2.111/8.370/18.435 Problem Set 2 Solutions

Due: September 31, 2015

Pauli matrices and eigenvectors:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : |{\rightarrow}\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, |{\leftarrow}\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} : |{\otimes}\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, |{\odot}\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : |{\uparrow}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |{\downarrow}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma_{\hat{i}} = \hat{i} \cdot \vec{\sigma}$$

1. **Verify the following properties of Pauli matrices:**
   - **Hermitian**: $\sigma_x^\dagger = \sigma_x$, $\sigma_y^\dagger = \sigma_y$, $\sigma_z^\dagger = \sigma_z$.
   - **Involutory**: $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$.
   - $\sigma_x\sigma_y = i\sigma_z$, $\sigma_y\sigma_z = i\sigma_x$, $\sigma_z\sigma_x = i\sigma_y$.
   - **Cyclic permutations and commutators:** $[\sigma_x, \sigma_y] \equiv \sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z$**,** $[\sigma_y, \sigma_z] = 2i\sigma_x$**,** $[\sigma_z, \sigma_x] = 2i\sigma_y$ **(Compactly,** $[\sigma_\mu, \sigma_\nu] = 2i\epsilon_{\mu\nu\delta}\sigma_\delta$**.** $\epsilon_{\mu\nu\delta}$**: Levi-Civita)**.
   - $\sigma_z|{\uparrow}\rangle = |{\uparrow}\rangle$**,** $\sigma_z|{\downarrow}\rangle = -|{\downarrow}\rangle$**, i.e.,** $|{\uparrow}\rangle$ **and** $|{\downarrow}\rangle$ **are eigenstates of** σ_z **with eigenvalues 1 and -1 respectively.**
   - $\sigma_x|{\rightarrow}\rangle = |{\rightarrow}\rangle$**,** $\sigma_z|{\leftarrow}\rangle = -|{\leftarrow}\rangle$**.**
   - $\sigma_y|{\otimes}\rangle = |{\otimes}\rangle$**,** $\sigma_y|{\odot}\rangle = -|{\odot}\rangle$**.**

   Use the matrix representation of Pauli matrices and their eigenvectors given above.

2. **Show that (up to a phase term)** $\sigma_z|{\rightarrow}\rangle = |{\leftarrow}\rangle, \sigma_z|{\leftarrow}\rangle = |{\rightarrow}\rangle, \sigma_y|{\uparrow}\rangle = |{\downarrow}\rangle, \sigma_y|{\downarrow}\rangle = |{\uparrow}\rangle, \sigma_x|{\otimes}\rangle = |{\odot}\rangle, \sigma_x|{\odot}\rangle = |{\otimes}\rangle$**.**

   Plug in the matrices given above. The phase terms are $1, 1, i, -i, i, -i$.

3. **Show that if A is Hermitian: its eigenvalues a_i are real, and the eigenvectors corresponding to distinct eigenvalues are orthogonal.**

   Let $\vec{i}$ be the normalized eigenvector of A with eigenvalue a_i. Then $\vec{i}^\dagger A\vec{i} = \vec{i}^\dagger a_i\vec{i} = a_i\vec{i}^\dagger\vec{i} = a_i$. Notice that $A = A^\dagger$, then $\vec{i}^\dagger A\vec{i} = \vec{i}^\dagger A^\dagger\vec{i} = (A\vec{i})^\dagger\vec{i} = a_i^*\vec{i}^\dagger\vec{i} = a_i^*$. Thus $a_i^* = a_i$, i.e., the eigenvalues are real.

   Let $\vec{j}, \vec{k}$ be eigenvectors of A respectively corresponding to eigenvalues a_j, a_k ($a_j \neq a_k$). Then $\vec{j}^\dagger A\vec{k} = a_k\vec{j}^\dagger\vec{k}$, and $\vec{k}^\dagger A\vec{j} = a_j\vec{k}^\dagger\vec{j}$. By $A = A^\dagger$, $\vec{j}^\dagger A\vec{k} = \vec{j}^\dagger A^\dagger\vec{k} = (\vec{k}^\dagger A\vec{j})^\dagger$. Thus $a_k\vec{j}^\dagger\vec{k} = (a_j\vec{k}^\dagger\vec{j})^\dagger = a_j^*\vec{j}^\dagger\vec{k} = a_j\vec{j}^\dagger\vec{k}$, or $(a_j - a_k)\vec{j}^\dagger\vec{k} = 0$. Since $a_j \neq a_k$, the only possibility is that $\vec{j}^\dagger\vec{k} = 0$, i.e., the eigenvectors corresponding to distinct eigenvalues are orthogonal.

4. **Show that** $\sigma_{\hat{i}}^2 = I$**.**

   $\sigma_{\hat{i}}^2 = (\hat{i} \cdot \vec{\sigma})^2 = |\hat{i}|^2 I + \text{anticommutators} = I$. The first term comes from the fact that Pauli matrices are involutory (p.1.2). The second term vanishes since Pauli matrices anticommute (p.1.4).

5. **Show that** $e^{-i\frac{\theta}{2}\sigma} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma$**.**

$$\begin{aligned}
e^{-i\frac{\theta}{2}\sigma} &= \sum_{n=0}^{\infty}\frac{(-i\frac{\theta}{2})^n}{n!}\sigma^n \\
&= \sum_{k=0}^{\infty}\frac{(-i\frac{\theta}{2})^{2k}}{(2k)!}\sigma^{2k} + \sum_{k=0}^{\infty}\frac{(-i\frac{\theta}{2})^{2k+1}}{(2k+1)!}\sigma^{2k+1} \\
&= \sum_{k=0}^{\infty}\frac{(-i\frac{\theta}{2})^{2k}}{(2k)!}I + \sum_{k=0}^{\infty}\frac{(-i\frac{\theta}{2})^{2k+1}}{(2k+1)!}\sigma \\
&= \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}\sigma.
\end{aligned}$$

Second line: separate even and odd n. Third line: $\sigma^2 = I$ (p.4).

6. **Show that:**

- $e^{-i\frac{\pi}{4}\sigma_z}|{\rightarrow}\rangle = \text{phase}\cdot|{\otimes}\rangle$**.**
  By p.5, $e^{-i\frac{\pi}{4}\sigma_z} = \frac{1}{\sqrt{2}}I - \frac{i}{\sqrt{2}}\sigma_z$. Using the matrix representation:

$$\begin{aligned}
e^{-i\frac{\pi}{4}\sigma_z}|{\rightarrow}\rangle &= \frac{1}{2}\begin{pmatrix}1-i & 0\\ 0 & 1+i\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \\
&= \frac{1-i}{2}\begin{pmatrix}1\\i\end{pmatrix} \\
&= \frac{1-i}{\sqrt{2}}|{\otimes}\rangle. \qquad (1)
\end{aligned}$$

- **Let** $\hat{j} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$**. Then** $e^{-i\frac{\pi}{2}\sigma_{\hat{j}}}|{\uparrow}\rangle = \text{phase}\cdot|{\rightarrow}\rangle$**,** $e^{-i\frac{\pi}{2}\sigma_{\hat{j}}}|{\downarrow}\rangle = \text{phase}\cdot|{\leftarrow}\rangle$**,** $e^{-i\pi\sigma_{\hat{j}}}|{\uparrow}\rangle = \text{phase}\cdot|{\uparrow}\rangle$**,** $e^{-i\pi\sigma_{\hat{j}}}|{\downarrow}\rangle = \text{phase}\cdot|{\downarrow}\rangle$**.**
  $e^{-i\frac{\pi}{2}\sigma_{\hat{j}}} = -i\sigma_{\hat{j}} = -\frac{i}{\sqrt{2}}(\sigma_x + \sigma_z)$, $e^{-i\pi\sigma_{\hat{j}}} = -I$. Then by p.1.5–7 and p.2:

$$\begin{aligned}
&e^{-i\frac{\pi}{2}\sigma_{\hat{j}}}|{\uparrow}\rangle = -\frac{i}{\sqrt{2}}(|{\downarrow}\rangle + |{\uparrow}\rangle) = -i|{\rightarrow}\rangle, \\
&e^{-i\frac{\pi}{2}\sigma_{\hat{j}}}|{\downarrow}\rangle = -\frac{i}{\sqrt{2}}(|{\uparrow}\rangle - |{\downarrow}\rangle) = -i|{\leftarrow}\rangle, \\
&e^{-i\pi\sigma_{\hat{j}}}|{\uparrow}\rangle = -|{\uparrow}\rangle, \\
&e^{-i\pi\sigma_{\hat{j}}}|{\uparrow}\rangle = -|{\downarrow}\rangle.
\end{aligned}$$

- **Consider rotation by** 2π **and** 4π **about the** z **axis.**
  $e^{-i\pi\sigma_z} = -I$, $e^{-i2\pi\sigma_z} = I$: Rotation by 2π gives a -1 phase to the state, whereas rotation by 4π preserves the state.