Optical Model and Inelastic Scattering

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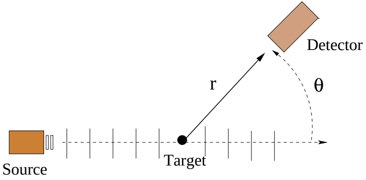
Experimental cross section



$$\Delta I = I_0 \, n_t \, \frac{d\sigma}{d\Omega} \Delta\Omega$$

- $\Delta\Omega$: solid angle of detector ($=\Delta A/r^2$)
- ΔI: detected particles per unit time in $\Delta\Omega$ (s^{-1})
- I_0: incident particles per unit time and unit area ($s^{-1}L^{-2}$)
- n_t: number of target nuclei within the beam
- $d\sigma/d\Omega$: differential cross section (L^2)

$$\frac{d\sigma}{d\Omega} = \frac{\text{flux of scattered particles through } dA = r^2 d\Omega}{\text{incident flux}}$$

Scattering amplitude and cross sections

Asymptotically, when the projectile and target are well far apart,

$$\boxed{\begin{aligned}\Psi^{(+)}_{\mathbf{K}_\alpha} &\xrightarrow{R_\alpha \gg} \Phi_\alpha(\xi_\alpha) e^{i\mathbf{K}_\alpha\cdot\mathbf{R}_\alpha} + \Phi_\alpha(\xi_\alpha) f_{\alpha,\alpha}(\theta)\frac{e^{iK_\alpha R_\alpha}}{R_\alpha} && \text{(elastic)}\\ &+ \sum_{\alpha'\neq\alpha}\Phi_{\alpha'}(\xi_\alpha) f_{\alpha',\alpha}(\theta)\frac{e^{iK_{\alpha'}R_\alpha}}{R_\alpha} && \text{(inelastic)}\\ \Psi^{(+)}_{\mathbf{K}_\alpha} &\xrightarrow{R_\beta \gg} \sum_\beta \Phi_\beta(\xi_\beta) f_{\beta,\alpha}(\theta)\frac{e^{iK_\beta R_\beta}}{R_\beta} && \text{(transfer)}\end{aligned}}$$

where the function $f_{\beta,\alpha}$ modulating the outgoing waves is called scattering amplitude

Cross sections:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\alpha\to\beta} = \frac{\mu_\alpha}{\mu_\beta}\frac{K_\beta}{K_\alpha}\left|f_{\beta,\alpha}(\theta)\right|^2} \qquad E = \frac{\hbar^2 K_\alpha^2}{2\mu_\alpha} + \varepsilon_\alpha = \frac{\hbar^2 K_\beta^2}{2\mu_\beta} + \varepsilon_\beta$$

Single-channel approach to elastic scattering: the optical model

Elastic scattering in the optical model (no spin case)

- Effective Hamiltonian:

$$H = T_{\mathbf{R}} + U(\mathbf{R}) \qquad (U(\mathbf{R}) \text{ complex!})$$

- Schrödinger equation:

$$\boxed{[T_{\mathbf{R}} + U(\mathbf{R}) - E_\alpha]\chi_0^{(+)}(\mathbf{K},\mathbf{R}) = 0} \qquad (E_\alpha = \text{incident energy in CM})$$

- Boundary condition: Plane wave plus spherical wave, multiplied by the scattering amplitude $f(\theta,\phi)$:

$$\chi_0^{(+)}(\mathbf{K},\mathbf{R}) \to e^{i\mathbf{K}\cdot\mathbf{R}} + f(\theta,\phi)\frac{e^{iKR}}{R} \qquad K = \frac{\sqrt{2\mu E_\alpha}}{\hbar}$$

- Elastic differential cross section:

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$$

Partial wave decomposition

- For a central potential $[U(\mathbf{R}) = U(R)]$, the scattering wavefunction can be expanded in spherical harmonics:

$$\boxed{\chi_0^{(+)}(\mathbf{K},\mathbf{R}) = \frac{1}{KR}\sum_{\ell} i^{\ell}\chi_{\ell}(K,R)(2\ell+1)P_{\ell}(\cos\theta)} \qquad (\theta = \text{scattering angle})$$

- The radial wavefuntions $\chi_{\ell}(K,R)$ satisfy the equation:

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dR^2} + \frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{R^2} + U(R) - E_0\right]\chi_{\ell}(K,R) = 0.$$

- For a zero potential ($U = 0$) the solution is just the plane wave:

$$\chi_0^{(+)}(\mathbf{K},\mathbf{R}) = e^{i\mathbf{KR}} \quad \Rightarrow \quad \chi_{\ell}(K,R) \to F_{\ell}(KR) = \frac{i}{2}[H_{\ell}^{(-)}(KR) - H_{\ell}^{(+)}(KR)]$$

where: $\quad F_{\ell}(KR) \to \sin(KR - \ell\pi/2) \quad ; \quad H_{\ell}^{(\pm)}(KR) \to e^{\pm i(KR-\ell\pi/2)}$

Non-zero potential: asymptotic behaviour

- For $R \gg \Rightarrow U(R) = 0 \Rightarrow \chi_\ell(K,R)$ will be a combination of F_ℓ and G_ℓ

$$\boxed{F_\ell(KR) \to \sin(KR - \ell\pi/2) \qquad G_\ell(KR) \to \cos(KR - \ell\pi/2)}$$

or their *outgoing/ingoing* combinations:

$$\boxed{H^{(\pm)}(KR) \equiv G_\ell(KR) \pm iF_\ell(KR) \to e^{\pm i(KR-\ell\pi/2)}}$$

- The specific combination is determined by the physical boundary condition:

$$\begin{array}{rlcll}
 & \chi_0^{(+)}(\mathbf{KR}) & \to & e^{i\mathbf{K}\cdot\mathbf{R}} & + \; f(\theta)\dfrac{e^{iKR}}{R} \\
 & \Downarrow & & \Downarrow & \quad \Downarrow \\
U = 0 & \chi_\ell(KR) & \to & F_\ell(KR) & + \; 0 \\
U \neq 0 & \chi_\ell(KR) & \to & F_\ell(KR) & + \; T_\ell H^{(+)}(KR)
\end{array}$$

☞ The coefficients T_ℓ are to be determined by numerical integration.

Numerical integration of Schrodinger equation

1. Fix a *matching radius*, R_m, such that $U(R_m) \approx 0$
2. Integrate $\chi_\ell(R)$ from $R = 0$ up to R_m, starting with the condition:

$$\lim_{R\to 0} \chi_\ell(K,R) = 0$$

3. At $R = R_m$ impose the boundary condition:

$$\begin{aligned} \chi_\ell(K,R) &\to F_\ell(KR) + T_\ell H_\ell^{(+)}(KR) \\ &= \frac{i}{2}[H_\ell^{(-)}(KR) - S_\ell H_\ell^{(+)}(KR)] \end{aligned}$$

☞ $S_\ell = 1 + 2iT_\ell$ = **S-matrix**

4. Phase-shifts:

$$\boxed{S_\ell \equiv e^{i2\delta_\ell}} \qquad \boxed{T_\ell = e^{i\delta_\ell}\sin(\delta_\ell)}$$

$$\boxed{\chi_\ell(K,R) \to e^{i\delta_\ell}\sin(KR + \delta_\ell - \ell\pi/2)}$$

The scattering amplitude

- Replace the asymptotic $\chi_\ell(K,R)$ in the general expansion:

$$\chi^{(+)}(\mathbf{K},\mathbf{R}) = \frac{1}{KR}\sum_\ell i^\ell(2\ell+1)\chi_\ell(K,R)P_\ell(\cos\theta)$$

$$\to \frac{1}{KR}\sum_\ell i^\ell(2\ell+1)\Big[F_\ell(KR) + T_\ell H^{(+)}(KR)\Big]P_\ell(\cos\theta)$$

- The scattering amplitude is the coefficient of e^{iKR}/R in $\chi^{(+)}(\mathbf{K},\mathbf{R})$:

$$\boxed{f(\theta) = \frac{1}{2iK}\sum_\ell(2\ell+1)(S_\ell-1)P_\ell(\cos\theta).}$$

- Elastic differential cross section:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

Coulomb plus nuclear case

Radial equation:

$$\boxed{\left[\frac{d^2}{dR^2} + K^2 - \frac{2\eta K}{R} + \frac{2\mu}{\hbar^2}U(R) + \frac{\ell(\ell+1)}{R^2}\right]\chi_\ell(K,R) = 0}$$

$$\boxed{\eta = \frac{Z_p Z_t e^2}{4\pi\epsilon_0 \hbar v} = \frac{Z_p Z_t e^2 \mu}{4\pi\epsilon_0 \hbar^2 K}}$$

(Sommerfeld parameter)

Asymptotic condition:

$$\chi^{(+)}(\mathbf{K},\mathbf{R}) \to e^{i[\mathbf{K}\cdot\mathbf{R}+\eta\log(kR-\mathbf{K}\cdot\mathbf{R})]} + f(\theta)\frac{e^{i(KR-\eta\log 2KR)}}{R}$$

$$\boxed{\chi_\ell(K,R) \to \frac{i}{2}e^{i\sigma_\ell}\left[H_\ell^{(-)}(\eta,KR) - S_\ell H_\ell^{(+)}(\eta,KR)\right]}$$

- ☞ $\sigma_\ell(\eta)$=Coulomb phase shift
- ☞ $F_\ell(\eta,KR)$=regular Coulomb wave
- ☞ $H_\ell^{(\pm)}(\eta,KR)$=outgoing/ingoing Coulomb wave

Coulomb plus nuclear case: scattering amplitude

Total scattering amplitude:

$$\boxed{f(\theta) = f_C(\theta) + \frac{1}{2iK}\sum_{\ell}(2\ell+1)e^{2i\sigma_\ell}(S_\ell - 1)P_\ell(\cos\theta)}$$

☞ $f_C(\theta)$ is the amplitude for pure Coulomb:

$$\boxed{\frac{d\sigma_R}{d\Omega} = |f_C(\theta)|^2 = \frac{\eta^2}{4K^2\sin^4(\frac{1}{2}\theta)} = \left(\frac{Z_p Z_t e^2}{16\pi\epsilon_0 E}\right)^2 \frac{1}{\sin^4(\frac{1}{2}\theta)}}$$

Classical interpretation of elastic scattering

Deflection function and classical cross section

- For a given projectile-target potential $V(r)$, the deflection function can be obtained for each impact parameter solving:

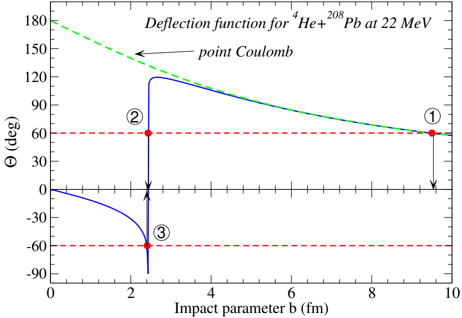
$$\boxed{\Theta(b) = \pi - 2\int_{r_0}^{\infty} \frac{b}{r^2} \frac{dr}{\sqrt{1 - \frac{V(r)}{E} - \frac{b^2}{r^2}}}}$$

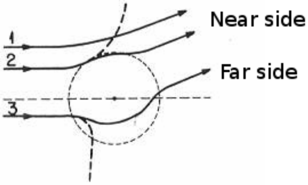
- The classical scattering cross section is a function of the deflection function (or scattering angle) according to:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{b}{\sin(\theta)} \left|\frac{db}{d\theta}\right|}$$

Coulomb + nuclear scattering: deflection function

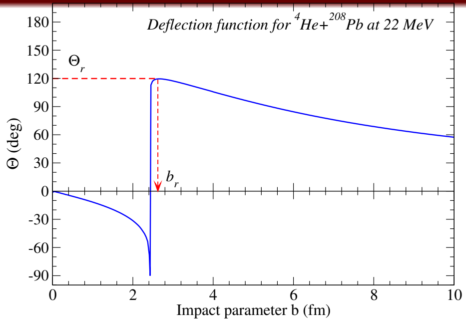
☞ For large values of b, the scattering is Coulombic (the projectile does not feel the nuclear potential).





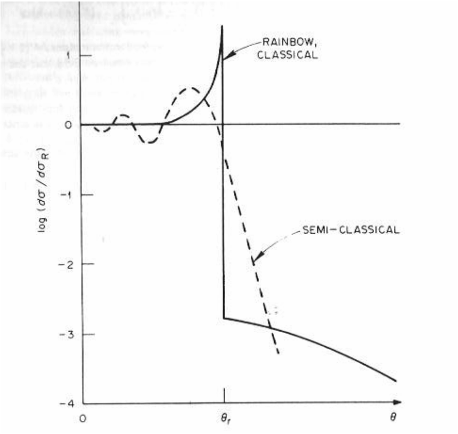
☞ For a given scattering angle θ there are in general 3 values of b contributing to this angle. (1) is the Coulomb trajectory, (2) is the nuclear near-side trajectory, and (3) is the nuclear far-side trajectory.

Coulomb + nuclear scattering: Rainbow



- The deflection function has a maximum at $b = b_r \to \Theta_r$ (rainbow angle)
- For $b = b_r$: $\frac{d\Theta}{db} = 0 \Rightarrow \frac{db}{d\Theta} = \infty \Rightarrow \frac{d\sigma}{d\Omega} \to \infty$
- In the vicinity of b_r, many trajectories give approximately the same scattering angle (Θ_r)
- For angles greater than the rainbow ($\theta > \Theta_r$), neither the Coulomb trajectories nor the nuclear nearside trajectories contribute to the cross section so, classically, there is a sharp decrease in the differential cross section for ($\theta > \Theta_r$). This is the "shadow region".

Coulomb + nuclear scattering: undulatory effects



☞ In a treatment beyond the classical limit, several trajectories may interfere, and the divergence at the rainbow is smoothed.

Elastic scattering phenomenology

Nucleus-nucleus scattering: Optical Potential

Optical potential: $\mathcal{V} \approx U(r) = U_{\rm nuc}(r) + V_{\rm coul}(r)$

- Coulomb potential: charge sphere distribution

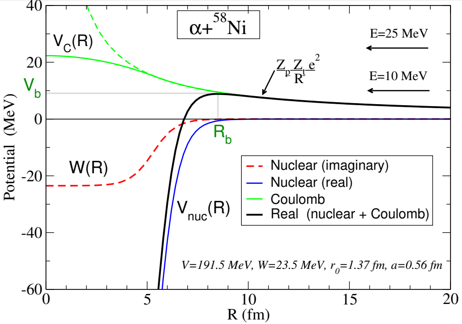
$$\boxed{V_{\rm coul}(r) = \begin{cases} \frac{Z_1 Z_2 e^2}{2R_c}\left(3 - \frac{r^2}{R_c^2}\right) & \text{if } r \leq R_c \\ \frac{Z_1 Z_2 e^2}{r} & \text{if } r \geq R_c \end{cases}}$$

- Nuclear potential (complex): Eg. Woods-Saxon parametrization

$$\boxed{U_{\rm nuc}(r) = V(r) + iW(r) = -\frac{V_0(E)}{1 + \exp\left(\frac{r-R_V}{a_V}\right)} - i\,\frac{W_0(E)}{1 + \exp\left(\frac{r-R_W}{a_W}\right)}}$$

- Potential parameters: 6, fitted to reproduce the elastic differential cross sections.
  - Depths $V_0(E), W_0(E)$;
  - Radii $R_{V,W} = r_{V,W}(A_p^{1/3} + A_t^{1/3})$. $r_V \approx r_W \sim 1.1 - 1.4$ fm.
  - Difuseness $a_V \approx a_W \sim 0.5 - 0.7$ fm

Nucleus-nucleus scattering: The Coulomb barrier



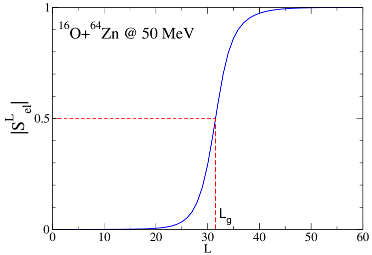
- The maximum of $V_N(r) + V_C(r)$ defines the Coulomb barrier. The radius of the barrier is R_b. The height of the barrier is $V_b = V_N(R_b) + V_C(R_b)$
- As a **rough approximation,**

$$\boxed{R_b \simeq 1.44(A_p^{1/3} + A_t^{1/3}) \text{ fm}}$$

$$\boxed{V_b \simeq \frac{Z_p Z_t e^2}{4\pi\epsilon_0 R_b} \approx \frac{Z_p Z_t}{(A_p^{1/3} + A_t^{1/3})} \text{ [MeV]}}$$

Nucleus-Nucleus Elastic scattering: Strong absorption

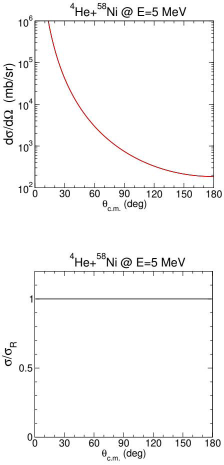
- The nuclear attraction is determined by the real part of the optical potential $V(r)$. Together with the Coulomb potential, determines the Coulomb barrier.
- The absorption, which corresponds to the removal of flux from the elastic channel, is determined by the imaginary part of the optical potential $W(r)$.
- Elastic scattering of heavy nuclei (beyond He) displays strong absorption. One can define a grazing angular momentum (ℓ_g), such that:
  - $|S_\ell| \approx 0$ when $\ell \ll \ell_g$ and $|S_\ell| \to 1$ when $\ell \gg \ell_g$.
  - A convenient quantitative definition of the grazing angular momentum (ℓ_g) is provided by the condition $|S(\ell_g)| \simeq \frac{1}{2}$



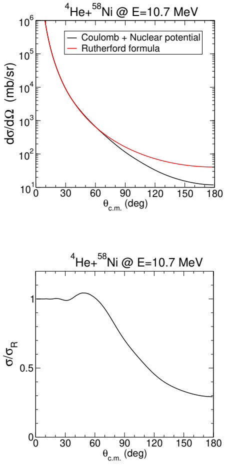
Patterns of elastic scattering: Energy dependence

- The semi-classical vs quantum character of the scattering can be given in terms of the Sommerfeld parameter: $\eta = \frac{Z_p Z_t e^2}{4\pi\epsilon_0 \hbar v}$
- The Coulomb vs nuclear relevance, in terms of the energy of the Coulomb barrier: $V_b \simeq \frac{Z_p Z_t}{A_p^{1/3} + A_t^{1/3}}$ [MeV]
- Three distinct patterns appear for the elastic cross sections
  - Nuclear relevant $E > V_b$, quantum $\eta \lesssim 1$ $\Rightarrow$ Fraunhofer scattering
  - Nuclear relevant $E > V_b$, semiclassical $\eta \gg 1$ $\Rightarrow$ Fresnel scattering
  - Coulomb-dominated $E < V_b$ $\Rightarrow$ Rutherford scattering

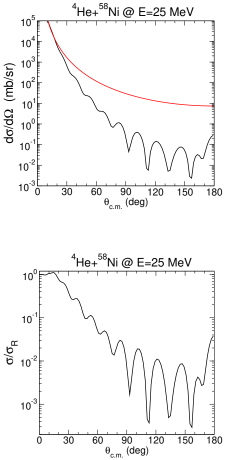
Patterns of elastic scattering: ^{4}He+^{58}Ni example



Rutherford scattering



Fresnel Scattering

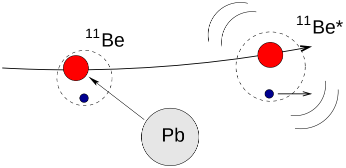


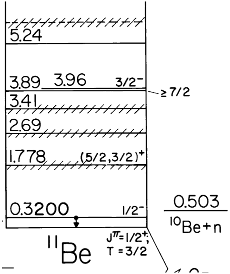
Fraunhöfer Scattering

Inelastic scattering

Inelastic scattering to bound states

- Nuclei are not inert or *frozen* objects; they do have an internal structure of protons and neutrons that can be modified (excited) during the collision.
- Quantum systems exhibit, in general, an energy spectrum with bound and unbound levels.





Formal treatment of inelastic scattering

Asymptotically, when the projectile and target are well far apart,

$$\Psi^{(+)}_{\mathbf{K}_\alpha} \xrightarrow{R_\alpha \gg} \Phi_\alpha(\xi_\alpha) e^{i\mathbf{K}_\alpha \cdot \mathbf{R}_\alpha} + \Phi_\alpha(\xi_\alpha) f_{\alpha,\alpha}(\theta) \frac{e^{iK_\alpha R_\alpha}}{R_\alpha} \qquad \text{(elastic)}$$

$$+ \sum_{\alpha' \neq \alpha} \Phi_{\alpha'}(\xi_\alpha) f_{\alpha',\alpha}(\theta) \frac{e^{iK_{\alpha'} R_\alpha}}{R_\alpha} \qquad \text{(inelastic)}$$

$$\Psi^{(+)}_{\mathbf{K}_\alpha} \xrightarrow{R_\beta \gg} \sum_{\beta} \Phi_\beta(\xi_\beta) f_{\beta,\alpha}(\theta) \frac{e^{iK_\beta R_\beta}}{R_\beta} \qquad \text{(transfer)}$$

where the function $f_{\beta,\alpha}$ modulating the outgoing waves is called scattering amplitude

Cross sections:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\alpha\to\beta} = \frac{\mu_\alpha}{\mu_\beta}\frac{K_\beta}{K_\alpha}\left|f_{\beta,\alpha}(\theta)\right|^2} \qquad E = \frac{\hbar^2 K_\alpha^2}{2\mu_\alpha} + \varepsilon_\alpha = \frac{\hbar^2 K_\beta^2}{2\mu_\beta} + \varepsilon_\beta$$

The coupled-channels method

The coupled-channels method for inelastic scattering

We need to incorporate explicitly in the Hamiltonian the internal structure of the nucleus being excited (e.g. projectile).

$$\boxed{H = T_R + h(\xi) + V(\mathbf{R}, \xi)}$$

- T_R: Kinetic energy for projectile-target relative motion.
- $\{\xi\}$: Internal degrees of freedom of the projectile (depend on the model).
- $V(\mathbf{R}, \xi)$: Projectile-target interaction.
- $h(\xi)$: Internal Hamiltonian of the projectile.

$$\boxed{h(\xi)\phi_n(\xi) = \varepsilon_n \phi_n(\xi)}$$

- $\phi_n(\xi)$: internal states of the projectile.

CC model wavefunction (target excitation)

The total wave function is expanded in a subset of internal states representing the adopted modelspace:

$$\boxed{\Psi_{\mathrm{model}}(\mathbf{R},\xi) = \phi_0(\xi)\chi_0(\mathbf{K}_0,\mathbf{R}) + \sum_{n>0} \phi_n(\xi)\chi_n(\mathbf{K}_n,\mathbf{R})}$$

and impose the boundary conditions for the (unknown) $\chi_n(\mathbf{R})$:

$$\chi_0^{(+)}(\mathbf{K}_0,\mathbf{R}) \to e^{i\mathbf{K}_0\cdot\mathbf{R}} + f_{0,0}(\theta)\frac{e^{iK_0R}}{R} \qquad \text{for n=0 (elastic)}$$

$$\chi_n^{(+)}(\mathbf{K}_n,\mathbf{R}) \to f_{n,0}(\theta)\frac{e^{iK_nR}}{R} \qquad \text{for n>0 (non-elastic)}$$

Calculation of $\chi_n^{(+)}(\mathbf{R})$: the coupled equations

- The model wavefunction must satisfy the Schrödinger equation:

$$[H - E]\Psi^{(+)}_{\text{model}}(\mathbf{R}, \xi) = 0$$

- Multiply on the left by each $\phi_n^*(\xi)$, and integrate over $\xi \Rightarrow$ coupled channels equations for $\{\chi_n(\mathbf{R})\}$:

$$\boxed{[E - \varepsilon_n - T_R - V_{n,n}(\mathbf{R})]\chi_n(\mathbf{R}) = \sum_{n' \neq n} V_{n,n'}(\mathbf{R})\chi_{n'}(\mathbf{R})}$$

- The structure information is embedded in the coupling potentials:

$$\boxed{V_{n,n'}(\mathbf{R}) = \int d\xi \phi^*_{n'}(\xi) V(\mathbf{R}, \xi)\phi_n(\xi)}$$

☞ *$\phi_n(\xi)$ will depend on the assumed structure model (collective, few-body, etc).*

Optical Model vs. Coupled-Channels method

Optical Model

- The Hamiltonian:
  $H = T_R + V(\mathbf{R})$
- Internal states: Just $\phi_0(\xi)$
- Model wavefunction:
  $\Psi_{\text{mod}}(\mathbf{R},\xi) \equiv \chi_0(\mathbf{K},\mathbf{R})\phi_0(\xi)$
- Schrödinger equation:
  $[H - E]\chi_0(\mathbf{K},\mathbf{R}) = 0$

Coupled-channels method

- The Hamiltonian:
  $H = T_R + h(\xi) + V(\mathbf{R},\xi)$
- Internal states:
  $h(\xi)\phi_n(\xi) = \varepsilon_n\phi_n(\xi)$
- Model wavefunction:
  $\Psi_{\text{model}}(\mathbf{R},\xi) = \phi_0(\xi)\chi_0(\mathbf{K},\mathbf{R}) + \sum_{n>0}\phi_n(\xi)\chi_n(\mathbf{K},\mathbf{R})$
- Schrödinger equation:

$$[H - E]\Psi_{\text{model}}(\mathbf{R},\xi) = 0$$

$$\Downarrow$$

$$[E - \varepsilon_n - T_R - V_{n,n}(\mathbf{R})]\chi_n(\mathbf{K},\mathbf{R}) = \sum_{n' \neq n} V_{n,n'}(\mathbf{R})\chi_{n'}(\mathbf{K},\mathbf{R})$$

The DWBA approximation for inelastic scattering

- Assume that we can write the p-t interaction as: $V(\mathbf{R},\xi) = V_0(R) + \Delta V(\mathbf{R},\xi)$
- Use central $V_0(R)$ part to calculate the (distorted) waves for p-t relative motion:

$$\left[\hat{T}_{\mathbf{R}} + V_0(R) - E_i\right]\chi_i^{(+)}(\mathbf{K}_i,\mathbf{R}) = 0 \qquad (E_i = \text{c.m. energy})$$

$$\left[\hat{T}_{\mathbf{R}} + V_0(R) - E_f\right]\chi_f^{(+)}(\mathbf{K}_f,\mathbf{R}) = 0 \qquad (E_f = E_i + Q = E_i - E_x)$$

- In first order of $\Delta V(\mathbf{R},\xi)$ (DWBA) :

$$\boxed{f_{i\to f}^{\mathrm{DWBA}}(\theta) = -\frac{\mu}{2\pi\hbar^2}\int \chi_f^{(-)*}(\mathbf{K}_f,\mathbf{R})\,\Delta V_{if}(\mathbf{R})\,\chi_i^{(+)}(\mathbf{K}_i,\mathbf{R})\,d\mathbf{R}}$$

with the transition potential:

$$\boxed{\Delta V_{if}(\mathbf{R}) \equiv \int \phi_f^*(\xi)\,\Delta V(\mathbf{R},\xi)\,\phi_i(\xi)\,d\xi}$$

Multipole expansion of the interaction: reduced matrix elements

- In actual calculations, the internal states will have definite spin/parity:

$$\phi_i(\xi) = |I_i M_i\rangle \qquad \text{and} \qquad \phi_f(\xi) = |I_f M_f\rangle$$

- The projectile-target interaction can be expanded in multipoles:

$$V(\mathbf{R},\xi) = \sum_{\lambda,\mu} V_{\lambda\mu}(R,\xi) Y_{\lambda\mu}(\hat{R}) \equiv V_0(\mathbf{R}) + \Delta V(\mathbf{R},\xi)$$

- In many practical (and important) situations:

$$\Delta V(\mathbf{R},\xi) = \sum_{\lambda>0} \underbrace{\mathcal{F}_\lambda(R)}_{\text{formfactor}} \sum_{\mu} \underbrace{\mathcal{T}_{\lambda,\mu}(\xi)}_{\text{structure}} Y_{\lambda\mu}(\hat{R})$$

- DWBA and CC calculations require the coupling potentials

$$\langle I_f M_f | \Delta V(\mathbf{R},\xi) | I_i M_i \rangle = \sum_{\lambda>0} \mathcal{F}_\lambda(R) \langle I_f M_f | \mathcal{T}_{\lambda\mu}(\xi) | I_i M_i \rangle Y_{\lambda\mu}(\hat{R})$$

- Wigner-Eckart theorem → reduced matrix elements (r.m.e.)*:

$$\boxed{\langle I_f M_f | \mathcal{T}_{\lambda\mu}(\xi) | I_i M_i \rangle = (2I_f+1)^{-1/2} \langle I_f M_f | I_i M_i \lambda\mu \rangle \underbrace{\langle I_f \| \mathcal{T}_\lambda(\xi) \| I_i \rangle_{\mathrm{BM}}}_{\text{r.m.e}}}$$

Coulomb and nuclear cases

- Form factors and r.m.e.:

Potential	$\mathcal{F}_\lambda(R)$	$\langle I_f \Vert \mathcal{T}_\lambda(\xi) \Vert I_i \rangle$
Coulomb	$\frac{4\pi Z_t e}{(2\lambda+1)R^{\lambda+1}}$	$\langle f; I_f \Vert \mathcal{M}(E\lambda,\mu) \Vert i; I_i \rangle$
Nuclear	$-\frac{dU}{dR}$	$\langle f; I_f \Vert \hat{\delta}_\lambda \Vert i; I_i \rangle$

☞ *This nuclear formfactor is only applicable for small deformations (small δ_λ.)*

- Relation to physical quantities (Coulomb case)

$$B(E\lambda; I_i \to I_f) = (2I_i+1)^{-1} \left| \langle f; I_f \Vert \mathcal{M}(E\lambda,\mu) \Vert i; I_i \rangle \right|^2 \qquad (I_i \neq I_f)$$

$$Q_2 = \sqrt{16\pi/5}\,(2I+1)^{-1/2} \langle II20|II\rangle \langle I \Vert M(E2) \Vert I \rangle \qquad (I_i = I_f \equiv I)$$

DWBA amplitude (nuclear case)

DWBA scattering amplitude:

$$\boxed{f(\mathbf{K}',\mathbf{K})_{iM_i\to fM_f} = -\frac{\mu}{2\pi\hbar^2}\langle f;I_fM_f|\hat{\delta}_{\lambda\mu}|i;I_iM_i\rangle\int d\mathbf{R}\,\chi_f^{(-)*}(\mathbf{K}',\mathbf{R})\frac{dV}{dR}Y_{\lambda\mu}(\hat{\mathbf{R}})\chi_i^{(+)}(\mathbf{K},\mathbf{R})}$$

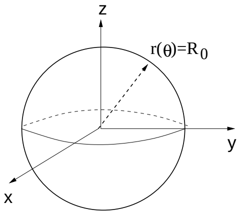
Cross sections:

$$\left(\frac{d\sigma(\theta)}{d\Omega}\right)_{iM_i\to fM_f} = \frac{K_f}{K_i}\left(\frac{\mu}{2\pi\hbar^2}\right)^2\left|\langle f;I_fM_f|\hat{\delta}_{\lambda\mu}|i;I_iM_i\rangle\right|^2 \times\left|\int d\mathbf{R}\,\chi_f^{(-)*}(\mathbf{K}',\mathbf{R})\frac{dV}{dR}Y_{\lambda\mu}(\hat{\mathbf{R}})\chi_i^{(+)}(\mathbf{K},\mathbf{R})\right|^2$$

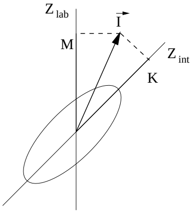
- *The differential cross section is proportional to the deformation parameters*
- *If the approximations are valid, the deformation parameters can be obtained comparing the magnitude of the inelastic cross sections with DWBA calculations.*

Collective excitations in the rotor model

Spherical nucleus ($\beta = 0$)



Deformed nucleus ($\beta \neq 0$)



$$r(\theta') = R_0 \left[1 + \beta_2 Y_{20}(\theta', 0)\right]$$

⇨ For a permanent deformed nucleus (or for a nucleus with surface vibrations) the nucleus-nucleus potential will no longer be central.

⇨ States are characterized by their spin (I) and its projection of the angular momentum along the symmetry axis (K).

Reduced matrix elements in the rotor model

➩ For axial deformation, the charge and matter deformations can be characterized by the parameters $M_n(E\lambda)$ and δ_λ.

1. **Coulomb excitation:**

$$\boxed{\langle f; K\ I_f \| \mathcal{M}(E\lambda) \| i; K\ I_i \rangle_{\rm BM} = \sqrt{2I_i+1}\langle IK\lambda 0|I_f K\rangle M_n(E\lambda) = \pm\sqrt{2I_i+1}\sqrt{B(E\lambda; I_i \to I_f)}}$$

2. **Nuclear excitation:**

$$\boxed{\langle f; KI_f \| \hat{\delta}_\lambda \| i; KI_i \rangle_{\rm BM} = (-1)^\lambda \langle I_i K\lambda 0|I_f K\rangle \delta_\lambda}$$

➩ In the case of even-even nucleus ($K=0$) and $I_i = 0 \to I_f$ transitions:

1. **Coulomb excitation:**

$$\boxed{\langle f; K\ I_f \| \mathcal{M}(E\lambda) \| i; K\ I_i \rangle_{\rm BM} = M_n(E\lambda) = \pm\sqrt{B(E\lambda; I_i \to I_f)}}$$

2. **Nuclear excitation:**

$$\boxed{\langle f; KI_f \| \hat{\delta}_\lambda \| i; KI_i \rangle_{\rm BM} = (-1)^\lambda \delta_\lambda}$$

Coulomb + nuclear potential

- We expect the Coulomb excitation to be more important when:
  - The projectile and/or target charges are large (i.e. large $Z_1 Z_2 \gg 1$)
  - At energies below the Coulomb barrier (where nuclear effects are less important).
  - At very forward angles (large impact parameters).

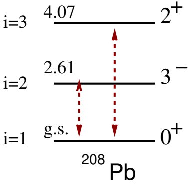
- If both Coulomb and nuclear contributions are important the scattering *amplitudes* for both processes should be added:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{i\to f} = \frac{K_f}{K_i}\left|f_{if}^{\mathrm{coul}} + f_{if}^{\mathrm{nucl}}\right|^2}$$

☞ *In this case, interferences effects will appear!*

Inelastic scattering example: collective excitations

Physical example: $^{16}\mathrm{O} + {}^{208}\mathrm{Pb} \to {}^{16}\mathrm{O} + {}^{208}\mathrm{Pb}(3^-, 2^+)$



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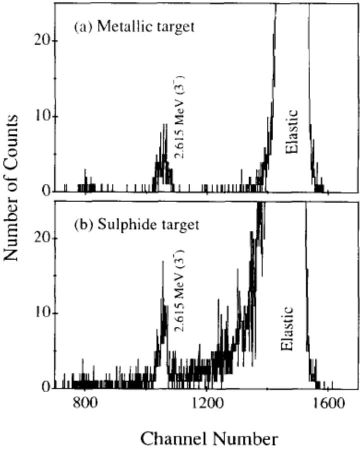
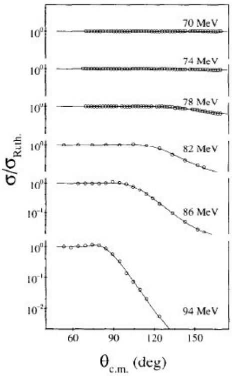
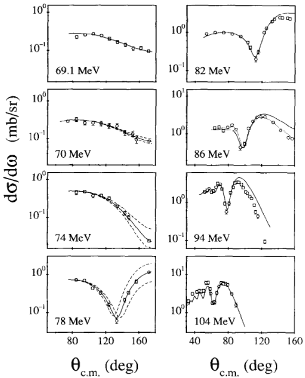
Outgoing ^{16}O energy:



Collective excitations: example





☞ Coulomb barrier:

$$V_{\rm barrier} = \frac{Z_p Z_t e^2}{R_b} \approx \frac{Z_p Z_t e^2}{1.44(A_p^{1/3} + A_t^{1/3})} \simeq 78 \text{ MeV}$$

$^{208}Pb(^{16}O,^{16}O)^{208}Pb$ inelastic scattering

Coulomb and Nuclear excitations can produce constructive or destructive interference: