Curvature and the Mean-Variance-ESG Frontier: A New Measure of Risk-Return-ESG Trade-offs

Abstract

This article develops a measure of portfolio return, variance, and ESG trade-offs, using the concept of curvature in differential geometry. This measure assesses the return reduction or risk expansion when seeking higher ESG scores for a given optimal portfolio. The application of curvature across four global markets indicates that (1) investors in the global minimum variance and the maximum Sharpe ratio portfolios can improve their ESG performance without a reduction in return or an expansion in risk; and (2) that investors with lower risk aversion have better opportunities to improve their portfolio’s ESG score without considerable reductions in return compared to those with higher risk aversion.

*Keywords:* Optimal portfolio, Curvature, Risk aversion, Efficient frontier

*JEL Classification :* G11, G15, C61, C65

1. Introduction

In recent times Environmental, Social, and Governance (ESG) has emerged as a fundamental investment criterion, besides return and risk. Various investment allocation strategies and methodologies incorporating ESG have been developed in the asset management industry.

In a parallel way, academic literature has witnessed significant interest in ESG as a criterion for financial investments (Höck et al., 2023; Capelli et al., 2024; Baek & Kang, 2024). One of the most explored questions in literature is the trade-offs between return, risk, and ESG in asset allocation. A growing body of these contributions highlights the existing compromise among them, i.e., one cannot improve one dimension without a cost in the other dimensions (Burchi, 2019; Utz et al., 2014; Pedersen et al., 2021; De Spiegeleer et al., 2021; Burchi & Włodarczyk, 2020; Prol & Kim, 2022; Cesarone et al., 2022; Boubaker et al., 2023; Steuer & Utz, 2023).

Although researchers have given considerable attention to the trade-offs between return, risk, and ESG, the literature addressing this question still lacks a general measure of these trade-offs.

This article addresses this gap by first proposing a methodology for constructing the Mean-Variance-ESG efficient surface. While based on existing approaches to efficient frontiers in higher dimensions, it focuses on the ESG dimension. Second, it provides visual insights into the trade-offs among return, risk, and ESG. Finally, it develops a general measure of portfolio and market ESG-integration concerning return and risk trade-offs, using the concept of curvature in differential geometry.

This paper is organized as follows: Section 2 presents the methodology and data. Section 3 examines the curvature findings using the constituents of four global indices, while Section 4 concludes the article.

1. Material and Methods
   1. Mean-Variance-ESG Analysis

First, let's develop some basic definitions. Suppose selecting a portfolio from N risky assets, where denotes the expected return of i-th asset and denotes the return covariance between i-th and j-th assets. Suppose is the expected ESG score of the i-th asset. A portfolio is defined with a vector of weights, with the following return, risk, and ESG measures in matrix notation:

(1)

(2)

(3)

With:

, , and

An optimal portfolio with a specified return and ESG score ( , ) can be obtained using weights defined as a solution to the following quadratic problem:

|  |  |
| --- | --- |
| : | (P1)[[1]](#footnote-1) |

To solve the quadratic problem (P1) analytically, the Lagrangian function can be formulated as:

(

Where represents the Lagrangian multipliers. The solution is based on:

(4)

Where is the gradient of the Lagrangian expression for the weight vector and the Lagrangian multipliers.

Equation (4) can be expressed as a system of equations with N+3 equations:

|  |  |
| --- | --- |
|  | (S1) |

Solving for W yields the following equation in matrix notation:

(5)

Where:

We can solve for by substituting the weights expression from (5) into the second through fourth equations of (S1):

|  |  |
| --- | --- |
|  | (S2) |

The final system (S2) can be rewritten in matrix notation as:

(6)

Where: ,

can be expressed as:

(7)

Substituting the expression for into the weight expression (5) yields:

(8)

The variance of an optimal portfolio in three-dimensional space is expressed as follows:

(9)

The Mean-Variance-ESG optimal portfolios can be generated using the following vector-valued function:

(10)

where

<Insert Figure 1 about here>

Figure 1[[2]](#footnote-2) illustrates the Mean-Variance-ESG surface alongside the traditional Mean-Variance efficient frontier (black curve). The yellow area of the surface highlights portfolios with high ESG scores. For clarity, the global minimum variance (GMV) and Maximum Sharpe Ratio (MSR) portfolios are marked with red and orange diamond markers. The GMV portfolio has a higher ESG score than the MSR portfolio. Portfolios on the Mean-Variance efficient frontier, including the GMV and MSR portfolios, cannot achieve higher ESG scores without accepting a reduction in return or an increase in risk.

* 1. Curvature:

The curvature gauges the extent to which a curve or surface deviates from a straight line or plane. It defines how much the direction of the curve, or surface changes locally over a small distance.

In our framework, curvature is used to measure the trade-offs between Mean, Variance, and ESG factors. How does mean or variance change with a small improvement in the ESG dimension? For example, an investor can evaluate the return reduction and/or risk increase associated with achieving higher ESG gains. Curvature can also help distinguish between portfolios and markets based on ESG integration. For example, portfolios and markets with strong ESG investment integration that offer opportunities across all criteria: return, risk, and ESG should exhibit lower curvature than portfolios on the Mean-Variance frontier.

To better understand this trade-off, Figure 2a illustrates the return reduction that GMV and MSR portfolio investors can accept while maintaining the same risk (Fixed Variance) and increasing their portfolio’s ESG score. For portfolios with fixed variance, all possible ESG levels trace out a quadratic curve (contour curve). Looking for ESG score improvement while maintaining the variance of the MSR portfolio will result in a reduction in return. For the GMV portfolio, this curve boils down to a single point, as the Mean-Variance-ESG surface represents an extreme point in this region. Likewise, Figure 2b illustrates the trade-off between return, risk, and ESG at a fixed return, showing the risk expansion that an investor must accept for ESG improvement while maintaining the same return level.

<Insert Figure 2a about here>

<Insert Figure 2b about here>

After discussing the geometry of curvature, we now derive its measure along the curves previously shown with fixed variance and return. Recall that the Mean-Variance-ESG optimal portfolios can be generated using the vector-valued function (10):

: (

Let matrix be:

Substituting the expression of in the variance formula (9) and computing the matrix multiplication, yields:

(11)

(12)

Equation (12) represents an implicit function whose parametric representation in (10) traces out the Mean-Variance-ESG surface.

A cross-section of the Mean-Variance-ESG surface is obtained by fixing one parameter either return, or risk, and allowing the ESG and the non-fixed parameter to change. This approach allows the curvature assessment of the cross-section along the ESG dimension.

These cross-sections represent the curves shown in Figures 2a and 2b, which intersect the mean-variance frontier. For example, if we fix the variance at a value , the derived curve corresponds to the contour of the Mean-Variance-ESG surface at that fixed variance. This curve is generated using the following function:

(13)

<Insert Figure 3a about here>

Figure 3a illustrates these contours for fixed variances corresponding to the GMV and MSR portfolios. It presents the same information as Figure 2a but in a 2D space defined by (E(R), E(ESG)).

Similarly, we can obtain a contour curve by fixing the return at a certain value . This curve is generated using the following function:

(14)

<Insert Figure 3b about here>

Figure 3b shows contours for fixed returns corresponding to GMV and MSR portfolios. It illustrates the same information in Figure 2b but now in 2D space defined by (V(R), E(ESG)).

To derive the curvature measure for fixed variance and return, we will use the definition from Goldman (2005). Goldman’s approach derives the curvature formula for implicit functions based on classical curvature concepts from differential geometry for parametric curves. The curvature is defined as follows:

(15)

Where and are the gradient components of the curve function, and is the Hessian matrix.

In our framework, the curvature is calculated using either the curve function (13) with fixed variance or function (14) with fixed return. Its value measures the inverse of the circle radius that best approximates the curve at a given point. Thus, a minimal value indicates a larger radius, suggesting that the curve at that point does not deviate from being a straight line. In the Mean-Variance-ESG framework, a zero-curvature value with fixed variance indicates that the ESG score of the portfolio could be increased without reducing return. However, very high curvature suggests that any attempts to improve the ESG score will significantly reduce the portfolio return. The curvature can also help assess the degree of ESG integration within both a portfolio and a market. For example, a market that effectively integrates ESG should exhibit lower portfolio curvature, thus offering investors a lower cost in terms of return reduction or risk expansion when seeking higher portfolio-ESG scores.

* 1. Data

The constituents of four major global stock indices are considered in this section: the S&P 500, the Nikkei 225, the MSCI Europe, and the Asia Ex Japan index. The data consists of monthly returns and ESG scores extracted from the Refinitiv Database, with 121 observations covering the period from September 2013 to September 2023. Table 1 summarizes the collected data.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | S&P 500 | Nikkei 225 | MSCI | ASIA EX JAPAN |
| Number of stocks[[3]](#footnote-3) | 255 | 213 | 297 | 443 |

Table 1: Collected Data

For the computation of a specific portfolio’s return, risk, ESG, and weights such as those for the MSR portfolio, long-term government bond yields were collected as a proxy for the risk-free rate.

1. Results
   1. GMV and MSR Curvature

Tables 2 and 3 report the results of curvature computations with variance fixed at that of the GMV and MSR portfolios and return fixed at that of the same portfolios.

First, the results indicate that in the four markets, the curvature values are minimal, suggesting that these markets exhibit good ESG integration. Furthermore, investors in GMV and MSR portfolios can improve their ESG performance without experiencing a significant reduction in return or a substantial expansion in risk. For example, in the Asia Ex-Japan market, the fixed variance curvature values for the GMV () and MSR () portfolios are low, demonstrating that ESG integration can be achieved with minimal impact on the portfolio’s overall return characteristic.

Next, the curvature computed using a fixed return for GMV and MSR portfolios is similar across each market. This suggests that seeking ESG improvements exposes investors in these portfolios to the same amount of risk expansion regardless of the target return. Investors can expect a certain amount of risk increase when targeting ESG scores, and this reduction is similar across different markets and portfolio types.

Lastly, MSR portfolios exhibit lower curvatures compared to GMV portfolios. This implies that MSR portfolio investors experience less decrease in the return or an increase in the risk compared to GMV portfolio investors. For instance, the S&P 500 MSR portfolio has a curvature of , much lower than the GMV portfolio’s . This suggests that the MSR strategy, which targets maximum Sharpe ratio, is more efficient in balancing risk and return while incorporating ESG factors.

It is important to note that MSR portfolios often exhibit negative ESG values. For example, the S&P 500 MSR portfolio shows an ESG value of . This result reflects the portfolio’s focus on maximizing risk-adjusted return rather than prioritizing ESG performance, which may lead to lower ESG scores in favor of higher financial returns.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| GMV Portfolio | S&P 500 | Nikkei 225 | MSCI Europe | Asia Ex-Japan |
| Fixed Variance Curvature | 5.95 | 0.21 | 0.519 | 0.162 |
| Fixed Return Curvature | 9.28 | 1.17 | 4.9 | 1.32 |
| V(R) | 3.58 | 4.27 | 0.198 | 4.02 |
| E(R) | 1.37 | 0.76 | 1.23 | 0.73 |
| E(ESG) | 55.05 | 72.82 | 80.11 | 51.39 |

Table 2: Curvature Results for GMV Portfolio with Fixed Variance and Return Across All Markets

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| MSR Portfolio | S&P 500 | Nikkei 225 | MSCI Europe | Asia Ex-Japan |
| Fixed Variance Curvature | 4.8 | 5.11 | 7.7 | 5.78 |
| Fixed Return Curvature | 9.28 | 1.17 | 4.9 | 1.32 |
| V(R) | 66.22 | 0.35 | 2.92 | 8.89 |
| E(R) | 46.06 | 0.68 | 17.68 | 28.02 |
| E(ESG) | -16.62 | -0.17 | -1.81 | -2.39 |

Table 3: Curvature Results for MSR Portfolio with Fixed Variance and Return Across All Markets

* 1. Curvature with different risk levels

This section examines the behavior of curvature as a function of portfolio risk. For each market, 100 optimal portfolios are generated and sorted from low to high risk. The curvature is then computed with fixed variance and return approaches. Figures 4 and 5 illustrate the average curvature through risk percentiles.

<Insert Figure 4 about here>

<Insert Figure 5 about here>

Figure 4 presents the average curvature with a fixed return. It exhibits a similar curvature value regardless of the risk taken by investors in all markets. This suggests that investors seeking improvements in ESG face the same expansion of risk regardless of the risk level. The amount of risk expansion required does not vary with the risk level chosen for the portfolio. In other words, whether investors are looking to take on low or high risk, the additional risk they face when pursuing ESG scores is similar across the risk range. This finding points to a certain stability in how ESG integration influences risk levels in portfolios, unrelated to the investor’s risk tolerance or return targets.

However, Figure 5 indicates that curvature decreases as portfolio risk increases. The last result implies that investors with lower risk aversion have more opportunities to improve their portfolio’s ESG score without a significant reduction in return.

This result has significant implications for portfolio construction. Investors who are less risk-averse and willing to take on higher levels of volatility may be able to achieve better ESG scores without the substantial financial costs (in terms of reduced returns) typically associated with ESG improvements. Therefore, the results suggest that higher-risk portfolios can optimally balance the integration of ESG factors without substantial financial return reduction.

1. Conclusions

This article develops a general measure of portfolio and market ESG integration in relation to return and risk trade-offs. The application of curvature in four leading global markets indicates that investors in GMV and MSR portfolios can improve their ESG performance without experiencing a significant reduction in return or a substantial expansion in risk. Additionally, the curvature decreases as portfolio risk increases, suggesting that investors with lower risk aversion have better opportunities to improve their portfolio’s ESG score without a significant reduction in return compared to those with higher risk aversion.

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1. This program allows short selling, which this article considers when computing the curvature of the Maximum Sharpe Ratio (MSR). If short selling is excluded, the MSR may not be reached in some applications. [↑](#footnote-ref-1)
2. Figure 1 was generated using artificial return and ESG data for three assets. [↑](#footnote-ref-2)
3. The number of stocks is fewer than the number of index constituents due to the filtering process used to remove missing data. [↑](#footnote-ref-3)