2.111/8.370/18.435 Problem Set 3 Solutions

Due: October 8, 2015

1. **Show that** $\operatorname{tr}\rho^2 = 1 \Rightarrow \rho = |\psi\rangle\langle\psi|$**.**

   A density matrix can always be decomposed as $\rho = \sum_i p_i |i\rangle\langle i|$ (specctral theorem), where p_i's are real numbers $\in [0,1]$ satisfying $\sum_i p_i = 1$ (a probability distribution), and $\{|i\rangle\}$ is an orthonormal basis ($\langle i|j\rangle = \delta_{ij}$). Then

   $$\rho^2 = \left(\sum_i p_i |i\rangle\langle i|\right)\left(\sum_j p_j |j\rangle\langle j|\right) = \sum_{i,j} p_i p_j \langle i|j\rangle |i\rangle\langle j| = \sum_i p_i^2 |i\rangle\langle i|.$$

   Thus $\operatorname{tr}\rho^2 = \sum_i p_i^2 \leq (\sum_i p_i)^2 = 1$, where the equality holds iff all cross terms vanish. The only possible case is that only one p_i is 1 while others are all 0, corresponding to $\rho = |i\rangle\langle i|$ (idempotent rank-one projectors: $\rho^2 = \rho$), i.e., pure states. (The other direction is obviously true. Otherwise $\operatorname{tr}\rho^2 < 1$, iff ρ is mixed.)

2. $\rho = p_\uparrow |\uparrow\rangle\langle\uparrow| + p_\downarrow |\downarrow\rangle\langle\downarrow|$**. Show that** $\operatorname{tr}\rho^2 = p_\uparrow^2 + p_\downarrow^2$**.**

   Notice that $\langle\uparrow|\downarrow\rangle = 0$:

   $$\rho^2 = p_\uparrow^2 \langle\uparrow|\uparrow\rangle|\uparrow\rangle\langle\uparrow| + p_\downarrow^2\langle\downarrow|\downarrow\rangle|\downarrow\rangle\langle\downarrow| + p_\uparrow p_\downarrow(\langle\uparrow|\downarrow\rangle|\uparrow\rangle\langle\downarrow| + \langle\downarrow|\uparrow\rangle|\downarrow\rangle\langle\uparrow|) = p_\uparrow^2|\uparrow\rangle\langle\uparrow| + p_\downarrow^2|\downarrow\rangle\langle\downarrow|.$$

   Then $\operatorname{tr}\rho^2 = p_\uparrow^2 + p_\downarrow^2$.

3. **In the** σ_z **eigenbasis, write out the 4 by 4 matrices corresponding to:** $\sigma_x^A \otimes I^B$, $I^A \otimes \sigma_x^B$, $\sigma_y^A \otimes I^B$, $I^A \otimes \sigma_y^B$, $\sigma_x^A \otimes \sigma_x^B$, $\sigma_y^A \otimes \sigma_y^B$, $\sigma_z^A \otimes \sigma_x^B$, $\sigma_x^A \otimes \sigma_y^B$, $\sigma_y^A \otimes \sigma_x^B$, $\sigma_y^A \otimes \sigma_z^B$**.**

   $$\sigma_x^A \otimes I^B = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

   $$I^A \otimes \sigma_x^B = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

   $$\sigma_y^A \otimes I^B = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

   $$I^A \otimes \sigma_y^B = \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\sigma_x^A \otimes \sigma_x^B = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_y^A \otimes \sigma_y^B = \begin{pmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_z^A \otimes \sigma_x^B = \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\sigma_x^A \otimes \sigma_y^B = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_y^A \otimes \sigma_x^B = \begin{pmatrix} 0 & -i\sigma_x \\ i\sigma_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_y^A \otimes \sigma_z^B = \begin{pmatrix} 0 & -i\sigma_z \\ i\sigma_z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}.$$

4. **Rewrite $|\Psi_-\rangle$ in the σ_y eigenbasis.**

   Recall that $|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\otimes\rangle + |\odot\rangle), |\downarrow\rangle = -\frac{i}{\sqrt{2}}(|\otimes\rangle - |\odot\rangle)$:

   $$|\Psi_-\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} = -\frac{i}{2\sqrt{2}}[(|\otimes\rangle + |\odot\rangle) \otimes (|\otimes\rangle - |\odot\rangle) - (|\otimes\rangle - |\odot\rangle) \otimes (|\otimes\rangle + |\odot\rangle)] = \frac{i(|\otimes\odot\rangle - |\odot\otimes\rangle)}{\sqrt{2}}.$$

5. **Rewrite $|\Psi_-\rangle$ in $\{|\nearrow\rangle, |\swarrow\rangle\}$.**

   $|\nearrow\rangle, |\swarrow\rangle$ are respectively $|\uparrow\rangle, |\downarrow\rangle$ rotated by $\pi/4$ along the y axis:

   $$|\nearrow\rangle = e^{-i\frac{\pi}{8}\sigma_y}|\uparrow\rangle = \begin{pmatrix} \cos\frac{\pi}{8} \\ \sin\frac{\pi}{8} \end{pmatrix},$$

   $$|\swarrow\rangle = e^{-i\frac{\pi}{8}\sigma_y}|\downarrow\rangle = \begin{pmatrix} -\sin\frac{\pi}{8} \\ \cos\frac{\pi}{8} \end{pmatrix}.$$

   Denote $\cos\frac{\pi}{8} \equiv c, \sin\frac{\pi}{8} \equiv s$, we have $|\uparrow\rangle = c|\nearrow\rangle - s|\swarrow\rangle, |\downarrow\rangle = s|\nearrow\rangle + c|\swarrow\rangle$. Thus

   $$|\Psi_-\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} = \frac{1}{2\sqrt{2}}[(c|\nearrow\rangle - s|\swarrow\rangle) \otimes (s|\nearrow\rangle + c|\swarrow\rangle) - (s|\nearrow\rangle + c|\swarrow\rangle) \otimes (c|\nearrow\rangle - s|\swarrow\rangle)] = \frac{|\nearrow\swarrow\rangle - |\swarrow\nearrow\rangle}{\sqrt{2}}.$$

6. **Rewrite $|\Psi_+\rangle$, $|\Phi_+\rangle$, $|\Phi_-\rangle$ in terms of $\{|\rightarrow\rangle, |\leftarrow\rangle\}$, $\{|\otimes\rangle, |\odot\rangle\}$. Identify the form of correlation along x, y axes.**

By similar calculations:

$$
\begin{aligned}
|\Psi_+\rangle &= \frac{1}{\sqrt{2}}(|{\rightarrow}{\rightarrow}\rangle - |{\leftarrow}{\leftarrow}\rangle) = -\frac{i}{\sqrt{2}}(|{\otimes}{\otimes}\rangle - |{\odot}{\odot}\rangle), &&\text{(correlated in } x, y\text{, anticorrelated in } z\text{)}\\
|\Phi_+\rangle &= \frac{1}{\sqrt{2}}(|{\rightarrow}{\rightarrow}\rangle + |{\leftarrow}{\leftarrow}\rangle) = \frac{1}{\sqrt{2}}(|{\otimes}{\odot}\rangle + |{\odot}{\otimes}\rangle), &&\text{(correlated in } x, z\text{, anticorrelated in } y\text{)}\\
|\Phi_-\rangle &= \frac{1}{\sqrt{2}}(|{\rightarrow}{\leftarrow}\rangle + |{\leftarrow}{\rightarrow}\rangle) = \frac{1}{\sqrt{2}}(|{\otimes}{\otimes}\rangle + |{\odot}{\odot}\rangle). &&\text{(correlated in } y, z\text{, anticorrelated in } x\text{)}
\end{aligned}
$$

All correlations above are maximal.

Remarks: More generally, there is no state that is maximally correlated in every basis. The singlet is maximally anticorrelated in every basis.

7. **What is $\sigma_z \otimes \sigma_z$ acting on the singlet and triplet states?**

$$
\sigma_z \otimes \sigma_z |\Psi_-\rangle = \frac{1}{\sqrt{2}}(\sigma_z|\uparrow\rangle \otimes \sigma_z|\downarrow\rangle - \sigma_z|\downarrow\rangle \otimes \sigma_z|\uparrow\rangle) = -\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = -|\Psi_-\rangle.
$$

Similarly,

$$
\begin{aligned}
\sigma_z \otimes \sigma_z|\Psi_+\rangle &= -\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = -|\Psi_+\rangle,\\
\sigma_z \otimes \sigma_z|\Phi_-\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) = |\Phi_-\rangle,\\
\sigma_z \otimes \sigma_z|\Phi_+\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) = |\Phi_+\rangle.
\end{aligned}
$$

8. **Show that the singlet state is invariant under transformations of the form $U_A \otimes U_B$, where $U_A = e^{-i\frac{\theta}{2}\sigma^A_{\hat{j}}}$, $U_B = e^{-i\frac{\theta}{2}\sigma^B_{\hat{j}}}$ (bilateral rotations).**

By p.5 in PS2, $U_A = U_B = U = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\sigma_{\hat{j}}$, $\hat{j} = j_x\hat{x} + j_y\hat{y} + j_z\hat{z}$, where $j_x^2 + j_y^2 + j_z^2 = 1$. Denote $\cos\frac{\theta}{2} \equiv c, \sin\frac{\theta}{2} \equiv s$. Thus

$$
\begin{aligned}
U \otimes U|\Psi_-\rangle &= \begin{pmatrix} c - isj_z & -isj_x - sj_y \\ -isj_x + sj_y & c + isj_z \end{pmatrix}^{\otimes 2} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}\\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} \cdot & (c - isj_z)(-isj_x - sj_y) & (-isj_x - sj_y)(c - isj_z) & \cdot \\ \cdot & (c - isj_z)(c + isj_z) & (-isj_x - sj_y)(-isj_x + sj_y) & \cdot \\ \cdot & (-isj_x + sj_y)(-isj_x - sj_y) & (c + isj_z)(c - isj_z) & \cdot \\ \cdot & (-isj_x + sj_y)(c + isj_z) & (c + isj_z)(-isj_x + sj_y) & \cdot \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}\\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ c^2 + s^2(j_x^2 + j_y^2 + j_z^2) \\ -c^2 - s^2(j_x^2 + j_y^2 + j_z^2) \\ 0 \end{pmatrix}\\
&= |\Psi_-\rangle.
\end{aligned}
$$

The first and fourth columns of the $U \otimes U$ matrix are omitted on the second line since they do not matter.

Remarks: There is no n-partite maximally entangled state that is invariant under n-lateral rotations, when $n \geq 3$.

9. **Show that the subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ spanned by the triplet states is invariant under the same set of transformations.**

Notice that the singlet and triplet states form an orthonormal complete basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$. Triplets span a 3d subspace T, which is orthogonal to the 1d singlet subspace. Suppose that T is not invariant under $U \otimes U$, i.e., $\exists |\psi\rangle \in T$ s.t. $\langle \Psi_- | U \otimes U |\psi\rangle \neq 0$. By p.8, $U' \otimes U' |\Psi_-\rangle = |\Psi_-\rangle$ for any U' of the same form ($U' = e^{-i\frac{\theta'}{2}\sigma_{\hat{j}'}}$). Let $U' = U^\dagger$ ($\theta' = -\theta, \hat{j}' = \hat{j}$), we obtain $\langle \Psi_- |\psi\rangle \neq 0$: contradiction. Thus T must be invariant under $U \otimes U$.

10. **Calculate partial trace:** $\mathrm{tr}_B |\Psi_\pm\rangle_{AB}\langle\Psi_\pm|$**,** $\mathrm{tr}_B |\Phi_\pm\rangle_{AB}\langle\Phi_\pm|$**?**

$$\mathrm{tr}_B |\Psi_-\rangle_{AB}\langle\Psi_-| = \langle\uparrow|_B(|\Psi_-\rangle_{AB}\langle\Psi_-|)|\uparrow\rangle_B + \langle\downarrow|_B(|\Psi_-\rangle_{AB}\langle\Psi_-|)|\downarrow\rangle_B = \frac{1}{2}(|\uparrow\rangle_A\langle\uparrow| + |\downarrow\rangle_A\langle\downarrow|) = \frac{I_A}{2}.$$

Similarly, you can verify that $\mathrm{tr}_B |\Psi_+\rangle_{AB}\langle\Psi_+| = \mathrm{tr}_B |\Phi_\pm\rangle_{AB}\langle\Phi_\pm| = I/2$.

Remarks: All reduced (local) states of Bell pairs are maximally mixed ($I/2$ in any basis).

11. **Given** $|\psi\rangle_{AB} = \sqrt{\frac{2}{3}}|\uparrow\rangle_A \otimes |\downarrow\rangle_B - \frac{i}{\sqrt{3}}|\downarrow\rangle_A \otimes |\uparrow\rangle_B$**. Calculate** $\mathrm{tr}_B|\psi\rangle_{AB}\langle\psi|, \mathrm{tr}_A|\psi\rangle_{AB}\langle\psi|$**.**

Notice that $\langle\psi| = \sqrt{\frac{2}{3}}\langle\uparrow\downarrow| + \frac{i}{\sqrt{3}}\langle\downarrow\uparrow|$:

$$|\psi\rangle\langle\psi| = \frac{2}{3}|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + \frac{\sqrt{2}i}{3}|\uparrow\downarrow\rangle\langle\downarrow\uparrow| - \frac{\sqrt{2}i}{3}|\downarrow\uparrow\rangle|\uparrow\downarrow\rangle + \frac{1}{3}|\downarrow\uparrow\rangle\langle\downarrow\uparrow|.$$

Thus

$$\mathrm{tr}_B|\psi\rangle_{AB}\langle\psi| = \frac{2}{3}|\uparrow\rangle_A\langle\uparrow| + \frac{1}{3}|\downarrow\rangle_A\langle\downarrow|,$$
$$\mathrm{tr}_A|\psi\rangle_{AB}\langle\psi| = \frac{1}{3}|\uparrow\rangle_B\langle\uparrow| + \frac{2}{3}|\downarrow\rangle_B\langle\downarrow|.$$