PARABOLC EQUATIONS: We consider the heat conduction equation

$$C\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \quad \text{(C being a constant)} \tag{1}$$

Let the (x,t) plane be divided into smaller rectangles by means of the sets of lines

$$x = ih, \qquad i = 0,\ 1,\ 2,\ \ldots$$
$$t = jk, \qquad j = 0,\ \mathrm{I},\ 2,\ \ldots$$

Using the approximations $\quad \dfrac{\partial u}{\partial t}=\dfrac{u_{i,j+1}-u_{i,j}}{k}$

and $\quad \dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{h^2}\left[u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right] \qquad (2)$

equation (1) can be replaced by the finite-difference analogue

$$\frac{C}{k}\left[u_{i,j+1}-u_{i,j}\right]=\frac{1}{h^2}\left[u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right]$$

which can be written as

$$u_{i,j+1}=u_{i,j}+r\left[u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right] \tag{3}$$

Where $r=\dfrac{k}{Ch^2}$.

This formula expresses the unknown function value at the $(i,\ j+1)$th interior point in terms of the known function values and hence it is called the *explicit formula*. It can be shown that this formula is valid only for $0<r\le\dfrac{1}{2}$.

For $r=\dfrac{1}{2}$ equation (3) reduces to $\boxed{u_{i,j+1}=\dfrac{1}{2}\left[u_{i-1,j}+u_{i+1,j}\right]} \qquad$ (3.b)

Which is called *Bender-Schmidt recurrence* relation.

In this formula (3), we have used the function values along the jth row only in the approximation of $\dfrac{\partial^2 u}{\partial x^2}$

Crank and Nicolson proposed a method in 1947 according to which $\dfrac{\partial^2 u}{\partial x^2}$ is replaced by the average of its finite-difference approximations on the jth and $(j+1)$th rows. Thus,

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{2}\left[\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+\frac{u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}}{h^2}\right]$$

and hence equation (1) is replaced by

$$\frac{C}{k}\left[u_{i,j+1}-u_{i,j}\right]=\frac{1}{2h^2}\left[u_{i-1,j}-2u_{i,j}+u_{i+1,j}+u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}\right]$$

which gives on rearranging

$$\boxed{-ru_{i-1,j+1}+(2+2r)u_{i,j+1}-ru_{i+1,j+1}=ru_{i-1,j}+(2-2r)u_{i,j}+ru_{i+1,j}} \tag{4}$$

where $r = \frac{k}{Ch^2}$

On the left side of (4) we have three unknowns and on the right side all the three quantities are known. Equation (4) which is an *implicit* scheme is called *Crank-Nicolson formula* and is convergent for all finite values of r.

If there are N internal mesh points on each row, then formula (4) gives N simultaneous equations for N unknowns in terms of the given boundary values. Similarly, the internal mesh points on all rows can be calculated.

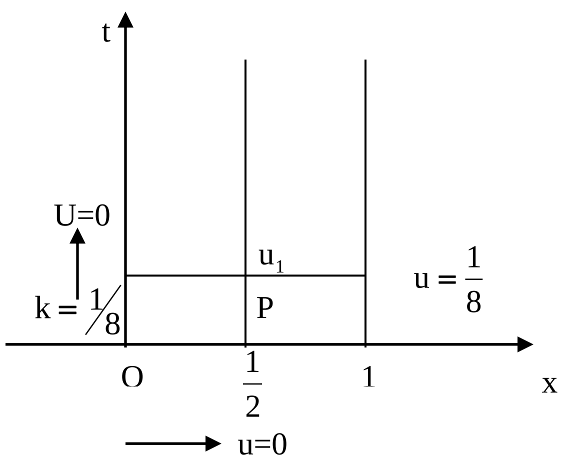
Example 1. We consider the solution of the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x,0) = 0$, $u(0,t) = 0$ and $u(1,t) = t$

(i) We first choose $k = \frac{1}{8}$ and $h = \frac{1}{2}$ so that for $r = \frac{k}{h^2} = \frac{1}{2}$. The Crank-Nicolson scheme (4) now becomes

$$-u_{i-1,j+1} + 6u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + 2u_{i,j} + u_{i+1,j} \qquad \text{(i)}$$

Let the value of u corresponding to $t = \frac{1}{8}$ and $x = \frac{1}{2}$, i.e., at the mesh point P be u_1. Appling the Crank-Nicolson scheme (i) given above at this point, we obtain

$$0 + 6u_1 - \frac{1}{8} = 0 \text{ which gives } \quad u_1 = 0.02083$$



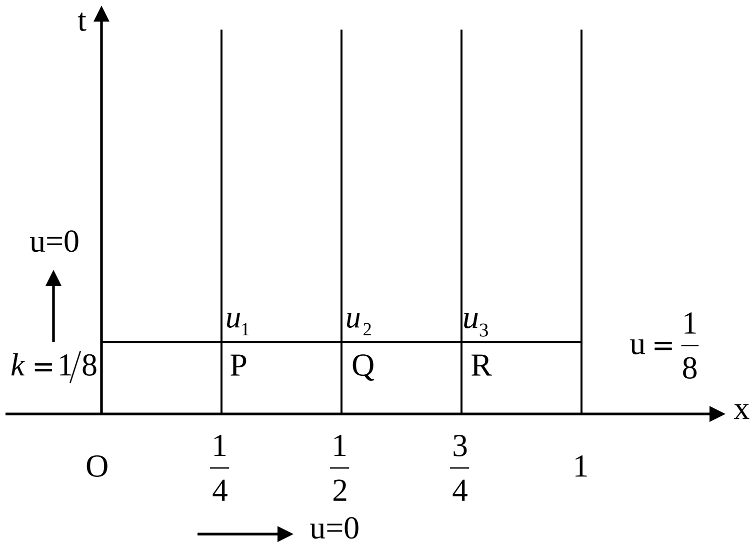
(ii) We now choose $k = \frac{1}{8}$, $h = \frac{1}{4}$ so that for $r = 2$. The Crank-Nicolson scheme corresponding to this value of r is given by

$$-u_{i-1,j+1} + 3u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} - u_{i,j} + u_{i+1,j} \qquad \text{(ii)}$$

Applying the above equation at the mesh point P, we obtain

$$0 + 3u_1 - u_2 = 0$$

i.e., $$3u_1 = u_2$$



Similarly, applying the same equation at the mesh points Q and R, we obtain the two equations1

$$-u_1 + 3u_2 - u_3 = 0$$

$$-u_2 + 3u_3 - \frac{1}{8} = 0$$

We have thus three equations in the three unknowns u_1, u_2, u_3 and the solution is $u_1 = 0.00595$, $u_2 = 0.01785$ and $u_3 = 0.04760$

(iii) As our final choice, we choose $k = \frac{1}{16}$, $h = \frac{1}{4}$ so that $r = 1$. This means that we propose to find our solution for $r = \frac{1}{8}$ in two steps instead of one as in (i) and (ii) above.

The Crank-Nicolson scheme corresponding to this value of r is now

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j} \qquad \text{(iii)}$$

Applying the scheme (iii) above at the mesh points P, Q and R, we obtain the three equations

$$4u_1 - u_2 = 0$$

$$-u_1 + 4u_2 - u_3 = 0$$

and

$$-u_2 + 4u_3 - \frac{1}{16} = 0$$

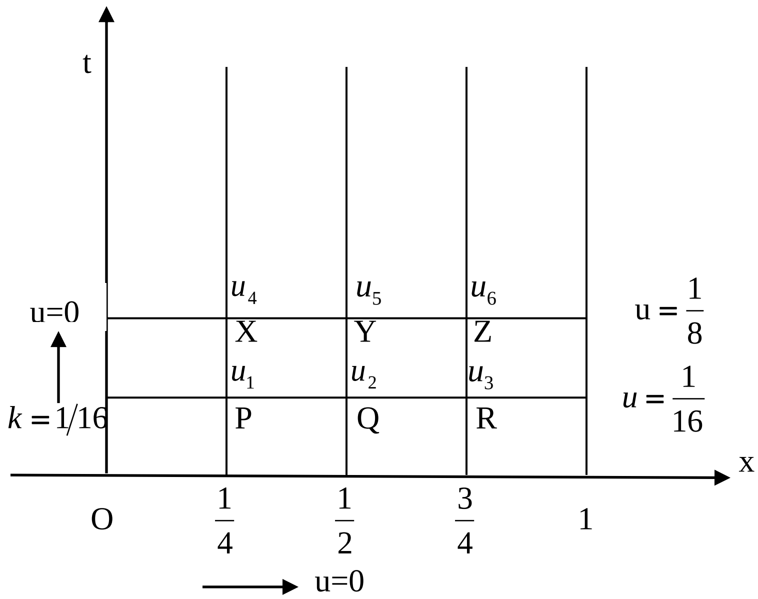
whose solution is $u_1 = \frac{1}{56 \times 16}$, $u_2 = \frac{1}{56 \times 4}$ and $u_3 = \frac{15}{56 \times 16}$

Again, applying the scheme (iii) at each of the mesh points X, Y, Z, we obtain the three equations:

$$4u_4 - u_5 = \frac{1}{4 \times 56}$$

$$-u_4 + 4u_5 - u_6 = \frac{1}{56}$$

and
$$-u_5 + 4u_6 - \frac{1}{8} = \frac{1}{4 \times 56} + \frac{1}{16}$$



The solution is $u_4 = 0.005899$ $u_5 = 0.019132$ $u_6 = 0.052771$

The exact solution of the problem is given by

$$u(x,t) = \frac{1}{6}\left(x^3 - x + 6xt\right) + \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} e^{-n^2\pi^2 r} \sin n\pi x$$

which gives $u\left(\frac{1}{4},\frac{1}{8}\right) = 0.00541$, $u\left(\frac{1}{2},\frac{1}{8}\right) = 0.01878$ and $u\left(\frac{3}{4},\frac{1}{8}\right) = 0.05240$.