Third referee's report on the paper
"PERSISTENCE IN FUNCTIONAL TOPOLOGY: EXISTENCE OF PERSISTENCE DIAGRAMS AND MORSE INEQUALITIES"
by Ulrich Bauer, Anibal M. Medina-Mardones and Maximilian Schmahl

I wish to thank the authors for the modifications they made in response to my two previous reports. The points I raised have been addressed properly. During the last round of modifications, certain new inaccuracies were introduced in the text, which I am listing below. Nevertheless, I am confident that these inaccuracies will be corrected and I now recommend the paper to be published.

1. The definition of a weakly upper-reducible function on page 19 differs from the one in [3]. Namely, in [3] the map φ is assumed to map $F_{\leq c-\eta}$ into itself. I am not sure why the authors decided to exclude this condition. It seems to me that it is indeed important for certain arguments. Precisely, I am not able to understand the sentence "G is also upper-reducible because the same is true for F" on page 24, line 15. It seems to me that the authors are using the fact that φ restricts to $F_{\leq t} \setminus S$, which, as far as I understand, follows from this extra condition of Morse and Tompkins. Another place where I think the same condition is used is in the proof of Lemma 5.11 on page 30 when it is stated that "For p with $F(p) < t$ the existence of such δ_p and φ_p is guaranteed by the assumption that F is weakly upper-reducible." Namely, without this condition, it is not clear why φ_p would satisfy property (2), (a).

2. In the proof of Lemma 5.14, [1] is cited for the fact that the natural map from singular to Čech homology is surjective for compact metric spaces. However, the space M to which this results is applied need not be compact. I tried to complete the proof by applying the statement to $F_{\leq t}$, but was not able to. I would like to ask the authors to clarify this point. While this is important in order for the statement and the proof of Theorem 5.3 to be precise, it does not influence applications to minimal surfaces, since the space Ω_g is contractible and thus $\dim \check{H}_0(\Omega_g) = 1$.

3. In the proof of Lemma 5.11 on page 31, it is not clear why property

(3), $\varphi(F_{\leq t}, 1) \subseteq F_{\leq t-\epsilon}$, of the concatenation φ, will be satisfied. Let me explain the problem on an example of two homotopies $\varphi_1(\cdot,\cdot)$ and $\varphi_2(\cdot,\cdot)$. Assume that $x \in B_{\frac{\delta_{p_2}}{3}}(p_2) \setminus B_{\frac{\delta_{p_1}}{3}}(p_1)$. Under the current assumptions, I think it can happen that $\varphi_1(x, 1) \in F_{>t-\epsilon} \setminus B_{\frac{\delta_{p_2}}{3}}(p_2)$ or even $\varphi_1(x, 1) \in F_{>t-\epsilon} \setminus B_{\frac{2\delta_{p_2}}{3}}(p_2)$. In other words, after the first homotopy, it may happen that x stays above $t-\epsilon$, but leaves the set of points which will be taken to $F_{\leq t-\epsilon}$ by the second homotopy. I was not able to find an immediate fix to this, so I would like to ask the authors to address it in some way. As far as I understand, in [2], this issue is resolved by first passing to a "related homotopy" and then bounding the distance between x and $\varphi_i(x, 1)$, see Lemmas 7.3, 7.4 and the proof of Lemma 8.1.

Below is a list of minor suggestions, remarks and errors I found in the latest version of the paper.

1. Page 12 lines 5 and 4 from below and page 26 the paragraph below Definition 5.16 - It seems that there is an inconsistency, since on page 12 it is claimed that Morse and Tompkins proved that compact filtration implies boundedness from below, while on page 26 it is claimed that Morse made this assumption.

2. Page 23, line 8 - "function a metric space" should be "function on a metric space".

3. Page 24, line 19 from the bottom - One "for" should be removed.

4. Page 25, line 16 - I think that "distinct critical sets" was meant to be "disjoint critical sets".

5. Page 25, line 22 - It seems that the argument implicitly uses that $t_1 \neq t_2$ since if $t_1 = t_2 = t$ the expression should be $\check{c}_0^\epsilon \geq \check{\alpha}_0^\epsilon(t) \geq \dim H_0(S_1 \sqcup S_2) \geq 2$. Perhaps this should be noted for clarity.

6. Page 25, line 24 - $\check{\omega}_0^\epsilon)(t)$ should be $\check{\omega}_0^\epsilon(t)$.

7. Page 29 - I think that Proposition B.1 holds without the weak upper-reducibility assumption, with the same proof.

8. Page 29, lines 12 and 18 - H_* should be H_d.

9. Page 30, proof of Lemma 5.11 - There is a slight inaccuracy in the proof. In line 2 from the bottom, there is a chain of inequalities $F(x) - t \geq F(x) - F(\varphi(x, 1)) \geq \epsilon > 0$, which is wrong (for example $F(x) - t \leq 0$). If I understand correctly, one is supposed to select q_n in such a way that $F(\varphi(q_n, 1)) \to t$ as well and then $t - F(\varphi(x, 1)) \geq F(x) - F(\varphi(x, 1)) \geq \epsilon > 0$ gives a contradiction, since $F(q_n), F(\varphi(q_n, 1)) \to t$ as $n \to \infty$.

10. Page 31, lines 1 and 11 - If $r \in (\frac{i}{n}, \frac{i+1}{n}]$ then p_i should go from $i = 0$ to $i = n - 1$.

11. Page 31, line 3 - max should be min.

12. Page 31, line 3 and 15 - It seems to me that perhaps it should be $\epsilon = \min_{i=1,\dots,n} \epsilon(p_i, \bar{B}_{\frac{2\delta_{p_i}}{3}}(p_i))$ in order to have $\tilde{\varphi}_i(\cdot, r) : F_{\leq s} \to F_{\leq s}$ for all r and all $s \in (t - \epsilon, t]$. This would then imply property (2) of φ. If we take minimum over smaller balls, it is not obvious to me why sublevel sets will be preserved for points in $B_{\frac{2\delta_{p_i}}{3}}(p_i) \setminus \bar{B}_{\frac{\delta_{p_i}}{3}}(p_i)$.

13. Page 31, line 11 - There is mistake in the formula for the concatenation of maps. It should be $\varphi(x, r) = \varphi_i(\varphi_{i-1} \circ \ldots \circ \varphi_0(x), n \cdot r - i)$, where I denoted $\varphi_j = \varphi_j(\cdot, 1)$.

References

[1] K. Eda, and K. Kawamura, *The surjectivity of the canonical homomorphism from singular homology to Čech homology.* Proc. Amer. Math. Soc. 128 (2000), no. 5, 1487-1495.

[2] M. Morse, *Functional topology and abstract variational theory.* eng. Gauthier-Villars, 1938.

[3] M. Morse and C. Tompkins, *The existence of minimal surfaces of general critical types.* Ann. of Math. (2) 40 (1939), no. 2, 443-472.