2.111/8.370/18.435 Problem Set 4 Solutions

Due: October 13, 2015

1. **Show that if $(\langle\phi|U^\dagger)(U|\psi\rangle) = \langle\phi|\psi\rangle$ for all $|\phi\rangle$ and $|\psi\rangle$, then $U^\dagger U = I$, U is unitary.**

   Given $(\langle\phi|U^\dagger)(U|\psi\rangle) = \langle\phi|\psi\rangle$ for all $|\phi\rangle$ and $|\psi\rangle$: Let $A = U^\dagger U - I$, then $\forall|\phi\rangle, |\psi\rangle : \langle\phi|A|\psi\rangle = 0$. Let $\{|i\rangle\}$ be an orthonormal basis of the Hilbert space of $|\phi\rangle, |\psi\rangle$. Then $\sum_i |i\rangle\langle i|A|\psi\rangle = A|\psi\rangle = 0$. That is, $\ker(A)$ is the whole space: $A = 0$. Thus, $U^\dagger U = I$.

2. **Show that no unitary can perform the following: $U_{ABC}|\psi\rangle_A|0\rangle_B|0\rangle_C = |\psi\rangle_A|\psi\rangle_B|\text{junk}\rangle_C$ ($|\text{junk}\rangle$ can be a function of $|\psi\rangle$).**

$$\begin{aligned}
\langle\phi|\psi\rangle &= \langle\phi|\langle 0|\langle 0||\psi\rangle|0\rangle|0\rangle \\
&= \langle\phi|\langle 0|\langle 0|U^\dagger U|\psi\rangle|0\rangle|0\rangle \\
&= \langle\phi|\langle\phi|\langle\text{junk}(\phi)||\psi\rangle|\psi\rangle|\text{junk}(\psi)\rangle \\
&= \langle\phi|\psi\rangle^2\langle\text{junk}(\phi)|\text{junk}(\psi)\rangle.
\end{aligned}$$

   Take the norm of both sides: $|\langle\phi|\psi\rangle| = |\langle\phi|\psi\rangle|^2|\langle\text{junk}(\phi)|\text{junk}(\psi)\rangle| \Rightarrow |\langle\phi|\psi\rangle||\langle\text{junk}(\phi)|\text{junk}(\psi)\rangle| = 1$, which only holds iff $\forall|\phi\rangle, |\psi\rangle : |\langle\phi|\psi\rangle| = |\langle\text{junk}(\phi)|\text{junk}(\psi)\rangle| = 1$: not true. Thus, such U doesn't exist.

   Remarks: This result indicates that even if we allow arbitrary quantum operations instead of just unitaries, the no-cloning theorem still holds. Furthermore, if we allow mixed states as the input, the theorem can also be easily proven by purification.

3. **What happens for a CNOT with $|\otimes\rangle, |\odot\rangle$ inputs?**

   Recall that $|\otimes\rangle = \frac{1}{\sqrt{2}}(1, i)^T = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|\odot\rangle = \frac{1}{\sqrt{2}}(1, -i)^T = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$. Applying regular CNOT on bipartite product states of $|\otimes\rangle, |\odot\rangle$:

$$\begin{aligned}
\text{CNOT}|\otimes\rangle|\otimes\rangle &= \frac{1}{2}\text{CNOT}(|00\rangle + i|01\rangle + i|10\rangle - |11\rangle) = \frac{1}{2}(|00\rangle + i|01\rangle - |10\rangle + i|11\rangle), \\
\text{CNOT}|\otimes\rangle|\odot\rangle &= \frac{1}{2}\text{CNOT}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle) = \frac{1}{2}(|00\rangle - i|01\rangle + |10\rangle + i|11\rangle), \\
\text{CNOT}|\odot\rangle|\otimes\rangle &= \frac{1}{2}\text{CNOT}(|00\rangle + i|01\rangle - i|10\rangle + |11\rangle) = \frac{1}{2}(|00\rangle + i|01\rangle + |10\rangle - i|11\rangle), \\
\text{CNOT}|\odot\rangle|\odot\rangle &= \frac{1}{2}\text{CNOT}(|00\rangle - i|01\rangle - i|10\rangle - |11\rangle) = \frac{1}{2}(|00\rangle - i|01\rangle - |10\rangle - i|11\rangle).
\end{aligned}$$

   All of them are not separable (cannot be factorized into $\cdot \otimes \cdot$), i.e., entangled.