2.111/8.370/18.435 Midterm Exam Solutions

November 13, 2015

1. (a) Fredkin/CSWAP; Toffoli/CCNOT
   (b) Deutsch-Jozsa
   (c) Bell/EPR; GHZ
   (d) entanglement; spooky action at a distance/spukhafte Fernwirkung
   (e) Hope; tensor; slots

   (1 point each)

2. (a) For rotation of $\pi/2$ about axis $\vec{j} = (0, 1/2, \sqrt{3}/2)$, we have that $\sigma_{\vec{j}} = \sigma_y/2 + \sqrt{3}\sigma_z/2$:

$$U\left(\vec{j}, \frac{\pi}{2}\right) = \exp\left(-\frac{i}{2}\frac{\pi}{2}\sigma_{\vec{j}}\right) = I\cos\left(\frac{\pi}{4}\right) - i\sigma_{\vec{j}}\sin\left(\frac{\pi}{4}\right).$$

   Using $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$, and:

$$\sigma_{\vec{j}} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2}i \\ \frac{1}{2}i & -\frac{\sqrt{3}}{2} \end{pmatrix},$$

$$U\left(\vec{j}, \frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 - i\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 + i\frac{\sqrt{3}}{2} \end{pmatrix}.$$

   (5 points)
   (Each instance of algebraic error results in a -1. Not getting the right expression of U, such as missing out a minus sign or the factor of 2 in the exponential, carries a penalty of -3.)

   (b) We have:

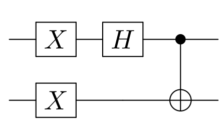
$$|{\rightarrow}\rangle = \frac{1}{\sqrt{2}}(|{\uparrow}\rangle - |{\downarrow}\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

   Doing the matrix multiplication, we get:

$$U\left(\vec{j}, \frac{\pi}{2}\right)|{\rightarrow}\rangle = \frac{1}{4}\begin{pmatrix} 3 - \sqrt{3}i \\ -1 - \sqrt{3}i \end{pmatrix}.$$

   (5 points)
   (Each instance of algebraic error results in a -1. Not getting the right expression for $|{\rightarrow}\rangle$ is a -3 as that is one of the main points of this part. If explicit expression of U in matrix form is not shown in part a, and the final answer in part b is wrong, additional 2 points will be taken off as the student did not show that he/she knows how to compute U.)

3. (a) Consider the following circuit C:



In this circuit, the Pauli-X gate can be implemented by a rotation of π about the x axis, and the Hadamard gate H can be implemented by a rotation of π about the axis $\frac{1}{\sqrt{2}}(1, 0, 1)$.

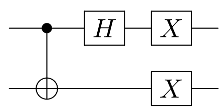
Action of C on computational basis states:

$$
\begin{aligned}
|00\rangle \to |11\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|1\rangle \to \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\Psi_-\rangle, \\
|01\rangle \to |10\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|0\rangle \to \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Phi_-\rangle, \\
|10\rangle \to |01\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|1\rangle \to \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi_+\rangle, \\
|11\rangle \to |00\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|1\rangle \to \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Phi_+\rangle.
\end{aligned}
$$

(8 points)

(Partial credit: 3–5 points if you give a circuit that outputs the Bell states given the computational basis states as inputs, but fail to map the $|00\rangle$ state to the singlet state.)

(b) Taking the inverse of the above circuit, we obtain:



(2 points)

4. (a) Notice that $\sigma_x|\pm\rangle = \pm|\pm\rangle$. One can encode 0 as $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, and 1 as $|\mp\rangle = (|0\rangle \mp |1\rangle)/\sqrt{2}$. (5 points)

   (b) Notice that $UU = |0\rangle\langle 0| \otimes I - |1\rangle\langle 1| \otimes I$, thus $UU|\Psi_\pm\rangle = |\Psi_\mp\rangle$, and $UU|\Phi_\pm\rangle = |\Phi_\mp\rangle$. For example, one can encode 0 as $|\Psi_+\rangle$ and 1 as $|\Phi_+\rangle$. After UU is applied 0 becomes $|\Psi_-\rangle$ and 1 becomes $|\Phi_-\rangle$. That is, if a Bell basis measurement returns $|\Psi_-\rangle$ or $|\Phi_-\rangle$ then one can tell the state is hacked. The recovery operation is simply $UU = \mathrm{diag}(1, 1, -1, -1)$. (5 points)

5. (a) Yes. (0.5 points)

   Since $|\uparrow\rangle|\otimes\rangle$ and $|\leftarrow\rangle|\odot\rangle$ are *orthonormal*, $\langle\psi|\psi\rangle = |\frac{1}{2}|^2 + |-\frac{\sqrt{3}}{2}|^2 = 1$. (0.5 points)

   (b) Relevant state vectors: $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\leftarrow\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, |\otimes\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, |\odot\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$.

   The joint density matrix reads

$$
|\psi\rangle\langle\psi| = \frac{1}{4}|\uparrow\otimes\rangle\langle\uparrow\otimes| - \frac{\sqrt{3}}{4}|\uparrow\otimes\rangle\langle\leftarrow\odot| - \frac{\sqrt{3}}{4}|\leftarrow\odot\rangle\langle\uparrow\otimes| + \frac{3}{4}|\leftarrow\odot\rangle\langle\leftarrow\odot|.
$$

   The reduced states are

$$
\begin{aligned}
\rho_A &= \mathrm{tr}_B|\psi\rangle\langle\psi| \\
&= \frac{1}{4}|\uparrow\rangle\langle\uparrow| + \frac{3}{4}|\leftarrow\rangle\langle\leftarrow| \\
&= \frac{5}{8}|\uparrow\rangle\langle\uparrow| - \frac{3}{8}|\uparrow\rangle\langle\downarrow| - \frac{3}{8}|\downarrow\rangle\langle\uparrow| + \frac{3}{8}|\downarrow\rangle\langle\downarrow| = \begin{pmatrix} \frac{5}{8} & -\frac{3}{8} \\ -\frac{3}{8} & \frac{3}{8} \end{pmatrix}. \qquad \text{(Computational basis)}
\end{aligned}
$$

Second line: cross terms vanish since $\langle\otimes|\odot\rangle = 0$.

$$\begin{aligned}
\rho_B &= \mathrm{tr}_A|\psi\rangle\langle\psi| \\
&= \frac{1}{4}|\otimes\rangle\langle\otimes| - \frac{\sqrt{3}}{4}\langle\leftarrow|\uparrow\rangle|\otimes\rangle\langle\odot| - \frac{\sqrt{3}}{4}\langle\uparrow|\leftarrow\rangle|\odot\rangle\langle\otimes| + \frac{3}{4}|\odot\rangle\langle\odot| \\
&= \frac{1}{4}|\otimes\rangle\langle\otimes| - \frac{\sqrt{6}}{8}|\otimes\rangle\langle\odot| - \frac{\sqrt{6}}{8}|\odot\rangle\langle\otimes| + \frac{3}{4}|\odot\rangle\langle\odot| = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{6}}{8} \\ -\frac{\sqrt{6}}{8} & \frac{3}{4} \end{pmatrix} \qquad (y \text{ basis}) \\
&= \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{6}}{8} & \frac{i}{4} \\ -\frac{i}{4} & \frac{1}{2} + \frac{\sqrt{6}}{8} \end{pmatrix}. \qquad \text{(Computational basis)}
\end{aligned}$$

Notice that the cross terms here do not vanish.
(7 points)
(Different wrong answers for this part reflect different levels of understanding. Partial credits are given accordingly.)

(c) Both reduced density matrices have eigenvalues $\frac{1}{2} \pm \frac{\sqrt{10}}{8}$. (2 points)

6. The joint state can be written as follows:

$$\begin{aligned}
|\psi\rangle|\Psi_-\rangle &= (\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\
&= \frac{1}{\sqrt{2}}(\alpha|001\rangle - \alpha|010\rangle + \beta|101\rangle - \beta|110\rangle).
\end{aligned}$$

(1 point)

We express the first two registers in Bell basis by plugging in

$$|00\rangle = (|\Phi_+\rangle + |\Phi_-\rangle)/\sqrt{2},$$
$$|11\rangle = (|\Phi_+\rangle - |\Phi_-\rangle)/\sqrt{2},$$
$$|01\rangle = (|\Psi_+\rangle + |\Psi_-\rangle)/\sqrt{2},$$
$$|10\rangle = (|\Psi_+\rangle - |\Psi_-\rangle)/\sqrt{2}.$$

We obtain

$$|\psi\rangle|\Psi_-\rangle = \frac{1}{2}|\Phi_+\rangle(\alpha|1\rangle - \beta|0\rangle) + \cdots,$$

(6 points)

meaning that if the first two registers are measured to be $|\Phi_+\rangle$, then the third register is left in the state $\alpha|1\rangle - \beta|0\rangle$. (An alternative method is to act $\langle\Phi_+|$ on $|\psi\rangle|\Psi_-\rangle$, and then normalize the state.) (3 points)