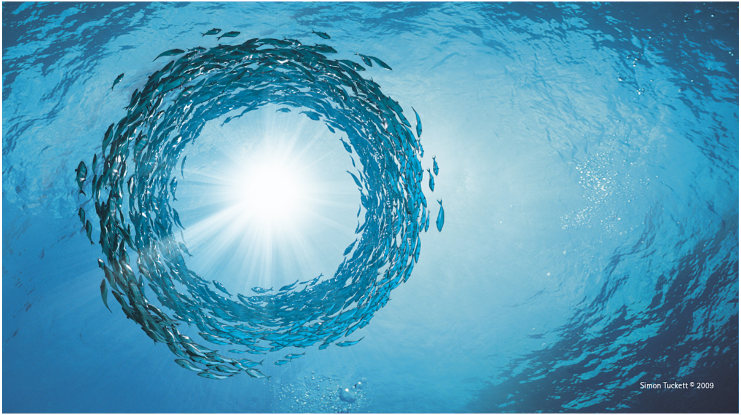
Hyperharmonic analysis

Complex systems



The whole is greater than the sum of its parts.

Complex systems



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Question: How to quantify this slogan?

Information signals

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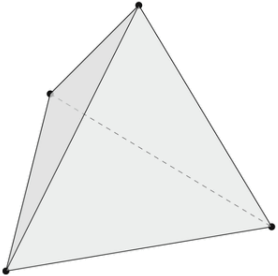
Problem: Number of subsets grows exponentially with N.

Western

Signals on a simplex



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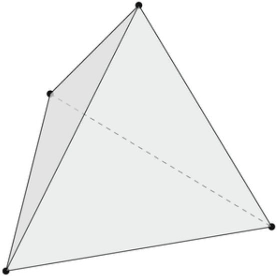

Weighted with structural information. (e.g. EEG node positions)

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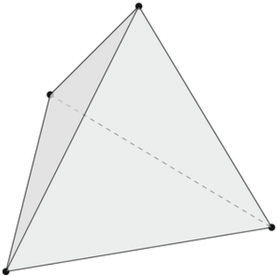

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Signals are the same as $\mathbb{R}$-valued functions on simplices. E.g.

Entropy : 0-simplices
Mutual information : 1-simplices
Interaction information : 2-simplices
Higher order signals : d-simplices

Hyperharmonic analysis

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Analogy: Listen to a few harmonics to tell a guitar and a piano apart.

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How to measure compressibility?

Give signal α, let $\{\alpha_i\}$ be its coeff's in a basis with $\alpha_1^2 \geq \alpha_2^2 \geq \cdots$

$$\mathsf{EV}_\alpha(k) = \frac{\alpha_k^2}{\sum_i \alpha_i^2} \qquad \text{and} \qquad \mathsf{CEV}_\alpha(k) = \sum_{1\leq i\leq k} \mathsf{EV}_\alpha(i),$$

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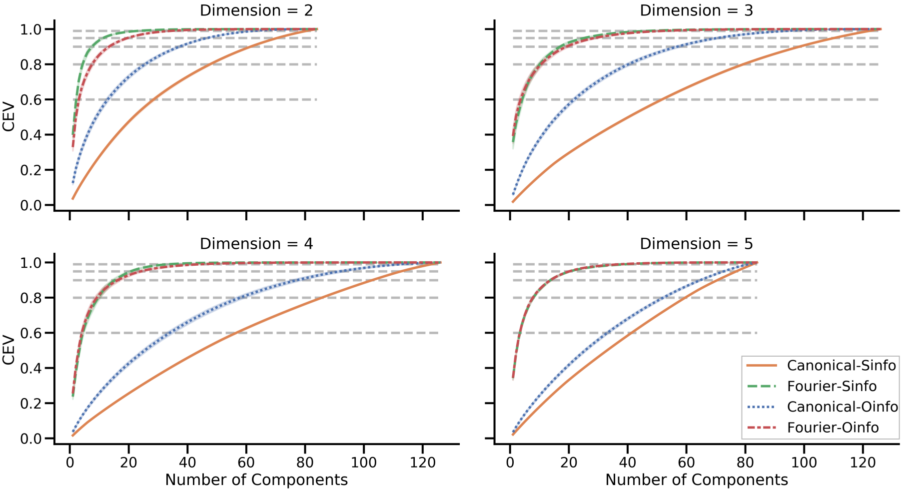

Thank you!

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