2.111/8.370/18.435 Problem Set 6 Solutions

Due: November 12, 2015

1. **Suppose $f(x) = e^{-i\omega x}$. What is the FFT of f?**

   FFT : $f(x) \mapsto g(y) = \sum_{x=0}^{2^n-1} e^{2\pi i x y/2^n} f(x)$. Plugging in $f(x) = e^{-i\omega x}$:

$$g(y) = \sum_{x=0}^{2^n-1} e^{i(2\pi y/2^n - \omega)x} = \begin{cases} \frac{e^{i(2\pi y - \omega 2^n)}-1}{e^{i(2\pi y/2^n-\omega)}-1} & \omega \neq \frac{2\pi y}{2^n} \\ 2^n & \omega = \frac{2\pi y}{2^n} \end{cases}. \tag{1}$$

   In conclusion:

$$g(y) = \begin{cases} 2^n \delta_{y\tilde{y}} & \omega = \frac{2\pi \tilde{y}}{2^n} \text{ for some } \tilde{y} \\ \frac{e^{i(2\pi y - \omega 2^n)}-1}{e^{i(2\pi y/2^n-\omega)}-1} & \text{Otherwise} \end{cases}. \tag{2}$$

2. **Show that U_{QFT} is unitary.**

   By definition

$$U_{\text{QFT}}|x\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{\frac{2\pi i x y}{N}} |y\rangle \tag{3}$$

   for every x. Then by inspection,

$$U_{\text{QFT}} = \frac{1}{\sqrt{N}} \sum_{x,y=0}^{N-1} e^{\frac{2\pi i x y}{N}} |y\rangle\langle x|. \tag{4}$$

   Thus

$$U_{\text{QFT}}^\dagger = \frac{1}{\sqrt{N}} \sum_{x',y'=0}^{N-1} e^{-\frac{2\pi i x' y'}{N}} |x'\rangle\langle y'|. \tag{5}$$

   Thus

$$U_{\text{QFT}}^\dagger U_{\text{QFT}} = \frac{1}{N} \sum_{x,y,x',y'=0}^{N-1} e^{\frac{2\pi i (xy - x'y')}{N}} |x'\rangle\langle y'|y\rangle\langle x| = \frac{1}{N} \sum_{x,y,x'=0}^{N-1} e^{\frac{2\pi i (x-x')y}{N}} |x'\rangle\langle x|. \tag{6}$$

   Notice that $\sum_{y=0}^{N-1} e^{\frac{2\pi i(x-x')y}{N}} = \delta_{xx'}$. Thus

$$U_{\text{QFT}}^\dagger U_{\text{QFT}} = \frac{1}{N} \sum_{x,y=0}^{N-1} |x\rangle\langle x| = \sum_{x=0}^{N-1} |x\rangle\langle x| = I. \tag{7}$$

   QED.

3. $\text{QFT}^{-1} : \sum_y g(y)|y\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{x,y} e^{-\frac{2\pi i x y}{N}} g(y)|x\rangle$**. Show that** $(\text{QFT}^{-1})(\text{QFT}) = I$**,** $U_{\text{QFT}^{-1}} = U_{\text{QFT}}^\dagger$**.**

   By Problem 2, $U_{\text{QFT}}^\dagger = U_{\text{QFT}}^{-1}$. So the two statements $U_{\text{QFT}^{-1}} U_{\text{QFT}} = I$ and $U_{\text{QFT}^{-1}} = U_{\text{QFT}}^\dagger$ are equivalent. Here we show the latter. The definition of QFT^{-1} indicates that

$$U_{\text{QFT}^{-1}} = \frac{1}{\sqrt{N}} \sum_{x,y=0}^{N-1} e^{-\frac{2\pi i x y}{N}} |x\rangle\langle y| = U_{\text{QFT}}^\dagger, \tag{8}$$

   by Eq. (5). QED.

4. **Show that** $\mathrm{QFT} : |x_n \cdots x_1\rangle \mapsto \frac{1}{2^{n/2}}(|0\rangle + e^{2\pi i\, 0.x_1}|1\rangle) \otimes (|0\rangle + e^{2\pi i\, 0.x_2 x_1}|1\rangle) \otimes \cdots \otimes (|0\rangle + e^{2\pi i\, 0.x_n \cdots x_1}|1\rangle)$**.**

   Nielsen and Chuang p. 218.

5. **Verify the circuit of** QFT**.**

   Nielsen and Chuang p. 218–219.

6. **Show that** $U_{\mathrm{QFT}}|0 \cdots 0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$**.**

   By plugging $x = 0$ into Eq. (3) we directly obtain the desired result.