Tensors:

A tensor is a multilinear thing with slots (that perches on a vector).

Notation:

$$|0\rangle \otimes |0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle \otimes |1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|1\rangle \otimes |0\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1\rangle \otimes |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Sometimes we write,

$$|0\rangle \otimes |0\rangle \equiv |0\rangle|0\rangle \equiv |00\rangle.$$

Tensors are multilinear: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$ means that

$$|\psi\rangle \otimes |\phi\rangle = \alpha\gamma|0\rangle \otimes |0\rangle + \alpha\delta|0\rangle \otimes |1\rangle + \beta\gamma|1\rangle \otimes |0\rangle + \beta\delta|1\rangle \otimes |1\rangle = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}.$$

A general normalized vector in the tensor product space $C^2 \otimes C^2$ can be written

$$|\Psi\rangle = \sum_{ij=0,1} \alpha_{ij}|i\rangle \otimes |j\rangle$$

where $\sum_{ij} |\alpha_{ij}|^2 = 1$. Note that in general it is not true that $|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle$ for some $|\psi\rangle, |\phi\rangle$ (count the parameters). When such a $|\Psi\rangle$ is not equal to the tensor product of two vectors it is said to be *entangled*.

We can take tensor products of matrices. When a tensor product of matrices is applied to a tensor product of vectors, each matrix is applied to the vector in the appropriate slot:

$$\sigma_x \otimes \sigma_z |\psi\rangle \otimes |\phi\rangle = (\sigma_x|\psi\rangle) \otimes (\sigma_z|\phi\rangle).$$

Similarly, using multilinearity,

$$\sigma_x \otimes \sigma_z |\Psi\rangle = \sum_{ij=0,1} \alpha_{ij} (\sigma_x |i\rangle) \otimes (\sigma_z |j\rangle).$$

We can also apply tensor products of bra vectors to tensor products of ket vectors:

$$(\langle\psi'| \otimes \langle\phi'|)|\psi\rangle \otimes |\phi\rangle = \langle\psi'|\psi\rangle \otimes \langle\phi'|\phi\rangle \equiv \langle\psi'|\psi\rangle\langle\phi'|\phi\rangle.$$

That is, the tensor product of two complex numbers is just equal to the product of the complex numbers.

We can mix up matrices and bra vectors in the slots:

$$(\sigma_x \otimes \langle\phi'|)|\psi\rangle \otimes |\phi\rangle = \sigma_x|\psi\rangle \otimes \langle\phi'|\phi\rangle = \langle\phi'|\phi\rangle(\sigma_x|\psi\rangle).$$

That is, we can use a bra vector to 'eliminate' a slot.

Now look at tensor products of Pauli matrices. Define σ_μ, $\mu = 0, 1, 2, 3$, by $\sigma_0 = I$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, and $\sigma_3 = \sigma_z$. Use the notation of special relativity so that $\sigma^0 = I$, $\sigma^i = -\sigma_i$, for $i = 1, 2, 3$. Define $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) = \eta^{\mu\nu}$, and use the convention that we sum over repeated indices, so that $\sigma^\mu = \eta^{\mu\nu}\sigma_\nu$. It is not hard to show that any two by two matrix can be written $M = \alpha^\mu \sigma_\mu$, for complex α^μ. Any Hermitian matrix can be written $A = a^\mu \sigma_\mu$ for real a^μ (this is a homework problem). Similarly, any matrix on $C^2 \otimes C^2$ can be written $N = \alpha^{\mu\nu}\sigma_\mu \otimes \sigma_\nu$, for complex $\alpha^{\mu\nu}$. Any Hermitian matrix on the same space can be written $B = a^{\mu\nu}\sigma_\mu \otimes \sigma_\nu$ (this is also a homework problem). Tensor products of Pauli matrices can also be written out in $C^4 \times C^4$, e.g.

$$\sigma_z \otimes \sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \sigma_x \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

which can be verified by looking at the action of the Pauli tensor products on states $|i\rangle \otimes |j\rangle \equiv |ij\rangle$. Try and see!