GEOMETRIC COHOMOLOGY

GREG FRIEDMAN, ANIBAL M. MEDINA-MARDONES, AND DEV SINHA

Abstract. We set up the theory of geometric cohomology.

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1. Introduction

The goal of this work is to develop a geometric approach to singular homology and cohomology on smooth manifolds. By this we mean that the homology and cohomology classes should be represented by smooth maps from manifolds to our target manifold of interest. Such a goal is reminiscent of very classic thinking about homology dating back to Poincaré and Lefschetz and of course is also classically impossible due to the work of Thom (REFERENCE), who showed that not all homology classes can be represented by pushing forward fundamental classes of manifolds. To get around this obstruction, one needs a broader notion of "manifold," and the one we utilize here is manifolds with corners. On the homology side, it is not so surprising that one can do this, for after all simplices and cubes are manifolds with corners, and it is well known that one can compute the homology of a smooth manifold using only smooth maps of simplices or cubes. What is more surprising, though still to be found in certain classical corners of 20th century geometric topology, is that *co*homology classes can also be represented by appropriate maps from manifolds with corners, more specifically, proper and co-oriented smooth maps from manifolds with corners. In particular, in such theories cochains are not (just) some kind of algebraic duals to chains but are geometric objects in their own right.

One benefit to such thinking is that the classical operations of algebraic topology, such as cup and cap products, can be described *at the level of cochains* by simple geometric operations without recourse to algebraic diagonals, Alexander-Whitney maps, or other such combinatorics. In fact, when our cochains are represented by embeddings, the cup product is simply the intersection. So, in fact, is the cap product. More generally, these products are represented by pullbacks or fiber products. This again is reminiscent of the original classical thinking about such products in terms of intersections[1]. The trade-off for such a pleasant description is that these intersections are not always defined; they require transversality. This limitation is also classically anticipated by the famous commutative cochain problem: loosely speaking, no chain algebra over $\mathbb{Z}$ who cohomology gives the singular cohomology ring can be both algebraically fully defined and (graded) commutative. Since the process of forming intersections is commutative, it cannot be fully defined. We will not let that stop us — no theory is perfect.

The idea of representing both homology and cohomology by some kind of geometric maps is also not new. Past examples include GORESKY'S REPRESENTATION OF PL COHOMOLOG and Kreck's "differential algebraic topology" [**?**], which provides homology and cohomology on smooth manifolds using maps from *stratifolds*, a certain kind of singular space; and present work in progress by Joyce [**?**], which utilizes manifolds with corners, but in a more elaborate setting with the goal of applications to symplectic geometry.

Our particular flavor of geometric homology and cohomology is originally due to Lipyanskiy in an incomplete and unpublished manuscript [**?**]. Lipyanskiy gives a fairly thorough account of geometric homology, but a much more lightly sketched account of geometric cohomology. Several of the major theorems are unproven or have arguments just hinted at. Some other expected results are not stated at all, including an isomorphism between geometric and singular cohomology or that the geometrically defined cup product coincides with the classical singular cup product. So one of our main goals is to give a thorough account, with detailed proofs, of geometric homology and cohomology, with our primary focus on geometric cohomology, both because Lipyanskiy's account is more deficient in this area and also because cohomology, with its richer algebraic structures, is of more immediate interest to the authors. In fact, geometric cohomology caught our attention while working on [**?**] and looking for a rigorous foundation to geometrically model the cup product and ultimately, we hope, the higher algebraic products that arise in forming Steenrod squares and other aspects of the E_∞-algebra structure of the cochain complex. So a second major goal, not really present in [**?**], is to consider the product structures not just in cohomology but at the level of the cochains themselves. To show that such products are ever well defined (even when

[1] A nice modern summary and treatment of such intersections in the PL category can be found in [**?**].

some transversality is present) requires some careful analysis of the structure of geometric cochains, which we will see our not actually simply maps of manifolds but certain equivalence classes of such, which is a necessity to obtain a theory that models singular cohomology.

Our main objects are manifolds with corners, and we are indebted to Joyce's [**?**], which not only contains specific and rigorous definitions of these objects but also works through the details of showing that the fiber product of two appropriate transverse maps of manifolds with corners is again a manifold with corners. To avoid the most complicated part of Joyce's theory, and also because our results about geometric cohomology would no longer be true with the greater generality[2], our target manifolds will always be smooth manifolds without boundary whenever we need transversality of maps from manifolds with corners.

Another important background piece that we could not find properly worked out in the literature (perhaps more due to our own ignorance than its non-existence) is co-orientation of smooth maps. Roughly speaking, a map of manifolds is co-oriented when a loop in the domain is orientation-preserving or orientation-reversing if and only if its image loop in the target has the same property. As we shall see, the co-orientability requirement allows maps of non-orientable manifolds to serve as geometric cochains, and they are important to the theory[3]. For example, the identity map of any smooth manifold, orientable or not, represents the cohomology class[4] 1. Co-orientations are more appropriate for cochains that orientations, as co-orientations "pull back" in a way that we will see orientations do not. We develop the notion and properties of co-orientation quite thoroughly, which we hope may be also of broader use.

Conventions. Throughout we will denote the dimension of a manifold represented by an upper case character by the corresponding lower case character. For example, $\dim(M) = m$, $\dim(V) = v$, etc.

Old Stuff:

NEEDS TO BE REWRITTEN SPECIFIC TO THIS NEW VERSION OF THE PAPER. WE SHOULD EMPHASIZE THAT WE'LL PAY MORE ATTENTION TO COHOMOLOGY THAN HOMOLOGY BECAUSE LIPYANSKIY'S TREATMENT OF HOMOLOGY SEEMS TO CARRY OVER FINE WITH OUR MODIFICATIONS AND IS FAIRLY THOROUGHLY WRITTEN, UNLIKE COHOMOLOGY WHICH IS WHERE WE REALLY HAVE NEW RESULTS

Over the integers, submanifolds and intersection in various settings provide geometrically meaningful cochains [**?**, **?**], much as forms and wedge product do over the real numbers.

We are setting the foundations of this theory, including new developments such as a cochain-level product.

To specify a cubical cochain in a fixed degree is to give an integer for each and every nondegenerate cube in that degree, which in practice can be an unwieldy amount of data. Submanifolds, which can be simple to describe in cases of interest, can encode such data through intersection.

The basic idea is classical, essentially an implementation of Poincaré duality at the chain and cochain level by using intersection with a submanifold in order to define a function on chains - see Definition 6.13. But there are two ways in which generalization is needed to implement this idea. First, submanifolds alone do not capture homology and cohomology, as Thom famously realized and can be seen in applications such as using Schubert varieties to represent cohomology of Grassmannians. So we generalize from intersection with a submanifold to pullback of a map from a manifold. Secondly, in order to model cohomology, we need manifolds with corners, which also arise immediately when taking intersections or pullbacks of manifolds with boundary, as we use to define our product.

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[2] We should give an example here or somewhere.

[3] While co-orientations are central to our approach, surprisingly they do not seem to be absolutely essential, as Kreck's cohomology theory in [**?**] does not utilize them. On the other hand, Kreck requires his targets to be oriented and handles non-orientable manifolds via a nice trick with double covers Double check and add specific references. He also does not work as directly with cochain level products, for which we find the use of co-orientations to be a bit more natural

[4] Perhaps this is a good point to observe that geometric cochain are indexed by co-dimension and not dimension like chains.

While there are a number of treatments of homology and cohomology which employ manifolds and their generalizations [?, ?, ?, ?, ?, ?, ?] the cochain theory most compatible with the differential topology we employ is geometric cohomology, developed by Lipyanskiy [?]. Geometric cohomology uses manifolds with corners, for which we follow the careful treatment by Joyce [?]. Lipyanskiy does not give, or utilize, the careful treatment of manifolds with corners contained in [?], and neither gives a complete treatment of co-orientations. So we fill in some gaps in the details of Lipyanskiy's use of manifolds with corners, especially as regards pullbacks and co-orientations, though to preserve space we do refer to Lipyanskiy for details wherever possible.

GBF: From the email I wrote on 10/27/21: I think probably the philosophy should be that at the level of (co)chains the intersection pairing of chains is just a subcase of the intersection pairing of cochains. As previously observed, the intersection of chains is only well defined in general when everything in sight is orientable. But if everything in sight is orientable, then co-orientations and orientations are the same thing, so an oriented intersection is also a co-oriented intersection, and if the domains are compact then the maps are automatically proper, so a well-defined intersection of chains *is* an intersection of cochains.

Now of course that's not the same as what happens in homology, but I bet we can also prove that in the totally oriented setting where homology products are defined then homology is the same thing as compactly supported cohomology, so again homology becomes largely disposable in the study of products, especially as in our setting it's not like the chains are really any simpler than the cochains like they are for singular homology/cohomology.

Acknowledgments

The authors thank Mike Miller, for pointing us to [?], and Dominic Joyce, for answering questions about his work.

2. Manifolds with corners

In this section, we provide an overview of manifolds with corners, which are the main geometric objects in the definitions of geometric chains and cochains. Our reference for this material is Joyce [?]. By **smooth** we always mean differentiable of all orders. Throughout the paper, all manifolds and maps will be in the smooth category unless noted otherwise.

Definition 2.1. If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are any subsets we say that $f\colon A \to B$ is **smooth** if it extends to a smooth map from an open neighborhood of A to $\mathbb{R}^m$. We say that f is a **diffeomorphism** if f is a homeomorphism and both f and f^{-1} are smooth[5].

With this definition, the notions of smooth charts and atlases for manifolds in the standard setting can be extended to define (smooth) manifolds with corners; see [?, Section 2]: An n-dimensional chart (U, ϕ) of the space W has domain U an open subset of $\mathbb{R}^n_k = [0,\infty)^k \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$ and ϕ is a homeomorphism from U to $\phi(U) \subset W$. Two n-dimensional charts (U, ϕ) and (V, ψ) are compatible if $\psi^{-1}\phi\colon \phi^{-1}(\phi(U) \cap \psi(V)) \to \psi^{-1}(\phi(U) \cap \psi(V))$ is a diffeomorphism of subsets of $\mathbb{R}^n$. An n-dimensional atlas for W is a family of pairwise compatible n-dimensional charts that cover W, and an n-dimensional manifold with corners is a paracompact Hausdorff space with a maximal n-dimensional atlas.

While essentially equivalent, we choose to work entirely with subspaces of $\mathbb{R}^\infty$ in order to have a set of such objects, so rather than directly employ Joyce's definition, we define manifolds with corners for the purposes of this text as follows.

[5] Joyce requires $n = m$ in his definition for f to be a diffeomorphism. But suppose f is a diffeomorphism as defined here and, without loss of generality, $n > m$. Let $\mathbb{R}^m \subset \mathbb{R}^n$ in the usual way. Then as f extends to a smooth map from a neighborhood of A to $\mathbb{R}^m$, we can certainly consider f as extending to a smooth map from a neighborhood of A to $\mathbb{R}^n$. Similarly, as f^{-1} extends to a smooth map from a neighborhood of B in $\mathbb{R}^m$ to $\mathbb{R}^n$, we can extend f^{-1} to a neighborhood B in $\mathbb{R}^n$ by precomposing with the projection $\mathbb{R}^n \to \mathbb{R}^m$. So diffeomorphisms in our sense can be made into diffeomorphisms in Joyce's sense.

Definition 2.2. An n-dimensional **manifold with corners** W is a subspace of some $\mathbb{R}^N \subset \mathbb{R}^\infty$ such that each point of W possesses a neighborhood diffeomorphic to an open subset of $\mathbb{R}^n_k$ for some k.

We note that these local diffeomorphisms provide charts $\phi\colon U \subset \mathbb{R}^n_k \to W$, and these charts are compatible, with $\psi^{-1}\phi\colon \phi^{-1}(\phi(U) \cap \psi(V)) \to \psi^{-1}(\phi(U) \cap \psi(V))$ being a diffeomorphism, even using Joyce's more restrictive notion of diffeomorphism. These charts collectively give an atlas, and every atlas extends to a unique maximal one. So our manifolds with corners are also manifolds with corners by Joyce's definition.

On the other hand, Joyce notes on page 231 of [**?**] that his manifolds with corners are what Melrose calls t-manifolds, and Melrose shows in [**?**, Proposition 1.15.1] that t-manifolds can be embedded into manifolds without boundary. These can then be embedded into Euclidean space by the Whitney Embedding Theorem. So any one of Joyce's manifolds with corners is diffeomorphic to a manifold with corners in our sense.

By modeling on $\mathbb{R}^n_k$, our category includes manifolds ($k = 0$ in all charts) and manifolds with boundary ($k \leq 1$ in all charts) as well as cubes and simplices, but not the octahedron, for example, as the cone on $[0, 1] \times [0, 1]$ is not modeled by any $\mathbb{R}^n_k$.

Definition 2.3. A map $f\colon W \to M$ between manifolds with corners is (weakly) **smooth** if whenever (U, ϕ), (V, ψ) are charts for W and M respectively then

$$\psi^{-1} f\phi\colon (f\phi)^{-1}\psi(V) \to V$$

is smooth.

The **tangent bundle** of a manifold with corners is the space of derivations of the ring of smooth real-valued functions. Analogously to smooth manifolds with boundary, if (U, ϕ) is a chart of W then $d\phi$ takes the tangent space to $\mathbb{R}^n$ at $x \in U$ isomorphically to the tangent spaces of W at $\phi(x)$. In particular, if W is n-dimensional and $x \in W$, then the tangent space at x, denoted T_xW, is isomorphic to $\mathbb{R}^n$.

In [**?**], Joyce reserves the word *smooth* for weakly smooth maps that also satisfy an additional condition concerning how they interact with the boundaries of their codomains. When the codomain is a manifold without boundary, which will be our primary situation of interest, the weaker and stronger notions coincide. In the few other cases in which we need to consider maps whose codomains have boundary, in particular boundary immersions and projections of pullbacks, the maps will also satisfy Joyce's stronger condition, though we will never need to utilize this explicitly. Thus we will feel justified in simply using the word "smooth" throughout, referring the reader to [**?**, Definition 3.1] for the full definition.

Notation. Our default notation for manifolds with corners will be capital letters with the corresponding lower case letter denoting the dimension. In other words, $\dim(V) = v$, $\dim(W) = w$, $\dim(M) = m$, etc. We generally reserve M for a manifold without boundary.

2.1. **Boundaries.** We next need to describe boundaries of manifolds with corners. Again see [**?**, Section 2] for further details.

Definition 2.4. A point x in an n-dimensional manifold with corners W has **depth** k if there is a chart from an open subset of $\mathbb{R}^n_k$ which sends the origin to x. The set $S^k(W) \subseteq W$ of elements having depth k is called the **stratum of depth** k. By [**?**, Proposition 2.4.], $S^k(W)$ is an $n - k$ manifold without boundary.

Example 2.5. If W is a smooth manifold with boundary in the classical sense, then $S^0(W)$ is its interior, $S^1(W)$ is its boundary, and $S^k(W) = \emptyset$ for $k > 1$. If $S^k(W) = \emptyset$ for all $k > 0$, then W is a manifold without boundary.

When W is a general manifold with corners, the boundary is more naturally a space equipped with a map to W, rather than a subspace of W. The reason can be seen, for example, in the teardrop space of Figure 1, where the boundary should be considered to be homeomorphic to the closed interval mapping both endpoints to the vertex of the tear drop. To explain in more detail, we have the following definitions.

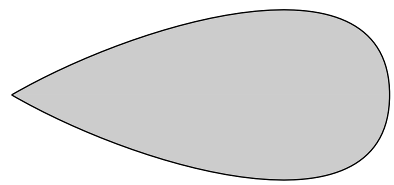


Figure 1. The 2-dimensional "teardrop" is a manifold with corners whose boundary inclusion is not injective.

Definition 2.6. A **local boundary component** β of W at $x \in W$ is a consistent choice of connected component $\mathbf{b}_V$ of $S^1(X) \cap V$ for any neighborhood V of x, with consistent meaning that $\mathbf{b}_{V \cap V'} \subset \mathbf{b}_V \cap \mathbf{b}_{V'}$.

Since this notion is local, the number of such components is determined by the depth. Considering the origin in $\mathbb{R}^n_k$, for any $k \geq 0$, points having depth k have exactly k local boundary components. As another example, letting $\mathbb{I} = [0,1]$ and so $\mathbb{I}^3$ the 3-cube, the set $S^1(\mathbb{I}^3)$ consists of the interiors of two-dimensional faces, $S^3(\mathbb{I}^3)$ is the set of eight corners, and any sufficiently small neighborhood of a corner intersects exactly three of the two-dimensional faces.

Definition 2.7. Let W be a manifold with corners. Define its **boundary** ∂W to be the space of pairs $(x, \mathbf{b})$ with $x \in W$ and $\mathbf{b}$ a local boundary component[6] of W at x. Define $i_{\partial W} \colon \partial W \to W$ by sending $(x, \mathbf{b})$ to x.

In the teardrop example, $i_{\partial W} \colon \partial W \to W$ takes $\mathbb{I}$ to the topological manifold boundary with both endpoints going to the unique point of $S^2(W)$. For $\mathbb{I}^3$, the boundary consists of six closed two-dimensional squares each mapping homeomorphically to a face of the cube. In general, $|i^{-1}_{\partial W}(x)|$ is equal to the depth of x.

As in Douady [**?**], ∂W is itself a manifold with corners, and the boundary map $i_{\partial W}$ is a smooth immersion [**?**, Theorem 3.4]. Note that $S^1(W)$ will always be diffeomorphic to the interior of ∂W, i.e. $S^0(\partial W) = S^1(W)$. Inductively, we let $\partial^k W$ denote $\partial(\partial^{k-1} W)$ with $\partial^0 W = W$, and we let $i_{\partial^k W}$ denote the composite of the boundary maps sending $\partial^k W$ to W. The map $i_{\partial^k W}$ takes $S^a(\partial^k W)$ onto $S^{a+k}(W)$, but not, in general, injectively.

Remark 2.8. If $f : V \to W$ is a diffeomorphism of manifolds with corners then it must, in particular, take components of $S^i(V)$ diffeomorphically onto corresponding components of $S^i(W)$, and, consequently, given a local boundary component $\mathbf{b}$ at a point $x \in V$, the map f picks out a corresponding local boundary component, say $f_*(\mathbf{b})$ in W. We obtain a diffeomorphism $f_\partial : \partial V \to \partial W$ by $(x, \mathbf{b}) \mapsto (f(x), f_*(\mathbf{b}))$ with $f i_{\partial V} = i_{\partial W} f_\partial$.

While for manifolds with boundary $\partial^2 W$ is empty, for general manifolds with corners it will not be. However, given a co-oriented map $W \to M$, $\partial^2 W$ does come equipped with a natural co-orientation reversing $\mathbb{Z}_2$ action. This will be explained below and is a key component in showing that ∂ is suitable for defining boundary maps of geometric chain and cochain complexes.

N.B. When no confusion is likely to occur, we will sometimes abuse notation and use ∂W also to refer to the image $i_{\partial W}(\partial W) \subset W$, which is the boundary of W as a topological manifold in the usual sense.

[6]Note that if $x \in S^0(W)$ then x does not have any local boundary components and so does not appear in such a pair.

The following is an example in which $\partial^2 W$ immerses into W in an interesting way.

Example 2.9. Consider the quotient Q of $S^n \times \mathbb{R}^2_2$ by the diagonal C_2 action, where C_2 (the group of order 2) acts antipodally on S^n and acts by permuting the two coordinates of $\mathbb{R}^2_2$. The projection of Q onto $S^n/C_2 = \mathbb{R}P^n$ defines a fiber bundle with fiber $\mathbb{R}^2_2$. Local coordinates can then be used to endow Q with the structure of a manifolds with corners.

The subspace $S^1(Q)$ is diffeomorphic to $S^n \times (0, \infty)$, and $S^2(Q)$ is diffeomorphic to $\mathbb{R}P^n$. The boundary ∂Q is the quotient of $S^n \times \partial\mathbb{R}^2_2$ by C_2, which is diffeomorphic to $S^n \times [0, \infty)$. Thus $\partial^2 Q$ is S^n, which maps to $S^2(Q)$ by the standard quotient by antipodal action.

In general, Proposition 2.9 of [**?**] identifies $\partial^k W$ with the set of points $(x, \mathbf{b}_1, \ldots, \mathbf{b}_k)$ with $x \in W$ and the $\mathbf{b}_i$ providing an ordered k-tuple of distinct local boundary components of W.

Special cases of manifolds with corners, including manifolds with faces or manifolds with embedded corners [**?**], steer clear of the interesting boundary phenomena of Example 2.9. Although, a more restrictive notion should suffice for our application, Lipyanskiy develops geometric cohomology in the current generality, and we appreciate Joyce's careful treatment of transversality in this category, so we use their definitions.

We conclude this section by observing that the product $V \times W$ of two manifolds with corners is naturally a manifold with corners with, by [**?**, Proposition 2.12],

$$\partial(V \times W) = (\partial V \times W) \sqcup (V \times \partial W).$$

2.2. **Transversality.** Transversality of smooth maps will play a key role, as this is the condition that assures that intersections or, more generally, pullbacks of manifolds (with corners) are also manifolds (with corners). Recall that in the classical setting if $f\colon V \to M$ and $g\colon W \to M$ are smooth maps of manifolds without boundary then we say that f and g are **transverse** if whenever $f(x) = g(y) = z$ for some $x \in V$, $y \in W$, and $z \in M$ we have $Df(T_x V) + Dg(T_y W) = T_z M$. We here discuss the extension of transversality to manifolds with corners, though only in the case where M is without boundary. More general versions of transversality can be found in [**?**, Section 6].

Definition 2.10. [**?**, Special case of Definition 6.1] Let $f\colon V \to M$ and $g\colon W \to M$ be smooth maps of manifolds with corners to a manifold without boundary. We say f and g are **transverse** if whenever $x \in S^j(X)$ and $y \in S^k(Y)$ with $f(x) = g(y) = z$ then $Df|_{S^j(V)}(T_x S^j(V)) + Dg|_{S^k(W)}(T_y S^k(W)) = T_z M$.

While this particular formulation of transversality is given in terms of the behavior of f and g on the strata $S^j(V)$ and $S^k(W)$, it is sometimes useful to have a reformulation in terms of the boundaries $\partial^j V$ and $\partial^k W$. This is the content of Lemma 2.11 below.

To establish notation, let $f\colon V \to M$ and $g\colon W \to M$ be maps from manifolds with corners to a manifold without boundary. We say that f and g are **plainly transverse** if they are transverse as maps of manifolds in the classical sense, without special consideration of strata or boundaries. To be explicit in the case that $x \in V - S^0(V)$ or $y \in W - S^0(W)$ with $f(x) = g(y)$, let (U_x, ϕ_x) and (U_y, ϕ_y) be charts with $\phi_x(0) = x$ and $\phi_y(0) = y$. By definition of smoothness, there exist neighborhoods N_x and N_y of 0 in $\mathbb{R}^v$ and $\mathbb{R}^w$, respectively, so that $f\phi_x$ and $g\phi_y$ extend to smooth maps $\psi_x\colon N_x \to M$ and $\psi_y\colon N_y \to M$. Then f and g are plainly transverse at x and y if $D\psi_x(T_0\mathbb{R}^v) + D\psi_y(T_0\mathbb{R}^w) = T_{f(x)}M$. This property is independent of the involved choices as $D\psi_x(T_0\mathbb{R}^v)$ is the limit of $D\psi_x(T_a\mathbb{R}^v)$ for a taken along any smooth path in U_x, and this does not depend on the choice of N_x or ψ_x, and similarly for ψ_y.

Lemma 2.11. *Let $f\colon V \to M$ and $g\colon W \to M$ be maps from manifolds with corners to a manifold without boundary. Then f and g are transverse if and only if $fi_{\partial^j}\colon \partial^j V \to M$ and $gi_{\partial^k}\colon \partial^k W \to M$ are plainly transverse for all j, k.*

Note that, a priori, the latter is a stronger condition as it imposes conditions not just on the interior of strata but on their closures.

Proof. First suppose fi_{∂^j} and gi_{∂^k} are plainly transverse for all j,k. Suppose $x \in S^j(V)$ and $y \in S^k(W)$ for some fixed j,k and $f(x)=g(y)$. The preimage of x under i_{∂^j} consists of $j!$ points in $S^0(\partial^j V)$, and i_{∂^j} maps a neighborhood of each such preimage point diffeomorphically to a neighborhood of x in $S^j(V)$, and similarly for y. Let ψ_x be the inverse diffeomorphism from a neighborhood of x in $S^j(V)$ to a neighborhood of one of the preimages. Then $f|_{S^j(V)} = fi_{\partial^j}\psi_x$ in a neighborhood of x, and similarly for y. Since fi_{∂^j} and gi_{∂^k} are plainly transverse, and ψ_x and ψ_y are diffeomorphisms, it follows that $f|_{S^j(V)}$ and $g|_{S^k(W)}$ are transverse at $f(x)=g(y)$.

Conversely, suppose $f|_{S^j(V)}$ and $g|_{S^k(W)}$ are transverse for all j,k, and suppose $x \in \partial^j V$ and $y \in \partial^k W$ for some fixed j,k with $fi_{\partial^j}(x) = gi_{\partial^k}(y) = z$. Furthermore, suppose $i_{\partial^j}(x) \in S^a(V)$ and $i_{\partial^k}(y) \in S^b(W)$, which implies $x \in S^{a-j}(\partial^j V)$. By focusing on local charts, there is a neighborhood of x in $\partial^j(V)$ whose intersection with $S^{a-j}(\partial^j V)$ maps diffeomorphically via i_{∂^j} onto a neighborhood of $i_{\partial^j}(x)$ in $S^a(V)$, and analogously for y. Thus, $f|_{S^a(V)} i_{\partial^j}|_{S^{a-j}(\partial^j V)}$ and the analogous $g|_{S^b(W)} i_{\partial^k}|_{S^{b-k}(\partial^k W)}$ are transverse at $f(x)=g(y)$ as they precompose transverse maps with local diffeomorphisms. But the image of $D_x(fi_{\partial^j})$ contains the image of $D_x(f|_{S^j(V)} i_{\partial^j}|_{S^{a-j}(\partial^j V)})$ and similarly for y, and thus the images of $D_x(fi_{\partial^j})$ and $D_y(gi_{\partial^k})$ must also span T_zM. Therefore, fi_{∂^j} and gi_{∂^k} are plainly transverse at $f(x)$. □

2.2.1. *Achieving transversality.* Throughout this text, we will need a series of increasingly more general results guaranteeing that we can make certain maps transverse to each other. We begin here with a relatively simple case that will be first used in Section 5 to show that geometric cohomology is contravariantly functorial with respect to continuous maps of manifolds without boundary. All of our transversality theorems will be built using some basic tools that can be found in [**?**, Section 2.3]. In particular, we record the following results, referring the reader to [**?**, Section 2.3] for the proofs[7]:

Theorem 2.12 (Transversality Theorem)**.** *Suppose $F\colon X \times S \to Y$ is a smooth map of manifolds, where only X has boundary, and let Z be any boundaryless submanifold of Y. If both F and $F|_{\partial X \times S}$ are transverse to X, then for almost every $s \in S$, both $F(-,s)\colon X \to Y$ and $F(-,s)|_{\partial X}\colon \partial X \to Y$ are transverse to Z.*

Theorem 2.13 (ϵ-Neighborhood Theorem)**.** *For a compact boundaryless manifold Y in $\mathbb{R}^M$ and a positive number ϵ, let Y_ϵ be the open set of points in $\mathbb{R}^M$ with distance less than ϵ from Y. If ϵ is sufficiently small, then each point $w \in Y_\epsilon$ possesses a unique closest point in Y, denoted $\pi(w)$. Moreover, the map $\pi\colon Y_\epsilon \to Y$ is a submersion. When Y is not compact, there still exists a submersion $\pi\colon Y_\epsilon \to Y$ that is the identity on Y, but now ϵ must be allowed to be a smooth positive function on Y, and Y_ϵ is defined as $\{w \in \mathbb{R}^m \mid |w-y| < \epsilon(y)$ for some $y \in Y\}$.*

These theorems are used in [**?**] to prove the following basic transversality result:

Theorem 2.14 (Transversality Homotopy Theorem)**.** *For any smooth map $f\colon X \to Y$ and any boundaryless submanifold Z of the boundaryless manifold Y, there exists a smooth map $g\colon X \to Y$ homotopic to f such that g is transverse to Z and $g|_{\partial X}$ is transverse to Z.*

Among other generalizations as we progress, we will extend these results to maps of manifolds with corners. We will also often require that the homotopies take a special form, i.e. that they are *universal* homotopies as defined in Section 6.4. See, for example, Proposition 7.24.

The following technical lemma will help us extend the results of [**?**] from transversality with respect to submanifolds to transversality with respect to maps.

Lemma 2.15. *Let $f\colon V \to M$ and $g\colon W \to M$ be smooth maps from manifolds with corners to a manifold without boundary. Let $e\colon W \to M \times \mathbb{R}^n$ be an embedding such that $\pi e = g$, where π is the projection $M \times \mathbb{R}^n \to M$. Then f and g are transverse if and only if e is transverse to $f \times \mathrm{id}_{\mathbb{R}^n}\colon V \times \mathbb{R}^n \to M \times \mathbb{R}^n$.*

[7]We rephrase the statements of these theorems slightly to better fit our context and notation.

Proof. It suffices to assume that V and W are without boundary. Otherwise we can apply the following argument to each pair of strata of V and W.

Suppose that f and g are transverse, i.e. that if $f(v) = g(w)$ then $Df(T_vV) + Dg(T_wW) = T_{f(v)}M$. For each $w \in W$, we can write $e(w) = (g(w), e_{\mathbb{R}}(w)) \in M \times \mathbb{R}^n$. Now suppose $w \in W$ and $(v, z) \in V \times \mathbb{R}^n$ such that $e(w) = (f \times \mathrm{id}_{\mathbb{R}^n})(v, z)$. Then we have $(g(w), e_{\mathbb{R}}(w)) = (f(v), z)$. The image of the derivative of $f \times \mathrm{id}_{\mathbb{R}^n}$ at such a point will span $Df(T_vV) \oplus T_z(\mathbb{R}^n) = Df(T_vV) \oplus \mathbb{R}^n$, while the derivative of e will take $a \in T_w(W)$ to $Dg(a) + De_{\mathbb{R}}(a)$. But the image of $D(f \times \mathrm{id}_{\mathbb{R}^n})$ already includes $0 \oplus \mathbb{R}^n$, so subtracting off the second summand, $D(f \times \mathrm{id}_{\mathbb{R}^n})(T_{(v,z)}(V \times \mathbb{R}^n)) + De(T_wW)$ contains $Dg(a)$. It follows that $D(f \times \mathrm{id}_{\mathbb{R}^n})(T_{(v,z)}(V \times \mathbb{R}^n)) + De(T_wW)$ contains $Df(T_vV) \oplus 0$, $Dg(T_wW) \oplus 0$, and $0 \oplus \mathbb{R}^n$. Since f and g are transverse and $D(f \times \mathrm{id}_{\mathbb{R}^n})(T_{(v,z)}(V \times \mathbb{R}^n)) + De(T_wW)$ is a vector space, it therefore contains all of $T_{f(v)}M \oplus \mathbb{R}^n = T_{e(w)}(M \oplus \mathbb{R}^n)$. So $f \times \mathrm{id}_{\mathbb{R}^n}$ and e are transverse.

Next suppose $f \times \mathrm{id}_{\mathbb{R}^n}$ and e are transverse and that $f(v) = g(w) \in M$. Suppose $e(w) = (g(w), z)$. Then $e(w) = (f \times \mathrm{id}_{\mathbb{R}^n})(v, z)$. So, by definition and assumption,

$$D(f \times \mathrm{id}_{\mathbb{R}^n})(T_{(v,z)}(V \times \mathbb{R}^n)) + De(T_wW) = T_{e(w)}(M \times \mathbb{R}^n) = T_{f(v)}M \oplus \mathbb{R}^n. \tag{1}$$

As π is a submersion, the image of this tangent space under $D\pi$ is all of $T_{f(v)}M$. But $(D\pi)(De) = D(\pi e) = Dg$, so $(D\pi \circ De)(T_wW) = Dg(T_wW)$. Furthermore, letting $\pi_V : V \times \mathbb{R}^n \to V$ be the projection, we have $(D\pi)(D(f \times \mathrm{id}_{\mathbb{R}^n})) = D(\pi(f \times \mathrm{id}_{\mathbb{R}^n})) = D(f\pi_V) = (Df)(D\pi_V)$, so, as $D\pi_V : T_{(v,z)}(V \times \mathbb{R}^n) \to T_vV$ is surjective, we have $(D\pi)(D(f \times \mathrm{id}_{\mathbb{R}^n}))(T_{(v,z)}(V \times \mathbb{R}^n)) = Df(T_vV)$. So applying $D\pi$ to equation (1), we get $Df(T_vV) + Dg(T_wW) = T_{f(v)}M$, and f is transverse to g. □

Theorem 2.16. *Let $f : M \to N$ be a continuous map from a manifold with (possibly empty) boundary to a manifold without boundary, and let $g : V \to N$ be a smooth map from a manifold with corners. Then there is a homotopy $h : M \times I \to N$ such that $h(-, 0) = f$, and $h(-, 1)$ is smooth and transverse to g. If $f|_{\partial M}$ is already smooth and transverse to g, then we can find such an h with $h(-, 1) = f$ on ∂M.*

Proof. Let $j : V \to \mathbb{R}^k$ be an embedding (we recall that such an embedding always exists by our definition of manifolds with corners). Then $e : V \to N \times \mathbb{R}^k$ given by $e(x) = (g(x), j(x))$ is an embedding that satisfies the hypotheses of Lemma 2.15, so by that lemma it suffices to find a homotopy such that $h(-, 0) = f$ and $h(-, 1) \times \mathrm{id}_{\mathbb{R}^k} : M \times \mathbb{R}^k \to N \times \mathbb{R}^k$ is smooth and transverse to $e(V)$.

We first find a homotopy from f to a smooth map that agrees with f on ∂M if $f|_{\partial M}$ is already smooth. This can be done by the smooth approximation theorem; see [**?**, Theorem III.2.5]. So we assume for the rest of the proof that this first homotopy has been completed and that f is a smooth map. We will first construct our homotopy to achieve transversality without consideration of whether or not f is already transverse to g on ∂M, and then we show how to modify the construction for that case.

We can think of N as properly embedded in some $\mathbb{R}^K$ by the Whitney Embedding Theorem [**?**, Section 1.8], and we let N_ϵ be an ϵ neighborhood of N in $\mathbb{R}^K$ with submersion $\pi : N_\epsilon \to N$. Let D be the open unit ball in $\mathbb{R}^K$. We define a composite map $H : M \times D \to \mathbb{R}^K \to N$ by

$$H(x, s) = \pi(f(x) + \epsilon(x)s).$$

We have $H(x, 0) = \pi(f(x)) = f(x)$, and, since $\epsilon(x) > 0$ for all x, the map $(x, s) \to f(x) + \epsilon(x)s$ is a submersion onto its image in $\mathbb{R}^K$. As π is also a submersion, so is H, and then $H \times \mathrm{id}_{\mathbb{R}^k} : M \times D \times \mathbb{R}^k \to N \times \mathbb{R}^k$ is a submersion. In particular, $H \times \mathrm{id}_{\mathbb{R}^k}$ is transverse to $e(V)$ and to $e(S^i(V))$ for each stratum $S^i(V)$ of W. We can now apply the Transversality Theorem (though with the parameter space D in a slightly nontraditional location in the ordering of factors) to obtain that $H(-, s) \times \mathrm{id}_{\mathbb{R}^k}$ is transverse to $e(V) \times \mathbb{R}^k$ for almost every $s \in D$. If W is a stratum of V, then this statement also holds replacing V with W. It also holds, for the same reasons, replacing M with ∂M. Since V has a finite number of strata and M and ∂M are only two space, and since the intersection of a finite number of dense subsets is dense, it holds for almost every $s \in D$, that both $H(-, s) \times \mathrm{id}_{\mathbb{R}^k}$ and its restriction to $\partial M \times \mathbb{R}^k$ are transverse to every stratum of V. Now let s_0 be one such s, and let $h\colon M\times I\to N$ be given by $h(x,t)=H(x,ts_0)$. Then $h(x,0)=H(x,0)=\pi(f(x))=f(x)$. While $h(x,1)=H(-,s_0)$ has the desired properties by construction.

If $f|_{\partial M}$ is already transverse to g, we modify the construction as follows: Let $\rho\colon M\to[0,1]$ be a smooth function that is 0 on ∂M and >0 on $M-\partial M$. Then we define

$$H(x,s)=\pi(f(x)+\epsilon(x)\rho(x)s).$$

Then when $x\in\partial M$, we have $H(x,s)=\pi(f(x))=f(x)$, so h is constant along ∂M. For $x\notin\partial M$, the argument goes through exactly as above. □

2.3. **Pullbacks and fiber products.** When two embedded submanifolds of a manifold meet transversely, their intersection is again a submanifold. More generally, if two smooth maps of manifolds are transverse, we can form their pullback, also called their fiber product, which is again a manifold. This construction extends to manifolds with corners mapping into a manifold without boundary.

Definition 2.17. Let $f\colon V\to M$ and $g\colon W\to M$ be transverse smooth maps from manifolds with corners to a manifold without boundary. Define the **pullback** or **fiber product** $V\times_M W$ by

$$V\times_M W=\{(x,y)\in V\times W\mid f(x)=g(y)\}.$$

There are canonical maps from $V\times_M W$ to V, W, and M that respectively take (x,y) to x, y, and $f(x)=g(y)$.

$$\begin{tikzcd} V\times_M W \arrow[r, "g^*"] \arrow[d, "f^*"'] & V \arrow[d, "f"] \\ W \arrow[r, "g"] & M \end{tikzcd}$$

We typically suppress the maps from the notation, though we sometimes label them as in the diagram and sometimes write $f\times_M g\colon V\times_M W\to M$. We also sometimes write f^* as π_W and g^* as π_V, as these maps are induced by restricting to $V\times_M W$ the projections from $V\times W$ to V and W.

We will generally use the term *pullback* when we want to emphasize $V\times_M W$ with its map to V or W, while the *fiber product* is to be considered as mapping to M. When treating $V\times_M W$ as a pullback, we also sometimes use the notation $g^*V\to W$ or $f^*W\to V$; this notation is consistent with the analogous notation for pullbacks of fiber bundles, which is a special case.

The following analysis of fiber products is standard – see for example Proposition 7.2.7 of [**?**].

Theorem 2.18. *Let $f\colon V\to M$ and $g\colon W\to M$ be transverse smooth maps from manifolds with corners to a manifold without boundary. Then $V\times_M W$ is a manifold with corners with*

$$S^i(V\times_M W)=\bigsqcup_{k+\ell=i}S^k(V)\times_M S^\ell(W).$$

Moreover, the maps from the fiber product to V, W, and M are weakly smooth.

To generalize this theorem when M is also a manifold with corners requires substantial additional hypotheses in the definition of transverse smooth maps. Such a generalization is a central result in [**?**].

There is a Leibniz rule for taking boundaries of fiber products of transverse maps [**?**, Proposition 6.7]:

$$\partial(V\times_M W)=(\partial V\times_M W)\sqcup(V\times_M\partial W),\tag{2}$$

recalling that if $g\colon W\to M$ then we interpret ∂W as equipped with the map $gi_{\partial W}\colon\partial W\to M$ and similarly for V. We will see versions of this formula below that take into account orientations and co-orientations.

2.3.1. *Some further properties of transversality and fiber products.* In this section we collect some well known, though not always easy to find, results about transversality and fiber products. We state these results mainly in the classical setting of manifolds without boundary, though they generally extend to the case of transverse maps of manifolds with corners mapping to a manifold without boundary, either by applying them to the pairwise transverse strata or by thinking in terms of "plain transversality" as defined above.

Lemma 2.19. *Let $f\colon V \to M$ and $g\colon W \to M$ be transverse smooth maps of manifolds without boundary. Suppose $x \in V$ and $y \in W$ with $f(x) = g(y) = z$ so that $(x, y) \in V \times_M W$. Then the tangent space of $V \times_M W$ at (x, y) as a subspace of $T_{(x,y)}(V \times W) = T_xV \oplus T_yW$ consists of those vectors $(\mathbf{v}, \mathbf{w})$ such that $Df(\mathbf{v}) = Dg(\mathbf{w}) \in T_zM$. In other words,*

$$T_{(x,y)}(V \times_M W) = T_xV \times_{T_zM} T_yW,$$

or "the tangent space of the fiber product is the fiber product of the tangent spaces."

A proof can be found in [**?**, Theorem 5.47]. Wedhorn proves this theorem for "premanifolds," which are essentially manifolds minus the Hausdorff and second countability conditions. These will be automatic in our setting, so Wedhorn's proof applies. In many circumstances, this lemma allows us to reduce arguments about fiber products of maps of manifolds to arguments about fiber products of linear maps.

Lemma 2.20. *Let $f\colon V \to M$ and $g\colon W \to M$ be transverse smooth maps of manifolds without boundary, and suppose f is an embedding so that V has a normal bundle νV in M. Then the pullback map $g^*V = V \times_M W \to W$ is an embedding and the normal bundle of g^*V in W is isomorphic to the pullback of νV. In other words,*

$$g^*(\nu V) \cong \nu(g^*V).$$

So "the pullback of the normal bundle is the normal bundle of the pullback."

In case g is also an embedding, this allows us to identify the restriction of νV to $V \cap W$ with a sub-bundle of TW over $V \cap W$.

Proof. See [**?**, Proposition IV.1.4]. □

Lemma 2.21. *Let $f\colon V \to M$ and $g\colon W \to M$ be transverse smooth maps of manifolds without boundary, and suppose f and g are both embeddings. Then, identifying V and W as submanifolds of M, the fiber product $V \times_M W$ is simply the intersection $V \cap W$, which is a smooth submanifold of M.*

Proof. By Lemma 2.20, the pullback is a smooth submanifold of W, but g is itself an embedding, so, identifying the manifolds with their embedded images, we obtain smooth submanifolds $V \times_M W \subset W \subset M$. That $V \times_M W = V \cap W$ in this situation follows immediately from the definition of the fiber product. □

The next lemma is a bit more specific but will be useful later in proving certain formulas.

Lemma 2.22. *Let $f\colon V \to M$, $g\colon W \to M$, and $h\colon X \to W$ be smooth maps of manifolds without boundary, and suppose f is transverse to g. Then $gh\colon X \to M$ is transverse to f if and only h is transverse to the pullback $\pi_W\colon g^*V \to W$.*

Proof. For simplicity of notation, let $P = g^*V = V \times_M W$, and let $\pi_V\colon P \to V$ and $\pi_W\colon P \to W$ be the maps induced by the projections from $V \times W$ to V and W. So we have the diagram

$$\begin{tikzcd}
 & P = V \times_M W \arrow[r, "\pi_V"] \arrow[d, "\pi_W"'] & V \arrow[d, "f"] \\
X \arrow[r, "h"] & W \arrow[r, "g"] & M.
\end{tikzcd}$$

First suppose h is transverse to the pullback $g^*V \to W$. Note that the existence of g^*V uses the assumption that f and g are transverse. Suppose $x \in X$ and $y \in V$ with $gh(x) = f(y)$. Then $(y, h(x)) \in P$, and by assumption we have both $Df(T_yV) + Dg(T_{h(x)}W) = T_{f(y)}M$ and $Dh(T_xX) + D\pi_W(T_{(y,h(x))}P) = T_{h(x)}W$. Applying Dg to the second formula, we have

$$Dg(T_{h(x)}W) = D(gh)(T_xX) + D(g\pi_W)(T_{(y,h(x))}P).$$

So from the first formula,

$$T_{f(y)}M = Df(T_yV) + D(gh)(T_xX) + D(g\pi_W)(T_{(y,h(x))}P).$$

But $g\pi_W = f\pi_V$, so $D(g\pi_W)(T_{(y,h(x))}P) = D(f\pi_V)(T_{(y,h(x))}P) \subset Df(T_yV)$. It follows that

$$T_{f(y)}M = Df(T_yV) + D(gh)(T_xX),$$

i.e. f is transverse to gh.

Next suppose f is transverse to gh and that $x \in X, p \in P$ with $h(x) = \pi_W(p)$. As $gh(x) = g\pi_W(p) = f\pi_V(p)$, we have $gh(x)$ in the image of f, so in particular $gh(x)$ is in the intersection of the images of V and W in M. Now suppose $\mathbf{w} \in T_{h(x)}W$. By assumption $T_{hg(x)}M = Df(T_{\pi_V(p)}V) + D(gh)(T_xX)$, so we can write $Dg(\mathbf{w}) = Df(\mathbf{a}) + Dgh(\mathbf{b})$ for some $\mathbf{a} \in T_{\pi_V(p)}V$ and $\mathbf{b} \in T_xX$. Now consider $\mathbf{z} = \mathbf{w} - Dh(\mathbf{b}) \in T_{h(x)}W$. Applying Dg to both sides, we have $Dg(\mathbf{z}) = Dg(\mathbf{w}) - Dgh(\mathbf{b}) = Df(\mathbf{a})$. So $(\mathbf{a}, \mathbf{z})$ is in the pullback $T_{\pi_V(p)}V \times_{T_{gh(x)}M} T_{h(x)}W$. But by Lemma 2.19, this is precisely the tangent space of $V \times_M W$ at $(h(x), \pi_V(p))$. Furthermore, we have $D\pi_W(\mathbf{a}, \mathbf{z}) = \mathbf{z}$. Thus $\mathbf{w} = \mathbf{z} + Dh(\mathbf{b}) = D\pi_W(\mathbf{a}, \mathbf{z}) + Dh(\mathbf{b})$. As $\mathbf{w}$ was arbitrary, $W = D\pi_W(T_{(h(x),\pi_V(p))}P) + Dh(T_xX)$, as desired. ☐

2.3.2. *Fiber products with more than three inputs.* Finally, we briefly consider transversality of more than two maps. Remark 2.23 highlights some of the issues and difficulties involved, while Proposition 2.24 shows that there are still some simplifications that can be observed.

Remark 2.23. In this long remark we briefly discuss transversality and fiber products that involve more than two maps. This is relevant, for example, when considering associativity of fiber products or pullbacks of fiber products. As transversality of maps of manifolds is defined in terms of the behavior of the maps of tangent spaces, it is useful to first recall some notions about transversality in the setting of linear maps of vector spaces.

If $f\colon V \to M$ and $g\colon W \to M$ are linear maps, then transversality of f and g can be expressed in a number of equivalent ways [**?**, Section 4.7]:

- $f(V)$ and $g(W)$ span M,
- the map $\Delta\colon V \times W \to M$ given by $\Delta(v, w) = f(v) - g(w)$ is surjective,
- $\dim(V \times_M W) = \dim(V) + \dim(W) - \dim(M)$.

The last two formulations easily generalize to n-tuples of maps $f_i\colon V_i \to M$. Such an n-tuple is considered transverse (as an n-tuple) when either of the following equivalent conditions hold:

- the map $\Delta\colon \prod V_i \to M^{n-1}$ given by $\Delta(v_1, \dots, v_n) = (f_2(v_2) - f_1(v_1), \dots, f_n(v_n) - f_{n-1}(v_{n-1}))$ is surjective,
- the fiber product given by $\{(v_1, \dots, v_n) \in V_1 \times \cdots \times V_n \mid f_1(v_1) = \cdots = f_n(v_n)\}$ has dimension $\sum_{i=1}^n \dim(V_i) - n\dim(M)$.

This version of transversality behaves very well in that this "n-transversality" is equivalent to the iterated transversality conditions that are required when taking fiber products two at a time. In other words, an n-tuple is transverse if and only if for any $1 \le i < j \le n$,

- the $j - i + 1$-tuple $\{f_i, \dots, f_j\}$ is transverse, and
- letting P denote the fiber product of the maps $\{f_i, \dots, f_j\}$, the $n - (j - i)$-tuple consisting of the fiber product map $P \to M$ and the f_k with $k \notin \{i, \dots, j\}$ is transverse.

Iterating this fact, we can see that this is equivalent to having V_1 transverse to V_2, then V_3 transverse to $V_1 \times_M V_2$ and so on. In particular, we can form the n-tuple fiber product if and only if we can form the *transverse* iterated fiber products such as $(((V_1 \times_M V_2) \times_M V_3) \times_M \ldots) \times_M V_n$ as well as in any other order of association. See[8] [**?**, Propositions 4-9 and 8-1].

Unfortunately, in the setting of maps of manifolds (for the moment without corners), the situation is less well behaved. Let now $f_i \colon V_i \to M$ be an n-tuple of smooth maps of manifolds. We say that this n-tuple is transverse when for any n-tuple $(v_1, \ldots, v_n) \in V_1 \times \cdots \times V_n$ such that $f_1(v_1) = \cdots = f_n(v_n)$ the linear maps $D_{v_i} f_i$ are transverse. In this case, the fiber product $P = \{(v_1, \ldots, v_n) \in V_1 \times \cdots \times V_n \mid f_1(v_1) = \cdots = f_n(v_n)\}$ is a smooth manifold of dimension $\sum \dim(V_i) - (n-1)\dim(M)$. Furthermore, at any such point, this general transversality has implications such as those above for transversality of subcollections.

However, the above relations between n-ary transversality and "iterated transversality" cannot hold in general, because the n-ary fiber product cannot necessarily know about all points in the iterated fiber product. For example, consider maps $f \colon V \to M$, $g \colon W \to M$, and $h \colon Z \to M$. Suppose that f and g are not transverse; in particular, we can suppose that the fiber product $V \times_M W$ is not a manifold. Further suppose that $h(Z)$ is disjoint from $f(V)$ and $g(W)$. In this case, the triple of maps is transverse (vacuously) and its 3-ary fiber product is well defined as $\emptyset$. However, the iterated fiber product $(V \times_M W) \times_M Z$ is not well defined in the category of smooth manifolds and maps. Given that we define fiber products only of transverse maps of smooth manifolds, in this case $V \times_M W$ is not properly defined in our category, and it is further impossible then to define $(V \times_M W) \times_M Z$ in this context.

The upshot of all this is that when considering situations involving fiber products of more than two maps, we shall have to be careful about the transversality assumptions.

Despite the preceding remark, the transversality conditions involved in associativity of fiber products are not completely independent, as the following proposition shows. In later sections, we will usually simply assume all needed transversality exists, but the following proposition can be useful in practice.

Proposition 2.24. *Let $f : V \to M$, $g : W \to M$, and $h : Z \to M$ be maps from manifolds with corners to a manifold without boundary. Suppose that W is transverse to Z and that V is transverse to W and to $W \times_M Z$. Then $V \times_M W$ is transverse to Z. In particular, if $V \times_M (W \times_M Z)$ and $V \times_M W$ are well defined, then so is $(V \times_M W) \times_M Z$.*

Proof. We must show that $V \times_M W$ is transverse to Z, so we consider points $(v, w) \in V \times_M W$ and $z \in Z$ such that $h(z)$ is equal to $(f \times_M g)(v, w)$, which by definition is equal to $f(v) = g(w)$. In other words, we consider $(v, w, z) \in V \times W \times Z$ such that $f(v) = g(w) = h(z)$.

So suppose (v, w, z) is such a triple, and denote the common image by $m \in M$. By the transversality assumptions, we know that the images of $D_w g : T_w W \to T_m M$ and $D_z h : T_z Z \to T_m M$ span $T_m M$, i.e. that $D_w g$ and $D_z h$ are transverse as linear maps, and similarly that $D_{(w,z)}(g \times_M h) : T_{(w,z)}(W \times_M Z) \to T_m M$ is transverse to $D_v f : T_v V \to T_m M$. Furthermore, by Lemma 2.19, the tangent space of a fiber product is the fiber product of the tangent spaces, so $T_{(w,z)}(W \times_M Z) = T_w W \times_{T_m M} T_z Z$ and $D_{(w,z)}(g \times_M h) = D_w g \times_{T_m M} D_z h$.

Now by [**?**, Propositions 4-9], the triple of linear maps $(D_v f, D_w g, D_z h)$ is transverse as a triple of maps, if and only if both $D_w g$ is transverse to $D_z h$ and $D_w g \times_{T_m M} D_z h$ is transverse to $D_v f$. As such statements are independent of how we order the terms, the transversality established in the preceding paragraph also implies that $D_v f$ and $D_w g$ are transverse (which already follows from the hypotheses of the proposition), and $D_v f \times_{T_m M} D_w g$ is transverse to $D_z h$. But this implies, again using Lemma 2.19, that h is transverse to $f \times_M g$, as desired. □

[8]Proposition 8-1 of [?] actually concerns a more general situation of "mixed associativity" in which the data consists of zig-zags

$$V_1 \xrightarrow{f_1} S_1 \xleftarrow{g_2} V_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} S_{n-1} \xleftarrow{g_n} V_2.$$

However, this reduces to our setting by taking $S_i = M$ and $g_i = f_i$ for all i.

Remark 2.25. The end of the preceding proof at first seems to imply that if g and h are transverse and $g \times_M h$ is transverse to f, then f is transverse to g and $f \times_M g$ is transverse to h. Indeed, [**?**, Propositions 4-9] says this is the case for the linear maps of the tangent spaces. Unfortunately, however, as [**?**, Propositions 4-9] applies only to linear maps, we can apply it only at those points $(v, w, z) \in V \times W \times Z$ where we know that $f(v) = g(w) = h(z)$ so that all three tangent space maps are well defined. So such a result would hold if the only intersections among the maps were such triple intersections. However, as noted in Remark 2.23, there could be pairs $(v, w) \in V \times W$ with $f(v) = g(w)$, but with this common image in M not in the image of h. At such points, [**?**, Propositions 4-9] cannot tell us anything about the transversality of V and W, and so $V \times_M W$ might not be well defined due to failure of transversality, even if $V \times_M (W \times_M Z)$ is.

3. Orientations and co-orientations

Manifolds with corners are, in particular, topological manifolds, and so they carry the standard notions of orientability and orientation. As in singular or simplicial homology, orientations carry sign information in geometric versions of homology theory. For geometric cohomology, however, it turns out that the natural structures to carry sign information are co-orientations, sometimes called orientations of maps. Unlike orientations, co-orientation can be "pulled back."

Co-orientations are less familiar than orientations, so it is helpful to keep the following central example in mind: If $W \to M$ is an immersion of manifolds, a co-orientation is equivalent to an orientation of the normal bundle of the image; see Definition 3.15 below. Notice that this condition does not require the orientability of either W or M. In fact, an important case is when neither W nor M is orientable but the monodromies of their orientation bundles around loops in W are either both orientation-preserving or both orientation-reversing; it is in this sense that we have a *co*-orientation.

The special case of (local) immersions is important both for intuition and in practice; for example, the only geometric cochains which evaluate non-trivially on fixed collections of chains through the intersection homomorphisms of [**?**] are local immersions. However, for complete definitions that cover all cases of interest, it is critical to co-orient more general maps and to do so in a way that provides key properties when forming pullbacks, such as a Leibniz rule when taking boundaries, graded commutativity of cochains, and a simple expression when the maps are immersions. While co-orientations can be found in many places in the literature, we could not find a careful treatment that guaranteed these key properties. Therefore, we develop co-orientations in depth in this (long) section. Some readers may prefer to skip ahead, either considering geometric cohomology only with mod-two coefficients in a first reading, or perhaps take orientation of the normal bundle as a temporary partial definition, coming back later to understand the general setting.

3.1. **Orientations.**

If W is a manifold with corners then it is a topological manifold with boundary and so in the interior of W, i.e. on $S^0(W)$, we can consider orientability and orientations for W in the usual sense, from either the topological or smooth manifold points of view, which are equivalent [**?**, Theorem VI.7.15]. Following standard conventions, we typically refer to an oriented manifold with corners W with the orientation tacit. If an orientation on W is understood, then $-W$ refers to W with the opposite orientation.

When W is orientable, so is its (topological) boundary [**?**, Lemma 6.9.1], and since $S^0(W) \cup S^1(W)$ is a smooth manifold with boundary, we can allow an orientation of W to determine an orientation for $S^1(W)$ using standard smooth manifold conventions. In particular, we choose the following convention, which agrees with that of Joyce [**?**, Convention 7.2.a]:

Convention 3.1. For a smooth oriented manifold with boundary N, we orient ∂N by stipulating that an outward normal vector followed by an oriented basis of ∂N yields an oriented basis for N.

When W is an oriented manifold with corners, we can identify $S^1(W)$ with the interior of ∂W, and this convention determines an orientation of ∂W.

When we wish to work with orientations symbolically, the following interpretation will be extremely useful.

Definition 3.2. Let $E \to M$ be a rank d real vector bundle. Define the **determinant line bundle** $\mathrm{Det}(E)$ to be $\bigwedge^d E$. If $d = 0$ this is interpreted to be the trivial rank one "bundle of coefficients." We call the principal $O(1) \cong C_2$ bundle associated to $\mathrm{Det}(E)$ the **orientation cover**.

An orientation of E is then a section of the orientation cover associated to $\mathrm{Det}(E)$ or, equivalently, an equivalence class of non-zero sections of $\mathrm{Det}(E)$ such that two sections are equivalent if they differ by multiplication by an everywhere positive scalar function.

In particular, we thus think of orientations of M^m as m-forms in $\mathrm{Det}(TM)$. We typically use the notation β_M to stand for such an m-form, and, as we are only ever interested in m-forms to represent orientations, we systematically abuse notation by not distinguishing between an m-form and its equivalence class. Thus all expressions such as $\beta_M = \beta_V \wedge \beta_W$ should be interpreted as equality of equivalence classes.

3.1.1. *Orientations of fiber products.* If V and W are oriented manifolds, we orient $V \times W$ in the standard way by concatenating oriented bases of tangent spaces of V with those of W. More generally, if $f : V \to M$ and $g : W \to M$ are transverse maps with V, W, and M all oriented and M without boundary, Joyce defines an orientation on the pullback $V \times_M W$ as follows [**?**, Convention 7.2b]. Consider the short exact sequence of vector bundles over $P = V \times_M W$ given by

$$0 \to TP \xrightarrow{D\pi_V \oplus D\pi_W} \pi_V^*(TV) \oplus \pi_W^*(TW) \xrightarrow{\pi_V^*(Df) - \pi_W^*(Dg)} (f\pi_V)^*TM \to 0. \tag{3}$$

Here π_V and π_W are the projections of $V \times_M W$ to V and W, and $D\pi_V$ is being treated as a map $TP \to \pi_V^* TV$ and similarly for $D\pi_W$. Analogously, $\pi_V^*(Df)$ is the pullback of the map $Df : TV \to TM$ obtained first by treating it as a map $TV \to f^*(TM)$ and then pulling back functorially by π_V^*, and similarly for $\pi_W^*(Dg)$. By choosing a splitting, this sequence determines an isomorphism $TP \oplus (f\pi_V)^*TM \cong \pi_V^*(TV) \oplus \pi_W^*(TW)$. The choices of orientations determine orientations on all summands in this expression except TP. The orientation on TP is then chosen so that the two direct sums differ in orientation by a factor of $(-1)^{\dim(W)\dim(M)}$.

Much more about the orientation of fiber products can be found in the technical report of Ramshaw and Basch [**?**]. While the focus there is on manifolds without boundary, and sometimes just fiber products of linear maps of vector spaces, the results about orientations extend to manifolds with corners by employing them on the top-dimensional stratum and utilizing their stability property, by which orientation properties of fiber products of linear maps extend to properties of fiber products of transverse manifolds (see [**?**, Sections 6.3, 9.1.2, and 9.3]). Their orientation of fiber products agrees with Joyce's. This can be checked directly from the definitions[9] or, as Joyce notes in [**?**, Remark 7.6.iii], axiomatically, as Ramshaw and Basch show that theirs is the unique choice of orientation convention satisfying certain basic expected properties. It is these properties that determine the sign in the definition. We state these properties in the following two propositions.

Proposition 3.3. *Let $f : V \to M$ and $g : W \to M$ be transverse maps from oriented manifolds with corners to an oriented manifold without boundary.*

1. *When M is a point, the oriented fiber product $V \times_M W$ is simply $V \times W$, and in this case the fiber product orientation is consistent with the basic concatenation rule for products.*
2. *When one of the maps is the identity $\mathrm{id}_M : M \to M$, the projection maps to the other factors are orientation preserving diffeomorphisms*

$$M \times_M V = V \quad \textit{and} \quad V \times_M M = V$$

[9]Their multiplicative "fudge factor" in [**?**, Theorem 9.14] at first appears to be different from Joyce's, but this is only because their conventions utilize what in our notation would be the map $\pi_W^*(Dg) - \pi_V^*(Df)$ rather than $\pi_V^*(Df) - \pi_W^*(Dg)$.

Proposition 3.4. *Let V, W, and Z be oriented manifolds with corners, and let M and N be oriented manifolds without boundary. Then the "mixed associativity" formula for oriented fiber products*

$$(V \times_M W) \times_N Z = V \times_M (W \times_N Z) \tag{4}$$

holds when given maps

$$V \xrightarrow{f} M \xleftarrow{g} W \xrightarrow{h} N \xleftarrow{k} Z$$

and assuming sufficient transversality for all the fiber products in (4) to be well defined (see Remark 2.23). In this case the map $V \times_M W \to N$ is given by composing the projection from $V \times_M W$ to W with h, and similarly for the map $W \times_N Z \to M$.

These propositions are evident at the level of spaces. When taking orientations into account, the first property in Proposition 3.3 is proven in [**?**, Sections 9.3.9] as the "concatenation axiom," and the second is proven in [**?**, Sections 9.3.5 and 9.3.6] as the "left and right identity axioms." The mixed associativity property is proven in [**?**, Sections 9.3.7]. An important special case of this associativity that we will need below occurs when $M = N$ and $g = h$, so that our initial data is three maps all to M. In this case we have the ordinary associativity

$$(V \times_M W) \times_M Z = V \times_M (W \times_M Z). \tag{5}$$

That these properties determine the orientation rule is the content of [**?**, Theorem 9-10]. Technically they require for uniqueness two other properties: an Isomorphism Axiom, which says that the construction is consistent across oriented homeomorphisms, and a Stability Axiom, which implies that the orientation can be determined pointwise in a globally consistent manner. These properties are both implicit in Joyce's global definition of the fiber product orientation.

There is also a commutativity rule proven in [**?**, Sections 9.3.8] that follows from the other properties:

Proposition 3.5. *Let $f : V \to M$ and $g : W \to M$ be transverse maps from oriented manifolds with corners to an oriented manifold without boundary. Then, as oriented manifolds,*

$$V \times_M W = (-1)^{(m-v)(m-w)} W \times_M V.$$

This means that the canonical diffeomorphism taking $(v, w) \in V \times_M W \subset V \times W$ to $(w, v) \in W \times_M V \subset W \times V$ takes a positively-oriented basis of the tangent space of $V \times_M W$ to a $(-1)^{(\dim M - \dim V)(\dim M - \dim W)}$-oriented basis of the tangent space of $W \times_M V$. We note that these signs, while note quite in line with the Koszul conventions, agree with those for the intersection product of homology classes in Dold [**?**, Section VIII.13].

Furthermore, with our convention for oriented boundaries, one obtains the following useful identity of oriented manifolds with corners; see [**?**, Propositions 7.4 and 7.5]

Proposition 3.6. *Let $f : V \to M$ and $g : W \to M$ be transverse maps from oriented manifolds with corners to an oriented manifold without boundary. Then, as oriented manifolds,*

$$\partial(V \times_M W) = ((\partial V) \times_M W) \sqcup (-1)^{m-v} (V \times_M \partial W)$$

Fiber products of immersions. The special case of fiber products with $f : V \to M$ and $g : W \to M$ embeddings or, a bit more generally, immersions, is always of particular interest, especially for developing intuition. In the case of embeddings, $V \times_M W$ is simply the intersection of V and W as submanifolds of M, and in the immersed case this is true locally, i.e. restricting attention to submanifolds of V and W on which f and g are embeddings. Let us try to understand the orientation of $V \times_M W$ in this setting.

For convenience of notation, let us assume f and g are embeddings and write $P = V \times_M W = V \cap W$. As orientations are defined via the tangent bundles and as the tangent bundle of a fiber product is the fiber product of the tangent bundles by Lemma 2.19, it suffices to consider f and g as linear embeddings of vector spaces. Then we can identify V, W, and P as subspaces and write $M = \nu W \oplus P \oplus \nu V$, where

$\nu W \subset V$ and $\nu V \subset W$ are complementary subspaces to P within V and W respectively. In particular, $V = P \oplus \nu W$ and $W = P \oplus \nu V$, and we think of νW and νV as representing normal bundles to P. With these identifications, the maps f and g are simply the identifications with the appropriate subspaces.

Using the decompositions, we can now write a splitting $M \to V \oplus W$ of the exact sequence (3) in terms of these decomposition as

$$\nu W \oplus P \oplus \nu V \to P \oplus \nu W \oplus P \oplus \nu V,$$

given by $(x, p, y) \to (0, x, -p, -y)$. The signs are necessary as, in our current linear setting, the surjective map of the short exact sequence if $f - g$. Our isomorphism $P \oplus M \to V \oplus W$ thus has block form

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ I & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{pmatrix},$$

which has determinant $(-1)^w$. By definition, we choose an orientation of P so that this matrix takes the concatenation of the orientation of P with an orientation of M to $(-1)^{wm}$ times the concatenation of the orientations of V and W.

Proposition 3.7. *Let β_V, β_W, and β_M be the given orientations for V, W, and M. Then the fiber product orientation for $P = V \times_M W$ is the unique the orientation β_P such that*[10] *if we choose orientations $\beta_{\nu W}$ and $\beta_{\nu V}$ for νW and νV such that $\beta_P \wedge \beta_{\nu W} = \beta_V$ and $\beta_P \wedge \beta_{\nu V} = \beta_W$ then $\beta_{\nu W} \wedge \beta_P \wedge \beta_{\nu V} = (-1)^{w(m+1)}\beta_M$ or, alternatively,*

$$\beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W} = \beta_M.$$

Proof. Note that if we replace β_P with its opposite orientation, then this must also reverse the orientations $\beta_{\nu W}$ and $\beta_{\nu V}$ and hence altogether we get the opposite orientation for $\beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W}$. Thus there is a unique such β_P as described.

It will be more convenient to prove the lemma in the first form, but the second form follows by observing that

$$(-1)^{w(m+1)} = (-1)^{wm+w} = (-1)^{wm-w^2} = (-1)^{w(m-w)},$$

and then

$$\beta_{\nu W} \wedge \beta_P \wedge \beta_{\nu V} = (-1)^{w(m-w)}\beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W}$$

as $P \oplus \nu V = W$ and $\dim(\nu W) = m - w$.

To prove the first statement, let $(p_1, \cdots, p_a)$ be an ordered basis for P consistent with the orientation described in the lemma; so we can write $\beta_P = p_1 \wedge \cdots \wedge p_a$. When we consider each p_i as a vector in V, W, or M, we write p_i^V, p_i^W, or p_i^M. We employ a similar convention with the other bases we will consider. Let $(x_1, \cdots, x_b)$ and $(y_1, \cdots, y_c)$ be corresponding ordered bases for νW and νV as described in the lemma, and we can write $\beta_{\nu W}$ and $\beta_{\nu V}$ analogously as for β_P. Then, we have β_M represented by $(-1)^{w(m+1)}\beta_{\nu W} \wedge \beta_P \wedge \beta_{\nu V}$. So our orientation of $P \oplus M$ obtained by concatenation is

$$(-1)^{w(m+1)} p_1 \wedge \cdots \wedge p_a \wedge x_1^M \wedge \cdots \wedge x_b^M \wedge p_1^M \wedge \cdots \wedge p_a^M \wedge y_1^M \wedge \cdots \wedge y_c^M.$$

When we apply our matrix above, we obtain the form in $V \oplus W$ given by

$$(-1)^{w(m+1)}(p_1^V + p_1^W) \wedge \cdots \wedge (p_a^V + p_a^W) \wedge x_1^V \wedge \cdots \wedge x_b^V \wedge (-p_1^W) \wedge \cdots \wedge (-p_a^W) \wedge (-y_1^W) \wedge \cdots \wedge (-y_c^W).$$

This expression simplifies to

$$(-1)^{wm} p_1^V \wedge \cdots \wedge p_a^V \wedge x_1^V \wedge \cdots \wedge x_b^V \wedge p_1^W \wedge \cdots \wedge p_a^W \wedge y_1^W \wedge \cdots \wedge y_c^W.$$

[10] Note that as P and νW are subspaces of V, we may consider β_P and $\beta_{\nu W}$ as forms over V, and similarly for the other expressions that follow.

But this is now precisely $(-1)^{wm}$ times the concatenation orientation of $V \oplus W$ as desired for definition of the fiber product orientation. $\square$

Example 3.8. As an example, let $M = \mathbb{R}^3$ oriented by the standard ordered basis (e_x, e_y, e_z). Let V be the $z = 0$ plane oriented by the ordered basis vectors (e_x, e_y), and let W be the $x = 0$ plane oriented by the ordered basis vectors (e_y, e_z). The intersection P is the y axis. We claim that P should be oriented by $-e_y$. Indeed, in this case we have $\beta_V = e_x \wedge e_y = -e_y \wedge e_x = (-e_y) \wedge e_x$, so $\beta_{\nu W} = e_x$, and $\beta_W = e_y \wedge e_z = (-e_y) \wedge (-e_z)$, so $\beta_{\nu V} = -e_z$. And then

$$\beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W} = -e_y \wedge (-e_z) \wedge e_x = e_x \wedge e_y \wedge e_z = \beta_M,$$

as required.

Corollary 3.9. *Suppose V and W have complementary dimensions so that they intersect in a point. Then the point is positively oriented if and only if $\beta_W \wedge \beta_V = \beta_M$.*

Proof. In this case, $\nu W = V$, $\nu V = W$, and $\beta_P = \pm 1 \in \mathbb{R}$. If $\beta_P = 1$, then the formula from Proposition 3.7 becomes exactly the formula of the corollary. $\square$

Example 3.10. Let $M = \mathbb{R}^2$ with the standard orientation that we can write $e_x \wedge e_y$. Let V be the x-axis with orientation e_x and W be the y-axis oriented by e_y. Then it is false that $e_x \wedge e_y = \beta_V \wedge \beta_W = (-1)^{w(m+1)}\beta_M = -e_x \wedge e_y$. So the orientation of the intersection point is the negative one. This runs against the standard convention for transverse intersections of manifolds of complementary dimension, but we nonetheless favor this overall convention for orienting fiber products due to the properties and uniqueness result of [**?**].

3.2. **Co-orientations.** To define co-orientations, we recall our definition of an orientation of a bundle from Definition 3.2 as an equivalence class, up to positive scalar multiplication, of an everywhere non-zero section of the top exterior power of the bundle. This motivates the following.

Definition 3.11. A **co-orientation** ω_g of a *continuous*[11] map $g : W \to M$ of manifolds with corners is an equivalence class, up to positive scalar multiplication, of a nowhere zero section of the line bundle $\mathrm{Hom}(\mathrm{Det}(TW), \mathrm{Det}(g^*TM)) \cong \mathrm{Hom}(\mathrm{Det}(TW), g^*\mathrm{Det}(TM))$. Equivalently, a co-orientation is a choice of isomorphism between the associated orientation cover $\mathrm{Det}(TW)$ and the pullback of the associated orientation cover $\mathrm{Det}(TM)$.

Thus, if W is connected and $g : W \to M$ is co-orientable, there are exactly two co-orientations, which are **opposite** to one another; we write the opposite of ω_g as $-\omega_g$. In particular, for connected W a choice of co-orientation at a single point determines a co-orientation globally when g is co-orientable (analogously to orientations). Also, just as most manifolds do not possess a preferred orientation, most maps $g : W \to M$ do not carry a natural choice of co-orientation.

Definition 3.12. An exception to the lack of a natural choice of co-orientation occurs when g is a diffeomorphism or, more generally, a codimension-0 immersion in which case the top exterior power of Dg provides a **tautological co-orientation**.

The local triviality of the determinant line bundle of a manifold means being able to choose a consistent basis vector over sufficiently small neighborhoods. We call such a choice of basis vector around a point in W a **local orientation**, and, as for global orientations, often denote the local orientation β_W. Again, abusing notation, we also often allow β_W to refer to its equivalence class up to multiplication by a positive scalar. We identify β_W with a local choice of (equivalence class of) non-zero $\dim(W)$-form[12] in $\bigwedge^{\dim(W)} TW$ in a

[11] We will most often be interested in the case of g smooth, but continuous co-orientable maps do come up in Section 5 where we consider covariant functoriality of geometric cohomology with respect to continuous maps.

[12] As usual, if $\dim(W) = 0$ we identify $\bigwedge^0 TW$ with the ground field $\mathbb{R}$, and, when forming exterior products, multiplication by a 0-form is treated as scalar multiplication.

neighborhood of a point x in W or, equivalently, a local smoothly varying ordered basis for the fibers of TW. We typically don't specify the point x, though when necessary we write $\beta_{W,x}$. We then use ordered-pair notation for co-orientation homomorphisms, with $\omega_g = (\beta_W, \beta_M)$ being the **local co-orientation** that sends the local orientation β_W at $x \in W$ to a local orientation β_M for $g^*(TM)$ up to a positive scalar. We will often further abuse notation by neglecting the pullback and treating β_M as a local orientation at $g(x)$ in M. We also write the opposite co-orientation $(\beta_W, -\beta_M) = (-\beta_W, \beta_M)$ as $-(\beta_W, \beta_M)$. As a co-orientation at a point completely determines the co-orientation of a co-orientable map for connected W, it is useful to cheat further and write $\omega_g = (\beta_W, \beta_M)$ for appropriate β_W and β_M with the chosen points x implicit.

A manifold is orientable if and only if the orientation cover is trivial. So if M is orientable, $\mathrm{Det}(g^*(TM))$ is trivial, and a co-orientation of $g : W \to M$ implies that W is orientable. Moreover, an orientation on M along with a co-orientation of g gives rise to an **induced orientation** of W. Explicitly, if β_M denotes the global orientation of M, then we orient W at each point by the β_W such that $\omega_g = (\beta_W, \beta_M)$. Conversely, if M and W are both oriented, say by β_M and β_W respectively, we have the **induced co-orientation** given by $\omega_g = (\beta_W, \beta_M)$ at each point of W. On the other hand, it is not true that if we have an orientation of W and a co-orientation of $g : W \to M$ then we obtain an orientation of M. For example, if W is orientable, any constant map to M is co-orientable, regardless of whether or not M is orientable.

More generally, recall that the fundamental group of a manifold acts on classes of local orientations as the deck transformations of the orientation cover. A map is co-orientable if a loop in W acts nontrivially on a local orientation of W if and only if its image in M acts nontrivially. Explicitly, if $g : W \to M$ is co-orientable the fundamental group can change the local orientation pair (β_W, β_M) to $(-\beta_W, -\beta_M)$, but these pairs define equivalent co-orientations. Similarly, to compare local constructions at different points, it is useful to use paths. Suppose $\gamma : I \to W$ is a path with $\gamma(0) = x$ and $\gamma(1) = y$. We can choose a lift $\tilde{\gamma}$ of γ to the complement of the 0-section of $\mathrm{Det}(TW)$ such that $\tilde{\gamma}(0)$ is in the equivalence class of $\beta_{W,x}$. We then define $\gamma_*\beta_W$ to be the equivalence class of $\tilde{\gamma}(1)$. Likewise, we define $(g\gamma)_*\beta_M$ by a lift of $g\gamma$ to the complement of the zero section of $\mathrm{Det}(TM)$. Then $\gamma_*\beta_W$ and $(g\gamma)_*\beta_M$ depend on γ, but if $g : W \to M$ is co-orientable the pair $(\gamma_*\beta_W, (g\gamma)_*\beta_M)$ is independent of γ as this data also determines a non-vanishing lift of γ in $\mathrm{Hom}(\mathrm{Det}(TW), g^*\mathrm{Det}(TM))$, which is trivial if g is co-orientable. In particular, if g is co-oriented and (β_W, β_M) represents the choice of co-orientation locally at x, then $\gamma_*(\beta_W, \beta_M) := (\gamma_*\beta_W, (g\gamma)_*\beta_M)$ will represent the same co-orientation locally at y.

Example 3.13. Let g be any map $g : S^1 \to S^2$. As S^1 and S^2 are orientable, g is co-orientable. If we choose a local orientation vector e_θ at any point $x \in S^1$ and latitude/longitude coordinates ϕ, ψ at $g(x)$ so that[13] $e_\phi\, e_\psi$ is a local orientation in a neighborhood of $g(x)$, then the two possible co-orientations for for g can be written $(e_\theta, e_\phi\, e_\psi)$ and $-(e_\theta, e_\phi\, e_\psi) = (-e_\theta, e_\phi\, e_\psi) = (e_\theta, -e_\phi\, e_\psi)$. While the notation explicitly references a local orientation at a point, this is sufficient to determine the co-orientation globally. In what follows we will often demonstrate properties of co-orientations by showing that they hold locally at an arbitrary point but do not depend on the choice of point.

As another example, consider the standard embedding $g : \mathbb{R}P^2 \hookrightarrow \mathbb{R}P^4$. Choosing local orientations $e_1\, e_2$ at some $x \in \mathbb{R}P^2$ and $f_1\, f_2\, f_3\, f_4$ at $g(x)$, the two co-orientations are $(e_1\, e_2, f_1\, f_2\, f_3\, f_4)$ and its opposite. If γ is a loop that reverses the orientation of $\mathbb{R}P^2$ then it also reverses the orientation of $\mathbb{R}P^4$, so $\gamma_*(e_1\, e_2, f_1\, f_2\, f_3\, f_4) = (-e_1\, e_2, -f_1\, f_2\, f_3\, f_4) = (e_1\, e_2, f_1\, f_2\, f_3\, f_4)$, reflecting that g is co-orientable.

By contrast, no embedding of the Möbius strip in $\mathbb{R}^3$ is co-orientable.

Remark 3.14. Co-oriented maps compose in an immediate way, forming a category. Namely, given $V \xrightarrow{f} W \xrightarrow{g} M$ and co-orientations $\mathrm{Det}(TV) \to \mathrm{Det}(f^*TW)$ and $\mathrm{Det}(TW) \to \mathrm{Det}(g^*TM)$, we simply compose the former with the pullback of the latter via f^*, recalling that $f^*(\mathrm{Det}(E)) = \mathrm{Det}(f^*E)$ in a natural way. We will refer to this simply as composing co-orientations and write the composition in

[13]Following [?] we will generally omit the exterior product symbol $\wedge$ when working with explicit coordinate choices.

symbols as $\omega_f * \omega_g$. Warning: note that we write the terms in the order $\omega_f * \omega_g$ for the map $g \circ f$. This is more convenient when writing out co-orientations using the local orientations as we obtain expressions such as $(\beta_V, \beta_W) * (\beta_W, \beta_M) = (\beta_V, \beta_M)$.

Notation. It will be useful in notation to sometimes leave the maps, codomains, and co-orientations all implicit once they have already been established and just write V to represent the co-oriented map $f : V \to M$. In this case we write $-V$ to refer to the same map with the opposite co-orientation.

3.3. **Normal co-orientations of immersions and co-orientations of boundaries.** As mentioned, a key example is when g is an immersion, which is co-orientable if and only if its normal bundle is orientable[14]. Specifically, if $g : W \to M$ is an immersion, letting νW denote the normal bundle, we have $TW \oplus \nu W \cong g^*TM$. So, taking $w = \dim(W)$ and $m = \dim(M)$, a co-orientation is a nowhere-zero map from $\bigwedge^w TW$ to $\bigwedge^m g^*TM = \bigwedge^m(TW \oplus \nu W) \cong \bigwedge^w TW \otimes \bigwedge^{m-w} \nu W$. Such a nowhere-zero map exists if and only $\bigwedge^{m-w} \nu W$ is a trivial line bundle.

Although such a co-orientation is a global object over W, given an orientation of νW we can specify a standard choice of "normal co-orientation" through the following local construction:

Definition 3.15. Let $g : W \to M$ be an immersion with normal bundle locally oriented by β_ν. Define the **normal co-orientation** associated to β_ν locally as[15] $\omega_\nu = (\beta_W, \beta_W \wedge \beta_\nu)$, where β_W is any choice of a local orientation of W.

This construction is independent of the choice of β_W, as reversing the orientation of β_W gives $(-\beta_W, -\beta_W \wedge \beta_\nu) = (\beta_W, \beta_W \wedge \beta_\nu)$. If the normal bundle to W is oriented globally on W then the construction is also independent of the point at which it is carried out since if γ is a path from x to y with β_W and β_ν constructed at x then

$$(\gamma_*\beta_{W,x}, (g\gamma)_*(\beta_{W,x} \wedge \beta_{\nu,x})) = (\gamma_*\beta_{W,x}, (\gamma_*\beta_{W,x}) \wedge (g\gamma)_*\beta_{\nu,x}) = (\gamma_*\beta_{W,x}, (\gamma_*\beta_{W,x}) \wedge \beta_{\nu,y}),$$

using that ν is assumed oriented and that, via the immersion, we can treat a path in W as a path in M. So if the normal bundle to W is oriented these local choices determine a global co-orientation of $W \to M$. If the normal bundle is orientable, one can conversely orient the normal bundle if one is given a co-orientation. Signs in the theory are highly dependent on choices. One such choice in this definition is whether to append the local normal orientation before or after the local tangent orientation.

3.3.1. *Quillen co-orientations.* There is a useful alternative, though equivalent, definition of co-orientations due to Quillen [**?**] that only involves orientations of normal bundles.[16]

Lemma 3.16. *A map $g : W \to M$ is co-orientable if and only if for some $N \in \mathbb{Z}_{\geq 0}$ it factors as the composition $W \hookrightarrow M \times \mathbb{R}^N \to M$ of an embedding and a projection such that the image of W in $M \times \mathbb{R}^N$ has an orientable normal bundle.*

Proof. We first note that such a smooth factorization always exists. For example, let $j : W \to \mathbb{R}^N$ be an embedding as guaranteed by Whitney's theorem. Then define $e : W \hookrightarrow M \times \mathbb{R}^N$ by $e(x) = (g(x), j(x))$ and $\pi : M \times \mathbb{R}^N \to M$ the projection. Clearly $\pi e = g$.

[14]Recall that technically all bundles in the conversation are over W, though our convention is to elide that in the notation. Hence we can consider W to have a normal bundle even if g is merely an immersion and not actually an embedding. The normal bundle can be identified with $g^*(TM)/TW$ after identifying TW with a subbundle of $g^*(TM)$ using the differential. In any case, locally in the neighborhood of any point of W one has the usual identification of the normal bundle with a tubular neighborhood of the image, which suffices for our purposes here.

[15]Whenever we form wedge products, if one of the terms is an element of $\bigwedge^0 TV$ for some V we treat that term as a 0-form and interpret $\wedge$ as the fiberwise scalar product.

[16]Quillen's context was slightly different. He assumed the normal bundles to have complex structures and so called these *complex orientations*.

Next we note that $T(M \times \mathbb{R}^N) \cong \pi^*(TM) \oplus \underline{\mathbb{R}}^N$ where $\underline{\mathbb{R}}^N$ is the trivial $\mathbb{R}^N$ bundle over $M \times \mathbb{R}^N$. As $\mathrm{Det}(\underline{\mathbb{R}}^N)$ is a trivial line bundle,

$$\mathrm{Det}(T(M \times \mathbb{R}^N)) \cong \mathrm{Det}(\pi^*(TM)) \otimes \mathrm{Det}(\underline{\mathbb{R}}^N) \cong \mathrm{Det}(\pi^* TM).$$

Thus π is always co-orientable. Furthermore, we see that

$$\mathrm{Det}(g^*TM) \cong \mathrm{Det}(e^*\pi^*TM) \cong e^*\mathrm{Det}(\pi^*TM) \cong e^*\mathrm{Det}(T(M \times \mathbb{R}^N)) \cong \mathrm{Det}(e^*T(M \times \mathbb{R}^N)).$$

So e is co-orientable if and only if g is co-orientable. But e is an immersion, and so it is co-orientable if and only if the normal bundle of the image is trivial by the discussion preceding Definition 3.15. □

Definition 3.17. If we orient $\mathbb{R}^N$ by its standard orientation $\beta_E := dx_1 \wedge \cdots \wedge dx_N$, then the projection $\pi : M \times \mathbb{R}^N \to M$ has a canonical co-orientation $(\beta_M \wedge \beta_E, \beta_M)$ that is well defined as $\mathbb{R}^N$ is contractible. If $g : W \to M$ is co-oriented and $e : W \hookrightarrow M \times \mathbb{R}^N$ is an immersion with normal bundle ν, we then define the **compatible normal orientation** or **Quillen normal orientation** of ν so that the composition of co-orientations $(\beta_W, \beta_W \wedge \beta_\nu)$ with $(\beta_M \wedge \beta_E, \beta_M)$ is the given co-orientation of g. In other words, if (β_W, β_M) is the given co-orientation of g, then the compatible normal orientation of ν is such that $\beta_W \wedge \beta_\nu = \beta_M \wedge \beta_E$ up to positive scalar multiple.

We sometimes speak of the entire structure $W \hookrightarrow M \times \mathbb{R}^N \to M$ with an orientation of ν as a **Quillen co-orientation** for $W \to M$ or as a "compatible Quillen co-orientation" if we have already specified a co-orientation for $W \to M$ and we wish to choose the Quillen co-orientation that agrees with it.

Remark 3.18. In particular, if $g : W \to M$ is an immersion, then, by taking $N = 0$, a co-orientation of g is equivalent to a Quillen orientation of the normal bundle ν of W in M. In particular, the co-orientation is given locally by (β_W, β_M) if and only if ν is oriented so that $\beta_W \wedge \beta_\nu = \beta_M$ up to positive scalar multiple. If g is a codimension-0 immersion, then ν will be 0-dimensional, and if the co-orientation is the tautological one then β_ν will be the positive orientation at each point.

Remark 3.19. Lipyanskiy's definition of co-orientation in [**?**] factors a proper map through a map which is surjective onto TM, rather than injective from TW as in the Quillen approach. An argument similar to the one just given establishes an equivalence between Lipyanskiy's definition and ours in that setting. [GBF: I'd suggest we give it, or at least sketch it.]

3.3.2. *Co-orientations of boundaries.* Given a co-oriented map $g : W \to M$ where W is a manifold with corners, we can use the normal co-orientation of ∂W in W together with the composition of co-orientations noted in Remark 3.14 to define "boundary co-orientations":

Definition 3.20. The **standard co-orientation of a boundary immersion** $i_{\partial W} : \partial W \hookrightarrow W$ is the normal co-orientation associated to the *inward*-pointing[17] orientation of $\nu_{\partial W \subset W}$.

If $g : W \to M$ is co-oriented, the **induced co-orientation** or **boundary co-orientation** of the composition $g i_{\partial W}$ is the composition of the standard co-orientation of $i_{\partial W}$ with the given co-orientation of $g : W \to M$. We write $\partial g : \partial W \to M$ to denote $g i_{\partial W}$ with its induced co-orientation.

We will use induced co-orientations on boundaries to define the differential in the geometric cochain complex.

The following examples will be important in Section 4.3 when we discuss creasing.

Example 3.21. Suppose $g : W \hookrightarrow M$ is the inclusion of a codimension-0 submanifold of M. In this case TW is the pullback of TM, and we have the tautological co-orientation of Definition 3.12 that, slightly abusing notation, we can write as (β_M, β_M). A particularly important pair of examples is given by the

[17] The outward-pointing normal would also work to provide a co-orientation convention for which the Leibniz formula of Section 3.5 holds. However, using an outward normal is not consistent with the intersection map $\mathcal{I}$ of Section 6.5 being a chain map with our other conventions, while using the inward normal does make $\mathcal{I}$ a chain map.

inclusions $g^- : (-\infty, 0] \hookrightarrow \mathbb{R}$ and $g^+ : [0, \infty) \hookrightarrow \mathbb{R}$, each tautologically co-oriented by (e_1, e_1), where e_1 is the standard unit vector in $\mathbb{R}$.

Next consider the submanifold consisting of the point $0 \in \mathbb{R}$. It has trivial determinant line bundle, and we can choose the basis element to be the 0-form $1 \in \bigwedge^0 T0 \cong \mathbb{R}$. By Definitions 3.15 and 3.20, the standard co-orientation of the boundary inclusion $\{0\} \hookrightarrow (-\infty, 0]$ is $(1, 1 \wedge -e_1) = (1, -e_1)$. The boundary co-orientation of the inclusion $\{0\} \to \mathbb{R}$ induced by the tautological co-orientation of the inclusion $g^- : (-\infty, 0] \to \mathbb{R}$ is then the composition of $(1, -e_1)$ with (e_1, e_1), which is again $(1, -e_1) = -(1, e_1)$ (all bases interpreted in the appropriate spaces). As the inward normal to $[0, \infty)$ at 0 is e_1, the inclusion $g^+ : [0, \infty) \to \mathbb{R}$ induces the opposite co-orientation $(1, e_1)$ on the inclusion $\{0\} \to \mathbb{R}$.

Example 3.22. An important generalization of the preceding example occurs when we have smooth maps $g : W \to M$ and $\varphi : M \to \mathbb{R}$ with 0 a regular value of φg. Consider the spaces $W^0 = (\varphi g)^{-1}(0)$, $W^- = (\varphi g)^{-1}((-\infty, 0])$, and $W^+ = (\varphi g)^{-1}([0, \infty))$, which are manifolds with corners. In fact, they are the fiber products of the maps $\varphi g : W \to \mathbb{R}$ and, respectively, the inclusions of $\{0\}$, $(-\infty, 0]$, and $[0, \infty)$ into $\mathbb{R}$. The required transversality is guaranteed by 0 being a regular value of φg. In this case, however, rather than consider the maps of these fiber products to $\mathbb{R}$, we consider their maps to M via restrictions of g.

In this setting, the normal bundle of W^0 in W has a natural orientation given by pulling back via φg the standard orientation e_1 of the normal bundle of 0 in $\mathbb{R}$, and this determines the **co-orientation of W^0 in W induced by** φ, which is given by the normal co-orientation $(\beta_{W^0}, \beta_{W^0} \wedge (\varphi g)^* e_1)$. We also have the boundary co-orientations of the inclusions of W^0 into $W^\pm$. Analogously to Example 3.21, the co-orientation on $W^0 \hookrightarrow W$ induced by φ is the opposite of co-orientation of $W^0 \hookrightarrow W$ as the (partial) boundary of W^- and agrees with the co-orientation of $W^0 \hookrightarrow W$ as the (partial) boundary of W^+.

Now suppose $g : W \to M$ is co-oriented with local representatives (β_W, β_M). By composing this co-orientation with the tautological co-orientations of $W^\pm$ in W as in the preceding example, we can canonically co-orient $g|_{W^\pm}$ by (β_W, β_M). Similarly, by composing the co-orientation of g with the ϕ-induced co-orientation of $W^0 \hookrightarrow W$ we obtain the **co-orientation of $g|_{W^0} : W^0 \to M$ induced by** φ. As above, the co-orientation of $g|_{W^0} : W^0 \to M$ induced by φ disagrees with the boundary co-orientation of $g|_{W^0} : W^0 \to M$ obtained by treating W^0 as part of the boundary of W^-, while it agrees with the co-orientation obtained by treating W^0 as part of the boundary of W^+.

In the special case of the identity map $\mathrm{id} : M \to M$ with the tautological co-orientation, we obtain submanifolds M^0 and $M^\pm$ of M with co-orientations of their embeddings induced as above. It is worth observing that then if we also have $g : W \to M$ then $W^0 = M^0 \times_M W$ and $W^\pm = M^\pm \times_M W$. This statement holds purely topologically. To ensure manifolds with corners, we observe that φg is transverse to 0 if and only if g is transverse to M^0. Our co-orientation computations here for W^0 and $W^\pm$ will later be seen to be consistent with a co-orientation we define for fiber products of co-oriented maps. In Corollary 3.58, we will use that technology to see that $\partial(W^0)$ and $(\partial W)^0$ agree as spaces but their maps to M have opposite co-orientations given our conventions, i.e. "$\partial W^0 = -(\partial W)^0$" eliding the maps.

3.3.3. *Co-orientations of boundaries of boundaries.* In order to form a chain complex of geometric cochains, we will need a result about co-orientations of $\partial^2 W$. Recall from Section 2.1 that Proposition 2.9 of [**?**] identifies $\partial^2 W$ with the set of points $(x, \mathbf{b}_1, \mathbf{b}_2)$ with $x \in W$ and the $\mathbf{b}_i$ encoding distinct local boundary components. The map $i_{\partial^2 W} : \partial^2 W \to W$ takes $(x, \mathbf{b}_1, \mathbf{b}_2)$ to x. The manifold with corners $\partial^2 W$ is equipped with a canonical diffeomorphism ρ defined by $(x, \mathbf{b}_1, \mathbf{b}_2) \to (x, \mathbf{b}_2, \mathbf{b}_1)$.

Lemma 3.23. *Suppose $i_{\partial^2 W} : \partial^2 W \to W$ is co-oriented via the composition of boundary co-orientations $\partial^2 W \to \partial W \to W$, and suppose $\rho : \partial W \to \partial W$ is given the co-orientation induced by $\bigwedge D\rho$ (see Section 3.2). Then $i_{\partial^2 W}$ and $i_{\partial^2 W}\rho$ have opposite co-orientations.*

Proof. It suffices to consider points $(x, \mathbf{b}_1, \mathbf{b}_2) \in \partial^2 W$ with $x \in S^2(W)$, as such points fill out the interior of $\partial^2 W$. In W such x have neighborhoods of the form $[0, \infty)^2 \times \mathbb{R}^{w-2}$ with x at the origin. We identify

$[0,\infty)^2$ with the first quadrant of $\mathbb{R}^2$, letting X and Y denote the non-negative x and y axes. We let $\mathbf{b}_X$ and $\mathbf{b}_Y$ be the corresponding local boundary components. Then the preimage in $\partial^2 W$ of a small neighborhood U of x in $S^2(W)$ consists of two copies of U that we can write $(U, \mathbf{b}_X, \mathbf{b}_Y)$ and $(U, \mathbf{b}_Y, \mathbf{b}_X)$. The notation indicates that we think of the first copy of U as embedding into $X \times \mathbb{R}^{w-2} \subset \partial W$ and the second as embedding into $Y \times \mathbb{R}^{w-2} \subset \partial W$. The map $i_{\partial W} : \partial W \to W$ then identifies the two copies. The map ρ simply interchanges them.

Let β_X and β_Y denote positively-directed tangent vectors in X and Y, and let β_U be an arbitrary local orientation of U. Abusing notation, we also write β_U for the corresponding local orientations of $(U, \mathbf{b}_X, \mathbf{b}_Y)$ and $(U, \mathbf{b}_Y, \mathbf{b}_X)$. The induced co-orientation on ρ can be written (β_U, β_U). Up to identifying neighborhoods in W with their local models, the boundary co-orientation of $i_{\partial^2 W}$ on $(U, \mathbf{b}_X, \mathbf{b}_Y)$ comes from first mapping it into $X \times \mathbb{R}^{w-2}$ and then into $X \times Y \times \mathbb{R}^{w-2}$. So from the definition of boundary co-orientations this co-orientation is $(\beta_U, \beta_U \wedge \beta_X \wedge \beta_Y)$. Analogously, the boundary co-orientation of $i_{\partial^2 W}$ on $(U, \mathbf{b}_Y, \mathbf{b}_X)$ is $(\beta_U, \beta_U \wedge \beta_Y \wedge \beta_X)$. By composition, the co-orientations of $i_{\partial^2 W}\rho$ on $(U, \mathbf{b}_X, \mathbf{b}_Y)$ and $(U, \mathbf{b}_Y, \mathbf{b}_X)$ are respectively $(\beta_U, \beta_U \wedge \beta_Y \wedge \beta_X)$ and $(\beta_U, \beta_U \wedge \beta_X \wedge \beta_Y)$ as first we interchange then embed. But $\beta_U \wedge \beta_X \wedge \beta_Y = -\beta_U \wedge \beta_Y \wedge \beta_X$, which establishes the lemma. □

Remark 3.24. A similar argument using Convention 3.1 shows that if W is oriented than $\partial^2 W$ possess an orientation reversing diffeomorphism. In this case we observe that of our two copies of U, one is oriented by Convention 3.1 so that $\beta_X \wedge \beta_Y \wedge \beta_U$ is the local orientation of W and the other is oriented so that $\beta_Y \wedge \beta_X \wedge \beta_U$ is the orientation of W. Thus the two copies of U have opposite orientations, and again the diffeomorphism simply interchanges them.

3.4. **Co-orientation of homotopies.** In this section we develop co-orientations related to homotopies. As the product of two spaces is the same as their fiber product over a point, we have by equation (2):

$$\partial(W \times I) = (\partial W \times I) \sqcup (W \times \partial I) = (W \times 1) \sqcup (W \times 0) \sqcup (\partial W \times I).$$

Now recall that in general if we have a map $f : V \to M$ then we write ∂f for the composition $\partial V \xrightarrow{i_{\partial V}} V \xrightarrow{f} M$. Adopting this convention also for pieces of the boundary we make the following definition:

Definition 3.25. If $G : W \times I \to M$ is a co-oriented map, we say that G is a **co-oriented homotopy** (or simply a **homotopy** when working with co-orientations is understood) from $g_0 : W \to M$ to $g_1 : W \to M$ if $\partial G = g_1 \amalg -g_0 \amalg H$, where g_1,$-g_0$, and H correspond respectively to the compositions of G with the inclusions into $W \times I$ of $W \times 1$, $W \times 0$, and $\partial W \times I$, taking each with its boundary co-orientation as in Definition 3.20.

Note that, by analogy with homotopies involving oriented manifolds, a homotopy from g_0 to g_1 involves the oppositely co-oriented $-g_0$ in the boundary formula. In the oriented case, this arises because if we orient W by, say, β_W then to orient $W \times I$ we consider $\beta_W \wedge \beta_I$. Then at one end of the cylinder β_W agrees with the boundary orientation of $\partial(W \times I)$ while at the other end it disagrees. The situation for co-orientations is analogous.

Although we will not need it, we note that by employing appropriate smoothing near the boundaries in order to accomplish transitivity, co-oriented homotopy can be shown to be an equivalence relation on maps $W \to M$.

In our most common use of homotopies, we begin with a co-oriented map $g : W \to M$ and want to construct a homotopic co-oriented map. For this the following lemma is useful.

Lemma 3.26. *Suppose $g : W \to M$ is co-orientable and that $G : W \times I \to M$ is a homotopy with $g = G(-, t_0)$ for some $t_0 \in I$. Then G is co-orientable. Conversely, if $G : W \times I \to M$ is co-orientable, then so is $g = G(-, t_0) : W \to M$ for any t_0.*

Proof. Over $W \times t_0$, we have $\mathrm{Det}(T(W \times I)) \cong \mathrm{Det}(TW \oplus TI) \cong \mathrm{Det}(TW) \otimes \mathrm{Det}(TI) \cong \mathrm{Det}(TW)$, while the restriction of $G^*\mathrm{Det}(TM)$ over $W \times t_0$ is isomorphic to $g^*\mathrm{Det}(TM)$. If G is co-orientable then there is a nowhere-vanishing map of line bundles $\mathrm{Det}(T(W\times I)) \to G^*\mathrm{Det}(TM))$, so restricting to $W\times t_0$ and using the above identifications we obtain a nowhere-vanishing map of line bundles $\mathrm{Det}(TW) \to g^*\mathrm{Det}(TM))$ over $W\times t_0$, hence g is co-orientable. Conversely, if g is co-orientable, there is a nowhere-vanishing map of line bundles $\mathrm{Det}(TW) \to g^*\mathrm{Det}(TM)$ over $W\times t_0$. By general bundle theory, any vector bundle E over $W\times I$ is isomorphic to $E_{t_0}\times I$, where E_{t_0} is the restriction of E to $W\times\{t_0\}$. So our nowhere-vanishing map of line bundles over $W\times t_0$ extends to a nowhere vanishing map of line bundles $\mathrm{Det}(T(W\times I)) \to G^*\mathrm{Det}(TM))$ over $W\times I$. This implies the co-orientability of G. □

Definition 3.27. Suppose $g_0 : W \to M$ is co-oriented and $G : W\times I \to M$ is a smooth homotopy with $G(-,0) = g_0$. Then by the above lemma G is co-orientable and clearly there is exactly one choice of co-orientation for G for which the $W\times 0$ component of ∂G is $-g_0$. We call this co-orientation the **co-orientation on G induced by g_0**. The map G then determines a co-oriented homotopy from g_0 to a co-oriented map $g_1 : W \to M$. We call this co-orientation on $g_1 = G(-,1)$ the **induced co-orientation on g_1.**

Remark 3.28. In the above scenario, if g_0 is co-oriented locally at $x\in W$ by (β_W,β_M), then at $(x,0)\in W\times I$, the corresponding local co-orientation of G that yields $-g_0$ as a boundary component of G is $(\beta_W\wedge-\beta_I,\beta_M)$, where β_I corresponds to the standard orientation of I. This follows from $(\beta_W,\beta_W\wedge\beta_I)$ being the boundary co-orientation of $W\times 0 \hookrightarrow W\times I$ as e_1 is the inward pointing normal at $0\in I$. As we can take $\beta_W\wedge-\beta_I$ to be a consistent orientation along the path given by $\gamma(t) = (x,t)$, we have $\gamma_*(\beta_W\wedge-\beta_I,\beta_M) = (\beta_W\wedge-\beta_I,\gamma_*\beta_M)$, and at this end of the homotopy the induced local co-orientation of g_1 at x is $(\beta_W,\gamma_*\beta_M)$. If G is stationary along $x\times I$, then the co-orientation for g_1 at x is again (β_W,β_M) so that the co-orientations of g_0 and g_1 agree at x. This observation will be useful below in showing that pullback co-orientations are well defined.

Lemma 3.29. *Suppose $G : W\times I \to M$ is a co-oriented homotopy from g_0 to g_1 so that $\partial G = g_1 \amalg -g_0 \amalg H$ as in Definition 3.25. Then H is a homotopy from $-\partial g_0$ to $-\partial g_1$.*

Proof. By definition H is co-oriented as a boundary component of the co-oriented map G, so it remains to check that the induced co-orientations of the ends of H have the expected signs. This could be done directly, but rather we use Lemma 3.23, noting that each copy of ∂W (at the top and bottom of the cylinder) can be considered to be a piece of $\partial^2(W\times I)$. In particular, applying this lemma we see the maps $\partial W \to M$ take opposite co-orientations depending on whether we think of them as first mapping ∂W into W and then identifying W as one end of the cylinder versus first including ∂W into $(\partial W)\times I$ and then mapping this to $W\times I$. In both cases we follow with the map G. As we think of g_1 as defined on $W\subset W\times I$ and let ∂g_1 denote its boundary, we see that the corresponding map from the top of the cylinder $(\partial W)\times I$ must be $-\partial g_1$. Similarly, the map at the bottom of the cylinder is ∂g_0. □

3.5. **Co-orientations of pullbacks and fiber products.** In this section we define a convention for co-orientations of pullbacks and fiber products. More specifically, if $f : V \to M$ and $g : W \to M$ are transverse smooth maps from manifolds with corners to a manifold without boundary and f is co-oriented, we define a co-orientation of the pullback $f^* : V\times_M W \to W$. This does not require g to be co-oriented, but if it is, we can compose to also get a co-orientation of the fiber product $V\times_M W \to M$. Ultimately this will allow us to define certain products of geometric cochains.

Recall that our canonical realization of the topological pullback $P = V\times_M W$ is defined to be $P = \{(x,y)\in V\times W \mid f(x) = g(y)\}$. By Joyce [**?**], the projections $P\to V$ and $P\to W$ are smooth, and hence so is $f\times_M g : P \to M$ given by $(x,y)\to f(x) = g(y)$. It is not obvious how to define the co-orientations of pullbacks and fiber products, and any such definition will depend on choices of convention. Our goal in this section is to provide a definition such that co-orientations of fiber products of co-oriented maps possess the following desirable properties:

(1) If f and g are transverse co-oriented embeddings, then their fiber product is just the (embedding of the) intersection of the images of V and W in M. If f and g are normally co-oriented (see Definition 3.15), then the intersection should be normally co-oriented with the orientation of the normal bundle of the intersection given by concatenating the orientation for the normal bundle of V followed by the orientation for the normal bundle of W.
(2) Graded commutativity: Letting $v = \dim(V)$, $w = \dim W$, and $m = \dim(M)$, we should have $V \times_M W = (-1)^{(m-v)(m-w)} W \times_M V$ as fiber product, using Notation 3.2.
(3) Leibniz rule: we should have $\partial(V \times_M W) = (\partial V \times_M W) \amalg (-1)^{m-v}(V \times_M \partial W)$, using Notation 3.2. This formula will hold for pullbacks as well as fiber products.

Before getting into the specifics of the construction, we need to make the following important observations.

Remark 3.30. While our canonical pullback P has been defined as $V \times_M W = \{(x, y) \in V \times W \mid f(x) = g(y)\}$, categorically the pullback P is technically only well defined up to canonical diffeomorphisms. In particular, if P and P' are two specific representatives of the pullback, we have commutative diagrams

$$\begin{tikzcd} P \arrow[rr, "\cong"] \arrow[dr] & & P' \arrow[dl] \\ & W. & \end{tikzcd} \tag{6}$$

But, as we have observed in Definition 3.12, diffeomorphisms come equipped with natural co-orientations, and so a co-orientation of $P \to M$ determines a unique co-orientation of $P' \to M$ by composition and vice versa. Thus when working with co-orientations of pullbacks, we typically think of selecting a fixed representative $P \to W$ to work with for computations, though not necessarily the canonical one. This observation shows that we are free to do so, and typically we will do so tacitly. This foreshadows the notion of isomorphic representatives of geometric chains and cochains; see Definition 4.2.

Remark 3.31. This is also a good place to point out exactly what we mean by writing $V \times_M W = (-1)^{(m-v)(m-w)} W \times_M V$ in our commutativity statement, as $V \times_M W \subset V \times W$ and $W \times_M V \subset W \times V$ are different spaces, though canonically identified via the map $\tau : V \times W \to W \times V$ that switches the coordinates. This map fits into a commutative diagram of fiber products

$$\begin{tikzcd} V \times_M W \arrow[rr, "\tau"] \arrow[dr] & & W \times_M V \arrow[dl] \\ & M. & \end{tikzcd} \tag{7}$$

Again, τ has an induced co-orientation from being a diffeomorphism, and so the statement means that co-orientation of the fiber product $V \times_M W \to M$ and the composite co-orientation of τ and then the co-orientation of $W \times_M W \to M$ should differ by the sign $(-1)^{(m-v)(m-w)}$.

Putting together this observation with Remark 3.30, we will generally be able to identify $V \times_M W$ and $W \times_M W$ via τ as spaces, and so we typically can leave the map τ itself tacit and just work with the local orientations on $V \times_M W$ and M.

3.5.1. *Co-orientability of pullbacks and fiber products.* Before defining pullback and fiber product co-orientations, we first want to ensure that pullbacks and fiber products of co-orientable maps are themselves co-orientable. The following argument about co-orientability will provide a roadmap to defining co-orientations such co-orientations. We also take the opportunity to observe that pullbacks of proper maps are proper.

Lemma 3.32. *Suppose $f : V \to M$ and $g : W \to M$ are transverse maps of manifolds with corners to a manifold without boundary. Then:*

(1) *If f is co-orienteable, the pullback $f^* : P = V \times_M W \to W$ is co-orientable.*

(2) *If f is proper, the pullback $f^* : P = V \times_M W \to W$ is proper.*

Note that g need not be co-orientable or proper for this lemma to apply.

Proof. We first show that the pullback is proper. Let us label our maps

$$\begin{tikzcd} P \arrow[r, "\pi_V"] \arrow[d, "f^* = \pi_W"'] & V \arrow[d, "f"] \\ W \arrow[r, "g"] & M. \end{tikzcd}$$

Suppose $K \subset W$ is compact. We have

$$\begin{aligned} \pi_W^{-1}(K) &= \{x \in P \mid \pi_W(x) \in K\} \\ &\subset \{x \in P \mid g\pi_W(x) \in g(K)\} \\ &= \{x \in P \mid f\pi_V(x) \in g(K)\} \\ &= \{x \in P \mid \pi_V(x) \in f^{-1}(g(K))\}. \end{aligned}$$

So $\pi_W^{-1}(K) \subset K \times f^{-1}(g(K)) \subset V \times W$. But this is a product of compact sets as f is proper. So π_W is proper.

For co-orientations, we use Quillen's definition from Lemma 3.16. We factor f as $V \hookrightarrow M \times \mathbb{R}^N \to M$, and then we have the pullback diagram

$$\begin{tikzcd} P \arrow[r, "\pi_V"] \arrow[d] & V \arrow[d, "e"] \\ W \times \mathbb{R}^N \arrow[r, "g \times \mathrm{id}"] \arrow[d] & M \times \mathbb{R}^N \arrow[d, "\pi_M"] \\ W \arrow[r, "g"] & M. \end{tikzcd} \tag{8}$$

The bottom square is evidently a pullback. Thus by elementary topology the top square is a pullback diagram if and only if the composite rectangle is a pullback diagram. So by letting the top square be a pullback diagram, we obtain the pullback P of $W \xrightarrow{g} M \xleftarrow{f} V$.

Since f is transverse to g, we have $g \times \mathrm{id}$ transverse to e. As e is an embedding, it follows that $P = (g \times \mathrm{id})^{-1}(e(V))$ is a submanifold of $W \times \mathbb{R}^N$. Furthermore, by Lemma 3.16, $e(V)$ has an orientable normal bundle in $M \times \mathbb{R}^N$, and since the pullback of the normal bundle is the normal bundle of the pullback, it follows that the normal bundle of P in $W \times \mathbb{R}^N$ is also orientable. Applying Lemma 3.16 again, the map $f^* = \pi_W : P \to W$ is co-orientable. ☐

Remark 3.33. As foreshadowed in Remark 3.30, we here use a different realization of $P = V \times_M W$. Thinking of the top square of the diagram as a pullback square, this P is concretely the subset $\{(v, (w, z)) \in V \times (W \times \mathbb{R}^N) \mid e(v) = (g(w), z)\}$. If $\pi_1 : M \times \mathbb{R}^N \to M$ and $\pi_2 : M \times \mathbb{R}^N \to \mathbb{R}^N$ are the projections, we know by definition that $\pi_1(e(v)) = f(v)$, and this is also $g(w)$, so the points in this realization of P also satisfy $f(v) = g(w)$. In fact, there is a canonical map between this realization of P and our standard realization $\{(v, w) \in V \times W \mid f(v) = g(w)\}$ given by $(v, (w, z)) \to (v, w)$ with inverse given by $(v, w) \to (v, (w, \pi_2(e(v))))$.

We also already observed in the above proof that P can be identified with $(g \times \mathrm{id})^{-1}(e(V)) \subset W \times \mathbb{R}^N$. Of course when we think of P as $\{(v, (w, z)) \in V \times (W \times \mathbb{R}^N) \mid e(v) = (g(w), z)\}$, the embedding into $W \times \mathbb{R}^N$ is just the map $(v, (w, z)) \to (w, z)$.

Corollary 3.34. *If $f : V \to M$ and $g : W \to M$ are transverse and co-orientable, their fiber product $V \times_M W \to M$ is also co-orientable.*

Proof. By the preceding lemma, the pullback $V \times_M W \to W$ is co-orientable, and the map $g : W \to M$ is co-orientable by assumption. Now choose co-orientations and compose to get a co-orientation of $P \to M$. □

3.5.2. *Co-orientations of pullbacks and fiber products.* The construction in the proof of Lemma 3.32 provides a roadmap to define the co-orientations of pullbacks and fiber products. For the following definition, refer again to Diagram (8).

Definition 3.35. Suppose $f : V \to M$ and $g : W \to M$ are transverse with f co-oriented and the normal bundle νV of $e(V) \subset M \times \mathbb{R}^N$ given its Quillen normal orientation as defined in Definition 3.17. Then the pullback $P = V \times_M W = (g \times \mathrm{id}_{\mathbb{R}^N})^{-1}(e(V)) \subset W \times \mathbb{R}^N$ has an oriented normal bundle that is the pullback of νV, which, by abuse of notation, we also label νV. Let β_P and β_W be local orientations of P and W, and let β_E be the standard orientation of $\mathbb{R}^N$. Define the **pullback co-orientation** on $P \to W$ to be the composition of the normal co-orientation $(\beta_P, \beta_P \wedge \nu V)$ with the canonical co-orientation $(\beta_W \wedge \beta_E, \beta_W)$. In other words, the pullback co-orientation is (β_P, β_W) if $\beta_P \wedge \beta_{\nu V} = \beta_W \wedge \beta_E$ up to positive scalar and $-(\beta_P, \beta_W)$ otherwise.

Following Definition 2.17, we sometimes write $f^* : P \to W$. We also sometimes write $P = g^*V$ to emphasize that P is the pullback of V by g to a manifold over W.

If g is co-oriented, define the **fiber product co-orientation** on $P \to M$ as the composition of the pullback co-orientation with the co-orientation of $g : W \to M$.

In the definition, note that the Quillen orientation of νV is determined by the co-orientation of f and the orientation β_E is taken to be canonically fixed across all instances. The other orientations appearing in the definition are β_P and β_W, but the co-orientation of the pullback $f^* : P \to W$ does not depend on the particular choices. For example, suppose we choose β_P and β_W so that $\beta_P \wedge \beta_{\nu V} = \beta_W \wedge \beta_E$ and hence the pullback co-orientation is (β_P, β_W). If we replace β_P with $\beta'_P = -\beta_P$, then $\beta'_P \wedge \beta_{\nu V} = -\beta_P \wedge \beta_{\nu V} = -\beta_W \wedge \beta_E$, so the co-orientation is $-(\beta'_P, \beta_W) = -(-\beta_P, \beta_W) = (\beta_P, \beta_W)$. So the co-orientation is unchanged. Similarly the definition is independent of our choice of β_W. We will show just below that the definition is independent of N and e as well.

Remark 3.36. It follows from the definition that reversing the co-orientation of $f : V \to M$ reverses the co-orientation of $f^* : V \times_M W \to W$. Furthermore, if $g : W \to M$ is co-oriented, then reversing the co-orientation of either f or g reverses the co-orientation of the fiber product $f \times_M g : V \times_M W \to M$.

It is also clear that the definition is consistent under restrictions to open sets. In other words if $x \in V$, $y \in W$ with $f(x) = g(y)$, then replacing V, W, and M with neighborhoods of x, y, and $f(x) = g(y)$ yields a co-orientation of the restriction of f^* to a neighborhood of $(x, y) \in V \times_M W$ that is consistent with the co-orientation of all of f^*, at least so long as we use the same N and a restriction of e, though we will now show independence of these choices as well.

Lemma 3.37. *The pullback and fiber product co-orientations do not depend on the choices of N, e, or local orientations of P, V, W, or M.*

Remark 3.38. The co-orientations *do* depend on the choice of the canonical orientation for $\mathbb{R}^N$, the choice of the standard co-orientation of the projection to be $(\beta_M \wedge \beta_E, \beta_M)$, etc., but these are all universal choices. The point is that the pullback and fiber product co-orientations only depend on f, g, and their co-orientations, after fixing such universal choices that do not depend on f or g.

Proof of Lemma 3.37. As the local orientations of P, V, W, and M used in the construction all come in pairs (e.g. β_P in $(\beta_P, \beta_P \wedge \nu_V)$), the construction is independent of those choices.

Next, suppose we are given an embedding $e : V \hookrightarrow M \times \mathbb{R}^N$ and extend it to $e' = (e, 0) : V \hookrightarrow M \times \mathbb{R}^N \times \mathbb{R}^n$. In the construction involving e, if we choose β_V, β_M so that (β_V, β_M) is the co-orientation for f, then by the definition of the Quillen orientation, νV will be such that $\beta_V \wedge \beta_{\nu V} = \beta_M \wedge \beta_E$ up to positive scalar. If we now increase the dimension of the Euclidean factor to $\mathbb{R}^{N+n}$ and write its canonical local orientation as $\beta_{E^N} \wedge \beta_{E^n}$ while extending e to e', we see that ν_V becomes $\nu_V \oplus \underline{\mathbb{R}}^n$ so that $\beta_{\nu V}$ becomes $\beta_{\nu V} \wedge \beta_{E^n}$. Pulling back over W we obtain the pullback co-orientation $(\beta_P, \beta_P \wedge \beta_{\nu V} \wedge \beta_{E^n}) * (\beta_W \wedge \beta_{E^N} \wedge \beta_{E^n}, \beta_W)$. This is (β_P, β_W) if and only if $\beta_P \wedge \beta_{\nu V} \wedge \beta_{E^n} = \beta_W \wedge \beta_{E^N} \wedge \beta_{E^n}$ up to positive scalar, but this condition is equivalent to having $\beta_P \wedge \beta_{\nu V} = \beta_W \wedge \beta_{E^N}$ up to positive scalar. So the pullback co-orientation is unchanged.

Next suppose that $e_0 : V \to M \times R^{N_0}$ and $e_1 : V \to M \times R^{N_1}$ are any two embeddings over f. By the preceding paragraph, by adding Euclidean factors we can assume $N_0 = N_1 = N$ for some sufficiently large N without changing the pullback co-orientations associated to e_0 and e_1. Let $\pi : M \times \mathbb{R}^N \to M$ be the projection to M. As $\pi e_0 = \pi e_1$, the maps e_0 and e_1 are homotopic over f, say by linear homotopies in the Euclidean fibers. Let $H : V \times I \to M \times \mathbb{R}^N$ be the chosen homotopy. Next, by the same argument by which embeddings e exist, there is an embedding $\tilde{H} : V \times I \hookrightarrow M \times \mathbb{R}^N \times \mathbb{R}^Q$ for some Q so that if $\tilde{\pi} : M \times \mathbb{R}^N \times \mathbb{R}^Q \to M \times \mathbb{R}^N$ is the projection then $\tilde{\pi}\tilde{H} = H$. If we let $\tilde{e}_0 = \tilde{H}(-, 0) : V \to M \times \mathbb{R}^N \times \mathbb{R}^Q$ then $\tilde{\pi}\tilde{e}_0 = e_0$. If we let $(e_0, 0) : V \to M \times \mathbb{R}^N \times \mathbb{R}^Q$ denote the map $x \to (e_0(x), 0)$, then there is a homotopy from $\tilde{e}_0$ to $(e_0, 0)$; in fact as e_0 is an embedding and $\tilde{\pi}\tilde{e}_0 = e_0$, we can let these homotopies be linear in the $\mathbb{R}^Q$ factor and constant in the other factors and this homotopy will be an embedding of $V \times I$. We can define $\tilde{e}_1$, $(e_1, 0)$, and an embedded homotopy between them similarly.

So we have a sequence of three embedded homotopies, say F_1, F_2, F_3 from $(e_0, 0)$ to $\tilde{e}_0$, from $\tilde{e}_0$ to $\tilde{e}_1$, and from $\tilde{e}_1$ to $(e_1, 0)$, respectively. Additionally, $\pi\tilde{\pi} : F_j(x, t) = f(x)$ so each homotopy is constant in I when projected to M, and in particular each of $(e_0, 0)$, $(e_1, 0)$, $\tilde{e}_0$, and $\tilde{e}_1$ is an embedding $V \hookrightarrow M \times \mathbb{R}^{N+Q}$ over $f : V \to M$. We know from above that the pullback co-orientation obtained from using $(e_0, 0)$ and $(e_1, 0)$ agree with those obtained from e_0 and e_1. So it suffices to use the homotopies to show successively that $(e_0, 0)$, $\tilde{e}_0$, $\tilde{e}_1$, and $(e_1, 1)$ all provide the same pullback co-orientation of f^*.

By Lemma 3.26 and Definition 3.27, we can use the embeddings F_j to co-orient each of our constant homotopies. In particular, using F_j in place of e in Definition 3.35 we obtain three co-orientations of the pullbacks $(V \times I) \times_M W \to W$. We will see below in Proposition 3.44, whose proof is independent of this one, that when accounting for co-orientations, pullback co-orientations satisfy a Leibniz rule and, in particular, two of the signed boundary components of each co-oriented $(V \times I) \times_M W \to W$ will be $(V \times \{0\}) \times_M W \to W$ and $(V \times \{1\}) \times_M W \to W$, occurring with opposite signs. In other words, with appropriate choices on the co-orientations of the homotopies, by Definition 3.25, we obtain three sequential co-oriented (constant) homotopies from f to itself. It now follows by applying Remark 3.28 sequentially that all four copies of f must have the same co-orientation. In particular, this is the case for the co-orientations of f obtained from the embeddings e_0 and e_1. □

Remark 3.39. The pullback co-orientation is determined locally in the sense that if U is an open subset of M then the pullback co-orientation of $f^{-1}(U) \times_U g^{-1}(U) \to g^{-1}(U)$ will just be the restriction of the pullback co-orientation of $V \times_M W \to W$. This is clear from the construction if we co-orient the local pullback using the Quillen co-orientation of $f^{-1}(U) \to U$ given by $f^{-1}(U) \xhookrightarrow{e|_{f^{-1}(U)}} U \times \mathbb{R}^N \to U$, the restriction of the Quillen co-orientation $V \overset{e}{\hookrightarrow} M \times \mathbb{R}^N \times M$ use to co-oriented $f : V \to M$. But Lemma 3.37 says that we are free to do so.

Remark 3.40. We have just shown that, after choosing conventions, the fiber product of two transverse co-oriented maps is co-oriented, and this will eventually lead us to the cup product of geometric cochains. Analogously, if $f : V \to M$ is co-oriented and W is oriented, then the pullback co-orientation $f^* : P =$

$V \times_M W \to W$ provides a way to orient P, namely if β_W is the given globally-defined orientation of W we can choose β_P so that (β_P, β_W) is the co-orientation of f^*. This observation will be utilized below in our construction of the cap product. However, somewhat surprisingly, the fiber product of two maps with oriented domains cannot necessarily be oriented, and so there is in general no product of geometric chains and hence, in general, no homology product. Such oriented fiber products can be formed if the the codomain M is oriented, as in this case there is an equivalence between orientations of domains and co-orientations of maps. But this is not always possible when M is not orientable. For example, we recall that the intersection of two orientable $\mathbb{R}P^3$s in the non-orientable $\mathbb{R}P^4$ can be a non-orientable $\mathbb{R}P^2$.

Functoriality of pullbacks. The co-oriented pullback construction is functorial in the following sense.

Proposition 3.41. *Suppose $f : V \to M$ is co-oriented. Then the pullback of f by the identity $\mathrm{id}_M : M \to M$ is (diffeomorphic to) $f : V \to M$ with the same co-orientation.*

Suppose further that W and M are manifolds without boundary, that $g : W \to M$ is transverse to f and that $h : X \to W$ is transverse to $V \times_M W \to W$ (or, equivalently by Lemma 2.22, that gh is transverse to f). Then $(gh)^*V \cong h^*g^*V$ as co-oriented manifolds over M.

Proof. We first note that there is a diffeomorphism between V and $\{(v, x) \in V \times M \mid f(v) = x\}$ given by $v \to (v, f(v))$ and $(v, x) \to v$. Then given compatible Quillen co-orientation of f, we can form the pullback diagram as

$$\begin{tikzcd}
V \arrow[r, "\mathrm{id}_V"] \arrow[d, "e"'] & V \arrow[d, "e"] \\
M \times \mathbb{R}^N \arrow[r, "\mathrm{id}_{M\times\mathbb{R}^N}"] \arrow[d, "\pi_M"'] & M \times \mathbb{R}^N \arrow[d, "\pi_M"] \\
M \arrow[r, "\mathrm{id}_M"] & M,
\end{tikzcd}$$

and the conclusion is evident.

For the second claim, there is a diffeomorphism between $\{(v, x) \in V \times X \mid f(v) = g(h(x))\}$ and $\{((v, w), x) \in (V \times_M W) \times X \mid w = h(x)\}$ given by $(v, x) \to (v, h(x), x)$ and $((v, w), x) \to (v, x)$. To see that the last map is well-defined notice that $f(v) = g(h(x))$ as $h(x) = w$, and $f(v) = g(w)$ from the assumption $(v, w) \in V \times_M W$. Compatibility of the co-orientations now follows by considering the following diagram and noting that it is equivalent to pull back the normal νV to $X \times \mathbb{R}^N$ either in two steps or all at once.

$$\begin{tikzcd}
(V \times_M W) \times_W X \arrow[r, "\pi_{V \times_M W}"] \arrow[d] & V \times_M W \arrow[r, "\pi_V"] \arrow[d] & V \arrow[d, "e"] \\
X \times \mathbb{R}^N \arrow[r, "h \times \mathrm{id}"] \arrow[d] & W \times \mathbb{R}^N \arrow[r, "g \times \mathrm{id}"] \arrow[d] & M \times \mathbb{R}^N \arrow[d, "\pi_M"] \\
X \arrow[r, "h"] & W \arrow[r, "g"] & M.
\end{tikzcd}$$

□

3.5.3. *Fiber products of immersions.* Pullbacks have particularly nice descriptions when on or both of the maps are embeddings or immersions. In addition, these special cases are good for building intuition about the more general situation.

Example 3.42. When $f : V \to M$ is a co-oriented embedding, the pullback $V \times_M W$ is particularly easy to describe. In this case, we know from Section 3.3 that the co-orientation is determined by an orientation $\beta_{\nu V}$ of the normal bundle to V in M. Then, as f is already an embedding, we can take $N = 0$ in Definition 3.35. So the pullback $V \times_M W$ is just the submanifold $g^{-1}(V) \subset W$, co-oriented by (β_P, β_W), where $\beta_P \wedge \beta_{\nu V} = \beta_W$, the νV here being the pullback of the normal bundle to $g^{-1}(V)$ in W. In other words, the co-orientation of the pullback is just the normal co-orientation corresponding to the pulled back orientation of νV. We can say that V and $V \times_M W = g^{-1}(V)$ have compatible normal co-orientations.

The case where g is embedded instead also has a nice description but requires some more technology. We will discuss that case below in **??**.

In the key example when both f and g are immersions, the fiber products will locally correspond to intersections of the images in M, and in this case our general definition of fiber product co-orientation is compatible with the approach to co-orientations via orientations of normal bundles.

Proposition 3.43. *Let $f : V \to M$ and $g : W \to M$ be transverse co-oriented immersions from manifolds with corners to a manifold without boundary. Let νV and νW denote the respective normal bundles. Choose local Quillen orientations $\beta_{\nu V}$ and $\beta_{\nu W}$ so that the normal co-orientations $(\beta_V, \beta_V \wedge \beta_{\nu V})$ and $(\beta_W, \beta_W \wedge \beta_{\nu W})$ agree with the given co-orientations of f and g. Then, decomposing the normal bundle of the fiber product $P = V \times_M W \to M$ at any point of intersection as $\nu V \oplus \nu W$ and giving it the orientation $\beta_{\nu V} \wedge \beta_{\nu W}$, the fiber product co-orientation agrees with the normal co-orientation, i.e.*

$$\omega_{f \times_M g} = (\beta_P, \beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W}).$$

That is, if one orients the normal bundle of the intersection by following an oriented basis of the normal bundle of V by one for W, the associated normal co-orientation is the fiber product co-orientation.

Proof. It suffices to demonstrate this property in the neighborhood of any intersection point, so we may assume that f and g are embeddings and consider $x \in V$, $y \in W$ with $f(x) = g(y) = z \in M$. Locally, for our Quillen co-orientation of f we can apply the definition of the pullback co-orientation with $N = 0$ and the embedding $e : V \hookrightarrow M \times \mathbb{R}^N$ to be simply f itself. As $N = 0$, in this case νV is itself the oriented normal bundle of $e(V) = f(V)$ in $M \times \mathbb{R}^N = M$. Pulling back via g to W, we obtain the oriented pullback of νV (which we also call νV) as the normal bundle of P in W. By definition, the co-orientation of $P \to W$ is then the composition of $(\beta_P, \beta_P \wedge \beta_{\nu V})$ with the standard co-orientation of the projection $W \times \mathbb{R}^N$ to W, which in this case is the identity. The co-orientation of the fiber product is thus the composition of

$(\beta_P, \beta_P \wedge \beta_{\nu V})$ with the co-orientation $(\beta_W, \beta_W \wedge \beta_{\nu W})$ of g. But this last co-orientation is independent of the choice of β_W, so we can take $\beta_W = \beta_P \wedge \beta_{\nu V}$. Thus we see that the fiber product co-orientation of $P \to M$ is $(\beta_P, \beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W})$, as desired. □

3.5.4. *The Leibniz rule.* We now verify the Leibniz rule. Note that in the proof we work with a single arbitrary but fixed $e : V \hookrightarrow M \times \mathbb{R}^N$ and its restriction to ∂V, and so the following theorem holds for any such choice in the definition of the pullback co-orientation. Consequently, this result completes the proof of Lemma 3.37.

Proposition 3.44 (Leibniz rule)**.** *Let $f : V \to M$ and $g : W \to M$ be transverse maps from manifolds with corners to a manifold without boundary, and suppose f co-oriented. Let $V \times_M W \to W$ be the co-oriented pullback. Then*

$$\partial(V \times_M W) = (\partial V) \times_M W \bigsqcup (-1)^{\dim(M)-\dim(V)} V \times_M (\partial W),$$

interpreting each of these pullback spaces as representing its co-oriented map to W; see Notation 3.2.[18] *If g is also co-oriented then this formula also holds as fiber products mapping to M.*

Establishing this directly for immersions, for which we can use the normal co-orientations, is a quick exercise. The general case requires more care.

Proof. The statement at the level of underlying manifolds with corners is [**?**, Proposition 6.7], so we focus on co-orientations. The second statement follows from the first by composing each map with the co-oriented map $g : W \to M$ and taking the composite co-orientations. We will write ∂P when considering the boundary co-orientation of $P = V \times_M W$ and $(\partial V) \times_M W$ or $V \times_M (\partial W)$ when considering these pullback co-orientations. In the following arguments, it suffices to consider points in the interiors of ∂V or ∂W as knowing a co-orientation at one point of each component determines it globally; in other words, we can avoid corners.

By Definition 3.35, the co-orientation ω_{f^*} of $P \to W$ is (β_P, β_W) if and only if $\beta_P \wedge \beta_{\nu V} = \beta_W \wedge \beta_E$ up to positive scalar, where νV is the pullback to P of the Quillen-oriented normal bundle of $e(V)$ in $M \times \mathbb{R}^N$. Recall that by Definition 3.20, if $\nu\partial P$ is the inward pointing normal of ∂P in P then $\partial P \to V$ is co-oriented by composing the boundary co-orientation $(\beta_{\partial P}, \beta_{\partial P} \wedge \beta_{\nu\partial P})$ with ω_{f^*}. Choosing $\beta_P = \beta_{\partial P} \wedge \beta_{\nu\partial P}$ (for an arbitrary choice of $\beta_{\partial P}$), we see $\omega_{\partial P \to W}$ is $(\beta_{\partial P}, \beta_W)$ if and only if $\beta_{\partial P} \wedge \beta_{\nu\partial P} \wedge \beta_{\nu V} = \beta_W \wedge \beta_E$.

Now, consider a point in the interior of $(\partial V) \times_M W \subset V \times_M W = P$. By choosing standard choices of coordinate charts in such neighborhoods, we see that we can take $\nu\partial V = \nu\partial P$ to be the same inward pointing normal vector. In the construction of the pullback co-orientation for $(\partial V) \times_M W$, we can take $e : \partial V \to M \times \mathbb{R}^N$ to be the restriction of the $e : V \to M \times \mathbb{R}^N$ used to co-oriented P. As we have already used $\nu\partial V$ for the normal of $\partial V \subset V$, to avoid confusing the notation, we write $\nu^s\partial V$ (s for stable) for the normal bundle of $e(\partial V) \subset M \times \mathbb{R}^N$. Note that $\nu^s\partial V \cong \nu\partial V \oplus \nu V$. The Quillen orientation $\beta_{\nu V}$ was chosen so that $\beta_V \wedge \beta_{\nu V} = \beta_M \wedge \beta_E$. If we choose $\beta_{\partial V}$ so that $\beta_V = \beta_{\partial V} \wedge \beta_{\nu\partial V}$ then the boundary co-orientation of $\partial V \to M$ will be $(\beta_{\partial V}, \beta_M)$ so we can then perform the construction of Definition 3.35 using $\beta_{\partial V}$. Note that as $\beta_{\nu V}$ is chosen so that $\beta_V \wedge \beta_{\nu V} = \beta_M \wedge \beta_E$, we will have also $\beta_M \wedge \beta_E = \beta_{\partial V} \wedge \beta_{\nu\partial V} \wedge \beta_{\nu V}$ and so the Quillen orientation of $\nu^s\partial V$ is $\beta_{\nu^s\partial V} = \beta_{\nu\partial V} \wedge \beta_{\nu V}$.

So now applying once again Definition 3.35 and writing $\beta_{(\partial V)\times_M W} = \beta_{\partial P}$ at our chosen point, the co-orientation is $(\beta_{\partial P}, \beta_W)$ if and only if $\beta_{\partial P} \wedge \beta_{\nu^s\partial V} = \beta_W \wedge \beta_E$ up to positive scalar. But $\beta_{\partial P} \wedge \beta_{\nu^s\partial V} = \beta_{\partial P} \wedge \beta_{\nu\partial V} \wedge \beta_{\nu V}$. So the co-orientation is $(\beta_{\partial P}, \beta_W)$ if and only if $\beta_{\partial P} \wedge \beta_{\nu\partial V} \wedge \beta_{\nu V} = \beta_W \wedge \beta_E$ up to positive scalar, which we established above to be precisely the description for this component of $\partial P \to W$.

Next, consider a point x in $V \times_M \partial W$. Again we can take $\nu\partial W = \nu\partial P$ to be the same inward pointing normal vector, and we write $\beta_{V\times_M W}$ and $\beta_{V\times_M \partial W}$ as β_P and $\beta_{\partial P}$ near our point. Furthermore, we can choose β_W, $\beta_{\partial W}$, and β_M so that (β_W, β_M) is the co-orientation of g and $(\beta_{\partial W}, \beta_W)$ co-orients $i_{\partial W}$, which

[18]Here we interpret $V \times \partial W \to W$ as the composition of the co-oriented pullback $V \times \partial W \to \partial W$ with the co-oriented boundary immersion $\partial W \to W$.

implies $\beta_W = \beta_{\partial W}\wedge\beta_{\nu\partial W}$. We also continue to choose $\beta_P = \beta_{\partial P}\wedge\beta_{\nu\partial P} = \beta_{\partial P}\wedge\beta_{\nu\partial W}$. By Definition 3.35, applied to $\partial g : \partial W \to M$, the co-orientation of the pullback $V\times_M \partial W \to \partial W$ is $(\beta_{\partial P}, \beta_{\partial W})$ if and only if $\beta_{\partial P}\wedge\beta_{\nu V} = \beta_{\partial W}\wedge\beta_E$ up to positive scalar as local orientations at the image of x in $\partial W\times\mathbb{R}^N$. Considering $\partial W\times\mathbb{R}^N \subset W\times\mathbb{R}^N$ locally (recalling that x is chosen in the interior of ∂W), $\beta_{\partial P}\wedge\beta_{\nu V} = \beta_{\partial W}\wedge\beta_E$ if and only if $\beta_{\partial P}\wedge\beta_{\nu V}\wedge\beta_{\nu\partial W} = \beta_{\partial W}\wedge\beta_E\wedge\beta_{\nu\partial W}$. But as $\dim(\nu\partial W) = 1$ and $\dim(\nu V) = m+N-v$,

$$\beta_{\partial P}\wedge\beta_{\nu V}\wedge\beta_{\nu\partial W} = (-1)^{m+N-v}\beta_{\partial P}\wedge\beta_{\nu\partial W}\wedge\beta_{\nu V} = (-1)^{m+N-v}\beta_{\partial P}\wedge\beta_{\nu\partial P}\wedge\beta_{\nu V},$$

and

$$\beta_{\partial W}\wedge\beta_E\wedge\beta_{\nu\partial W} = (-1)^N\beta_{\partial W}\wedge\beta_{\nu\partial W}\wedge\beta_E = (-1)^N\beta_W\wedge\beta_E.$$

So the co-orientation of $V\times_M\partial W\to W$ is $(\beta_{\partial P},\beta_{\partial W})$, and hence by composition the co-orientation of $V\times_M\partial W\to\partial W$ is $(\beta_{\partial P},\beta_W)$, if and only if $\beta_{\partial P}\wedge\beta_{\nu\partial P}\wedge\beta_{\nu V} = (-1)^{m-v}\beta_W\wedge\beta_E$ up to positive scalar. Comparing with $\omega_{\partial P\to W}$, we thus see that $\omega_{V\times_M\partial W\to W}$ differs from it by a sign of $(-1)^{m-v}$ as desired. $\square$

3.5.5. *Graded commutativity.* We demonstrate here graded commutativity of fiber product co-orientations. We do so only for proper maps, but this will suffice for the purposes of geometric chains and cochains below. The reader should recall Remark 3.31 for a precise explanation of the statement of the proposition.

Proposition 3.45. *Suppose $f : V \to M$ and $g : W \to M$ are transverse co-oriented maps from manifolds with corners to a manifold without boundary. Then as co-oriented fiber products over M we have $\omega_{g\times_M f} = (-1)^{(m-v)(m-w)}\omega_{f\times_M g}$, or, using Notation 3.31,*

$$V\times_M W = (-1)^{(m-v)(m-w)}W\times_M V.$$

Before proving the proposition, we put it to use in the following example.

Example 3.46. In Example 3.42, we considered pullback co-orientations $V\times_M W\to W$ when $V\hookrightarrow M$ was embedded. In this example, we discuss the case where W is embedded.

Let $f : V\to M$ and $g : W\to M$ be transverse maps from manifolds with corners to a manifold without boundary. Suppose f is co-oriented and $g : W\to M$ is an embedding. Let $(x,y)\in V\times_M W$. Even though g might not be co-oriented, let us choose a contractible neighborhood U of y in W and an arbitrary co-orientation (β_U,β_M) on the restriction of g to U. For the remainder of the argument, we fix this local orientation β_M at $f(x)\in M$, and we choose β_V at x so that (β_V,β_M) is the co-orientation of V at x.

Although we are interested in $V\times_M U$, the definition of the pullback co-orientation makes it easier to work with $U\times_M V$ when U is embedded; see Example 3.42. As we have chosen a co-orientation for $U\hookrightarrow M$ and $V\to M$ comes with a co-orientation, we can consider the fiber product $U\times_M V\to M$. If we choose an orientation $\beta_{\nu U}$ of the normal bundle to U at y so that $\beta_U\wedge\beta_{\nu U} = \beta_M$, then this fiber product is $P_U = f^{-1}(U) = f^{-1}(U)\xrightarrow{f} M$ co-oriented by (β_P,β_M), where β_P is chosen so that $\beta_P\wedge\beta_{\nu U} = \beta_V$, as usual letting νU here also stand for its pullback as a normal bundle of P_U in V. At this point, if we had chosen the opposite co-orientation for $U\hookrightarrow M$, we would have the opposite fiber product co-orientation.

Now, we apply Proposition 3.45. Technically we need $g|_U$ to be proper, but we can achieve this by assuming that U is the intersection of W with some open neighborhood $\mathcal{U}$ of y in M. Then we can replace the co-oriented embedding $U\hookrightarrow M$ with the proper co-oriented embedding $U\hookrightarrow\mathcal{U}$ and f with its restriction to $f^{-1}(\mathcal{U})$. As pullback co-orientations are determined locally as noted in Remark 3.39, this will suffice. For simplicity, though, we maintain our original notations.

By Proposition 3.45, we have $U\times_M V = (-1)^{(m-v)(m-w)}V\times_M U$ as co-oriented fiber products over M. So the fiber product $V\times_M U\to M$ is represented by $f^{-1}(U)\xrightarrow{f} M$ with co-orientation at (x,y) given by $(-1)^{(m-v)(m-w)}(\beta_P,\beta_M)$. But decomposing the fiber product as the pullback $V\times_M U\to U$ and the inclusion $U\hookrightarrow M$, we can also decompose the co-orientation as $(-1)^{(m-v)(m-w)}(\beta_P,\beta_M) = (-1)^{(m-v)(m-w)}(\beta_P,\beta_U)*(\beta_U,\beta_M)$. As (β_U,β_M) is our co-orientation for $g|_U$, the pullback co-orientation must be $(-1)^{(m-v)(m-w)}(\beta_P,\beta_U)$. Furthermore, if we had chosen the opposite co-orientation for $g|_U$, that would reverse the signs of both (β_U, β_M) and of the fiber product, so in either case the pullback co-orientation is $(-1)^{(m-v)(m-w)}(\beta_P, \beta_U)$. In other words, this description of the co-oriented pullback $V \times_M U \to U$ is independent of our choice of co-orientation for $g|_U$, and so it extends globally to $V \times_M W \to W$. And explicitly it can be described as follows: $V \times_M W \to W$ is just the inclusion $f^{-1}(W) \hookrightarrow W$ co-oriented by $(-1)^{(m-v)(m-w)}(\beta_P, \beta_W)$, where, if we write the co-oriented of f locally by (β_V, β_M) and choose any local orientation $\beta_{\nu W}$ for the normal bundle of W in M, then $\beta_P \wedge \beta_{\nu W} = \beta_V$ and $\beta_W \wedge \beta_{\nu W} = \beta_M$. In other words, we can say that $W \subset M$ and $V \times_M W = f^{-1}(W) \subset V$ have compatible local normal co-orientations up to the sign $(-1)^{(m-v)(m-w)}$.

It is a nice exercise to confirm that this agrees with the computation of Proposition 3.43 when V and W are both embedded.

Proposition 3.45 will be proven through a sequence of lemmas. One common theme is to first observe that the theorem will be true if it is true at any one point in each connected component of P. As fiber product co-orientations are well defined globally, if they agree or disagree at any one point then they must agree or disagree on the entire connected component. By Remark 3.31 we can also generally identify $V \times_M W$ with $W \times_M V$ as spaces in what follows and focus only on how the two constructions specify co-orientations.

Lemma 3.47. *If there exists $x \in V$ and $y \in W$ such that $f(x) = g(y)$ and f and g are respectively immersions at x and y, then the theorem holds for the connected component containing $(x, y) \in V \times_M W$.*

Proof. As observed just above, it suffices to prove the desired identity at (x, y). By Proposition 3.43, we have $\omega_{f \times_M g} = (\beta_P, \beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W})$, noting in the second term that β_P is technically the image of the local orientation β_P of P now considered as a submanifold of M via the embedding. Similarly, Proposition 3.43 gives $\omega_{g \times_M f} = (\beta_P, \beta_P \wedge \beta_{\nu W} \wedge \beta_{\nu V})$. So the two fiber products differ by $(-1)^{\dim(\nu W)\dim(\nu V)} = (-1)^{(m-v)(m-w)}$ as required. □

Lemma 3.48. *If there exists x in the interior of V and $y \in W$ such that $f(x) = g(y)$, f is a submersion at x, and g is an immersion at y, then the theorem holds for the connected component containing $(x, y) \in V \times_M W$.*

Proof. By Remark 3.36 and the above observations, it suffices to consider a neighborhood of x and a chart around $f(x)$ with respect to which f agrees locally with the projection $M \times \mathbb{R}^N \to M$. For simplicity of notation, we can take all of M to be this chart. We can choose the ordering of the coordinates in $\mathbb{R}^N$ so that $(\beta_M \wedge \beta_E, \beta_M)$ agrees with the co-orientation for f. In this case we let $e : V \hookrightarrow M \times \mathbb{R}^N$ be the diffeomorphism realizing the neighborhood of x as $M \times \mathbb{R}^N$. The normal bundle ν_V in this case is 0-dimensional and positively oriented. The pullback to P is all of $W \times \mathbb{R}^N$, and if we also choose β_W at y so that $\omega_g = (\beta_W, \beta_M)$, then by Definition 3.35 the fiber product co-orientation for $f \times_M g$ is (β_P, β_M) if and only if $\beta_P = \beta_W \wedge \beta_E$.

Now consider instead $W \times_M V$, constructed using the same neighborhoods. In this case we take $e = g : W \to M \times \mathbb{R}^0$. The map f is still our projection $M \times \mathbb{R}^N \to M$, and so the pullback is $f^{-1}(W) = W \times \mathbb{R}^N$. The normal bundles are ν_W. We continue to assume coordinates so that f is co-oriented by $(\beta_M \wedge \beta_E, \beta_M)$. By Definition 3.35 the fiber product co-orientation for $g \times_M f$ is (β_P, β_M) if and only if $\beta_P \wedge \beta_{\nu W} = \beta_V$. But if $\beta_P = \beta_W \wedge \beta_E$ consistent with the preceding case and $\beta_V = \beta_M \wedge \beta_E$, then we have

$$\begin{aligned}\beta_P \wedge \beta_{\nu W} = \beta_W \wedge \beta_E \wedge \beta_{\nu W} &= (-1)^{N \dim(\nu W)} \beta_W \wedge \beta_{\nu W} \wedge \beta_E \\ &= (-1)^{N \dim(\nu W)} \beta_M \wedge \beta_E = (-1)^{N \dim(\nu W)} \beta_V.\end{aligned}$$

Here we have used that $\beta_M = \beta_W \wedge \beta_{\nu W}$ by the choice of Quillen local orientation for νW in Definition 3.35, taking into account $N = 0$. As $\dim(\nu W) = m - w$ and $N = v - m$, the lemma is follows. □

Lemma 3.49. *If there exist x and y in the respective interiors of V and W such that $f(x) = g(y)$ and f and g are respective submersions at x and y, then the theorem holds for the connected component containing $(x, y) \in V \times_M W$.*

Proof. As in the preceding lemma, we can choose local coordinates such that f is the projection $M \times \mathbb{R}^N \to M$ and g is the projection $M \times \mathbb{R}^n \to M$. Let β_E and β_F be local orientations of $\mathbb{R}^N$ and $\mathbb{R}^n$ such that $\omega_f = (\beta_V, \beta_M) = (\beta_M \wedge \beta_E, \beta_M)$ and $\omega_g = (\beta_M \wedge \beta_F, \beta_M)$. To compute $\omega_{f \times_M g}$, we identify V and $M \times \mathbb{R}^N$ as in the preceding lemma, so the Quillen normal bundle is trivial and positive. The pullback is $W \times \mathbb{R}^N \cong M \times \mathbb{R}^n \times \mathbb{R}^N$. And by Definition 3.35, taking $\beta_P = \beta_W \wedge \beta_E$, we have $\omega_{f \times_M g} = (\beta_W \wedge \beta_E, \beta_M) = (\beta_M \wedge \beta_F \wedge \beta_E, \beta_M)$. Analogously, $\omega_{g \times_M f} = (\beta_V \wedge \beta_F, \beta_M) = (\beta_M \wedge \beta_E \wedge \beta_F, \beta_M)$. The lemma follows as $N = v - m$ and $n = w - m$. $\square$

Corollary 3.50. *If there exists x and y in the respective interiors of V and W such that $f(x) = g(y)$ and if f and g are of full rank at x and y then Proposition 3.45 holds for the connected component containing $(x, y) \in V \times_M W$.*

Proof. If $f : V \to M$ is of maximal rank at $x \in V$, it is an immersion or submersion at that point, and similarly for g. Thus the corollary follows directly from Lemmas 3.47, 3.48, and 3.49. Not that while the statement of Lemma 3.48 assumes f is the submersion and g the immersion, we obtain the opposite case by reversing the roles of g and f in the statement of Proposition 3.45. $\square$

We now show that for arbitrary transverse intersecting f and g there are always homotopies that maintain these properties while taking f and g each to maps of full rank at the intersection point.

Lemma 3.51. *Let $f : V \to M$ and $g : W \to M$ be transverse proper maps from manifolds with corners to a manifold without boundary. Suppose $x \in V$ and $y \in W$ such that $f(x) = g(y)$. Then there is a smooth homotopy $F : V \times I \to M$ such that $F(-, 0) = f$, $F(x, t) = f(x)$ for all $t \in I$, $F(-, t)$ is transverse to g for all $t \in I$, and $DF_{(x,1)}$ has maximal rank at x.*

Proof. If Df already has maximal rank at x we can take $F(-, t) = f(-)$. So suppose Df does not have maximal rank at x. We will construct a homotopy fixed outside a small neighborhood of x, so we work in local charts, identifying neighborhoods of x and $f(x)$ with $\mathbb{R}^v$ and $\mathbb{R}^m$ so that x and $f(x)$ are at the respective origins. We may also choose the charts so that $Df_x(e_i) = e_i$ for $1 \leq i \leq k$ for some $k < \dim(V) = v$ and $Df_x(e_i) = 0$ for $i > k$. Let $\eta : \mathbb{R}^v \to \mathbb{R}$ be a smooth function that is 0 outside a compact neighborhood of the origin and 1 on a neighborhood of the origin. Let $z = (z_1, \dots, z_v)$ be the coordinates of $\mathbb{R}^v$, and let

$$H(z_1, \dots, z_v, t) = f(z) + t\eta(z) \left(\sum_{i=k+1}^{\min(v,m)} z_i \right).$$

Then $H(z, t) = f(z)$ outside a compact neighborhood of the origin in $\mathbb{R}^v$, and so H extends to a homotopy defined on all of $V \times I$. Furthermore, $H(0, t) = f(0)$, i.e. $H(x, t) = f(x)$. Also, $DH_{(0,t)} e_i = e_i$ for $i \leq k$ and te_i for $k < i \leq \min(v, m)$. Thus $DH(-, t)$ has full rank at x for all $t > 0$. Finally, as f and g are proper and transverse, there is an ϵ so that for $0 \leq t \leq \epsilon$, $H(-, t)$ and g will be transverse by stability of transversality[19]. Now define $F(-, t) = H(-, \epsilon t)$. $\square$

Proof of Proposition 3.45. As noted above, it suffices to verify the claim at one point of each connected component of P. As f and g are transverse, if a component P_0 of $V \times_M W$ is nonempty, by Theorem 2.18 we can find $x \in V$, $y \in W$, each point in the interior, so that $(x, y) \in P_0$. By Lemma 3.51, we can first perform a homotopy $F : V \times I \to M$ of f to $f' : V \to M$ and then a homotopy $G : W \times I \to M$ of g to $g' : W \to M$ so that f' and g' have full rank at x and y respectively.

[19] A proof is given within our proof of Theorem 6.5, below.

First consider F. As f is co-oriented, F is co-orientable by Lemma 3.26 and can be co-oriented by Definition 3.27. So $(V \times I) \times_M W$ can be co-oriented, and by Proposition 3.44 two of its boundary components will be $f \times_M g : (V \times 0) \times_M W = V \times_M W \to M$ and $f' \times_M g : (V \times 1) \times_M W = V \times_M W \to M$, topologically, with the co-orientations given by composing the boundary co-orientations with the co-orientation of $F \times_M g$. Similarly, two of the boundary components of $W \times_M (V \times I)$ will be $g \times_M f : W \times_M (V \times 0) = W \times_M V \to M$ and $g \times_M f' : W \times_M (V \times 1) = W \times_M V \to M$ topologically, with the co-orientations given by $(-1)^{m-w}$ times the compositions of the boundary co-orientations with the co-orientation of $g \times_M F$. But now the point is that $(V \times I) \times_M W$ and $W \times_M (V \times I)$ are diffeomorphic with compatible maps to M, and so their co-orientations either agree or disagree, and correspondingly the co-orientations of the corresponding boundary components will agree or disagree. In other words, identifying these pullback as in Diagram (6), we see that regardless of the actual co-orientations of $F \times_M g$ and $g \times_M F$ we have $\omega_{f \times_M g} = (-1)^{(m-v)(m-w)} \omega_{g \times_M f}$ if and only if $\omega_{f' \times_M g} = (-1)^{(m-v)(m-w)} \omega_{g \times_M f'}$. Analogously using G, we have $\omega_{f' \times_M g} = (-1)^{(m-v)(m-w)} \omega_{g \times_M f'}$ if and only if $\omega_{f' \times_M g'} = (-1)^{(m-v)(m-w)} \omega_{g' \times_M f'}$. But this last equality is true by Corollary 3.50. □

3.5.6. *Codimension* 0 *and* 1 *pullbacks.* The results in this section should be compared with, and justify, the discussion and choices in Examples 3.21 and 3.22. They will be useful when working with the creasing construction for geometric cochains.

Proposition 3.52. *Let V be an embedded codimension 0 submanifold with corners in the manifold without boundary M, and let $f : V \to M$ be the embedding, co-oriented by $(\beta_V, f_*\beta_V)$ at each point of V. Let W be a manifold with corners and suppose $g : W \to M$ is transverse to f. Then the co-oriented pullback $f^* : V \times_M W \to W$ is the inclusion of the codimension 0 manifold with corners $g^{-1}(V) \hookrightarrow W$, co-oriented by the normal co-orientation of the immersion, which in this case is (β_P, β_P), identifying the tangent spaces of $V \times_M W$ and W. Consequently, if g is co-oriented the co-oriented fiber product $V \times_M W \to M$ is just the restriction of the co-oriented map g to the embedded codimension 0 submanifold with corners $g^{-1}(V)$.*

Proof. It is clear topologically that the pullback is $g^{-1}(V)$, and it must be a codimension 0 manifold with corners of W by Joyce [**?**]. So we consider co-orientations. As f is an embedding we may choose $N = 0$ and $e = f$ in Definition 3.35. Identifying β_V with β_M via the embedding, the Quillen normal bundle of V in M is the positively oriented $\mathbb{R}^0$-bundle. So then the definition says that the pullback co-orientation is (β_P, β_W) when $\beta_P = \beta_W$ up to positive scalar. □

Corollary 3.53. *Let $f : V \to M$ be a co-oriented map from a manifold with corners to a manifold without boundary, and let $\mathrm{id}_M : M \to M$ be the identity. Then both co-oriented fiber products $V \times_M M \to M$ and $M \times_M V \to M$ are again $f : V \to M$.*

Proof. The first case follows immediately from the preceding lemma, and the other then follows from the graded commutativity of the fiber product, Proposition 3.45. □

Example 3.54. Let $\varphi : M \to \mathbb{R}$ be a smooth function from a manifold without boundary to $\mathbb{R}$, and let $M^+ = \varphi^{-1}([0, \infty)) = [0, \infty) \times_{\mathbb{R}} M$, $M^- = \varphi^{-1}((-\infty, 0]) = (-\infty, 0] \times_{\mathbb{R}} M$, and $M^0 = \varphi^{-1}(0) = 0 \times_{\mathbb{R}} M$. Then $M^\pm$ are embedded codimension 0 manifolds with corners in M and $\partial M^\pm = M^0$. Suppose the embeddings $M^\pm \hookrightarrow M$ are given the tautological co-orientations of Definition 3.12, and let $g : W \to M$ be transverse to $M^\pm$, which in this case is equivalent to being transverse to M^0. Then let $W^\pm := g^{-1}(M^\pm) = M^\pm \times_M W$, and the pullback co-orientations of $W^\pm \hookrightarrow W$ given by Proposition 3.52 are again just the tautological co-orientations of the codimension 0 embeddings. The compositions $W^\pm \hookrightarrow W \to M$ are then the fiber products, and the co-orientations of Proposition 3.52 agree with those described in Example 3.22.

Proposition 3.55. *Suppose $V \subset M$ is a codimension 1 submanifold with corners with oriented normal bundle ν in the manifold without boundary M. Suppose the embedding $f : V \hookrightarrow M$ is co-oriented by*

$(\beta_V, \beta_V \wedge \beta_\nu)$. Let W be a manifold without corners and suppose $g : W \to M$ is transverse to f so that $g^{-1}(V) = W^0$ is a codimension 1 submanifold with corners with oriented pullback normal vector bundle ν_0. Then the co-oriented pullback $f^* : P = V \times_M W \to W$ is the embedding $W^0 \hookrightarrow W$, co-oriented by $(\beta_P, \beta_P \wedge \beta_{\nu_0})$.

Proof. It is again standard that the pullback is $g^{-1}(V) = W^0$ embedded in W with codimension 1. For the co-orientation, if we take $N = 0$ and $e = f$ in Definition 3.35, then ν is the same as in that definition and ν_0 is simply the pullback that we just again called ν there. Then by definition the pullback co-orientation is (β_P, β_W) if and only if $\beta_P \wedge \beta_{\nu_0} = \beta_W$ up to positive scalar, as claimed. □

Example 3.56. Continuing to assume $\varphi : M \to \mathbb{R}$, let $V = M^0 = \varphi^{-1}(0)$ with normal bundle oriented by the pullback of the standard (positive-direction) orientation of the normal bundle of $0 \in \mathbb{R}$, which determines a co-orientation of the embedding $M^0 \to M$. Then by Proposition 3.55 the pullback co-orientation of $W^0 = M^0 \times_M W \hookrightarrow W$ agrees with the φ-induced co-orientation of W^0 defined in Example 3.22. We can also confirm now using the Leibniz rule that

$$\partial(W^-) = \partial(M^- \times_M W) = (\partial M^-) \times_M W \bigsqcup M^- \times \partial W = M^0 \times_M W \bigsqcup M^- \times_M \partial W = (-W^0) \bigsqcup (\partial W)^-,$$

using that the orientation of the normal bundle to M^0 is outward pointing from M^- and so disagrees with the inward-pointing normal used to co-orient the boundary inclusion. Analogously,

$$\partial(W^+) = W^0 \bigsqcup (\partial W)^+,$$

using that the orientation of the normal bundle to M^0 is inward pointing for M^+.

Remark 3.57. The formulas in this example are mnemonically convenient, with either all terms involving the symbol $-$ or all terms involving the symbol $+$, though implicitly in the case of the W^0 in the formula for $\partial(W^+)$.

We can now prove the claim from the end of Example 3.22. We express the following corollary using Notation 3.2.

Corollary 3.58. *Suppose the hypotheses and notation of Proposition 3.55 and suppose V is without boundary. Then $\partial(W^0) = -(\partial W)^0$ as co-oriented maps to M, with W^0 and $(\partial W)^0 = (gi_{\partial W})^{-1}(V)$ co-oriented as in Proposition 3.55 as the pullbacks $V \times_M W \to W$ and $V \times_M \partial W \to W$.*

Proof. By Proposition 3.44, we have that

$$\partial(W^0) = \partial(V \times_M W) = (-1)^{\dim(M) - \dim(V)} V \times_M \partial W = -V \times_M \partial W = -(\partial W)^0$$

as spaces mapping M. □

3.6. **Exterior products.** In this section we consider products of maps that will eventually become the exterior products in geometric homology and cohomology. While fiber products, which will eventually be used to define cup and intersection products, require special transversality conditions in order to be defined, these exterior products are always fully defined. We consider first products of oriented manifolds and then products of co-oriented maps of manifolds.

3.6.1. *Exterior products of maps of oriented manifolds.* If we begin with maps $f : V \to M$ and $g : W \to N$, then we have the usual product map $f \times g : V \times W \to M \times N$ that takes $(v, w) \in V \times W$ to $(f(v), g(w)) \in M \times N$. If V and W are oriented, say by β_V and β_W, then $V \times W$ possesses the standard product orientation via concatenation of oriented bases that can be described at each point by $\beta_V \wedge \beta_W$. More technically, $T(M \times N) = \pi_M^*(TM) \oplus \pi_N^*(TN)$ with π_M, π_N being the projections, but we elide the pullbacks in our notation for local orientations. By Proposition 3.3, this convention is consistent with the observation that the exterior product is isomorphic to the fiber product of the unique maps $V, W \to pt$.

We observe the following interplay between fibered and exterior products of maps of oriented manifolds more generally.

Proposition 3.59. *Suppose $f : V \to M$ and $g : W \to M$ are transverse maps of oriented manifolds with corners to an oriented manifold without boundary and similarly for $h : X \to N$ and $k : Y \to N$. Then*

$$(V \times X) \times_{M\times N} (W \times Y) = (-1)^{(m-w)(n-y)}(V \times_M W) \times (X \times_N Y)$$

as oriented manifolds.

Proof. We first note that the transversality assumptions ensure also that $f \times h$ will be transverse to $g \times k$. It is straightforward to verify that these are diffeomorphic spaces, so we focus on the orientations. For simplicity, let us write $P = (V \times X) \times_{M\times N} (W \times Y)$ and $P' = (V \times_M W) \times (X \times_N Y)$. We then write orientation symbolically as β_P, etc. By definition, and omitting the pullbacks from the notation, P is oriented so that

$$\beta_P \wedge \beta_{M\times N} = (-1)^{(w+y)(m+n)}\beta_{V\times X} \wedge \beta_{W\times Y},$$

or, as (non-fiber) products are oriented by concatenation,

$$\beta_P \wedge \beta_M \wedge \beta_N = (-1)^{(w+y)(m+n)}\beta_V \wedge \beta_X \wedge \beta_W \wedge \beta_Y.$$

Here we identify $T(M \times N)$ as a summand by splitting $D(f \times h) - D(g \times k)$. Similarly, we have

$$\beta_{V\times_M W} \wedge \beta_M = (-1)^{wm}\beta_V \wedge \beta_W$$
$$\beta_{X\times_N Y} \wedge \beta_N = (-1)^{yn}\beta_X \wedge \beta_Y,$$

using the splittings of $Df - Dg$ and $Dh - Dk$. In particular, the signs of Df, Dg, Dh, and Dk in the splitting formulas are consistent in computing the orientations for P and P'. Wedging the two sides, we have

$$\beta_{V\times_M W} \wedge \beta_M \wedge \beta_{X\times_N Y} \wedge \beta_N = (-1)^{wm+ny}\beta_V \wedge \beta_W\beta_X \wedge \beta_Y.$$

As $\beta_{P'} = \beta_{V\times_M W} \wedge \beta_{X\times_N Y}$, we have

$$\begin{aligned}
\beta_{P'} \wedge \beta_M \wedge \beta_N &= (-1)^{m(x+y-n)}\beta_{V\times_M W} \wedge \beta_M \wedge \beta_{X\times_N Y} \wedge \beta_N \\
&= (-1)^{m(x+y-n)+wm+ny}\beta_V \wedge \beta_W \wedge \beta_X \wedge \beta_Y \\
&= (-1)^{m(x+y-n)+wm+ny+xw}\beta_V \wedge \beta_X \wedge \beta_W \wedge \beta_Y
\end{aligned}$$

So β_P differs from $\beta_{P'}$ by -1 to the power

$$m(x + y - n) + wm + ny + xw - ((w + y)(m + n)).$$

An elementary computation now shows that this is $(m - w)(n - x)$ as desired. □

3.6.2. *Exterior products of co-oriented maps and applications to fiber products.* Next we define and study an exterior product for co-orientable maps. In the next subsection, we will see that such products are intimately related to fiber products, just as cup products and exterior products are related in cohomology. In fact, this relationship will allow us to easily proof some properties about fiber products that we have delayed.

Before getting to co-orientations, we first show that a product of proper maps is proper.

Lemma 3.60. *If $f : V \to M$ and $g : W \to N$ are proper maps of spaces then the product map $f \times g : V \times W \to M \times N$ is proper.*

Proof. Let π_M, π_N be the projections of $M\times N$ to M and N, and similarly for π_V, π_W. Let K be a compact subspace of $M \times N$. Then $\pi_M(K)$ and $\pi_N(K)$ are compact, and hence so is $\pi_M(K) \times \pi_N(K) \subset M \times N$, and this set contains K. So

$$(f \times g)^{-1}(K) \subset (f \times g)^{-1}(\pi_M(K) \times \pi_N(K)) = f^{-1}(\pi_M(K)) \times g^{-1}(\pi_N(K)).$$

But now $f^{-1}(\pi_M(K))$ and $g^{-1}(\pi_N(K))$ are compact as f and g are proper and so $(f\times g)^{-1}(K)$ is a closed subset of a compact set, hence compact. □

Lemma 3.61. *If $f : V \to M$ and $g : W \to N$ are co-orientable maps of manifolds with corners then the product map $f \times g : V \times W \to M \times N$ is co-orientable.*

Proof. We recall that, by definition, a co-orientation of f is equivalent to a choice of isomorphism between the orientation cover $\mathrm{Det}(TV)$ and the pullback $f^*\mathrm{Det}(TM)$ of the orientation cover $\mathrm{Det}(TM)$, and similarly for g.

If we let π_V, π_W denote the projections of $V\times W$ to V and W, then $T(V\times W) \cong \pi_V^*(TV) \oplus \pi_W^*(TW)$, and so

$$\mathrm{Det}(T(V\times W)) \cong \mathrm{Det}(\pi_V^*(TV)) \otimes \mathrm{Det}(\pi_W^*(TW)) \cong \pi_V^*\mathrm{Det}(TV) \otimes \pi_W^*\mathrm{Det}(TW).$$

Similarly

$$\begin{aligned}(f\times g)^*T(M\times N) &\cong (f\times g)^*(\pi_M^*(TM)\oplus \pi_N^*(TN))\\ &\cong (f\times g)^*\pi_M^*(TM)\oplus (f\times g)^*\pi_N^*(TN)) \cong \pi_V^* f^*(TM) \oplus \pi_W^* g^*(TN),\end{aligned}$$

using that $\pi_M(f\times g) = f\pi_V : V\times W \to M$ and $\pi_N(f\times g) = g\pi_W : V \times W \to N$. So

$$\begin{aligned}(f\times g)^*\mathrm{Det}(T(M\times N)) &\cong \mathrm{Det}((f\times g)^*T(M\times N))\\ &\cong \mathrm{Det}(\pi_V^* f^*TM)\otimes \mathrm{Det}(\pi_W^* g^* TN) \cong \pi_V^* f^*\mathrm{Det}(TM)\otimes \pi_W^* g^*\mathrm{Det}(TN).\end{aligned}$$

Thus if $\mathrm{Det}(TV)\cong f^*\mathrm{Det}(TM)$ and $\mathrm{Det}(TW)\cong g^*\mathrm{Det}(TN)$, we can construct isomorphisms $\mathrm{Det}(T(V\times W))\cong (f\times g)^*\mathrm{Det}(T(M\times N))$. □

Definition 3.62. If $f : V \to M$ and $g : W \to N$ are co-oriented maps of manifolds with corners with co-orientations given by isomorphisms $\phi : \mathrm{Det}(TV) \to f^*\mathrm{Det}(TM)$ and $\psi : \mathrm{Det}(TW) \to g^*\mathrm{Det}(TN)$, we define the **product co-orientation** of $f\times g : V\times W \to M\times N$ by the isomorphism $(-1)^{(m-v)w}\pi_V^*\phi\otimes\pi_W^*\psi$. In particular, if at $x\in V$ the co-orientation of f is given locally by (β_V, β_M) and at $y \in W$ the co-orientation of g is given locally by (β_W, β_N), then up to standard abuses of notation the product co-orientation is locally represented at (x, y) by

$$(-1)^{(m-v)w}(\beta_V \wedge \beta_W, \beta_M \wedge \beta_N).$$

Following our standard conventions, we often denote this product manifold over $M\times N$ by simply $V\times W$.

Remark 3.63. The sign in the definition is not at first obvious, though it will be justified in the following lemmas. One way to think of it is as follows: If we we take V and W as immersed submanifolds co-oriented by orienting their normal bundles as in Definition 3.15, then $V\times W$ is also immersed, and at an image point we have $T(M\times N)\cong TV\oplus \nu V\oplus TW\oplus \nu W$. The sign $(-1)^{(m-v)w}$ is the sign needed in the orientation to permute this to $TV\oplus TW\oplus \nu V\oplus \nu W$ so that we can properly utilize the normal co-orientation for $\nu(V\times W)\cong \nu V\oplus \nu W$. While this argument is essentially heuristic, it is borne out in the computations below.

Example 3.64. Let S^p and S^q be oriented spheres with $p, q > 0$. Let $V = W = pt$, and let $f : V \to S^p$ and $g : W \to S^q$ be embeddings to points $x \in S^p$, $y \in S^q$. Let f be co-oriented by $(1, \beta_{S^p})$; in other words V is normally co-oriented by orientation of its normal bundle that agrees with the orientation of S^p. Let g be co-oriented similarly. Then $V\times W$ is represented by the embedding of the point to $(x, y) \in S^p\times S^q$ with normal bundle oriented consistently with the product orientation of $S^p\times S^q$. There is no extra sign in this case as $\dim(W) = 0$.

Proposition 3.65. *Let $f : V \to M$ be a co-oriented map of manifolds with corners, and let $g : pt \to pt$ be the unique map with the canonical co-orientation. Then $f\times g : V\times pt \to M\times pt$ and $g\times f : pt\times V \to pt\times M$ are each isomorphic as co-oriented maps of manifolds with corners to $f : V \to M$.*

Proof. This is obvious ignoring co-orientations. Considering co-orientations, if f is co-oriented at a point by (β_V, β_M), then the co-orientation of $f \times g$ is simply $(\beta_V \wedge \mathbb{R}, \beta_M \wedge \mathbb{R}) = (\beta_V, \beta_M)$, noting that the sign $(-1)^{(m-v)\cdot 0} = 1$ in this case. The case $g \times f$ is similar, though due to the transposition the sign is now $(-1)^{(0-0)v}$, which is again 1. □

Proposition 3.66. *Let $f : V \to M$ and $g : W \to N$ be co-oriented maps of manifolds with corners and suppose $f \times g : V \times W \to M \times N$ is given the product co-orientation. Then the boundary co-orientation of $V \times W$ as co-oriented maps to $M \times N$ is*

$$\partial(V \times W) = (\partial V) \times W \bigsqcup (-1)^{m-v} V \times \partial W.$$

Proof. We know that this expression is an identity ignoring co-orientations, so we must establish the agreement of the co-orientations for each component. As usual, it suffices to consider points in the top dimensional strata of $\partial(V \times W)$. In what follows, let (β_V, β_M), (β_W, β_N), and $(-1)^{(m-v)w}(\beta_V \wedge \beta_W, \beta_M \wedge \beta_N)$ denote the co-orientations of V, W, and $V \times W$ at the point under consideration.

Let ν denote an inward pointing normal to $V \times W$ at such a point. Then the inclusion $\partial(V \times W) \to V \times W$ is co-oriented at that point by $(\beta_{\partial(V\times W)}, \beta_{\partial(V\times W)} \wedge \beta_\nu)$ for any $\beta_{\partial(V\times W)}$. If we choose $\beta_{\partial(V\times W)}$ so that $(\beta_{\partial(V\times W)} \wedge \beta_\nu, \beta_M \wedge \beta_N)$ represents the co-orientation of $V \times W \to M \times N$, then from the definition of the boundary co-orientation, the boundary $\partial(V \times W) \to M \times N$ is co-oriented by $(\beta_{\partial(V\times W)}, \beta_M \wedge \beta_N)$. We fix such a choice in what follows.

Now suppose our point is more specifically in the top-dimensional stratum of $(\partial V) \times W$. If we choose $\beta_{\partial V}$ so that $\beta_{\partial V} \wedge \beta_\nu = \beta_V$, then $\partial V \to M$ is co-oriented by $(\beta_{\partial V}, \beta_M)$ and so $(\partial V) \times W$ is co-oriented by $(-1)^{(m-v+1)w}(\beta_{\partial V} \wedge \beta_W, \beta_M \wedge \beta_N)$. On the other hand, the co-orientation of $V \times W$ can then be written $(-1)^{(m-v)w}(\beta_{\partial V} \wedge \beta_\nu \wedge \beta_W, \beta_M \wedge \beta_N) = (-1)^{(m-v)w+w}(\beta_{\partial V} \wedge \beta_W \wedge \beta_\nu, \beta_M \wedge \beta_N)$, so the boundary co-orientation of $\partial(V \times W)$ is $(-1)^{(m-v)w+w}(\beta_{\partial V} \wedge \beta_W, \beta_M \wedge \beta_N)$, which agrees with our co-orientation for $(\partial V) \times W$.

Next consider a point in the top-dimensional stratum of $V \times \partial W$. If we choose $\beta_{\partial W}$ so that $\beta_{\partial W} \wedge \beta_\nu = \beta_W$ then we have ∂W co-oriented by $(\beta_{\partial W}, \beta_N)$ and so $V \times \partial W$ is co-oriented by $(-1)^{(m-v)(w-1)}(\beta_V \wedge \beta_{\partial W}, \beta_M \wedge \beta_N)$. On the other hand, the co-orientation of $V \times W$ can now be written $(-1)^{(m-v)w}(\beta_V \wedge \beta_{\partial W} \wedge \beta_\nu, \beta_M \wedge \beta_N)$, so the boundary co-orientation of $\partial(V \times W)$ is $(-1)^{(m-v)w}(\beta_V \wedge \beta_{\partial W}, \beta_M \wedge \beta_N)$, which differs from that of $V \times \partial W$ by a factor of $(-1)^{m-v}$. □

Proposition 3.67. *Let $f : V \to M$, $g : W \to N$, and $h : X \to Q$ be co-oriented maps of manifolds with corners. Then the co-orientations of $(V \times W) \times X \to M \times N \times Q$ and $V \times (W \times X) \to M \times N \times Q$ agree. In other words, forming co-oriented products is associative.*

Proof. Let $v = \dim(V)$, etc. If f, g, h are co-oriented by (β_V, β_M), etc., then both products are co-oriented up to sign by $(\beta_V \wedge \beta_W \wedge \beta_X, \beta_M \wedge \beta_N \wedge \beta_P)$. In forming $(V \times W) \times X$ we first have the sign $(-1)^{(m-v)w}$ from $V \times W$, then taking the product with X on the right multiplies by $(-1)^{(m+n-v-w)x}$. So the total sign is $(-1)^{(m-v)w+(m+n-v-w)x}$. Alternatively, forming $W \times X$ has the sign $(-1)^{(n-w)x}$ and then taking the product with V on the left contributes $(-1)^{(m-v)(w+x)}$. So the total sign is $(-1)^{(n-w)x+(m-v)(w+x)}$. One readily verifies that these signs agree. □

Dev and Anibal, please check the following arguments carefully as I'm not 100% confident in it. It gives the "right" answer but I'm a little uncomfortable divorcing the order of the orientation terms from the order of the manifold terms. Of course this happens all the time - even if we think of $\mathbb{R}^2$ as $\mathbb{R}_x \oplus \mathbb{R}_y$ we can still think about the two-form $y \wedge x$, but I'm still a little nervous about maybe having missed a sign somewhere. I'm also a little nervous about my trick of taking a and b to be even so that they won't contribute, but the earlier work says that this should be allowable. Presumably if I didn't do this there would be a bunch of extra signs that miraculous cancel out, but I'm not so excited about trying that out in detail to see.

The following lemma provides a nice description of the Quillen co-orientation of product of co-oriented maps. Among other things, it will help us to next demonstrate a commutativity property for exterior products of co-oriented maps.

Lemma 3.68. *Let $f : V \to M$ and $g : W \to N$ be co-oriented maps from manifolds with corners to manifolds without boundary. Consider Quillen co-orientations representing f and g via embeddings $e_V : V \hookrightarrow M \times \mathbb{R}^a$ and $e_W : W \hookrightarrow N \times \mathbb{R}^b \to N$ with a and b even. Denote the normal bundles of V and W in $M \times \mathbb{R}^a$ and $N \times \mathbb{R}^b$ by νV and νW. Let $T : M \times \mathbb{R}^a \times N \times \mathbb{R}^b \to M \times N \times \mathbb{R}^{a+b}$ be the diffeomorphism that interchanges the middle two factors. Then $T(e_V \times e_W)$ gives an embedding $V \times W \to M \times N \times \mathbb{R}^{a+b}$ with normal bundle isomorphic to the sum of the pullbacks of νV and νW by the projections of $V \times W \to M \times N \times \mathbb{R}^a \times \mathbb{R}^b$ to either the first and third factor or the second and fourth factors. For simplicity, we simply write $\nu V \oplus \nu W$.*

Then the Quillen co-orientation of $f \times g : V \times W \to M \times N$ has Quillen co-orientation given by

$$\beta_{\nu V \oplus \nu W} = \beta_{\nu V} \wedge \beta_{\nu W},$$

suitably interpreting the relevant coordinates in $M \times N \times \mathbb{R}^a \times \mathbb{R}^b$.

Proof. Let β_a and β_b denote the standard orientations for $\mathbb{R}^a$ and $\mathbb{R}^b$. By definition, νV and νW are oriented so that $\beta_V \wedge \beta_{\nu V} = \beta_M \wedge \beta_a$ and $\beta_W \wedge \beta_{\nu W} = \beta_N \wedge \beta_b$.

By definition, the Quillen orientation of $\nu V \oplus \nu W$ corresponding to the product co-orientation of $V \times W$ is the orientation $\beta_{\nu V \oplus \nu W}$ such that

$$(\beta_{V \times W}, \beta_{V \times W} \wedge \beta_{\nu V \oplus \nu W}) * (\beta_{M \times N} \wedge \beta_{a+b}, \beta_{M \times N}) = (-1)^{(m-v)w}(\beta_V \wedge \beta_W, \beta_M \wedge \beta_N).$$

Taking $\beta_{M \times N} = \beta_M \wedge \beta_N$ and $\beta_{V \times W} = \beta_V \wedge \beta_W$ and noting $\beta_{a+b} = \beta_a \wedge \beta_b$, this formula becomes

$$(\beta_V \wedge \beta_W, \beta_V \wedge \beta_W \wedge \beta_{\nu V \oplus \nu W}) * (\beta_M \wedge \beta_N \wedge \beta_a \wedge \beta_b, \beta_M \wedge \beta_N) = (-1)^{(m-v)w}(\beta_V \wedge \beta_W, \beta_M \wedge \beta_N).$$

We also have

$$\beta_M \wedge \beta_N \wedge \beta_a \wedge \beta_b = \beta_M \wedge \beta_a \wedge \beta_N \wedge \beta_b,$$

as a is even, so using $\beta_V \wedge \beta_{\nu V} = \beta_M \wedge \beta_a$ and $\beta_W \wedge \beta_{\nu W} = \beta_N \wedge \beta_b$, we require

$$(\beta_V \wedge \beta_W, \beta_V \wedge \beta_W \wedge \beta_{\nu V \oplus \nu W}) * (\beta_V \wedge \beta_{\nu V} \wedge \beta_W \wedge \beta_{\nu W}, \beta_M \wedge \beta_N) = (-1)^{(m-v)w}(\beta_V \wedge \beta_W, \beta_M \wedge \beta_N).$$

But now

$$\beta_V \wedge \beta_{\nu V} \wedge \beta_W \wedge \beta_{\nu W} = (-1)^{(m-v)w} \beta_V \wedge \beta_W \wedge \beta_{\nu V} \wedge \beta_{\nu W},$$

so, after all that, we see that the Quillen orientation of the normal bundle to $V \times W$ is simply

$$\beta_{\nu V \oplus \nu W} = \beta_{\nu V} \wedge \beta_{\nu W},$$

suitably interpreting the relevant coordinates in $M \times N \times \mathbb{R}^a \times \mathbb{R}^b$. □

Proposition 3.69. *Let $f : V \to M$ and $g : W \to N$ be co-oriented maps from manifolds with corners to manifolds without boundary and suppose $f \times g : V \times W \to M \times N$ is given the product co-orientation. Let $\tau : N \times M \to M \times N$ be the diffeomorphism that interchanges coordinates. Then the pullback of $V \times W$ by τ is*

$$\tau^*(V \times W) = (-1)^{(m-v)(n-w)} W \times V,$$

where $m = \dim M$, etc.

We assume M and N to be without corners so that we can properly use the pullback construction, which requires transversality, in the proposition and its proof. However, the pullback is by a diffeomorphism, so this result should extend without problem to more general settings.

Proof. Let (β_V, β_M) and (β_W, β_N) be local representations of the co-orientations at some points. The product co-orientation of $V \times W \to M \times N$ is $(-1)^{(m-v)w}(\beta_V \wedge \beta_W, \beta_M \wedge \beta_N)$.

As in Lemma 3.68,we consider Quillen co-orientations representing f and g via embeddings $e_V : V \hookrightarrow M \times \mathbb{R}^a$ and $e_W : W \hookrightarrow N \times \mathbb{R}^b \to N$ with a and b even. This is sufficient as we know that the pullback construction is independent of a and b for sufficiently large dimensions by Lemma 3.37. Assuming the other notation from Lemma 3.68, the lemma tells us that $f \times g : V \times W \to M \times N$ has Quillen co-orientation given by

$$\beta_{\nu V \oplus \nu W} = \beta_{\nu V} \wedge \beta_{\nu W},$$

suitably interpreting the relevant coordinates in $M \times N \times \mathbb{R}^a \times \mathbb{R}^b$.

Now using this Quillen co-orientation for $f \times g$, we pull back by the diffeomorphism $\tau : N \times M \to M \times N$, obtaining the composition we can write $W \times V \hookrightarrow N \times M \times \mathbb{R}^a \times \mathbb{R}^b \to N \times M$. The pulled back normal bundle is still oriented in each fiber as $\beta_{\nu V} \wedge \beta_{\nu W}$ (though of course the order of actual local coordinates have now been jumbled around). By definition, the pullback co-orientation is $(\beta_W \wedge \beta_V, \beta_N \wedge \beta_M)$ if and only if

$$\beta_W \wedge \beta_V \wedge \beta_{\nu V} \wedge \beta_{\nu W} = \beta_N \wedge \beta M \wedge \beta_{a+b},$$

and as a and b are even this last expression is equal to $\beta_N \wedge \beta_b \wedge \beta_M \wedge \beta_a$. But by the previous choices, $\beta_V \wedge \beta_{\nu V} = \beta_M \wedge \beta_a$ and $\beta_W \wedge \beta_{\nu W} = \beta_N \wedge \beta_b$. So

$$\begin{aligned}
\beta_N \wedge \beta_b \wedge \beta_M \wedge \beta_a &= \beta_W \wedge \beta_{\nu W} \wedge \beta_V \wedge \beta_{\nu V} \\
&= (-1)^{v(n+b-w)} \beta_W \wedge \beta_V \wedge \beta_{\nu W} \wedge \beta_{\nu V} \\
&= (-1)^{v(n+b-w)+(m+a-v)(n+b-w)} \beta_W \wedge \beta_V \wedge \beta_{\nu V} \wedge \beta_{\nu W} \\
&= (-1)^{v(n-w)+(m-v)(n-w)} \beta_W \wedge \beta_V \wedge \beta_{\nu V} \wedge \beta_{\nu W},
\end{aligned}$$

where again we use that a and b are even.

Therefore, the pullback co-orientation is $(-1)^{v(n-w)+(m-v)(n-w)}(\beta_W \wedge \beta_V, \beta_N \wedge \beta_M)$, which is $(-1)^{(m-v)(n-w)}$ times the product co-orientation of $W \times V$, as claimed. $\square$

The next result relates co-oriented products in which one map is the identity with pullbacks by projections.

Proposition 3.70. *Let $f : V \to M$ be a co-oriented map from a manifolds with corners to a manifold without boundary, and let $\mathrm{id}_N : N \to N$ be the identity map of a manifold with corners with the canonical co-orientation.*

1. *The co-oriented pullback of V by the projection $\pi_1 : M \times N \to M$ is $f \times \mathrm{id}_N : V \times N \to M \times N$ with its product co-orientation, i.e. $\pi_1^* V = V \times N$.*
2. *The co-oriented pullback of V by the projection $\pi_2 : N \times M \to M$ is $\mathrm{id}_N \times f : N \times V \to N \times M$ with its product co-orientation, i.e. $\pi_2^* V = N \times V$.*

Proof. As the projections are submersions, the required transversality conditions to ensure the existence of the pullbacks are met. These claims are then clear concerning maps of topological spaces, so we need only verify the co-orientations.

As in the preceding argument, we start again with an embedding $e : V \hookrightarrow M \times \mathbb{R}^a$ to establish a Quillen co-orientation for f. We again may assume a be even for simplicity. We write the co-orientation of f as (β_V, β_M), and we let νV denote the normal to $e(V)$ and orient νV so that $\beta_V \wedge \beta_{\nu V} = \beta_M \wedge \beta_a$, writing β_a for the standard orientation of $\mathbb{R}^a$.

For the second statement, the product co-orientation of $\mathrm{id}_N \times f : N \times V \to N \times M$ is $(\beta_N \wedge \beta_V, \beta_N \wedge \beta_M)$, as the domain and codomain of id_N have the same dimension. The pullback by the projection $N \times M \to M$ gives us the embedding/projection sequence $N \times V \xhookrightarrow{\mathrm{id}_N \times e} N \times M \times \mathbb{R}^a \to N \times M$, and the oreintations of the pullback of the normal bundle νV by $\pi_2 \times \mathrm{id}_{\mathbb{R}^a}$ is again $\beta_{\nu V}$ at each point of $N \times e(V)$. So now from the definition, the pullback has the product co-orientation if and only if $\beta_N \wedge \beta_V \wedge \beta_{\nu V} = \beta_N \wedge \beta_M \wedge \beta_a$. But $\beta_V \wedge \beta_{\nu V} = \beta_M \wedge \beta_a$ by assumption, so this holds.

For the first statement, the product co-orientation of $f \times \mathrm{id}_N : V \times N \to M \times N$ is $(-1)^{(m-v)n}(\beta_V \wedge \beta_N, \beta_M \wedge \beta_N)$. The pullback by the projection $M \times N \to M$ gives us an embedding/projection sequence $V \times N \hookrightarrow M \times N \times \mathbb{R}^a \to M \times N$ (where the first arrow is the composition of $e \times \mathrm{id}_N$ with a permutation of coordinates), and the orientation of the pullback of the normal bundle νV by $\pi_1 \times \mathrm{id}_{\mathbb{R}^a}$ is again $\beta_{\nu V}$. So now from the definition, the pullback has the product co-orientation if and only if $\beta_V \wedge \beta_N \wedge \beta_{\nu V} = (-1)^{(m-v)n}\beta_M \wedge \beta_N \wedge \beta_a$. But $\beta_V \wedge \beta_{\nu V} = \beta_M \wedge \beta_a$ by assumption, so

$$\begin{aligned}\beta_V \wedge \beta_N \wedge \beta_{\nu V} &= (-1)^{(m+a-v)n}\beta_V \wedge \beta_{\nu V} \wedge \beta_N \\ &= (-1)^{(m+a-v)n}\beta_M \wedge \beta_a \wedge \beta_N \\ &= (-1)^{(m-v)n}\beta_M \wedge \beta_N \wedge \beta_a. \qquad \square\end{aligned}$$

The next proposition shows that the exterior product construction is natural.

Proposition 3.71. *Let $f : V \to M$ and $g : W \to N$ be co-oriented maps of manifolds with corners with M and N having no boundaries. Let $h : X \to M$ and $k : Y \to N$ be maps of manifolds with corners that are transverse to f and g respectively. Then $(h \times k)^*(V \times W) = h^*V \times k^*W$ spaces with co-oriented maps to $X \times Y$.*

Proof. It is easy to show that $h \times k$ is transverse to $f \times g$ so we focus on co-orientation. As in the preceding proofs, we write the Quillen orientation of the normal to the image of $V \times W \hookrightarrow M \times N \times \mathbb{R}^a \times \mathbb{R}^b$ as $\beta_{\nu V \oplus \nu W} = \beta_{\nu V} \wedge \beta_{\nu W}$. Then the pullback $P = (h \times k)^*(V \times W)$ is co-oriented by $(\beta_P, \beta_{X \times Y})$ if and only if we choose β_P and $\beta_{X \times Y}$ so that

$$\beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W} = \beta_{X \times Y} \wedge \beta_{a+b}.$$

On the other hand, we know h^*V is co-oriented by (β_{h^*V}, β_X) if an only if $\beta_{h^*V} \wedge \beta_{\nu V} = \beta_X \wedge \beta_a$ and k^*W is co-oriented by (β_{k^*W}, β_Y) if an only if $\beta_{k^*W} \wedge \beta_{\nu W} = \beta_Y \wedge \beta_b$. Assuming these hold, then $h^*V \times k^*W$ is co-oriented by

$$(-1)^{(x-(v+x-m))(w+y-n)}(\beta_{h^*V} \wedge \beta_{k^*W}, \beta_X \wedge \beta_Y) = (-1)^{(m-v)(w+y-n)}(\beta_{h^*V} \wedge \beta_{k^*W}, \beta_X \wedge \beta_Y).$$

Now, continuing to assume the equalities of the last paragraph and taking a and b even as usual, we have

$$\begin{aligned}\beta_{h^*V} \wedge \beta_{k^*W} \wedge \beta_{\nu V} \wedge \beta_{\nu W} &= (-1)^{(w+y-n)(m-v)}\beta_{h^*V} \wedge \beta_{\nu V} \wedge \beta_{k^*W} \wedge \beta_{\nu W} \\ &= (-1)^{(w+y-n)(m-v)}\beta_X \wedge \beta_a \wedge \beta_Y \wedge \beta_b \\ &= (-1)^{(w+y-n)(m-v)}\beta_X \wedge \beta_Y \wedge \beta_a \wedge \beta_b.\end{aligned}$$

So if we take $\beta_P = (-1)^{(w+y-n)(m-v)}\beta_{h^*V} \wedge \beta_{k^*W}$ and $\beta_{X \times Y} = \beta_X \times \beta_Y$, then this also gives us $\beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W} = \beta_{X \times Y} \wedge \beta_{a+b}$. Therefore, $(h \times k)^*(V \times W)$ is also co-oriented by

$$(\beta_P, \beta_X \wedge \beta_Y) = ((-1)^{(w+y-n)(m-v)}\beta_{h^*V} \wedge \beta_{k^*W}, \beta_X \wedge \beta_Y).$$

We conclude $(h \times k)^*(V \times W) = h^*V \times k^*W$. $\square$

3.6.3. *The relation between co-oriented exterior products and co-oriented fiber products.* Having established these elementary properties for our exterior product, we can now relate co-oriented exterior products to co-oriented fiber products. These relationships correspond to those in singular cohomology between the exterior and cup products of cochains, though one very nice feature is that in our context these relationships all hold "on the nose" at the cochain level and can be proven without any need for Alexander-Whitney maps or any other approximations to the diagonal. This will also be useful for proving some properties of the co-oriented fiber product that we have deferred so far.

Proposition 3.72. *Suppose $f : V \to M$ and $g : W \to M$ are transverse co-oriented maps from manifolds with corners to a manifold without boundary. Let $\mathbf{d} : M \to M \times M$ be the diagonal map $\mathbf{d}(x) = (x, x)$. Then $f \times g$ is transverse to $\mathbf{d}$, and the pullback of $V \times W \to M \times M$ by $\mathbf{d}$ is the co-oriented fiber product $V \times_M W \to M$, i.e.*

$$V \times_M W = \mathbf{d}^*(V \times W).$$

Proof. It is standard that if f and g are transverse then $f \times g$ is transverse to the diagonal map. We briefly recall the argument. Suppose $f(x) = g(y) = z \in M$. Then $Df(T_xV) + Dg(T_yW) = T_zM$. Now suppose $(a, b) \in T_{(z,z)}(M \times M) \cong T_zM \oplus T_zM$. Write $a = v_a + w_a$ with $v_a \in Df(T_xV)$ and $w_a \in Dg(T_yW)$. Similarly, write $v = v_b + w_b$. Then

$$\begin{aligned}(a, b) &= (v_a + w_a, v_b + w_b)\\ &= (v_a - v_b + v_b + w_a, v_b + w_a - w_a + w_b)\\ &= (v_a - v_b, -w_a + w_b) + (v_b + w_a, v_b + w_a),\end{aligned}$$

which is in $D(f \times g)(T_xV \times T_yW) + D\mathbf{d}(T_zM)$. Hence the pullback by the diagonal is a manifold with corners.

This pullback consists precisely of those $(v, w, z) \in V \times W \times M$ such that $(f(v), g(w)) = (z, z)$, which is diffeomorphic to $V \times_M W = \{(v, w) \in V \times W \mid f(v) = g(w)\}$ via the projection $(v, w, z) \to (v, w)$ with inverse $(v, w) \to (v, w, f(v))$. So we consider the co-orientations.

Proceeding as in the proof of Proposition 3.69 (with now $M = N$), if f and g are co-oriented (at appropriate points) by (β_V, β_M) and (β_W, β_M) and if we we take Quillen co-orientations coming from $V \overset{e_v}{\hookrightarrow} M \times \mathbb{R}^a \to M$ and $W \xrightarrow{e_W} M \times \mathbb{R}^b \to M$ (with a and b assumed even) by orienting ν_V and ν_W in $M \times \mathbb{R}^a$ and $M \times \mathbb{R}^b$, then the co-orientation of the product $f \times g$ has Quillen co-orientation with the normal bundle to $V \times W$ in $M \times M \times \mathbb{R}^{a+b}$ oriented $\beta_{\nu V} \wedge \beta_{\nu W}$.

Pulling back by the diagonal, we thus obtain from the definition that the pullback P is co-oriented by (β_P, β_M) if and only if $\beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W} = \beta_M \wedge \beta_{a+b}$.

On the other hand, using the Quillen co-orientation for V, the co-orientation of the fiber product $P = V \times_M W \to W \to M$ is (β_P, β_M) if and only if $\beta_P \wedge \beta_{\nu V} = \beta_W \wedge \beta_a$.

Note that even though these local orientation forms do not all live on the same spaces, they are essentially just statements about the relative orders of coordinates on the spaces where those coordinates occur; this point of view allows us to elide the pullback maps that would be necessary needed (and messy) to formally pull all our comparisons onto the same space. With these thoughts in mind, we can conclude as follows.

If we assume that we do have $\beta_P \wedge \beta_{\nu V} = \beta_W \wedge \beta_a$ then

$$\begin{aligned}\beta_P \wedge \beta_{\nu V} \wedge \beta_{\nu W} &= \beta_W \wedge \beta_a \wedge \beta_{\nu W}\\ &= \beta_W \wedge \beta_{\nu W} \wedge \beta_a\\ &= \beta_M \wedge \beta_b \wedge \beta_a\\ &= \beta_M \wedge \beta_{a+b}.\end{aligned}$$

So the two co-orientations for $P \to M$ are either both (β_P, β_M) or both the opposite co-orientation. In other words, they agree. □

Corollary 3.73. *Suppose $f : V \to M$ and $g : W \to M$ are transverse co-oriented maps of manifolds with corners to a manifold without boundary, that N is a manifold with corners, and that $h : N \to M$ is transverse to f, g, and $f \times_M g$. Then*

$$h^*(V \times_M W) = h^*V \times_N h^*W$$

as manifolds with co-oriented maps to N.

Proof. Using Propositions 3.72, 3.41, and 3.71, and that $\mathbf{d}_M h = (h \times h) \circ \mathbf{d}_N$, we compute

$$\begin{aligned} h^*(V \times_M W) &= h^* \mathbf{d}_M^*(V \times W) \\ &= (\mathbf{d}_M h)^*(V \times W) \\ &= ((h \times h) \circ \mathbf{d}_N)^*(V \times W) \\ &= \mathbf{d}_N^*(h \times h)^*(V \times W) \\ &= \mathbf{d}_N^*(h^*V \times h^*W) \\ &= h^*V \times_N h^*W. \end{aligned}$$

To apply Proposition 3.41 in the second line, we note that h is transverse to $\mathbf{d}_M^*(V \times W) = V \times_M W$ by assumption. For the fourth line we observe that $h \times h$ is transverse to $f \times g$ because h is transverse to f and g, and the composite $(h \times h) \circ \mathbf{d}_N = \mathbf{d}_M h$ is transverse to $f \times g$ by our assumptions and Lemma 2.22. □

Corollary 3.74 (Associativity of co-oriented fiber products)**.** *Suppose $f : V \to M$, $g : W \to M$, and $h : X \to M$ are co-oriented maps from manifolds with corners to a manifold without boundary such that the following pairs are transverse (see Remark 2.23): (V, W), (W, X), $(V \times_M W, X)$, and $(V, W \times_M X)$. Then*

$$(V \times_M W) \times_M X = V \times_M (W \times_M X)$$

as co-oriented fiber products mapping to M.

Proof. We compute using Propositions 3.72, 3.41, 3.67, and 3.71 and that $(\mathrm{id}_M \times \mathbf{d})\mathbf{d} = (\mathbf{d} \times \mathrm{id}_M)\mathbf{d}$:

$$\begin{aligned} (V \times_M W) \times_M X &= \mathbf{d}^*(\mathbf{d}^*(V \times W) \times X) \\ &= \mathbf{d}^*(\mathbf{d} \times \mathrm{id}_M)^*((V \times W) \times X) \\ &= ((\mathbf{d} \times \mathrm{id}_M)\mathbf{d})^*((V \times W) \times X) \\ &= ((\mathrm{id}_M \times \mathbf{d})\mathbf{d})^*(V \times (W \times X)) \\ &= \mathbf{d}^*(\mathrm{id}_M \times \mathbf{d})^*(V \times (W \times X)) \\ &= \mathbf{d}^*(V \times \mathbf{d}^*(W \times X)) \\ &= V \times_M (W \times_M X). \end{aligned}$$

□

Corollary 3.75. *Suppose $f : V \to M$ and $g : W \to M$ are transverse co-oriented maps from manifolds with corners to a manifold without boundary and similarly for $h : X \to N$ and $k : Y \to N$. Then*

$$(V \times X) \times_{M \times N} (W \times Y) = (-1)^{(m-w)(n-x)}(V \times_M W) \times (X \times_N Y)$$

as co-oriented fiber products over $M \times N$. Add a version for pullbacks (as opposed to fiber products)? Might have some extra signs to figure out. Not really needed anywhere I don't think.

Proof. With our given transversality assumptions, $f \times h$ is transverse to $g \times k$, so the expression on the left is well defined. We then compute using Propositions 3.72, 3.41, 3.67, 3.69, and 3.71 and letting τ here be the interchange of the interior N and M in the quadruple product:

$$\begin{aligned}(V\times X)\times_{M\times N}(W\times Y) &= \mathbf{d}^*_{M\times N}(V\times X\times W\times Y)\\ &= \mathbf{d}^*_{M\times N}(\mathrm{id}\times\tau^*\times\mathrm{id})^*(-1)^{(n-x)(m-w)}(V\times W\times X\times Y)\\ &= (\mathbf{d}_M\times\mathbf{d}_N)^*(-1)^{(n-x)(m-w)}(V\times W\times X\times Y)\\ &= (-1)^{(n-x)(m-w)}\mathbf{d}^*_M(V\times W)\times\mathbf{d}^*_N(X\times Y)\\ &= (-1)^{(m-w)(n-x)}(V\times_M W)\times(X\times_N Y).\end{aligned}$$

For the third equality, we use that $\mathbf{d}_M\times\mathbf{d}_N=(\mathrm{id}_M\times\tau\times\mathrm{id}_N)\mathbf{d}_{M\times N}$. □

Corollary 3.76. *Let $V\to M$ and $W\to N$ be maps from manifolds with corners to manifolds without boundary. Let $\pi_M: M\times N\to M$ and $\pi_N: M\times N\to N$ be the projections. Then*

$$V\times W=\pi^*_M(V)\times_{M\times N}\pi^*_N(W)$$

as co-oriented manifolds mapping to $M\times N$.

Proof. By Proposition 3.70, $\pi^*_M(V)=V\times N$ and $\pi^*_N(W)=M\times W$, so these are transverse in $M\times N$. Then by Corollaries 3.75 and 3.53, we have

$$\begin{aligned}\pi^*_M(V)\times_{M\times N}\pi^*_N(W) &= (V\times N)\times_{M\times N}(M\times W)\\ &= (V\times_M M)\times(N\times_N W)\\ &= V\times W.\end{aligned}$$ □

3.7. **Mixed properties.** In this section we study properties that involve both orientations and co-orientations. In particular, we are mostly interested in the pullback of a co-oriented map $V\to M$ by a map $W\to M$ with W oriented, in which case the co-orientation of the pullback $V\times_M W\to W$ together with the orientation of W produces an induced orientation on $V\times_M W$ as described in Section 3.2. As $V\times_M W\to M$ with this orientation will eventually correspond to the cap product when we get to geometric homology and cohomology, we will here refer to this orientation as the **cap orientation**.

The following results all concern cap orientations on $V\times_M W$. We note that, by construction, this oriented manifold comes equipped with a map to W and, by composing with the given map $W\to M$, a map to M.

We start with the next theorem, which is yet another Leibniz formula. It will be used in Section 6.5 to demonstrate that our intersection map $\mathcal{I}$, relating geometric cochains to cubical cochains of a cubulation, is a chain map. This map is a critical component in relating geometric cohomology to other cohomology theories and is also central to the main result about cup products in [**?**].

Proposition 3.77. *Let $f: V\to M$ and $g: W\to M$ be transverse maps of manifolds with corners to a manifold without boundary. Suppose f is co-oriented and W is oriented. Then*[20]

$$\partial(V\times_M W)=\left[(-1)^{\dim(V\times_M W)}(\partial V)\times_M W\right]\bigsqcup V\times_M\partial W$$

as oriented manifolds, giving $V\times_M W$, $(\partial V)\times_M W$, and $V\times_M\partial W$ each their cap orientations.

Proof. We compute and compare these orientations by first considering the pullback co-orientations as defined in Definition 3.35. We proceed by analogy to the proof of the Leibniz rule for the pullback of co-oriented maps in Proposition 3.44, utilizing the computations already performed there.

[20]Note that the signs here agree with those for the boundary of a chain-level cap product in Spanier [**?**, Section 5.6.15] and Munkres [**?**, Section 66].

Recall, in brief, from Definition 3.35 that to co-oriented the pullback $P = V \times_M W \to W$ we first construct a composition $V \overset{e}{\hookrightarrow} M \times \mathbb{R}^N \to M$ and find a Quillen orientation for the normal bundle νV of $e(V) \subset M \times \mathbb{R}^N$ as determined by the co-orientation of $V \to M$. Then we pull back via $W \times \mathbb{R}^N \to M \times \mathbb{R}^N$ to obtain a normal bundle, also labeled νV, of $P \subset W \times \mathbb{R}^N$. Then we co-orient $P \to W$ locally by (β_P, β_W) so that $\beta_P \wedge \beta_{\nu V} = \beta_W \wedge \beta_E$, where β_E represents the standard orientation of $\mathbb{R}^N$. In the case at hand, we can assume β_W to represent the global orientation of W, and then β_P becomes a global orientation for P. This is the orientation given to $V \times_M W$ in Definition 4.21.

Let $\nu\partial P$ denote an outward pointing normal vector in the tangent bundle to P at a boundary point of P, and let $\beta_{\nu\partial P}$ denote the corresponding orientation. Then, by definition, ∂P is oriented at that point by $\beta_{\partial P}$ so that $\beta_{\nu\partial P} \wedge \beta_{\partial P} = \beta_P$. In other words, with β_P, β_W, $\beta_{\nu V}$, and β_E given, ∂P is oriented by $\beta_{\partial P}$ such that $\beta_{\nu\partial P} \wedge \beta_{\partial P} \wedge \beta_{\nu V} = \beta_W \wedge \beta_E$.

Now, recall that $\partial(V \times_M W) = (\partial V) \times_M W \bigsqcup V \times_M \partial W$ as spaces and consider a point in $(\partial V) \times_M W$. By Proposition 3.44, at such a point the pullback co-orientation of $(\partial V) \times_M W \to W$ agrees with boundary co-orientation of the pullback $P = V \times_M W \to W$, as described again in the preceding paragraph. So continuing to let (β_P, β_W) denote the pullback co-orientation of $P \to W$ and recalling that the boundary co-orientation utilizes the *inward* normal, the boundary co-orientation of $(\partial V) \times_M W \to W$ is the composite $(\beta_{\partial P}, \beta_{\partial P} \wedge -\beta_{\nu\partial P}) * (\beta_P, \beta_W)$ for any $\beta_{\partial P}$. But if we choose $\beta_{\partial P}$ to represent the orientation of ∂P found above by orienting P and then taking its boundary orientation, we have $\beta_P = \beta_{\nu\partial P} \wedge \beta_{\partial P} = (-1)^{\dim(\partial P)} \beta_{\partial P} \wedge \beta_{\nu\partial P}$. So the boundary co-orientation of $(\partial V) \times_M W \to W$ is the composite

$$(\beta_{\partial P}, \beta_{\partial P} \wedge -\beta_{\nu\partial P}) * ((-1)^{\dim(\partial P)} \beta_{\partial P} \wedge \beta_{\nu\partial P}, \beta_W) = (-1)^{\dim(\partial P)+1} (\beta_{\partial P}, \beta_W).$$

Thus the resulting orientation of $(\partial V) \times_M W$ is $(-1)^{\dim(P)}$ times the orientation of ∂P obtained by taking the oriented boundary of $V \times_M W$.

Next we consider a point in $V \times_M \partial W$. From the Leibniz rule computation for co-orientations, the co-orientation of the pullback $V \times_M \partial W \to \partial W$ is $(\beta_{\partial P}, \beta_{\partial W})$ if and only if $\beta_{\partial P} \wedge \beta_{\nu V} = \beta_{\partial W} \wedge \beta_E$. If $\nu\partial P$ is an outward normal then this is equivalent to $\beta_{\nu\partial P} \wedge \beta_{\partial P} \wedge \beta_{\nu V} = \beta_{\nu\partial P} \wedge \beta_{\partial W} \wedge \beta_E = \beta_W \wedge \beta_E$. But taking β_P and $\beta_{\partial P}$ as found above using our given orientation of β_W, we have $\beta_{\nu\partial P} \wedge \beta_{\partial P} = \beta_P$, and so the condition is equivalent to $\beta_P \wedge \beta_{\nu V} = \beta_W \wedge \beta_E$, which holds by our choice above of β_P. So the orientation of $V \times_M \partial W$ agrees with the orientation of ∂P and we obtain overall

$$\partial(V \times_M W) = \left[(-1)^{\dim(P)} (\partial V) \times_M W\right] \bigsqcup V \times_M \partial W.$$

Finally, we use that $\dim(P) = \dim(V) + \dim(W) - \dim(M)$. □

Next we describe the cap orientations when $V \to M$ and $W \to M$ are embeddings. As we've observed in the cases where either both maps are oriented or both maps are co-oriented, this is often an instructive and important example.

Proposition 3.78. *Let $f : V \to M$ and $g : W \to M$ be transverse embeddings from manifolds with corners to a manifold without boundary. Suppose f is co-oriented and W is oriented. Then $P = V \times_M W$ is just the intersection of V and W in M. If β_W is the orientation of W and $\beta_{\nu V}$ is the Quillen orientation of the normal bundle to V in M, which at points of P we can identify*[21] *with the normal bundle to P in W, then the cap orientation β_P of P satisfies $\beta_P \wedge \beta_{\nu V} = \beta_W$. If f and g are immersions, then this description holds locally.*

Proof. As f is an embedding, we can take $N = 0$ in the definition of the pullback co-orientation, Definition 3.35. Then the pullback is just the inclusion of $P = g^{-1}(V) = V \cap W$ into W, and by definition the pullback co-orientation has the form (β_P, β_W), where $\beta_P \wedge \beta_{\nu V} = \beta_W$ and νV here is the pullback of the normal bundle of V in M to be the normal bundle of $V \cap W$ in W. Furthermore, if we take β_W to be

[21] See Lemma 2.20.

the given orientation of W, then β_P is the induced orientation on the intersection by definition. The last statement about immersions follows as we can compute the co-orientations locally. □

The following corollary is particularly important and follows immediately from Proposition 3.78.

Corollary 3.79. *Let $f : V \to M$ and $g : W \to M$ be transverse embeddings from manifolds with corners to a manifold without boundary. Suppose f is co-oriented, W is oriented, and $\dim(V) + \dim(W) = \dim(M)$. Then $V \times_M W$ is the union of intersection points of V and W. Such a point $x \in V \cap W$ is positively oriented if and only if the Quillen orientation of the normal bundle νV of V at x agrees with the orientation of W at x, identifying the fiber of νV at x with $T_x W$. If f and g are immersions, then this description holds locally.*

The next two propositions will eventually correspond to the unital identities for the cap product for geometric chains and cochains. The analogues for singular chains and cochains are the cap product with the cochain 1 and the cap product with a chain representing the fundamental class, though in this case our underlying spaces do not need to be compact.

Proposition 3.80. *Let $g : W \to M$ be a map from an oriented manifold with corners to a manifold without boundary, and consider $M \to M$ as the identity with the tautological co-orientation. Then $M \times_M W = W$ as oriented manifolds.*

Proof. By definition, there is a Quillen co-orientation for M consisting of the sequence of identity maps $M \hookrightarrow M \to M$ with the normal bundle to M in itself being the 0-dimensional vector bundle, which we consider to have positive orientation at each point. It follows from the definition of the pullback that the corresponding Quillen co-orientation for $M \times_M W$ comes from the sequence $g^{-1}(M) = W \hookrightarrow W \to W$ with W also having a 0-dimensional positively-oriented normal bundle in itself. Consequently, the pullback co-orientation for $W \to W$ is the tautological one, and so the induced orientation on W is the given one. □

Proposition 3.81. *Let $f : V \to M$ be a co-oriented map from a manifold with corners to an oriented manifold without boundary, and consider M equipped with its identity map $M \to M$. Then $V \times_M M$ is V with its induced orientation.*

Proof. This follows directly from Corollary 3.53 and the definitions. □

The next property relates products of pullbacks with pullbacks of products.

Proposition 3.82. *Let $f : V \to M$ and $g : X \to N$ be co-oriented maps from manifolds with corners to manifolds without boundary, and let $h : W \to M$ and $k : Y \to N$ be maps with W and Y compacted oriented manifolds with corners. Suppose that V is transverse to W and that X is transverse to Y. Then*[22]*,*

$$(V \times X) \times_{M\times N} (W \times Y) = (-1)^{(x+y-n)(m-v)} (V \times_M W) \times (X \times_N Y),$$

as oriented manifolds mapping to $M \times N$.

Proof. Let β_W and β_Y denote the orientations of W and Y. Then $W \times Y$ is oriented by $\beta_W \wedge \beta_Y$.

Now let $P = V \times_M W$ and $P' = X \times_N Y$. By definition, P and P' are oriented by the orientations β_P and $\beta_{P'}$ such that (β_P, β_W) and $(\beta_{P'}, \beta_Y)$ are the pullback co-orientations for $P \to W$ and $P' \to Y$. Then $P \times P'$ is oriented by $\beta_P \wedge \beta_{P'}$.

Furthermore, using our construction of pullback co-orientations, β_P and $\beta_{P'}$ are such that $\beta_P \wedge \beta_{\nu V} = \beta_W \wedge \beta_a$ and $\beta_{P'} \wedge \beta_{\nu X} = \beta_Y \wedge \beta_b$, where β_a and β_b are the standard orientations Euclidean spaces $\mathbb{R}^a$ and $\mathbb{R}^b$ and we are free to take a and b to be even integers.

By Lemma 3.68, we have that the Quillen co-orientation of $V \times X \to M \times N$ is represented by an embedding $V \times X \hookrightarrow M \times N \times \mathbb{R}^a \times \mathbb{R}^b$ with normal bundle $\nu V \oplus \nu X$. So letting $Q = (V \times X) \times_{M\times N} (W \times Y)$,

[22]Again our signs agree with the cap product formulas in Spanier, in this case [**?**, Section 5.6.21].

the orientation β_Q is the one such that $(\beta_Q, \beta_W \wedge \beta_Y)$ is the pullback co-orientation, i.e. the one such that $\beta_Q \wedge \beta_{\nu V} \wedge \beta_{\nu X} = \beta_W \wedge \beta_Y \wedge \beta_a \wedge \beta_b$. But then we compute, using a and b even,

$$\begin{aligned}\beta_W \wedge \beta_Y \wedge \beta_a \wedge \beta_b &= \beta_W \wedge \beta_a \wedge \beta_Y \wedge \beta_b \\ &= \beta_P \wedge \beta_{\nu V} \wedge \beta_{P'} \wedge \beta_{\nu X} \\ &= (-1)^{|P'||\nu V|} \beta_P \wedge \beta_{P'} \wedge \beta_{\nu V} \wedge \beta_{\nu X} \\ &= (-1)^{(x+y-n)(m-v)} \beta_P \wedge \beta_{P'} \wedge \beta_{\nu V \oplus \nu X}.\end{aligned}$$

So $\beta_Q = (-1)^{(x+y-n)(m-v)} \beta_P \wedge \beta_{P'} = (-1)^{(x+y-n)(m-v)} \beta_{P \times P'}$. □

The following technical lemma will be used to prove Proposition 3.84, which will eventually become the associativity relation among cup and cap products, i.e. $(a \cup b) \cap x = a \cap (b \cap x)$.

Lemma 3.83. *Let $f : V \to M$ and $g : W \to M$ be transverse maps from manifolds with corners to a manifold without boundary. Suppose that f is co-oriented and that W and M are oriented, with respective (global) orientations β_W and β_M. Suppose we co-orient g by (β_W, β_M). Then the cap orientation of $V \times_M W$ (i.e. that induced from the the pullback co-orientation of $V \times_M W \to W$ and the orientation of W) is the same as the orientation induced on $V \times_M W$ by the fiber product co-orientation of $V \times_M W \to M$ and the orientation of M .*

Proof. By definition, the orientation of $V \times_M W$ induced from the orientation of W and the pullback co-orientation of $V \times_M W \to W$ is the orientation β_P such that (β_P, β_W) is the pullback co-orientation. But then the fiber product co-orientation is the composite $(\beta_P, \beta_W) * (\beta_W, \beta_M) = (\beta_P, \beta_M)$. So the orientation induced by the orientation of M and the composite co-orientation is again β_P. □

Proposition 3.84 (Mixed associativity)**.** *Let $f : V \to M$ and $g : W \to M$ be co-oriented maps from manifolds with corners to a manifold without boundary. Let $h : Z \to M$ be a map with Z oriented. Then, assuming sufficient transversality for all terms to be defined (see Remark 2.23),*

$$(V \times_M W) \times_M Z = V \times_M (W \times_M Z),$$

as oriented manifolds mapping to M.

Note that both expressions are well defined: On the left, $V \times_M W$ has a fiber product co-orientation, so we can form the cap orientation of the co-oriented $V \times_M W \to M$ over the oriented Z. On the right we first give $W \times_M Z$ its cap orientation and then use that to form the cap orientation of $V \times_M (W \times_M Z)$.

Proof. First suppose M is orientable and that we have given it an arbitrary, but fixed, orientation. Then applying Lemma 3.83, the cap orientation of $(V \times_M W) \times_M Z$ is the same as the orientation induced by the orientation of M and the fiber product co-orientation $(V \times_M W) \times_M Z \to M$, after co-orienting $Z \to M$ with the co-orientation induced by the orientations of Z and M. Similarly, applying Lemma 3.83 twice, the cap orientation of $V \times_M (W \times_M Z)$ is the same as that induced by the orientation of M and the iterated fiber product co-orientation $V \times_M (W \times_M Z) \to M$ again coming from the canonical co-orientation of $Z \to M$. But now these co-oriented fiber products are the same by Corollary 3.74.

Next, suppose M is not necessarily orientable. We know from their constructions that $(V \times_M W) \times_M Z$ and $V \times_M (W \times_M Z)$ are oriented manifolds, and it is not difficult to see that they are diffeomorphic, both being diffeomorphic to $\{(v, w, z) \in V \times W \times Z \mid f(v) = g(w) = h(z)\}$. So it suffices to consider these as identical spaces and to show that their induced orientations agree at any arbitrary point. If (v, w, z) is such a point, consider its image $a = f(v) = g(w) = h(z) \in M$. Let U be a Euclidean neighborhood of a, and consider the restrictions of f, g, and h to $f^{-1}(U)$, $g^{-1}(U)$, and $h^{-1}(U)$. The resulting products over U give us the pieces of $(V \times_M W) \times_M Z$ and $V \times_M (W \times_M Z)$ over U, and the resulting orientations will be compatible with those of the full manifolds $(V \times_M W) \times_M Z$ and $V \times_M (W \times_M Z)$, as orientations and co-orientations of fiber products are determined locally (see Remark 3.39 and the construction of fiber product orientations). But as U is orientable, the preceding argument shows that these orientations must agree with each other. □

The following property will eventually manifest itself in geometric (co)homology as the familiar naturality formula for cap products $f_*(f^*(\alpha) \frown x)) = \alpha \frown f_*(x)$.

Proposition 3.85. *Let $f : V \to M$ and $h : N \to M$ be transverse maps with h co-oriented, V a manifold with corners and M and N manifolds without boundary. Furthermore, let $g : W \to N$ be a map from an oriented manifold with corners that is transverse to the co-oriented pullback $V \times_M N \to N$. Then the cap orientation induced on $(V \times_M N) \times_N W$ by pulling back the co-oriented map $V \times_M N \to N$ over $W \to N$ is the same as the cap orientation obtained by pulling back $V \to M$ by the composite $hg : W \to M$. In other words, $(h^*V) \times_N W = V \times_M W$ as oriented manifolds.*

Proof. Note that V is transverse to $hg : W \to M$ by Lemma 2.22, so both expressions are defined. It follows directly from Proposition 3.41 that the two pullback co-orientations we have described for $(V \times_M N) \times_N W \to W$ agree. Therefore, the induced orientations on $(V \times_M N) \times_N W$ agree. □

3.7.1. *Comparing the oriented and co-oriented fiber products.* Suppose $f : V \to M$ and $g : W \to M$ are two transverse co-oriented maps from manifolds with corners to a manifold without boundary. Further, suppose M oriented. Then we know there is a bijection between co-orientations of f and orientations of V: an orientation of V induces a co-orientation of f and vice versa. Of course the same is true of W and g. In this scenario, we have two different ways to orient $V \times_M W$, depending on whether we start by thinking of V and W as oriented or by thinking of f and g as co-oriented. If we think of V and Was oriented, we have the fiber product orientation of $V \times_M W$ discussed in Section 3.1. Alternatively, if we think of f and g as co-oriented, we can form the fiber product co-orientation of $V \times_M W \to M$ as in Section 3.5.2 and then consider the induced orientation given the orientation of M.

Our goal in this section is to compare these two orientations. To attempt to avoid confusion, we will write $V \times^o_M W$ for the fiber product orientation of Section 3.1 and $V \times^c_M W$ for the co-oriented fiber product or, equivalently, the resulting induced orientation.

The reader might have noticed that a third way to orient $V \times_M W$ is to consider f to be co-oriented and W to be oriented and then form the cap orientation that we studied in detail in the preceding section. However, we already know this to be identical to $V \times^c_M W$ by Lemma 3.83. By contrast, these are not always the same as $V \times^o_M W$.

When we move on to geometric homology, $V \times^o_M W$ will correspond to the classical intersection product of homology classes, as described for example in [**?**, Section VI.11], while $V \times^c_M W$ will correspond to the cup product of cohomology classes. When M is closed and oriented, switching between thinking of V and W as oriented vs. co-oriented will be the Poincaré duality isomorphism. So this proposition will ultimately demonstrate that the intersection product is Poincaré dual to the cup product, up to a sign; see Section 7.7.3.

Proposition 3.86. *Let $f : V \to M$ and $g : W \to M$ be transverse co-oriented maps from manifolds with corners to an oriented manifold without boundary or, equivalently, suppose V, W, and M all oriented. Then*

$$V \times^o_M W = (-1)^{(m-v)(m-w)} V \times^c_M W$$

as oriented manifolds with corners.

The proof will take a bit of work. Our strategy will be as follows: First, we will prove in Lemma 3.87 that the result holds when f and g are immersions. Then we will show, first for co-orientations and then for orientations, that we can replace the fiber product of

$$V \xrightarrow{f} M \xleftarrow{g} W$$

with the fiber product

$$V \times \mathbb{R}^{2b} \xhookrightarrow{e\times \mathrm{id}_{\mathbb{R}^{2b}}} M \times \mathbb{R}^{2a} \times \mathbb{R}^{2b} \xhookleftarrow{W} \times \mathbb{R}^{2a},$$

where e is the embedding $V \hookrightarrow M \times \mathbb{R}^{2a}$ of a Quillen co-orientation of f and the leftward arrow is the identity between the $\mathbb{R}^{2a}$ factors and takes the W factor into $M \times \mathbb{R}^{2b}$ by the embedding map of a Quillen co-orientation of g. By "replace," we mean in the orientated case that the two fiber products are canonically oriented diffeomorphic. In the co-oriented case we mean that we have a canonical oriented diffeomorphism between the domains of the two co-oriented fiber products, oriented with their induced orientations coming respectively from the orientation of M and from the concatenation orientation of $M \times \mathbb{R}^{2a} \times \mathbb{R}^{2b}$ using the standard orientations of the Euclidean terms. The maps of this second fiber product are embeddings for which the proposition holds by Lemma 3.87, and so the general case will follow. The even dimensions of the Euclidean factors are chosen to avoid some extraneous signs in the arguments below.

Before proceeding, let us explain in more detail what we mean by "canonical" here and below. Recall that, as a space, $P = V \times_M W$ can be identified with $\{(v, w) \in V \times W \mid f(v) = g(w)\}$. Below we will see various fancier embeddings of P in spaces of the form $V \times X \times W \times Y$, with X and Y Euclidean or I. For each such embedding, the projection to $V \times W$ will take the embedding of P back to P. In this way, all versions of P can be canonically identified, and it is these identifications that will yield our orientation preserving diffeomorphisms. Such identifications have already been discussed in Remarks 3.30 and 3.33. In the latter remark, we provide exactly such a canonical identification between our standard realization of $V \times_M W$ as a subset of $V \times W$ and the version used for co-orienting pullbacks and fiber products.

We begin with the case where V and W are immersions and then work toward the general case.

Lemma 3.87. *If f and g are transverse immersions of oriented manifolds with corners into an oriented manifold without boundary, then $V \times^o_M W = (-1)^{(m-v)(m-w)} V \times^c_M W$.*

Proof. It suffices to consider small neighborhoods on which f and g are embeddings. Let β_V, β_W, and β_M denote the orientations of V, W, and M, respectively. Assuming f and g to be co-oriented with the compatible co-orientations, we have the resulting Quillen orientations $\beta_{\nu V}$ and $\beta_{\nu W}$ of the normal bundles of V and W. Recall that these are defined so that $(\beta_V, \beta_V \wedge \beta_{\nu V})$ and $(\beta_W, \beta_W \wedge \beta_{\nu W})$ are the co-orientations of f and g, respectively. In this scenario, with β_V and β_W fixed as the orientations of V and W, this is equivalent to requiring $\beta_V \wedge \beta_{\nu V} = \beta_M$ and $\beta_W \wedge \beta_{\nu W} = \beta_M$. Again to keep the contexts clear, for the remainder of the argument we will write $\beta^c_{\nu V}$ and $\beta^c_{\nu W}$ for the orientations of νV and νW.

By Proposition 3.43, the co-orientation of the fiber product $V \times^c W$ is $(\beta_P, \beta_P \wedge \beta^c_{\nu V} \wedge \beta^c_{\nu W})$. In particular, the induced orientation of $V \times^c W$ is the orientation β^c_P such that $\beta^c_P \wedge \beta^c_{\nu V} \wedge \beta^c_{\nu W} = \beta_M$.

On the other hand, in Proposition 3.7, β^o_P is such that if $\beta^o_P \wedge \beta^o_{\nu W} = \beta_V$ and $\beta^o_P \wedge \beta^o_{\nu V} = \beta_W$ then $\beta^o_P \wedge \beta^o_{\nu V} \wedge \beta^o_{\nu W} = \beta_M$. A priori these may be different orientations of νV and νW than those of the preceding paragraph, so we use these alternate labels. In fact, let us suppose β^o_P, $\beta^o_{\nu V}$, and $\beta^o_{\nu W}$ chosen so that these expressions all hold. Then we have

$$\beta_M = \beta^o_P \wedge \beta^o_{\nu V} \wedge \beta^o_{\nu W} = \beta_W \wedge \beta^o_{\nu W},$$

so $\beta^o_W = \beta^c_W$. Similarly,

$$\beta_M = \beta^o_P \wedge \beta^o_{\nu V} \wedge \beta^o_{\nu W} = (-1)^{(m-v)(m-w)} \beta^o_P \wedge \beta^o_{\nu W} \wedge \beta^o_{\nu V} = (-1)^{(m-v)(m-w)} \beta^o_V \wedge \beta^o_{\nu V},$$

so $\beta^0_{\nu V} = (-1)^{(m-v)(m-w)} \beta^c_{\nu V}$.

Thus

$$\beta_M = \beta^o_P \wedge \beta^o_{\nu V} \wedge \beta^o_{\nu W} = (-1)^{(m-v)(m-w)} \beta^o_P \wedge \beta^c_{\nu V} \wedge \beta^c_{\nu W}.$$

We conclude that $\beta^c_P = (-1)^{(m-v)(m-w)} \beta^o_P$. ∎

.

Now we show how to replace a general co-oriented fiber product with a co-oriented fiber product whose maps are embeddings. In our remaining constructions in this section we take all introduced Euclidean spaces to be even-dimensional to simplify the signs.

Lemma 3.88. *Let $f : V \to M$ and $g : W \to M$ be transverse co-oriented maps from manifolds with corners to an oriented manifold without boundary. Let $V \overset{e}{\hookrightarrow} M \times \mathbb{R}^{2a} \to M$ be a Quillen co-orientation for f.*

Then $V \times_M W$ (with its orientation induced by the fiber product co-orientation and the orientation of M) is canonically orientated diffeomorphic to $V \times_{M\times\mathbb{R}^{2a}} (W \times \mathbb{R}^{2a})$ (with its orientation induced by the fiber product co-orientation and the orientation of $M \times \mathbb{R}^{2a}$). Here the maps for the second fiber product are $e : V \to M \times \mathbb{R}^{2a}$ and $g \times \mathrm{id}_{\mathbb{R}^{2a}} : W \times \mathbb{R}^{2a} \to M \times \mathbb{R}^{2a}$. As in the construction of the Quillen co-orientation (Definition 3.17), we assume $e : V \hookrightarrow M \times \mathbb{R}^{2a}$ to be co-oriented so that its composition with the canonical co-orientation $(\beta_M \wedge \beta_a, \beta_M)$ is the co-orientation of f. We also take $g \times \mathrm{id}_{\mathbb{R}^{2a}}$ to be co-oriented by the product co-orientation $(\beta_W \wedge \beta_a, \beta_M \wedge \beta_a)$ if (β_W, β_M) is the co-orientation of g. Finally, $M \times \mathbb{R}^{2a}$ is given the product orientation with $\mathbb{R}^{2a}$ having the standard orientation.

Proof. By the definitions of the induced orientation and the fiber product co-orientation, the orientation of the fiber product $V \times_M W \to M$ is β_P where $\beta_P \wedge \beta_{\nu V} = \beta_M \wedge \beta_a$. We recall that the νV in this formula is actually the pullback of the normal bundle of $e(V) \subset M \times \mathbb{R}^{2a}$ via the map $g \times \mathrm{id}_{\mathbb{R}^{2a}} : W \times \mathbb{R}^{2a} \to M \times \mathbb{R}^{2a}$. But this is also precisely the description of the induced fiber product orientation from the co-oriented fiber product $V \times_{M\times\mathbb{R}^{2a}} (W \times \mathbb{R}^{2a})$, treating $e : V \hookrightarrow M \times \mathbb{R}^{2a}$ as its own Quillen co-orientation with $N = 0$ in Definition 3.35. ☐

Corollary 3.89. *Let $f : V \to M$ and $g : W \to M$ be transverse co-oriented maps from manifolds with corners to an oriented manifold without boundary. Let $V \overset{r}{\hookrightarrow} M \times \mathbb{R}^{2a} \to M$ and $W \overset{s}{\hookrightarrow} M \times \mathbb{R}^{2b} \to M$ be Quillen co-orientations compatible with f and g. Then the fiber product $V \times_M W$ (with its orientation induced from the co-oriented fiber product) is canonically oriented diffeomorphic to the fiber product $(V \times \mathbb{R}^{2b}) \times_{M\times\mathbb{R}^{2a}\times\mathbb{R}^{2b}} (W \times \mathbb{R}^{2a})$ (with its orientation induced from the co-oriented fiber product), in which the first map is $r \times \mathrm{id}_{\mathbb{R}^{2b}}$ and the second map takes $(w, z) \in W \times \mathbb{R}^{2a}$ to $s(w)$ in the first and third coordinates and z in the second coordinate.*

Proof. By the preceding lemma we have $V \times_M W$ canonically oriented diffeomorphic to $V \times_{M\times\mathbb{R}^{2a}} (W \times \mathbb{R}^{2a})$ with $V \to M \times \mathbb{R}^{2a}$ an embedding. By Proposition 3.45, the transposition map from this space to $(W \times \mathbb{R}^{2a}) \times_{M\times\mathbb{R}^{2a}} V$ is $(-1)^{(m-v)(m-w)}$-orientation preserving, using that all our Euclidean spaces are taken even-dimensional. We next observe the map described for $W \times \mathbb{R}^{2a} \to M \times \mathbb{R}^{2a} \times \mathbb{R}^{2b}$ is an embedding whose composition with the projection to $M \times \mathbb{R}^{2a}$ gives a Quillen co-orientation for $g \times \mathrm{id}_{\mathbb{R}^{2a}}$. Now we can apply the lemma again and then transpose again. ☐

Next we show how to replace a general oriented fiber product with an oriented fiber product whose maps are embeddings. Again, we take all introduced Euclidean spaces to be even-dimensional to simplify the signs.

Lemma 3.90. *Let $f : V \to M$ and $g : W \to M$ be transverse maps from oriented manifolds with corners to an oriented manifold without boundary. Let $V \overset{e}{\hookrightarrow} M \times \mathbb{R}^{2a} \to M$ be a factorization of f with e an embedding. Then $V \times_M W$ is canonically oriented diffeomorphic to the oriented fiber product $V \times_{M\times\mathbb{R}^{2a}} (W \times \mathbb{R}^{2a})$ of $e : V \to M \times \mathbb{R}^{2a}$ and $g \times \mathrm{id}_{\mathbb{R}^{2a}} : W \times \mathbb{R}^{2a} \to M \times \mathbb{R}^{2a}$.*

Proof. We will first show that $V \times_M W$ is canonically oriented diffeomorphic to the fiber product $V \times_{M\times\mathbb{R}^{2a}} (W \times \mathbb{R}^{2a})$ with the map $V \to M \times \mathbb{R}^{2a}$ being the composition of f with the inclusion $M = M \times \{0\} \hookrightarrow M \times \mathbb{R}^{2a}$. We will write this composite as f_0. It is clear that f_0 and $g \times \mathrm{id}_{\mathbb{R}^{2a}}$ are transverse as f and g are transverse in M and the $\mathrm{id}_{\mathbb{R}^{2a}}$ factor takes care of the $\mathbb{R}^{2a}$ factor of the tangent spaces. We also observe that the two fiber products are canonically the same as spaces, as $V \times_M W = \{(v, w) \in V \times W \mid f(v) = g(w)\}$, while the other fiber product is $\{(v, w, 0) \in V \times W \times \mathbb{R}^a \mid f(v) = g(w)\}$.

Now we consider the orientations. As the tangent space of the pullback is the pullback of the tangent space, it suffices to assume that all spaces and maps are linear; in the general case we replace all the following spaces with their bundles and all maps with their derivatives. For $P = V \times_M W$, we look at the map

$$\Phi : P \oplus M \to V \oplus W \tag{9}$$

given by $((v, w), x) \to (v, w) + s(x)$, where s is a splitting of the map $V \oplus W \to M$ that takes (v, w) to $f(v)-g(w)$. We recall that the orientation of P is chosen so that Φ is an orientation preserving isomorphism up to a sign of $(-1)^{\dim(W)\dim(M)}$.

In the case of $V \times_{M\times\mathbb{R}^{2a}} (W \times \mathbb{R}^{2a})$, if we abuse notation and again write P for the fiber product, then we have instead a map

$$\Psi : P \oplus M \oplus \mathbb{R}^{2a} \to V \oplus W \oplus \mathbb{R}^{2a}, \tag{10}$$

with Ψ again the restriction to P of the projections to V and $W \oplus \mathbb{R}^{2a}$, while the restriction to M must be a splitting of the map $\Upsilon : V \oplus W \oplus \mathbb{R}^{2a} \to M \oplus \mathbb{R}^{2a}$ that is f_0 on the first factor and $-(g \oplus \mathrm{id}_{\mathbb{R}^{2a}})$ on the last two factors. We claim that we can take $\Psi((v, w, 0), x, z) = (v, w, 0) + (s(x), 0) - (0, 0, z)$. This is certainly correct on the P factor. For the $M \oplus \mathbb{R}^{2a}$ factor, we must show $\Upsilon\Psi(0, x, z) = (x, z)$. We have $\Psi(0, x, z) = (s(x), 0) - (0, 0, z)$, noting that $s(x) \in V \oplus W$. If we write $s(x) = (s_V(x), s_W(x))$, then by definition $f(s_V(x)) - g(s_W(x)) = x$. So we have

$$\begin{aligned}
\Upsilon((s(x), 0) - (0, 0, z)) &= \Upsilon(s_V(x), s_W(x), z) \\
&= f_0(s_V(x)) - (g(s_W(x)), z) \\
&= (f(s_V(x)), 0) - (g(s_W(x)), z) \\
&= (f(s_V(x)) - g(s_W(x)), z) \\
&= (x, z).
\end{aligned}$$

So our definition of Ψ is correct.

As Φ and Ψ agree in the first two factors and the dimension of $\mathbb{R}^{2a}$ is even, we see that Ψ is orientation-preserving if and only if Φ is. Furthermore, we have $(-1)^{\dim(W)\dim(M\times\mathbb{R}^{2a})} = (-1)^{\dim(W)\dim(M)}$, so the two fiber product orientations agree in this case.

Next we must generalize from $f_0 : V \to M\times\mathbb{R}^{2a}$ to the general case of $e : V \to M\times\mathbb{R}^{2a}$. By assumption, $f : V \to M$ is the composition of e with the projection $M\times\mathbb{R}^{2a} \to M$, so we may write $e(v) = (f(v), e_{\mathbb{R}}(v))$, and there is a fiberwise homotopy $H : V\times I \to M\times\mathbb{R}^{2a}$ from e to f_0 given by $H(v, t) = (f(v), te_{\mathbb{R}}(v))$. We note that H and e are each transverse to $g\times\mathrm{id}_{\mathbb{R}^{2a}}$. Indeed, if $e(v) = (g\times\mathrm{id}_{\mathbb{R}^{2a}})(w, z)$, then $(f(v), e_{\mathbb{R}}(v)) = (g(w), z)$. The image of the derivative of $g \times \mathrm{id}_{\mathbb{R}^{2a}}$ at such a point will span $Dg(T_wW) \oplus T(\mathbb{R}^{2a}) = Dg(T_wW) \oplus \mathbb{R}^{2a}$, while the derivative of e will have the form $(Df, De_{\mathbb{R}})$. But the image of $D(g \times \mathrm{id}_{\mathbb{R}^{2a}})$ already spans $0 \oplus \mathbb{R}^{2a}$, so $De + D(g \times \mathrm{id}_{\mathbb{R}^{2a}}) = (Df, 0) + (Dg, 0) + (0, \mathrm{id}_{\mathbb{R}^{2a}}) = T(M \times \mathbb{R}^{2a})$. The same argument holds for $H(-, t)$ for any fixed t, replacing $De_{\mathbb{R}}$ with $tDe_{\mathbb{R}}$. But if each $H(-, t)$ is transverse to $g \times \mathrm{id}_{\mathbb{R}^{2a}}$ then so is H.

It follows that we can form the oriented fiber product of H and $g \times \mathrm{id}_{\mathbb{R}^{2a}}$ over $M \times \mathbb{R}^{2a}$. In fact, this fiber product is diffeomorphic to $P \times I$: Noting that we have $H(v, t) = (f(v), te_{\mathbb{R}}(v)) = (g(w), z)$ if and only if $f(v) = g(w)$ and $te_{\mathbb{R}}(v) = z$, we obtain a diffeomorphism $P \times I \to (V \times I) \times_{M\times\mathbb{R}^{2a}} (W \times \mathbb{R}^{2a})$ given by $((v, w), t) \to ((v, t), (w, te_{\mathbb{R}}(v))$ with inverse $((v, t), (w, z)) \to ((v, w), t)$. The two ends of the cylinder correspond to our two versions of $V \times_{M\times\mathbb{R}^{2a}} (W \times \mathbb{R}^{2a})$, one mapping V by f_0 and the other by e. Hence these spaces are oriented diffeomorphic, and canonically so by our construction.

Putting this canonical diffeomorphism together with the one constructed above gives the desired oriented canonical diffeomorphism with the original $V \times_M W$. $\square$

Corollary 3.91. *Let $f : V \to M$ and $g : W \to M$ be transverse maps from oriented manifolds with corners to an oriented manifold without boundary. Let $V \overset{r}{\hookrightarrow} M \times \mathbb{R}^{2a} \to M$ and $W \overset{s}{\hookrightarrow} M \times \mathbb{R}^{2b} \to M$ be factorizations of f and g with r and s embeddings. Then the oriented fiber product $V \times_M W$ is canonically oriented diffeomorphic to the fiber product $(V \times \mathbb{R}^{2b}) \times_{M \times \mathbb{R}^{2a} \times \mathbb{R}^{2b}} (W \times \mathbb{R}^{2a})$ in which the first map is $r \times \mathrm{id}_{\mathbb{R}^{2b}}$ and the second map takes $(w, z) \in W \times \mathbb{R}^{2a}$ to $s(w)$ in the first and third coordinates and z in the second coordinate.*

Proof. As in the proof of Corollary 3.89, we apply the preceding lemma to get $V \times_M W$ canonically oriented diffeomorphic to $V \times_{M \times \mathbb{R}^{2a}} (W \times \mathbb{R}^{2a})$. Then we use the graded commutativity rule for oriented fiber product as given by Proposition 3.5, by which the transposition map to $(W \times \mathbb{R}^{2a}) \times_{M \times \mathbb{R}^{2a}} V$ is $(-1)^{(m-v)(m-w)}$-orientation preserving, using that all our Euclidean spaces are taken even-dimensional. Then we observe the map described for $W \times \mathbb{R}^{2a} \to M \times \mathbb{R}^{2a} \times \mathbb{R}^{2b}$ is an embedding whose composition with the projection to $M \times \mathbb{R}^{2a}$ is $g \times \mathrm{id}_{\mathbb{R}^{2b}}$. Now we apply the lemma again and then transpose again. □

Proof of Proposition 3.86. By Corollaries 3.89 and 3.91, the proposition reduces to Lemma 3.87. □

4. Geometric chains and cochains

Geometric homology and cohomology are homology/cohomology theories for smooth manifolds defined through submanifolds or, more generally, maps from manifolds with corners. They agree with singular homology and cohomology, but having different representatives at the chain/cochain level, they provide geometric approaches to both theory and calculations. They are thus akin to de Rham theory in the sense that chains and cochains are defined through the smooth structure rather than continuous maps or open sets, but they are defined over the integers and not just the real numbers.

Our definitions of geometric chains and cochains will be a modification of that given by Lipyanskiy in [**?**], though his results will continue to hold with our modified definitions. As Lipyanskiy primarily focuses on geometric chains and geometric homology, our focus, where the theories diverge, will primarily be on geometric cochains and cohomology, for which Lipyanskiy's account is much less complete despite the cohomological setting having its own subtleties. We also take the opportunity to fill in some of the details missing from Lipyanskiy's account more generally, especially utilizing the more thorough foundations on manifolds with corners provided by Joyce [**?**].

4.1. **Preliminary definitions.** We first identify certain types of "manifolds over M." In future sections M will most typically be a manifold without boundary, as that is the case where we obtain agreement between geometric (co)homology and singular (co)homology, but in this section (and anywhere else we do not require transversality of maps to M), we allow it to be a manifold with corners anywhere that the extra generality may be useful but so long as it does not create more technical work (such as when utilizing smoothing, as happens below). In this case, we may interpret "smooth" to mean either "smooth" or "weakly smooth" in the sense of [**?**], so long as we are consistent in this interpretation.

Definition 4.1. A **manifold over M** is a manifold with corners W with a smooth map $r_W : W \to M$, called the **reference map**. We freely and almost always abuse notation by using the domain W to refer to the manifold over M, not r_W or some other symbol, letting context determine whether we are referring to the entire data or the domain.

We say a manifold over M is

- **compact** if the domain W is compact,
- **proper** if the reference map is proper (if $K \subset M$ is compact then $r_W^{-1}(K)$ is compact),
- **oriented** if W is oriented,
- **co-oriented** if the reference map r_W is co-oriented.

If W is a manifold over M then so is ∂W using the reference map $r_W \circ i_{\partial W} : \partial W \to M$. If W is oriented or co-oriented, then ∂W inherits an orientation or co-orientation as in Convention 3.1 or Definition 3.20.

By [Lemma 2.8]Joy12, $i_{\partial W}$ is proper, so, as the composition of proper maps is proper, if W is a proper manifold over M then so is ∂W.

If W is oriented or co-oriented we write $-W$ to refer to the manifold over M with the opposite orientation or co-orientation; it should always be clear from context which structure we are replacing with its opposite.

Geometric chains and cochains will be equivalence classes of manifolds over M under an equivalence relation we define using the following concepts, which are taken from or modify the definitions of [**?**].

Definition 4.2. Let V, W be manifolds over M with reference maps r_V and r_W.

- If V and W are oriented, we say they are **(oriented) isomorphic** if there is an orientation-preserving preserving diffeomorphism $\phi\colon W \to V$ such that $r_V \circ \phi = r_W$. If V and W are co-oriented, we say they are **(co-oriented) isomorphic** if there is a diffeomorphism $\phi\colon W \to V$ such that $r_V \circ \phi = r_W$ and the composition of the co-orientation induced by ϕ with the co-orientation of r_V agrees with the co-orientation of r_W.
- If W is oriented then W is **(oriented) trivial** if there is an orientation-reversing diffeomorphism $\rho\colon W \to W$ such that $r_W \circ \rho = r_W$. If W is co-oriented then it is **(co-oriented) trivial** if there is a diffeomorphism $\rho : W \to W$ such that $r_W \circ \rho = r_W$ and the composite of the co-orientation induced by ρ with the co-orientation of r_W is the opposite of the co-orientation of r_W. We call such a map ρ **co-orientation reversing**.
- W has **small rank** if the differential Dr_W is less than full rank everywhere.
- If W is (co-)oriented, then it is **((co-)oriented) degenerate** if it has small rank and ∂W is the disjoint union of a trivial (co-)oriented manifold over M and one with small rank.

Note that isomorphism (oriented or co-oriented) is an equivalence relation and that is preserves triviality and degeneracy (oriented or co-oriented, respectively).

Rather than small rank, Lipyanskiy uses the condition of **small image**. In our notation, a map $r_W : W \to M$ has small image if there is another map $r_T : T \to M$ such that $r_T(T) \supset r_W(W)$ but $\dim(T) < \dim(W)$. However, the small rank condition turns out to be more manageable for our purposes while still providing geometric homology and cohomology theories that are equivalent to singular homology and cohomology. Roughly speaking, the geometric chains and cochains of a manifold M will consist of isomorphism classes of oriented or proper co-oriented manifolds with corners over M modulo the trivial and degenerate chains and cochains. The most obvious use of the notion of triviality will be to ensure that $\partial^2 = 0$ so that we have a chain complex. The degeneracy comes into play in ensuring that geometric homology and cohomology satisfy the dimension axiom; see Example 4.18 and Remark 4.19.

Example 4.3. Let S^1 be the unit circle in the plane with the standard counterclockwise orientation. Let $\pi : S^1 \to \mathbb{R}$ be the projection of the circle onto the x-axis. This map is trivial. Indeed, if ρ is the reflection of the circle across the x-axis, then ρ is orientation-reversing and $\pi\rho = \pi$.

Alternatively, let $\pi : S^1 \to \mathbb{R}$ be co-oriented by (e_θ, e_x), where e_x is the standard positively-directed unit vector in $\mathbb{R}$ and e_θ is the counterclockwise tangent vector in S^1. This map is trivial as a co-oriented map, again via the reflection ρ across the x-axis. Indeed, we still have $\pi\rho = \pi$, and the co-orientation induced by ρ is $(e_\theta, -e_\theta)$ so that the composite co-orientation of $\pi\rho$ is $-(e_\theta, e_x)$.

Example 4.4. If $r_V : V \to M$ is any oriented or co-oriented map and $-r_V : V \to M$ is the same map with the opposite orientation or co-orientation, then $r_V \sqcup -r_V$ is trivial, taking ρ to be the map that switches the two copies of V. To be technically accurate, we should not think of a single instance of V as being the domain for both maps in order for $r_V \sqcup -r_V$ to have a well-defined domain that is a submanifold of some $\mathbb{R}^N$. However, recall that we are always free to replace $r_V : V \to M$ with an isomorphic map whose domain is another, diffeomorphic, “copy” of V in $\mathbb{R}^N$.

Example 4.5. Any co-oriented map of the interval to a point has small rank, but its boundary does not have small rank; it is nonetheless (co-oriented) degenerate because its boundary is trivial.

Example 4.6. Let V be the 2-simplex in $\mathbb{R}^2$ with vertices at $(1,0)$, $(-1,0)$, and $(0,1)$, and let $\pi : V \to \mathbb{R}$ be the projection to the x-axis. This map has small rank, but the boundary does not have small rank and is not trivial.

Example 4.7. Let $V = W = M = \mathbb{R}^1$. Let $r_W : W \to M$ be the identity map of $\mathbb{R}^1$ with the canonical co-orientation, which we can write (e_1, e_1), letting e_1 be a positively-oriented tangent vector to $\mathbb{R}^1$. Let $r_V : V \to M$ be given by $g(t) = -t$ with co-orientation $(-e_1, e_1)$. Let $\phi : W \to V$ be given by $\phi(t) = -t$. Then the canonical co-orientation of ϕ is $(e_1, -e_1)$. Then $r_V\phi = r_W$ as co-oriented maps, so ϕ provides a co-oriented equivalence between r_W and r_V even though they are very different maps.

4.2. **Geometric homology and cohomology.**

Definition 4.8. Denote by $PC_*^\Gamma(M)$ the set of oriented isomorphism classes of compact oriented manifolds over M, $r_W : W \to M$, graded by dimension $\dim(W)$. Denote by $PC^*_\Gamma(M)$ the set of co-oriented isomorphism classes of proper co-oriented manifolds over M, $r_W : W \to M$, graded by **codimension** $\dim(M) - \dim(W)$. We declare the empty manifold of each dimension to be orientable and the empty maps from the empty manifolds to M to be co-orientable.

As per Definition 4.1, we will often write $W \in PC_*^\Gamma(M)$ or $W \in PC^*_\Gamma(M)$, letting the reference map be tacit. In these respective cases we write $-W$ for W with the opposite orientation or co-orientation.

We will sometimes refer to elements of $PC_*^\Gamma(M)$ or $PC^*_\Gamma(M)$ as **prechains** and **precochains** respectively, and we tend to abuse notation by letting a given map $r_W : W \to M$ stand for its isomorphism class.

For the following definition only we will write isomorphism classes as $[W]$. In what follows we will abuse notation and refer to such an equivalence class only by a chosen representative.

Definition 4.9. If V, W are compact oriented manifolds over M (respectively proper co-oriented manifolds over M), then define $[V] \sqcup [W] = [V' \sqcup W']$, where on the right $\sqcup$ denotes disjoint union and V' and W' are compact oriented manifolds over M (respectively proper co-oriented manifolds over M) such that V' and W' are disjoint in $\mathbb{R}^\infty$ (cf. Definition 2.2).

The last clause in the definition is due to our requirement that all manifolds with corners be subsets of $\mathbb{R}^\infty$. As given, the definition allows constructions like $[W] \sqcup [W]$ to be well defined. We observe that this definition is well defined in general, since if V'' and W'' are two other manifolds over M isomorphic to V and W (in the appropriate sense), then we can compose the diffeomorphisms $V' \xleftarrow{\phi'} V \xrightarrow{\phi''} V''$ and $W' \xleftarrow{\psi'} W \xrightarrow{\psi''} W''$ to obtain an isomorphism $\phi''(\phi')^{-1} \sqcup \psi''(\psi')^{-1} : V' \sqcup W' \to V'' \sqcup W''$.

With $\sqcup$, $PC_*^\Gamma(M)$ and $PC^*_\Gamma(M)$ become commutative monoids in each degree with the empty maps $r_\emptyset : \emptyset \to M$ as the identities.

We now return to denoting isomorphism classes by their representatives, noting again that triviality and small rank are properties of the isomorphism classes.

Definition 4.10. Let $Q_*(M) \subset PC_*(M)$ denote the set of (isomorphism classes of) compact oriented manifolds over M of the form $V \sqcup W$ with V trivial and W degenerate. Let $Q^*(M) \subset PC^*(M)$ denote the set of (isomorphism classes of) proper co-oriented manifolds over M of the form $V \sqcup W$ with V trivial and W degenerate. In either case V or W may be empty.

We will sometimes write $W \in Q(M)$ to mean $W \in Q_*(M)$ or $W \in Q^*(M)$ for arguments that are analogous in the two cases. When we do so, we assume a consistent choice of $Q_*(M)$ or $Q^*(M)$ throughout the discussion; see for instance Lemma 4.12 and its proof.

The following useful basic properties are proven in [**?**]; for completeness we provide versions of the arguments here, occasionally augmenting those of [**?**]. In each case, "isomorphic," "trivial," or "degenerate" should be read consistently to refer to the compact oriented case or the proper co-oriented case. Lipyanskiy's proofs assume small image rather than small rank, butthe proofs are equivalent.

Lemma 4.11 (Lipyanskiy Lemma 10). *If V is trivial and $V \sqcup W$ is trivial, then W is trivial.*

Proof. We can write W as the disjoint union of a (possibly infinite) number of isomorphism classes of connected components and then group the isomorphism classes together up to (co-)orientation as $W = W_1 \sqcup W_2 \sqcup \cdots$ so that all connected components of each W_i are isomorphic (ignoring (co-)orientations) for each i and so that no connected component of W_i is isomorphic to a connected component of W_j for $i \neq j$. As either W is compact or r_W is proper, each W_i is the union of a finite number of connected components. Any automorphism of W preserves the decomposition into W_i and so W is trivial if and only if for all i either W_i has zero components when counting with (co-)orientation or each component of W_i has a (co-)orientation reversing automorphism.

Similarly, since V is trivial, V can be decomposed into unions of isomorphism classes, up to (co-)orientation, of connected components with 0 components when counting with sign or with all components having (co-)orientation reversing automorphisms. In particular, forming $V \sqcup W$ adds to W zero components when counting with sign or components with (co-)orientation reversing automorphisms, so if $V \sqcup W$ is trivial, W must have already been trivial. □

Lemma 4.12 (Lipyanskiy Lemma 11). *If W is in $Q(M)$, then $\partial W \in Q(M)$.*

Proof. We first check that if W is trivial then so is its boundary $r_{\partial W} = r_W i_{\partial W} : \partial W \to M$. If $\rho : W \to W$ is (co-)orientation reversing, then we can consider ρ_∂ as defined in Remark 2.8. As $r_W \rho = r_W$ we also have $r_W i_{\partial W} = r_W \rho i_{\partial W} = r_W i_{\partial W} \rho_\partial$. Thus we only need see that ρ_∂ is orientation reversing. It is sufficient to consider what happens at points on the interior of ∂W so working locally we may identify such points of ∂W with points of W itself and similarly identify ρ_∂ with ρ. In the oriented case, the orientation of W determines orientations of $T_x W$ and $T_{\rho(x)} W$, and by assumption $D\rho : T_x W \to T_{\rho(x)} W$ takes the orientation of $T_x W$ to the opposite of the orientation of $T_{\rho(x)} W$. But also $D\rho$ must preserve inward/outward pointing vectors. Thus $D\rho$ must restrict to a map $T_x(\partial W) \to T_{\rho(x)}(\partial W)$ that also reverses the orientation. The co-oriented situation is analogous using local orientation pairs $\left(\beta_{W,x}, \beta_{M,r_W(x)}\right)$ and $\left(\beta_{W,\rho(x)}, \beta_{M,r_W \rho(x)}\right)$ and taking $\beta_{M,r_W \rho(x)} = \beta_{M,r_W(x)}$ as ρ is a diffeomorphism over M.

Now suppose W degenerate. By definition $\partial W = A \sqcup B$ with A trivial and B of small rank. Then $\partial^2 W = \partial A \sqcup \partial B$. As noted, ∂A is trivial, and $\partial^2 W$ is trivial for all W by Lemma 3.23. It follows from Lemma 4.11 that ∂B is trivial, and so B is degenerate. Thus ∂W is degenerate. □

Lemma 4.13 (Lipyanskiy Lemma 12). *If V and $V \sqcup W$ are in $Q(M)$, then $W \in Q(M)$.*

Proof. As in the proof of Lemma 4.11, decompose W as $W = W_1 \sqcup W_2 \sqcup \cdots$ and V as $V = V_1 \sqcup V_2 \sqcup \cdots$. As V and $V \sqcup W$ are in $Q(M)$, each V_i has small rank or, also as in the proof of Lemma 4.11, V_i is trivial, and similarly for $V \sqcup W$, from which it follows again by counting with signs in the trivial components that each W_i is either trivial or has small rank. By grouping terms of the decomposition we can write $W = A \sqcup B$ with A trivial and B of small rank.

We then have ∂V and $\partial(V \sqcup W) = \partial V \sqcup \partial A \sqcup \partial B$ in $Q(M)$ by Lemma 4.12, and also ∂A is trivial as the boundary of a trivial manifold over M by Lemma 4.12. Now by the same argument as in the previous paragraph, replacing V with $\partial V \sqcup \partial A$ and W with ∂B, we have that ∂B can be decomposed into the disjoint union of a trivial manifold over M and one with small rank. But this shows B is degenerate, so $W \in Q(M)$. □

Lemma 4.14 (Lipyanskiy Lemma 13). *The relation given by $V \sim W$ if $V \sqcup -W$ is in $Q_*(M)$ (respectively $Q^*(M)$) is an equivalence relation on $PC_*^\Gamma(M)$ (respectively $PC^*_\Gamma(M)$).*

Proof. Reflexivity: For any W, we have $W \sqcup -W$ trivial via the map that interchanges the two copies of W.

Symmetry: If $V \sqcup -W$ is the union of trivial and degenerate manifolds over M then certainly so is $W \sqcup -V = -(V \sqcup -W)$.

Transitivity: If $V \sqcup -W$ and $W \sqcup -U$ are in $Q_*(M)$ (or $Q^*(M)$), then so is $V \sqcup -W \sqcup W \sqcup -U \cong V \sqcup -U \sqcup W \sqcup -W$. We know $W \sqcup -W$ is trivial, and so $V \sqcup -U$ is in $Q_*(M)$ (or $Q^*(M)$) by Lemma 4.13. $\square$

These lemmas allow us to follow Lipyanskiy in defining geometric chains and cochains. We will show that the claims of the following definition hold in Lemma 4.16 just below.

Definition 4.15. The **geometric chains** of M, denoted $C_*^\Gamma(M)$, are the $\sim$ equivalence classes in $PC_*^\Gamma(M)$. The **geometric cochains** of M, denoted $C_\Gamma^*(M)$, are the $\sim$ equivalence classes in $PC_\Gamma^*(M)$. These are chain complexes under the operation $\sqcup$ and with boundary map ∂.

In either case we denote the equivalence class of W by $\underline{W}$, and $\underline{W} = 0$ in $C_*^\Gamma(M)$ (respectively $C_\Gamma^*(M)$) if and only if W is in $Q_*(M)$ (respectively $Q^*(M)$).

We define the **geometric homology** of M, written $H_*^\Gamma(M)$, to be $H_*(C_*^\Gamma(M))$, and we define the **geometric cohomology** of M, written $H_\Gamma^*(M)$ to be $H^*(C_\Gamma^*(M))$.

The notation says that if V and W represent equivalence classes $\underline{V}$ and $\underline{W}$, then in $C_*^\Gamma(M)$ (or $C_\Gamma^*(M)$), we have $\underline{V} + \underline{W} = \underline{V \sqcup W}$. In particular every chain or cochain can be represented by a single map $r_W : W \to M$ for appropriate W. Similarly, $V \in Q_*(M)$ (or $Q^*(M)$) if and only if $\underline{V} = 0$.

Lemma 4.16. *The chain complexes $C_\Gamma^*(M)$ and $C_\Gamma^*(M)$ are well defined with $\underline{W} = 0$ in $C_*^\Gamma(M)$ (respectively $C_\Gamma^*(M)$) if and only if W is in $Q_*(M)$ (respectively $Q^*(M)$).*

Proof. This is a consequence of the preceding lemmas. For simplicity we work with $Q_*(M)$ and $C_*^\Gamma(M)$, but the identical arguments hold with $Q^*(M)$ and $C_\Gamma^*(M)$.

Addition is well defined because if $\underline{V}, \underline{W} \in C_*^\Gamma(M)$ with V and V' in the class $\underline{V}$ and W and W' in the class $\underline{W}$, then $\underline{V} + \underline{W}$ is well defined as the class $\underline{V \sqcup W}$ because $(V \sqcup W) \sqcup -(V' \sqcup W') = (V \sqcup -V') \sqcup (W \sqcup -W')$: By assumption $V \sqcup -V' \in Q_*(M)$ is the disjoint union of a trivial manifold over M and a degenerate manifold over M, and similarly for $W \sqcup -W'$, so $(V \sqcup -V') \sqcup (W \sqcup -W') \in Q_*(M)$.

The identity in each degree is represented by $\emptyset$ with the unique empty map to M (with either orientation or co-orientation). In fact, every element of $Q_*(M)$ represents 0 in $C_*^\Gamma(M)$ as elements of $Q_*(M)$ are all equivalent to $\emptyset$. Conversely, if $\underline{W} = 0$ then $W \in Q_*(M)$, as if $\underline{V} + \underline{W} = \underline{V}$ then $V \sqcup W \sqcup -V \in Q_*(M)$. But $V \sqcup -V \in Q_*(M)$, so by Lemma 4.13, $W \in Q_*(M)$. We also see that the additive inverse of $\underline{W}$ is $\underline{-W}$, as $W \sqcup -W$ is trivial.

That the boundary map is well defined with $\partial \underline{W} = \underline{\partial W}$ is due to Lemma 4.12 and [**?**, Lemma 2.8], which implies that ∂W is proper. That $\partial^2 = 0$ in the case of cochains follows from Lemma 3.23, which shows that $\partial^2 W$ is always trivial. Similarly, to obtain $\partial^2 = 0$ for chains see Remark 3.24. $\square$

In Section 6 of [**?**], Lipyanskiy shows that the homology theory based on geometric chains satisfies some of the Eilenberg-Steenrod axioms. This is enough to state in Section 10 of [**?**] that geometric homology is isomorphic to singular homology on the fixed manifold M, though we provide our own proof below. Unfortunately, Lipyanskiy does not provided a detailed treatment of geometric cohomology, which is different from geometric homology in several respects, though we will also show that it is isomorphic to singular cohomology.

First, though, we present here some computational examples, mostly without proofs, to aid the reader's intuition.

Example 4.17. For any manifold, the map of an oriented point to M is a generator of $H_0^\Gamma(M)$. Any two points with the same orientation mapping to the same component of M represent the same element of $H_0^\Gamma(M)$ as can be seen by joining them with a smooth path.

The generators of $H_\Gamma^0(M)$ are the inclusions of the connected components with the tautological co-orientations.

If M is close and oriented with $\dim(M) = m$ then $H_m^\Gamma(M)$ is generated by the inclusions of connected components.

Elements of $C^m_\Gamma(M)$ are represented by co-oriented maps of points to M. If M is compact such a map is a generator of a nontrivial cohomology class. If the point maps to a non-compact component of M then it represents 0 in $H^m_\Gamma(M)$ because any proper path $(-\infty, 0] \to M$ with the restriction to 0 being our cochain representative gives a null-cohomology with appropriate choices of co-orientation.

In $\mathbb{R}^3 - \{0\}$, the inclusion of the positive x-axis with either co-orientation is a generator of $H^2_\Gamma(M)$, while the embedding the oriented unit 2-sphere containing the origin is a generator of $H_2^\Gamma(M)$.

If M is compact and oriented then tautologically $C_*^\Gamma(M) = C_\Gamma^{\dim(M)-*}(M)$. This is a strong form of Poincaré duality.

Example 4.18 (Dimension axiom)**.** If M is a point, then every $r_W : W \to M$ with $\dim(W) > 0$ is degenerate: If $\dim(W) > 1$, both W and ∂W must have small rank. If $\dim(W) = 1$ then W is a union of closed intervals and circles, so it has small rank, while ∂W is trivial, consisting of pairs of maps $pt \to pt$ with opposite (co-)orientations. Thus $C_*^\Gamma(pt) = C_\Gamma^{-*}(pt) = 0$ unless $* = 0$, and so $H_*^\Gamma(pt) = H_\Gamma^{-*}(pt) = 0$ unless $* = 0$. When $* = 0$, the identity $pt \to pt$ is a cycle (or cocycle) that does not (co)bound, and as in ordinary bordism theory we have $H_0^\Gamma(pt) \cong H^0_\Gamma(pt) \cong \mathbb{Z}$.

Remark 4.19. It is in the dimension axiom that we most obviously see the need to include degenerate chains and cochains in $Q(M)$. On the other hand, the formulation of degeneracy as given will create some difficult for us in Section 7 when it comes to consider notions of transversality for geometric chains and cochains. The reason is that degeneracy causes much of the problematic ambiguity in choosing representatives for chains and cochains. For example, consider a connected prechain $V \in PC_*(M)$ with small rank but a boundary that is not in $Q_*(M)$. If V' is any other such prechain with small rank and $\partial V = \partial V'$, then $V \sqcup -V' \in Q_*(M)$, so V and V' represent the same chain but could behave wildly differently aside from their boundaries. By contrast, we will see in Section 7 that trivial chains are less of an issue (they can generally be ignored) and that non-trivial components without small rank are "essential" in a sense we will make precise in Definition 7.3; essential components appear in any representative of the same geometric chain or cochain.

Given the headaches thus caused by the degenerate chains and cochains, it is tempting to ask for a simpler definition of degeneracy. One variant that comes to mind would be defining degeneracy so that each individual component must have a boundary consisting of trivial and small rank (co)chains. This would seem to be sufficient for the dimension axiom and would eliminate the difficulty described above. Unfortunately, with such an alternative definition of degeneracy, it will not generally be true that if $V \in Q(M)$ then fiber products $V \times_M W$ are also in $Q(M)$. This is an important property that will arise in the next section and then be needed both to construct cup and cap products and to prove the existence of Mayer-Vietoris sequences. See Remark 4.23 for further discussion of this point.

The following algebraic property will be useful below as we consider homological algebra with geometric cochains:

Lemma 4.20. *Each $C_i^\Gamma(M)$ or $C^i_\Gamma(M)$ is torsion-free and hence flat as a $\mathbb{Z}$-module.*

Proof. The second statement follows from the first as $\mathbb{Z}$ is a Dedekind domain. The first statement is proven for geometric chains in [**?**, Lemma 34]. The proof for geometric cochains, even accounting for our different definition of degeneracy, is the same. □

4.2.1. *Products of manifolds over M.* In this section we define various products of elements of $PC_*^\Gamma(M)$ and $PC^*_\Gamma(M)$, all coming from the external products of fiber products defined above. These products will ultimately become our cup, cap, intersection, and exterior products, but we introduce them here as products on $PC_*^\Gamma(M)$ and $PC^*_\Gamma(M)$ and derive some further properties as we will need some of this material in the next section to define creasing. The first time reader can fairly safely skip this section for now and return to it later as needed.

Definition 4.21. Given a manifold without boundary M, the fiber product $(V, W) \to V \times_M W$, with the appropriate corresponding fiber product orientation or co-orientation, determines partially-defined products of the following forms:

$$PC^*_\Gamma(M) \times PC^*_\Gamma(M) \to PC^*_\Gamma(M)$$
$$PC^*_\Gamma(M) \times PC_*^\Gamma(M) \to PC_*^\Gamma(M).$$

If, furthermore, M is oriented, then there is also a partially-defined product

$$PC_*^\Gamma(M) \times PC_*^\Gamma(M) \to PC^*_\Gamma(M).$$

In each case, the product is defined when the reference maps $r_V : V \to M$ and $r_W : W \to M$ are transverse.

To define these products, in each case we suppose transverse maps $r_V : V \to M$ and $r_W : W \to M$ representing elements of $PC^*_\Gamma(M)$ or $PC_*^\Gamma(M)$ and form the fiber product $V \times_M W \to M$. Then the first product is well defined by the definition of the fiber product co-orientation, Lemma 3.32, and that the composition of proper maps is proper. For the second product, we have the pullback co-orientation $V \times_M W \to W$, and W is compact and oriented by definition. So $V \times_M W$ is compact by Lemma 3.32 as a proper map to a compact space must have compact domain. We also recall that given a smooth map of manifolds with oriented codomain, an orientation of the domain determines a co-orientation of the map and vice versa; see the discussion following Definition 3.11. So $V \times_M W$ is compact, and if we give it the orientation we obtain from the pullback co-orientation of $V \times_M W \to W$, we obtain an element of $PC_*^\Gamma(M)$. For the third map, we use the convention of Joyce recalled in Section 3.1 to oriented a fiber product of oriented manifolds with corners and observe that $V \times_M W$ is compact as a closed subset of the compact $V \times W$. Thus we obtain an element $PC_*^\Gamma(M)$. In this last case, we really need M to be oriented in general, as we have observed in Remark 3.40 that if M is not orientable the fiber product of orientable manifolds over M may be non-orientable.

In the cases where r_V and r_W are transverse embeddings, these products are represented by just taking intersections, with the orientations or co-orientations given explicitly in Propositions 3.43, 3.78, and 3.7. If r_V and r_W are immersions, these descriptions hold locally.

The next Lemma will be critical in Section 7 toward showing that these products extend to well-defined, though only partially-defined, products of geometric chains and cochains. It will also be needed much sooner to show that the creasing construction is well defined. This construction is used, in turn, to demonstrate the existence of Mayer-Vietoris sequences.

Lemma 4.22. *For any of the products above, if either V or W is in $Q(M)$, then $V \times_M W \in Q(M)$. In fact, if V or W is trivial then $V \times_M W$ is trivial, and if V or W has small rank then $V \times_M W$ has small rank.*

Proof. We provide the proof if $V \in Q(M)$; the other case is similar. By assumption V is the disjoint union of trivial and degenerate chains or cochains, so it suffices to consider independently the possibilities that V is trivial or degenerate.

If ρ is a (co-)orientation reversing diffeomorphism of V over M, then $\rho \times_M \mathrm{id}_W$ is a (co-)orientation reversing diffeomorphism of $V \times_M W$, by Joyce's construction in the oriented case and by Remark 3.36 in the co-oriented case.

Next assume that V is degenerate, so in particular it has small rank. Recall that the tangent bundle of a fiber product is the fiber product of the tangent bundles [**?**, Theorem 5.47], and so the derivative is the fiber product of derivatives. Note that the fiber product of two linear maps, one with a nontrivial kernel, must also have a nontrivial kernel: If A, B are linear maps with a common codomain and $v \in \ker(A)$, then $(v, 0)$ is in the kernel of the fiber product of A and B. So if the differential of r_W has non-trivial kernel everywhere so will the derivative of any fiber product with r_W. Thus $V \times_M W$ has small rank.

Now we recall that $\partial(V \times_M W)$ is, up to (co-)orientations, the union of $(\partial V) \times_M W$ and $V \times_M (\partial W)$. We have just shown that $V \times_M (\partial W)$ must have small rank. As V is degenerate, ∂V is a disjoint of trivial and small rank manifolds over M, and so by the preceding arguments $(\partial V) \times_M W$ will be a union of trivial and small rank manifolds over M. Altogether, $V \times_M W$ is degenerate. $\square$

Remark 4.23. As noted in Remark 4.19, it is this lemma that fails if we attempt to simplify the definition of degeneracy by requiring each connected component of a degenerate prechain or precochain to have small rank and boundary that is a union of trivial and small rank pre(co)chains. In fact, it is possible to construct a V and W such that V is a non-trivial prechain that is degenerate in this stronger sense but such that $V \times_M W$ has multiple components that are each non-trivial and of small rank but such that the boundary of each component is nontrivial and not of small rank. So $V \times_M W$ would not be in a version of $Q(M)$ defined using this stronger, but simpler, notion of degeneracy. Of course it is in $Q(M)$ with our actual definitions by the preceding lemma.

We have a similar result for the exterior products studied in Section 3.6, which are always fully defined:

Lemma 4.24. *Suppose $f : V \to M$ and $g : W \to N$ are respectively in $PC_*(M)$ and $PC_*(N)$ or that they are respectively in $PC^*(M)$ and $PC^*(N)$. Suppose $V \in Q(M)$ or $W \in Q(N)$. Then $f \times g : V \times W \to M \times N$ is in $Q(M \times N)$.*

Proof. We provide the proof if both maps are in PC^* and $V \in Q^*(M)$; the other cases are similar. By assumption V is the disjoint union of trivial and degenerate chains or cochains, so it suffices to consider independently the possibilities that V is trivial or degenerate.

If ρ is a (co-)orientation reversing diffeomorphism of V over M, then $\rho \times \mathrm{id}_W$ is a (co-)orientation reversing diffeomorphism of $V \times_M W$. So if V is trivial so is $V \times W$.

Next assume that V is degenerate, so in particular it has small rank. The derivative of $f \times g$ is matrix with Df and Dg on the block diagonals, so $f \times g$ has small rank. Now we recall that $\partial(V \times W)$ is, up to co-orientations, the union of $(\partial V) \times_M W$ and $V \times_M (\partial W)$. We have just shown that $V \times_M (\partial W)$ must have small rank. As V is degenerate, ∂V is a disjoint of trivial and small rank manifolds over M, and so by the preceding arguments $(\partial V) \times_M W$ will be a union of trivial and small rank manifolds over M. Altogether, $V \times_M W$ is degenerate. $\square$

4.3. **Creasing.** We now review the creasing construction of [**?**, Section 2.4], which is essential for Mayer-Vietoris sequences and excision, though we use different orientation conventions and also consider versions involving co-orientations. Given a real-valued function $\varphi\colon W \to \mathbb{R}$ with 0 a regular value, define W^+ to be $\varphi^{-1}[0, \infty)$, define W^- to be $\varphi^{-1}(-\infty, 0]$, and define $W^0 = \varphi^{-1}(0)$. We have already seen this construction in Example 3.22 and the examples in Section 3.5.6. As observed in those examples, $W^\pm$ and W^0 are manifolds with corners as they are also the fiber products $M^\pm \times_M W$ and $M^0 \times_M W$. The idea of creasing is to, for example, show that if W is a cycle (or cocycle) over M then it is homologous (or cohomologous) to $W^+ + W^-$. This is the analog in geometric homology/cohomology of subdivision of singular simplices.

More precisely, by Lemma 9 of [**?**], there is a manifold-with-corners structure on the topological space $W \times [0, 1]$, called the **creasing homotopy** of W at 0 and denoted $\mathrm{Cre}(W)$, such that $\partial \mathrm{Cre}(W) = W \sqcup -W^+ \sqcup -W^- \sqcup -\mathrm{Cre}(\partial W)$. We will also see that if $W \in Q(M)$ then $\mathrm{Cre}(W) \in Q(M)$, which will provide the claimed (co)homology.

By using pullbacks we provide an arguably simpler description of $\mathrm{Cre}(W)$ than found in [**?**]. Let D be the semi-open pentagonal region of the plane given by

$$D = \{(x, y) \mid -1 < x < 1, 0 \leq y \leq 2 - |x|\}.$$

Then D is a manifold with corners with a smooth proper projection map $\pi : D \to (-1, 1)$ given by $\pi(x, y) = x$. We see that ∂D has three pieces, say D_x, D_+, and D_-, corresponding respectively to the intersection of D with the x-axis, the graph of $y = 2 - x$ over $[0, 1)$, and the graph of $y = 2 + x$ over

$(-1,0]$. We orient all three pieces by rightward pointing tangent vectors in the plane and give D itself the standard planar orientation so that π restricts to oriented diffeomorphisms from D_x, D_+, and D_- onto their images in $(-1,1) = D_x$. Then as oriented manifolds with corners,

$$\partial D = D_x \sqcup -D_- \sqcup -D_+.$$

To obtain analogous boundary formulas for creasings of cochains, we let the projections of D_x, D_+, and D_- to $(-1,1)$ be co-oriented by taking the rightward orientations to the rightward orientations, and $\pi : D \to (-1,1)$ to be co-oriented by taking the standard planar orientation to the rightward orientation. In this case, as co-oriented manifolds with corners over $(-1,1)$, we have

$$\partial D = D_x \sqcup -D_- \sqcup -D_+.$$

The projection $\pi : D \to (-1,1)$ restricts to a submersion from D_x, D_+, and D_- onto their images, and so a map $\varphi : W \to (-1,1)$ from a manifold with corners is transverse to π if and only if it is transverse to the map from the point at the tip of the pentagon to $(-1,1)$. This is equivalent to the requirement that the restriction of φ to every stratum of W have 0 as a regular value. In this case we will simply say that "φ has a regular value at 0."

We can now define creasing.

Definition 4.25. Suppose given a smooth map[23] $\varphi : W \to (-1,1)$ with a regular value at 0. Then define the **creasing homotopy** to be the pullback

$$\mathrm{Cre}(W) = D \times_{(-1,1)} W \to W.$$

We note that $\mathrm{Cre}(W)$ does depend on the map φ, though we typically omit it from the notation. When necessary for clarity, we may write $\mathrm{Cre}_\varphi(W)$.

Typically in practice W will arise with a map $r_W : W \to M$ representing an element of $PC_*^\Gamma(M)$ or $PC^*_\Gamma(M)$, and in this case our map $\varphi : W \to (-1,1)$ will generally be given as a composition $W \xrightarrow{r_W} M \xrightarrow{\phi} (-1,1)$ for some smooth $\phi : M \to (-1,1)$. When $r_W : W \to M$ represents an element of $PC_*^\Gamma(M)$, we treat $D \times_{(-1,1)} W$ as oriented by the pullback conventions, using our given orientation of D, and we will show in a moment that if W is compact so is $D \times_{(-1,1)} W$. Thus composing the pullback map with r_W we obtain another element of $PC_*^\Gamma(M)$ given by $D \times_{(-1,1)} W \to W \xrightarrow{r_W} M$. Similarly, when $r_W : W \to M$ represents an element of $PC^*_\Gamma(M)$, we treat $D \times_{(-1,1)} W \to W$ as co-oriented by the pullback conventions, using our given co-orientation of $D \to (-1,1)$, and this is a proper map by Lemma 3.32. Thus composing the pullback map with r_W we obtain another element of $PC^*_\Gamma(M)$ given by $D \times_{(-1,1)} W \to W \xrightarrow{r_W} M$. In either case, we denote the compositions $D \times_{(-1,1)} W \to W \xrightarrow{r_W} M$ by $r_{\mathrm{Cre}(W)}$.

We will regularly abuse notation by allowing $\mathrm{Cre}(W)$ to refer to the space $D \times_{(-1,1)} W$, the pullback $D \times_{(-1,1)} W \to W$, or the element of $PC_*^\Gamma(M)$ or $PC^*_\Gamma(M)$ formed by the preceding constructions. It should usually be clear from context which is meant at any point.

To justify the notion of creasing as a type of homotopy, at least topologically, we have the following lemma and corollary:

Lemma 4.26. *Suppose given a projection $X \times Y \to X$ and a map $g : W \to X$. Then the fiber product $(X \times Y) \times_X W$ is homeomorphic to $Y \times W$.*

Proof. We have $(X \times Y) \times_X W = \{(x,y,w) \in X \times Y \times W \mid x = g(w)\}$. A homeomorphism $(X \times Y) \times_X W \to Y \times W$ is then given by $(x,y,w) \to (y,w)$ with inverse given by $(y,w) \to (g(w),y,w)$. $\square$

[23] Given instead a smooth $\varphi : W \to \mathbb{R}$ with regular value at 0 as in the preceding examples, we may alter φ to have image in $(-1,1)$ by composing with a diffeomorphism, for example $x \to \frac{2\arctan(x)}{\pi}$. This does not change W^0, W^+, W^-, and, up to diffeomorphism, the creasing construction does not depend on the choice of diffeomorphism $\mathbb{R} \to (-1,1)$ so long as $(-\infty,0]$ maps to $(-1,0]$ and $[0,\infty)$ maps to $[0,1)$. Similarly, there is nothing special about the value 0 or the interval $(-1,1)$; given any $\varphi : W \to \mathbb{R}$ with regular value p, we could construct a version of $\mathrm{Cre}(W)$ that creases along $\varphi^{-1}(p)$ rather than $\varphi^{-1}(0)$.

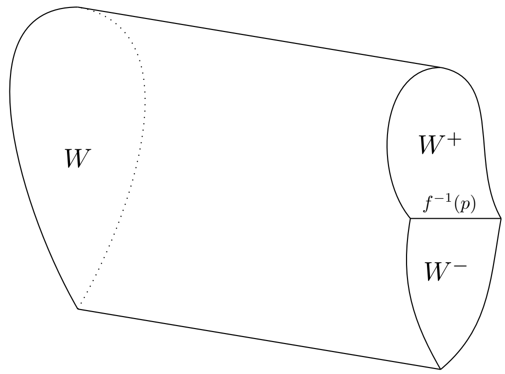


FIGURE 2. Creasing homotopy

Corollary 4.27. $\mathrm{Cre}(W)$ *is homeomorphic to* $I \times W$.

Proof. This follows from the preceding lemma by observing that D is homeomorphic to $(-1,1) \times I$ via a homeomorphism that preserves π by taking (x,y) to $\left(x, \frac{y}{2-|x|}\right)$. $\square$

In particular, this demonstrates that if W is compact so is $\mathrm{Cre}(W)$, as claimed above.

See Figure 2 for a sketch of a creasing homotopy of the teardrop manifold.

If either $W \in PC_*^\Gamma(M)$ or $W \in PC^*_\Gamma(M)$, we have

$$\partial(\mathrm{Cre}(W)) \cong (D_x \sqcup -D_- \sqcup -D_+) \times_{(-1,1)} W \bigsqcup -D \times_{(-1,1)} \partial W = W \sqcup -W^- \sqcup -W^+ \sqcup -\mathrm{Cre}(\partial W), \tag{11}$$

again interpreting these formulas in $PC_*^\Gamma(M)$ or $PC^*_\Gamma(M)$, respectively, by composing the pullback maps to W with $r_W : W \to M$. This first equality comes from our boundary formulas for D and our Leibniz rules from Section 3.1 and Proposition 3.44. For the second equality we use, for example, that π is an orientation preserving diffeomorphism from $D_+ \cong [0,1)$ onto its image in $(-1,1)$, and so we have an orientation-preserving diffeomorphism $D_+ \times_{(-1,1)} W \cong [0,1) \times_{(-1,1)} W = W^+$, and similarly in the co-oriented case.

Convention 4.28. In order to crease an element of $PC_*^\Gamma(M)$ or $PC^*_\Gamma(M)$ represented by $r_W : W \to M$, we need a smooth map $\varphi : W \to (-1,1)$ so that the composite $\varphi : W \xrightarrow{r_W} M \xrightarrow{\phi} (-1,1)$ has 0 as a regular value. In this case M^0 and $M^\pm$ are fiber products and thus manifolds with corners as in Example 3.22. Also as observed there, in this case φ has 0 as a regular value if and only if r_W is transverse to M^0, in which case we also have $W^0 = M^0 \times_M W$ and $W^\pm = M^\pm \times_M W$. Unless noted, we will always assume this situation when working with creasing, i.e. that we have a fixed $\phi : M \to (-1,1)$ with 0 a regular value and that creasing is defined with respect to the composition $\varphi = \phi r_W$ with r_W transverse to M^0.

To next promote the construction $\mathrm{Cre}(-)$ to an operator on $C_*^\Gamma(M)$ or $C^*_\Gamma(M)$, we will need Lemma 4.22 along with its following corollary.

Corollary 4.29. *Let M be a manifold without boundary. Suppose given $\phi : M \to (-1,1)$ with 0 a regular value and $r_T : T \to M$ transverse to M^0. If $T \in Q(M)$ then so are T^+, T^-, T^0, and* $\mathrm{Cre}(T)$.

Proof. As observed in Convention 4.28, with our assumptions about φ the spaces $T^\pm$ and T^0 are fiber products over M of T with $M^\pm$ and M^0. So in this case the claim follows from the Lemma 4.22.

For $\mathrm{Cre}(T)$, the pullback projection $\pi : \mathrm{Cre}(T) \to T$ is of small rank, as $\dim(\mathrm{Cre}(T)) > \dim(T)$, from which it follows that $r_{\mathrm{Cre}(T)} = r_T\pi$ is of small rank. We also have $\partial\mathrm{Cre}(T) = T \sqcup -T^+ \sqcup -T^- \sqcup -\mathrm{Cre}(\partial T)$.

By assumption and the preceding paragraph, $T, T^{\pm} \in Q(M)$ and by the preceding sentence $\mathrm{Cre}(\partial T)$ is of small rank. Thus all components of $\partial\mathrm{Cre}(T)$ are trivial or of small rank, and so $\mathrm{Cre}(T)$ is degenerate. $\square$

Proposition 4.30. *Let M be a manifold without boundary. Suppose given $\phi : M \to (-1, 1)$ with 0 a regular value. If V and W are any two representatives of $\underline{W} \in C^{\Gamma}_*(M)$ whose reference maps are transverse to M^0, then $\mathrm{Cre}(V)$ and $\mathrm{Cre}(W)$ represent the same element of $C^{\Gamma}_*(M)$. Thus if the equivalence class $\underline{W}$ contains any representative that is transverse to M^0, there is a well-defined element $\underline{\mathrm{Cre}(W)} \in C^{\Gamma}_*(M)$. Similarly for $C^*_{\Gamma}(M)$.*

Proof. The proofs for chains and cochains are the same, so we provide that with chains.

If $r_V : V \to M$ and $r_W : W \to M$ represent the same class in $C^{\Gamma}_*(M)$ then $V \sqcup -W \in Q_*(M)$, and if V and W are both transverse to M^0 then so is $V \sqcup -W$. So $\mathrm{Cre}(V \sqcup -W)$ is defined, and by Corollary 4.29, $\mathrm{Cre}(V \sqcup -W) \in Q_*(M)$. Using our convention that all creasing maps are compositions of the fixed $\phi : M \to (-1, 1)$ with the reference maps and the properties of fiber product orientations, we have $\mathrm{Cre}(V \sqcup -W) = \mathrm{Cre}(V) \sqcup \mathrm{Cre}(-W) = \mathrm{Cre}(V) \sqcup -\mathrm{Cre}(W)$. Thus $\mathrm{Cre}(V)$ and $\mathrm{Cre}(W)$ represent the same element of $C^{\Gamma}_*(M)$. $\square$

Finally, we come to the punchline of creasing:

Theorem 4.31. *Let M be a manifold without boundary. Suppose $\underline{W} \in H^{\Gamma}_*(M)$ and that $\underline{W}$ has a representative $r_W : W \to M$ that is transverse to M^0, defined with respect to some $\phi : M \to (-1, 1)$ with 0 a regular value. Then $\underline{W} = \underline{W^+} + \underline{W^-} \in H^{\Gamma}_*(M)$. Similarly for $H^*_{\Gamma}(M)$.*

Proof. Again the proofs for homology and cohomology are the same so we focus on homology.

We have seen that, with our assumptions, $\underline{W}$ yields a well-defined element $\underline{\mathrm{Cre}(W)}$ represented by $\mathrm{Cre}(W)$. Computing in $C^{\Gamma}_*(M)$ we have

$$\begin{aligned}\partial\underline{\mathrm{Cre}(W)} &= \underline{\partial\mathrm{Cre}(W)}\\ &= \underline{W \sqcup -W^+ \sqcup -W^- \sqcup -\mathrm{Cre}(\partial W)}\\ &= \underline{W} - \underline{W^+} - \underline{W^-}.\end{aligned}$$

In the last line we have used that $\partial\underline{W} = 0$ so that $\partial W \in Q_*(M)$ and hence $\mathrm{Cre}(\partial W) \in Q_*(M)$ by Corollary 4.29 and $\underline{\mathrm{Cre}(\partial W)} = 0 \in C^{\Gamma}_*(M)$. The theorem follows. $\square$

5. Basic properties of geometric (co)homology and equivalence with singular (co)homology

In this section we establish the basic properties of geometric homology and cohomology, eventually showing that they are equivalent to singular homology and cohomology using a result of Kreck and Singhof. In later sections we will provide an alternative argument for geometric cohomology.

5.1. **Functoriality and homotopy properties.**

Given a continuous map of manifolds $f : N \to M$, in this section we define the induced maps $f_* : H^{\Gamma}_*(N) \to H^{\Gamma}_*(M)$ and $f^* : H^*_{\Gamma}(M) \to H^*_{\Gamma}(N)$ and show that they are independent of f up to homotopy. We also treat the case that f is proper, in which case it induces a covariant map $f_* : H^*_{\Gamma}(N) \to H^*_{\Gamma}(M)$ independent of proper homotopy. In the covariant cases M and N may both be manifolds with corners. In the contravariant case we need to use transversality, and so M will be without boundary.

In the homology case this is relatively straightforward and can essentially be found in [**?**, Section 6], though we provide additional details. The covariant case of cohomology is similar to that for homology. For the contravariant case of cohomology there is slightly more work, and our argument there parallels Kreck's in [**?**], which is slightly different than the sketch in Lipyanskiy [**?**, Section 6] in that we choose to perturb f rather than the reference map for the cohomology class to obtain transversality.

As part of the constructions, we will see that if $F : N \times I \to M$ is a homotopy (proper or co-oriented as need be) and $r_W : W \to N$ is a cycle or cocyle then the compositions $F \circ (r_W \times \mathrm{id}_I)$ provide homologies or cohomologies from $F(-,0)r_W : W \to M$ to $F(-,1)r_W : W \to M$. However, we remark that, somewhat contrary to intuition, homologies $W \times I \to M$ will not generally generate homologies or cohomologies as membership in $Q(M)$ is not generally preserved by homotopies of this form.

5.1.1. *Covariant functoriality of geometric homology and cohomology.* In this section we consider the covariant behavior of geometric chains and cochains under maps and homotopies. In this case, both M and N may have corners unless noted otherwise.

If $r_W : W \to M$ is in $PC^{\Gamma}_*(M)$ and $f : M \to N$ is a smooth map, then the composition $fr_W : W \to N$ is in $PC^{\Gamma}_*(N)$. Consistent with our notation of writing $W \in PC^{\Gamma}_*(M)$, we write the image in $PC^{\Gamma}_*(N)$ as $f(W)$. Similarly, if f is proper and co-oriented we obtain a map $PC^*_{\Gamma}(M) \to PC^*_{\Gamma}(N)$ that we also write as $W \to f(W)$. Functoriality is clear, and both $\partial(f(W))$ and $f(\partial W)$ are represented by $fr_W i_{\partial W}$. To obtain chain maps $C^{\Gamma}_*(M) \to C^{\Gamma}_*(N)$ and $C^*_{\Gamma}(M) \to C^*_{\Gamma}(N)$ (that we also write as f) it suffices to show that if $W \in Q(M)$ then $f(W) \in Q(N)$. This is the content of the following lemma.

Lemma 5.1. *If $r_W : W \to M$ represents an element of $Q_*(M)$ (or $Q^*(M)$) and $f : M \to N$ is any (co-oriented proper) smooth map, then $fr_W : W \to N$ is in $Q_*(N)$ (or $Q^*(N)$, co-orienting fr_W with the composition co-orientation.).*

Proof. First consider $W \in Q_*(M)$. By assumption, W is the disjoint union of a trivial manifold over M given by $r_T : T \to M$ and a degenerate manifold over M given by $r_D : D \to M$. If $\rho : T \to T$ is an orientation-reversing diffeomorphism of W such that $r_T\rho = r_T$ then also $fr_T\rho = fr_T$, so $fr_T : T \to N$ is also trivial. Furthermore, if r_D has small rank then certainly so does fr_D. Similarly, $\partial(fr_D)$ is a union of trivial and small rank manifolds over M, and so $fr_D : D \to M$ is degenerate.

The proof for $W \in Q^*(M)$ is the same using that the composition of proper maps is proper and the composition of co-oriented maps is co-oriented. $\square$

Corollary 5.2 (Lipyanskiy's Theorem 4)**.** *Given a smooth map $f : M \to N$ of manifolds (possibly with corners), there is an induced chain map $C^{\Gamma}_*(M) \to C^{\Gamma}_*(N)$ given by $\underline{W} \to \underline{f(W)}$, and this construction gives a functor from the category of smooth manifolds and smooth maps to chain complexes over $\mathbb{Z}$.*

Corollary 5.3. *Given a smooth proper co-oriented map $f : M \to N$ of manifolds (possibly with corners), there is an induced chain map $C^*_{\Gamma}(M) \to C^*_{\Gamma}(N)$ given by $\underline{W} \to \underline{f(W)}$, and this construction gives a functor from the category of smooth manifolds and smooth maps to cochain complexes over $\mathbb{Z}$.*

Homotopies. Next we consider behavior with respect to homotopies.

Convention 5.4. If $r_W : W \to M$ is co-oriented, then we co-orient $r_W \times \mathrm{id}_I : W \times I \to M \times I$ so that if (β_W, β_M) is the co-orientation at $x \in W$ then $(\beta_W \wedge \beta_I, \beta_M \wedge \beta_I)$ is the co-orientation at (x,t) for any $t \in I$, with β_I being the standard orientation of the interval.

Lemma 5.5. *Let $W \in PC^{\Gamma}_*(M)$ (or $W \in PC^*_{\Gamma}(M)$). Let $F \colon M \times I \to N$ be a smooth homotopy; if $W \in PC^*_{\Gamma}(M)$ suppose further that F is proper and co-oriented. Then $F \circ (r_W \times \mathrm{id}_I) : W \times I \to N$ is in $PC^{\Gamma}_*(N)$ (or $PC^*_{\Gamma}(N)$). Furthermore, if $r_W : W \to M$ is trivial, of small rank, or degenerate then so are $F \circ (r_W \times id_I) \colon W \times I \to N$ and $F(-,1) \circ r_W : W \to N$.*

Proof. The last statement concerning $F(-,1) \circ r_W : W \to N$ holds by Lemma 5.1 and its proof.

We first suppose $W \in PC^{\Gamma}_*(M)$. In this case it is clear that $F \circ (r_W \times \mathrm{id}_I) \in PC^{\Gamma}_*(N)$ using the product orientation on $W \times I$.

If $\rho \colon W \to W$ is an orientation reversing diffeomorphism such that $r_W \circ \rho = r_W$, then $\rho \times \mathrm{id}_I \colon W \times I \to W \times I$ is an orientation-reversing self-diffeomorphism of $W \times I$ such that $F \circ (r_W \times id_I) \circ (\rho \times \mathrm{id}_I) = F \circ (r_W \times id_I)$. So $W \times I \to N$ is trivial.

Next, if the derivative of the reference map Dr_W has non-trivial kernel at a point of W then so does the derivative $D(r_W \times \mathrm{id}_I)$, and thus so will $D(F \circ (r_W \times id_I)) = DF \circ D(r_W \times id_I)$. Thus $W \times I \to N$ has small rank.

By definition, if $W \to M$ is degenerate then it has small rank and $\partial W = T \cup S$ with T trivial and S small rank. We've shown that $W \times I \to N$ has small rank, so its suffices to consider its boundary, which is the union (up to signs) of $\partial W \times I$ and $W \times \partial I$. But $\partial W \times I = (T \times I) \sqcup (S \times I)$, which by our previous arguments are trivial and of small rank respectively. And $W \times \partial I$ maps to N by maps which factor through our original $W \to M$, and thus are of small rank.

If $W \in PC^*_\Gamma(M)$, then F is proper and co-oriented by assumption, and we noted above our co-orientation convention for $r_W \times \mathrm{id}_I$. To see the latter is proper, if K is compact in $M \times I$, then $K \subset \pi_M(K) \times I$, which is also compact. Then $(r_W \times \mathrm{id}_I)^{-1}(K) \subset (r_W \times \mathrm{id}_I)^{-1}(\pi_M(K) \times I) = r_W^{-1}(\pi_M(K)) \times I$, which is compact as r_W is proper. As compositions of proper co-oriented maps are proper and co-oriented, $F \circ (r_W \times \mathrm{id}_I) : W \times I \to N$ is well defined in $PC^*_\Gamma(N)$. The remaining arguments are analogous to the compact oriented case. □

This lemma shows that a homotopy $M \times I \to N$ cannot "promote" a manifold over M that is in $Q(M)$ to one that is not in $Q(N)$.

Corollary 5.6. *Let $V, W \in PC^\Gamma_i(M)$ (or $V, W \in PC^i_\Gamma(M)$). Let $F \colon M \times I \to N$ be a smooth homotopy; if $V, W \in PC^i_\Gamma(M)$ suppose further that F is proper and co-oriented. If V, W represent the same element of $C^\Gamma_i(M)$, then $F(-,1) \circ r_V : V \to N$ and $F(-,1) \circ r_W : W \to N$ represent the same element of $C^\Gamma_i(N)$ and $F \circ (r_V \times id_I) \colon V \times I \to N$ and $F \circ (r_W \times id_I) \colon W \times I \to N$ represent the same elements of $C^\Gamma_{i+1}(M)$. The analogous fact holds for $V, W \in PC^i_\Gamma(M)$).*

Proof. We know $V, W \in PC^\Gamma_i(M)$ represent the same element of $C^\Gamma_i(M)$ if and only if $V \sqcup -W \in Q_*(M)$, so the corollary follows from Lemma 5.5. Similarly for cochains. □

Corollary 5.7. *Suppose $F : M \times I \to N$ is a smooth (co-oriented) homotopy between maps $f, g : M \to N$.*

If $\underline{W} \in C^\Gamma_*(M)$ is a cycle then $\underline{f(W)}$ and $\underline{g(W)}$ are homologous cycles in N.

If $\underline{W} \in C^*_\Gamma(M)$ is a cocycle and F if proper and co-oriented, then $\underline{f(W)}$ and $\underline{g(W)}$ are cohomologous cocycles in N.

Proof. By the preceding corollary we may work with any representative $r_W : W \to M$ of $\underline{W}$. First suppose $W \in PC^\Gamma_*(M)$. By Leibniz rule recalled in Section 3.1 (treating W and I as manifolds over a point), we have $\partial(W \times I) = \partial W \times I \coprod (-1)^w W \times \partial I$. As W is a cycle, $\partial W \in Q_*(M)$, and hence so is $F \circ ((\partial r_W) \times \mathrm{id}_I) : \partial W \times I \to N$ by the preceding lemma. Thus in $C^\Gamma_*(N)$ the boundary of $F \circ (r_W \times \mathrm{id}_I) : W \times I \to N$ is represented up to signs by the restriction of $F \circ (r_W \times \mathrm{id}_I)$ to $W \times \partial I = W \times (\{1\} - \{0\})$. As $F \circ (r_W \times \mathrm{id}_I)|_{W \times 1} = gr_W$ and $F \circ (r_W \times \mathrm{id}_I)|_{W \times 0} = fr_W$, we see that up to the overall sign $(-1)^w$, the boundary of $F \circ (r_W \times \mathrm{id}_I)$ is represented in $C^\Gamma_*(N)$ by the disjoint union of $gr_W : W \to N$ and $fr_W : -W \to N$. Thus $fr_W : W \to N$ and $gr_W : W \to N$ represent homologous cycles.

Next consider the case of W a cocycle. The map $F \circ ((\partial r_W) \times \mathrm{id}_I) : \partial W \times I \to N$ is in $PC^*_\Gamma(N)$ by Lemma 5.5. In this case, again up to a global sign, we have

$$\partial(F \circ (r_W \times \mathrm{id}_I)) = gr_W \amalg -fr_W \amalg H,$$

where H is the homotopy $F \circ (r_W \times \mathrm{id}_I) \circ i_{\partial W \times \mathrm{id}_I} = F \circ (r_{\partial W} \times \mathrm{id}_I) : \partial W \times I \to N$. But $\partial W \in Q^*(M)$, and hence so is this manifold over N by the Lemma 5.5. Thus we have that $fr_W : W \to N$ and $gr_W : W \to N$ represent cohomologous cocycles. □

Proposition 5.8 (Lipyanskiy's Theorem 5)**.** *Given a continuous map $f : M \to N$ of manifolds without boundary*[24]*, it induces a map $f_* : H^\Gamma_*(M) \to H^\Gamma_*(N)$ that depends only on the homotopy class of f. If f is*

[24] Possibly we can still let M and N be manifolds with corners and obtain a true statement, but we simplify our assumption here to avoid treating the question of smooth approximations in that setting.

proper and co-oriented, it also induces $f_*: H^*_\Gamma(M) \to H^*_\Gamma(N)$ that depends only on the proper homotopy class and co-orientation of f.

Proof. First consider the case of homology. Let g be any smooth approximation to f. Then by the preceding proposition g induces a chain map $C^\Gamma_*(M) \to C^\Gamma_*(N)$ and hence a map $H^\Gamma_*(M) \to H^\Gamma_*(N)$. Let h be any other smooth map homotopic to f (and so also homotopic to g). The continuous homotopy from g to h can be smoothly approximated by a smooth homotopy $H : M \times I \to N$ from g to h [**?**, Theorem III.2.5]. The cycles represented by $g(W)$ and $h(W)$ are homologous by Corollary 5.7.

The cohomological case is the same by taking proper smooth approximations and co-orienting the homotopies using Lemma 3.26. □

5.1.2. *Contravariant functoriality of geometric cohomology.* In this section we assign to a continuous map $f : M \to N$ of manifolds without boundary a map $H^*_\Gamma(N) \to H^*_\Gamma(M)$.

First suppose $W \in PC^*_\Gamma(N)$ is represented by a proper co-oriented map $r_W : W \to N$, and let $g : M \to N$ be a *smooth* (not necessarily co-oriented or proper) map such that g is transverse to r_W. Then we define $g^*(W)$ to be the pullback $r_W \times_N g : W \times_N M \to M$, co-oriented as described in Definition 3.35. Note that the pullback is proper by Lemma 3.32. Also, the pullback has dimension $\dim(W) + \dim(M) - \dim(N) = \dim(M) - (\dim(N) - \dim(W)) = \dim(M) - \operatorname{codim}_N(W)$ by [**?**, Theorem 6.4] so that the cohomological index is preserved by the pullback construction.

The following lemma is similar to Lemma 4.22, as are their proofs, though in that context the analog of M was permitted to have corners so that $Q^*(M)$ would not technically be defined. We also here continue to assume that $g : M \to N$ is transverse to the cochain so that the pullback of the cochain has the same codimension as the original cochain. GBF: Might want to try to combine those into a single lemma somewhere at some point, but it looks like it might be less messy, if a bit redundant, not to.

Lemma 5.9. *Suppose N is a manifold without boundary and $r_S : S \to N$ is trivial or has small rank. Let M be a manifold with corners, and let $g : M \to N$ be a smooth map transverse to r_S. Then $S \times_N M \to M$ is also trivial or has small rank, respectively. Consequently, if $S \in Q^k(N)$ then $S \times_N M \to M$ is in $Q^k(M)$.*

Proof. If ρ is a co-orientation reversing diffeomorphism of S over N, then $\rho \times_N \mathrm{id}_M$ is a co-orientation reversing diffeomorphism of $S \times_N M$ over M by Remark 3.36.

Similarly, suppose $r_S : S \to N$ has small rank. Suppose at the point $x \in S$ we have $v \in \ker(D_x r_S)$. The tangent space of the pullback is the pullback of the tangent spaces [**?**, Theorem 5.47], so $T_{(x,y)}(S \times_N M)$ is the pullback $T_{(x,y)}P = T_xS \times_{T_{(r_S(x),g(y))}N} T_yM$, and we are interested in the map from this space under the derivative of the projection $\pi : S \times M \to M$. As $v \in \ker D_x r_S$, we have $(v, 0) \in T_{(x,y)}P$, and this projects to 0 in T_yM. So $(v, 0) \in \ker(D_{(x,y)}\pi)$. This shows that the pullback has small rank.

The final statement is then a consequence of the definitions as in the proof of Lemma 5.1. □

The requirement that $g : M \to N$ must be transverse to W means that even with the preceding lemma we cannot define a chain map $g^* : C^*_\Gamma(N) \to C^*_\Gamma(M)$ because for any fixed g there may be geometric cochains none of whose representatives are transverse to g. Nonetheless, given any continuous $f : M \to N$ of manifolds without boundary we can define a map in geometric cohomology $f^* : H^*_\Gamma(N) \to H^*_\Gamma(M)$ as follows.

Definition 5.10. Suppose $f : M \to N$ is a continuous (not necessarily smooth, co-oriented, or proper) map of manifolds without boundary and $\underline{W} \in H^*_\Gamma(N)$ is represented by $r_W : W \to N$. Let $g : M \to N$ be any smooth approximation of f that is transverse to r_W (which we know exists by Theorem 2.16). Then we define $f^*(\underline{W}) \in H^*_\Gamma(M)$ as the cohomology class represented by $r_W \times_N g : W \times_N M \to M$.

Proposition 5.11. *Suppose $f : M \to N$ is a continuous (not necessarily smooth, co-oriented, or proper) map of manifolds without boundary. The map $f^* : H^k_\Gamma(N) \to H^k_\Gamma(M)$ is well defined and depends only on the homotopy class of f.*

Proof. First, let $r_W : W \to N$ represent a cocycle, and suppose we have two maps $g_0, g_1 : M \to N$ that are smooth, homotopic to f, and transverse to r_W. Let $G : M \times I \to N$ be a homotopy from g_0 to g_1, which exists by Theorem 2.16. We can find such a G that is smooth and transverse to r_W. Consider the composition $W \times_N (M \times I) \to M \times I \to M$ of the pullback r_W by G and the projection $M \times I \to M$. As r_W is proper and co-oriented so is the pullback, and the the projection $M \times I \to M$ has its standard co-orientation $(\beta_M \wedge \beta_I, \beta_M)$ and is proper. Its boundary is the sum of $\partial W \times_N (M \times I) \to M \times I \to M$ and, up to sign,

$$W \times_N (M \times \{1\}) - W \times_N (M \times \{0\}) \to M \times I \to M.$$

As W is assumed to be a cocycle, $\partial W \in Q^*(N)$. So $\partial W \times_N (M \times I) \to M \times I$ is in[25] $Q^*(M \times I)$ by Lemma 5.9, and it is then easy to see that the projection to M preserve triviality and small rank. The other terms correspond to the pullbacks of W via g_0 and g_1. Thus f^* is independent of the choice of g within the homotopy class of f.

Next, as we have noted that f^* is not a fully-defined chain map, we must show that f^* as a cohomology map does not depend on the representative of $\underline{W}$. So suppose $r_{W_0} : W_0 \to N$ and $r_{W_1} : W_1 \to N$ represent the same cohomology class. In this case there will be a $Z \in C^*_\Gamma(N)$ with $\partial Z = W_0 - W_1 + T$ with $T \in Q^*(N)$. We may choose a smooth g homotopic to f that is transverse to Z (and hence also W_0, W_1, T). Then considering $Z \times_N M \to M$ we have, as M is without boundary,

$$\partial(Z \times_N M) = \partial Z \times_N M = W_1 \times_N M - W_2 \times_N M + T \times_N M.$$

By Lemma 5.9 the last term will be in $Q^*(M)$, and so $Z \times_N M$ provides a cohomology between $W_0 \times_N M$ and $W_1 \times_N M$ over M, demonstrating independence of the choice of representative for $\underline{W}$.

Thus f^* does not depend on our choices and depends on f only through its homotopy class. □

Remark 5.12. Analogously, given $f : M \to N$, one could also define a contravariant pullback functor of homology groups $f^* : H^\Gamma_*(N) \to H^\Gamma_*(M)$ if f is proper and M and N are both oriented (so that the pullback can be oriented). The following lemma shows that the pullback of a compact map by a proper map is compact. For the orientation, we observe that given an oriented map $r_V : V \to N$ that represents $\underline{V}$ and a map $g : M \to N$ that is properly homotopic to f and transverse to r_V, the orientations of V and N induce a co-orientation on r_V and so a co-orientation on the pullback $V \times_N M \to M$ by Definition 3.35. This in turn induces an orientation on $V \times_N M$ given the orientation of M.

Lemma 5.13. *Suppose $f : M \to N$ is a proper map of spaces and $r_V : V \to N$ is a map with V compact. Then $V \times_N M$ is compact.*

Proof. As V is compact, so is $r_V(V)$, and as f is proper, $f^{-1}(r_V(V))$ is compact. Now we observe that we must have $V \times_N M \subset V \times f^{-1}(r_V(V)) \subset V \times M$. So $V \times_N M$ is compact. □

5.2. **Mayer-Vietoris sequences.** In this sections we consider Mayer-Vietoris sequences for homology and cohomology. As in the previous section, we present covariant homology and cohomology sequences and a contravariant cohomology sequence.

A key tool in proving the existence of these sequences will be creasing; see Section 4.3. The following definition will be useful.

Definition 5.14. Let U, V be open subsets of a manifold M. We will call a smooth non-constant[26] function $\phi : U \cup V \to [-1, 1]$ a **separating function** for U and V if $\phi(U \setminus V) = -1$ and $\phi(V \setminus U) = 1$.

[25] Note that even if M is without boundary, $M \times I$ will have boundary (unless M is empty). This instance is the main reason we have allowed such constructions as $PC^*(-)$ and $Q^*(-)$ to take manifolds with non-empty boundary as inputs up to this point. In this particular case, since N does not have boundary, all "smooth" maps to N are automatically smooth in the strong sense of [?], as are then $W \times_N (M \times I) \to M \times I$ and $M \times I \to M$ as these are both pullbacks in the smooth category. Thus we obtain an element of $Q^*(M)$ whether one prefers to define $Q^*(-)$ in general using smooth or weakly smooth maps.

[26] The extra condition that ϕ not be constant is required if $U \setminus V$ or $V \setminus U$ is empty, i.e. if $U \subset V$ or $V \subset U$.

Such a function can always be found by an application of Urysohn's Lemma to find a continuous function with this property and then applying the Smooth Approximation Theorem [**?**, Theorem III.2.5].

Suppose now ϕ is a separating function for U and V and that we have a map $r_W : W \to U \cup V$. The subset of $(-1,1)$ of regular values for the composite ϕr_W is dense in $(-1,1)$ by Sard's Theorem. We will say that such a regular value p is a **separating point** for W. In this case we may perform creasing of W along $(\phi r_W)^{-1}(p)$. By composing with an orientation-preserving diffeomorphism of $(-1,1)$ we may always assume $p = 0$ for notational convenience.

5.2.1. *Covariant Mayer-Vietoris sequences and cohomology with support.* For the homology Mayer-Vietoris sequences, the maps $H_*^\Gamma(U) \to H_*^\Gamma(M)$, etc., induced by inclusion will be those defined in the preceding section. For cohomology, however, the map of the preceding section will not work, as the inclusion of an open set into a manifold is not generally proper. Rather, for the covariant cohomology sequence we will need to use the following variant of geometric cohomology supported on open subsets:

Definition 5.15. Let M be a manifold without boundary and $U \subset M$ an open subset. Define $C_\Gamma^*(M|_U) \subset C_\Gamma^*(M)$ to be the subcomplex consisting of elements of $C_\Gamma^*(M)$ that can be represented by $r_W : W \to M$ in $PC_\Gamma^*(M)$ with the image of W contained in U. Let $H_\Gamma^*(M|_U) = H^*(C_\Gamma^*(M|_U))$.

In general, $C_\Gamma^*(M|_U)$ does not agree with the image of $C_\Gamma^*(U)$ in $C_\Gamma^*(M)$ as defined in the preceding section, as the inclusion $U \hookrightarrow M$ is not generally proper.

It is easy to observe that $C_\Gamma^*(M|_U)$ is a chain complex as it is closed under addition and taking boundaries using that the sum $W_1 + W_2$ of two elements of $PC_\Gamma^*(M)$ with image in U can be represented by the disjoint union $W_1 \sqcup W_2$ with image in U, and similarly the boundary of a map with image in U has image in U. In fact, we can reformulate this definition as follows.

Lemma 5.16. *Let $PC_\Gamma^*(M|_U) \subset PC_\Gamma^*(M)$ be the subset consisting of $r_W : W \to M$ with image in U, and let $Q^*(M|_U)$ be the elements of $Q^*(M)$ with image in U. Then $C_\Gamma^*(M|_U)$ is isomorphic to the set of equivalence class of $PC_\Gamma^*(M|_U)$ under the relation $V \sim_U W$ if $V \sqcup -W \in Q^*(M|_U)$.*

Proof. We first observe that $\sim_U$ is an equivalence relation on $PC_\Gamma^*(M|_U)$ by the same proof as Lemma 4.14, assuming all maps have image in U. Clearly if $V, W \in PC_\Gamma^*(M|_U)$ and $V \sim_U W$ then $V \sim W$ in the sense of Lemma 4.14. So letting $\hat{C}_\Gamma^*(M|_U)$ temporarily denote the equivalence classes under $\sim_U$, we have a well-defined map $f : \hat{C}_\Gamma^*(M|_U) \to C_\Gamma^*(M|_U)$. By definition, elements of $C_\Gamma^*(M|_U)$ can be represented by elements of $PC_\Gamma^*(M|_U)$, so f is surjective. Now suppose $W_1, W_2 \in PC_\Gamma^*(M|_U)$ represent the same element of $C_\Gamma^*(M|_U)$. Then $W_1 \sim W_2$ in the sense of Lemma 4.14, i.e. $W_1 \sqcup -W_2 \in Q^*(M)$, but W_1 and W_2 each have support in U, so $W_1 \sqcup -W_2 \in Q^*(M|_U)$ and $W_1 \sim_U W_2$. So f is injective. $\square$

The cohomology groups $H_\Gamma^*(M|_U)$ are functorial in the sense that if $U \subset W \subset M$ are open sets then we have $C_\Gamma^*(M|_U) \subset C_\Gamma^*(M|_W) \subset C_\Gamma^*(M|_M) = C_\Gamma^*(M)$, and these induce maps $H_\Gamma^*(M|_U) \to H_\Gamma^*(M|_W) \to H_\Gamma^*(M)$.

Theorem 5.17. *Let M be a manifold without boundary. For any pair of open sets U and V in M there are Mayer-Vietoris exact sequences Fix tikz arrows?*

$$\cdots \longrightarrow H_k^\Gamma(U \cap V) \longrightarrow H_k^\Gamma(U) \oplus H_k^\Gamma(V) \longrightarrow H_k^\Gamma(U \cup V) \longrightarrow H_{k-1}^\Gamma(U \cap V) \longrightarrow \cdots.$$

$$\cdots \longrightarrow H_\Gamma^k(M|_{U\cap V}) \longrightarrow H_\Gamma^k(M|_U) \oplus H_\Gamma^k(M|_V) \longrightarrow H_\Gamma^k(M|_{U\cup V}) \longrightarrow H_\Gamma^{k+1}(M|_{U\cap V}) \longrightarrow \cdots.$$

Proof. The proof parallels standard proofs of Mayer-Vietoris for singular homology, and is in fact simpler than those relying on barycentric subdivision since "creasing subdivision" only needs to be applied once. The proofs for homology and cohomology are analogous, so we give the cohomological case, which is slightly more exotic.

Let S^* denote the quotient of $C^*_\Gamma(M|_U) \oplus C^*_\Gamma(M|_V)$ by the image of $C^*_\Gamma(M|_{U\cap V})$ under the map $(i_U, -i_V)$, with i_U and i_V being the inclusions. Then we have a short exact sequence

$$0 \longrightarrow C^*_\Gamma(M|_{U\cap V}) \xrightarrow{(i_U,-i_V)} C^*_\Gamma(M|_U) \oplus C^*_\Gamma(M|_V) \longrightarrow S^* \longrightarrow 0. \tag{12}$$

The theorem will follow from showing there is a quasi-isomorphism $\psi : S^* \to C^*_\Gamma(M|_{U\cup V})$. Our quasi-isomorphism will be induced by the map $C^*_\Gamma(M|_U) \oplus C^*_\Gamma(M|_V) \to C^*_\Gamma(M|_{U\cup V})$ that takes $\underline{W}_1 \oplus \underline{W}_2$ to $\underline{W}_1 + \underline{W}_2$. This is well defined as it clearly takes elements in the image of $(i_U, -i_V)$ to 0. To establish the quasi-isomorphism, we use creasing.

First suppose a cocycle $\underline{W} \in C^*_\Gamma(M|_{U\cup V})$ represented by $r_W : W \to U \cup V$. Let $\phi : U \cup V \to (-1, 1)$ be a separating function with 0 a separating point. Then $W^- \in C^*_\Gamma(M|_U)$ and $W^+ \in C^*_\Gamma(M|_V)$, and by Theorem 4.31, we have $\underline{W} = \underline{W^-} + \underline{W^+} \in H^*_\Gamma(M)$. But by the creasing construction if W has image in $U \cup V$ then so does $\mathrm{Cre}(W)$, so also $\underline{W} = \underline{W^-} + \underline{W^+} \in H^\Gamma_*(M|_{U\cup V})$. We have $\partial W \in Q^*(U \cup V)$ by assumption that W is a cocycle. By the computation in Example 3.56, we have $\partial(W^-) = (-W^0) \sqcup (\partial W)^-$, and $\partial(W^+) = W^0 \sqcup (\partial W)^+$. AS $\partial W \in Q^*(M)$ with image in $U \cup V$, we have $(\partial W)^\pm \in Q^*(M)$ with respective images in U and V by Corollary 4.29, i.e. $(\partial W)^- = 0 \in C^*_\Gamma(M|_U)$ and similarly $(\partial W)^+ = 0 \in C^*_\Gamma(M|_V)$. Thus (W^-, W^+) represents an element of $C^*_\Gamma(M|_U) \oplus C^*_\Gamma(M|_V)$ whose boundary is $(-W^0, W^0)$, which is in the image of $(i_U, -i_V)$. Therefore, (W^-, W^+) represents an element of $H^*(S^*)$ that maps to $\underline{W} \in H^*_\Gamma(M_{U\cup V})$. Thus ψ is surjective.

Next, suppose $\underline{W}$ is a cocyle in S^*, represented by $(W_1, W_2) \in PC^*_\Gamma(M|_U) \oplus PC^*_\Gamma(M|_V)$, that maps to zero in $H^*_\Gamma(M|_{U\cup V})$. Thus there is some $Z \in PC^*_\Gamma(M|_{U\cup V})$ such that $\partial Z = W_1 + W_2 + T$ for some $T \in Q^*(M|_{U\cup V})$. Now let us choose a separating function and consider $(Z^- + \mathrm{Cre}(W_1), Z^+ + \mathrm{Cre}(W_2)) \in PC^*_\Gamma(M|_U) \oplus PC^*_\Gamma(M|_V)$; note that as W_1 and W_2 are in the boundary of Z, a separating function for Z can also be used for creasing W_1 and W_2. We compute using again Example 3.56 and Equation (11):

$$\begin{aligned}
\partial(Z^- + \mathrm{Cre}(W_1), Z^+ + \mathrm{Cre}(W_2)) &= (\partial(Z^-) + \partial\mathrm{Cre}(W_1), \partial(Z^+) + \partial\mathrm{Cre}(W_2)) \\
&= (-Z^0 + W_1^- + W_2^- + T^- + W_1 - W_1^- - W_1^+ - \mathrm{Cre}(\partial W_1), \\
&\qquad Z^0 + W_1^+ + W_2^+ + T^+ + W_2 - W_2^- - W_2^+ - \mathrm{Cre}(\partial W_2)) \\
&= (-Z^0 + W_2^- + T^- + W_1 - W_1^+ - \mathrm{Cre}(\partial W_1)), \\
&\qquad Z^0 + W_1^+ + T^+ + W_2 - W_2^- - \mathrm{Cre}(\partial W_2))) \\
&= (W_1, W_2) + (-Z^0, Z^0) + (W_2^- - W_1^+, W_1^+ - W_2^-) + (T^-, T^+) \\
&\qquad - (\mathrm{Cre}(\partial W_1), \mathrm{Cre}(\partial W_2)).
\end{aligned}$$

The second and third terms are in $\mathrm{im}(i_U, -i_V)$, while $(T^-, T^+) \in Q^*(M|_U) \oplus Q^*(M|_V)$ by Corollary 4.29. The last term is obtained by applying the creasing construction to the terms of $(\partial W_1, \partial W_2)$, which by assumption is 0 in S^*. In other words, $(\partial W_1, \partial W_2)$ can be represented as the union of an element of $Q^*(M|_U) \oplus Q^*(M|_V)$ with an element in the image of $(i_U, -i_V)$. But creasing preserves both support and membership in Q^* by definition and by Corollary 4.29, and if $(A, -B) = (i_U, -i_V)(C) = (i_U(C), -i_V(C))$, then $(\mathrm{Cre}(A), \mathrm{Cre}(-B)) = (i_U(\mathrm{Cre}(C)), -i_V(\mathrm{Cre}(C)))$. Altogether, we obtain

$$(W_1, W_2) = \partial(Z^- + \mathrm{Cre}(W_1), Z^+ + \mathrm{Cre}(W_2))$$

in S^*. So ψ is injective. □

Remark 5.18. The connecting homomorphism in this long exact sequence, as defined through this proof, is given by taking a cocycle in $M|_{U\cup V}$, using the creasing construction to write it as a sum of cochains in $C^*_\Gamma(M|_U)$ and $C^*_\Gamma(M|_V)M$, and taking the boundary of the cochain in M_U. In short, it takes a cocycle represented by $W \to U \cup V$ and sends it to $-W^0 \to U \cap V$ determined by an appropriate separating function and separating point.

The description for the homology sequence is identical.

Mayer-Vietoris sequences are the engines of inductive arguments, especially for isomorphisms. To show that in many cases of interest, geometric cohomology coincides with singular cohomology, the induction starts with Euclidean balls.

Proposition 5.19. *Let M be an m-dimensional smooth manifold, and $U \subset M$ an open set that is diffeomorphic to an open ball and whose closure in M is compact. Then $H^n_\Gamma(M|_U)$ is $\mathbb{Z}$ if $n = m$ and is zero otherwise.*

This observation is akin to the dimension axiom or the Poincaré lemma, but with the cohomology concentrated in top degree rather than degree zero; cf. the standard dimension axiom in Example 4.18.

Proof of Proposition 5.19. If $r_W : W \to M$ is proper and has image in U, then W, as the preimage of the closure of U, must be compact. Also, U is orientable, and so by choosing an orientation of U every co-oriented $r_W : W \to U$ determines an orientation of W as explained in Section 3.2. Thus $C^*_\Gamma(M|_U) \cong C^\Gamma_{m-*}(U)$, and the result follows from Example 4.18 (the dimension axiom), homotopy functoriality of geometric homology, and that U is homotopy equivalent to a point. $\square$

5.2.2. *Contravariant Mayer-Vietoris sequence.* In this section we show that geometric cohomology possesses a contravariant Mayer-Vietoris sequence on manifolds. This does not seem possible by deriving a long exact sequence from a short exact sequence. For example, in general restriction maps $C^*_\Gamma(M) \to C^*_\Gamma(U)$ will not be surjective. Consequently, this sequence will take more work than the covariant ones.

Instead, we proceed with a direct analysis at each term of the exact sequence as in Kreck's proof of the Mayer-Vietoris sequence for his cohomology theory using stratifolds in [**?**]. In fact, our argument for exactness parallel's Kreck's fairly closely. However, our proof that the connecting map is well defined is more complicated than the analogous proof in Kreck (ignoring the extra complications in [**?**] arising from considerations of collars that we do not need). This is because Kreck is able to define the splitting functions directly on his representing objects as $\varphi : W \to [0, 1]$, which then makes it possible to compare two different φs using a cylinder $W \times I$ with different versions of φ on each end. Here, however, we must always use splitting maps that factor through M or risk that even if W is trivial $W^\pm$ may not be. This complicates the proof of Proposition 5.21 below.

Definition 5.20. Suppose $U, V \subset M$ are open subsets and $\underline{W} \in H^k_\Gamma(U \cap V)$ represented by $r_W : W \to U \cap V$. Of course we can also consider r_W to have image in $U \cup V$. Let $\phi : M \to [-1, 1]$ be a separating function for U and V such that 0 is a separating point for W. Define $\delta(\underline{W}) \in H^{k+1}_\Gamma(U \cup V)$ to be represented by $-W^0 = -\phi^{-1}(0) \times_{U \cup V} W$.

The choice to use $-W^0$ rather than W^0 in the definition is explained by Remark 5.18. An alternative convention that uses W^0 rather than $-W^0$ would be to use $(-i_U, i_V)$ rather than $(i_U, -i_V)$ in our covariant Mayer-Vietoris sequences and the map $\underline{W} \to (-\underline{W}|_U, \underline{W}|_V)$ rather than $\underline{W} \to (\underline{W}|_U, -\underline{W}|_V)$ in our contravariant Mayer-Vietoris sequence below.

Proposition 5.21. *The map $H^k_\Gamma(U \cap V) \xrightarrow{\delta} H^{k+1}_\Gamma(U \cup V)$ is well defined.*

Proof. Note that the $\phi^{-1}(0) \to U \cup V$ is a closed inclusion and so proper. Thus $r_{W^0} : W^0 \to U \cup V$ is proper by Lemma 3.32 as the composition of $r^*_W : \phi^{-1}(0) \times_{U \cup V} W \to \phi^{-1}(0)$ and $\phi^{-1}(0) \hookrightarrow U \cup V$.

We next fix ϕ and show independence of the choices of separating point and representative for $\underline{W}$.

Suppose p, p' are separating points for W. Without lose of generality, we assume $p < p'$. Then $\phi^{-1}([p, p'])$ will be transverse to $r_W : W \to M$, and we have $\phi^{-1}([p, p']) \subset U \cap V$. As $\phi^{-1}([p, p'])$ is closed in $U \cup V$, its inclusion is proper, and the inclusion can be tautologically co-oriented as a codimension 0 embedding (see Definition 3.12). Thus

$$\partial(\varphi^{-1}([p, p']) \times_{U \cup V} W = (\varphi^{-1}(p') \times_{U \cup V} W) - (\varphi^{-1}(p) \times_{U \cup V} W) + \varphi^{-1}([p, p']) \times_{U \cup V} \partial W.$$

As W is a cocycle, the last term is in $Q^*(U\cup V)$ (with support in $U\cap V$) by Lemma 4.22, so $\varphi^{-1}(p')\times_{U\cup V} W$ and $\varphi^{-1}(p)\times_{U\cup V} W$ are cohomologous. This shows independence of choice of separating point.

Next suppose W_1 and W_2 cohomologous in $U\cap V$, so there is a Z in $U\cap V$ with $\partial Z = W_1 - W_2 + T$ and $T\in Q^*(U\cap V)$. Let p be a separating point for all of Z, W_1, W_2, T. Then (continuing to write, e.g., W^0 even when the separating point is p), by Corollary 3.58

$$\partial(Z^0) = -(\partial Z)^0 = -W_1^0 + W_2^0 - T^0,$$

with the last term in $Q^*(U\cup V)$, and so W_1^0 and W_2^0 are cohomologous. This establishes independence of the representing object for $\underline{W}$.

Finally, we must show independence of the choice of ϕ. This is clear in the case that we vary ϕ by postcomposing with an orientation-preserving diffeomorphism ψ of $[-1,1]$ and replace the splitting point p by $\psi(p)$. Suppose more generally we have two separating functions $\phi_1,\phi_2 : U\cup V\to[-1,1]$ and corresponding p_1,p_2 that are separating for W.

Next suppose there exists q with $-1<q<p_2$ such that $\phi_1^{-1}(p_1)\subset\phi_2^{-1}([-1,q])$. By postcomposing ϕ_1 with a diffeomorphism of $[-1,1]$, we may suppose that $p_1<q$. Let us choose u_1,u_2 such that $-1<u_1<p_1<u_1<q$. Using the Urysohn lemma we can find a continuous $\hat{\Phi}$ on $U\cup V$ such that:

(1) $\hat{\Phi}$ is equal to ϕ_1 on $\phi_1^{-1}[u_1,u_2]$,
(2) $\hat{\Phi}$ is equal to ϕ_2 on $\phi_2^{-1}([q,1])$,
(3) $\hat{\Phi}$ takes $U\setminus V$ to -1,
(4) $\hat{\Phi}^{-1}(p_1)=\phi_1^{-1}(p_1)$,
(5) $\hat{\Phi}^{-1}(p_2)=\phi_2^{-1}(p_2)$.

Furthermore, by a sufficiently small homotopy $\hat{\Phi}$ can be approximated by a smooth function Φ that preserves the last two properties and is still separating. Thus by using Φ and applying the previous case for a single separating function but two different separating points, we see that splitting at $\phi_1^{-1}(p_1)$ and $\phi_2^{-1}(p_2)$ produce the same image of δ. While this argument is written for $\phi_1^{-1}(p_1)$ "below some $q<p_2$," clearly an analogous argument holds for $\phi_1^{-1}(p_1)$ "above some $q>p_2$," or with the roles of ϕ_1 and ϕ_2 reversed.

Lastly, we have to consider the case where there is not a q as in the preceding paragraph that allows us to separate $\phi_1^{-1}(p_1)$ from $\phi_2^{-1}(p_2)$. In this case we will modify ϕ_2 in an appropriate way. Let use choose r with $p_2<r<1$. Define $\hat{\Phi}$ as follows:

(1) On $\phi_2^{-1}([-1,r])$, take $\hat{\Phi}=\phi_2$,
(2) On $\phi_2^{-1}([r,1])$ use the Urysohn Lemma to construct a continuous function $\phi_2^{-1}([r,1])\to[r,1]$ that takes $\phi_2^{-1}(r)\cup[\phi_2^{-1}([r,1])\cap\phi_1^{-1}(p_1)]$ to r and $V\setminus U$ to 1; let $\hat{\Phi}$ be this constructed function on $\phi_2^{-1}([r,1]$.

Now, we can modify $\hat{\Phi}$ by an ϵ-homotopy fixing $\phi_2^{-1}([-1,r])$ and $\Phi^{-1}(1)$ to a smooth Φ with $\Phi^{-1}(p_2)=\phi_2^{-1}(p_2)$ and such that there is a q, $r<q<1$, with $\phi_1^{-1}(p_1)\subset\Phi^{-1}([-1,q])$. We can now choose a p_2' with $q<p_2'<1$ that separates W via Φ. From our previous cases, we know $\Phi^{-1}(p_2')\times_{U\cup V} W$ is cohomologous to $\Phi^{-1}(p_2)\times_{U\cup V} W=\phi_2^{-1}(p_2)\times_{U\cup V} W$, but $\Phi^{-1}(p_2')$ and $\phi_1^{-1}(p_1)$ are now related as in the preceding case.

Altogether we have shown that δ is independent of choices. □

Lemma 5.22. *The connecting map δ is natural, i.e. for a continuous map of triples $f:(M;U',V')\to(N;U,V)$ the following diagram commutes for all k:*

$$\begin{tikzcd} H^k_\Gamma(U\cap V) \arrow[r, "\delta"] \arrow[d, "f^*"'] & H^{k+1}_\Gamma(U\cup V) \arrow[d, "f^*"'] \\ H^k_\Gamma(U'\cap V') \arrow[r, "\delta"] & H^{k+1}_\Gamma(U'\cup V') \end{tikzcd}$$

Proof. Consider $\underline{W} \in H^k_\Gamma(U\cap V)$ represented by $r_W : W \to U\cap V$. Let $\phi : U\cup V \to [-1,1]$ be a separating function for U, V. We may assume that $0 \in [-1,1]$ is a separating point for W, by postcomposing ϕ with a diffeomorphism of $[-1,1]$ if necessary.

Next we note that ϕf separates U' and V' but it may not be smooth. This can be remedied by a small homotopy of f to g. This homotopy may not preserve the property that $f(U') \subset U$ and $f(V') \subset V$ but we may choose the homotopy small enough that $(\phi g)^{-1}([-1/3,1/3]) \subset (\phi f)^{-1}(-1/2,1/2)$, which will be sufficient to ensure that $(\phi g)^{-1}([-1/3,1/3]) \subset U'\cap V'$. We may further assume that the homotopy has been chosen so that g is transverse to $r_{W^0} : W^0 \to N$.

We claim that the transversality of g with r_{W^0} implies that 0 is a regular value for the composition $\phi g r^*_W$, where r^*_W is the pullback of W to M via g, i.e. $r^*_W : W\times_N M \to M$. To see this, recall that the tangent bundle of the pullback is the pullback of the tangent bundles [**?**, Theorem 5.47], so at a point (x,y) of $(\phi g r^*_W)^{-1}(0)$ with $x\in W$, $y \in M$, the tangent space of $W\times_N M$ is $T_xW \times_{T_{(r_W(x),g(y))}} T_yM$. As 0 is a regular value for ϕr_W, there must be a vector $\xi \in T_xW$ with $D_x(\phi r_W)(\xi) \neq 0$. Consider $D_x r_W(\xi)$, which must also be nonzero. As g is assumed transverse to W^0, there must be $\alpha \in T_xW^0$ and $\beta \in T_yM$ such that $(D_xr_W)(\xi) = (D_xr_W)(\alpha) + (D_yg)(\beta)$. Rewriting, $D_yg(\beta) = D_xr_W(\xi-\alpha)$. As $\alpha \in T_xW^0$, we have

$$D_{r_W(x)}\phi(D_yg(\beta)) = D_{r_W(x)}\phi \circ D_xr_W(\xi-\alpha) = D_x(\phi r_W)(\xi) - D_x(\phi r_W)(\alpha) = D_x(\phi r_W)(\xi) \neq 0.$$

So, recalling that Dr^*_W is simply the projection to TM, the pair $(\xi-\alpha, \beta)$ is a non-zero vector in $T_xW \times_{T_{(r_W(x),g(y))}} T_yM$ that maps by $D(\phi g r^*_W)$ to a non-zero vector in $T_0[-1,1]$. As (x,y) was an arbitrary point of $(\phi g r^*_W)^{-1}(0) \subset W\times_N M$, this shows that 0 is a regular value for $\phi g r^*_W$.

Now we have from the definitions that $f^*\delta(W)$ is represented by $(-W^0)\times_N M = -(N^0\times_N W)\times_N M = -(\phi^{-1}(0)\times_N W)\times_N M$, where $(\phi^{-1}(0)\times_N W)$ is a fiber product over M and the whole expression is the pullback of this fiber product to N via $g : M \to N$. On the other hand, $\delta f^*(W) = -(W\times_N M)^0 = -M^0 \times_M (W\times_N M) = -(\phi g)^{-1}(0)\times_M (W\times_N M)$. Here $W\times_N M$ is the pullback to M and then we take the fiber product with $(\phi g)^{-1}(0)$. In both cases these correspond to pairs of points $(x,y) \in W\times M$ with $r_W(x) = g(y)$ and $\phi(r_W(x)) = \phi(g(y)) = 0$. In other words, as spaces these are both precisely the limit of the following diagram together with its map to M:

$$\begin{tikzcd} & M \arrow[d, "g"'] & & \\ W \arrow[r, "r_W"] & N \arrow[r, "\phi"] & {[-1,1]} & 0. \arrow[l, hook'] \end{tikzcd}$$

Thus $f^*\delta(\underline{W})$ and $\delta f^*(\underline{W})$ are represented by the same map, and we only need to check co-orientations.

We return to the definitions of the pullback and fiber product co-orientations. It suffices to compare $(W\times_N M)^0$ and $W^0\times_N M$ at an arbitrary point. We first consider the co-orientation in the form $(W\times_N M)^0$. For the pullback $W\times_N M \to M$, choose $e : W \hookrightarrow N\times\mathbb{R}^K$. At our chosen point, fix β_N and choose β_W so that $\omega_{r_W} = (\beta_W, \beta_N)$. Let ν be the Quillen-oriented normal bundle of W in $N\times\mathbb{R}^K$ so that $\beta_W \wedge \beta_\nu = \beta_N \wedge \beta_E$, where β_E is the standard orientation of $\mathbb{R}^K$. Then by definition the pullback map from $P = W \times_N M \subset M \times \mathbb{R}^K$ to M is co-oriented by (β_P, β_M) if we choose β_P and β_M such that $\beta_P \wedge \beta_\nu = \beta_M \times \beta_E$, recalling that we let β_ν also denote the pulled back normal bundle of P in $M \times \mathbb{R}^K$. We suppose we have chosen such β_P and β_M. Next, let $M^0 \subset M$ have normal co-orientation β_ϕ determined by pulling back the standard orientation from $(-1,1)$, and similarly let β_ϕ denote the pullback normal co-orientation to $(W \times_N M)^0$. Then by Proposition 3.55, the co-orientation of the pullback $(W \times_N M)^0 = M^0 \times_M (W \times_N M) \to W \times_N M$ is $(\beta_Q, \beta_Q \wedge \beta_\phi)$ for any β_Q. The fiber product $(W \times_N M)^0 \hookrightarrow M$ is then co-oriented by the composition $(\beta_Q, \beta_Q \wedge \beta_\phi) * (\beta_P, \beta_M)$. So if we choose β_Q so that $\beta_Q \wedge \beta_\phi = \beta_P$ (or equivalently β_Q and β_M so that $\beta_Q \wedge \beta_\phi \wedge \beta_\nu = \beta_M \wedge \beta_E$), the co-orientation is (β_Q, β_M).

On the other hand, consider $(N^0 \times_N W) \times_N M = W^0 \times_N M \to M$. One again we fix β_N and β_W so that $\omega_{r_W} = (\beta_W, \beta_N)$. Again by Proposition 3.55, the co-orientation of the pullback $N^0 \times_N W \to W$ is $(\beta_{W^0}, \beta_{W^0} \wedge \beta_\phi)$ for any β_{W^0}, continuing to let β_ϕ denote any normal co-orientation pulled back via ϕ. If we choose β_{W^0} so that $\beta_{W^0} \wedge \beta_\phi = \beta_W$ then we have $r_{W^0} : W^0 \to N$ co-oriented by (β_{W^0}, β_N). As $W^0 \subset W$, we can embed W^0 in $N \times \mathbb{R}^K$ via the composition $W^0 \hookrightarrow W \overset{e}{\hookrightarrow} N \times \mathbb{R}^K$, using the same e and K as above. As $\beta_W \wedge \beta_\nu = \beta_N \wedge \beta_E$ and $\beta_W = \beta_{W^0} \wedge \beta_\phi$, we have $\beta_{W^0} \wedge \beta_\phi \wedge \beta_\nu = \beta_N \wedge \beta_E$ so that $\beta_\phi \wedge \beta_\nu$ is the Quillen orientation for the normal bundle of W^0 in $N \times \mathbb{R}^K$. Using this to pull back $W^0 \to N$ to $W^0 \times_N M \to M$, by definition the pullback co-orientation is (β_Q, β_M) when β_Q and β_M are chosen so that $\beta_Q \wedge \beta_\phi \wedge \beta_\nu = \beta_M \wedge \beta_E$. But this is exactly the same co-orientation we arrived at in the preceding paragraph. Thus the co-orientations of the two constructions agree.

We conclude that the diagram of the lemma commutes. ∎

Notation. If $f : U \hookrightarrow M$ is the inclusion of an open subset and $\underline{W} \in C^*_\Gamma(M)$, we may also write $f^*(\underline{W})$ as $\underline{W}|_U$. As such an inclusion U is necessarily transverse to any other map to M, we also obtain a well-defined map $PC^*_\Gamma(M) \to PC^*_\Gamma(U)$ that we also write $W \to W|_U$. By Proposition 3.52, $W|_U$ is simply the restriction of $r_W : W \to M$ to $r_W^{-1}(U)$.

Theorem 5.23. *Let $U, V \subset M$ be open subsets. There is a long exact Mayer-Vietoris sequence*

$$\cdots \to H^k_\Gamma(U \cup V) \xrightarrow{i} H^k_\Gamma(U) \oplus H^k_\Gamma(V) \xrightarrow{j} H^k_\Gamma(U \cap V) \xrightarrow{\delta} H^{k+1}_\Gamma(U \cup V) \to \cdots$$

with $i(\underline{W}) = (\underline{W}|_U, -\underline{W}|_V)$, $j(\underline{W}_1, \underline{W}_2) = \underline{W}_1|_{U\cap V} + \underline{W}_2|_{U\cap V}$, and δ as above.

Proof. Notationally, we always assume $\underline{W}$ is represented by W, etc. In the following we typically assume an appropriate separating function $\phi : U \cup V \to [-1,1]$ such that $\phi(U \setminus V) = -1$ and $\phi(V \setminus U) = 1$ and choose an appropriate separating point without further comment. We observe that in this case $(U \cap V)^+ = \phi^{-1}([0,1]) \cap (U \cap V)$ is a closed subspace of both $U \cap V$ and of U, as $\phi|_U^{-1}([0,1]) = \phi^{-1}([0,1]) \cap (U \cap V)$ given that $\phi(U \setminus V) = -1$. So the inclusions $(U \cap V)^+ \hookrightarrow U \cap V$ and $(U \cap V)^+ \hookrightarrow U$ are both proper maps. Thus if $W \in PC^*_\Gamma(U \cap V)$, in which case in particular $r_W : W \to U \cap V$ is proper, both the fiber products $(U \cap V)^+ \times_{U \cap V} W \to U \cap V$ and $(U \cap V)^+ \times_U W \to U$ will be proper maps. In fact, they both have the same domain, which is just W^+, and we will write W^+ for the corresponding elements of $PC^*_\Gamma(U \cap V)$ or $PC^*_\Gamma(U)$, determining which via context. Similarly, we obtain W^- in $PC^*_\Gamma(U \cap V)$ or $PC^*_\Gamma(V)$. See Figure 3. On the other hand, $(U \cap V)^- = \phi^{-1}([-1,0]) \cap (U \cap V)$ is not generally closed in U and so we do not obtain a W^- in $PC^*_\Gamma(U)$.

Similarly, continuing with our assumptions about $\phi : U \cup V \to [-1,1]$, we have $\phi^{-1}([-1,0]) = \phi|_U^{-1}([-1,0])$ so that $\phi^{-1}([-1,0])$ is a closed subset of U and of $U \cup V$. Therefore, the inclusions $\phi^{-1}([-1,0]) \hookrightarrow U$ and $\phi^{-1}([-1,0]) \hookrightarrow U \cup V$ are proper so that if $W \in PC^*_\Gamma(U)$, then W^- is well-defined in both $PC^*_\Gamma(U)$ and $PC^*_\Gamma(U \cup V)$. See Figure 4. Analogously, if $W \in PC^*_\Gamma(V)$ then we obtain W^+ in both $PC^*_\Gamma(V)$ and $PC^*_\Gamma(V^+)$.

Finally, we also note that $\phi^{-1}(0)$ is closed in $U \cap V$, U, V, and $U \cup V$, so for any W in $PC^*_\Gamma(-)$ for any of these spaces, W^0 is also in $PC^*_\Gamma(-)$ for all these spaces.

These observations will be used freely in the remainder of the proof.

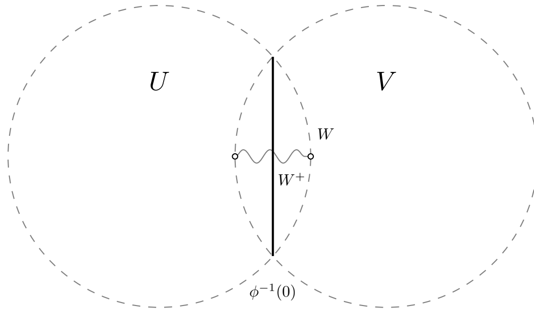


FIGURE 3. For a proper map $r_W : W \to U \cap V$, the restriction to W^+ is proper into U, while the restriction to W^- is proper into V.

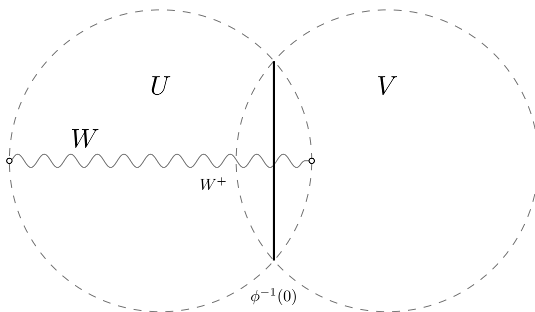


FIGURE 4. For a proper map $r_W : W \to U$, the restriction to W^- is proper into $U \cup V$.

Exactness at $H^k_\Gamma(U \cup V)$. Let $\underline{W} \in H^{k-1}(U \cap V)$. Then $i\delta(\underline{W})$ is represented by $(-W^0, W^0)$. By the above discussion, we have $W^+ \in PC^*_\Gamma(U)$, and by applying the same discussion to $\partial(W^+)$ we also have in U that $\partial(W^+) = W^0 + (\partial W)^+$ via Example 3.56. But $(\partial W)^+ \in Q^*(U)$ by Corollary 4.29 as W is a cocycle. Thus W^0 represents $0 \in H^k_\Gamma(U)$. Similarly, W^0 represents $0 \in H^k_\Gamma(V)$ using W^-. So $i\delta = 0$.

Next suppose $\underline{W} \in H^k_\Gamma(U \cup V)$ with $i(\underline{W}) = 0$. Representing $\underline{W}$ by W, this means that $W|_U$ and $W|_V$ each bound, in U and V respectively. This means there exist $A \in PC^*_\Gamma(U)$ and $S \in Q^*(U)$ with $\partial A = W|_U + S$ and $B \in PC^*_\Gamma(V)$ and $T \in Q^*(V)$ with $\partial B = W|_V + T$. We choose a common separating point for A and B and consider A^- and B^+, which are both well defined in $U \cup V$ by the discussion above. We compute

$$\begin{aligned}\partial(A^- + B^+) &= -A^0 + (\partial A)^- + B^0 + (\partial B)^+ \\ &= -A^0 + (W|_U)^- + S^- + B^0 + (W|_V)^+ + T^+.\end{aligned}$$

But $(W|_U)^- + (W|_V)^+ = W^- + W^+$, which through the creasing construction is cohomologous to W. Further, $S^- + T^+ \in Q^*(U \cup V)$, so we see that W is cohomologous to $A^0 - B^0$. But $A^0 - B^0 = \delta(B|_{U\cap V} - A|_{U\cap V})$.

Exactness at $H^k_\Gamma(U) \oplus H^k_\Gamma(V)$. It is immediate that the composition $ji = 0$.

Now suppose $(\underline{W}_1, \underline{W}_2) \in \ker j \subset H^k_\Gamma(U) \oplus H^k_\Gamma(V)$. Using representatives W_1, W_2, this means that there is a $Z \in C^k_\Gamma(U \cap V)$ with $\partial Z = W_1|_{U\cap V} + W_2|_{U\cap V} + T$, with $T \in Q^*(U \cap V)$. Choose a separating point for all Z and hence automatically W_1, W_2, and T. We claim that $\gamma = W_1^- - Z^0 - W_2^+$ represents an element of $H^k_\Gamma(U \cup V)$ whose image under i is $(\underline{W}_1, \underline{W}_2)$. We compute

$$\begin{aligned}\partial\gamma &= \partial(W_1^- - Z^0 - W_2^+)\\ &= -W_1^0 + (\partial W_1)^- + (\partial Z)^0 - W_2^0 - (\partial W_2)^+\\ &= -W_1^0 + (\partial W_1)^- + W_1^0 + W_2^0 + T^0 - W_2^0 - (\partial W_2)^+\\ &= (\partial W_1)^- + T^0 - (\partial W_2)^+.\end{aligned}$$

As W_1 and W_2 are cycles, $(\partial W_1)^-$, $(\partial W_2)^+$, and T^0 are in $Q^*(U \cup V)$. So this boundary is 0 in $C^*_\Gamma(U \cup V)$, and γ represents an element of $H^k_\Gamma(U \cup V)$.

Next we show that $\gamma|_U$ is cohomologous to W_1 in U. In fact, in U we have

$$\partial(Z^+) = Z^0 + (\partial Z)^+ = Z^0 + W_1^+ + W_2^+|_U + T^+.$$

So $\partial(Z^+) + \gamma|_U = W_1^- + W_1^+ + T^+$, and we see that $\gamma|_U$ is cohomologous in U to $W_1^- + W_1^+$, which is cohomologous to W_1 in U via creasing. Similarly, in V we have

$$\partial(Z^-) = -Z^0 + (\partial Z)^- = -Z^0 + W_1^-|_V + W_2^- + T^-.$$

So $\partial(Z^-) - \gamma|_V = W_2^- + W_2^+ + T^-$, and $-\gamma|_V$ is cohomologous to $W_2^- + W_2^+$, which is cohomologous to W_2 in V.

So $(\underline{W}_1, \underline{W}_2) = i(\underline{\gamma})$.

Exactness at $H^k_\Gamma(U \cap V)$. Consider a cocycle W in $C^k_\Gamma(U)$. Then $\delta j(\underline{W})$ is represented by $-W^0$. By the discussion above, $W^- \in PC^*_\Gamma(U \cup V)$, and

$$\partial(W^-) = -W^0 + (\partial W)^-.$$

As W is a cocycle, $\partial W \in Q^*(U)$, and so $(\partial W)^- \in Q^*(U \cup V)$ via Corollary 4.29. So $\delta j(\underline{W}) = 0 \in H^*_\Gamma(U \cup V)$, and similarly for elements of $H^k_\Gamma(V)$.

Now suppose $\underline{W} \in H^k_\Gamma(U \cap V)$ and $\delta(\underline{W}) = 0$. Representing $\underline{W}$ by W and choosing a separating function and separating point, this means there is a Z in $U \cup V$ such that $\partial Z = -W^0 + T$ with $T \in Q^*(U \cup V)$. Let $A = Z|_U + W^+ \in PC^*_\Gamma(U)$ and $B = -Z|_V + W^- \in PC^*_\Gamma(V)$. Then

$$\begin{aligned}\partial A &= \partial Z|_U + \partial(W^+) = -W^0 + T|_U + W^0 + (\partial W)^+ = T|_U + (\partial W)^+\\ \partial B &= -\partial Z|_V + \partial(W^-) = W^0 - T|_V - W^0 + (\partial W)^- = -T|_V + (\partial W)^-.\end{aligned}$$

As $\partial W \in Q^*(U \cap V)$ and $T \in Q^*(U \cup V)$, their pullbacks and restrictions are also in the appropriate Q^*s, so (A, B) represents an element of $H^k_\Gamma(U) \oplus H^k_\Gamma(V)$.

We then have

$$\begin{aligned}j(A, B) &= A|_{U\cap V} + B|_{U\cap B}\\ &= Z|_{U\cap V} + W^+ - Z|_{U\cap V} + W^-\\ &= W^+ + W^-,\end{aligned}$$

which represents $\underline{W} \in H^k(U \cap V)$ via creasing. □

5.3. **Geometric homology and cohomology are singular homology and cohomology.** In this section we apply a theorem of Kreck and Singhof to show that geometric homology and cohomology are isomorphic to singular homology and cohomology on smooth manifolds.

Theorem 5.24. *On the category of smooth manifolds (without boundary) and continuous maps, geometric homology and cohomology are respectively isomorphic to singular homology and cohomology with integer coefficients, i.e.* $H^\Gamma_* \cong H_*(\cdot; \mathbb{Z})$ *and* $H^*_\Gamma \cong H^*(\cdot; \mathbb{Z})$ *as functors.*

Proof. This is a consequence of [**?**, Theorem 10] once we verify that H_*^Γ and H_Γ^* are respectively an ordinary homology theory and an ordinary cohomology theory on the category of smooth manifolds as defined in [**?**]. This requires the following axioms:

(1) H_*^Γ is a covariant functor on the category of smooth manifolds (without boundary) and continuous maps between them, and H_Γ^* is a contravariant homotopy functor on the same category.
(2) For each triple $(M;U,V)$ with M a smooth manifold and U,V open subsets such that $U\cup V=M$ there are exact (homological or cohomological) Mayer-Vietoris sequences with natural connecting maps δ.
(3) For all M, $H_k^\Gamma(M)=H_\Gamma^k(M)=0$ for $k<0$.
(4) The Dimension Axiom: $H_k^\Gamma(pt)=H_\Gamma^k(pt)=0$ for $k\neq 0$ and $H_0^\Gamma(pt)\cong H_\Gamma^0(pt)\cong\mathbb{Z}$.
(5) H_*^Γ and H_Γ^* are additive: for a manifold M of dimension 0, each $H_k^\Gamma(M)$ is canonically isomorphic to $\oplus_{x\in M}H_k^\Gamma(x)$ and each $H_\Gamma^k(M)$ is canonically isomorphic to $\prod_{x\in M}H_\Gamma^k(x)$.

Axiom 1 holds from the definitions and Propositions 5.8 and 5.11. We have Mayer-Vietoris sequences by Theorems 5.17 and 5.23. The connecting map for the cohomology sequence is natural by Lemma 5.22. The connecting map for the homology sequence is natural just as in the standard argument for singular homology: given a map of triples $(M;U',V')\to(N;U,V)$ there is a map of Mayer-Vietoris sequences induced by a map of short exactly sequences of chain complexes of the form of diagram (12) (replacing supported cochains complexes with chain complexes), itself induced by functoriality from the maps of chain complexes $C_*^\Gamma(U')\to C_*^\Gamma(U)$ and similarly for V and $U\cap V$. This map of Mayer-Vietoris sequences in particular shows that the connecting map is natural.

Axiom 3 holds trivially for homology as there are no chains of degree <0. It also holds for cohomology because for $k<0$ any representing cocycle must have small rank and boundary in $Q^*(M)$. Thus any such cocycle must be $0\in C_\Gamma^k(M)$.

The Dimension Axiom has been proven in Example 4.18.

The Additivity Axiom is apparent. ☐

Example 5.25. Let us consider $H_\Gamma^*(\mathbb{R}P^2)$. We know from the standard computations of $H^*(\mathbb{R}P^2)$ and the above theorem that

$$H_\Gamma^i(\mathbb{R}P^2)=\begin{cases}\mathbb{Z}_2, & i=2,\\ \mathbb{Z}, & i=0,\\ 0, & \text{otherwise.}\end{cases}$$

From the viewpoint of geometric cohomology, $H^0(\mathbb{R}P^2)$ is generated by the identity map $\mathbb{R}P^2\to\mathbb{R}P^2$, which is co-oriented at each point by (β,β) for any local orientation β.

In degree 1, cochains are represented by co-oriented maps from (unions of) closed intervals or circles. For a map from the circle to be co-oriented it cannot represent the non-trivial element $\alpha\in\pi_1(\mathbb{R}P^2)$, and so it must be contractible. Any map from a disk must be co-orientable, as the disk is contractible (this can be considered an extension of Lemma 3.26). Thus by smooth approximation, any smooth co-oriented map from the circle to $\mathbb{R}P^2$ is the boundary of a smooth co-oriented map from the disk, and so represents 0 in cohomology. The same is true for any "circle of intervals" that does not represent $\alpha\in\pi_2(\mathbb{R}P^2)$; via homotopy and creasing, such a "circle" is the boundary of a polygon. On the other hand, consider a map $g:\mathbb{I}\to\mathbb{R}P^2$ that does represent α. As $\mathbb{I}$ is contractible, such a map can always be co-oriented. We can let e represent the standard unit vector of $\mathbb{I}$ and suppose that g is co-oriented so that at $0\in\mathbb{I}$ the co-orientation is represented by (e,β) for some local orientation β of $\mathbb{R}P^2$ at $f(0)$. Traversing the path, the representation for the co-orientation at 1 is then $(e,-\beta)$. Recalling our boundary conventions, the boundary of f therefore consists of the maps $f|_0:0\to\mathbb{R}P^2$ co-oriented by $(1,e)*(e,\beta)=(e,\beta)$ and $f|_1:1\to\mathbb{R}P^2$ co-oriented by $(1,-e)*(e,-\beta)=(e,\beta)$. Thus with these co-orientations, $f|_0:0\to\mathbb{R}P^2$ and $f|_1:1\to\mathbb{R}P^2$ represent isomorphic manifolds over $\mathbb{R}P^2$. As a co-oriented map from any single point to $\mathbb{R}P^2$ is a cocycle, we see by choosing the image to be the basepoint for $\pi_1(\mathbb{R}P^2)$ that twice any such map is a boundary. We leave it to the reader to verify that any two such points generate the same cohomology class, and so we have verified that $H^1_\Gamma(\mathbb{R}P^2) = 0$ and $H^2_\Gamma(\mathbb{R}P^2) \cong \mathbb{Z}_2$.

5.3.1. *A more direct comparison of singular and geometric homology.* While it was convenient to cite simultaneously the homology and cohomology versions of the Kreck-Singhof theorem, we can provide a second proof in the homology case that says a bit more. In particular, as simplices are manifolds with corners and as the standard model simplex comes equipped with an orientation, if we let $S^{sm}_*(M)$ denote the chain complex of smooth singular chains, there is an obvious inclusion map $S^{sm}_*(M) \hookrightarrow PC^\Gamma_*(M)$. It is not hard to check consistency of the boundary orientations so that this induces a chain map $S^{sm}_*(M) \to C^\Gamma_*(M)$. We will see that this is a quasi-isomorphism. Combined with the well-known fact that the inclusion $S^{sm}_*(M) \hookrightarrow S_*(M)$ is a chain homotopy equivalence if $S_*(M)$ is the full singular chain complex on M [**?**, Theorem 18.7], this provides a concrete chain of isomorphisms $H_*(M) \cong H^\Gamma_*(M)$.

It will also be useful below to have the analogous result for cubical singular homology in addition to the more common simplicial singular homology. Details of cubical singular homology can be found, for example, in (see, e.g., [**?**] or [**?**, Section 8.3]). Rather than maps of simplices $\Delta^k \to M$, the cubical singular chain complex $SK_*(M)$ is generated by maps $\mathbb{I}^k \to M$ with $\mathbb{I}$ being the standard interval $\mathbb{I} = [0,1]$. The boundary formula is defined so that if $\sigma : \mathbb{I}^k \to M$ is a singular cube, then

$$\partial\sigma = \sum_{i=1}^{k} (-1)^i (\sigma\delta_i^0 - \sigma\delta_i^1), \tag{13}$$

where for $\epsilon \in \{0,1\}$, the map $\delta_i^\epsilon : \mathbb{I}^{k-1} \to \mathbb{I}^k$ is defined by

$$\delta_i^\epsilon(x_1, \dots, x_k) = (x_1, \dots, \epsilon, \dots, x_k)$$

with ϵ in the ith slot. The homology of the chain complex $SK_*(M)$ is not isomorphic to singular homology as it does not satisfy the dimension axiom, so one instead forms the normalized complex $NK_*(M)$ by quotienting out the subcomplex of degenerate singular cubes, generated by singular cubes $\sigma : \mathbb{I}^k \to M$ such that σ does not depend on at least one of the variables. In other words, the degenerate singular cubes are those maps $\sigma : \mathbb{I}^k \to M$ that factor through one of the standard projections $\mathbb{I}^k \to \mathbb{I}^{k-1}$. It then holds that $NK_*(M)$ is chain homotopy equivalent to $S_*(M)$. In fact this holds for M any space and not just a manifold [**?**, Theorem 8.4.7]. Of course we will need the smooth version $NK^{sm}_*(M)$ generated by smooth singular cubes and modulo degenerate smooth singular cubes. We defer to an appendix to this section the proof that $NK^{sm}_*(M) \hookrightarrow NK_*(M)$ is a chain homotopy equivalence.

As the cubes $\mathbb{I}^k$ are compact manifolds with corners equipped with their standard orientations, we have inclusion $SK^{sm}_i(M) \hookrightarrow PC^\Gamma_i(M)$ for all i.

Lemma 5.26. *The inclusion $SK^{sm}_*(M) \hookrightarrow PC^\Gamma_*(M)$ determines a chain map $NK^{sm}_*(M) \to C^\Gamma_*(M)$.*

Proof. Any degenerate singular cube $\sigma : \mathbb{I}^k \to M$ is also degenerate in the sense of **??????**. In fact, it will have small rank as it filters through a projection. Furthermore, if that projection collapses the ith coordinate then each face $\sigma\delta_j^\epsilon$ for $j \neq i$ will also be a degenerate small cube and so have small rank, while the term $\pm(\sigma\delta_i^0 - \sigma\delta_i^1)$ will be trivial with the trivializing map ρ being the interchange of the two faces. Thus, the degenerate smooth singular cubes are elements of $Q_*(M)$, and our map is well defined in each degree.

We check compatibility of boundary orientations. Consider the $n-1$ face F of $\mathbb{I}^n$ given by $x_i = j$ with $j \in \{0,1\}$. Then we have an outward pointing vector given by $(-1)^{j+1}e_i$, where e_i is the vector in the positive ith direction. In general if we let β_k denote the positive orientation corresponding to the kth coordinate, then the boundary orientation for β_F is the one such that $(-1)^{j+1}\beta_i \wedge \beta_F$ is the orientation $\beta_1 \wedge \cdots \wedge \beta_n$ of $\mathbb{I}^n$. Thus the boundary orientation is $(-1)^{j+1+i-1}\beta_1 \wedge \cdots \wedge \hat{\beta}_i \wedge \cdots \beta_n$. On the other hand, $\beta_1 \wedge \cdots \wedge \hat{\beta}_i \wedge \cdots \beta_n$ is precisely the standard orientation of F when considering $\mathbb{I}^n$ as a cubical complex. So the boundary orientation of F is $(-1)^{i+j}$ times its orientation in the cubical complex. But this corresponds precisely to the formula for the boundary in $K_*^{sm}(M)$ coming from equation (13). $\square$

Theorem 5.27. *The maps $H_*(S_*^{sm}(M)) \to H_*^\Gamma(M)$ and $H_*(NK_*^{sm}(M)) \to H_*^\Gamma(M)$ obtained by treating smooth singular simplices and smooth singular cubes as elements of $C_*^\Gamma(M)$ are isomorphisms.*

Proof. Both $H_*(S_*^{sm}(M))$ and $H_*(NK_*^{sm}(M))$ are isomorphic to the standard singular homology groups, so in the following we simply write H_* for either of these theories and provide a uniform proof.

We apply Theorem [**?**, 5.1.1], which is based on standard Mayer-Vietoris techniques. In particular, on the category consisting of the open sets of M, the maps $\Phi : H_*(-) \to H_*^\Gamma(-)$ provide a natural transformation of functors. We need to check the following three properties:

1. On $\emptyset$ or $U \subset M$ with U homeomorphic to $\mathbb{R}^m$, the map $\Phi : H_*(U) \to H_*^\Gamma(U)$ is an isomorphism. As both H_* and $H_*^\Gamma(-)$ are homotopy functors, we know from the respective Dimension Axioms (see Example 4.18) that in this case $H_k(U) = H_k^\Gamma(U) = 0$ for $k \neq 0$, while for $k = 0$ we have the commutative diagram

$$\begin{tikzcd} H_0(pt) \arrow[r] \arrow[d, "\Phi"'] & H_0(U) \arrow[d, "\Phi"] \\ H_0^\Gamma(pt) \arrow[r] & H_0^\Gamma(U). \end{tikzcd}$$

The horizontal maps are isomorphisms because these are homotopy functors, and the lefthand vertical map is an isomorphism because Φ takes a generator of $H_0(pt) \cong \mathbb{Z}$ to a generator of $H_0^\Gamma(pt) \cong \mathbb{Z}$; see again Example 4.18. So the righthand map is also an isomorphism.

2. Φ induces a commutative diagram of long exact Mayer-Vietoris sequences. This follows from basic homological algebra given the commutativity of the following diagram and its analogue for singular cubical chains

$$\begin{tikzcd} S_*(U \cap V) \arrow[r, hook, "{(i_U, -i_V)}"] \arrow[d, "\Phi"'] & S_*(U) \oplus S_*(V) \arrow[d, "\Phi \oplus \Phi"] \\ C_*^\Gamma(U \cap V) \arrow[r, hook, "{(i_U, -i_V)}"] & C_*^\Gamma(U) \oplus C_*^\Gamma(V). \end{tikzcd}$$

3. If $\{U_\alpha\}$ is an increasing collection of open submanifolds of M such that $\Phi : H_*(U_\alpha) \to H_*^\Gamma(U_\alpha)$ is an isomorphism for all α, then $\Phi : H_*(\cup_\alpha U_\alpha) \to H_*^\Gamma(\cup_\alpha U_\alpha)$ is an isomorphism. This argument is standard given that both singular (simplicial or cubical) chains and geometric chains are represented by compact spaces: If W represents a cycle in $C_*^\Gamma(\cup_\alpha U_\alpha)$, then $W \to \cup_\alpha U_\alpha$ factors through some particular U_β, so, as $H_*(U_\beta) \xrightarrow{\Phi} H_*^\Gamma(U_\beta)$ is an isomorphism, $\underline{W}$ is in the image of $H_*(U_\beta) \xrightarrow{\Phi} H_*^\Gamma(U_\beta) \to H_*^\Gamma(\cup_\alpha U_\alpha)$. But then $\underline{W}$ is in the image of composition $H_*(U_\beta) \to H_*(\cup_\alpha U_\alpha) \to H_*^\Gamma(\cup_\alpha U_\alpha)$, so $\Phi : H_*(\cup_\alpha U_\alpha) \to H_*^\Gamma(\cup_\alpha U_\alpha)$ is surjective. Similarly, if $\Phi : H_*(\cup_\alpha U_\alpha) \to H_*^\Gamma(\cup_\alpha U_\alpha)$ maps a class represented by a singular cycle ξ to 0, then ξ bounds as a geometric cycle, say $\partial W = \xi + T$ for some $T \in Q_*(U)$. But by compactness, there is some β so that W, T, and ξ all have image in U_β. So ξ represents a class in $H_*(U_\beta)$ that maps to 0 in $H_*^\Gamma(U_\beta)$. As Φ is assumed an isomorphism on U_β, it must be that ξ represents 0 in $H_*(U_\beta)$, and so it also represents 0 in $H_*(\cup_\alpha U_\alpha)$.

It now follows from Theorem [**?**, 5.1.1] that $\Phi : H_*(M) \to H_*^\Gamma(M)$ is an isomorphism. $\square$

Theorem 5.27 is claimed without proof in [**?**, Section 10]. Lipyanskiy states "The fact that the natural maps induce isomorphisms follow from the standard Mayer-Vietoris arguments." However, these arguments are not given and, in fact, no Mayer-Vietoris sequence is proven to exist in [**?**], though the main required tool, creasing, is provided.

Unfortunately, providing a direct comparison for cohomology theories is not so straight forward as there is no obvious map between $C^*_\Gamma(M)$ and $S^*(M) = \mathrm{Hom}(S_*(M), \mathbb{Z})$. It will take some work in the following sections to develop a geometric connection between these cohomology theories.

5.4. Appendix: smooth singular cubes.

Proposition 5.28. *The inclusion* $\psi : NK^{sm}_*(M) \hookrightarrow NK_*(M)$ *is a chain homotopy equivalence.*

Proof. The proof is analogous to the simplicial case as given in detail in [**?**, Theorem 18.7], though we need to take care with degenerate cubes, which is not an issue in the simplicial case. To account for this, we sketch the proof in [**?**] but provide some detailed modifications.

We first observe that the map $SK^{sm}_*(M) \to NK_*(M)$ takes a smooth singular cubical chain to 0 only if all of its cubes (with non-zero coefficient) are degenerate, and so we do have an injection $NK^{sm}_*(M) \hookrightarrow NK_*(M)$. We will define cube-wise a chain homotopy inverse $s : NK_*(M) \to NK^{sm}_*(M)$ by starting with a map $\tilde{s} : SK_*(M) \to SK^{sm}_*(M)$ and passing to quotients.

Recall from Section 5.3 that we write δ^ϵ_i for the face inclusions of the standard cubes. If $\sigma : \mathbb{I}^k \to M$ is a singular cube, we define homotopies $H_\sigma : \mathbb{I}^k \times \mathbb{I} = \mathbb{I}^{k+1} \to M$ so that the following properties hold:

(1) H_σ is a homotopy from σ to a smooth map $\tilde{\sigma} : \mathbb{I}^k \to M$.
(2) $H_{\sigma\delta^\epsilon_i} = H_\sigma \circ (\delta^\epsilon_i \times \mathrm{id}_{\mathbb{I}})$ so that the construction is compatible along faces. More explicitly, $H_{\sigma\delta^\epsilon_i}(x,t) = H_\sigma(\delta^\epsilon_i(x), t)$.
(3) If σ is smooth then $H_\sigma(x,t) = \sigma(x)$, i.e. the homotopy is constant.
(4) If σ is independent of the coordinate x_i then so is H_σ.

The last condition, which we have added for cubes, ensures that if σ is degenerate so will be H_σ and $\tilde{\sigma}$.

The construction is by induction on dimension. If σ is a 0-cube, then we define $H_\sigma(x,t) = \sigma(x)$, the constant homotopy. This satisfies the conditions. We then assume H_σ defined with these properties for all cubes of dimension $< k$ and extend the definition to k-cubes. If σ is already smooth, then the constant homotopy $H_\sigma(x,t) = \sigma(x)$ satisfies the conditions, noting that if σ is smooth then so is each $\sigma \circ \delta^\epsilon_i$. If σ is not smooth, we consider separately the two cases when σ is degenerate or nondegenerate.

First suppose σ is not degenerate. By the induction hypothesis and Condition (2), H_σ is determined on $(\mathbb{I}^k \times 0) \cup (\partial\mathbb{I}^k \times \mathbb{I})$. One can check as in the proof of [**?**, Lemma 18.8] that Condition (2) guarantees that the faces glue to form a continuous map. As $\partial\mathbb{I}^k \hookrightarrow \mathbb{I}^k$ is a cofibration, there is a retraction $\mathbb{I}^k \times \mathbb{I} \to (\mathbb{I}^k \times 0) \cup (\partial\mathbb{I}^k \times \mathbb{I})$, and the composition determines a homotopy $F : \mathbb{I}^k \times \mathbb{I} \to M$ such that $F(-,1)$ is smooth on each $k-1$ face of $\partial\mathbb{I}^k$. In fact, this implies that $F(-,1)$ is smooth on all of $\partial\mathbb{I}^k$ by a minor modification of [**?**, Lemma 18.9]. So by the Whitney Approximation Theorem [**?**, Theorem 6.26], there is a homotopy rel $\partial\mathbb{I}^k$ from $F(-,1)$ to a smooth map $\tilde{\sigma} : \mathbb{I}^k \to M$; we denote this homotopy G. Finally, let $u : \mathbb{I}^k \to (0,1]$ be a continuous function that takes $\partial\mathbb{I}^k$ to 1 and the interior of the cube to $(0,1)$. Then we can define

$$H_\sigma(x,t) = \begin{cases} F\left(x, \frac{t}{u(x)}\right), & x \in \mathbb{I}^k, 0 \le t \le u(x), \\ G\left(x, \frac{t-u(x)}{1-u(x)}\right), & x \in \mathrm{Int}(\mathbb{I}^k), u(x) \le t \le 1. \end{cases}$$

One can check as in the proof of [**?**, Lemma 18.8] that this is a continuous homotopy that satisfies the first two conditions above, as required.

Next suppose σ is degenerate, i.e. there is some coordinate x_i so that σ does not depend on x_i. Let $\pi_i : \mathbb{I}^k \to \mathbb{I}^{k-1}$ be given by $\pi_i(x_1, \dots, x_k) = (x_1, \dots, \hat{x}_i, \dots, x_k)$ with the x_i term omitted. In this case we let $H_\sigma(x,t) := H_{\sigma\delta^0_i}(\pi_i(x), t) = H_{\sigma\delta^1_i}(\pi_i(x), t)$. We claim that if there are multiple coordinates of which σ is independent then this definition is independent of the choice of such coordinate. This is clear for

1-cubes for which there is only one possible coordinate. Suppose then the claim proven in dimensions $< k$ and that $\sigma : \mathbb{I}^k \to M$ is independent of x_i and x_j with $j < i$. Since σ is independent of x_j, so is $\sigma \circ \delta_i^0$, so inductively $H_{\sigma\delta_i^0}(\pi_i(x), t) = H_{\sigma\delta_i^0\delta_j^0}(\pi_j\pi_i(x), t)$. Similarly, using that the ith coordinate of the cube is the $i-1$-st coordinate of the jth faces, we have $H_{\sigma\delta_j^0}(\pi_j(x), t) = H_{\sigma\delta_j^0\delta_{i-1}^0}(\pi_{i-1}\pi_j(x), t)$. But $\delta_i^0\delta_j^0$ and $\delta_j^0\delta_{i-1}^0$ determine the same $k-2$ face of $\mathbb{I}^k$, and $\pi_j\pi_i(x) = \pi_{i-1}\pi_j(x)$. So both constructions give the same H_σ.

In this case, Conditions (1) and (4) hold by construction and by induction. We must verify Condition (2). If σ is independent of x_i, the condition is clear by construction for the faces $\sigma\delta_i^0$ and $\sigma\delta_i^1$. For $j \neq i$, first suppose $i < j$. As σ is independent of x_i, so is $\sigma\delta_\epsilon^j$, so

$$\begin{aligned}
H_{\sigma\delta_j^\epsilon}(x,t) &= H_{\sigma\delta_j^\epsilon\delta_i^0}(\pi_i(x), t) \\
&= H_{\sigma\delta_i^0\delta_\epsilon^{j-1}}(\pi_i(x)), t) \\
&= H_{\sigma\delta_i^0}(\delta_{j-1}^\epsilon\pi_i(x)), t) \\
&= H_{\sigma\delta_i^0}(\pi_i(\delta_j^\epsilon(x)), t) \\
&= H_\sigma(\delta_j^\epsilon(x), t).
\end{aligned}$$

Here the first equality uses our definition of $H_{\sigma\delta_j^\epsilon}$ as $\sigma\delta_j^\epsilon$ is independent of x_i. The second equality is an identity for cubical face inclusions. The third is Condition (2) for $H_{\sigma\delta_i^0}$, which holds by induction hypothesis. The fourth equality is another cubical identity, and the last is the definition of H_σ.

Similarly, if $j < i$, then $\sigma\delta_j^\epsilon$ is independent of its $i-1$-st coordinate, and we compute analogously:

$$\begin{aligned}
H_{\sigma\delta_j^\epsilon}(x,t) &= H_{\sigma\delta_j^\epsilon\delta_{i-1}^0}(\pi_{i-1}(x), t) \\
&= H_{\sigma\delta_i^0\delta_j^\epsilon}(\pi_{i-1}(x)), t) \\
&= H_{\sigma\delta_i^0}(\delta_j^\epsilon\pi_{i-1}(x)), t) \\
&= H_{\sigma\delta_i^0}(\pi_i(\delta_j^\epsilon(x)), t) \\
&= H_\sigma(\delta_j^\epsilon(x), t).
\end{aligned}$$

This completes our construction of the homotopies H_σ. We can now define $\tilde{s} : SK_*(M) \to SK_*^{sm}(M)$ by $\tilde{s}(\sigma) = H_\sigma(-, 1)$. Then if $\tilde{\psi} : SK_*^{sm}(M) \to SK_*(M)$ is the inclusion, we have by definition that $\tilde{s}\tilde{\psi} = \mathrm{id}$. We show that $\tilde{\psi}\tilde{s}$ is chain homotopic to the identity[27]. Indeed, if σ is a singular k-cube then treating H_σ as a singular $k+1$ cube we have

$$\begin{aligned}
\partial H_\sigma &= \sum_{i=1}^{k+1} (-1)^i \left(H_\sigma\delta_i^0 - H_\sigma\delta_i^1\right) \\
&= \left(\sum_{i=1}^{k} (-1)^i \left(H_\sigma(\delta_i^0 \times \mathrm{id}_{\mathbb{I}}) - H_\sigma(\delta_i^1 \times \mathrm{id}_{\mathbb{I}})\right)\right) + (-1)^{k+1}(H_\sigma(-,0) - H_\sigma(-,1)) \\
&= \left(\sum_{i=1}^{k} (-1)^i \left(H_{\sigma\delta_i^0} - H_{\sigma\delta_i^1}\right)\right) + (-1)^{k+1}(\sigma(-) - \tilde{\psi}\tilde{s}(\sigma)).
\end{aligned}$$

[27]Here, finally, is a step that is easier in the cubical setting as we do not need to subdivide prisms into simplices.

So if we define $\tilde{J}(\sigma) = (-1)^{k+1}H_\sigma$, we obtain

$$\begin{aligned}(-1)^{k+1}\partial\tilde{J}(\sigma) &= \left(\sum_{i=1}^{k}(-1)^i\left((-1)^k\tilde{J}(\sigma\delta_i^0) - (-1)^k\tilde{J}(\sigma\delta_i^1)\right)\right) + (-1)^{k+1}\left(\sigma(-) - \tilde{\psi}\tilde{s}(\sigma)\right)\\ &= (-1)^k\tilde{J}(\partial\sigma) + (-1)^{k+1}(\sigma(-) - \tilde{\psi}\tilde{s}(\sigma)),\end{aligned}$$

so

$$\partial\tilde{J}(\sigma) + \tilde{J}(\partial\sigma) = \sigma(-,0) - \tilde{\psi}\tilde{s}(\sigma),$$

which shows that $\tilde{\psi}\tilde{s}$ is chain homotopic to the identity.

Finally, we note that, by definition and construction, $\tilde{\psi}$, $\tilde{s}$, and $\tilde{J}$ all take degenerate simplices to degenerate simplices so that these descend to chain maps $\psi : NK_*^{sm}(M) \to NK_*(M)$ and $s : NK_*(M) \to NK_*^{sm}(M)$ with $s\psi = \mathrm{id}$ and a chain homotopy $J : NK_*(M) \to NK_{*+1}(M)$. □

6. Interaction with cubical structures

In this section we bring in some auxiliary structures that will help us further develop geometric cohomology and its connections to singular cohomology. In particular, we equip our manifolds with smooth cubulations. Many of our results would apply just as well with the more familiar smooth triangulations, but we find cubulations to be more convenient. In particular, in [**?**] we have considered geometric cochains in the presence of cubulations, demonstrating how to obtain a fully-defined cochain-level cup product via intersection using certain flows developed in terms of the cubulation. Cup products in geometric cohomology will be discussed in the following section.

Smooth cubulations are analogous to smooth triangulations in that they involve a homeomorphism $M \cong |X|$ between a manifold M and the geometric realization $|X|$ of a cubical complex X such that the restriction to each cubical face is a smooth embedding. What is slightly different, aside from substituting cubes (i.e. copies of $\mathbb{I}^k$) for simplices, is that cubical complexes are required to have a bit more structure than simplicial complexes, which can always be constructed just by gluing together simplices along faces. The issue is that any simplicial complex can be given a total ordering of its vertices, and this ordering provides a canonical identification between any simplicial face and the standard model simplex of the same dimension. By contrast, the natural combinatorial structure on the vertices of the standard cube is not a total ordering but rather a partial ordering. In particular, if we take the standard cube to be $\mathbb{I}^k = [0,1]^k \subset \mathbb{R}^k$, then we have $v \leq w$ for two vertices if each coordinate of v is less than or equal to the corresponding coordinate of w. There turn out to be spaces obtained from naively gluing cubes that do not support compatible partial orderings of this type. So rather when we speak of cubical complexes we will restrict ourselves to complexes that do admit such combinatorial data. Consequently, each cubical k-face comes equipped with an identification with the standard k-cube, and hence also a standard orientation. As we will note below, smooth cubulations of this form exist for any smooth manifold. In the remainder of this work, "cubulation" will always mean a smooth cubulation.

Also analogously to simplicial complexes, cubical complexes possess algebraic cubical chain and cochain complexes and so cubical homology and cohomology that coincides with singular homology and cohomology[28]. Our primary goal in this section is to see that there are direct geometrically-defined isomorphisms between cubical (co)homology nd geometric (co)homology.

For this, we first provide some background on cubical complexes and cubical homology and cohomology in Sections 6.1 and 6.2. Then in Section 6.3 we show that the obvious map that takes a face of a cube complex to its corresponding geometric chain induces an isomorphism from cubical homology to geometric homology. Next, in Section 6.4, we consider those geometric cochains that are transverse to a given

[28]We will show below that cubical homology coincide with singular cubical homology, which coincide with simplicial singular homology by [**?**]. As all of the involved chain complexes are free, the corresponding cohomologies are also isomorphic by basic homological algebra [**?**, Theorem 45.5].

cubulation and show that their cohomology agrees with the geometric cohomology obtained without that constraint.

The motivation for our interest in cochains that are transverse to the cubulation is that they allow us to define an *intersection map* $\mathcal{I}$ from these transverse geometric cochains to the cubical cochains. If F is a face of the cubulation, F^* its dual cochain, and W is a geometric cochain of complementary dimension to F, then the coefficient of F^* in $\mathcal{I}(W)$ is simply the geometric intersection number of W with F. This intersection map is defined in Section 6.5, which also contains our proof that the intersection map induces a cohomology isomorphism when $H^*(M)$ is finitely generated in each degree. To implement this proof, we include in Section 6.5.1 a discussion of what we call "central subdivisions" of cubical complexes, which are analogous to barycentric subdivisions of simplicial complexes and allow us to construct the cubical dual cells to faces of the cubulation.

6.1. **Cubical complexes and cubulations.** We begin by recalling some notation from [**?**]. In the context of cubical complexes we write the unit interval as $\mathbb{I} = [0, 1]$ and define the **standard n-cube** to be

$$\mathbb{I}^n = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1\big\}.$$

Denote $\{1, \ldots, n\}$ by $\overline{n}$.

Given a partition $F = (F_0, F_{01}, F_1)$ of $\overline{n}$, it determines a **face** of $\mathbb{I}^n$ given by

$$\{(x_1, \ldots, x_n) \in \mathbb{I}^n \mid \forall \varepsilon \in \{0, 1\},\ i \in F_\varepsilon \Rightarrow x_i = \varepsilon\}.$$

We abuse notation and write F for both the partition and its associated face. We refer to coordinates x_i with $i \in F_{01}$ as **free** and to the others as **bound**. The **dimension** of F is its number of free coordinates, and as usual the faces of dimension 0 and 1 are called vertices and edges, respectively. The set of vertices of $\mathbb{I}^n$ is denoted by $\mathrm{Vert}(\mathbb{I}^n)$.

For $\varepsilon \in \{0, 1\}$ and $i \in \overline{n}$, we define maps $\delta_i^\varepsilon \colon \mathbb{I}^{n-1} \to \mathbb{I}^n$ by

$$\delta_i^\varepsilon(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1}).$$

Any composition of these is referred to as a **face inclusion map**.

For $v \in \mathrm{Vert}(\mathbb{I}^n)$ all coordinates are bound – that is, $v_{01} = \emptyset$. Thus v is determined by the partition of $\overline{n}$ into v_0 and v_1, so we have a bijection from the set of vertices of $\mathbb{I}^n$ to the power set $\mathcal{P}(\overline{n})$ of $\overline{n}$, sending v to v_1. The inclusion relation in the power set induces a poset structure on $\mathrm{Vert}(\mathbb{I}^n)$ given explicitly by

$$v = (\epsilon_1, \ldots, \epsilon_n) \leq w = (\eta_1, \ldots, \eta_n) \iff \forall i,\ \epsilon_i \leq \eta_i.$$

We will freely use the identification of these two posets. Face embedding maps induce order-preserving maps at the level of vertices.

An **interval subposet** of $\mathcal{P}(\overline{n})$ is one of the form $[v, w] = \{u \in \mathcal{P}(\overline{n}) \mid v \leq u \leq w\}$ for a pair of vertices $v \leq w$. There is a canonical bijection between faces of $\mathbb{I}^n$ and such subposets, associating to $[v, w]$ the face F defined by $F_\varepsilon = \{i \in \overline{n} \mid v_i = w_i = \varepsilon\}$ for $\varepsilon \in \{0, 1\}$.

The posets $\{\mathcal{P}(\overline{n})\}_{n \geq 1}$ play the role for cubical complexes that finite totally ordered sets play for simplicial complexes. Recall for comparison that one definition of an abstract ordered simplicial complex is as a pair (V, X), where V is a poset and X is a collection of subsets of V, each with an induced total order, such that all singletons are in X and subsets of sets in X are also in X. We have the following cubical analogue.

Definition 6.1. A **cubical complex** X is a collection $\{\sigma\}$ of finite non-empty subsets of a poset $\mathrm{Vert}(X)$, together with, for each $\sigma \in X$, an order-preserving bijection $\iota_\sigma \colon \sigma \to \mathcal{P}(\overline{n})$ for some n, such that:

(1) For all $v \in \mathrm{Vert}(X)$, $\{v\} \in X$,

(2) For all $\sigma \in X$ and all $[u, w] \subset \mathcal{P}(\overline{n})$ the set $\rho = \iota_\sigma^{-1}([u, w]) \in X$ and the following commutes

$$\begin{tikzcd}
\sigma \arrow[rr, "\iota_\sigma"] & & \mathcal{P}(\overline{n}) \\
& {[u,w]} \arrow[ur, hook] & \\
\rho \arrow[uu, hook] \arrow[rr, "\iota_\rho"'] & & \mathcal{P}(\overline{m}). \arrow[uu, dashed] \arrow[ul, "\cong"']
\end{tikzcd}$$

We refer to an element $\sigma \in X$ as a **cube** or **face** of X, refer to $\iota_\sigma \colon \sigma \to \mathcal{P}(\overline{n})$ as its **characteristic map**, and refer to n as its **dimension**. If $\rho \subseteq \sigma \in X$, we say that ρ is a **face** of σ in X. We identify elements in $\mathrm{Vert}(X)$ with the singleton subsets in X, referring to them as vertices.

In analogy with the usual terminology in the simplicial setting, one could call these "ordered cubical complexes," but we only work with these and have seen little use elsewhere for the unordered version. Our definition sits between cubical sets [**?**] and cellular subsets of the cubical lattice of $\mathbb{R}^\infty$ [**?**], analogously to the way that abstract ordered simplicial complexes sit between simplicial sets and simplicial complexes. The following geometric realization construction makes our definition and the cubical lattice definition essentially equivalent. Just as is the case for simplicial complexes, faces in cubical complexes are completely determined by their vertices.

Consider the category defined by the inclusion poset of a cubical complex X and the subcategory `Cube` of the category `Top` of topological spaces whose objects are the n-cubes, identified with $\mathbb{I}^n$, and whose morphisms are face inclusions. The characteristic maps of X define a functor from its poset category to `Cube`, and we define its **geometric realization** as the colimit of this functor. A **cubical structure** or **cubulation** on a space S is a homeomorphism $h \colon |X| \to S$ from the geometric realization of a cubical complex. We abuse notation and write $h \circ \iota_{|\sigma|}$ simply as ι_σ for any $\sigma \in X$ when a cubical structure $h \colon |X| \to S$ is understood.

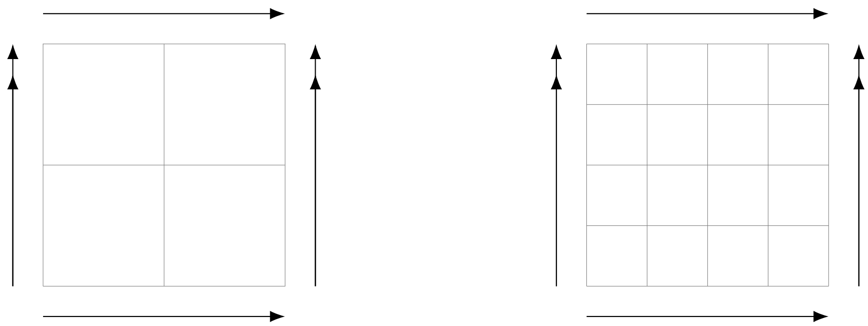
A smooth cubulation is one for which characteristic maps are smooth maps of manifolds with corners. Smooth cubulations exist for any smooth manifold, as in the following construction of [**?**]. Start with a smooth triangulation (see for example [**?**, Theorem 10.6] for the existence of such). Consider the cell complex that is dual to its barycentric subdivision. Intersecting those dual cells with each simplex in the triangulation provides a subdivision of the simplex into cells that are linearly isomorphic to cubes. Moreover, starting with an ordered triangulation – obtained for example by taking a barycentric subdivision – such a cubical decomposition embeds cellularly into the cubical lattice of $\mathbb{R}^\infty$, and thus it is the geometric realization of a cubical complex.

6.2. **Cubical chains and cochains.** We can also define an "algebraic realization" for a cubical complex in analogy to its geometric realization. Let $K_*(\mathbb{I}^1)$ be the usual cellular chain complex of the interval with integral coefficients. Explicitly, $K_0(\mathbb{I}^1)$ is generated by the vertices $[\underline{0}]$ and $[\underline{1}]$, and $K_1(\mathbb{I}^1)$ is generated by the unique 1-dimensional face, denoted $[\underline{0}, \underline{1}]$ in the interval subposet notation. The boundary map is $\partial[\underline{0}, \underline{1}] = [\underline{1}] - [\underline{0}]$.

Let $K_*(\mathbb{I}^n) = K_*(\mathbb{I}^1)^{\otimes n}$, with differential defined by the graded Leibniz rule. Given a face inclusion $\delta_i^\varepsilon \colon \mathbb{I}^n \to \mathbb{I}^{n+1}$ the natural chain map $K_*(\delta_i^\varepsilon) \colon K_*(\mathbb{I}^1)^{\otimes n} \to K_*(\mathbb{I}^1)^{\otimes n+1}$ is defined on basis elements by

$$x_1 \otimes \cdots \otimes x_n \mapsto x_1 \otimes \cdots \otimes [\underline{\varepsilon}] \otimes \cdots \otimes x_n.$$

Regarding a cubical complex X as a functor to `Cube`, we can compose it with the chain functor above to obtain a functor to chain complexes. The complex of **cubical chains** of X, denoted $K_*(X)$, is defined to be the colimit of this composition. As one would expect, in each degree it is a free abelian group generated by the cubes of that dimension, and its boundary homomorphism sends the generator associated to a cube to a sum of generators associated to its codimension-one faces with appropriate signs. By abuse, we use the same notation and terminology for an element in X, its geometric realization in $|X|$, and the corresponding basis element in $K_*(X)$. Most commonly we will write F and refer simply to a "face of X."



(A) **Not** a cubulation of the torus (B) A cubulation of the torus

Figure 5. The first cellular decomposition of a torus pictured above does not represent the geometric realization of a cubical complex, as each square has the same set of vertices. On the right, each square can be coherently identified with the standard square with initial vertex in the lower left corner and final vertex in the upper right corner. Therefore, (B) depicts a cubical structure on the torus.

We note that for each $\mathbb{I}^n$ we have the ordered set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ where $\mathbf{e}_i = \frac{\partial}{\partial x_i}$. For any face F of $\mathbb{I}^n$, the ordered subset $\beta_F = \{\mathbf{e}_i \mid i \in F_{01}\}$ defines the **canonical orientation** of F. In forming the cubical complex X, these orientations are preserved, and so each face of X carries an orientation. These orientations are compatible with our standard generators of $K_*(X)$ in the sense that if we identify $[0,1]^{\otimes k}$ with $\mathbb{I}^k$ with its standard orientation then, in the boundary formula, $k-1$ faces appear sign 1 or -1 according to whether or not their standard orientations agree with the boundary orientation of $\mathbb{I}^k$ as a manifold with corners.

The **cubical cochain complex** of X (with $\mathbb{Z}$ coefficients) is the chain complex $K^*(X) = \mathrm{Hom}_{\mathbb{Z}}(K_*(X), \mathbb{Z})$. If F is a face of X, and correspondingly a generator of $K_*(X)$, then we will write F^* for the dual, i.e. the generator of $K^*(X)$ such that $F^*(F) = 1$ and $F^*(E) = 0$ for all other faces $E \neq F$ of X. We will use the convention as in [**?**] that

$$(dF^*)(\xi) = F^*(\partial \xi).$$

6.3. **Comparing cubical and geometric homology.** Suppose $h : |X| \to M$ is a smooth cubulation. As the cubes of X are compact oriented manifolds with corners, the composition of the inclusion of a cube into X with the map h gives an element of $PC_*^\Gamma(M)$ and hence an element of $C_*^\Gamma(M)$. Furthermore, the boundary formula for cubes in $K_*(X)$ agrees with the geometric boundary formula, so the cubes of X generate a subcomplex $K_*^X(M) \subset C_*^\Gamma(M)$ that is canonically isomorphic to $K_*(X)$. As expected this gives the standard homology:

Theorem 6.2. *The map $\mathcal{J} : K_*(X) \cong K_*^X(M) \to C_*^\Gamma(M)$ induces an isomorphism of homology groups $H_*(K_*(X)) \to H_*^\Gamma(M)$.*

Proof. The proof is analogous to the proof of [**?**, Proposition V.8.3], which provides an isomorphism between simplicial and singular homology.

Let $NK_*(M)$ be the normalized singular cubical chain complex of M as in recalled in Section 5.3, and let $NK_*^{sm}(M)$ be the subcomplex generated by smooth cubes. By Proposition 5.28 and Theorem 5.27, there are quasi-isomorphisms $NK_*(M) \xrightarrow{\psi} NK_*^{sm}(M) \to C_*^\Gamma(M)$, the latter induced by observing that smooth singular cubes are elements of $PC_*^\Gamma(M)$ and that degenerate cubes are elements of $Q_*(M)$.

Next we observe that there is a map $\eta : K_*^X(M) \to NK_*^{sm}(M)$ that takes each cube into its embedding to M (recall that we assume the cubulation is smooth) and that the composition $\phi\eta$ is the map $\mathcal{J} : K_*^X(M) \to C_*^\Gamma(M)$ of the theorem statement. So it suffices to show that η is an isomorphism.

For this, we have the diagram

$$\begin{tikzcd}
H_*(K_*^X(M)) \arrow[r, "\eta_*"] \arrow[d, "\cong"'] & H_*(NK_*^{sm}(M)) \arrow[r, "\psi_*", "\cong"'] & H_*(NK_*(M)) \\
H_*(CW_*(M)), \arrow[urr, "\cong", "\Theta"'] & &
\end{tikzcd}$$

in which $CW_*(M)$ is the CW chain complex of M corresponding to the CW complex structure given by the cubulation and Θ is the standard isomorphism between CW homology and singular homology as in Dold [**?**, Proposition V.1.9]. The isomorphism in Dold is developed using simplicial singular homology, but as simplicial singular and cubical singular homology are isomorphic, the argument goes through identically using singular cubes. The map on the left is an isomorphism at the chain level as there is an evident isomorphism in this case between the cubical chain complex and the CW chain complex that takes an embedding of a k-cube to the corresponding generator of $CW_k(M) = H_k(X^k, X^{k-1})$ (where we assume the expression on the right is singular cubical homology). As in Dold, the map Θ takes a class in $H_k(CW_*(M))$ represented to by a k-cycle z in $CW_k(M)$ to the class in $H_k(NK_*(M))$ represented by a singular (cubical) cycle in $NK_k(M)$ that represents the same class as z in $H_k(X^k, X^{k-1})$. But all cycles in the image of the vertical map of the diagram are already represented by singular cycles in $NK_k^{sm}(M)$, so the diagram commutes, and it follows that η_* is an isomorphism.

The isomorphism of the theorem is now obtained by composing the maps $K_*^X(M) \to NK_*^{sm}(M) \to C_*^\Gamma(M)$. □

As a corollary of the proof, we have the following useful result concerning the cohomology groups of the complexes

$$\begin{aligned}
NK^*(M)) &:= \mathrm{Hom}(NK_*(M), \mathbb{Z}) \\
NK^*_{sm}(M) &:= \mathrm{Hom}(NK_*^{sm}(M), \mathbb{Z}) \\
K^*_X(M) &:= \mathrm{Hom}(K_*^X(M), \mathbb{Z}).
\end{aligned}$$

Corollary 6.3. *The following maps on cohomology induced by restrictions are isomorphisms:*

$$H^*(NK^*(M)) \to H^*(NK^*_{sm}(M)) \to H^*(K^*_X(M)).$$

Proof. This follows from basic homological algebra [**?**, Theorem 45.5] as $NK_*(M)$, $NK_*^{sm}(M)$, and $K_*^X(M)$ are all free chain complexes, observing that even though $NK_*(M)$ are $NK_*^{sm}(M)$ are defined by taking the quotients of the groups of all singular cubes, $SK_*(M)$ and $SK_*^{sm}(M)$, by the subgroups of degenerate cubes, the degenerate cubes correspond to generators of $SK_*(M)$ and $SK_*^{sm}(M)$, and so each $NK_i(M)$ and $NK_i^{sm}(M)$ is freely generated by the nondegenerate, respectively nondegenerate and smooth, singular i-cubes. □

6.4. **Cubically transverse geometric cohomology.** In this section we consider the cochains on M represented by maps $W \to M$ that are transverse to a given cubulation of M.

Definition 6.4. Let M be equipped with a smooth cubulation $|X| \to M$. We say that $r_W : W \to M$ is **transverse** to X if $r_W : W \to M$ is transverse to each characteristic map of the cubulation. In particular, this implies by Lemma 2.11 that each induced $\partial^k W \to M$ is plainly transverse to each face of the cubulation. If $r_W : W \to M$ is transverse to X then the same is true for any $r_V : V \to M$ isomorphic to $r_W : W \to M$, and so we can define $PC^*_{\Gamma \pitchfork X}(M)$ to be the subset of $PC^*_\Gamma(M)$ consisting of those precochains with reference maps transverse to X.

We let $Q^*_{\Gamma \pitchfork X}(M) = PC^*_{\Gamma \pitchfork X}(M) \cap Q^*(M)$ and note that the equivalence relation of Lemma 4.14 descends to an equivalence relation on $PC^*_{\Gamma \pitchfork X}(M)$ such that $V \sim W$ if and only if $V \sqcup -W \in Q^*_{\Gamma \pitchfork X}(M)$. The **geometric cochains of** M **transverse to** X, denoted $C^*_{\Gamma \pitchfork X}(M)$, are the equivalence classes in $PC^*_{\Gamma \pitchfork X}(M)$. The set $C^*_{\Gamma \pitchfork X}(M)$ is a chain complexes under the operation $\sqcup$ and with boundary map ∂. The **geometric cohomology transverse to** X is $H^*_{\Gamma \pitchfork X}(M) := H^*(C^*_{\Gamma \pitchfork X}(M))$.

When the specific cubulation X is understood, we sometimes simplify the notation to $PC^*_{\Gamma \pitchfork}(M)$, $C^*_{\Gamma \pitchfork}(M)$, and $H^*_{\Gamma \pitchfork}(M)$.

The proof of Lemma 4.16 continues to hold for transverse cochains, and so $r_W : W \to M$ represents 0 in $C^*_{\Gamma \pitchfork X}(M)$ if and only if it is in $Q^*_{\Gamma \pitchfork X}(M)$. Therefore, the evident map $C^*_{\Gamma \pitchfork X}(M) \to C^*_\Gamma(M)$, which takes the element of $C^*_{\Gamma \pitchfork X}(M)$ represented by $r_W : W \to M$ to the element $C^*_\Gamma(M)$ represented by the same map, is a monomorphism of chain complexes, for such an r_W is transverse to X by definition and if it is also in $Q^*(M)$ then it is in $Q^*_{\Gamma \pitchfork X}(M)$. Thus we will think of $C^*_{\Gamma \pitchfork X}(M)$ as a subcomplex of $C^*_\Gamma(M)$. A key technical result, which will take the remainder of this section to prove, is that this inclusion induces a cohomology isomorphism. In other words, the cochains that are transverse to X are sufficient to compute the cohomology of M.

Theorem 6.5. *The inclusion $C^*_{\Gamma \pitchfork X}(M) \hookrightarrow C^*_\Gamma(M)$ is a quasi-isomorphism.*

To show that the inclusion $C^*_{\Gamma \pitchfork X}(M) \hookrightarrow C^*_\Gamma(M)$ is a quasi-isomorphism, it will be necessary to consider the following scenario. Suppose we have a map $r_V : V \to M$ with V a manifold with corners and M a manifold with a cubulation. Let $\partial V = W$, which will also be transverse to the cubulation. We construct a homotopy $h : V \times I \to M$ such that $g(-,0) = r_V$, $g(-,1)$ is transverse to the cubulation, and so the restriction of h to $W \times I$ is transverse to the cubulation. However, cochains are not generally well behaved with respect to homotopy. For example, $r_V : V \to M$ might be degenerate but a map homotopic to r_V may not. To solve this problem, we launder our homotopies through homotopies of M, i.e. we only use homotopies of the form $V \times I \xrightarrow{r_V \times \mathrm{id}} M \times I \xrightarrow{H} M$ with $H(-,0) = \mathrm{id}$. This ensures that components of V or W in $Q^*(M)$ remain in $Q^*(M)$ and, in general, that our homotopies generate cohohomologies of cocycles via Section 5.1.1. We co-orient h using the co-orientation induced by $\mathrm{id} : M \to M$; see Definition 3.27. Given such an h, we will call the resulting homotopy from r_V to $H(-,1) \circ r_V$ a **universal homotopy** $V \times I \to M$.

The technique for constructing such homotopies will be modeled on a variety of results in [**?**]. We use the Transversality Theorem and Transversality Homotopy Theorem of [**?**, Section 2.3] as stated. However, for the Stability Theorem of [**?**, Section 1.6] we will provide details of the proof because the proof is only sketched in [**?**] and we will need the result to be generalized in several ways. Also, the Stability Theorem is not completely correct as stated in [**?**, Section 1.6]; the requirement that the submanifolds of the target be closed sets is omitted[29]. The needed versions of these results is established in the following proposition:

[29] As stated in [**?**], the claim is that if $f : X \to Y$ is transverse to any submanifold Z of Y then this property is stable under small homotopies of f; more specifically that if $f_t : X \times I \to Y$ is a homotopy with f_0 transverse to Z then there is an $\epsilon > 0$ such that f_t is transverse to Z for all $t \in [0, \epsilon)$. Here is a counterexample:

In the plane $\mathbb{R}^2$, let $Z = \{(x,y) | y = x^2, x \neq 0\}$ and consider maps $g_t : \mathbb{R} \to \mathbb{R}^2$ with $g_t(x) = (x, t^2 + 2t(x-t))$. For each fixed t, the image is the line given by $y - t^2 = 2t(x-t)$, which has slope $2t$ and passes through the point (t, t^2). So the map g_0 embeds $\mathbb{R}$ as the x-axis, and as the image does not intersect Z, the map g_0 is transverse to Z. But for all $t \neq 0$, g_t takes $\mathbb{R}$ to a line that is tangent to Z, and so g_t is not transverse to Z for $t \neq 0$, violating the Stability Theorem as stated on page 35 of [**?**].

The error in the proof comes from considering only what happens in neighborhoods of points x such that $f(x) \in Z$ but not points x with $f(x) \notin Z$. As we can see, the claim breaks down when $f(x) \notin Z$ but every neighborhood of $(x, 0)$ in $X \times I$ has a point with image in Z. However, this can be avoided if Z is a closed set in Y.

Proposition 6.6. *Suppose $r_V : V \to M$ is a proper map from a manifold with corners to a cubulated manifold without boundary. Then there is a proper universal homotopy $h : V \times I \to M$ such that:*

(1) $h(-,0) = r_V$,
(2) $h(-,1) : V \to M$ *is transverse to the cubulation,*
(3) *if $i_W : W \to V$ is the inclusion of a union of boundary components of V with $r_W = r_V i_W : W \to M$ transverse to the cubulation then $h \circ (i_W \times \mathrm{id}) : W \times I \to M$ is transverse to the cubulation.*

Before proving the proposition, which is somewhat technical, we use it to prove Theorem 6.5, which states that $H^*(C^*_{\Gamma \pitchfork X}(M)) \to H^*(C^*_\Gamma(M))$ is an isomorphism.

Proof of Theorem 6.5. The idea of the argument that $H^*(C^*_{\Gamma \pitchfork X}(M)) \to H^*(C^*_\Gamma(M))$ is a surjection is contained already in the proof of [**?**, Lemma 15], which involves constructing a homotopy to move a cycle into transverse position. We elaborate upon that argument.

Suppose $\underline{V} \in C^*_\Gamma(M)$ is a cocycle represented by $r_V : V \to M$. By Proposition 6.6, there is a proper universal homotopy $h : V \times I \to M$ from r_V to $r_{V'} : V' \to M$ with V' transverse to the cubulation. By Corollary 5.7, r_V and $r_{V'}$ represent the same cohomology class in $H^*_\Gamma(M)$, but the class represented by $r_{V'}$ is in the image of $H^*(C^*_{\Gamma \pitchfork X}(M))$.

For injectivity, suppose $W \in PC^*_{\Gamma \pitchfork X}(M)$ is transverse to the cubulation and represents zero in $H^*(C^*_\Gamma(M))$. Then by definition there is a $V \in PC^*_\Gamma(M)$ with $\partial V = W + T$ for some $T \in Q^*(M)$. By Proposition 6.6 there is a proper universal homotopy $h : V \times I \to M$ such that $h(-,1)$ and $h \circ (i_W \times \mathrm{id})$ are both transverse to the cubulation. Let $V', W', T' \in PC^*_\Gamma(M)$ be W, V, and T but with reference maps given respectively by $h(-,1)$, $h(-,1)i_W$, and $h(-,1)i_T$, where $i_W : W \to V$ and $i_T : T \to V$ are the boundary inclusion maps restricted to the components of W and T, respectively. As $h \circ (i_W \times \mathrm{id})$ is transverse to the cubulation, W and W' represent the same element of $H^*_{\Gamma \pitchfork X}(M)$ by arguments analogous to the proof of Corollary 5.7. But we also have $\partial V' = W' + T'$ with V' in $C^*_{\Gamma \pitchfork X}(M)$ and, by Lemma 5.1, $T' \in Q^*(M)$. So W' represents $0 \in H^*(C^*_{\Gamma \pitchfork X}(M))$. $\square$

It remains to prove Proposition 6.6, which will requires the following technical lemma that is also useful below in the proof of Theorem 6.20.

Lemma 6.7. *Let M be a manifold without boundary, and let $\mathcal{U} = \{U_j\}$ be a locally finite open covering such that each $\bar{U}_j$ is compact. Suppose given $\varepsilon_j > 0$ for each j. Then there exists a smooth function $\phi : M \to \mathbb{R}$ such that $0 < \phi(x) < \varepsilon_j$ if $x \in \bar{U}_j$.*

Proof. Let $\eta_j = \min\{\varepsilon_k \mid \bar{U}_j \cap \bar{U}_k \neq \emptyset\}$. By the local finiteness and compactness conditions, $\{k \mid \bar{U}_j \cap \bar{U}_k \neq \emptyset\}$ is a finite sets and so the η_j are well defined. Let $\{\psi_j\}$ be a partition of unity subordinate to $\mathcal{U}$ and let $\phi_1 = \sum \eta_j \psi_j$. For any $x \in M$, this sum is positive. If $x \in \bar{U}_j$ then $\phi_1(x) = \sum_{\{k \mid \bar{U}_j \cap \bar{U}_k \neq \emptyset\}} \eta_k \psi_k$. But for any such k, we have $\eta_k \leq \varepsilon_j$. Thus $\phi_1(x) \leq \varepsilon_j$. Now take $\phi = \frac{1}{2}\phi_1$. $\square$

We can now prove Proposition 6.6. In the following D^N is the open unit ball in $\mathbb{R}^N$ and, more generally, D^N_r the open ball of radius r.

Proof of Proposition 6.6. We begin with the case that V is compact, and then we will show how to use the arguments of the compact case to obtain the general case. We first construct $F : M \times D^N \to M$, for some N, such that

(1) $F(-,0) = \mathrm{id} : M \to M$,
(2) for almost all $s \in D^N$ the composition $V \xrightarrow{r_V} M \xrightarrow{F(-,s)} M$ is transverse to the cubulation,
(3) there is a ball neighborhood D^N_r of 0 in D^N such that for all $s \in D^N_r$ the composition $W \xrightarrow{i_W} V \xrightarrow{r_V} M \xrightarrow{F(-,s)} M$ is transverse to the cubulation.

This will suffice in the compact case as then we can let s_0 be any point in D^N_r such that the composition $V \xrightarrow{r_V} M \xrightarrow{F(-,s_0)} M$ is transverse to the cubulation and define $h(-,t) = F(-,ts_0)r_V$, i.e. $h(x,t) = F(r_V(x), ts_0)$. The first property of the proposition holds since $F(-,0) = \mathrm{id}$. The second property holds by our choice of s_0. The last property then holds as $ts_0 \in D^N_r$ for all $t \in I$; thus each $h(-,ts_0)i_W$ is transverse to the cubulation, which then implies that $h \circ (i_W \times \mathrm{id})$ is transverse to it as well. Furthermore, for V compact any map and homotopy are proper, and this homotopy is universal as it can be decomposed into $r_V : V \times I \to M \times I$ and the homotopy $M \times I \to M$ taking (z,t) to $F(z, ts_0)$.

The construction of F is a small variation of the construction in the Transversality Homotopy Theorem of [**?**, Section 2.3]: Let M_ϵ be an ϵ-neighborhood of M in some $\mathbb{R}^N$ in the sense of the ϵ-Neighborhood Theorem of [**?**, Section 2.3]; in particular, M_ϵ is an ϵ-neighborhood of a proper embedding of M into $\mathbb{R}^N$ that possesses a submersion $\pi : M_\epsilon \to M$. If M is not compact, then ϵ is a smooth positive function of M and $M_\epsilon = \{z \in \mathbb{R}^N \mid |z-y| < \epsilon(y) \text{ for some } y \in M\}$. Let $f : M \times D^N \to M_\epsilon$ be given by $f(y,s) = y + \epsilon(y)s$; as $\epsilon(y) > 0$, this is clearly a submersion (onto its image) at all points. We let $F : M \times D^N \to M$ be the composition $M \times D^N \xrightarrow{f} M_\epsilon \xrightarrow{\pi} M$. Furthermore, the map $H : V \times D^N \to M$ given by the composition

$$V \times D^N \xrightarrow{r_V \times \mathrm{id}} M \times D^N \xrightarrow{F} M$$

as well as all the restrictions $H|_{S^k(V)}$ are submersions. In particular, each $H|_{S^k(V)}$ is transverse to any submanifold of M, so it follows by the Transversality Theorem of [**?**, Section 2.3] that for any fixed submanifold Z of M, each $H|_{S^k(V)}(-,s)$ is transverse to Z for almost all $s \in D^N$. In particular, we may take Z to be the interior of any cube E (of any dimension) of the cubulation. However, there are countably many cubes in the cubulation of M and finitely many manifolds $S^k(V)$. As the countable union of measure zero sets has measure zero, for almost all $s \in D^N$ we have all $H|_{S^k(V)}(-,s) = F(-,s)r_V|_{S^k(V)}$ transverse to all cubical faces.

It is clear that $F(-,0) = \mathrm{id}_M$, so next we show that if W is a union of boundary components of V with $r_W = r_V i_W : W \to M$ transverse to the cubulation then $F(-,s)r_V i_W = H(-,s)i_W$ is transverse to the cubulation for all s in some neighborhood U of 0 in D^N. It is here that we need to generalize the Stability Theorem of [**?**, Section 1.6]. As the Stability Theorem is not necessarily true when the manifolds involved are not closed submanifolds, compact, or controlled in some other way, it is more convenient here to work with the closed cubical faces of the cubulation and with $\partial^k V$ rather than $S^k(V)$. We recall that by Lemma 2.11, to prove that two maps of manifolds with corners are transverse it is sufficient to show that their compositions with all pairs of boundary inclusions are plainly transverse.

Let H_k denote the composition $H_k : \partial^k V \times D^N \xrightarrow{i_{\partial^k V} \times \mathrm{id}} V \times D^N \xrightarrow{r_V \times \mathrm{id}} M \times D^N \xrightarrow{F} M$. As W consists of boundary components of V, we must consider the H_k, $k \geq 1$. We provide the details for H_1, the other cases being similar. For simplicity of notation, we also assume for the remainder of the proof that $W = \partial V$; in case W is a union of only some of the components of ∂V, we can restrict H_k, $k > 0$, in the following to just those components of W.

Let E be a (closed) face of the cubulation and let $x \in W$. As $r_W = H_1(-,0)$ is transverse to the cubulation, for any $x \in W$, either $r_W(x) \notin E$ or r_W is transverse to E at $r_W(x)$. In the former case, as E is closed, there is an open neighborhood A_x of $(x,0) \in W \times D^N$ such that $H_1(A_x) \cap E = \emptyset$. Now suppose that $r_W(x) \in E$ and is transverse there. By appealing to charts, we can suppose without loss of generality (as least locally in a neighborhood of $r_W(x)$) that $M = \mathbb{R}^m$ with $m = \dim(M)$ and $r_W(x) = 0$ and that $E = \{(y_1, \ldots, y_m) \mid y_i \geq 0 \text{ for } i \leq \dim E \text{ and } y_i = 0 \text{ for } i > \dim(E)\}$. The transversality assumption means that the composition of $D_x r_W : T_x W \to T_{r_W(x)} M$ with the projection to the last $m - \dim(E)$ coordinates is a linear surjection. As this is an open condition on the Jacobian matrix of r_W at x, it follows again that there is an open neighborhood A_x of $(x,0) \in W \times D^N$ such that for each (x', s) in the neighborhood $H_1(-,s)$ is transverse to E at x' (it is possible that $H_1(x',s)$ no longer intersects E, but this is fine). Taking the union of the A_x over all $x \in W$ gives a neighborhood B_E of $W \times 0$ in $W \times D^N$, and by the Tube Lemma, as W is compact there is a neighborhood of $W \times 0$ of the form $W \times U_E \subset B_E$. For each $s \in U_E$, we have $H_1(-,s)$ transverse to E. Now let $D^N_{1/2}$ be the open ball of radius $1/2$ and $\bar{D}^N_{1/2}$ its closure. As $W \times \bar{D}^N_{1/2}$ is compact, its image under H_1 can intersect only a finite number of faces of the cubulation of M; call this collection $\mathcal{E}$. Then let U_1 be the finite intersection $U_1 = D^N_{1/2} \cap \bigcap_{E \in \mathcal{E}} U_E$. Then $W \times U_1 \subset W \times D^N$ is a neighborhood of $W \times 0$ on which $H_1(-,s)$ is transverse to every cubical face that its image intersects. Let U_k be defined similarly for each $k \geq 1$. As W has finite depth, $U = \cap U_k$ is a neighborhood of 0 in D^N. Let $r > 0$ be such that $D^N_r \subset U$. Then for every $s \in D^N_r$ we have $H_k(-,s) : \partial^k V \to M$ transverse to all E for all k as required.

This completes the proof of the proposition for V compact.

Next suppose that V is no longer necessarily compact. We show how to apply and extend the preceding arguments. We will define a new homotopy $\hat{h} : V \times I \to M$ with the desired three properties of the proposition.

To begin we will construct $F : M \times I \to M$ exactly as above, as its definition did not depend on the compactness of V. For V not compact, the first two properties listed above for F will continue to hold, but the third relied on compactness and so will not hold any long in general. However, let $K \subset W$ be compact and E be a closed cube. Taking the union of the A_x over all $x \in K$ and intersecting with $K \times D^N$ gives an open neighborhood B_E of $K \times 0$ in $K \times D^N$, such that $H_1(-,s)$ is transverse to E at all $x \in K$. Furthermore, as $K \times \bar{D}^N_{1/2}$ is compact, its image under H_1 can intersect only a finite number of faces of the cubulation of M, so again by applying the tube lemma and then intersecting tubular neighborhoods, we find an open ball $D^N_{r,K} \subset D^N$ centered at 0 such that for all $x \in K$ and $s \in D^N_{r,K}$ we have $H_1(-,s)$ transverse to the cubulation at x. Furthermore, as the inclusions $\partial^k V \hookrightarrow W$ for $k \geq 1$ are all proper, we can similarly find $D^N_{r,K}$ so that for all $s \in D^N_{r,K}$ and all $k \geq 1$, we have $H_k(-,s) : \partial^k V \to M$ transverse to the cubulation for any $x \in \partial^k V$ whose image in W is in K.

Let $\{\mathcal{U}_j\}$ be a locally finite covering of M such that each $\bar{\mathcal{U}}_j$ is compact. As r_V and $r_W = r_V \circ i_W$ are proper, each $r_W^{-1}(\bar{\mathcal{U}}_j)$ is compact in W. Proceeding as just above with $r_W^{-1}(\bar{U}_j)$ in place of W, we can find for each j an $\varepsilon_{j,1} \leq 1$ so that for every $s \in D^N_{\varepsilon_{j,1}}$ we have $H_1(-,s)$ transverse to all cubical faces of M at every $x \in r_W^{-1}(\bar{\mathcal{U}}_j)$. Analogously, we have $\varepsilon_{j,k}$ for all $k \geq 1$ using $(r_V i_{\partial^k V})^{-1}(\bar{\mathcal{U}}_j)$. Let $\varepsilon_j = \min\{\varepsilon_{j,k} \mid k \geq 1\}$. These minima exist as V has finite depth.

Now, using Lemma 6.7, we choose a smooth function $\phi : M \to \mathbb{R}$ such that for all $x \in M$ we have $0 < \phi(x) < \epsilon_j$ if $x \in \bar{\mathcal{U}}_j$. Let $M \times_\phi D^N = \{(y,s) \in M \times D^N \mid |s| < \phi(y)\}$. By our construction, $H_k(-,s) : \partial^{k-1} W \to M$ is transverse to the cubulation at each x such that $(x,s) \in (r_W i_{\partial^{k-1} W} \times \mathrm{id})^{-1}(M \times_\phi D^N) = \{(x,s) \in \partial^{k-1} W \times I \mid |s| < \phi(r_W i_{\partial^k W}(x))\}$. Unfortunately, however, while $M \times_\phi D^N$ is a neighborhood of $M \times 0$ in $M \times D^N$, there is not necessarily a $U \in D^M$ so that $M \times U \subset M \times_\phi D^M$. Thus, we cannot construct h from F as above using a fixed s_0 as there may be no single $s_0 \neq 0$ so that $W \times s_0 \subset M \times_\phi D^N$.

To account for this, we modify our functions above as follows: Let $\hat{f} : M \times D^N \to M_\epsilon$ be given by $\hat{f}(y,s) = y + \phi(y)\epsilon(y)s$; as $\phi(y)\epsilon(y) > 0$, this is again a submersion onto its image at all points. Let $\hat{F} : M \times D^N \to M$ be the composition $M \times D^N \xrightarrow{\hat{f}} M_\epsilon \xrightarrow{\pi} M$, and let $\hat{H}_k$ be the composition $\partial^k V \times D^N \xrightarrow{i_{\partial^k V} \times \mathrm{id}} V \times D^N \xrightarrow{r_V \times \mathrm{id}} M \times D^N \xrightarrow{\hat{F}} M$ for $k \geq 0$. Once again by the Transversality Theorem of [**?**, Section 2.3], for almost all $s \in D^N$ we have $\hat{H}_k(-,s)$ transverse to all cubical faces for all $k \geq 0$. Letting s_0 be any such point we define $\hat{h} : V \times I \to M$ to be $\hat{h}(x,t) = \hat{H}(x, ts_0)$, and we claim that this $\hat{h}$ satisfies the conditions of the proposition.

The map $\hat{h}$ is proper, and the first two conditions of the proposition follow immediately from the construction. It remains to verify that $\hat{h}(i_W \times \mathrm{id}) : W \times I \to M$ is transverse to the cubulation, and similarly for the higher boundaries. Note: we are not claiming an analogue of the third condition above holds for $\hat{F}$, nor do we need to. As we already know from the second condition of the proposition that

$\hat{h}(-,1)$ is transverse to the cubulation and from the hypotheses that $\hat{h}(-,0)$ is transverse to the cubulation on W, it suffices to demonstrated transversality to the cubulation on the restriction of $\hat{h}$ to $W \times (0,1)$.

We begin by observing that for $(x,t) \in W \times I$ we can write $\hat{h}(x,t) \circ (i_W \times \mathrm{id}_I) : W \times I \to M$ explicitly as

$$\hat{h}(x,t) = \pi(r_W(x) + \phi(r_W(x))\epsilon(r_W(x))ts_0).$$

So, alternatively, we can observe that $\hat{h}(x,t) \circ (i_W \times \mathrm{id}_I)$ is the composition

$$W \times I \xrightarrow{\Phi} W \times I \xhookrightarrow{\Psi} W \times D^N \xrightarrow{r_W \times \mathrm{id}} M \times D^N \xrightarrow{F} M, \tag{14}$$

with $\Phi(x,t) = (x, \phi(r_W(x))t)$, $\Psi(x,t) = (x, ts_0)$, and noting that on the right we do mean our original F and not $\hat{F}$.

The first map Φ is a diffeomorphism onto its image, which is a neighborhood of $W \times 0$ in $W \times I$, and the map Ψ embeds this linearly into $W \times D^N$. The composition of the last two maps is just our earlier map H_1. By construction, the map $r_W \times \mathrm{id}$ now takes the image of $\Psi\Phi$ into $M \times_\phi D^N$, and so at each point (z,s) in the image of $\Psi\Phi$ if we fix s and consider $H_1(-,s)$ we get by construction a map on W that is transverse to the cubulation. But as Φ is a diffeomorphism onto its image and Ψ is an embedding that is the identity with respect to W, we see $\Psi\Phi$ takes a neighborhood of any $(x,t) \in W \times (0,1)$ to a neighborhood of its image in $W \times \mathbb{R}s_0$, where $\mathbb{R}s_0$ is the line in $\mathbb{R}^N$ spanned by s_0. In particular, the derivative of $\Psi\Phi$ maps the tangent space to $W \times (0,1)$ at (x,t) onto $T_xW \times \mathbb{R}s_0 \subset T_{\Psi\Phi(x,t)}(W \times D^N)$. In particular, this image contains $T_xW \times 0$, and by construction DH_1 takes this tangent space to a tangent subspace in M at $\hat{h}(x,t)$ that is transverse to the tangent space there of any face of the cubulation containing $\hat{h}(x,t)$. The same holds for $k > 1$ replacing W in with $\partial^{k-1}W$ in (14) and r_W with $r_{\partial^{k-1}W}$. So we see that $\hat{h}$ satisfies all the requirements of the proposition. $\square$

6.5. **The intersection map and the isomorphism between cubical and geometric cohomology.** To define the intersection map, we introduce an augmentation map as in singular homology theory. For this we first need a quick lemma.

Lemma 6.8. *If $W \in Q_0(M)$, then W has the same number of positively and negatively oriented points.*

Proof. As elements of $PC_0^\Gamma(M)$ cannot be degenerate, if $W \in Q_0$ then W must be trivial, and so there is an orientation-reversing diffeomorphism ρ of W such that $r_W\rho = r_W$. But a compact 0-manifold has an orientation-reversing diffeomorphism if and only there are the same number of points with each orientation. $\square$

Definition 6.9. We define the **augmentation map** $\mathbf{a} : PC_0^\Gamma(M) \to \mathbb{Z}$ as follows: If $W \in PC_0^\Gamma(M)$ then W is the disjoint union of a finite number of points, each with orientation denoted 1 or -1. We let $\mathbf{a}(W)$ be the sum of the orientations of the points in W, interpreting 1 and -1 as integers. By Lemma 6.8, an element of $PC_0^\Gamma(M)$ can be in $Q_0(M)$ only if this sum is 0, so the augmentation descends to a homomorphism $\mathbf{a} : C_0^\Gamma(M) \to \mathbb{Z}$. Furthermore, if $W \in PC_1^\Gamma(M)$ then, as usual, $\mathbf{a}(\partial W) = 0$, so $\mathbf{a}$ further descends to a homomorphism $\mathbf{a} : H_0^\Gamma(M) \to \mathbb{Z}$.

Later, we will construct in general a partially-defined intersection map $C_\Gamma^*(M) \otimes C_*^\Gamma(M) \to C_*^\Gamma(M)$. In general, this is delicate as geometric chains and cochains do not have fixed representatives. However, at the moment we do not need this full generality to define the intersection map we will need to compare geometric cohomology and cubical cohomology. This is reflected in the following more limited definition:

Definition 6.10. Suppose M is an m-manifold without boundary and $W \in PC_\Gamma^i(M)$ and $N \in PC_i^\Gamma(M)$ are transverse. We define the *intersection number* $I_M(W,N)$ (or simply $I(W,N)$ if M is clear from context) by

$$I_M(W,N) = \mathbf{a}(W \times_M N),$$

with $W \times_M N$ as defined in Definition 4.21.

We observe that this definition makes sense as W and N are transverse with complementary dimensions and $W \times_M N$ is an element of $PC_0^\Gamma(M)$. In fact, in this case in order for transversality to hold the maps $r_W : W \to M$ and $r_N : N \to M$ must have full rank at each $x \in W$ and $y \in N$ such that $r_W(x) = r_N(y)$. As having full rank is an open condition, the maps will also have full rank on neighborhoods of these points. In particular, by the Implicit Function Theorem, they must be immersions on neighborhoods of these points. So, locally, the orientation of N determines an orientation of T_yN, which we can consider to be a subspace of T_yM, slightly abusing notation to identify y and $r_N(y)$ via the local immersion. Furthermore, in a neighborhood of x the co-orientation of r_W determines an orientation of the normal bundle of the local immersion of W, and we can take the fiber of the normal bundle at $r_W(x)$ to be T_yN.

Lemma 6.11. *The intersection number $I_M(W, N)$ is equal to signed count of intersection points of W and N, counting an intersection point with $+1$ if the normal co-orientation of W agrees with the orientation of N and -1 otherwise.*

Proof. This follows directly from Corollary 3.79. □

Lemma 6.12. *Suppose $W \in PC^i_\Gamma(M)$ and $N \in PC_i^\Gamma(M)$ are transverse and that $W \in Q^i(M)$. Then $I(W, N) = 0$.*

Proof. By Lemma 4.22, we know $W \times_M N \in Q_0(M)$, so $I(W, N) = \mathbf{a}(W \times_M N) = 0$ by definition and Lemma 6.8. □

Definition 6.13. Given the manifold without boundary M cubulated by X, we define the **intersection map** $\mathcal{I} : C^*_{\Gamma \pitchfork X}(M) \to K^*(X)$ by

$$\mathcal{I}(\underline{W}) = \sum_F I_M(W, F)F^*,$$

where the sum is taken over faces F of the cubulation X such that $\dim(F) + \dim(W) = \dim(M)$ and the W on the righthand side is any element of $PC^*_{\Gamma \pitchfork X}(M)$ representing $\underline{W}$.

In particular, for a face F of dimension $\dim(M) - \dim(W)$, we have

$$\mathcal{I}(\underline{W})(F) = I_M(W, F) = \mathbf{a}(W \times_M F).$$

Proposition 6.14. *The intersection map $\mathcal{I}$ is a well-defined chain map.*

Proof. If $W, W' \in PC^*_{\Gamma \pitchfork X}(M)$ are two representatives of $\underline{W}$ then $W \sqcup -W' \in Q^*(M)$ and it is transverse to X. So for any face F we have $\mathbf{a}(W \times_M F) - \mathbf{a}(W' \times_M F) = \mathbf{a}((W \sqcup -W') \times_M F) = 0$ by Lemma 6.12. So $\mathcal{I}(\underline{W})$ does not depend on the choice of W.

To see that $\mathcal{I}$ is a chain map, let W be transverse to the cubulation and representing $\underline{W}$. Then we compute for any face f of X that

$$\begin{aligned}\mathcal{I}(\partial \underline{W})(f) &= \mathbf{a}((\partial W) \times_M f) \\ &= \mathbf{a}(W \times_M \partial f) \\ &= \mathcal{I}(\underline{W})(\partial f).\end{aligned}$$

For the second equality, we use Proposition 3.77 together with the facts that the augmentations are both trivial unless $\dim(W \times_M f) = 1$ and that $\mathbf{a}(\partial(W \times_M F)) = 0$. □

Our goal now is to show that the intersection map $\mathcal{I}$ induces an isomorphism $H^i_{\Gamma \pitchfork X}(M) \to H^i(K^*(M))$ whenever $H^i_\Gamma(M)$ and $H^i(K^*(M))$ are finitely generated. Recall that we already know these groups are abstractly isomorphic by Theorems 6.5 and 5.24 and the footnote on page 82. We begin in the next section by using the cubical structure to start building an inverse map.

6.5.1. *Dualization in cubes.* Analogous to barycentric subdivisions of simplices, we will need to consider standard subdivisions of cubes. For this we let $\mathbb{J}$ denote the interval $\mathbb{I} = [0, 1]$ thought of as the (non-disjoint) union $[0, 1/2] \cup [1/2, 1]$. We can then write $\mathbb{J}^n = ([0, 1/2] \cup [1/2, 1])^n$, with the idea being that we consider $\mathbb{I}^n$ as the union of 2^n *subcubes* of side length $1/2$, each of which is the product within $\mathbb{I}^n$ of n factors, each factor is equal to either $[0, 1/2]$ or $[1/2, 1]$. We refer to $\mathbb{J}^n$ with this structure as the **central subdivision** of $\mathbb{I}^n$.

Analogously to the case with $\mathbb{I}^n$, each cube S of $\mathbb{J}^n$ possesses faces (some of which it shares with other n-cubes) consisting of the subsets of S in which some variables have been bound to the values 0, 1, or 1/2. In general we refer to such faces as **faces of** $\mathbb{J}^n$. If no variable of such a face is bound to 0 or 1, we say that we have an **internal face of** $\mathbb{J}^n$, otherwise we call it an **external** face. External faces are all subsets of $\partial\mathbb{I}^n$; internal faces are not subsets of $\partial\mathbb{I}^n$.

To each face F of $\mathbb{I}^n$, we refer to the point $\hat{F}$ at which all its free variables are equal to $1/2$ as the **center** of F. Each vertex is its own center. To each face F of $\mathbb{I}^n$ we define its **dual face in** $\mathbb{I}^n$, or simply its **dual**, to be the face $F^\vee$ of $\mathbb{J}^n$ determined as follows:

- If $i \in F_{01}$ (i.e. the coordinate x_i is free in F), then $x_i = 1/2$ in $F^\vee$.
- If $i \in F_0$ (i.e. the coordinate x_i is bound to 0 in F), then x_i is free in $[0, 1/2]$ in $F^\vee$.
- If $i \in F_1$ (i.e. the coordinate x_i is bound to 1 in F), then x_i is free in $[1/2, 1]$ in $F^\vee$.

It is clear that F and $F^\vee$ have complementary dimensions and that they intersect plainly transversely in the center of F.

Lemma 6.15. *The set function $F \to F^\vee$ is a bijection between the faces of $\mathbb{I}^n$ and the internal faces of $\mathbb{J}^n$.*

Proof. Injectivity is clear as two different faces of $\mathbb{I}^n$ will have different partitions of $\bar{n}$.

Next consider an internal face of $\mathbb{J}^n$. By definition this is a set in which some set of variables A has been bound to $1/2$ while two other sets of variables, B and C, are free on $[0, 1/2]$ or $[1/2, 1]$, respectively. But this is $F^\vee$ for the face F with partition (B, A, C). So our function is surjective. □

As each face F of $\mathbb{I}^n$ carries a natural orientation β_F determined by the order of its free variables (or the orientation 1 for vertices), this provides $F^\vee$ with the corresponding normal co-orientation. In other words, $F^\vee$ is co-oriented at all points by the co-orientation $(\beta_{F^\vee}, \beta_{F^\vee} \wedge \beta_F)$, where $\beta_{F^\vee}$ is an arbitrary orientation of $F^\vee$. Assigning $F^\vee$ this co-orientation, we interpret its embedding in $\mathbb{I}^n$ as representing a cochain in $\mathbb{I}^n$ of index $\dim(F)$. We define a map $\Psi : K^*(\mathbb{I}^n) \to PC^*_\Gamma(\mathbb{I})$ by $\Psi(F^*) = F^\vee$.

We introduce one more piece of notation for the following lemma. If f is a face in $\mathbb{J}^n$ co-oriented and considered as an element of $PC^*_\Gamma(\mathbb{I}^n)$, we let $\partial_{\mathrm{int}} f$ denote the union of the internal faces of ∂f and $\partial_{\mathrm{ext}} f$ denote the union of the external faces of f. If f is co-oriented, then we interpret the terms of $\partial_{\mathrm{int}} f$ and $\partial_{\mathrm{out}} f$ as co-oriented with the boundary co-orientations. We extend ∂_{int} to a linear operator in the obvious way.

Lemma 6.16. *For any face F of $\mathbb{I}^n$, we have*

$$\Psi(dF^*) = \partial_{int}\Psi(F^*) = \partial_{int}F^\vee.$$

Proof. We first show that there is a bijection between the interior faces of $\Psi(dF^*)$ and the set of $f^\vee$ such that the corresponding f^* have non-zero coefficients in dF^*. Then we will return to carefully consider the co-orientations.

So let F be a face of $\mathbb{I}^n$. Recall that F is determined by the partition (F_0, F_{01}, F_1) of $\overline{n}$ corresponding respectively to variables set to 0, free variables of F, and variables set to 1. By definition, $(dF^*)(f) = F^*(\partial f)$ and so the only faces participating in dF^* are those that have F as a boundary, in other words those faces whose free variables are those in F_{01} plus one more variable from F_0 or F_1.

Now let us consider again $F^\vee$, which we recall is obtained by setting all variables in F_{01} to $1/2$ and letting the variables in F_0 and F_1 become free variables on $[0, 1/2]$ or $[1/2, 1]$ respectively. Each boundary face of $F^\vee$ is then obtained by either setting one of the variables in F_0 to 0 or 1/2 or one of the variables in F_1 to 1/2 or 1. Furthermore, the internal faces are those where the variable in F_0 or F_1 has been set to 1/2. So, in summary, an internal face of $F^\vee$ has all of the variables in F_{01} as well as exactly one other variable set to 1/2 and the rest remain free over the appropriate domains, namely $[0, 1/2]$ for those in F_0 and $[1/2, 1]$ for those in F_1. But, from the definition of dualization $f \to f^\vee$, this exactly describes the duals $f^\vee$ of the faces participating in dF^*, which is sufficient due to the bijection of Lemma 6.16.

It remains now to consider the signs. We recall that the boundary formula for cubes has the form

$$\partial f = \sum_{i=1}^{k} (-1)^i (f\delta_i^0 - f\delta_i^1),$$

where the δs denote the embeddings of the faces. In $K_*(\mathbb{I}^n)$, we can shorten this notation to

$$\partial f = \sum_{i=1}^{k} (-1)^i (f_i^0 - f_i^1),$$

letting f_i^j, $j \in \{0,1\}$, denote the ith "front or back face" according to $j = 1$ or $j = 0$. In particular, we note that there are two factors, both i and j, affecting sign.

So now let us again fix a face F of $\mathbb{I}^n$ and let f be a face of dimension $\dim(F)+1$ that includes F in its boundary as $F = f_i^j$. Representing f as $f = (f_0, f_{01}, f_1)$, we obtain F by setting the ith variable of f_{01} to j. From the coboundary formula, we have

$$(dF^*)(f) = F^*(\partial f) = (f_i^j)^* \left(\sum_{i=1}^{k} (-1)^i (f_i^0 - f_i^1) \right) = (f_i^j)^*((-1)^{i+j} f_i^j) = (-1)^{i+j}.$$

So f^* occurs in dF^* with coefficient $(-1)^{i+j}$. Thus we must show that the coefficient of $f^\vee$ in $\partial_{\mathrm{int}}(F^\vee)$ is $(-1)^{i+j}$.

Now consider $F^\vee$. We can write its normal co-orientation as $(\beta_{F_0\cup F_1}, \beta_{F_0\cup F_1} \wedge \beta_{F_{01}})$. Let $k \in F_0 \cup F_1$ be the unique index that is free in f but bound in F and so also free in $F^\vee$. Then $f^\vee$ is the boundary component of $F^\vee$ with x_k set to 1/2. In this case the boundary co-orientation of $f^\vee$ in $F^\vee$ is $(\beta_{F_0\cup F_1-\{k\}}, \beta_{F_0\cup F_1-\{k\}} \wedge (-1)^{j+1}\beta_{e_k})$, where e_k is the unit vector in the direction of increasing x_k. The sign $(-1)^{j+1}$ is because if $j = 1$ then in $F^\vee$ the variable x_k is free on $[1/2, 1]$ so the inward normal vector at 1/2 points in the direction of e_k, and the opposite if $j = 0$. So the boundary co-orientation for $f^\vee$ in $\mathbb{I}^n$ as a piece of $\partial F^\vee$ is the composition

$$(\beta_{F_0\cup F_1-\{k\}}, \beta_{F_0\cup F_1-\{k\}} \wedge (-1)^{j+1}\beta_{e_k}) * (\beta_{F_0\cup F_1}, \beta_{F_0\cup F_1} \wedge \beta_{F_{01}}).$$

Meanwhile, the natural co-orientation for $f^\vee$ is

$$(\beta_{f_0\cup f_1}, \beta_{f_0\cup f_1} \wedge \beta_{f_{01}}).$$

But now we observe that $f_0 \cup f_1 = F_0 \cup F_1 - \{k\}$. So

$$\begin{aligned}
&(\beta_{F_0\cup F_1-\{k\}}, \beta_{F_0\cup F_1-\{k\}} \wedge (-1)^{j+1}\beta_{e_k}) * (\beta_{F_0\cup F_1}, \beta_{F_0\cup F_1} \wedge \beta_{F_{01}}) \\
&\qquad = (-1)^{j+1}(\beta_{f_0\cup f_1}, \beta_{f_0\cup f_1} \wedge \beta_{e_k}) * (\beta_{f_0\cup f_1} \wedge \beta_{e_k}, \beta_{f_0\cup f_1} \wedge \beta_{e_k} \wedge \beta_{F_{01}}) \\
&\qquad = (-1)^{j+1}(\beta_{f_0\cup f_1}, \beta_{f_0\cup f_1} \wedge \beta_{e_k} \wedge \beta_{F_{01}}) \\
&\qquad = (-1)^{j+i}(\beta_{f_0\cup f_1}, \beta_{f_0\cup f_1} \wedge \beta_{f_{01}})
\end{aligned}$$

In the second express after the first equal sign we have used that the expression for the co-orientation of $F^\vee$ is independent of the choice of local orientation for $F^\vee$. So we replace $\beta_{F_0\cup F_1}$ with $\beta_{f_0\cup f_1} \wedge \beta_{e_k}$. But we know that k is the ith variable of f_{01}, so $\beta_{e_k} \wedge \beta_{F_{01}} = (-1)^{i-1}\beta_{f_{01}}$, which we use in the last equality. $\square$

Lemma 6.17. *Suppose for any $n-1$ face E of $\mathbb{I}^n$ with $F < E$ we let $F^\vee_E$ denote the dual of F in E (ignoring co-orientation). Then the exterior faces $\partial_{ext}F^\vee$ of $F^\vee$ correspond exactly to the $F^\vee_E$ as E ranges over the $n-1$ faces of $\mathbb{I}^n$ containing F. Furthermore, if $F^\vee$ is given the normal co-orientation induced by F as above, then the co-orientation of $F^\vee_E \to F^\vee \to \mathbb{I}^n$ as a piece of the boundary of $F^\vee$ is given by $(\beta_{F^\vee_E}, \beta_{F^\vee_E} \wedge \beta_v \wedge \beta_F)$, where $\beta_{F^\vee_E}$ is any arbitrary orientation for $F^\vee_E$, v is an inward pointing normal at E, and β_F is the orientation of F.*

Proof. The $n-1$ face E contains F if and only if there is an $i \in F_0$ such that $i \in E_0$ or an $i \in F_1$ such that $i \in E_1$. Now from the definitions, a face f of $F^\vee$ is an external face if and only if there is an $i \in F_0$ such that x_i is bound to 0 in f or an $i \in F_1$ such that x_i is bound to 1 in f. Thus any f in $\partial_{ext}F^\vee$ must be contained in an E that contains F. Conversely, if $F < E$ and $i \in F_0$ with $x_i = 0$ defining E, then there is an external face of $F^\vee$ for which $x_i = 0$, $x_k = 1/2$ for all $k \in F_{01}$, and all other variables are free. Similarly, if $F < E$ and $i \in F_1$ with $x_i = 1$ defining E, then there is an external face of $F^\vee$ for which $x_i = 1$, $x_k = 1/2$ for all $k \in F_{01}$, and all other variables are free. So we have a bijection between external faces of $\partial F^\vee$ and the $n-1$ faces E containing F. Furthermore, after we have set x_i to 0 or 1 as appropriate, the behavior in the remaining variables shows that such a face has precisely the form of $F^\vee_E$.

It remains to consider the co-orientations. Suppose f is an external face of $F^\vee$ in the $n-1$ face E, i.e. $f = F^\vee_E$. The co-orientation of $F^\vee$ in $\mathbb{I}^n$ and the co-orientation of f in E are both the normal co-orientations determined by the orientation of F. As f is part of $\partial F^\vee$, its orientation in $\mathbb{I}^n$ is therefore the composite

$$(\beta_f, \beta_f \wedge \beta_v) * (\beta_f \wedge \beta_v, \beta_f \wedge \beta_v \wedge \beta_F) = (\beta_f, \beta_f \wedge \beta_v \wedge \beta_F),$$

where β_f is any arbitrary orientation for f, v is the inward pointing normal of f in $F^\vee$, which in this case is also an outward pointing normal of E in $\mathbb{I}^n$, and we use $\beta_f \wedge \beta_v$ as a convenient orientation for $F^\vee$. $\square$

6.5.2. *Dualization in cubical complexes.* Now suppose that X is any cubical complex. We obtain the **centrally subdivided cubical complex** $sd(X)$ by replacing each cube $\mathbb{I}^n$ with $\mathbb{J}^n$. Now suppose that M is an n-manifold without boundary cubulated by the cubical complex X via $\phi : |X| \to M$. For a given face F of an n-cube B of X, we can extend the above definitions, and abuse notation, by letting F refer also to the image of F in M and letting $F^\vee_B$ also denote the composition $F^\vee_B \hookrightarrow \mathbb{I}^n \hookrightarrow |X| \xrightarrow{\phi} M$, where $F^\vee_B$ is the dual of F in B. Similarly, we consider this version of $F^\vee_B$ in M to be co-oriented by the normal co-orientation obtained from the orientation of F. Moreover, in the cubulated M we extend our earlier definition and write $F^\vee = \sqcup_{F<B} F^\vee_B$, where the union is taken over all n-cubes B having F as a face. We similarly extend Ψ so that $\Psi(F^*) = F^\vee = \sum_{F<B} F^\vee_B$.

Lemma 6.18. *Ψ gives a chain map $K^*_X(M) \to C^*_\Gamma(M)$.*

Proof. We will show that for any face F of the cubulation X of M we have $\Psi(\partial F^*) = \partial \Psi(F^*)$.

By Lemma 6.16 and the definition of the extended Ψ, we know that $\Psi(dF^*) = \sum_{F<B} \partial_{\text{int}} F^\vee_B$, so we must only show that $\sum_{F<B} \partial(F^\vee_B) = \sum_{F<B} \partial_{\text{int}}(F^\vee_B)$, in other words that all of the external faces of the terms of $\Psi(dF^*)$ cancel in the sum $\sum_{F<B} \partial(F^\vee_B)$.

For this, let us fix an n-cube B containing F, and let E be an $n-1$ face of B containing F. As M is an n-manifold, there is exactly one other n-cube, say B_1, in the cubulation that shares the face E with B. Furthermore, by Lemma 6.17, E contains external faces of $\partial(F^\vee_B)$ and $\partial(F^\vee_{B_1})$ and they both correspond geometrically to $F^\vee_E$. Also by the lemma, the co-orientation as a boundary piece of $F^\vee_B$ is $(\beta_{F^\vee_E}, \beta_{F^\vee_E} \wedge \beta_{v_B} \wedge \beta_F)$, where v_B is an inward pointing vector of B at E, and the co-orientation as a boundary piece of $F^\vee_{B_1}$ is $(\beta_{F^\vee_E}, \beta_{F^\vee_E} \wedge \beta_{v_{B_1}} \wedge \beta_F)$, where v_{B_1} is an inward pointing vector of B_1 at E. Since a normal vector to E that is inward pointing with respect to B_1 is outward pointing with respect to B, and vice versa, these are opposite co-orientations, so these two boundary pieces cancel in $\partial\Psi(F^*)$. $\square$

Morally speaking, Ψ will be our inverse to the intersection map $\mathcal{I}$, but unfortunately the elements of $PC^*_\Gamma(M)$ that represent the image of Ψ are not transverse to the cubulation. So we need another step.

6.5.3. *Pushing the dual cubulation.* To remedy the problem with the image of Ψ consisting of elements of $PC^*_\Gamma(M)$ that are not transverse to the cubulation, we construct a homotopy of M to itself that pushes these cochains into transverse position with respect to the cubulation, which we regard as fixed. Moreover, we want to do so in a way such that the intersection number of the pushed $F^\vee$ with F will be 1, as expected. Constructing such a homotopy is the purpose of the following technical lemma. As we will see in the proof, the actual construction is a bit fiddly in order to ensure all the needed properties. The payoff is in Theorem 6.20, which can be found in the next subsection.

Lemma 6.19. *Suppose M is a manifold without boundary with a cubulation X. Then there is a smooth map $h : M \to M$ such that for each face F of X the following hold:*

1. $h|_{F^\vee} : F^\vee \to M$ *is transverse to* X
2. *the only* $\dim(F)$*-face of* X *that intersects* $h(F^\vee)$ *is* F
3. $I_M(h(F^\vee), F) = 1$.

Proof. We will construct our homotopy in multiple steps. The basic idea is first to construct small isotopies near the centers $\hat{F}$ of the faces F that push the corresponding $F^\vee$ into transverse position with F and so that the intersection number of the shifted $F^\vee$ with F is 1, giving the third condition. These will also be designed to ensure enough transversality in neighborhoods of the $\hat{F}$. Then we perform a more global shift to push the rest of $sd(X)$ into transverse position with X to achieve the first condition, while leaving fixed the isotopies already constructed near the $\hat{F}$. We also ensure the global shift is small enough to provide the second condition and also not disrupt the third.

Throughout we may assume that M is embedded properly in some $\mathbb{R}^N$ with an ϵ-neighborhood M^ϵ (with ϵ a function of $x \in N$) and a submersion $\pi : M^\epsilon \to M$; see [**?**, Section 2.3]. We can then consider M as a metric space with the subspace metric.

To create a template and explain the basic idea, we first consider the standard $m = \dim(M)$-cube $\mathbb{I}^m = [0,1]^m \subset \mathbb{R}^m$. Let e_i denote the unit tangent vector of $\mathbb{R}^m$ pointing in the positive ith direction. Let $f = (f_0, f_{01}, f_1)$ be a face of $\mathbb{I}^m$, let $\hat{f}$ be its center, and define the vector v_f by $v = \sum_{i \in f_1} e_i - \sum_{i \in f_0} e_i$ (if $F_0 = F_1 = \emptyset$, let $v = 0$). At points of f, the vector v_f points outward from the cube. The vector v_f is also tangent to $f^\vee$, and for small $\delta > 0$ the points $\hat{f} - \delta v_f$ lie in the interior of the face $f^\vee$. So, if we were to translate $f^\vee$ by δv_f for a sufficiently small positive δ, we would obtain a translate of $f^\vee$ within the $m - \dim(f)$ plane containing it. As this plane is orthogonal to f and intersects it at the single point $\hat{f}$, the only intersection of f with the translated $f^\vee + \delta v_f$ is at $\hat{f}$ with a point in the image of the interior of $f^\vee$. As the translation preserve the tangent plane of $f^\vee$, this is a transverse intersection, and the translation preserves the normal co-orientation so that the intersection number of $f^\vee + \delta v_f$ with f is 1. Furthermore, these properties are preserved under a small perturbation of v_f. But now we note that by the Transversality Theorem of [**?**, GuPo74]hat for almost all vectors u in $\mathbb{R}^m$, translation by u takes all open faces of $\mathbb{J}^m$ into transverse position with respect to all open faces of $\mathbb{I}^m$. In particular, there is such a vector z_f arbitrarily close to δv_f.

Next, suppose we are given a small ball B^f_r centered at $\hat{f}$ in $\mathbb{R}^m$ with $r < 1/4$ so that B^f_r does not intersect any faces of $\mathbb{I}^m$ that do not intersect f. Let r_1 be such that $0 < r_1 < r$, and let $B^f_{r_1}$ similarly be the ball of radius r_1 centered at $\hat{f}$. We choose a smooth function $\eta : \mathbb{R}^m \to [0,1]$ such that $\eta(x) = 1$ on $B^f_{r_1}$ and $\eta(x) = 0$ outside of B^f_r. Now choose $\delta \ll r_1$ and let Z be the vector field on $\mathbb{R}^m$ given by $Z(x) = \eta(x) z_f$, with $|z_f - \delta v_f| \ll \delta$. Let $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ be the map obtained by flowing by Z from $t = 0$ to $t = 1$. Outside of B^F_r, the map Φ is the identity. If we take δ and $|z_f - \delta v_f|$ sufficiently small with respect to r_1 then there is a closed ball neighborhood W_f of $\hat{f}$ in $B^f_{r_1}$ containing both $\hat{f}$ and $\hat{f} - z_f$ on which the map Φ acts identically to the translation of the preceding paragraph. Later we will also need further closed ball neighborhood W^f_1 and W^f_2 of $\hat{f}$ with $W^f_1 \subset int(W^f)$, $W^f_2 \subset int(W^f_1)$, and such that $\hat{f}, \hat{f} - z_f \in W^f_2$. This is again possible by taking δ and the small perturbation z_f of δv_f sufficiently small.

We also observe that the distance between the set $\Phi(f^\vee - (f^\vee \cap W_f^2))$ and the union of $\dim(F)$-faces of $\mathbb{I}^m$ is positive.

We now translate this procedure to the cubulation of M. Suppose F is a face of the cubulation X, and choose an m-cube E of the cubulation containing F as a face. By assumption there is a smooth diffeomorphism ϕ from $\mathbb{I}^m$ to E that is compatible with the cubical structure. Let f be the face of $\mathbb{I}^m$ that maps to F. Then ϕ takes $\hat{f}$ to the center $\hat{F}$ and $f^\vee_{\mathbb{I}^m}$ to one of the components of $F^\vee$. By the definition of smooth maps, there is a neighborhood U of $\hat{f}$ in $\mathbb{R}^m$ on which there is defined a smooth map $\phi_U : U \to M$ and such that ϕ_U agrees with ϕ on $U \cap \mathbb{I}^m$. As ϕ is an embedding at $\hat{f}$, by the Inverse Function Theorem we may choose U such that ϕ_U is a diffeomorphism from U onto its image, and by making U smaller if necessary, we can also assume U to be an open Euclidean ball B_r^F centered at $\hat{f}$ whose image in M intersections only faces of X that contain F. Identifying this ball in $\mathbb{R}^m$ with its image in M via ϕ, we can perform a flow Φ as in the preceding paragraph to push $F_E^\vee$ into transverse position with F (note: we keep F fixed under the flow) and with intersection number 1. Furthermore, this procedure pushes the other components $F_C^\vee$ for $C \neq E$ away from F (slightly), and so $I(\Phi(F^\vee), F) = 1$ and also the distance between the set $\Phi(F^\vee - (F^\vee \cap W_F^2))$ and the union of $\dim(F)$-faces of $\mathbb{I}^M$ is positive. By rechoosing our z_f if necessary (among almost all vectors arbitrarily close to δv_f, we can ensure that the restriction of Φ to the intersection of any face of $sd(X)$ with W_F^2 is transverse to the cubulation.

Next, we can apply this procedure at all faces F simultaneously by choosing a sufficiently small B_r^F neighborhood of each $\hat{F}$ so that they are all disjoint; we may let each r, r_1, and δ depend on F though we do not include this in the notation. We then generalize Φ by allowing a corresponding flow on all balls simultaneously. This provides a smooth map $\Phi : M \to M$ that satisfies the second two conditions of the theorem, as we can also choose the balls small enough that the ball around $\hat{F}$ does not intersect any other $\dim(F)$ face of X. We define $W = \cup_F W_F$, $W^1 = \cup_F W_1^F$, and $W^2 = \cup_F W_2^F$. By construction, the restriction of Φ to the intersection of W with any face of $sd(X)$ is transverse to all faces of X. We must further modify Φ to ensure transversality in general.

As in the proof of Theorem 6.5, we next follow the construction in [**?**, Section 2.3]. We have M embedded in some Euclidean space $\mathbb{R}^N$ with an ϵ-neighborhhood M^ϵ and a submersion $\pi : M^\epsilon \to M$. As we are happy with the map Φ as constructed so far on W, we let $\rho : M \to [0, 1)$ be a smooth function that is 0 on W^1 and > 0 on $M - W_1$. We will fine tune ρ a bit more soon. Let S be the unit ball in $\mathbb{R}^N$. We now consider the map $H : M \times S \to M$ defined by $H(x, s) = \pi(\Phi(x) + \rho(x)\epsilon s)$. At all points (x, s) such that $\rho(x) > 0$, i.e. on $M - W_1$, this is a submersion (and so transverse to all faces of X), and for all (x, s) such that $\rho(x) = 0$, i.e. on W_1, we have $H(x, s) = \Phi(x)$.

Now let G be the interior of any face of the cubical subdivision $sd(X)$. At any point $x \in G \cap W_1$, we already have that for any fixed $s_0 \in S$ the map $H|_G(-, s_0) = \Phi|_G(-)$ is transverse at x to any face of X. Furthermore, by the Transversality Theorem of [**?**, **?**, GuPo74] for almost every $s_0 \in S$ and for any face F of X, the map $H|_G(-, s_0)$ is transverse to the interior of F at all points on the submanifold $G - G \cap W_1$ of G. But there are a finite number of faces of X, so for almost every $s_0 \in S$ and for every face F of X, the map $H|_G(-, s_0)$ is transverse to the interior of every face of F, which implies it is transverse to every face of F. But there are also only a finite number of faces of $sd(X)$, and so for almost every $s_0 \in S$, $H|_G(-, s_0)$ is transverse to F for every G and every F. In particular, each $H|_{F^\vee}$ determines an element of $PC^*_{\Gamma \pitchfork X}(M)$ (that $H|_{F^\vee}$ is the output of an ambient isotopy in a neighborhood of $\hat{F}$ allows us to co-orient the components of H via orientations of their tubular neighborhoods, which are also pushed around isotopically).

To complete the proof we must do one last thing: we must fine tune ρ to ensure that in forming $H(-, s_0)$ to ensure all of the required transversality we have not pushed any $F^\vee$ so far as to create new intersections with faces of X of complementary dimensions beyond the single intersection of $H(F^\vee, s_0)$ with F inside W. For this we do the following.

Recall as noted above that, by our construction, if F is any face of X then the distance between $\Phi(F^\vee - (F^\vee \cap W^2))$ and the union of $\dim(F)$-faces of X is positive. Now suppose $x \in M - W_2$ and consider a compact neighborhood $\bar{U}_x$ of x in $M - W_2$. By the above, if $\Phi(\bar{U}_x \cap F^\vee)$, then there is a positive distance $\varepsilon_{x,F}$ between $\bar{U}_x \cap F^\vee \neq \emptyset$ and the $\dim(F)$-skeleton of the cubulation. As $\bar{U}_x$ is compact, it intersects only a finite number of $F^\vee$ as F ranges over all faces of X, and we let

$$\varepsilon_x = \min\{\varepsilon_{x,F} \mid \bar{U}_x \cap F^\vee \neq \emptyset\}.$$

So, by construction, any map $g : M \to M$ such that $d(z, g(z)) < \varepsilon_x$ for all $z \in \bar{U}_x$ satisfies the property that if $z \in F^\vee \cap \bar{U}_x$ then $g(z)$ is not contained in a $\dim(F)$ face of X.

Now, suppose we have constructed such a ε_x for all x in $M - W_2$. Then the interiors U_x of the $\bar{U}_x$ cover $M - W_2$, and we can take a locally finite refinement $\mathcal{U}$. By Lemma 6.7, we can find a smooth function $\rho_1 : M - W_2 \to \mathbb{R}$ so that $0 < \rho_1(z) < \varepsilon_x$ if $z \in \bar{U}_x$ for $U_x \in \mathcal{U}$. Let $\rho_2 = \rho\rho_1 : M \to \mathbb{R}$. This is smooth and well defined on all of M as $\rho(x) = 0$ for all $x \in W_1$ and $W_2 \subset int(W_1)$. We also see that $0 < \rho_2(x) < \rho_1(x)$ for all $x \in M - W_2$. So now if we replace H with $H_2 : M \times S \to M$ defined by $H_2(x, s) = \pi(\Phi(x) + \rho_2(x)\epsilon s)$, then for any s_0 and for any $x \in (M - W_2) \cap F^\vee$, we have that $H_2(x, s)$ does not intersect any face of dimension $\dim(F)$ as desired. We can now set $h(-) = H_2(-, s_0)$ for almost all s_0 to achieve all three required conditions, noting that the conclusions of the preceding paragraphs did not depend on the choice of ρ. □

6.5.4. *The intersection map is an isomorphism for finitely-generated cohomology groups.*

Theorem 6.20. *If M is a manifold without boundary cubulated by X, the intersection map $\mathcal{I} : H^i_{\Gamma\pitchfork X}(M) \to H^i(K^*_X(M))$ is an isomorphism whenever $H^i(M)$ is finitely generated. It is a surjection even if $H^i(M)$ is not finitely generated.*

Proof. Above we constructed a chain map $\Psi : K^*_X(M) \to C^*_\Gamma(M)$ by taking F^* to the geometric cochain represented by the inclusion of $F^\vee$ into M, but the image did not lie in $C^*_{\Gamma\pitchfork X}(M)$. We now modify that construction to define $\psi : K^*_X(M) \to C^*_{\Gamma\pitchfork X}(M)$ by taking F^* to the element of $C^*_{\Gamma\pitchfork X}(M)$ represented by the composition of the inclusion of $F^\vee$ into M with h. To see that this is a chain map, we observed that Ψ commutes with boundaries in the proof of Lemma 6.18, and we know that h commutes with boundaries by the discussion leading up to Proposition 5.8, co-orienting h using that it is homotopic to the identity. Furthermore, by construction $\psi(F^*)$ represents a map that is transverse to the cubulation for each generator F^* of $K^*_X(M)$. So ψ determined a chain map to $C^*_{\Gamma\pitchfork X}(M)$ as desired.

Furthermore, we see by the preceding lemma and the definition that $\mathcal{I}\psi$ is the identity map. In particular, then, $\mathcal{I}$ induces a cohomology surjection $H^*_{\Gamma\pitchfork X}(M) \twoheadrightarrow H^*(K^*_X(M))$.

Now, we know from Theorem 5.24 and the isomorphism between cubical and singular cubical cohomologies that these groups are both isomorphic to $H^*(M)$, and a surjective map of isomorphic finitely-generated abelian groups is an isomorphism (as $\mathbb{Z}$-is Noetherian). □

We conjecture that $\mathcal{I}$ is an isomorphism in the general case but have not been able to prove this.

Remark 6.21. Putting together the isomorphism $\mathcal{I} : H^i_{\Gamma\pitchfork X}(M) \to H^i(K^*_X(M))$ with the inverse of the isomorphism $H^*_{\Gamma\pitchfork X}(M) \to H^*_\Gamma(M)$, it is sometimes useful to abuse notation and speak of the intersection-induced isomorphism $H^*_\Gamma(M) \to H^i(K^*_X(M))$. Of course this map is given by taking a cohomology class representative that is transverse to the cubulation and applying the intersection map $\mathcal{I}$ to find a cubical cocycle representing the target cohomology class.

7. Products of geometric chains and cochains

In this section we consider products of geometric chains and cochains, first simply as chains and cochains and then as pairings on homology and cohomology. These pairings are all built from the fiber products and exterior products of maps as defined in Section 3. However, while the exterior products were fully defined, the fiber products required transversality of $f : V \to M$ and $g : W \to M$ in order for $V \times_M W$ to be a well-defined manifold with corners possessing an oriented or co-oriented map to M. Consequently, the fiber products do not induce fully-defined chain- and cochain-level products such as a pairing $C^*_\Gamma(M) \otimes C^*_\Gamma(M) \to C^*_\Gamma(M)$. At best we can hope for a partially-defined (co)chain-level pairing, though even this is not clear once we take into account that a geometric chain or cochain is not represented simply by a single map $V \to M$ but is rather an equivalence class of such mappings up to triviality and degeneracy.

In Section 7.1, we address this issue and show that there is a natural notion of transversality among chains and cochains, despite the ambiguity in representing chains and cochains. We use this to provide partially-defined cup, cap, and intersection pairings among geometric chains and cochains. We consider it important to have such pairings, even when only partially defined, as cochain algebras contain much information that is lost on passage to cohomology. For example, the singular cochains of a space are what carry the E_∞-algebra structure, while passage to cohomology often contains just shadows of this structure, such as the Steenrod squares. We also provide fully-defined chain and cochain exterior products, though these give us less trouble. In Section 7.2, we collect the various properties of these partially-defined products, mostly based on properties we have established for fiber products of maps in earlier sections.

In Section 7.3, we then turn to the resulting products in geometric homology and cohomology, which we show are fully defined, providing cup, cap, intersection, and exterior products. We also utilize this machinery along with our cubulations to construct a universal coefficient theorem when the cohomology is finitely generated. In Section 7.5, we show that the cup product in geometric cohomology is isomorphic to the singular cohomology cup product.

7.1. **Chain- and cochain-level products and transversality.** We begin by defining a simple version of transversality for geometric chains and cochains and the resulting (co)chain-level cup, cap, and intersection products. Geometric chains and (co)chains $\underline{V}$ and $\underline{W}$ will be called **simply transverse** if they possess transverse representing pre(co)chains. This allows us to define cup, cap, and intersection products of transverse chains and cochains via their fiber products. After a thorough study of this case, we will consider some more complex situations. For example, one might consider the case where $\underline{V}$ and $\underline{W}$ can be represented by disjoint unions of pre(co)chains $V = \bigsqcup V_i$ and $W = \bigsqcup W_j$ such that the pairs (V_i, W_j) are not necessarily transverse but such that for each such pair there are alternative representatives, say (V'_{ij}, W'_{ji}), depending on both i and j, with $\underline{V_i} = \underline{V'_{ij}}$, $\underline{W_j} = \underline{W'_{ji}}$, and V'_{ij} transverse to W'_{ji}. Then one would like to have products of $\underline{V}$ and $\underline{W}$ using these representatives, but we must show independence of the choices involved. Such examples, and even more general ones, are critical for obtaining a version of multilinearity for our partially-defined products. But we begin with a more basic scenario.

Definition 7.1. We say that $\underline{V}, \underline{W} \in C^*_\Gamma(M)$ are **simply transverse** as geometric cochains if there exist representatives $V, W \in PC^*_\Gamma(M)$ such that V and W are transverse as manifolds with corners mapping to M. We call the data of such a pair (V, W) a **simple transverse representation** for the pair $(\underline{V}, \underline{W})$.

We define simple transversality similarly if $\underline{V} \in C^*_\Gamma(M)$ and $\underline{W} \in C^\Gamma_*(M)$ or if M is oriented and $\underline{V}, \underline{W} \in C^\Gamma_*(M)$.

Definition 7.2. For $\underline{V}, \underline{W} \in C^*_\Gamma(M)$ simply transverse, we define the **cup product** $\underline{V} \uplus \underline{W} \in C^*_\Gamma(M)$ to be the geometric cochain represented by the fiber product $V \times_M W$ for some simple transverse representation (V, W) of $(\underline{V}, \underline{W})$.

Analogously, if $\underline{V} \in C^*_\Gamma(M)$ and $\underline{W} \in C^\Gamma_*(M)$ are simply transverse, we define the **cap product** $\underline{V} \Cap \underline{W} \in C^\Gamma_*(M)$ by $V \times_M W$ for some simple transverse representation (V, W). If M is oriented and $\underline{V}, \underline{W} \in C^\Gamma_*(M)$ are simply transverse, we define the **intersection product** $\underline{V} \bullet \underline{W} \in C^\Gamma_*(M)$ by $V \times_M W$ for some simple transverse representation (V, W).

In each context, the given product $V \times_M W$ is as defined in Definition 4.21, as V and W are transverse by assumption.

The first main result of this section is to show that these products are well defined as operations on transverse geometric chains or cochains, independent of the decomposition chosen. This is not immediately clear, as a geometric cochain $\underline{V}$ has in general an infinite number of representatives which may or may not be transverse to any other given element of $PC(M)$, and it is not evident that the difference can be entirely swept into $Q(M)$. For example, consider a cochain represented by an element $V \in PC(M)$ that has a single connected component and small rank but whose boundary is not trivial or degenerate. Suppose further that V is transverse to some $W \in PC(M)$. Then (V, W) is a simply transverse pair. But now if we replace V with another connected element V' that also has small rank but with $\partial V = \partial V'$, then $\underline{V} = \underline{V'}$ as $V \sqcup -V' \in Q(M)$, and V' may or may not be transverse to W, though V' may be transverse to some other W' with $\underline{W} = \underline{W'}$. There is some finer work to be done in showing that $V \times_M W$ and $V' \times_M W'$ represent the same cochain.

This will be accomplished using the following decomposition of prechains and precochains. To write the definition simultaneously for prechains and precochains, we adopt the following conventions. Let W be a connected manifold with corneres. We say that "V is isomorphic to $\pm W$" if V and W are either both elements of $PC_*^\Gamma(M)$ or both elements of $PC^*_\Gamma(M)$ and V is oriented or co-oriented isomorphic[30] either to W or to W with its opposite orientation or co-orientation, as appropriate. We will say that "V has n components isomorphic to W, counting without sign" if V has exactly n components that are isomorphic to $\pm W$. We will say that "V has n components isomorphic to W, counting with sign" if n is the number of components of V isomorphic to W minus the number of components of V isomorphic to $-W$. We will only count with sign in contexts in which W is not isomorphic to $-W$, so there should be no ambiguity. We also note that W being isomorphic to $-W$ is the same as W being trivial.

Definition 7.3. Suppose $V \in PC(M)$ and V_1 is a connected component of V.

- We call V_1 **trivially inessential** if either V_1 is trivial or if the number of components of V isomorphic to $\pm V_1$ is 0, counting with signs.
- We call V_1 **nontrivially inessential** if it is not trivially essential but it is of small rank.
- We call V_1 **essential** if it is not (trivially or nontrivially) inessential.

Each connected component of V falls into exactly one of these categories by definition.

The **essential decomposition** of V is the unique decomposition of V into essential, trivially inessential, and nontrivially inessential components, written $V = V_E \sqcup V_{TI} \sqcup V_{NI}$.

In an essential decomposition, $V_{TI} \in Q(M)$, and in fact V_{TI} is trivial as it decomposes as a union of connected trivial components and pairs of the form $V_1 \sqcup -V_1$. On the other hand, V_{NI} may or may not be in $Q(M)$ depending on whether or not ∂V_{NI} contains any essential components. Nonetheless, a component V_1 of V_{NI} is "inessential" in the sense that if V_1' is another small rank co-oriented manifold over M with $\partial V_1 = \partial V_1'$ then $\underline{V_1} = \underline{V_1'}$ in $C^*_\Gamma(M)$ or C_*^Γ as $V_1 \sqcup -V_1' \in Q(M)$. Thus V_1 is not required to appear as a component in a representative of $\underline{V}$; in any representation of $\underline{V}$ we could replace each V_1 with V_1'. By contrast, Lemma 7.6, which we will prove momentarily, shows that essential components really are essential in that they appear in any representation of $\underline{V}$.

Example 7.4. Let V be any connected oriented manifold with corners that does not possess an orientation reversing diffeomorphism, and let $r_V : V \to M$ be any map that is not of small rank. Then $V = V_E$ is essential.

Example 7.5. Consider Example 4.6, which consisted of the projection of the 2-simplex $V \subset \mathbb{R}^2$ with vertices at $(1, 0)$, $(-1, 0)$, and $(0, 1)$ to the x-axis. This map has small rank, but its boundary is not trivial or of small rank. In this case $V = V_{NI}$.

We can make this example a bit more interesting as follows. Instead of the codomain being $\mathbb{R}^1$, we let the codomain be $\mathbb{R}^2$. We continue to let most of the map $r_V : V \to \mathbb{R}^2$ be the projection to the x-axis, but

[30] Recall **??????**.

let us choose a Euclidean ball in the interior of the simplex and draw its image out into a 1-dimensional "thread" in the plane, analogously to how one changes basepoints for an element of some $\pi_2(M)$. With some care, this can be done smoothly. Any two such maps represent the same chain or cochain (depending on whether we assign orientations or co-orientations), but depending on the choice of thread, one such example may be transverse to some other (co)chain W while another such example may not be transverse to W.

Lemma 7.6. *Let $\underline{V} = \underline{W} \in C^*_\Gamma(M)$ (or $\underline{V} = \underline{W} \in C_*^\Gamma(M)$) be represented by $V, W \in PC(M)$, and suppose V_1 is an essential component of V. Then the number of components isomorphic to V_1 in V_E is equal to the number of such components in W_E, counting with sign. In particular, $V_E \sqcup -W_E$ is trivial and each component of $V_E \sqcup -W_E$ appears an even number of times when counting without sign. Furthermore, if V_1 is an essential component of V then it is also an essential component of W.*

Proof. We first note that if a component V_1 of V is in V_E then every component of V isomorphic to $\pm V_1$ must also be in V_E.

Let V_1 be an essential component of V. Then V_1 is also a a component of $V \sqcup -W \in Q^(M)$. We write $V \sqcup -W = T \sqcup D$, where T is trivial and D degenerate. As V_1 does not have small rank, it cannot be a component of D; thus V_1 must be a component of T. Moreover, it cannot possess an orientation-reversing self-diffeomorphism ρ such that $r_{V_1} \circ \rho = r_{V_1}$ or it would not be essential. Therefore, T must contain a number of components isomorphic to $\pm V_1$ that is 0 when counted with sign (compare with the proof of Lemma 4.11) and so an even number when counted without sign. Since V_1 is essential in V, all components of V isomorphic to $\pm V_1$ must also be contained in V_E and the number of such components cannot be 0 when counted with sign in V alone or V_1 would not be essential. So there must be isomorphic components (up to sign) in W, and the number of such components in W also cannot be 0 when counted with sign. Thus all such components in W must be in W_E.

Altogether, we have shown that for each component V_1 of V_E, there are isomorphic components (up to sign) contained in W_E and the total number of components isomorphic to $\pm V_1$ in $V_E \sqcup -W_E$, counted with sign, is 0. As this applies for all components of V_E, and similarly for all components of W_E, we see that $V_E \sqcup -W_E$ is trivial, and that the number of components isomorphic to V_1, counted with sign, in V_E must equal the number, counted with sign, in W_E. It follows that V_1 appears an even number of times in $V_E \sqcup -W_E$ when counted without sign. $\square$

Corollary 7.7. *Suppose $\underline{V} = \underline{V}' \in C^*_\Gamma(M)$ (or $C_*^\Gamma(M)$) and that V_E is transverse to some $r_W : W \to M$. Then V'_E is also transverse to W.*

Proof. By the lemma, any component of V_E is also a component of V'_E and vice versa. $\square$

Lemma 7.8. *If $\underline{V} = \underline{W} \in C^*_\Gamma(M)$ (or $C_*^\Gamma(M)$) then $\underline{V_{NI}} = \underline{W_{NI}}$ and $\underline{\partial V_{NI}} = \underline{\partial W_{NI}}$.*

Proof. By assumption $V \sqcup -W = (V_E \sqcup V_{TI} \sqcup V_{NI}) \sqcup -(W_E \sqcup W_{TI} \sqcup W_{NI})$ is in $Q(M)$. We know $V_{TI} \sqcup -W_{TI} \in Q(M)$ as each summand is in $Q(M)$, and $V_E \sqcup -W_E \in Q(M)$ by Lemma 7.6. Thus $V_{NI} \sqcup -W_{NI} \in Q(M)$ by Lemma 4.13, and the second equality follows similarly after taking boundaries. $\square$

Theorem 7.9. *Given simply transverse $\underline{V}$ and $\underline{W}$, the cup, cap, or intersection products of Definition 7.2 are well defined, independent of choice of transverse decomposition.*

Proof. For simplicity, we just give the proof for $\uplus$; the other arguments are identical. We also write $V + W$ and $V - W$ rather than $V \sqcup W$ and $V \sqcup -W$ to make the following easier to read.

Let $\underline{V}, \underline{W}$ be transverse geometric cochains with simply transverse representatives V and W as in Definition 7.1. Similarly, let (V', W') be another transverse representation. By assumption, $V - V', W - W' \in Q(M)$. For each of these precochains, we have their essential decompositions $V = V_E \sqcup V_{TI} \sqcup V_{NI}$, etc.

We must show that $V \times_M W$ and $V' \times_M W'$ represent the same element of $C^*_\Gamma(M)$, i.e. that

$$[(V_E + V_{TI} + V_{NI}) \times_M (W_E + W_{TI} + W_{NI})] - [(V'_E + V'_{TI} + V'_{NI}) \times_M (W'_E + W'_{TI} + W'_{NI})]$$

is in $Q^*(M)$. Writing out, we have

$$\begin{aligned}(V_E + V_{TI} + V_{NI}) \times_M (W_E + W_{TI} + W_{NI}) =& V_E \times_M W_E + V_E \times_M W_{NI} + V_{NI} \times_M W_E + V_{NI} \times_M W_{NI}\\ &+ V_{TI} \times_M (W_E + W_{TI} + W_{NI}) + (V_E + V_{NI}) \times_M W_{TI}.\end{aligned}$$

As $V_{TI}, W_{TI} \in Q(M)$, the terms on the second line are all in $Q(M)$ by Lemma 4.22. A similar decomposition holds for the primed versions, so we must show that

$$(V_E \times_M W_E + V_E \times_M W_{NI} + V_{NI} \times_M W_E + V_{NI} \times_M W_{NI}) - (V'_E \times_M W'_E + V'_E \times_M W'_{NI} + V'_{NI} \times_M W'_E + V'_{NI} \times_M W'_{NI}) \tag{15}$$

is in $Q(M)$.

From Corollary 7.7 we know that if V_E is transverse to W, and in particular W_E, then so is V'_E, and so we can form $V'_E \times_M W$. Thus, up to the element $-V'_E \times_M W_E + V'_E \times_M W_E \in Q(M)$, we have

$$\begin{aligned}V_E \times_M W_E - V'_E \times_M W'_E &= V_E \times_M W_E - V'_E \times_M W_E + V'_E \times_M W_E - V'_E \times_M W'_E\\ &= (V_E - V'_E) \times_M W_E + V' \times_M (W_E - W'_E).\end{aligned} \tag{16}$$

By Lemma 7.6, $V_E - V'_E$ and $W_E - W'_E$ are trivial, and the transverse intersection of any cochain with a trivial cochain is in $Q^*(M)$ by Lemma 4.22.

The remaining terms of (15) each involve an element of small rank and so are of small rank by Lemma 4.22. So to see that what's left is an element of $Q(M)$, it suffices to show that the boundary of the remaining terms consists of cochains that are trivial or of small rank. Letting $m = \dim M$ and $v = \dim V$, this boundary is

$$\begin{aligned}&(\partial V_E) \times_M W_{NI} + (-1)^{m-v} V_E \times_M \partial W_{NI} + (\partial V_{NI}) \times_M W_E + (-1)^{m-v} V_{NI} \times_M \partial W_E\\ &\quad + (\partial V_{NI}) \times_M W_{NI} + (-1)^{m-v} V_{NI} \times_M \partial W_{NI} - (\partial V'_E) \times_M W'_{NI} - (-1)^{m-v} V'_E \times_M \partial W'_{NI}\\ &\qquad - (\partial V'_{NI}) \times_M W'_E - (-1)^{m-v} V'_{NI} \times_M \partial W'_E - (\partial V'_{NI}) \times_M W'_{NI} - (-1)^{m-v} V'_{NI} \times_M \partial W'_{NI}.\end{aligned} \tag{17}$$

Every term except for the pairs $(-1)^{m-v} V_E \times_M \partial W_{NI} - (-1)^{m-v} V'_E \times_M \partial W'_{NI}$ and $(\partial V_{NI}) \times_M W_E - (\partial V'_{NI}) \times_M W'_E$ involves an intersection with a summand of small rank and so is of small rank by Lemma 4.22.

For $V_E \times_M \partial W_{NI} - V'_E \times_M \partial W'_{NI}$, we recall that the data of two maps being transverse includes the assumption of transversality on boundaries. So as in our argument employed above to show $V_E \times_M W_E - V'_E \times_M W'_E$ is trivial, we know that V_E and V'_E must each be transverse to both ∂W_{NI} and $\partial W'_{NI}$ by Corollary 7.7. So by an analogous computation we arrive at

$$V_E \times_M \partial W_{NI} - V'_E \times_M \partial W'_{NI} = (V_E - V'_E) \times_M \partial W_{NI} + V'_E \times_M (\partial W_{NI} - \partial W'_{NI}),$$

up to an element of $Q(M)$. We then have that $V_E - V'_E$ is trivial by Lemma 7.6, and $\partial W_{NI} - \partial W'_{NI}$ is in $Q(M)$ by Lemma 7.8. Thus these are in $Q(M)$ by Lemma 4.22, and so $V_E \times_M \partial W_{NI} - V'_E \times_M \partial W'_{NI} \in Q(M)$. By an analogous argument $(\partial V_{NI}) \times_M W_E - (\partial V'_{NI}) \times_M W'_E \in Q(M)$. In particular, these can be written as disjoint unions of components that are trivial or of small rank.

Therefore, the boundary (17) can be completely decomposed into cochains that are trivial or have small rank, and this completes our proof that $V \times_M W - V' \times_M W' \in Q(M)$. □

Now, suppose a geometric cochain $\underline{V}$ is simply transverse to two other geometric cochains of the same degee, $\underline{W_1}$ and $\underline{W_2}$. This means we can form $\underline{V} \uplus \underline{W}_1 + \underline{V} \uplus \underline{W}_2$, and we would like for this to equal $\underline{V} \uplus (\underline{W}_1 + \underline{W}_2)$. The problem is that it is not apparent from the definitions whether or not $\underline{V}$ is simply transverse to $\underline{W}_1 + \underline{W}_2$ as the transversality of the pairs $(\underline{V}, \underline{W_1})$ and $(\underline{V}, \underline{W_2})$ might be realize by representatives

$V_1, V_2, W_1, W_2 \in PC^*_\Gamma(M)$ with $\underline{V_1} = \underline{V_2}$ so that V_1 and W_1 are transverse as spaces mapping into to M and V_2 and W_2 are transverse as spaces mapping to M, but neither V_1 nor V_2 is transverse to $W_1 \sqcup W_2$. It is also not apparent how to find a V_3 representing $\underline{V}$ that is transverse to W_1, W_2, and $W_1 \sqcup W_2$. The simplest solution would then seem to be to just define $\underline{V} \uplus (\underline{W_1} + \underline{W_2})$ to be represented by $(V_1 \times_M W_1) \sqcup (V_2 \times_M W_2)$. To do so, however, we must make sure that such a construction is independent of the choices involved. This is what we turn to now. We first build toward showing in Proposition 7.12 that the products can be made linear in one variable as just described, and then we use that to provide a more general multilinearlity in Theorem 7.14.

Lemma 7.10. *Suppose $\bigsqcup_i W_i \in Q(M)$, and let $W_i = W_{i,E} \sqcup W_{i,TI} \sqcup W_{i,NI}$ be the essential decomposition for each W_i. Then $\bigsqcup_i W_{i,E}$ is trivial and $\bigsqcup_i W_{i,NI} \in Q(M)$.*

Proof. We write the proof for cochains, but the argument for chains is equivalent. We know each $W_{i,TI}$ is in $Q(M)$, so by Lemma 4.13, $\bigsqcup_i (W_{i,E} \sqcup W_{i,NI}) \in Q(M)$. By definition, each $W_{i,NI}$ has small rank, while each component of $W_{i,E}$ does not have small rank. So in any decomposition of $\bigsqcup_i (W_{i,E} \sqcup W_{i,NI})$ into a trivial precochain and a degenerate precochain, which is possible as we know this is in $Q(M)$, the components of the $W_{i,E}$ must be part of the trivial precochain (the components of $W_{i,NI}$ may be part of either the trivial precochain or the degenerate precochain). As in the proof of Lemma 4.11, this implies that each connected component, say $\mathfrak{W}$, appearing in one of the $W_{i,E}$ either has a co-orientation reversing automorphism or appears zero times in all of in $\bigsqcup_i (W_{i,E} \sqcup W_{i,NI})$ when counting with co-orientation. If $\mathfrak{W}$ has a co-orientation reversing automorphism, then $\mathfrak{W}$ is trivial. Otherwise, for each occurrence of $\mathfrak{W}$ in some $W_{i,E}$, there is an occurrence of $-\mathfrak{W}$ in some $W_{j,E}$, and $\mathfrak{W} \sqcup -\mathfrak{W}$ is trivial. So altogether, $\bigsqcup_i W_{i,E}$ decomposes into a union of trivial precochains. It follows that $\bigsqcup_i W_{i,E} \in Q(M)$, and so again by Lemma 4.13, $\bigsqcup_i W_{i,NI} \in Q(M)$. □

Corollary 7.11. *If W_1 and W_2 represent the same element of $C^*_\Gamma(M)$ or $C_*^\Gamma(M)$ then there is some Z so that $W_{1,E} = Z \sqcup T_1$ and $W_{2,E} = Z \sqcup T_2$ with T_1 and T_2 trivial.*

Proof. By Lemma 7.10, $W_{1,E} \sqcup -W_{2,E}$ is trivial. Let $\mathfrak{W}$ be a connected component of $W_{1,E} \sqcup -W_{2,E}$. If $\mathfrak{W}$ is itself trivial, then we assign it to T_1 or T_2 according to whether it is in $W_{1,E}$ or $W_{2,E}$. If there is a pair $\mathfrak{W} \sqcup -\mathfrak{W}$ in $W_{1,E}$, then we assign the pair to T_1 and similarly for $W_{2,E}$. This leaves only pairs $\mathfrak{W} \sqcup -\mathfrak{W}$ so that $\mathfrak{W} \in W_{1,E}$ and $-\mathfrak{W} \in W_{2,E}$ (reversing the sign of $\mathfrak{W}$ if necessary). We let Z be the disjoint union of the $\mathfrak{W}$ in this last category, and corollary follows. □

Proposition 7.12. *Suppose $\sum_i \underline{W_i} = \sum_j \underline{W'_j} \in C^a_\Gamma(M)$ and that all $\underline{W_i}$ and $\underline{W'_j}$ are simply transverse to $\underline{V} \in C^b_\Gamma(M)$. Then*

$$\sum_i \underline{V} \uplus \underline{W_i} = \sum_j \underline{V} \uplus \underline{W'_j}.$$

Analogous statements hold with the sum in the first factor and for the cap and intersection products.

Proof. We provide the proof for the cup product, the other arguments being analogous.

We suppose each $\underline{W_i}$ represented by $W_i \in PC^*_\Gamma(M)$, and similarly for each $\underline{W'_j}$. Let V_i and V'_j all be representatives of $\underline{V}$ with V_i transverse to W_i and V'_j transverse to W'_j. We must show that $\bigsqcup_i V_i \times_M W_i$ and $\bigsqcup_j V'_j \times_M W'_j$ represent the same cochain, i.e. that

$$\left(\bigsqcup_i V_i \times_M W_i\right) \sqcup \left(-\bigsqcup_j V'_j \times_M W'_j\right) \in Q^*(M).$$

As our assumption is that $\bigsqcup_i W_i$ and $\bigsqcup_j W'_j$ represent the same geometric cochain, it suffices to reformulate the lemma as follows: Suppose $\bigsqcup_k W_k \in Q^*(M)$ and V_k is transverse to W_k with all V_k representing the same geometric cochain, then we must show

$$\bigsqcup_k V_k \times_M W_k \in Q^*(M).$$

For each W_k, we consider its essential decomposition

$$W_k = W_{k,E} \sqcup W_{k,TI} \sqcup W_{k,NI}.$$

As each $W_{k,TI} \in Q^*(M)$, we have each $V_k \times_M W_{k,TI} \in Q^*(M)$ by Lemma 4.22.

We next consider $\bigsqcup_k V_k \times_M W_{k,E}$, which is trivial by Lemma 7.10. So as in the proof of Lemma 4.11, each connected component, say $\mathfrak{W}$, appearing in one of the $W_{k,E}$ either has a co-orientation reversing automorphism or appears zero times in all of in $\bigsqcup_k W_{k,E}$ when counting with co-orientation. If $\mathfrak{W}$ has a co-orientation reversing automorphism, then $\mathfrak{W}$ is trivial and $V_k \times \mathfrak{W}$ is trivial for any V_k transverse to $\mathfrak{W}$. Otherwise, for each occurrence of $\mathfrak{W}$ in some $W_{k,E}$, there is an occurrence of $-\mathfrak{W}$ in some $W_{\ell,E}$. By Theorem 7.9, $V_k \times_M \mathfrak{W}$ and $V_\ell \times \mathfrak{W}$ represent the same cochain, so

$$(V_k \times_M \mathfrak{W}) \sqcup -(V_\ell \times_M \mathfrak{W}) = (V_k \times_M \mathfrak{W}) \sqcup (V_\ell \times_M -\mathfrak{W}) \in Q^*(M).$$

Continuing this way with pairs of oppositely-co-oriented components of $\bigsqcup_k W_{k,E}$, and noting as in the proof of Lemma 4.11 that each $\mathfrak{W}$ occurs only a finite number of times, we see that $\bigsqcup_k V_k \times_M W_{k,E} \in Q^*(M)$.

It remains to show that $\bigsqcup_k V_k \times_M W_{k,NI} \in Q^*(M)$. By Lemma 7.10, we have $\bigsqcup_k W_{k,NI} \in Q^*(M)$. By definition, we can write $\bigsqcup_k W_{k,NI} = \mathcal{W}_{tr} \sqcup \mathcal{W}_d$ as the disjoint union of a trivial precochain and a degenerate precochain. By the same procedure as just above, if $\mathfrak{W}$ is a connected component of $\mathcal{W}_{tr}$, then either it has a co-orientation reversing automorphism or it appears zero times in all $\mathcal{W}_{tr}$ counting with co-orientation. Thus either either $V_k \times_M \mathfrak{W} \in Q^*(M)$, assuming $\mathfrak{W}$ is a component of $W_{k,NI}$, or we can have pairs V_k and V_ℓ with $(V_k \times_M \mathfrak{W}) \sqcup (V_\ell \times_M -\mathfrak{W}) \in Q^*(M)$. So it remains to consider the case of $\bigsqcup_k V_k \times_M Z_k$ where $\mathcal{W}_d = \bigsqcup_k Z_k$ with Z_k being the contribution to $\mathcal{W}_d$ from $W_{k,NI}$.

In this case, each Z_k has small rank, hence so does each $V_k \times_M Z_k$ by Lemma 4.22, so $\bigsqcup_k (V_k \times_M Z_k)$ has small rank and it suffices to show that $\partial(\bigsqcup_k V_k \times_M Z_k)$ is the union of a trivial precochain and one of small rank. By Proposition 3.44,

$$\partial\left(\bigsqcup_k V_k \times_M Z_k\right) = \bigsqcup_k ((\partial V_k) \times_M Z_k) \sqcup (-1)^{m-v}(V_k \times_M \partial Z_k).$$

We note that these terms are defined as the transversality of V_k and Z_k includes transversality with the boundaries. As the Z_k have small rank, each $(\partial V_k) \times_M Z_k$ has small rank by Lemma 4.22. As $\bigsqcup_k Z_k$ is degenerate, $\partial(\bigsqcup_k Z_k) = \bigsqcup_k \partial Z_k$ can be written as $\bigsqcup_k \partial Z_k = A_{tr} \sqcup A_{sm}$ with A_{tr} trivial and A_{sm} of small rank. But then if $\mathfrak{A}$ is a component of ∂Z_k in A_{sm}, we have $V_k \times \mathfrak{A}$ of small rank, and for the connected components of A_{tr} we can once again recognize that either $\mathfrak{A}$ has a co-orientation reversing automorphism or appears zero times in all of A_{tr} counting with co-orientation. So again repeating our earlier argument either $V_k \times_M \mathfrak{A}$ is trivial or we can find pairs of components $\mathfrak{A}$ in ∂Z_k and $-\mathfrak{A}$ in ∂Z_ℓ with $(V_k \times_M \mathfrak{A}) \sqcup (V_\ell \times_M -\mathfrak{A}) \in Q^*(M)$. So in particular this expression is a union of a trivial precochain and one of small rank by definition of $Q^*(M)$. Continuing in this way through all connected components, all of $\bigsqcup_k (V_k \times_M \partial Z_k)$ can be partitioned into trivial precochains and precochains of small rank. □

We now generalize yet again to a fully multilinear version of the cup product. Again, this requires enough transversality for all fiber products to be defined, but now we allow cochain representatives in both the first and second factors to vary.

Definition 7.13. Suppose $\underline{V} = \sum_i \underline{V_i} \in C^*_\Gamma(M)$, $\underline{W} = \sum_j \underline{W_j} \in C^*_\Gamma(M)$, and each pair $(\underline{V_i}, \underline{W_j})$ is simply transverse. Then we say that $\underline{V}$ and $\underline{W}$ are **transverse** and define $\underline{V} \uplus \underline{W}$ as

$$\underline{V} \uplus \underline{W} = \sum_{i,j} \underline{V_i} \uplus \underline{W_j},$$

where the cup products on the right are those of Definition 7.2, which are well defined by Theorem 7.9. We extend the definition of the cap and intersection products analogously.

In particular, if $(\underline{V_i}, \underline{W_j})$ is a simply transverse pair, then there are transverse representatives V_{ij} of $\underline{V_i}$ and W_{ji} of $\underline{W_j}$, and $\underline{V} \uplus \underline{W}$ (or $\underline{V} \Cap \underline{W}$ or $\underline{V} \bullet \underline{W}$) is represented by $\sum_{ij} V_{ij} \times_M W_{ji}$.

Theorem 7.14. *The products of Definition 7.13 are well defined. In particular, they do not depend on the decompositions of $\underline{V}$ and $\underline{W}$ into sums of geometric chains or cochains.*

Proof. The proof is essentially a more complicated version of the proof of Proposition 7.12, though at one point here we will want to use Proposition 7.12, so it would not be convenient to try to fuse these into a single proof and thus we go through the details again.

We provide the argument for the cup product, the other proofs being analogous. Suppose $\underline{V} = \sum_i \underline{V_i} = \sum_k \underline{V_k'}$ and $\underline{W} = \sum_j \underline{W_j} = \sum_\ell \underline{W_\ell'}$. Suppose the pairs $(\underline{V_i}, \underline{W_j})$ and $(\underline{V_k'}, \underline{W_\ell'})$ are simply transverse. We must show that $\sum_{i,j} \underline{V_i} \uplus \underline{W_j} = \sum_{k,\ell} \underline{V_k'} \uplus \underline{W_\ell'}$. The assumptions mean that for each pair $(\underline{V_i}, \underline{W_j})$, there are transverse representatives we can choose and call (V_{ij}, W_{ji}), and similarly for the primed versions. Then we must show that

$$\left(\bigsqcup_{i,j} V_{ij} \times_M W_{ji}\right) \sqcup \left(-\bigsqcup_{k,\ell} V'_{k\ell} \times_M W'_{\ell k}\right) \in Q^*(M).$$

For each W_{ji}, we have its essential decomposition

$$W_{ji} = W_{ji,E} \sqcup W_{ji,TI} \sqcup W_{ji,NI},$$

and by Lemma 7.6, we know that for any i, a the precochains $W_{ji,E}$ and $W_{ja,E}$ agree up to the disjoint union with some trivial terms. As fiber products involving trivial factors are trivial, it is safe to ignore these trivial precochains and, abusing notation, consider $W_{ji,E}$ and $W_{ja,E}$ as agreeing exactly. We call this common precochain $W_{j,E}$.

Next, let us choose for each j and ℓ particular representatives W_j and W_ℓ' for $\underline{W_j}$ and $\underline{W_\ell'}$. By the same argument as just above, the essential component of this W_j is the same as that of the other $W_{ji,E}$ up to trivial terms, so, again abusing notation but with no negative impact, we can take the essential component of W_j to also be $W_{j,E}$, and the notation is consistent. Similarly for the W_ℓ'.

We have $(\bigsqcup_j W_j) \sqcup (-\bigsqcup_\ell W_\ell') \in Q^*(M)$ by assumption, so by Lemma 7.10, $\left(\bigsqcup_j W_{j,E}\right) \sqcup \left(\bigsqcup_\ell W'_{\ell,E}\right)$ must be trivial. Therefore, as in the proof of Lemma 4.11, each connected component, say $\mathfrak{W}$, appearing in one of the $W_{j,E}$ or $W'_{\ell,E}$ either has a co-orientation reversing automorphism or appears zero times in all of $(\bigsqcup_j W_{j,E}) \sqcup (-\bigsqcup_\ell W'_{\ell,E})$ when counting with co-orientation. If $\mathfrak{W}$ has a co-orientation reversing automorphism, then $\mathfrak{W}$ is trivial and $V_{ij} \times \mathfrak{W}$ is trivial for any V_{ij} transverse to $\mathfrak{W}$. Otherwise, for each occurrence of $\mathfrak{W}$ in $(\bigsqcup_j W_{j,E}) \sqcup (-\bigsqcup_\ell W'_{\ell,E})$, there is an occurrence of $-\mathfrak{W}$. Suppose $\mathfrak{W} \in W_{a,E}$ and $-\mathfrak{W} \in W_{b,E}$. We know $\sqcup_i V_{ia}$ and $\sqcup_i V_{ib}$ represent the same element of $C^*_\Gamma(M)$ and are transverse to $\mathfrak{W}$. So

$$\left(\bigsqcup_i V_{ia} \times_M \mathfrak{W}\right) \sqcup \left(-\bigsqcup_i V_{ib} \times_M \mathfrak{W}\right) \in Q^*(M)$$

by Proposition 7.12, and similarly if one or both occurrences of $\pm\mathfrak{W}$ are components of one of the $W'_{\ell,E}$.

Continuing in this way, all of

$$\left(\bigsqcup_{i,j} V_{ij} \times W_{j,E}\right) \sqcup \left(-\bigsqcup_{k,\ell} V'_{k\ell} \times_M W'_{\ell,E}\right)$$

is in $Q^*(M)$. Also, each $V_{ij} \times_M W_{ji,TI} \in Q^*(M)$ by Lemma 4.22, as $W_{ji,TI} \in Q^*(M)$. So it remains to show that

$$\left(\bigsqcup_{i,j} V_{ij} \times W_{ji,NI}\right) \sqcup \left(-\bigsqcup_{k,\ell} V'_{k\ell} \times_M W'_{\ell k,NI}\right) \tag{18}$$

is in $Q^*(M)$. As each $W_{j,NI}$ and $W'_{\ell,NI}$ has small rank, each component of (18) is of small rank by Lemma 4.22. So it suffices to show that the boundary of (18) is a union of trivial and small rank precochains. The boundary terms of the form $(\partial V_{ij}) \times W_{ji,NI}$ and $(\partial V'_{k\ell}) \times_M W'_{\ell k,NI}$ all have small rank by Lemma 4.22. So we consider

$$\left(\bigsqcup_{i,j} V_{ij} \times \partial W_{ji,NI}\right) \sqcup \left(-\bigsqcup_{k,\ell} V'_{k\ell} \times_M \partial W'_{\ell k,NI}\right)$$

(we can ignore the sign, as all terms are multiplied by the same sign $(-1)^{m-v}$ in taking the boundary).

We now consider the essential decompositions of the $\partial W_{ji,NI}$ and $\partial W'_{\ell i,NI}$. By Lemma 4.22, any fiber product involving a TI component will be trivial and any fiber product involving an NI component will have small rank. So we must consider the terms $V_{ij} \times (\partial W_{ji,NI})_E$. By Lemma 7.8, since W_{ji} and W_{ja} represent the same cochain for any i, a, we have that $\partial W_{ji,NI}$ and $\partial W_{ja,NI}$ represent the same cochain (and similar for the W'). So by Lemma 7.6, these have the same essential part (up to trivial components or pairs of components), which we can write as $(\partial W_{j,NI})_E$. So if we can show that

$$\left(\bigsqcup_{j} \partial W_{j,NI}\right) \sqcup \left(\bigsqcup_{\ell} \partial W'_{\ell,NI}\right)$$

is in $Q^*(M)$ then we can conclude by the same argument we used above to show that any components coming from the $W_{j,E}$ and $W'_{\ell,E}$ are trivial or occur in canceling pairs. But, again, we know that $(\bigsqcup_j W_j) \sqcup (-\bigsqcup_\ell W'_\ell) \in Q^*(M)$, so by Lemma 4.13 again

$$\left(\bigsqcup_{j} (W_{j,E} \sqcup W_{j,NI})\right) \sqcup \left(-\bigsqcup_{\ell} (W'_{\ell,E} \sqcup W'_{\ell,NI})\right) \in Q^*(M),$$

as the $W_{j,TI}$ and $W'_{\ell,TI}$ are in $Q^*(M)$. Thus by Lemma 7.10,

$$\left(\bigsqcup_{j} W_{j,NI}\right) \sqcup \left(-\bigsqcup_{\ell} W'_{\ell,NI}\right) \in Q^*(M).$$

Thus by Lemma 4.12, the boundary

$$\left(\bigsqcup_{j} \partial W_{j,NI}\right) \sqcup \left(-\bigsqcup_{\ell} \partial W'_{\ell,NI}\right)$$

is in $Q^*(M)$ as required. □

7.1.1. *Pullbacks of cochains and the Kronecker pairing.* In Section 5.1.2, we showed that a continuous map $f : M \to N$ of manifolds without boundary yields a well-defined cohomology map $h^* : H^*_\Gamma(N) \to H^*_\Gamma(M)$. In this section we utilize the notions of the preceding section to consider f^* as a partially-defined map of cochain complexes $C^*_\Gamma(N) \to C^*_\Gamma(M)$. Then, by similar arguments, we consider a partially-defined Kronecker-type evaluation $C^*_\Gamma(M) \to \mathrm{Hom}(C^\Gamma_*(M), \mathbb{Z})$.

7.1.2. *Pullbacks of cochains.*

Definition 7.15. Let $h : M \to N$ be a smooth map of manifolds without boundary, and let $\underline{V} \in C^*_\Gamma(N)$. We will say that $\underline{V}$ is **transverse to** h if $\underline{V}$ has a representative $f : V \to N$ such that f is transverse to h. In this case we define the pullback $h^*(\underline{V})$ to be $\underline{h^*(V)} \in C^*_\Gamma(M)$.

We will write the set of cochains transverse to h as $C^*_{\Gamma \pitchfork h}(N)$.

We notice that the transversality situation here is simpler than the more general ones in the preceding section, as h is a fixed map.

Proposition 7.16. *Given a smooth map of manifolds without boundary $h : M \to N$, the set $C^*_{\Gamma\pitchfork h}(N)$ is a subcomplex of $C^*_\Gamma(N)$, and the map $h^* : C^*_{\Gamma\pitchfork h}(N) \to C^*_\Gamma(M)$ is a well-defined chain map.*

Proof. If $\underline{V}, \underline{W} \in C^*_\Gamma(N)$ are represented by $f : V \to N$ and $g : W \to N$ that are transverse to h, then $\underline{V} + \underline{W}$ can be represented by $V \sqcup W$, which will also be transverse to h. So $C^*_{\Gamma\pitchfork h}(N)$ is closed under addition. If $f : V \to N$ is transverse to h then so is $-f$, i.e. f with the opposite co-orientation, so $C^*_{\Gamma\pitchfork h}(N)$ is closed under negation. The empty map $\emptyset \to N$ is always transverse to h (since there are no points at which to check the tangent space condition), and so $0 \in C^*_{\Gamma\pitchfork h}(N)$. Finally, if $\underline{V}$ is represented by $f : V \to N$ transverse to h, then by definition $\partial V \to N$ is transverse to h, so $\partial\underline{V} \in C^*_{\Gamma\pitchfork h}(N)$. Therefore, $C^*_{\Gamma\pitchfork h}(N)$, is a subcomplex of $C^*_\Gamma(N)$.

To how that h^* is well defined on $C^*_{\Gamma\pitchfork h}(N)$ we must show that it does not depend on the choice of representative V. Suppose V and V' both represent $\underline{V}$ and are transverse to h. Then $V \sqcup -V'$ is transverse to h and an element of $Q^*(N)$. So by Lemma 5.9, $h^*(V \sqcup -V')$, which is by definition $(V \sqcup -V') \times_N M = (V \times_N M) \sqcup (-V' \times_N M)$ mapping to M, is an element of $Q^*(M)$. So $h^*(V)$ and $h^*(V')$ represent the same element of $C^*_\Gamma(M)$. Thus h^* is well defined.

To see that h^* is a homomorphism, let $V, W \to N$ represent elements of $C^*_\Gamma(N)$ that are transverse to h. Then

$$h^*(\underline{V} + \underline{W}) = \underline{h^*(V \sqcup W)} = \underline{h^*(V) \sqcup h^*(W)} = \underline{h^*(V)} + \underline{h^*(W)},$$

using the definitions, obvious properties of the pullback and Lemma 4.16. Furthermore, h^* is a chain map by Proposition 3.44, as $\partial M = \emptyset$. ∎

Remark 7.17. While $C^*_{\Gamma\pitchfork h}(N)$ is a subcomplex, it is not closed under taking cup products, even when they are well defined. As an example, let $h : M \to N$ be the inclusion of the x-axis into the plane $\mathbb{R}^2$. Let V be represented by an embedding of $\mathbb{R}$ into $\mathbb{R}^2$ as the line $y = x$, and let W similarly correspond to $y = -x$, with any co-orientations. Then $\underline{V} \uplus \underline{W}$ is represented by the embedding of the origin into $\mathbb{R}^2$, but this is not transverse to h, even though both V and W are transverse to h.

7.1.3. *Kronecker pairing.* The partially-defined cap product becomes a partially-defined pairing

$$C^i_\Gamma(M) \times C_i^\Gamma(M) \xrightarrow{\Cap} C_0^\Gamma(M) \xrightarrow{\mathbf{a}} \mathbb{Z},$$

where $\mathbf{a} : C_0^\Gamma(M) \to \mathbb{Z}$ is the augmentation map of Definition 6.9. We consider here the extent to which this pairing corresponds to a function $C^i_\Gamma(M) \to \operatorname{Hom}(C_i^\Gamma(M), \mathbb{Z})$. The situation is very similar to the preceding discussion of pullbacks.

Definition 7.18. Let $\underline{V} \in C^i_\Gamma(M)$ be a geometric cochain. We write $C_i^{\Gamma\pitchfork\underline{V}}(M)$ for the subgroup of $C_i^\Gamma(M)$ generated by geometric i-chains transverse to $\underline{V}$.

Proposition 7.19. *Given a geometric cochain $\underline{V} \in C^i_\Gamma(M)$, the map $\mathbf{a}(\underline{V} \Cap -) : C_i^{\Gamma\pitchfork\underline{V}}(M) \to \mathbb{Z}$ is a well-defined homomorphism.*

Proof. We first observe that $\underline{V} \Cap -$ is defined on all elements of $C_i^{\Gamma\pitchfork\underline{V}}(M)$. If $\underline{W} \in C_i^\Gamma(M)$ can be written as a sum $\underline{W} = \sum \underline{W_i}$ with each $\underline{W_i}$ transverse to $\underline{V}$, then $\underline{V} \Cap \underline{W}$ is well defined as $\sum \underline{V} \Cap \underline{W_i}$ by Definition 7.13 and Theorem 7.14. This is also consistent with Theorem 7.14. The element $0 \in C_i^\Gamma(M)$, as represented by the empty map, is transverse to $\underline{V}$ with $\underline{V} \pitchfork 0 = 0$, and if $\underline{W}$ is transverse to $\underline{V}$ then so is $-\underline{W}$. Definition 7.13 and Theorem 7.14 also imply that $\underline{V} \pitchfork -$ is a homomorphism. We know that **a** is a homomorphism, so the proposition follows. $\square$

So given $\underline{V} \in C^i_\Gamma(M)$, we obtain an element of $\mathrm{Hom}(C_i^{\Gamma \pitchfork \underline{V}}(M), \mathbb{Z})$, but of course we will not in general obtain an element of $\mathrm{Hom}(C_i^\Gamma(M), \mathbb{Z})$ due to transversality requirements.

7.1.4. *Exterior products.* We observe here that the exterior products defined in Section 3.6 give rise to well-defined products for geometric chains and cochains, and, in fact, these are fully defined, unlike the cup, cap, and intersection products.

Definition 7.20. Suppose $\underline{V} \in C_*^\Gamma(M)$ and $\underline{W} \in C_*^\Gamma(N)$ represented by $V \in PC_*^\Gamma(M)$ and $W \in PC_*^\Gamma(N)$. Then we define the **exterior chain product (or chain cross product)**

$$\times : C_*^\Gamma(M) \times C_*^\Gamma(N) \to C_*^\Gamma(M \times N)$$

by $\underline{V} \times \underline{W} = \underline{V \times W}$.

Similarly, suppose $\underline{V} \in C^*_\Gamma(M)$ and $\underline{W} \in C^*_\Gamma(N)$ represented by $V \in PC^*_\Gamma(M)$ and $W \in PC^*_\Gamma(N)$. Then we define the **exterior cochain product (or cochain cross product)**

$$\times : C^*_\Gamma(M) \times C^*_\Gamma(N) \to C^*_\Gamma(M \times N)$$

by $\underline{V} \times \underline{W} = \underline{V \times W}$.

As is standard for singular homology and cohomology, we use the symbol $\times$ for both products, allowing context to determine which product is meant.

Proposition 7.21. *The exterior chain and cochain products are well defined.*

Proof. As in Section 3.6, we give the product of oriented manifolds the standard product orientation, while co-orientations of products of co-oriented maps are defined in Definition 3.62. Lemma 3.60 establishes that the product of proper maps is proper.

It remains to show that if V' and W' are alternative representatives of V and W then $(V \times W) \sqcup -(V' \times W') \in Q(M \times N)$. We will show that $(V \times W) \sqcup -(V' \times W) \in Q(M \times N)$, then the general case follows from an equivalent argument with W. But we need only observe that $(V \times W) \sqcup -(V' \times W) = (V \sqcup -V') \times W$ and then apply Lemma 4.24. $\square$

7.2. **Properties of the chain and cochain products.** This whole section needs better formatting

Now that we have defined cup, cap, intersection, and exterior product and shown that they are well defined, at least when the necessary transversality and orientation conditions hold, they immediately inherit many of the properties demonstrated in Section 3. We provide below some tables listing these properties and the locations of the previous results that support them. The references are typically to results that involve only transversality of a pair of prechains or precochains, but in the chain and cochain setting they generalize to the more general products of Definition 7.13 by applying them to each summand.

For example, suppose $\underline{V}, \underline{W} \in C^*_\Gamma(M)$ are transverse. This means we can write $\underline{V} = \sum_i \underline{V_i} \in C^*_\Gamma(M)$, $\underline{W} = \sum_j \underline{W_j} \in C^*_\Gamma(M)$ with each pair $(\underline{V_i}, \underline{W_j})$ simply transverse. And this means that there are representatives $V_{ij}, W_{ji} \in PC^*_\Gamma(M)$ such that for all i and j, we have $\underline{V_{ij}} = \underline{V_i}$, $\underline{W_{ji}} = \underline{W_j}$, and V_{ij} transverse to W_{ji}. We then have $\underline{V} \uplus \underline{W}$ represented by $\sum_{i,j} V_{ij} \times_M W_{ji}$. By Proposition 3.45, we have

$$\sum_{i,j} V_{ij} \times_M W_{ji} = \sum_{i,j} (-1)^{(m-v)(m-w)} W_{ji} \times_M V_{ij} = (-1)^{(m-v)(m-w)} \sum_{i,j} W_{ji} \times_M V_{ij},$$

and the sum on the right represents $\underline{W} \uplus \underline{V}$. So we obtain the cup product commutativity formula

$$\underline{V} \uplus \underline{W} = (-1)^{(m-v)(m-w)} \underline{W} \uplus \underline{V}$$

for transverse cochains.

The more complicated exception to this inheritance of properties from the pre(co)chain properties concern associativity and naturality, which we will address below in a separate section.

In the tables that follow, we assume to hold all transversality required for each expression to be defined. For intersection products, we assume that the underlying manifold is oriented. Unless stated otherwise, our default notations for cup, cap, and intersection products will have manifolds with corners V and W mapping to a manifold without boundary M. Our default notations for chain and cochain cross products will assume $V \to M$ and $W \to N$. We explain the further assumptions and notations prior to each table of formulas

7.2.1. *Boundary formulas.* For our first table, with formulas involving boundaries, we also invoke the well-definedness of boundaries of geometric chains and cochains, see Lemma 4.16. The cup, cap, and intersection products require transversality of $\underline{V}$ and $\underline{W}$; the exterior products have no transversality requirements.

Chain cross product	$\partial(\underline{V} \times \underline{W}) = (\partial\underline{V}) \times \underline{W} + (-1)^v \underline{V} \times \partial\underline{W}$	Standard
Cochain cross product	$\partial(\underline{V} \times \underline{W}) = (\partial\underline{V}) \times \underline{W} + (-1)^{m-v} \underline{V} \times \partial\underline{W}$	Proposition 3.66
Cup product	$\partial(\underline{V} \uplus \underline{W}) = (\partial\underline{V}) \uplus \underline{W} + (-1)^{m-v} \underline{V} \uplus \partial\underline{W}$	Proposition 3.44
Cap product	$\partial(\underline{V} \Cap \underline{W}) = (-1)^{v+w-m}(\partial\underline{V}) \Cap \underline{W} + \underline{V} \Cap \partial\underline{W}$	Proposition 3.77
Intersection product	$\partial(\underline{V} \bullet \underline{W}) = (\partial\underline{V}) \bullet \underline{W} + (-1)^{m-v} \underline{V} \bullet \partial\underline{W}$	Proposition 3.6

7.2.2. *Commutativity formulas.* For the commutativity properties listed below, τ is the transposition map $\tau : N \times M \to M \times N$. The cup and intersection products require transversality of $\underline{V}$ and $\underline{W}$; the exterior products have no transversality requirements.

Chain cross product	$\tau(\underline{V} \times \underline{W}) = (-1)^{vw} \underline{W} \times \underline{V}$	Standard
Cochain cross product	$\tau^*(\underline{V} \times \underline{W}) = (-1)^{(m-v)(n-w)} \underline{W} \times \underline{V}$	Proposition 3.69
Cup product	$\underline{V} \uplus \underline{W} = (-1)^{(m-v)(m-w)} \underline{W} \uplus \underline{V}$	Proposition 3.45
Intersection product	$\underline{V} \bullet \underline{W} = (-1)^{(m-v)(m-w)} \underline{W} \bullet \underline{V}$	Proposition 3.5

7.2.3. *Unital properties.* For the following unital properties, we write pt to refer to the point with its positive orientation. We will write $\underline{pt}$ for the geometric chain given by $\mathrm{id}_{pt} : pt \to pt$ or for the geometric cochain given by the canonically co-oriented identity map $\mathrm{id}_{pt} : pt \to pt$. Similarly, $\underline{M}$ represents the geometric chain or cochain corresponding to $\mathrm{id}_M : M \to M$, canonically co-oriented in the cochain case. Technically, M must be compact for id_M to represent a chain, but the corresponding formulas hold more broadly at the referenced locations and so these identities could be taken as statements involving a broader class of geometric chains. Note that, as a cochain, $\underline{M} \in C^0_\Gamma(M)$, and Proposition 3.70 shows that these correspond to the singular cochain 1. We also let $\pi_1 : M \times N \to M$ and $\pi_2 : N \times M \to M$ denote the projections. In the first formula for the cap product with $\underline{M}$, M is oriented, and the first $\underline{V}$ is $V \to M$ as a cochain while the second is $V \to M$ as a chain with the induced orientation on V; see Proposition 3.81. In the second cap product formula, both instances of $\underline{V}$ are as chains.

As id_M is transverse to all other maps, the following hold for all $\underline{V}$.

Identity for chain cross product	$\underline{V} \times \underline{pt} = \underline{pt} \times \underline{V} = \underline{V}$	Obvious
Identity for chain cross product	$\underline{V} \times \underline{pt} = \underline{pt} \times \underline{V} = \underline{V}$	Proposition 3.65
Cochain cross product with 1	$\pi_1^* \underline{V} = \underline{V} \times \underline{N}$ $\pi_2^* \underline{V} = \underline{N} \times \underline{V}$	Proposition 3.70
Cup product with 1	$\underline{V} \uplus \underline{M} = \underline{M} \uplus \underline{V} = \underline{V}$	Corollary 3.53
Cap product with $\underline{M}$	$\underline{V} \Cap \underline{M} = \underline{V}$	Proposition 3.81
Cap with product with 1	$\underline{M} \Cap \underline{V} = \underline{V}$	Proposition 3.80
Intersection product with $\underline{M}$	$\underline{M} \bullet \underline{V} = \underline{V} \bullet \underline{M} = \underline{V}$	Proposition 3.3

7.2.4. *Mixed properties.* The next grouping concerns properties that involve multiple products. We recall that $\mathbf{d} : M \to M \times M$ is the diagonal map. For these properties we assume maps $V, W \to M$ and $X, Y \to N$. We also have projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$. The last formula follows from Proposition 3.86 by observing from the previous list of properties that, when M is oriented, the cap product with $\underline{M}$, which represents the chain $\mathrm{id}_M : M \to M$, takes a cochain to the chain with the orientation induced by the cochain co-orientation and the orientation of M.

The first and last properties require that $\underline{V}$ and $\underline{W}$ be transverse. The second holds for all $\underline{V}, \underline{W}$. The next three require that $\underline{V}$ be transverse to $\underline{W}$ and that $\underline{X}$ be transverse to $\underline{Y}$.

Cup from cross	$\underline{V} \uplus \underline{W} = \mathbf{d}^*(\underline{V} \times \underline{W})$	Proposition 3.72
Cross from cup	$\underline{V} \times \underline{X} = \pi_M^*(\underline{V}) \uplus \pi_N^*(\underline{X})$	Corollary 3.76
Cup of crosses	$(\underline{V} \times \underline{X}) \uplus (\underline{W} \times \underline{Y}) = (-1)^{(m-w)(n-x)}(\underline{V} \uplus \underline{W}) \times (\underline{X} \uplus \underline{Y})$	Corollary 3.75
Cap of crosses	$(\underline{V} \times \underline{X}) \Cap (\underline{W} \times \underline{Y}) = (-1)^{(x+y-n)(m-v)}(\underline{V} \Cap \underline{W}) \times (\underline{X} \Cap \underline{Y})$	Proposition 3.82
Intersection of crosses	$(\underline{V} \times \underline{X}) \bullet (\underline{W} \times \underline{Y}) = (-1)^{(m-w)(n-x)}(\underline{V} \bullet \underline{W}) \times (\underline{X} \bullet \underline{Y})$	Proposition 3.59
Cup and intersection	$(\underline{V} \uplus \underline{W}) \Cap \underline{M} = (-1)^{(m-v)(m-w)}(\underline{V} \Cap \underline{M}) \bullet (\underline{W} \Cap \underline{M})$	Proposition 3.86

7.2.5. *Immersion formulas.* While geometric chains and cochains do not have unique representatives by maps of manifolds, if the two terms can be represented by transverse immersions, then we recall that we have nice computational formulas for the cup, cap, and intersection products; see Propositions 3.43, 3.78, and 3.7, respectively. For cap and intersection products, the special cases where the (co)chains have complementary dimensions are further specified in Corollaries 3.79 and 3.9.

7.2.6. *Naturality and associativity formulas.* Formulas for associativity of geometric chain and cochain products are more delicate than our preceding formulas because they require sufficient transversality of more than two objects. This would require some careful assumptions even for just maps of manifolds. But the ambiguity in representation of geometric chains and cochains makes the situation even more problematic. For example, if $\underline{V}$ and $\underline{W}$ are transverse, then by definition we can write $\underline{V} = \sum_i \underline{V_i}$ and $\underline{W} = \sum_j \underline{W_j}$ and then find transverse representatives V_{ij} of $\underline{V_i}$ and W_{ji} of $\underline{W_j}$. Then $\underline{V} \uplus \underline{W}$ is represented by $\sum_{ij} V_{ij} \times_M W_{ji}$. Now suppose Z is transverse to $\underline{V} \uplus \underline{W}$. Then there must be similar decompositions of Z and $\underline{V} \uplus \underline{W}$ into simply transverse pairs, but it is not clear that this condition can necessarily be written in terms of the V_{ij} and W_{ji} so that we can advantage of Corollary 3.74. So, rather than attempt to pursue the most general case, we impose some extra restrictions so that we can utilize Corollary 3.74 and its analogues for the intersection product and the cap product with a cup product.

Similar concerns arise for our naturality formula, as pulling back a cup product by a map h requires that h be transverse to the cup product, so we again have an interaction of three maps, leading to similar concerns.

Naturality. As noted above, naturality of cup and cap products requires some extra care to ensure not just that chains and cochains are appropriately transverse but that there are also the appropriate transversalities with respect to the maps we pull back by. This requires a good number of further assumptions; see Remark 2.23. So suppose $h : M \to N$ is a map of manifolds without boundaries and that $\underline{V}, \underline{W} \in C^*_\Gamma(N)$. For naturality of cup products we assume not just that $\underline{V}$ and $\underline{W}$ are transverse, but we require decompositions $\underline{V} = \sum_i \underline{V_i} \in C^*_\Gamma(M)$, $\underline{W} = \sum_j \underline{W_j} \in C^*_\Gamma(M)$ such that each pair $(\underline{V_i}, \underline{W_j})$ has representatives V_{ij} and W_{ji} such that

- V_{ij} and W_{ji} are transverse and
- V_{ij}, W_{ji}, and $V_{ij} \times_M W_{ji}$ are all transverse to h.

Similarly for cap products we require decompositions such that V_{ij} and W_{ji} satisfy the transversality requirements of Proposition 3.85.

The exterior products are simpler. The naturality of the chain cross product requires no assumption, while the naturality of the cochain cross product requires only that V and W have representatives that are respectively transverse to h and k.

Chain cross product	$(h \times k)(\underline{V} \times \underline{W}) = h(\underline{V}) \times k(\underline{W})$	Obvious
Cochain cross product	$(h \times k)^*(\underline{V} \times \underline{W}) = h^*(\underline{V}) \times k^*(\underline{W})$	Proposition 3.71
Cup product	$h^*(\underline{V} \uplus \underline{W}) = h^*(\underline{V}) \uplus h^*(\underline{W})$	Corollary 3.73
Cap product	$\underline{V} \Cap h(\underline{W}) = h(h^*(\underline{V}) \Cap \underline{W})$	Proposition 3.85

Associativity. For the associativity formulas we add a manifold with corners X either mapping to M for the cup, cap, and intersection products or to a third target manifold Q for the cross products. Once again, there are no special requirements for the exterior products. For the other products, while this might not encompass the most general possibility, in order to ensure associativity we assume decompositions $\underline{V} = \sum_i \underline{V_i}$, $\underline{W} = \sum_j \underline{W_j}$, and $\underline{Z} = \sum_k \underline{Z_k}$ such that for each triple $(\underline{V_i}, \underline{W_j}, \underline{Z_k})$ there are representatives $V_{i,jk}$, $W_{j,ik}$, and $Z_{k,ij}$ such that the following pairs are transverse: $(V_{i,jk}, W_{j,ik})$, $(W_{j,ik}, Z_{k,ij})$, $(V_{i,jk} \times_M W_{j,ik}, Z_{k,ij})$, and $(V_{i,jk}, W_{j,ik} \times_M Z_{k,ij})$.

We leave the reader to formulate associativity for products of larger collections of maps.

Chain cross product	$(\underline{V} \times \underline{W}) \times \underline{X} = \underline{V} \times (\underline{W} \times \underline{X})$	Evident
Cochain cross product	$(\underline{V} \times \underline{W}) \times \underline{X} = \underline{V} \times (\underline{W} \times \underline{X})$	Proposition 3.67
Cup product	$(\underline{V} \uplus \underline{W}) \uplus \underline{X} = \underline{V} \uplus (\underline{W} \uplus X)$	Corollary 3.74
Cup/cap	$(\underline{V} \uplus \underline{W}) \Cap \underline{X} = \underline{V} \Cap (\underline{W} \Cap \underline{X})$	Proposition 3.84
Intersection product	$(\underline{V} \bullet \underline{W}) \bullet \underline{X} = \underline{V} \bullet (\underline{W} \bullet \underline{X})$	Proposition 3.4

We note that with our definitions, these triple products exhibit linearity in each variable, assuming as usual that all necessary transversality conditions are met.

7.3. **Homology and cohomology products.** In this section we observe that the partially-defined cup, cap, and intersection products of geometric chains and cochains give rise to fully-defined products of geometric homology and cohomology. Similarly, we obtain external homology and cohomology products, although this is more evident as external products are already fully defined for geometric chains and cochains.

Theorem 7.22. *Let M and N be manifolds without boundary. The chain cross product, cochain cross product, cup product, cap product, and, if M is oriented, intersection product induce fully-defined bilinear maps*

$$\begin{aligned}
\times :&H_*^\Gamma(M) \otimes H_*^\Gamma(N) \to H_*^\Gamma(M \times N)\\
\times :&H^*_\Gamma(M) \otimes H^*_\Gamma(N) \to H^*_\Gamma(M \times N)\\
\uplus :&H^*_\Gamma(M) \otimes H^*_\Gamma(M) \to H^*_\Gamma(M)\\
\Cap :&H^*_\Gamma(M) \otimes H_*^\Gamma(M) \to H_*^\Gamma(M)\\
\bullet :&H_*^\Gamma(M) \otimes H_*^\Gamma(M) \to H_*^\Gamma(M).
\end{aligned}$$

We will prove Theorem 7.22 in a moment, but first we note the following immediate consequence.

Theorem 7.23. *The homology cross product, cohomology cross product, cohomology cup product, cohomology/homology cap product, and, if M is oriented, homology intersection product satisfy the properties enumerated in Section 7.2, except for the boundary formulas.*

We now set to proving Theorem 7.22. We first prove it up to some technical lemmas and propositions, which follow just below.

Proof of Theorem 7.22. For the exterior products, by Section 7.1.4 we already have full-defined maps

$$\begin{aligned}
C_*^\Gamma(M) \times C_*^\Gamma(N) &\to C_*^\Gamma(M \times N)\\
C^*_\Gamma(M) \times C^*_\Gamma(N) &\to C^*_\Gamma(M \times N).
\end{aligned}$$

These are easily seen to be bilinear and $\mathbb{Z}$-balanced (i.e. they satisfy $(r\underline{V}) \times \underline{W} = r(\underline{V} \times \underline{W}) = \underline{V} \times r\underline{W}$ for any $r \in \mathbb{Z}$). Moreover, these are chain maps: for the chain cross product this follows from the standard boundary formula for oriented products and our boundary conventions, Convention 3.1, and for the cochain cross product this follows from Proposition 3.66, recalling our indexing convention for cochains. The existence of the homology and cohomology cross products now follows from standard homological algebra.

For the other products, we must show that if we are given homology or cohomology classes (depending on the particular product), then they can be represented by chains or cochains $\underline{V}$ and $\underline{W}$ that are transverse and that the product does not depend on such a choice. The basic steps of the proof are relatively standard and analogous to the proof of Theorem 6.5. We provide the general details here modulo a technical lemma that we will prove below.

First we define the products. Given (co)homology classes represented by $r_V : V \to M$ and $r_W : W \to M$, we will find a universal homotopy[31] of r_V to a map $r'_V : V \to M$ that is transverse to W. Then r_V and r'_V represent (co)homologous (co)chains, and we can represent the product by the (oriented or co-oriented) fiber product of r'_V and r_W.

To show that this gives a well-defined (co)homology class, we can suppose that $r''_V : W \to M$ is another map transverse to r_W representing the same (co)homology class as $r_V : V \to M$ with $r_Z : Z \to M$ providing the (co)homology. For notational purposes, let us write $r'_V : V' \to M$ and $r''_V : V'' \to M$ with $V = V' = V''$. Then $\partial Z = V' \sqcup -V'' \sqcup T$ with $T \in Q(M)$. We must show that $V' \times_M W$ and $V'' \times_M W$ are (co)homologous. To do so, we find a universal homotopy $H : Z \times I \to M$ from r_Z to an $r'_Z : Z' \to M$ with

[31] Recall from Section 6.4 that a universal homotopy will be one of the form $V \times I \xrightarrow{r_V \times \mathrm{id}} M \times I \xrightarrow{h} M$ and that such homotopies generate (co)homologies.

$Z = Z'$ such that the restrictions of H to $V' \times I$ and $V'' \times I$ are transverse to r_W and r'_Z is also transverse to r_W. So $H|_{V' \times I}$ and $H|_{V'' \times I}$ give (co)homologies from the (co)chains represented by V' and V'' to (co)chains that are (co)homologous via r'_Z. Altogether, this gives a (co)homology from V' to V'' that is transverse to r_W. Call the (co)chain representing the (co)homology $\mathfrak{V}$. Now we can consider $\mathfrak{V} \times_M W$, and we have

$$\partial(\mathfrak{V} \times_M W) = \pm(\partial\mathfrak{V}) \times_M W \pm \mathfrak{V} \times_M \partial W$$

via the appropriate boundary formulas (see Propositions 3.6, 3.44, and 3.77). As W represents a (co)cycle, $\partial W \in Q(M)$, so $\mathfrak{V} \times_M \partial W \in Q(M)$ by Lemma 4.22. Meanwhile, as $\mathfrak{V}$ is a (co)homology from V' to V'', we have

$$\partial\mathfrak{V} \times_M W = (V' \sqcup -V'' \sqcup T') \times_M W = (V' \times_M W) \sqcup -(V'' \times_M W) \sqcup (T' \times_M W),$$

with $T' \in Q(M)$. So $T' \times_M W \in Q(M)$ by Lemma 4.22, and we see that $V' \times_M W$ and $V'' \times_M W$ are (co)homologous.

So to finish the proof, we need an analogue of Proposition 6.6 that allows us to construct the homotopy H. This is the content of Proposition 7.24 below. $\square$

Proposition 7.24. *Suppose $f : V \to M$ and $g : W \to M$ are proper maps from manifolds with corners to a manifold without boundary. Then there is a proper universal homotopy $h : V \times I \to M$ such that:*

1. *$h(-, 0) = f$,*
2. *$h(-, 1) : V \to M$ is transverse to g,*
3. *if $i_X : X \to V$ is the inclusion of a union of boundary components of V with $r_X = fi_X : X \to M$ transverse to g then $h \circ (i_X \times \mathrm{id}) : X \times I \to M$ is transverse to g.*

This proposition is analogous to Proposition 6.6 with the difference being that instead of making a map transverse to the faces of a cubulation we must make a map transverse to another map. We will explain how to modify the proof of Proposition 6.6 to accomplish this. This requires some lemmas.

We have already seen the first relevant lemma as Lemma 2.15. Recall that this lemma allows us to replace transversality of arbitrary maps with transversality in which one map is an embedding.

The next lemma says, roughly speaking, that $e(W)$ does not run off to infinity in the $\mathbb{R}^n$ factors over compact subsets of M.

Lemma 7.25. *Let $g : W \to M$ be a proper map from a manifold with corners to a manifold without boundary, let $\pi_M : M \times \mathbb{R}^n \to M$ be the projection, and let $e : W \to M \times \mathbb{R}^n$ be an embedding such that $\pi_M e = g$. Then if $L \subset M$ is compact, there exists a close ball $\bar{B}^n_L \subset \mathbb{R}^n$ such that $e(g^{-1}(L)) \subset L \times \bar{B}^n_L$.*

Proof. As L is compact and g is proper, $g^{-1}(L)$ is compact. So $e(g^{-1}(L))$ is compact, as is its image under the projection $\pi_{\mathbb{R}^n} : M \times \mathbb{R}^n \to \mathbb{R}^n$. Let $\bar{B}^n_L \subset \mathbb{R}^n$ be a closed ball containing this projection. Then $\pi_M e(g^{-1}(L)) = g(g^{-1}(L)) \subset L$ and $\pi_{\mathbb{R}^n} e(g^{-1}(L)) \subset \bar{B}^n_L$. So $e(g^{-1}(L)) \subset L \times \bar{B}^n_L$. $\square$

We can now use these two lemmas to augment the proof of Proposition 6.6 to a proof of Proposition 7.24.

Proof of Proposition 7.24. Suppose we have an embedding $e : W \to M \times \mathbb{R}^n$ such that $\pi e = g$, with $\pi : M \times \mathbb{R}^n \to M$ the projection. Such an embedding always exists by the proof of Lemma 3.16. By Lemma 2.15, it suffices to show that there is a proper universal homotopy $h : V \times I \to M$ such that

1. $h(-, 0) = f$,
2. $h(-, 1) \times \mathrm{id}_{\mathbb{R}^n} : V \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is transverse to $e : W \to M \times \mathbb{R}^n$, and
3. $(h \circ (i_X \times \mathrm{id}_I)) \times \mathrm{id}_{\mathbb{R}^n} : X \times I \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is transverse to e.

To do so, we will run through the proof of Proposition 6.6 again, adapting it to this altered situation and referring back to that proof for some of the details. Note to Dev and Anibal: I know this is a bit redundant and Proposition 6.6 is arguably a special case with $n = 0$ (though there are a few other differences), but the proof of Proposition 6.6 is already very complicated so I didn't want to just do this more general version earlier.

As in the proof of Proposition 6.6, we begin with the case where V is compact and construct $F : M \times D^N \to M$, with D^N the unit ball centered at 0 in $\mathbb{R}^N$ for some N, such that

(1) $F(-,0) = \mathrm{id} : M \to M$,
(2) for almost all $s \in D^N$ the composition $V \times \mathbb{R}^n \xrightarrow{f \times \mathrm{id}_{\mathbb{R}^n}} M \times \mathbb{R}^n \xrightarrow{F(-,s)\times \mathrm{id}_{\mathbb{R}^n}} M \times \mathbb{R}^n$ is transverse to $e : W \to M \times \mathbb{R}^n$,
(3) there is a ball neighborhood D^N_r of 0 in D^N such that for all $s \in D^N_r$ the composition $X \times \mathbb{R}^n \xrightarrow{i_X \times \mathrm{id}_{\mathbb{R}^n}} V \times \mathbb{R}^n \xrightarrow{f \times \mathrm{id}_{\mathbb{R}^n}} M \times \mathbb{R}^n \xrightarrow{F(-,s)\times \mathrm{id}_{\mathbb{R}^n}} M \times \mathbb{R}^n$ is transverse to e.

Given such an F, we let s_0 be any point in D^N_r such that $V \times \mathbb{R}^n \xrightarrow{f \times \mathrm{id}_{\mathbb{R}^n}} M \times \mathbb{R}^n \xrightarrow{F(-,s_0)\times \mathrm{id}_{\mathbb{R}^n}} M \times \mathbb{R}^n$ is transverse to $e : W \to M \times \mathbb{R}^n$. Then let $h(-,t) = F(-,ts_0)f$, i.e. $h(x,t) = F(f(x),ts_0)$. The first required property for h holds since $F(-,0) = \mathrm{id}$. The second property holds by our choice of s_0. The last property then holds as $ts_0 \in D^N_r$ for all $t \in I$; thus each $(h(-,ts_0) \times \mathrm{id}_{\mathbb{R}^n})(i_X \times \mathrm{id}_{\mathbb{R}^n}) = (h(-,ts_0)i_x) \times \mathrm{id}_{\mathbb{R}^n}$ is transverse to e, which then implies that $(h \circ (i_X \times \mathrm{id}_I)) \times \mathrm{id}_{\mathbb{R}^n}$ is transverse to e as well. Furthermore, h is a universal homotopy by construction, and it is automatically proper as we are assuming V to be compact.

We now claim that we can construct F almost exactly as in Proposition 6.6. Recall that we let M_ϵ be an ϵ-neighborhood of a proper embedding of M into some $\mathbb{R}^N$ in the sense of the ϵ-Neighborhood Theorem of [**?**, Section 2.3], with ϵ a smooth positive function of M and $M_\epsilon = \{z \in \mathbb{R}^N \mid |z-y| < \epsilon(y) \text{ for some } y \in M\}$. Let $\pi_\epsilon : M_\epsilon \to M$ be the submersion. We define $\theta : M \times D^N \to M_\epsilon$ by $\theta(y,s) = y + \epsilon(y)s$, which is a submersion onto its image. Then we let $F : M \times D^N \to M$ be the submersion $M \times D^N \xrightarrow{\theta} M_\epsilon \xrightarrow{\pi} M$ and let $H : V \times D^N \to M$ be the composition $F \circ (f \times \mathrm{id}_{D^N})$. The map H is also a submersion onto its image, as are its restrictions $H|_{S^k(V)\times D^N}$. Consequently the maps

$$S^k(V) \times D^N \times \mathbb{R}^n \xrightarrow{H \times \mathrm{id}_{\mathbb{R}^n}} M \times \mathbb{R}^n$$

are also submersions onto their images. Thus they are all transverse to all the strata of W, embedded by e into $M \times \mathbb{R}^n$. It follows by the Transversality Theorem of [**?**, Section 2.3] that for any such stratum $S^j(W)$ of W, each $H|_{S^k(V)}(-,s) \times \mathrm{id}_{\mathbb{R}^n}$ is transverse to $S^j(W)$ for almost all $s \in D^N$; note that D^N remains our parameter space for invoking the Transversality Theorem, though we no longer write it as the last factor. However, there are finitely many strata of W and V. As the finite union of measure zero sets has measure zero, for almost all $s \in D^N$ we have all $H|_{S^k(S)}(-,s) \times \mathrm{id}_{\mathbb{R}^n}$, $k \geq 0$, transverse to all $S^j(W)$. So $H(-,s) \times \mathrm{id}_{R^n}$ is transverse to e for almost all $s \in D^N$.

Once again it is clear that $F(-,0) = \mathrm{id}_M$, so it remains to show that if X is a union of boundary components of V with $r_X = fi_X : X \to M$ transverse to g (or, equivalently, $r_X \times \mathrm{id}_{\mathbb{R}^n} : X \times \mathrm{id}_{R^n} \to M \times \mathbb{R}^n$ transverse to e) then $(H(-,s) \circ i_X) \times \mathbb{R}^n : X \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is transverse to e for all s in some neighborhood U of 0 in D^N. For simplicity of notation, we assume for the remainder of the proof that $X = \partial V$; in case X is a union of only some of the components of ∂V, we can restrict the following arguments to just the relevant components of X. Let H_k denote the composition $H_k : \partial^k V \times D^N \xrightarrow{i_{\partial^k V} \times \mathrm{id}} V \times D^N \xrightarrow{f \times \mathrm{id}} M \times D^N \xrightarrow{F} M$. As X consists of boundary components of V, we must consider the H_k, $k \geq 1$. We provide the details for H_1, the other cases being similar.

We must start with two observations that were not needed in the proof of Proposition 6.6.

First, as we are currently assuming that V is compact, $L = H(V \times \bar{D}^N_{1/2}) \subset M$ is compact, and so by Lemma 7.25 we have $e(g^{-1}(L)) \subset M \times \bar{B}^n_L$ for some closed ball $\bar{B}^n_L$. In particular, this implies that for $|s| \leq 1/2$ only points in the compact set $V \times \bar{B}^n_k$ can be taken by $H(-,s) \times \mathrm{id}_{\mathbb{R}^n}$ to points of $e(W)$ in $M \times \mathbb{R}^n$.

Second, we also have to be more careful here about the map $e : W \times \mathbb{R}^n$, as its behavior can be more complicated than the embedding of a closed face of a cubulation. Let $W_j = ei_{\partial^j W}(\partial^j W)$, the image of $\partial^j W$ in $M \times \mathbb{R}^n$. As the maps $i_{\partial^j W}$ are not necessarily embeddings, it will not generally be the case that $W_j \cong \partial^j W$. However, by [**?**, Lemma 2.8], the $i_{\partial^j W}$ are proper maps, so W_j is a closed subset of $M \times \mathbb{R}^n$ (recall that proper maps are closed — see [**?**, Section I.10]).

Now, suppose $(x, z) \in X \times \bar{B}^n_L$. As $f \times \mathrm{id}_{\mathbb{R}^n} = H_1(-, 0) \times \mathrm{id}_{\mathbb{R}^n}$ is transverse $e : W \to M \times \mathbb{R}^n$, either $(r_X(x), z) \notin W_j$ or $r_X \times \mathrm{id}_{\mathbb{R}^n}$ is (plainly) transverse to $ei_{\partial^j W}$ at $(r_X(x), z)$. In the former case, as W_j is closed, there is an open neighborhood $A_{(x,z)}$ of $(x, 0, z) \in X \times D^N \times \mathbb{R}^n$ such that $(H_1 \times \mathrm{id}_{\mathbb{R}^n})(A_x) \cap W_j = \emptyset$. Now suppose that $(r_X(x), z) \in W_j$ and is transverse there to $ei_{\partial^j W} : \partial^j W \to M \times \mathbb{R}^n$. As e is an embedding, the preimage of $(r_X(x), z)$ in $\partial^j W$ is the preimage of a point of W under the boundary map $i_{\partial^j W}$, which is a finite set of points. Let $a \in \partial^j W$ be one point of the preimage. As the boundary maps are immersions, there is a neighborhood C_a of a in $\partial^j W$ on which $ei_{\partial^j W}$ restricts to an embedding from a $w - j$ dimensional manifold with corners into $M \times \mathbb{R}^n$. In fact, by choosing a chart around a and using the definition of a smooth map of manifolds with corners, $ei_{\partial^j W}$ (composed with the chart map) extends to a smooth immersion of a neighborhood of $0 \in \mathbb{R}^{w-j}$ into $M \times \mathbb{R}^n$. By further appealing to charts and local diffeomorphisms, we can identify a neighborhood of $(r_X(x), z)$ in $M \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ and the image of the extension of $ei_{\partial^j W}$ with $\mathbb{R}^{w-j} \times 0 \subset \mathbb{R}^{m+n}$. The transversality assumption means that the composition of $D(r_X \times \mathrm{id}_{\mathbb{R}^n}) : T_x X \times T_z \mathbb{R}^n \to T_{(r_X(x),z)}(M \times \mathbb{R}^n)$ with the projection to the last $m + n - (w - j)$ coordinates is a linear surjection. As this is an open condition on the Jacobian matrix of $r_X \times \mathrm{id}_{\mathbb{R}^n}$ at (x, z), it follows again that there is an open neighborhood $A_{(x,t),a}$ of $(x, 0, z)$ in $X \times D^N_{1/2} \times \mathbb{R}^n$ such that for each (x', s, z') in the neighborhood $H_1(-, s) \times \mathrm{id}_{\mathbb{R}^n}$ is transverse to $ei_{\partial^j W}$ in a neighborhood of a. As there are a finite number of possible points a and the transversality assumptions must hold for all of them, by taking $A_{(x,z)} = \cap_a A_{(x,z),a}$ with the finite intersection running over all points of $\partial^j W$ that map to $(r_X(x), z)$, we obtain a neighborhood $A_{(x,z)}$ of $(x, 0, z)$ in $W \times D^N_{1/2} \times \mathbb{R}^n$ such that for each (x', s, z') in the neighborhood, $H_1(-, s) \times \mathrm{id}_{\mathbb{R}^n}$ is transverse to $ei_{\partial^j W}$ at (x', z').

Now, taking the union of the $A_{(x,z)}$ over all $(x, z) \in X \times \bar{B}^n_L$ gives a neighborhood G_j of $X \times 0 \times \bar{B}^n_L$ in $X \times D^N_{1/2} \times \bar{B}^n_L$, and by the Tube Lemma, as $X \times \bar{B}^n_L$ is compact, there is a neighborhood of $X \times 0 \times \bar{B}^n_L$ of the form $X \times U_j \times \bar{B}^n_L$ in G_j. For each $s \in U_j$, we have $H_1(-, s) \times \mathrm{id}_{\mathbb{R}^n} : X \times \bar{B}^n_L \to M \times \mathbb{R}^n$ transverse to $ei_{\partial^j W}$. Furthermore, by the choice of $\bar{B}^n_L$, the map $H_1(-, s) \times \mathrm{id}_{\mathbb{R}^n}$ takes no point of $X \times \mathbb{R}^n$ that is in the complement of $X \times \bar{B}^n_L$ to the image of W. Repeating the argument for all of the finite j such that $\partial^j W \neq \emptyset$ and taking $U = \cap_j U_j$, we obtain a neighborhood of 0 in $D^N_{1/2}$ on which $H_1(-, s) \times \mathrm{id}_{\mathbb{R}^n} : X \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is (plainly) transverse to all $ei_{\partial^j W}$. By the same argument, we can then find an even smaller U such that $H_k(-, s) \times \mathrm{id}_{\mathbb{R}^n} : \partial^k V \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is (simply) transverse to all $ei_{\partial^j W}$ for all $k \geq 1$.

This completes the proof of the proposition for V compact.

Next suppose that V is no longer necessarily compact. We construct a homotopy $\hat{h} : V \times I \to M$ that fulfills the proposition.

We will utilize $F : M \times I \to M$ as defined above, which did not rely on V being compact. For V not compact, the first two properties listed above for F will continue to hold, but the third relied on compactness and so need not hold any long in general. However, let $K \subset X$ be compact, and let $L = H_1(i_X(K) \times \bar{D}^N_{1/2}) \subset M$. Choosing a W_j as above and taking the union of the resulting $A_{(x,z)} \subset X \times D^N_{1/2} \times \mathbb{R}^n$ over all $(x, z) \in K \times \bar{B}^n_L$ and intersecting with $K \times D^N \times \bar{B}^n_L$ gives an open neighborhood G_j of $K \times 0 \times \bar{B}^n_L$ in $K \times D^N \times \bar{B}^n_L$, such that $H_1(-, s) \times \mathrm{id}_{\mathbb{R}^n}$ is transverse to $ei_{\partial^j W}$ for all $(x, z) \in K \times \bar{B}^n_L$. Again we can use the Tube Lemma to find a neighborhood U_j of 0 in D^N so that $K \times U_j \times \bar{B}^n_L \subset G_j$, and then for every $s \in U_k$, we know that $H_1(-, s) \times \mathrm{id}_{\mathbb{R}^n}$ is transverse to $ei_{\partial^j W}$ at every point of $K \times \mathbb{R}^n$, as points of $K \times \mathbb{R}^n$ outside of $K \times \bar{B}^n_L$ do not intersect W_j. Now, as above, by ranging over all of the finite options for j and k, we can then find a neighborhood U of 0 in $D^N_{1/2}$ such that for all $s \in U$ we have

$H_k(-,s)\times \mathrm{id}_{\mathbb{R}^n}$ (plainly) transverse to all $ei_{\partial^j W}$ at all points of $\partial^k V\times\mathbb{R}^n$, $k\geq 1$, that map to $K\times\mathbb{R}^n$ via the boundary immersions.

Let $\{\mathcal{U}_\ell\}$ be a locally finite covering of M such that each $\bar{\mathcal{U}}_\ell$ is compact. As $f: V\to M$ and $r_X = f\circ i_X$ are proper, each $r_X^{-1}(\bar{\mathcal{U}}_\ell)$ is compact in X. Proceeding as just above with $r_X^{-1}(\bar{U}_\ell)$ in place of K, we can find for each ℓ an $\varepsilon_{\ell,1}\leq 1$ so that for every $s\in D^N_{\varepsilon_{\ell,1}}$ we have $H_1(-,s)\times\mathrm{id}_{\mathbb{R}^n}$ transverse to all $ei_{\partial^j W}$ at every $(x,z)\in r_X^{-1}(\bar{\mathcal{U}}_\ell)\times\mathbb{R}^n$. Analogously, we have $\varepsilon_{\ell,k}$ for all $k\geq 1$ so that $H_k(-,s)\times\mathrm{id}_{\mathbb{R}^n}$ is transverse to all $ei_{\partial^j W}$ at every (x,z) in $(r_V i_{\partial^k V})^{-1}(\bar{\mathcal{U}}_\ell)\times\mathbb{R}^n$. Let $\varepsilon_\ell = \min\{\varepsilon_{\ell,k}\mid k\geq 1\}$. These minima exist as V has finite depth.

Now, using Lemma 6.7, we choose a smooth function $\phi: M\to\mathbb{R}$ such that for all $x\in M$ we have $0<\phi(x)<\epsilon_\ell$ if $x\in\bar{\mathcal{U}}_\ell$. Let $M\times_\phi D^N = \{(y,s)\in M\times D^N \mid |s|<\phi(y)\}$. By our construction, for all $k\geq 1$ we have $H_k(-,s)\times\mathrm{id}_{\mathbb{R}^n}:\partial^k V\times\mathbb{R}^n\to M\times\mathbb{R}^n$ transverse to $ei_{\partial^j W}$ for each s and at each (x,z) such that $(x,s)\in(r_X i_{\partial^k V}\times\mathrm{id})^{-1}(M\times_\phi D^N) = \{(x,s)\in\partial^k V\times I\mid |s|<\phi(r_X i_{\partial^k V}(x))\}$.

Let $\hat\theta: M\times D^N\to M_\epsilon$ be given by $\hat\theta(y,s) = y+\phi(y)\epsilon(y)s$; as $\phi(y)\epsilon(y)>0$, this is again a submersion onto its image at all points. Let $\hat F: M\times D^N\to M$ be the composition $M\times D^N\xrightarrow{\hat\theta} M_\epsilon\xrightarrow{\pi_\epsilon} M$, and let $\hat H_k$ be the composition $\partial^k V\times D^N\xrightarrow{i_{\partial^k V}\times\mathrm{id}} V\times D^N\xrightarrow{f\times\mathrm{id}} M\times D^N\xrightarrow{\hat F} M$ for $k\geq 0$. Once again by the Transversality Theorem of [**?**, Section 2.3], for almost all $s\in D^N$ we have $\hat H_k(-,s)\times\mathrm{id}_{\mathbb{R}^n}$ transverse to all $ei_{\partial^j W}$ for all $k\geq 0$. Letting s_0 be any such point we define $\hat h: V\times I\to M$ to be $\hat h(x,t)=\hat H(x,ts_0)$, and we claim that this h satisfies the conditions of the proposition.

The map $\hat h$ is proper, and the first two conditions of the proposition follow immediately from the construction. It remains to verify that

$$(\hat h\circ(i_X\times\mathrm{id}_I))\times\mathrm{id}_{\mathbb{R}^n}: X\times I\times\mathbb{R}^n\to M\times\mathbb{R}^n$$

is transverse to e. As we already know from the second condition of the proposition that $\hat h(-,1)\times\mathrm{id}_{\mathbb{R}^n}$ is transverse to e and from the hypotheses that $\hat h(-,0)$ is transverse to e, it suffices to demonstrated transversality to e of the restriction of $\hat h\times\mathrm{id}_{\mathbb{R}^n}$ to $X\times(0,1)\times\mathbb{R}^n$. From here, the argument is essentially the same as the end of the proof of Proposition 6.6 with the $\mathbb{R}^n$ factor just along for the ride.

In detail, for $(x,t)\in X\times I$ we can write $\hat h\circ(i_X\times\mathrm{id}_I): X\times I\to M$ explicitly as

$$\hat h(i_X(x),t) = \pi_\epsilon(r_X(x)+\phi(r_X(x))\epsilon(r_X(x))ts_0).$$

So, alternatively, we can observe that $\hat h\circ(i_X\times\mathrm{id}_I)$ is the composition

$$X\times I\xrightarrow{\Phi} X\times I\overset{\Psi}{\hookrightarrow} X\times D^N\xrightarrow{r_X\times\mathrm{id}} M\times D^N\xrightarrow{F} M, \tag{19}$$

with $\Phi(x,t)=(x,\phi(r_X(x))t)$, $\Psi(x,t)=(x,ts_0)$, and noting that on the right we do mean our original F and not $\hat F$.

The first map Φ is a diffeomorphism onto its image, which is a neighborhood of $X\times 0$ in $X\times I$, and the map Ψ embeds this linearly into $X\times D^N$. The composition of the last two maps is just our earlier map H_1. By construction, the map $r_X\times\mathrm{id}_{D^N}$ now takes the image of $\Psi\Phi$ into $M\times_\phi D^N$, and so at each point (x,s,z) in the image of $\Psi\Phi\times\mathrm{id}_{\mathbb{R}^n}$ if we fix s and consider $H_1(-,s)\times\mathrm{id}_{\mathbb{R}^n}$ we get by construction a map on $X\times\mathbb{R}^n$ that is transverse to e. But as Φ is a diffeomorphism onto its image and Ψ is an embedding that is the identity with respect to X, we see $\Psi\Phi$ takes a neighborhood of any $(x,t)\in X\times(0,1)$ to a neighborhood of its image in $X\times\mathbb{R}s_0$, where $\mathbb{R}s_0$ is the line in $\mathbb{R}^N$ spanned by s_0. In particular, the derivative of $\Psi\Phi$ maps the tangent space to $X\times(0,1)$ at (x,t) onto $T_xX\times\mathbb{R}s_0\subset T_{\Psi\Phi(x,t)}(X\times D^N)$. So for any $(x,t,z)\in X\times(-1,1)\times\mathbb{R}^n$, the image of $T_{(x,t,z)}(X\times(-1,1)\times\mathbb{R}^n)$ under $D(\Psi\Phi\times\mathrm{id}_{\mathbb{R}^n})$ contains $T_xX\times 0\times\mathbb{R}^n$. By our construction, DH_1 takes this tangent space to a tangent subspace in $M\times\mathbb{R}^n$ at $\hat h(x,t)$ that is transverse to the images of all $D(ei_{\partial^j W})$. The same holds for $k>1$ replacing X with $\partial^k V$ and r_X with $r_{\partial^k V}$. So we see that $\hat h$ satisfies all the requirements of the proposition. $\square$

7.4. **The Kronecker pairing and the Universal Coefficient Theorem for geometric cohomology.** When applying the cap product to a chain and cochain of the same degree, we can compose with the augmentation map $\mathbf{a} : H_0^\Gamma(M) \to \mathbb{Z}$ of Definition 6.9 to obtain a bilinear Kronecker pairing

$$H^i_\Gamma(M) \otimes H_i^\Gamma(M) \xrightarrow{\cap} H_0^\Gamma(M) \to \mathbb{Z}.$$

Taking the adjunct then provides a map

$$\alpha : H^i_\Gamma(M) \to \mathrm{Hom}(H_i^\Gamma(M), \mathbb{Z}).$$

Tracing through the definitions, this maps acts by counting the intersection number between a geometric chain and a geometric cochain in the sense of Definitions 6.10 and 6.10.

When $H^i_\Gamma(M)$ is finitely generated, this map fits into a short exact sequence, just as for singular cohomology.

Theorem 7.26. *If $H^i_\Gamma(M)$ is finitely generated, there is a short exact sequence*

$$0 \longrightarrow Ext\big(H_{i-1}^\Gamma(M), \mathbb{Z}\big) \longrightarrow H^i_\Gamma(M) \xrightarrow{\alpha} \mathrm{Hom}\big(H_i^\Gamma(M), \mathbb{Z}\big) \longrightarrow 0.$$

Remark 7.27. The existence of a Universal Coefficient exact sequence holds even if $H^i_\Gamma(M)$ is not finitely generated, as we know by Theorem 5.24 that $H^i_\Gamma(M) \cong H^i(M)$, and then we have the usual singular cohomology Universal Coefficient Theorem. We can further identify $\mathrm{Hom}(H_i(M), \mathbb{Z})$ and $\mathrm{Ext}(H_{i-1}(M), \mathbb{Z})$ with $\mathrm{Hom}(H_i^\Gamma(M), \mathbb{Z})$ and $\mathrm{Ext}(H_{i-1}^\Gamma(M), \mathbb{Z})$, also using Theorem 5.24. What we lose in this general case is that this approach does not allow us to identify the map $H^i_\Gamma(M) \to \mathrm{Hom}(H_i^\Gamma(M), \mathbb{Z})$ as given by counting intersection numbers.

Proof of Theorem 7.26. Let M have a cubulation X, let $\mathcal{I} : C^*_{\Gamma \pitchfork X}(M) \to K^*(X)$ be the intersection map of Definition 6.13, and let $\mathcal{J} : K_*(X) \to C_*^\Gamma(M)$ be the map inducing the homology isomorphism of Theorem 6.2. We consider the diagram

$$\begin{tikzcd}
H^i_\Gamma(M) \arrow[r, "\alpha"] & \mathrm{Hom}(H_i^\Gamma(M), \mathbb{Z}) \arrow[dd, "\cong"', "\mathcal{J}^*"] \\
H^i(C^*_{\Gamma \pitchfork X}(M)) \arrow[u, "\cong"'] \arrow[d, "\mathcal{I}", "\cong"'] & \\
H^i(K^*(X)) \arrow[r] & \mathrm{Hom}(H_*(K_*(X)), \mathbb{Z}).
\end{tikzcd}$$

The vertical maps on the left are isomorphisms by Theorems 6.5 and 6.20, while the righthand vertical map is an isomorphism by Theorem 6.2. We claim the diagram commutes. In fact, let $V \in PC^i_\Gamma(M)$ represent an element of $H^i(C^*_{\Gamma \pitchfork X}(M))$. Then V is transverse to the cubulation, and by definition the path clockwise around the diagram takes $\underline{V}$ to a map that acts on an element ξ of $H_*(K_*(X))$ represented by a $\mathbb{Z}$-linear combination of cubical faces $\sum_j c_j E_j$ by treating each E_j as a geometric chain and forming

$$\mathbf{a}\left(V \times_M \sum_j c_j E_j\right) = \sum_j c_j \mathbf{a}(V \times_M E_j).$$

On the other hand, by Definition 6.13, the composition counterclockwise takes $\underline{V}$ to a map that acts on ξ by $\sum_j I_M(V, E_j)$. But $I_M(V, E_j)$ is precisely $\mathbf{a}(V \times_M E_j)$ by Definition 6.10. So the diagram commutes.

We know that the $K_i(X)$ is a free abelian group, so the bottom map of the diagram is a surjection by the algebraic Universal Coefficient Theorem, with kernel $\mathrm{Ext}(H_{i-1}(X), \mathbb{Z})$. The commutativity of the diagram thus implies that the top map of the diagram is a surjection with isomorphic kernel. To complete the proof, we again invoke Theorem 6.2 to observe $\mathrm{Ext}(H_{i-1}(X), \mathbb{Z}) \cong \mathrm{Ext}(H^{\Gamma}_{i-1}(X), \mathbb{Z})$. □

Remark 7.28. Note that while we obtain the expected Universal Coeficient Theorem relating geometric cohomology and homology, we do not claim to have either an isomorphism or a quasi-isomorphism between $C^i_\Gamma(M)$ and $\mathrm{Hom}(C^\Gamma_i(M), \mathbb{Z})$. In fact, as we do not know $C^\Gamma_*(M)$ to be a complex of free abelian groups (which we leave as an open question), it is not clear $\mathrm{Hom}(C^\Gamma_i(M), \mathbb{Z})$ fits into a short exact Universal Coefficient-type sequence at all.

7.5. **The geometric cup product is the usual cup product.** In this section we show that the geometric cup product agrees with the singular cup product. Our proof is based on an axiomatic characterization of the cup product on manifolds due to Kreck and Singhof.

Theorem 7.29. *On the category of smooth manifold without boundary and continuous maps, there is for each $p \geq 0$ a natural isomorphism of functors $\Phi_p : H^p(-) \to H^p_\Gamma(-)$ from singular cohomology to geometric cohomology that is also compatible with cup products. In other words, for each manifold without boundary M there is a commutative diagram*

$$\begin{tikzcd}
H^p(M) \otimes H^q(M) \arrow[r, "\smile"] \arrow[d, "\Phi_p \otimes \Phi_q"'] & H^{p+q}(M) \arrow[d, "\Phi_{p+q}"] \\
H^p_\Gamma(M) \otimes H^q_\Gamma(M) \arrow[r, "\uplus"] & H^{p+q}_\Gamma(M).
\end{tikzcd}$$

The proof will rely on work of Kreck and Singhof [**?**, Proposition 12]. As the proof of this proposition is only sketched in [**?**], we first fill in the details, restricting ourselves to $\mathbb{Z}$ coefficients and changing Kreck and Singhof's notation a bit to avoid conflicts with our earlier notation. Before stating the result, we establish some further notation.

In this section we assume the S^p, $p > 0$, to each have a fixed orientation. We also want these orientations to be compatible in the sense that the composition $\nu : S^p \times S^q \to S^p \wedge S^q \cong S^{p+q}$ is orientation preserving away from the subspace that is collapsed to form the wedge product. In particular, if $[S^p]$ and $[S^q]$ are the corresponding fundamental classes, the quotient should take $[S^p] \times [S^q]$ to $[S^{p+q}]$. This can be arranged, for example, by modeling our spheres as the standardly-oriented cubes with their boundaries collapsed. For each p, we let $s_p \in H^p(S^p) \cong \mathrm{Hom}(H_p(S^p), \mathbb{Z})$ be the cohomology class that evaluates to 1 on $[S^p]$. Let $\pi_1 : S^p \times S^q \to S^p$ and $\pi_2 : S^p \times S^q \to S^q$ be the projections.

Similarly, let $K_p = K(\mathbb{Z}, p)$, $p > 0$, be the Eilenberg-MacLane space, and let $\iota_p \in H^p(K_p)$ denote the fundamental class such that if the p-skeleton of K_p is S^p and $\phi_p : S^p \to K_p$ is the inclusion, then $\phi_p^*(\iota_p) = s_p$. As the $p+1$ skeleton of K_p can be taken to be the image of S^p under ϕ_p, it is standard that ϕ_p^* is an isomorphism. We also let $\mu : K_p \times K_q \to K_{p+q}$ be the unique-up-to-homotopy map that extends the collapse map ν.

Proposition 7.30 (Kreck and Singhof, Proposition 12 of [**?**])**.** *Consider singular cohomology $H^*(-)$ as a cohomology theory on smooth manifolds*[32]*. Suppose $\star$ is a natural multiplication on $H^*(-)$ such that if M is connected and $\lambda \in H^0(M) \cong \mathbb{Z}$ (generated by the class $1 \in C^0(M)$) then $\lambda \star \alpha = \alpha \star \lambda = \lambda\alpha$ for*

[32]As defined in [**?**]; see the proof of Theorem 5.24 above.

all $\alpha \in H^*(M)$ (and with the obvious extension when M is not connected). Then if[33] $\pi_1^*(s_p) \star \pi_2^*(s_q) = s_p \times s_q \in H^{p+q}(S^p \times S^q)$ *for all $p, q \geq 1$, the product $\star$ is the cup product.*

Proof. For a smooth manifold M, let $\alpha \in H^p(M)$ and $\beta \in H^q(M)$. The condition that $\lambda \star \alpha = \alpha \star \lambda = \lambda\alpha$ whem M is connected already guarantees that $\star$ is the cup product when p or q is 0, so we can suppose $p, q > 0$. As H^* is ordinary singular cohomology, we know that α and β can be represented by maps $\bar{\alpha} : M \to K_p$ and $\bar{\beta} : M \to K_q$ with $\alpha = \bar{\alpha}^*(\iota_p)$ and $\beta = \bar{\beta}^*(\iota_q)$. Furthermore, $\alpha \smile \beta$ is the pullback of ι_{p+q} by the composition

$$M \xrightarrow{\mathbf{d}} M \times M \xrightarrow{\bar{\alpha}\times\bar{\beta}} K_p \times K_q \xrightarrow{\mu} K_{p+q}; \tag{20}$$

while similarly $s_p \times s_q$ is the pullback of ι_{p+q} by

$$S^p \times S^q \xrightarrow{\phi_p \times \phi_q} K_p \times K_q \xrightarrow{\mu} K_{p+q}; \tag{21}$$

see [**?**, Section 4.3].

As we will want to apply the naturality of $\star$ in the category of smooth manifolds, we will choose manifold replacements for K_p, K_q, and K_{p+q}. In particular, suppose we realize K_p as a CW complex by the standard constructions and let K_p^N be the N-skeleton of K_p with N much larger than $\dim(M)$. Then K_p is homotopy equivalent to a finite simplicial complex, and we can embed it simplicially into some Euclidean space and take an open regular neighborhood to get a smooth manifold $\mathcal{K}_p$ homotopy equivalent to the N-skeleton of K_p. We define $\mathcal{K}_q$ and $\mathcal{K}_{p+q}$ analogously, using a large enough skeleton K_{p+q}^M of K_{p+q} for the restriction of μ to $K_p^N \times K_q^N \to K_{p+q}^M$ to be defined. Abusing notation, we continue to write $\bar{\alpha}$, $\bar{\beta}$, ϕ_p, μ, etc. for the maps involving these manifold replacements of the K_*.

Next we make two more preliminary observations. The first is that it follows from $\star$ being natural with respect to pullbacks that, when f^* is an isomorphism, the product $\star$ is also natural with respect to $(f^*)^{-1}$, as we see by applying f^* to the claimed identity $(f^*)^{-1}(x) \star (f^*)^{-1}(y) = (f^*)^{-1}(x \star y)$. The second is that there is an evident commutative diagram

$$\begin{tikzcd} H^p(\mathcal{K}_p) \arrow[r, "\pi_1^*"] \arrow[d, "\phi_p^*"'] & H^p(\mathcal{K}_p \times \mathcal{K}_q) \arrow[d, "(\phi_p \times \phi_q)^*"] \\ H^p(S^p) \arrow[r, "\pi_1^*"] & H^p(S^p \times S^q), \end{tikzcd} \tag{22}$$

and similarly for π_2, abusing notation to write π_1 and π_2 for the projections to the first and second factors for both pairs of spaces. Now we compute:

[33] Rather than $s_p \times s_q$, Kreck and Singhof require $\pi_1^*(s_p) \star \pi_2^*(s_q)$ to be the element of $H^{p+q}(S^p \times S^q)$ that evaluates to 1 on the fundamental class of $S^p \times S^q$, but with our conventions that tensor products of cochains act by $(\alpha \otimes \beta)(x \otimes y) = \alpha(x)\beta(y)$, these are the same cohomology class (c.f. [**?**, page 245] and [**?**, Section 3B].

$$\begin{aligned}
\alpha \smile \beta &= \mathbf{d}^*(\bar{\alpha} \times \bar{\beta})^* \mu^*(\iota_{p+q}) && \text{see (20)}\\
&= \mathbf{d}^*(\bar{\alpha} \times \bar{\beta})^*((\phi_p \times \phi_q)^*)^{-1}(\phi_p \times \phi_q)^* \mu^*(\iota_{p+q}) &&\\
&= \mathbf{d}^*(\bar{\alpha} \times \bar{\beta})^*((\phi_p \times \phi_q)^*)^{-1}(s^p \times s^q) && \text{see (21)}\\
&= \mathbf{d}^*(\bar{\alpha} \times \bar{\beta})^*((\phi_p \times \phi_q)^*)^{-1}(\pi_1^*(s_p) \star \pi_2^*(s_q)) && \text{by assumption}\\
&= \mathbf{d}^*(\bar{\alpha} \times \bar{\beta})^*(((\phi_p \times \phi_q)^*)^{-1}\pi_1^*(s_p) \star ((\phi_p \times \phi)^*)^{-1}\pi_2^*(s_q)) && \text{by naturality}\\
&= \mathbf{d}^*(\bar{\alpha} \times \bar{\beta})^*(\pi_1^*(\phi_p^*)^{-1}(s_p) \star \pi_2^*(\phi_q^*)^{-1}(s_q)) && \text{by diagram (22)}\\
&= \mathbf{d}^*(\bar{\alpha} \times \bar{\beta})^*(\pi_1^*(\iota_p) \star \pi_2^*(\iota_q)) &&\\
&= (\mathbf{d}^*(\bar{\alpha} \times \bar{\beta})^*\pi_1^*(\iota_p)) \star (\mathbf{d}^*(\bar{\alpha} \times \bar{\beta})^*\pi_2^*(\iota_q)) && \text{by naturality}\\
&= \bar{\alpha}^*(\iota_p) \star \bar{\beta}^*(\iota_q) &&\\
&= \alpha \star \beta && \text{by definition.}
\end{aligned}$$

For the penultimate equality, we have used that the composition of maps

$$M \xrightarrow{\mathbf{d}} M \times M \xrightarrow{\bar{\alpha}\times\bar{\beta}} K_p \times K_q \xrightarrow{\pi_1} K_p$$

is just $\bar{\alpha}$, and similarly for $\bar{\beta}$. □

Proof of Theorem 7.29. Recall that in the Proof of Theorem 5.24, which established an isomorphism between geometric and singular cohomology, we applied [**?**, Theorem 10]. That theorem of Kreck-Singhof shows that there is a natural isomorphism of these cohomology theories on the category of smooth manifolds and continuous maps. In fact, it shows there is such an isomorphism extending any given isomorphism of coefficients $\Phi_0 : H^0(pt) \to H^0_\Gamma(pt)$ to natural isomorphisms $\Phi_p : H^p(-) \to H^p_\Gamma(-)$ for all $p \geq 0$. For the argument below, we would like to arrange that for all $p \geq 1$ we have $\Phi_p(s_p) = s_p^\Gamma$, where $s_p \in H^p(S^p)$ is our preferred generator described just above and $s_p^\Gamma \in H^p_\Gamma(S^p)$ is the generator represented by an embedded point with normal co-orientation agreeing with our chosen orientation of S^p. This will not necessarily be the case for the Φ_p output by Theorem 5.24. However, part of the data for a cohomology theory in the Kreck-Singhof theory consists of the natural connecting maps δ of the Mayer-Vietoris sequence, and part of the output of the theorem is that the isomorphisms Φ_p commute with these connecting maps. Let us write the connecting map for a cohomology theory h^* more explicitly as $h^p(U \cap V) \xrightarrow{\delta_p} h^{p+1}(U \cup V)$; we will generally write δ_p for the connecting map independent of which cohomology theory we are discussing. Of course if we replace a given δ_p by $-\delta_p$ for all spaces, then we still have a natural connecting map and we will not have affected the exactness of the Mayer-Vietoris sequence. If we make such a change, we technically have a new cohomology theory with the same cohomology groups, but [**?**, Theorem 10] will output different isomorphisms Φ_p. As our present theorem does not particularly care about the signs of the connecting maps in the Mayer-Vietoris sequence, we will feel free to thus tinker with the connecting maps in order to arrange that $\Phi_p(s_p) = s_p^\Gamma$ for all $p \geq 1$. Then we will show the resulting Φ_p satisfy Theorem 7.29.

Let

$$\begin{aligned}
U_p &= \{(x_1, \ldots, x_{p+1}) \in S^p \mid x_{p+1} > -1/2\}\\
V_p &= \{(x_1, \ldots, x_{p+1}) \in S^p \mid x_{p+1} < 1/2\}.
\end{aligned}$$

Then $U_p \cap V_p$ is homotopy equivalent to S^{p-1}. For $p \geq 1$, we now choose the sign of $\delta_p : H^p(U_{p+1} \cap V_{p+1}) \cong H^p(S^p) \to H^{p+1}(S^{p+1})$ so that $\delta_p(s_p) = s_{p+1}$, and similarly for geometric cohomology so that $\delta_p(s_p^\Gamma) = s_{p+1}^\Gamma$. For $p = 0$, to avoid confusion let us write $z_- = -1 \in \mathbb{R}$ and $z_+ = 1 \in \mathbb{R}$. We let s_0 be the element of $H^0(S^0) \cong \mathbb{Z}^2$ that restricts to $1 \in H^0(z_+)$ and $0 \in H^0(z_-)$. Similarly, let $s_0^\Gamma \in H^0_\Gamma(S^0)$ be represented by the identity map of z_+ with its canonical co-orientation. Then we choose the signs of δ_0 so that $\delta_0(s_0) = s_1$ and $\delta_0(s_0^\Gamma) = s_1^\Gamma$. Finally, let Φ_0 take $1 \in H^0(pt)$ to the element of $H^0_\Gamma(pt)$ represented by the identity with canonical co-orientation.

Taking $H^*(-)$ and $H^*_\Gamma(-)$ with these Mayer-Vietoris maps and this $\Phi_0 : H^0(pt) \to H^0_\Gamma(pt)$, [**?**, Theorem 10] gives natural isomorphisms $\Phi_p : H^p(-) \to H^*_\Gamma(-)$ extending Φ_0 on a point. The naturality implies that $\Phi_0(s_0) = s_0^\Gamma$. It now follows by induction using the following diagram due to the commutativity of Φ_* with the connecting maps, that $\Phi_p(s_p) = s_p^\Gamma$ for all p:

$$\begin{tikzcd}
H^p(S^p) \cong H^p(U_{p+1} \cap V_{p+1}) \arrow[r, "\delta_p"] \arrow[d, "\Phi_p"'] & H^{p+1}(U_{p+1} \cup V_{p+1}) = H^{p+1}(S^{p+1}) \arrow[d, "\Phi_{p+1}"'] \\
H^p_\Gamma(S^p) \cong H^p_\Gamma(U_{p+1} \cap V_{p+1}) \arrow[r, "\delta_p"] & H^{p+1}_\Gamma(U_{p+1} \cup V_{p+1}) = H^{p+1}_\Gamma(S^{p+1}).
\end{tikzcd}$$

For each M, we can now define pairings $\star$ on $H^p(-) \otimes H^q(-) \to H^{p+q}(-)$ by the composition

$$H^p(M) \otimes H^q(M) \xrightarrow{\Phi_p \otimes \Phi_q} H^p_\Gamma(M) \otimes H^q_\Gamma(M) \xrightarrow{\uplus} H^{p+q}_\Gamma(M) \xrightarrow{\Phi^{-1}_{p+q}} H^{p+q}(M).$$

This pairing is natural, as all the maps are natural. We will apply Proposition 7.30 to show that this is really the cup product, which will prove the theorem.

We first show that if M is connected and $\lambda \in H^0(M) \cong \mathbb{Z}$ (generated by the class of $1 \in C^0(M)$) then $\lambda \star \alpha = \alpha \star \lambda = \lambda\alpha$ for all $\alpha \in H^*(M)$. Let pt be an arbitrary point in M and consider the diagram

$$\begin{tikzcd}
H^0(M) \arrow[r] \arrow[d, "\Phi_0"'] & H^0(pt) \arrow[d, "\Phi_0"] \\
H^0_\Gamma(M) \arrow[r] & H^0_\Gamma(pt).
\end{tikzcd}$$

The vertical maps are isomorphisms by our application of [**?**, Theorem 10], and it is standard that the top map is an isomorphism. In fact, we can consider $H^0(M)$ as generated by the cochain 1_M, and this pulls back to the generator $1_{pt} \in H^0(pt)$ (each represented by the map that takes a positively oriented point considered as a singular 0-chain to 1). It follows that the bottom map is an isomorphism. Consider the generator $\underline{M} \in H^0_\Gamma(M)$ given by the identity map of M with its canonical co-orientation (β_M, β_M). This has normal orientation given by the positively-oriented 0-dimensional normal bundle. By the pullback construction, the pullback to $H^0_\Gamma(pt)$ is similarly represented by $\underline{pt}$, the identity map of pt with its canonical co-orientation. As we know that $\Phi_0(1_{pt}) = \underline{pt}$, it follows from the commutativity that $\Phi_0(1_M) = \underline{M}$.

So for $\alpha \in H^p(M)$, we have

$$\begin{aligned}
\lambda \star \alpha &= \Phi_p^{-1}(\lambda \underline{M} \uplus \Phi_p(\alpha)) \\
&= \Phi_p^{-1}(\lambda \Phi_p(\alpha)) \\
&= \lambda\alpha,
\end{aligned}$$

using the unital property of $\uplus$ — see Section 7.2. The same argument holds for $\alpha \star \lambda$. If M has multiple components, then these properties clearly hold component-wise, as needed.

To apply Proposition 7.30, it remains to show that $\pi_1^*(s_p) \star \pi_2^*(s_q) = s_p \times s_q$ for all $p, q \geq 1$. We have

$$\begin{aligned}
\Phi_{p+q}(\pi_1^*(s_p) \star \pi_2^*(s_q)) &= \Phi_{p+q}\Phi_{p+q}^{-1}(\Phi_p(\pi_1^*(s_p)) \uplus \Phi_q(\pi_2^*(s_q))) && \text{by definition of } \star \\
&= \pi_1^*\Phi_p(s_p) \uplus \pi_2^*\Phi_q(s_q) && \text{by naturality of the } \Phi \\
&= \pi_1^*(s_p^\Gamma) \uplus \pi_2^*(s_q^\Gamma) && \text{by the preceding} \\
&= (s_p^\Gamma \times \underline{S^q}) \uplus (\underline{S^p} \times s_q^\Gamma) && \text{by Prop. 3.70} \\
&= (s_p^\Gamma \uplus \underline{S^p}) \times (\underline{S^q} \uplus s_q^\Gamma) && \text{by Cor. 3.75} \\
&= s_p^\Gamma \times s_q^\Gamma && \text{by Cor. 3.53.}
\end{aligned}$$

So $\pi_1^*(s_p) \star \pi_2^*(s_q) = \Phi_{p+q}^{-1}(s_p^\Gamma \times s_q^\Gamma)$, and it remains to show that this is $s_p \times s_q$, i.e. that $\Phi_{p+q}(s_p \times s_q) = s_p^\Gamma \times s_q^\Gamma$.

To do so, consider the following diagram, which commutes by the naturality of Φ_{p+q}:

$$\begin{tikzcd}
H^{p+q}(S^p \times S^q) \arrow[d, "\Phi_{p+q}"'] & H^{p+q}(S^{p+q}) \arrow[l, "\nu^*"'] \arrow[d, "\Phi_{p+q}"] \\
H_\Gamma^{p+q}(S^p \times S^q) & H_\Gamma^{p+q}(S^{p+q}). \arrow[l, "\nu^*"']
\end{tikzcd}$$

Let s_{p+q}^Γ be represented by the embedding of a point at $y \in S^{p+q}$, normally co-oriented consistently with the orientation of S^{p+q}. By possibly rechoosing the point, we can choose a smooth map homotopic to ν that maps a Euclidean neighborhood of some point $x \in S^p \times S^q$ by an orientation-preserving diffeomorphism to a neighborhood of y, taking x to y and the complement of a neighborhood of x to the complement of a neighborhood of y. Then, from the definitions, the pullback of s_{p+q}^Γ is the embedding of x into $S^p \times S^q$ with normal co-orientation corresponding to the orientation of $S^p \times S^q$. By Example 3.64, this is exactly $s_p^\Gamma \times s_q^\Gamma$, i.e. $\nu^*(s_{p+q}^\Gamma) = s_p^\Gamma \times s_q^\Gamma$. So, recalling that $\Phi_p(s_{p+q}) = s_{p+q}^\Gamma$, we have $\nu^*\Phi_{p+q}(s_{p+q}) = s_p^\Gamma \times s_q^\Gamma$. Thus, from the commutativity of the diagram and Φ_{p+q} being an isomorphism, it suffices to show that $\nu^*(s_{p+q}) = s_p \times s_q$.

For this, consider the commutative diagram

$$\begin{tikzcd}
H^{p+q}(\mathcal{K}_p \times \mathcal{K}_q) \arrow[d, "(\phi_p \times \phi_q)^*"'] & H^{p+q}(\mathcal{K}_{p+q}) \arrow[l, "\mu^*"'] \arrow[d, "\phi_{p+q}^*"] \\
H^{p+q}(S^p \times S^q) & H^{p+q}(S^{p+q}). \arrow[l, "\nu^*"']
\end{tikzcd}$$

As the $p+q+1$ skeleton of $K_p \times K_q$ can be taken to be $S^p \times S^q$, the vertical maps are isomorphisms. And we know from (21) and the definitions that

$$(\phi_p \times \phi_q)^*\mu^*(\phi_{p+q}^*)^{-1}(s_{p+q}) = (\phi_p \times \phi_q)^*\mu^*(\iota_{p+q}) = s_p \times s_q,$$

so we must have $\nu^*(s_{p+q}) = s_p \times s_q$, as needed □

7.6. **Künneth theorems.** Now that we know that the geometric cup product is naturally isomorphic to the singular chain cup product, we can use this to compare cohomology cross products and obtain the geometric cohomology Künneth Theorem. We begin with the homology Künneth Theorem, which is simpler, and then address the cohomology one.

Recall from Theorem 5.27 that we have an isomorphism $H_*(NK^{sm}_*(M)) \xrightarrow{\cong} H^\Gamma_*(M)$, where $NK^{sm}_*(M)$ is the complex of normalized smooth singular cubical chains. As elements of $NK^{sm}_*(M)$ are represented by linear combinations of smooth maps from cubes and the cross product is represented by taking geometric products, we have the following immediate compatibility of chain cross products.

Lemma 7.31. *Let M and N be manifolds without boundary. Then the following diagram commutes:*

$$\begin{tikzcd} NK^{sm}_*(M) \otimes NK^{sm}_*(N) \arrow[r, "\times"] \arrow[d] & NK^{sm}_*(M \times N) \arrow[d] \\ C^\Gamma_*(M) \otimes C^\Gamma_*(N) \arrow[r, "\times"] & C^\Gamma_*(M \times N). \end{tikzcd}$$

This induces the commutative diagram

$$\begin{tikzcd} H_*(NK^{sm}_*(M)) \otimes H_*(NK^{sm}_*(N)) \arrow[r, "\times"] \arrow[d, "\cong"'] & H_*(NK^{sm}_*(M \times N)) \arrow[d, "\cong"] \\ H^\Gamma_*(M) \otimes H^\Gamma_*(N) \arrow[r, "\times"] & H^\Gamma_*(M \times N). \end{tikzcd}$$

Theorem 7.32 (Künneth Theorem)**.** *Let M and N be manifolds without boundary. There are natural short exact sequences*

$$0 \longrightarrow \oplus_{p+q=a} H^\Gamma_p(M) \otimes H^\Gamma_q(N) \xrightarrow{\times} H^\Gamma_{p+q}(M \times N) \to \oplus_{p+q=a-1} H^\Gamma_p(M) * H^\Gamma_q(N) \longrightarrow 0$$

that split (non-naturally).

Proof. We know there exists such an exact sequence in (not necessarily smooth) singular cubical cohomology [**?**, Chapter XI], and Lemma 7.31, together with its analogue based on the maps quasi-isomorphisms $NK^{sm}_*(-) \to NK_*(-)$ of Proposition 5.28, provides an isomorphism between the left three terms of the two short exact sequences. It follows that there is an isomorphism between the quotient terms of the two sequences, and $\oplus_{p+q=a-1} H_p(NK_*(M)) * H^q(NK_*(N)) \cong \oplus_{p+q=a-1} H^\Gamma_p(M) * H^\Gamma_q(N)$, also via these isomorphisms. Since the exact sequences are isomorphic, they both split. $\square$

We now turn to cohomology. Recall that for $\underline{V} \in H^*_\Gamma(M)$ and $\underline{W} \in H^*_\Gamma(N)$ we have the relation $\underline{V} \times \underline{W} = \pi^*_M(\underline{V}) \uplus \pi^*_N(\underline{W})$, which follows from Corollary 3.76, while the same relation is well known to hold in singular cohomology [**?**, Corollary 5.6.14]. So the following is immediate from the naturality of our comparison maps Φ and Theorem 7.29

Proposition 7.33. *On the category of smooth manifold without boundary and continuous maps, the isomorphisms Φ_p from singular cohomology to geometric cohomology are compatible with cross products. In other words, for manifolds without boundary M and N there are commutative diagrams*

$$\begin{tikzcd} H^p(M) \otimes H^q(N) \arrow[r, "\times"] \arrow[d, "\Phi_p \otimes \Phi_q"', "\cong"] & H^{p+q}(M \times N) \arrow[d, "\cong"', "\Phi_{p+q}"] \\ H^p_\Gamma(M) \otimes H^q_\Gamma(N) \arrow[r, "\times"] & H^{p+q}_\Gamma(M \times N). \end{tikzcd}$$

Theorem 7.34 (Künneth Theorem)**.** *If either $H^i_\Gamma(M)$ is finitely generated for all i or $H^i_\Gamma(N)$ is finitely generated for all i, then there are natural short exact sequences*

$$0 \longrightarrow \oplus_{p+q=a} H^p_\Gamma(M) \otimes H^q_\Gamma(N) \xrightarrow{\times} H^{p+q}_\Gamma(M \times N) \to \oplus_{p+q=a+1} H^p_\Gamma(M) * H^q_\Gamma(N) \longrightarrow 0$$

that split (non-naturally).

Proof. We know there exists such an exact sequence in singular cohomology [**?**, Theorem 60.5], and Proposition 7.33 provides an isomorphism between the left three terms of the two short exact sequences. It follows that there is an isomorphism between the quotient terms of the two sequences, and $\oplus_{p+q=a+1} H^p(M) * H^q(N) \cong \oplus_{p+q=a+1} H^p_\Gamma(M) * H^q_\Gamma(N)$ via the maps $\Phi_p * \Phi_q$. Since the exact sequences are isomorphic, they both split. $\square$

We can also approach the Künneth theorem with a more explict comparison map by once again utilizing cubulations.

Proposition 7.35. *Let M and N be cubulated manifolds without boundary, and let $M \times N$ have the product cubulation. Let K^* denote the complex of cubical cochains for the appropriate cubulation. The following diagram commutes*

$$\begin{tikzcd}
C^*_\Gamma(M) \otimes C^*_\Gamma(N) \arrow[r, "\times"] & C^*_\Gamma(M \times N) \\
C^*_{\Gamma\pitchfork}(M) \otimes C^*_{\Gamma\pitchfork}(N) \arrow[r, "\times"] \arrow[u, hook] \arrow[d, "\mathcal{I} \otimes \mathcal{I}"'] & C^*_{\Gamma\pitchfork}(M \times N) \arrow[u, hook] \arrow[d, "\mathcal{I}"] \\
K^*(M) \otimes K^*(N) \arrow[r, "\times"] & K^*(M \times N).
\end{tikzcd}$$

Proof. The top square certainly commutes as the vertical maps are inclusions. Note that the product of two maps transverse to the cubulation will be transverse to the product cubulation, so the bottom map of the top square is well defined.

Let V and W represent elements of $C^*_{\Gamma\pitchfork}(M)$, and let E and F be cubes of the cubulation. We check that the two ways around the bottom square evaluate the same on $E \times F$. Applying $\mathcal{I}(V \times W)$ to $E \times F$ gives $I_{M\times N}(V \times W, E \times F) = \mathbf{a}((V \times W) \times_{M\times N} (E \times F))$ by Definitions 6.13 and 6.10, while going the other way around the diagram and applying the result to $E \times F$ yields, with our conventions, $I_M(V, E)I_N(W, F) = \mathbf{a}(V \times_M E)\mathbf{a}(W \times_N F)$.

We can now compute

$$\begin{aligned}
\mathbf{a}((V \times W) \times_{M\times N} (E \times F)) &= (-1)^{(w+f-m)(m-v)}\mathbf{a}((V \times_M E) \times (W \times_N F)) \\
&= (-1)^{(w+f-n)(m-v)}\mathbf{a}(V \times_M E)\mathbf{a}(W \times_N F).
\end{aligned}$$

The first equality is due to Proposition 3.82. Note that if either $V \times_M E$ or $W \times_N F$ is not 0-dimensional then also $(V \times W) \times_{M\times N} (E \times F)$ is not 0-dimensional, and all three expressions above are 0. Otherwise, the second equality is apparent as the product of a $(-1)^a$-oriented point with a $(-1)^b$-oriented point is a $(-1)^{a+b}$-oriented point. In this case we also have $w + f = n$ so that $(-1)^{(w+f-n)(m-v)} = 1$. $\square$

The following corollary is now immediate from Proposition 7.35 and Theorems 6.5 and 6.20.

Corollary 7.36. *If M and N are cubulated manifolds without boundary and each $H^i(M)$ and $H^j(N)$ is finitely generated, then the cubical cohomology cross product is isomorphic to the geometric cohomology cross product. In particular, we have the following diagram with all vertical maps isomorphisms, letting H^*_{cub} stand for the appropriate cubical cohomology groups:*

$$\begin{tikzcd}
H^*_\Gamma(M) \otimes H^*_\Gamma(N) \arrow[r, "\times"] & H^*_\Gamma(M \times N) \\
H^*_{\Gamma\pitchfork}(M) \otimes H^*_{\Gamma\pitchfork}(N) \arrow[r, "\times"] \arrow[u, "\cong"'] \arrow[d, "\mathcal{I}\otimes\mathcal{I}"', "\cong"] & H^*_{\Gamma\pitchfork}(M \times N) \arrow[u, "\cong"] \arrow[d, "\cong"', "\mathcal{I}"] \\
H^*_{cub}(M) \otimes H^*_{cub}(N) \arrow[r, "\times"] & H^*_{cub}(M \times N).
\end{tikzcd}$$

Utilizing these maps rather than the more abstract Φ_p provides an alternative proof of Theorem 7.34.

7.7. **The geometric cap product is the usual cap product.** Our goal in this section is to use the intersection map $\mathcal{I}$ to relate the geometric cap product with the classical singular cap product, using the cubical cap product as an intermediary. We first discuss formulas for the cubical cup and cap products, relying on known formulas for the singular cubical products. Then we apply the cubical formulas to the geometric world.

In Theorem 7.37 and Corollary 7.38 we show that the cubical cap product (and hence the singular cap product) determines the geometric cap product in general, while the geometric cap product determines the cubical cap product if all $H^i(M)$ are finitely generated. This last condition is needed because Theorem 6.20 only tells us that the intersection map induces cohomology isomorphisms $\mathcal{I} : H^i_{\Gamma\pitchfork X}(M) \to H^i(K^*_X(M))$, for some cubulation X, when $H^i(M)$ is finitely generated. Unlike the situation with Theorem 7.29, for which we used the Kreck-Singhof theorem for cup products to show that the geometric and singular cup products are always isomorphic, we do not know of an analogue of the Kreck-Singhof theorem for cap products that would allow us to provide compatibility of cap products in full generality.

7.7.1. *Cubical cup and cap products.* In this section we discuss cup and cap products for cubical and singular cubical homology and cohomology. This will be needed below for comparing the geometric cap product with the classical cap products.

We first recall from Massey [**?**, Chapter XI] some results about the normalized singular cubical chain complexes, which we have been denoting $NK_*(-)$, though we utilize some different notation from Massey. Just as for the more familiar singular simplicial chains, there is an Eilenberg-Zilber theorem that provides a chain homotopy equivalence between $NK_*(X) \otimes NK_*(Y)$ and $NK_*(X \times Y)$ for any spaces X and Y. Explicit constructions of such homotopy inverse maps are given in [**?**, Section XI.5]. The map $\zeta : NK_*(X) \otimes NK_*(Y) \to NK_*(X \times Y)$ is simply the cross product that takes $S \otimes T$ for representative singular cubes $S : \mathbb{I}^m \to X$ and $T : \mathbb{I}^n \to Y$ to the product $S \times T : \mathbb{I}^m \times \mathbb{I}^n = \mathbb{I}^{m+n} \to X \times Y$. If S or T is degenerate, so is $S \times T$, so this product is well defined for the normalized complexes. The homotopy inverse map[34] $\xi : NK_*(X \times Y) \to NK_*(X) \otimes NK_*(Y)$ takes $S : \mathbb{I}^n \to X \times Y$ to

$$\xi(S) = \sum \rho_{H,K} A_H(\pi_1 S) \otimes B_K(\pi_2 S),$$

[34] Massey sometimes writes this map as η.

where π_i is the projection to the ith factor. The precise definitions of $\rho_{H,K}$, A_H, and B_K will not need to concern us except to note that H and K are complementary subsets of $\{1,\ldots,n\}$, the sum is over all such partitions, $\rho_{H,K}$ is either 1 or -1 (in fact it is the sign of permutation HK), and A_H and B_K are cubical faces of various dimensions of the singular cubes $\pi_1 S$ and $\pi_2 S$. Again, this construction is sufficiently compatible with degeneracies to be well defined for the normalized singular cube complexes. We also observe that if X is a smooth manifold and our input singular cubes are smooth, then all other cubes appearing in the constructions are smooth.

As usual, one then defines cup and cap products (up to one's favorite sign conventions) as follows (using our current sign conventions). If $\alpha,\beta \in NK^*(X) = \mathrm{Hom}(NK_*(X),\mathbb{Z})$, then $\alpha \smile \beta \in \mathrm{Hom}(NK_*(X),\mathbb{Z})$ acts on a normalized singular cube S by

$$(\alpha \smile \beta)(S) = (\alpha \otimes \beta)(\xi(\mathbf{d}S)), \tag{23}$$

with $\mathbf{d}$ the diagonal map $X \to X \times X$, while the cap product $\alpha \frown S$ is given by

$$\alpha \frown S = (\mathrm{id} \otimes \alpha)(\xi(\mathbf{d}S)), \tag{24}$$

identifying $NK_*(X) \otimes \mathbb{Z}$ with $NK_*(X)$.

Now suppose that X is a cubical complex, and let $K_*(X)$ and $K^*(X) = \mathrm{Hom}(K_*(X),\mathbb{Z})$ be the cubical chain and cochain complexes. If $E,F \in K_*(X)$ are any cubical faces, then $E \times F$ is also a cubical face. Furthermore, abusing notation by conflating E with its embedding into the cubical complex (thought of as a space), we have $\pi_1\mathbf{d}(E) = \pi_2\mathbf{d}(E) = E$, and every face of E is also a cube in the complex. So now if M is a cubulated manifold without boundary and we abuse notation by using K_* for the cubical chain complex $K_*(M)$ corresponding to the cubulation X and $K_*(M \times M)$ for the cubical complex coming from the product cubulation $X \times X$, we have the following commutative diagrams with the upward arrows being inclusions (note that all complexes are free) and the lower horizontal maps being the restrictions of the top horizontal maps:

$$\begin{tikzcd}
NK_*(M) \otimes NK_*(M) \arrow[r, "\zeta"] & NK_*(M \times M) \\
NK_*^{sm}(M) \otimes NK_*^{sm}(M) \arrow[r, "\zeta"] \arrow[u, hook] & NK_*^{sm}(M \times M) \arrow[u, hook] \\
K_*(M) \otimes K_*(M) \arrow[r, "\zeta"] \arrow[u, hook] & K_*(M \times M). \arrow[u, hook]
\end{tikzcd}$$

and

$$\begin{tikzcd}
NK_*(M) \otimes NK_*(M) & NK_*(M \times M) \arrow[l, "\xi"'] \\
NK_*^{sm}(M) \otimes NK_*^{sm}(M) \arrow[u, hook] & NK_*^{sm}(M \times M) \arrow[l, "\xi"'] \arrow[u, hook] \\
K_*(M) \otimes K_*(M) \arrow[u, hook] & K_*(M \times M). \arrow[l, "\xi"'] \arrow[u, hook]
\end{tikzcd}$$

The top map in each diagram is a homotopy equivalences, the vertical maps are all quasi-isomorphisms by Proposition 5.28 and the proof of Theorem 6.2, and the complexes are all free, so the horizontal maps are all chain homotopy equivalences [**?**, Theorem 46.2].

Putting this all together, for both the complexes of normalized smooth singular cubical chains and the cubical complexes K_* coming from the smooth cubulations, we may define cup and cap products again by the formulas (23) and (24). Note that in the case of a geometric cube E coming from a cubulation, $\mathbf{d}E$ is not a cube in the cubical complex, but that does not matter as in the end formula for $\xi(\mathbf{d}E)$ we work with $\mathbf{d}E$ only through its projections $\pi_1(\mathbf{d}E) = \pi_2(\mathbf{d}E) = E$. These products are then compatible with the constructions for normalized singular cubical chains and cochains, i.e. the restriction of the cup product is the cup product of the restriction and the appropriate mixed functoriality version of that statement holds for cap products.

7.7.2. *Relating geometric and cubical cap products via the intersection map.* In this section we show that the geometric and cubical cap products are compatible in the sense given below in Theorem 7.37 and Corollary 7.38. Recall that in Remark 6.21 we extended the definition of the intersection map $\mathcal{I}$ to give us a map $H^*_\Gamma(M) \to H^i(K^*_X(M))$ for the manifold M cubulated by X, though we leave X tacit in the following. We again let $\mathcal{J} : K_*(X) \cong K^X_*(M) \to C^\Gamma_*(M)$ be the map that takes a cubical face of X to its embedding into M; see Theorem 6.2.

Theorem 7.37. *Let M be a smoothly cubulated manifold without boundary. Let $\underline{V} \in H^*_\Gamma(M)$ and $W \in H_*(K_*(M))$. Then*

$$\underline{V} \Cap \mathcal{J}(W) = \mathcal{J}(\mathcal{I}(\underline{V}) \frown W).$$

Here the cap product on the left is our geometric cap product and the cap on the right is the cubical cap product defined in Theorem 6.2.

Before proving the theorem, we note the following corollary.

Corollary 7.38. *The cubical cap product (and hence the singular cap product) determine the geometric cap product. If all $H^i(M)$ are finitely generated, then the geometric cap product determines the cubical cap product.*

Proof. Let $\underline{V} \in H^*_\Gamma(M)$ and $\underline{W} \in H^\Gamma_*(M)$. Choose a cubulation of M. As $\mathcal{J} : H_*(K_*(M)) \to H^\Gamma_*(M)$ is an isomorphism by Theorem 6.2, we have by Theorem 7.37

$$\underline{V} \Cap \underline{W} = \underline{V} \Cap \mathcal{J}(\mathcal{J}^{-1}(\underline{W})) = \mathcal{J}(\mathcal{I}(\underline{V}) \frown \mathcal{J}^{-1}(\underline{W})).$$

On the other hand, suppose $V \in H^*(K^*(M))$ and $W \in H_*(K_*(M))$. Then $\mathcal{I} : H^*_\Gamma(M) \to H^*(K^*(M))$ is an isomorphism by Theorem 6.20 and Remark 6.21 when all $H^i(M)$ are finitely generated. So then

$$V \frown M = \mathcal{J}^{-1}(\mathcal{J}(\mathcal{I}\mathcal{I}^{-1}(V) \frown W)) = \mathcal{J}^{-1}(\mathcal{I}^{-1}(V) \Cap \mathcal{J}(W)).$$

□

We will approach the proof of Theorem 7.37 through a series of lemmas. The first two concern transversality. Then we have a series of lemmas that essentially consist of various reformulations of the cap products, eventually linking together the two terms of Theorem 7.37. Once we have all the lemma established, we explain how to tie them all together to prove the theorem.

Lemma 7.39. *Let M be a cubulated manifold without boundary. Let $V \in PC^*_\Gamma(M)$, and suppose $W \in PC_*(M \times M)$ is represented by a collection of embeddings. Then there is a proper universal homotopy $h : V \times I \to M$ such that $h(-,1) : V \to M$ is transverse to the product cubulation and $M \times V \xrightarrow{\mathrm{id}_M \times h(-,1)} M \times M$ is transverse W in $M \times M$.*

Proof. As in the proof of Proposition 6.6, we use the transversality techniques of [**?**, Section 2.3]. Consider M as embedded in some $\mathbb{R}^N$ with an ϵ-neighborhood M_ϵ and proper submersion $\pi : M_\epsilon \to M$. We define H as in proof of Proposition 6.6 so that $H : V \times D^N \to M$ is the proper universal homotopy given by $H(x, s) = \pi(r_V(x) + \epsilon(x)s)$. Then H is a submersion and so transverse to each face of the cubulation, and also $\mathrm{id}_M \times H : M \times V \times D^N \to M \times M$ is a submersion and hence transverse to W. So now by the Transversality Theorem of [**?**, Section 2.3], for any face E of the cubulation $H(-, s)$ is transverse to E for almost all $s \in D^N$ and, similarly, $\mathrm{id}_M \times H(-, s)$ is transverse W for almost all $s \in D^N$ (and similarly for each stratum of W). As the cubulation must have a countable number of faces and W has a finite number of strata, there is an $s_0 \in D^N$ such that $H(-, s_0)$ is transverse to the cubulation and $\mathrm{id}_M \times H(-, s_0)$ is transverse W. Now let $h(x, t) = H(x, ts_0)$. This is a proper universal homotopy, and $h(-, 1)$ has the required transversality properties. □

Lemma 7.40. *Suppose M is a manifold without boundary, $V \in PC^*_\Gamma(M)$, and V is transverse to $W \in PC^\Gamma_*(M)$. Then $\mathrm{id}_M \times r_V : M \times V \to M \times M$ is transverse to $\mathbf{d}r_W : W \to M \times M$ in $M \times M$, where $\mathbf{d} : M \to M \times M$ is the diagonal map. In particular, $\mathrm{id}_M \times r_V : M \times V \to M \times M$ is transverse to $\mathbf{d} : M \to M \times M$.*

Proof. By assumption, if $r_V(x) = r_W(y)$, then $Dr_V(T_xV) + Dr_W(T_yW) = T_{r_V(x)}M$. Now suppose that $(z, x) \in M \times V$ maps to $\mathbf{d}r_W(y)$ with $y \in W$. This is equivalent to $r_V(x) = r_W(y) = z$. At any $(z, x) \in M \times V$, the image of $D(\mathrm{id}_M \times r_V)$ acting on $T_{(z,x)}(M \times V) = T_zM \oplus T_xV$ is $T_z(M) \oplus Dr_V(T_xV)$. While the image of $D(\mathbf{d}r_W)$ acting on T_yW is $Dr_W(T_yW) \oplus Dr_W(T_yW)$. As we know $Dr_V(T_xV) + Dr_W(T_yW) = T_{r_V(x)}M$, when $r_V(x) = r_W(y) = z$ these images together span $T_{(r_W(y), r_W(y))}M \times M$.

The last statement follows by taking $r_W : W \to M$ to be $\mathrm{id}_M : M \to M$, which is certainly transverse to any V. □

For the next lemmas, we make the following definitions.

Definition 7.41. For a cubulation of M, let $\Delta : K_*(M) \to K_*(M \times M)$ be the chain map given by $\Delta(E) = \zeta\xi(\mathbf{d}E)$ (see Section 7.7.1). This is a chain map because it is the restriction of a chain map of the singular cubical complexes to subcomplexes.

Definition 7.42. Let $V \in PC^*_\Gamma(M)$. Below we write $M \times V$ for the element of $PC^*_\Gamma(M \times M)$ given by the co-oriented exterior product of V with the identity $\mathrm{id}_M : M \times M$ given its canonical co-orientation.

Lemma 7.43. *Suppose M is a cubulated manifold without boundary and that $K_*(M)$ is the cubical chain complex with respect to some fixed cubulation. Let $V \in PC^*_{\Gamma\pitchfork}(M)$ represent a cocycle, and let $W \in K_*(M)$. Suppose $\mathrm{id}_M \times r_V : M \times V \to M \times M$ is transverse to $\mathcal{J}(\Delta(W))$. Then*

$$\mathcal{J}(\mathcal{I}(V) \frown W) = \pi_1((M \times V) \times_{M \times M} \mathcal{J}(\Delta(W))) \in C^\Gamma_*(M),$$

where $\pi_1 : M \times M \to M$ is the projection to the first factor.

Proof. For a cubical face E representing an element of $K_*(M)$, let us write$\xi(\mathbf{d}(E)) = \sum_i E_{1i} \otimes E_{2i}$, analogously to Sweedler notation. By definition, at the chain/cochain level $\mathcal{I}(V) \frown E$ is given by

$$(1 \otimes \mathcal{I}(V))(\xi(\mathbf{d}(E))) = (1 \otimes \mathcal{I}(V)) \left(\sum_i E_{1i} \otimes E_{2i} \right) = \sum_i E_{1i} \otimes \mathcal{I}(V)(E_{2i}) = \sum_i \mathcal{I}(V)(E_{2i}) \cdot E_{1i},$$

where $\mathcal{I}(V)(E_{2i})$ is the intersection number of V with E_{2i} by Definition 6.13. So $\mathcal{J}(\mathcal{I}(V) \frown W)$ is just the geometric cochain represented by $\sum_i I(V, E_{2i})E_{1i} = \sum_i \mathbf{a}(V \times_M E_{2i})E_{1i}$, identifying the cubical face E_{1i} with its embedding into M. Note that we have $I(V, E_{2i}) = 0$ if V and E_{2i} do not have complementary dimension in M, so we can take the sum $\sum_i I(V, E_{2i})E_{1i}$ to be over those i such that E_{2i} has complementary dimension to V.

On the other hand, $\mathcal{J}(\Delta(E))$ is the geometric chain corresponding to $\sum_i E_{1i} \times E_{2i}$, and, applying our transversality assumption, we have

$$\begin{aligned}
&(M \times V) \times_{M \times M} \mathcal{J}(\zeta\Delta(E)) \\
&\qquad = (M \times V) \times_{M \times M} \left(\sum_i E_{1i} \times E_{2i} \right) \\
&\qquad = \sum_i (-1)^{(v+e_{2i}-m)(m-v)} (M \times_M E_{1i}) \times (V \times_M E_{2i}) && \text{by Proposition 3.82} \\
&\qquad = \sum_i (-1)^{(v+e_{2i}-m)(m-v)} E_{1i} \times (V \times_M E_{2i}) && \text{by Proposition 3.80.}
\end{aligned}$$

We now consider cases depending on the dimension of $V \times_M E_{2i}$. If $\dim(V) + \dim(E_{2i}) < \dim(M)$, then $V \times_M E_{2i} = \emptyset$, and the corresponding terms in the above formula vanish. Similarly if $\dim(V)+\dim(E_{2i}) \geq \dim(M)$ but V and E_{2i} do not intersect. For the remaining cases, suppose $V \times_M E_{2i} \neq \emptyset$.

If V and E_{2i} have complementary dimension, then $V \times_M E_{2i}$ is 0 dimensional, and $\pi_1(E_{1i} \times (V \times_M E_{2i}))$ is simply $\mathbf{a}(V \times_M E_{2i})E_{1i}$. Furthermore, $(-1)^{(v+e_{2i}-m)(m-v)} = 1$.

If $\dim(V \times_M E_{2i}) \geq 2$, then when we take the projection, $\pi_1(E_{1i} \times (V \times_M E_{2i}))$ has small rank. In this case, $\dim(\partial(V \times_M E_{2i})) \geq 1$ (or is empty) and so

$$\partial(\pi_1(E_{1i} \times (V \times_M E_{2i}))) = \pi_1(\partial E_{1i} \times (V \times_M E_{2i})) \pm \pi_1(E_{1i} \times \partial(V \times_M E_{2i}))$$

also has small rank, and so these terms are degenerate and vanish in $C_*^\Gamma(M)$.

Finally, suppose $\dim(V \times_M E_{2i}) = 1$. Then again $\pi_1(E_{1i} \times (V \times_M E_{2i}))$ has small rank, as does the boundary term $\pi_1(\partial E_{1i} \times (V \times_M E_{2i}))$. The second boundary summand $\pm\pi_1(E_{1i} \times \partial(V \times_M E_{2i}))$ may not have small rank. However, since $\dim(V \times_M E_{2i}) = 1$, it must consist of mappings of circles and compact intervals, and, therefore, its boundary consists of (maps to M of) pairs of oppositely oriented points. So $E_{1i} \times \partial(V \times_M E_{2i})$ consists of pairs of oppositely oriented copies of E_{1i} mapping to $M \times M$, and once we project via π_1, these pairs become trivial elements of $C_*^\Gamma(M)$. So $\pi_1(E_{1i} \times (V \times_M E_{2i}))$ is also degenerate in this case, and these terms are also 0 in $C_*^\Gamma(M)$.

We conclude that $\pi_1((M \times V) \times_{M \times M} \mathcal{J}(\Delta(E)))$ as an element of $C_*^\Gamma(M)$ can be represented as the sum $\sum_i \mathbf{a}(V \times_M E_{2i})E_{1i}$ over only those i with E_{2i} of complementary dimension to V. But this is the same formula we derived for $\mathcal{J}(\mathcal{I}(V) \frown E)$. ☐

Lemma 7.44. *Let M be a manifold without boundary. Let $V \in PC_\Gamma^*(M)$ and $W \in PC_*^\Gamma(M)$ be transverse. Let $\pi_1 : M \times M \to M$ be the projection on the first factor. Then*

$$V \times_M W = \pi_1((M \times V) \times_{M \times M} \mathbf{d}(W)).$$

Proof. By Lemma 7.40, $\mathrm{id}_M \times r_V : M \times V \to M \times M$ is transverse to $\mathbf{d}r_W : W \to M \times M$ in $M \times M$, so both expressions are defined. We also have $V = M \times_M V = \mathbf{d}^*(M \times V)$ by Corollary 3.53 and Proposition 3.72.

So we can compute

$$\begin{aligned} V \times_M W &= \pi_1 \mathbf{d}(V \times_M W) && \text{since } \pi_1 \mathbf{d} = \mathrm{id}_M \\ &= \pi_1 \mathbf{d}(\mathbf{d}^*(M \times V) \times_M W) && \text{by the above} \\ &= \pi_1((V \times M) \times_{M \times M} \mathbf{d}(W)) && \text{by naturality of cap products.} \end{aligned}$$

For the last equality, see Proposition 3.85 and its interpretation in terms of naturality of the cap product in Section 7.2. Proposition 3.85 requires $\mathrm{id}_M \times r_V : M \times V \to M \times M$ be transverse to $\mathbf{d} : M \to M \times M$ and $W \to M$ be transverse to the pullback of $M \times V$ by $\mathbf{d} : M \to M \times M$ to $(M \times V) \times_{M \times M} M \to M$. The first requirement holds by Lemma 7.40. For the second transversality requirement, Lemma 2.22 says that in the presence of the first transversality condition, this is equivalent to requiring $\mathrm{id}_M \times r_V : M \times V \to M \times M$ to be transverse to $\mathbf{d} r_W : W \to M \times M$. But this also holds by Lemma 7.40 as V and W are transverse. □

Lemma 7.45. *Let M be a cubulated manifold without boundary. Let W be a cycle in $K_*(M)$, and let $V \in PC^*_\Gamma(M)$ be a cocycle such that $\mathrm{id}_M \times r_V : M \times V \to M \times M$ is transverse to $\mathbf{d}(\mathcal{J}(W))$ and $\mathcal{J}(\Delta(W))$. Then*

$$\underline{\pi_1((M \times V) \times_{M \times M} \mathbf{d}(\mathcal{J}(W)))} = \underline{\pi_1((M \times V) \times_{M \times M} \mathcal{J}(\Delta(W)))} \in H^\Gamma_*(M).$$

Proof. If we consider W as an element of $NK^{sm}_*(M)$, then the geometric chain $\mathbf{d}(\mathcal{J}(W))$ is represented by the singular cubical chain $\mathbf{d}(W)$ and $\mathcal{J}(\Delta(M))$ is represented by the singular cubical chain $\zeta\xi\mathbf{d}(W)$.

As $\zeta\xi : NK^{sm}_*(M) \to NK^{sm}_*(M)$ is chain homotopic to the identity, $\mathbf{d}(W)$ and $\zeta\xi\mathbf{d}(W)$ must be homologous in $NK^{sm}_*(M)$, and so they are also homologous as geometric chains. In particular, $\mathbf{d}(\mathcal{J}(W))$ and $\mathcal{J}(\Delta(M))$ represent the same element of $H^\Gamma_*(M \times M)$.

As V is a cocycle, so is $M \times V$. It now follows from Theorem 7.22 that $(M \times V) \times_{M \times M} \mathbf{d}(\mathcal{J}(W))$ and $(M \times V) \times_{M \times M} \mathcal{J}(\Delta(W))$ represent the same geometric homology class, and so their images under π_1 represent the same geometric homology class. □

Proof of Theorem 7.37. Let us first choose a cubical cycle W representing our given cubical homology class. By Lemma 7.39, we can choose a representative V of our geometric cohomology class such that V is transverse to the cubulation (and hence to W) and $M \times V$ is transverse to $\mathbf{d}(\mathcal{J}(W)) \sqcup \mathcal{J}(\Delta(W))$, which is also represented by a union of embeddings. Then by Lemma 7.43, we have

$$\underline{\mathcal{J}(\mathcal{I}(V) \frown W)} = \underline{\pi_1((M \times V) \times_{M \times M} \mathcal{J}(\Delta(W)))} \in H^\Gamma_*(M),$$

and by Lemma 7.45 this equals $\underline{\pi_1((M \times V) \times_{M \times M} \mathbf{d}(\mathcal{J}(W)))}$. Then by Lemma 7.44, $\pi_1((M \times V) \times_{M \times M} \mathbf{d}(\mathcal{J}(W))) = V \times_M \mathcal{J}(W) \in PC^\Gamma_*(M)$. Finally, $V \times_M \mathcal{J}(W)$ represents $\underline{V} \Cap \mathcal{J}(W)$ by definition. □

7.7.3. *Poincaré duality.* In Example 4.17, we noticed that geometric homology and cohomology satisfy a very strong form of Poincaré duality, as for a closed oriented manifold M we in fact have chain-level identities $C^{m-i}_\Gamma(M) = C^\Gamma_i(M)$ obtained by identifying co-oriented cochains with their corresponding orientated chains, using the orientations induced by the orientation of M. Theorem 7.37 allows us to observe that this strong version of geometric Poincaré is compatible with the classical Poincaré duality:

Corollary 7.46 (Poincaré duality)**.** *Let M be a closed oriented cubulated manifold. Let $\underline{M} \in C^\Gamma_m(M)$ be represented by the orientation-preserving identity map $\mathrm{id}_M : M \to M$, and let $[M] \in K_*(M)$ represent the cubical fundamental class. Then there is a commutative diagram of isomorphisms*

$$\begin{tikzcd}
H_\Gamma^{n-i}(M) \arrow[r, "\pitchfork \underline{M}"] \arrow[d, "\mathcal{I}"'] & H_i^\Gamma(M) \\
H_{cub}^{n-i}(M) \arrow[r, "\frown {[M]}"] & H_i^{cub}(M). \arrow[u, "\mathcal{J}"']
\end{tikzcd}$$

The proof follows immediately from Theorem 7.37 and the following lemma.

Lemma 7.47. *Let M be closed, oriented, cubulated, and connected. Then $\mathcal{J}([M]) = \underline{M} \in H_m^\Gamma(M)$.*

Proof. Let $\underline{V} \in H_\Gamma^m(M)$ be represented by a map $V = pt \hookrightarrow M$ taking the point to the center of an m-cube of the cubulation, co-oriented so that its normal co-orientation agrees with the orientation of M. By Proposition 3.81, as $\underline{M}$ and $\mathcal{J}([M])$ are both represented by embeddings with the same orientation in a neighborhood of the embedded point V, the cap products $\underline{V} \pitchfork \underline{M}$ and $\underline{V} \pitchfork \mathcal{J}([M])$ in $H_0^\Gamma(M)$ are each represented by the same point with its induced orientation (which by Proposition 3.78 will be the positive orientation). This is a generator of $H_0^\Gamma(M) \cong \mathbb{Z}$, as we can see, for example, via our homology isomorphism $H_*(NK_*(M)) \to H_*^\Gamma(M)$. As $H_m^\Gamma(M) \cong H_0^\Gamma(M) \cong \mathbb{Z}$ by the isomorphisms between geometric and singular homology and cohomology, $\underline{V} \pitchfork : H_m^\Gamma(M) \to H_0^\Gamma(M)$ must be injective as we have shown it is not the 0 map. Since we have shown $\underline{V} \pitchfork \underline{M} = \underline{V} \pitchfork \mathcal{J}([M])$, we have $\underline{M} = \mathcal{J}([M])$. □

With $\pitchfork \underline{M}$ as our Poincaré duality map, the relation of Proposition 3.86, which in Section 7.2.4 became the chain/cochain formula

$$(\underline{V} \uplus \underline{W}) \pitchfork \underline{M} = (-1)^{(m-v)(m-w)} (\underline{V} \pitchfork \underline{M}) \bullet (\underline{W} \pitchfork \underline{M}) = (\underline{W} \pitchfork \underline{M}) \bullet (\underline{V} \pitchfork \underline{M}),$$

demonstrates the usual relationship between intersection products and cup products that is well known for homology classes represented by embedded manifolds, cf. [**?**, Section VI.11]. Here we see that this relationship extends not just for intersections of embedded manifolds but to all homology classes. Of course this is always possible if one takes the above formula as a defining formula for the intersection product, but here we see that the intersection product can always be defined geometrically in terms of fiber products.

7.7.4. *Umkehr maps.* Corollary 7.46 allows us to make some remarks about umkehr maps, also known as wrong-way or transfer maps, associated to maps of closed oriented manifolds $f : N \to M$. These are maps

$$f^! : H_\Gamma^{n-i}(N) \to H_\Gamma^{m-i}(M)$$
$$f_! : H^\Gamma_{m-i}(M) \to H^\Gamma_{n-i}(N),$$

typically defined by taking a homology or cohomology class, dualizing using Poincaré duality, applying f or f^*, and then dualizing again; see [**?**, Definition VI.11.2]. We will show that when M and N are closed and oriented, these transfer maps correspond to the pullbacks and pushforwards already encountered in Section 5.1, where we only required for homology pushforwards that f be proper and co-oriented and for homology pullbacks that f be proper and that M and N be oriented.

Proposition 7.48. *Let $f : N \to M$ be a map of closed oriented manifolds. We may consider f co-oriented via the orientations of M and N. Then the following diagrams commute:*

$$\begin{tikzcd}
H_\Gamma^{n-i}(N) \arrow[r, "f"] \arrow[d, "\pitchfork \underline{N}"'] & H_\Gamma^{m-i}(M) \arrow[d, "\pitchfork \underline{M}"] & H_\Gamma^{i}(M) \arrow[r, "f^*"] \arrow[d, "\pitchfork \underline{M}"'] & H_\Gamma^{i}(N) \arrow[d, "\pitchfork \underline{N}"] \\
H_i^\Gamma(N) \arrow[r, "f"] & H_i^\Gamma(M) & H_{m-i}^\Gamma(M) \arrow[r, "f^*"] & H_{n-i}^\Gamma(N).
\end{tikzcd}$$

Proof. We start with the diagram on the left. Let $\underline{V} \in H_\Gamma^{n-i}(N)$ represented by a co-oriented map $r_v : V \to N$. Then $\underline{V} \pitchfork \underline{N}$ is represented by the same map to N with its induced orientation; see Section 3.2. In particular, if $x \in V$ then V is oriented at x by the local orientation β_V such that (β_V, β_N) gives the co-orientation of r_V. The path down then right is then the composition fr_V, considering V with its orientation induced by r_V and the orientation of N. On the other hand, by the definition in Section 5.1.1, the element $f(\underline{V})$ in $H_\Gamma^{m-i}(M)$ is represented by fr_V co-oriented by composing the co-orientations of r_V and f. So if the co-orientation of r_V is again (β_V, β_N), the co-orientation of fr_V representing $f(\underline{V})$ is $(\beta_V, \beta_N) * (\beta_N, \beta_M) = (\beta_V, \beta_M)$. So $f(\underline{V}) \pitchfork \underline{M}$ is represented by fr_V with V oriented again by β_V. Thus the diagram commutes.

For the second diagram, let $r_V : V \to M$ represent $\underline{V} \in H_\Gamma^i(M)$. We can assume up to a homotopy that f is smooth and transverse to r_V. Then $f^*(\underline{V})$ is represented by the co-oriented pullback $V \times_M N \to N$, and $f^*(\underline{V}) \pitchfork \underline{N}$ is represented by this map with the orientation on $V \times_M N$ induced by the pullback co-orientation and the orientation of N. Meanwhile, $\underline{V} \pitchfork \underline{M}$ is represented by r_V with the orientation consistent with the given co-orientation and the orientation of M. Applying f^* then gives the pullback the same orientation just described for $f^*(\underline{V}) \pitchfork \underline{N}$ by the definition in Remark 5.12. □

Corollary 7.49. *If $f : N \to M$ is a map of closed oriented manifolds then $f^! = f : H^{n-i}(N) \to H^{m-i}(M)$ and*

8. Questions

(1) Is $H_\Gamma^*(X) \cong H^*(X)$ when not finitely generated?
(2) Is the cap product $H_\Gamma^i(M) \otimes H_{i+j}^\Gamma(M) \to H_j^\Gamma(M)$ isomorphic to the singular co(homology) cap product when $H^*(X)$ is not finitely generated?
(3) What can we say about geometric homology/cohomology with coefficients?
(4) Can we develop theories of relative geometric homology/cohomology?
(5) If we consider only compact objects in $C_\Gamma^*(M)$ is the resulting cohomology theory isomorphic to $H_c^*(M)$?
(6) If we allow objects in $C_\Gamma^*(M)$ to be noncompact but with proper maps to M, is the resulting homology theory isomorphic to the standard $H_*^\infty(M)$?
(7) Are the cup and cap products we obtain working with $H_c^*(M)$ and $H_*^\infty(M)$ isomorphic to the standard ones?
(8) Can we show that our convention for co-orientating fiber products is unique for some nice set of properties?
(9) Can we express the map $\mathrm{Ext}(H_{i-1}^\Gamma(M), \mathbb{Z}) \to H_\Gamma^i(M)$ of Theorem 7.26 in terms of linking numbers?

Appendix A. Lipyanskiy's co-orientations

In [**?**], Lipyanskiy uses a different notion of co-orientation from the one we have used to define geometric cochains. We here discuss Lipyanskiy's co-orientation, which he initially refers to as *orientations of maps*, and show that when we have smooth maps $f : M \to N$, it is equivalent to our definition, up to possible sign conventions. In other words, we show that a smooth map is co-orientable in our sense if and only if it is co-orientable in Lipyanskiy's sense. We will not explore the precise differences between the specific co-orientation conventions.

To define, co-orientations, Lipyanskiy utilizes the determinant line bundles of Donaldson and Kronheimer in [**?**, Section 5.2.1]. A key point throughout our discussion will be the following lemma, which is presented without proof in [**?**]. For the statement, recall our definition of $\mathrm{Det}(V)$ in Definition 3.2.

Lemma A.1. *Given an exact sequence of vector bundles*

$$0 \to V_1 \to \cdots \to V_m \to 0,$$

there is a canonical isomorphism

$$\underset{i\ odd}{\otimes} \mathrm{Det}(V_i) \cong \underset{i\ even}{\otimes} \mathrm{Det}(V_i).$$

Proof. UGH - TRY TO FIND THIS SOMEWHERE: STILL NEED TO FIX/ADD THIS We first consider the case of a short exact sequence $0 \to V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} V_3 \to 0$. As short exact sequences of vector bundles always split (REF!), we have a map $e : V_3 \to V_2$ such that $d_1 \oplus e : V_1 \oplus V_3 \to V_2$ is an isomorphism. So $\mathrm{Det}(V_1 \oplus V_3) \xrightarrow{\mathrm{Det}(d_1 \oplus e)} \mathrm{Det}(V_2)$ is an isomorphism, as is the canonical map (REF OR PROVE) $\mathrm{Det}(V_1) \otimes \mathrm{Det}(V_3) \to \mathrm{Det}(V_1 \oplus V_3 \to V_2)$. The composite isomorphism appears a priori to depend on the splitting e, but if e' is another such splitting then the image of $e - e'$ lies in $\ker(d_2) = \mathrm{im}(d_1)$, so $\mathrm{Det}(d_1 \otimes e) - \mathrm{Det}(d_1 \otimes e') = \mathrm{Det}(d_1 \otimes (e - e')) = 0$ as the exterior product of a top form of $\mathrm{im}(d_1)$ with anything else in $\mathrm{im}(d_1)$ must be 0. REF TO THESE MECHANICS So the isomorphism is independent of the choice of splitting. THIS PROOF NEEDS A LOT OF FIXING. ☐

We can now define the Donaldson-Kronheimer determinant line bundles as in [**?**, Section 5.2.1]. Donaldson and Kronheimer work in a more general setting, but we will confine ourselves to considering a map of vector bundles $F : E \to E'$ over M. At first, we also assume that $\ker(F)$ and cok(F) are well-defined vector bundles. Then the Donaldson-Kronheimer determinant line bundle is defined to be

$$\mathrm{Det}(\ker(F)) \otimes \mathrm{Det}(\mathrm{cok(F)})^*,$$

where the $*$ over Det(cok(F)) denotes the dual line bundle. Below we will consider that $\ker(F)$ and cok(F) are not always vector bundles, but for now we see that the determinant bundle is morally related to the index of an operator. We refer to [**?**, Section 5.2.1] for a more precise statement of the relationship.

To relate the Donaldson-Kronheimer determinant line bundle to our notion of co-orientation, consider the exact sequence of vector bundles

$$0 \to \ker(F) \to E \to E' \to \mathrm{cok(F)} \to 0.$$

Applying Lemma A.1, we have $\mathrm{Det}(\ker(F)) \otimes \mathrm{Det}(E') \cong \mathrm{Det}(E) \otimes \mathrm{Det}(\mathrm{cok(F)})$. Next we use that for a line bundle L we have $L \otimes L^* \cong \underline{\mathbb{R}}$, the trivial line bundle. So multiplying both sides by $\mathrm{Det}(\mathrm{cok(F)})^*$ and $\mathrm{Det}(E')^*$, we get

$$\mathrm{Det}(\ker(F)) \otimes \mathrm{Det}(\mathrm{cok(F)})^* \cong \mathrm{Det(E)} \otimes \mathrm{Det(E')}^*.$$

The latter is isomorphic to $\mathrm{Hom}(\mathrm{Det}(E'), \mathrm{Det}(E))$, which is dual to $\mathrm{Hom}(\mathrm{Det}(E), \mathrm{Det}(E'))$. In particular, $\mathrm{Hom}(\mathrm{Det}(E), \mathrm{Det}(E'))$ is trivial, and so admits a non-zero section, if and only if the Donaldson-Kronheimer determinant bundle $\mathrm{Det}(\ker(F)) \otimes \mathrm{Det}(\mathrm{cok(F)})^*$ is trivial.

In the setting of a smooth map $f : M \to N$, we can think of the derivative Df as a map $Df : TM \to f^*(TN)$, and then the above demonstrates that $\mathrm{Hom}(\mathrm{Det}(TM), \mathrm{Det}(f^*(TN)))$ is trivial if and only if the determinant bundle $\mathrm{Det}(\ker(Df)) \otimes \mathrm{Det}(\mathrm{cok(Df)})^*$ is trivial. We recall that the triviality of $\mathrm{Hom}(\mathrm{Det}(TM), \mathrm{Det}(f^*(TN)))$ is the condition for co-orientability of f in the sense of Definition 3.11. Our co-orientations in this setting are equivalence classes of non-zero sections of $\mathrm{Hom}(\mathrm{Det}(TM), \mathrm{Det}(f^*(TN)))$ up to positive scalars or, equivalently, orientations of this line bundle. Lipyanskiy's co-orientations are orientations of $\mathrm{Det}(\ker(Df)) \otimes \mathrm{Det}(\mathrm{cok(Df)})^*$. As orientations of line bundles exist if and only if the line bundle is trivial, the two notions of co-orientability coincide. We leave it to the reader to define the isomorphisms in sufficient detail to carry a particular co-orientation as defined in Section 3.2 to one of Lipyanskiy's co-orientations.

The problem with the preceding analysis is that in general $\ker(Df)$ and $\mathrm{cok(Df)}$ do not necessarily have the same dimensions from fiber to fiber, and so $\ker(Df)$ and $\mathrm{cok(Df)}$ are not necessarily well defined as vector bundles. The solution is to reframe the definition of the determinant line bundle as in [?] so that it is always well defined and such that it is isomorphic to $\mathrm{Det}(\ker(F)) \otimes \mathrm{Det}(\mathrm{cok(F)})^*$ when it is also well defined.

For this, let $\underline{\mathbb{R}}^n$ be the trivial $\mathbb{R}^n$ bundle over M, and suppose we have a map $\psi : \underline{\mathbb{R}}^n \to E'$ such that $F \oplus \psi : E \oplus \underline{\mathbb{R}}^n \to E'$ is surjective[35]. This will always be true in our setting, as we observed in Section 2 that work of Joyce and Melrose implies that smooth manifolds with corners can always be embedded in finite dimensional Euclidean space. Hence tangent bundles are subbundles of trivial bundles and so the images of projections of trivial bundles (or, up isomorphism, quotients of the trivial bundle by their orthogonal complements after endowing the trivial bundle with a Riemannian structure). The gain is that $F \oplus \psi$ now has trivial cokernel and a kernel that is a vector bundle, as now the fibers of the kernel have a fixed dimension. We then define the determinant line bundle to be

$$L = \mathrm{Det}(\ker(F \oplus \psi)) \otimes \mathrm{Det}(\underline{\mathbb{R}}^n)^* \cong \mathrm{Det}(\ker(F \oplus \psi)).$$

In the case where $\ker(F)$ and $\mathrm{cok(F)}$ were already vector bundles, L is isomorphic to the earlier Donaldson-Kronheimer determinant line bundle using Lemma A.1 and the following lemma:

Lemma A.2. *If $F : E \to E'$ and $\psi : \underline{\mathbb{R}}^n \to E'$ are bundle maps with $F \oplus \psi : E \oplus \underline{\mathbb{R}}^n \to E'$ surjective and $\ker(F)$ and $\mathrm{cok(F)}$ well-defined vector bundles, then the following sequence is exact*[36]*:*

$$0 \longrightarrow \ker(F) \longrightarrow \ker(F \oplus \psi) \longrightarrow \underline{\mathbb{R}}^n \longrightarrow \mathrm{cok(F)} \longrightarrow 0.$$

Proof. This exact sequence is simply the snake lemma exact sequence obtained from the commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, "F"'] & E \oplus \underline{\mathbb{R}}^n \arrow[r] \arrow[d, "F \oplus \psi"'] & \underline{\mathbb{R}}^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E' \arrow[r, "="] & E' \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

The category of vector bundles over a space is not technically an abelian category, but one can check by hand for this diagram that, with our assumptions, all the maps of the exact sequence are well defined and the exactness then holds fiberwise by the classical snake lemma. In particular, the map $\underline{\mathbb{R}}^n \to \mathrm{cok(F)}$ is the composition of the splitting map $\underline{\mathbb{R}}^n \to E \oplus \underline{\mathbb{R}}^n$, the map $F \oplus \psi$, and the projection E' to $\mathrm{cok(F)}$. $\square$

Combining this lemma with Lemma A.1 gives us an isomorphism

$$\mathrm{Det}(\ker(F)) \otimes \mathrm{Det}(\underline{\mathbb{R}}^n) \cong \mathrm{Det}(\ker(F \oplus \psi)) \otimes \mathrm{Det}(\mathrm{cok(F)}).$$

Multiplying both sides by $\mathrm{Det}(\underline{\mathbb{R}}^n)^* \otimes \mathrm{Det}(\mathrm{cok(F)})^*$ and using again that for a line bundle L we have $L \otimes L^* \cong \underline{\mathbb{R}}$, we obtain

$$\mathrm{Det}(\ker(F)) \otimes \mathrm{Det}(\mathrm{cok(F)})^* \cong \mathrm{Det}(\ker(\mathrm{F} \oplus \psi)) \otimes \mathrm{Det}(\underline{\mathbb{R}}^{\mathrm{n}})^*.$$

So, as promised, the two definitions agree (up to canonical isomorphisms) when $\ker(F)$ and $\mathrm{cok(F)}$ are defined.

[35] Donaldson and Kronheimer work with complex vector bundles, so [?] features $\underline{\mathbb{C}}^n$ rather than $\underline{\mathbb{R}}^n$.

[36] This exact sequence appears incorrectly in [?] with the ψ in place of F in the first and last terms.

Finally, we should observe that the construction of L is independent, at least up to isomorphism, of the choice of ψ and n. Clearly $\mathrm{Det}(\underline{\mathbb{R}}^n) \cong \mathrm{Det}(\underline{\mathbb{R}}^n)^* \cong \underline{\mathbb{R}}$ for all n, so we must only show that if $\psi_1 : \underline{\mathbb{R}}^n \to E'$ and $\psi_2 : \underline{\mathbb{R}}^m \to E'$ are two maps satisfying the requirement of the definition then $\mathrm{Det}(\ker(F \oplus \psi_1)) \cong \mathrm{Det}(\ker(F \oplus \psi_2))$. Adapting an argument in [**?**, Section 5.1.3], we note that the bundles maps

$$\begin{aligned} F \oplus \psi_1 \oplus 0 :& E \oplus \underline{\mathbb{R}}^{n+m} \to E' \\ F \oplus 0 \oplus \psi_2 :& E \oplus \underline{\mathbb{R}}^{n+m} \to E' \end{aligned}$$

are each homotopic through fiberwise linear homotopies to $F \oplus \psi_1 \oplus \psi_2$, and so they are homotopic to each other. Furthermore, these are homotopies through surjective bundle maps, so we can write the homotopy between the two maps as a surjective bundle map $(E \oplus \underline{\mathbb{R}}^{n+m}) \times I \to E' \times I$ over $M \times I$. Thus we have a well-defined kernel bundle over $M \times I$, which implies that the kernel bundles over $M \times \{0\}$ and $M \times \{1\}$ are isomorphic. But, up to reordering the summands, these are, respectively, $\ker(F \oplus \psi_1) \oplus \underline{\mathbb{R}}^m$ and $\ker(F \oplus \psi_2) \oplus \underline{\mathbb{R}}^n$. Therefore,

$$\mathrm{Det}(\ker(F \oplus \psi_1) \oplus \underline{\mathbb{R}}^m) \cong \mathrm{Det}(\ker(F \oplus \psi_2) \oplus \underline{\mathbb{R}}^n).$$

But

$$\mathrm{Det}(\ker(F \oplus \psi_1) \oplus \underline{\mathbb{R}}^m) \cong \mathrm{Det}(\ker(F \oplus \psi_1)) \otimes \mathrm{Det}(\underline{\mathbb{R}}^m) \cong \mathrm{Det}(\ker(F \oplus \psi_1)),$$

and similarly for the other bundle. So $\mathrm{Det}(\ker(F \oplus \psi_1)) \cong \mathrm{Det}(\ker(F \oplus \psi_2))$.

To-Do

(1) Is there a place where we don't use integer coefficients? Should Or(E) just be $\wedge^n(E)$?
(2) Add an argument that our definition of co-orientation agrees with Lipyanskiy's?
(3) Thom isomorphism (what does this mean in this setting - we can't have a singular space, and we don't have relative cohomology)
(4) anibal: diagrams in diagrams.sty are sensitive to having a blank space before and after. Make this choice homogeneous all around.
(5) anibal: consider a command `\ie` to make homogeneous the use of i.e. since some use a `\ ` after.

To-Do elsewhere

Department of Mathematics, Texas Christian University
Email address: g.friedman@tcu.edu

LAGA, Université Sorbonne Paris Nord
Email address: medina-mardones@math.univ-paris13.fr

Mathematics Department, University of Oregon
Email address: dps@uoregon.edu