Proposition 0.1. *Let $f : V \to M$, $g : W \to M$, and $h : Z \to M$ be maps from manifolds with corners to a manifold without boundary. Suppose that W is transverse to Z and that V is transverse to W and to $W \times_M Z$. Then $V \times_M W$ is transverse to Z. In particular, if $V \times_M (W \times_M Z)$ and $V \times_M W$ are well defined, then so is $(V \times_M W) \times_M Z$.*

Proof. We must show that $V \times_M W$ is transverse to Z, so we consider points $(v, w) \in V \times_M W$ and $z \in Z$ such that $h(z)$ is equal to $(f \times_M g)(v, w)$, which by definition is equal to $f(v) = g(w)$. In other words, we consider $(v, w, z) \in V \times W \times Z$ such that $f(v) = g(w) = h(z)$.

So suppose (v, w, z) is such a triple, and denote the common image by $m \in M$. By the transversality assumptions, we know that the images of $D_w g : T_w W \to T_m M$ and $D_z h : T_z Z \to T_m M$ span $T_m M$, i.e. that $D_w g$ and $D_z h$ are transverse as linear maps, and similarly that $D_{(w,z)}(g \times_M h) : T_{(w,z)}(W \times_M Z) \to T_m M$ is transverse to $D_v f : T_v V \to T_m M$. Furthermore, by Lemma **??**, the tangent space of a fiber product is the fiber product of the tangent spaces, so $T_{(w,z)}(W \times_M Z) = T_w W \times_{T_m M} T_z Z$ and $D_{(w,z)}(g \times_M h) = D_w g \times_{T_m M} D_z h$.

Now by [**?**, Propositions 4-9], the triple of linear maps $(D_v f, D_w g, D_z h)$ is transverse as a triple of maps, if and only if both $D_w g$ is transverse to $D_z h$ and $D_w g \times_{T_m M} D_z h$ is transverse to $D_v f$. As such statements are independent of how we order the terms, the transversality established in the preceding paragraph also implies that $D_v f$ and $D_w g$ are transverse (which already follows from the hypotheses of the proposition), and $D_v f \times_{T_m M} D_w g$ is transverse to $D_z h$. But this implies, again using Lemma **??**, that h is transverse to $f \times_M g$, as desired. $\square$

Remark 0.2. The end of the preceding proof at first seems to imply that if g and h are transverse and $g \times_M h$ is transverse to f, then f is transverse to g and $f \times_M g$ is transverse to h. Indeed, [**?**, Propositions 4-9] says this is the case for the linear maps of the tangent spaces. Unfortunately, however, as [**?**, Propositions 4-9] applies only to linear maps, we can apply it only at those points $(v, w, z) \in V \times W \times Z$ where we know that $f(v) = g(w) = h(z)$ so that all three tangent space maps are well defined. So such a result would hold if the only intersections among the maps were such triple intersections. However, as noted in Remark **??**, there could be pairs $(v, w) \in V \times W$ with $f(v) = g(w)$, but with this common image in M not in the image of h. At such points, [**?**, Propositions 4-9] cannot tell us anything about the transversality of V and W, and so $V \times_M W$ might not be well defined due to failure of transversality, even if $V \times_M (W \times_M Z)$ is.

References