2.111/8.370/18.435 Problem Set 5 Solutions

Due: October 22, 2015

1. **Show that $\tilde{f} : (x, y) \mapsto (x, y \oplus f(x))$ is invertible even is $f(x)$ is not. Exhibit its inverse.**

   The inverse is $\tilde{f}$ itself, i.e., $\tilde{f}^{-1} = \tilde{f}$:

$$\tilde{f}(\tilde{f}(x, y)) = \tilde{f}(x, y \oplus f(x)) = (x, y \oplus f(x) \oplus f(x)) = (x, y).$$

2. **Deutsch-Jozsa algorithm.**

   We start with $|0\rangle^{\otimes n}|1\rangle$ and Hadamard every qubit to obtain

$$|\phi_1\rangle = H \otimes \cdots \otimes H|0\rangle^{\otimes n}|1\rangle = \frac{1}{\sqrt{2^{n+1}}} \sum_{x\in\{0,1\}^n} |x\rangle(|0\rangle - |1\rangle).$$

   Then we apply f oracle:

$$|\phi_2\rangle = f|\phi_1\rangle = \frac{1}{\sqrt{2^{n+1}}} \sum_{x\in\{0,1\}^n} |x\rangle \left(|f(x)\rangle - |1 \oplus f(x)\rangle\right) = \frac{1}{\sqrt{2^{n+1}}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)}|x\rangle(|0\rangle - |1\rangle).$$

   Ignoring the last qubit and Hadamard the first n qubits left, we obtain

$$\frac{1}{2^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} \sum_{y\in\{0,1\}^n} (-1)^{x\cdot y}|y\rangle = \frac{1}{2^n} \sum_{x,y\in\{0,1\}^n} (-1)^{f(x)+x\cdot y}|y\rangle,$$

   where $x \cdot y = x_0y_0 \oplus \cdots x_{n-1}y_{n-1}$ (bitwise). If $f(x)$ is constant, then the $(-1)^{f(x)}$ term comes out as a trivial global phase and the sum yields $|0^n\rangle$ only (constructive interference), since for every $y \neq 0^n$, $\sum_{x\in\{0,1\}^n}(-1)^{x\cdot y} = 0$. Otherwise if $f(x)$ is balanced, the amplitudes of $|0^n\rangle$ now sum to zero (destructive interference), implying that the final measurement never yields $|0^n\rangle$. Or you may directly notice that the probability of measuring $|0^n\rangle$ is $\frac{1}{2^{2n}} \left|\sum_{x\in\{0,1\}^n}(-1)^{f(x)}\right|^2 = 1$ if $f(x)$ is constant, or 0 if $f(x)$ is balanced.

3. **GHZ state.**

   Given $|\mathrm{GHZ}\rangle \equiv (|0\rangle_A|0\rangle_B|0\rangle_C - |1\rangle_A|1\rangle_B|1\rangle_C)/\sqrt{2}$. Recall that σ_x flips $|0\rangle$ and $|1\rangle$:

$$(\sigma_x^A \otimes \sigma_x^B \otimes \sigma_x^C)|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(\sigma_x \otimes \sigma_x \otimes \sigma_x)(|000\rangle - |111\rangle) = \frac{1}{\sqrt{2}}(|111\rangle - |000\rangle) = -|\mathrm{GHZ}\rangle,$$

   or equivalently,

$$\langle\mathrm{GHZ}|\sigma_x^A \otimes \sigma_x^B \otimes \sigma_x^C|\mathrm{GHZ}\rangle = -1.$$

   That is, the GHZ state is the -1 eigenstate of XXX. This result implies if the three players all measure X locally, the product of the outcomes is -1.

   Likewise, recall that $\sigma_y|0\rangle = i|1\rangle$, $\sigma_y|1\rangle = -i|0\rangle$, we obtain

$$\langle\mathrm{GHZ}|\sigma_x^A \otimes \sigma_y^B \otimes \sigma_y^C|\mathrm{GHZ}\rangle = \langle\mathrm{GHZ}|\sigma_y^A \otimes \sigma_x^B \otimes \sigma_y^C|\mathrm{GHZ}\rangle = \langle\mathrm{GHZ}|\sigma_y^A \otimes \sigma_y^B \otimes \sigma_x^C|\mathrm{GHZ}\rangle = 1,$$

i.e., GHZ state is the $+1$ eigenstate of XYY, YXY, YYX: the outcomes of these local measurement multiply to 1.

Remarks: The key point is that the measurement outcomes cannot be predetermined as classical probability distributions, contrasting the local hidden variable theory. This can be seen by multiplying the four equations of measurement outcomes as if they are classical variables ± 1, and obtain $x_A^2 x_B^2 x_C^2 y_A^2 y_B^2 y_C^2 = -1$, which is impossible.