The two - point Gauss-Legendre method is given by

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

The three - point Gauss-Legendre method is given by

$$\int_{-1}^{1} f(x)dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

Problem: *Evaluate the integral* $I = \int_0^1 \frac{dx}{1+x}$ *using Gauss- Legendre three-point formula.*

Solution: First we transform the interval [0, 1] to the interval [-1, 1]. Let t= ax+b. We have

-1= b, 1= a + b.

or

a=2, b= -1, and $t = 2x - 1$

$$I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{dt}{t+3}$$

Using Gauss-Legendre three-point rule (corresponding to n=2), we get

$$I = \frac{1}{9}\left[8\left(\frac{1}{0+3}\right) + 5\left(\frac{1}{3+\sqrt{3/5}}\right) + 5\left(\frac{1}{3-\sqrt{3/5}}\right)\right]$$

$$= \frac{131}{189} = 0.693147.$$

The exact solution is $I = \ln 2 = 0.693147.$

Problem: *Evaluate the integral* $I = \int_1^2 \frac{2xdx}{1+x^4}$, *using the Gauss-Legendre 1-point, 2-point and 3-point quadrature rules. Compare with the exact solution*

$$I = \tan^{-1}(4) - (\pi/4).$$

Solution: To use the Gauss-Legendre rules, the interval [1, 2] is to be reduced to [-1, 1]. Writing $x = at + b$, we get

$$1 = -a + b, 2 = a + b$$

Whose solution is b=3/2, a=1/2. Therefore, $x = (t+3)/2, dx = dt/2$ and

$$I = \int_{-1}^{1} \frac{8(t+3)dt}{[16+(t+3)^4]} = \int_{-1}^{1} f(t)dt.$$

Using the 1-point rule, we get

$$I = 2f(0) = 2\left[\frac{24}{16+81}\right] = 0.4948$$

Using the 2-point rule, we get

$$I = \left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right] = 0.3842 + 0.1592 = 0.5434.$$

Using the 3-point rule, we get

$$I = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[5(0.4393) + 8(0.2474) + 5(0.1379)\right] = 0.5406$$

The exact solution is I=0.5404.

Problem: *Evaluate the integral*

$$I = \int_{-1}^{1} (1-x^2)^{3/2} \cos x dx$$

using the Gauss- Chebyshev 1-point, 2-point and 3-point quadrature rules. Evaluate it also using the Gauss-Legendre 3-point formula.

Solution: We write the integral as $I = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx$

where $f(x) = (1-x^2)^2 \cos x$

Using the 1-point Gauss- Chebyshev formula, we get

$I = \pi f(0) = \pi = 3.14159.$

Using the 2- point Gauss-Chebyshev formula, we get

$$I = \frac{\pi}{2}\left[f\left(-\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right)\right] = \frac{\pi}{2}\left[2\left(\frac{1}{4}\right)\cos\left(\frac{1}{\sqrt{2}}\right)\right] = 0.59709$$

Using the 3- point Gauss-Chebyshev formula, we get

$$I = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{\pi}{3}\left[2\left(\frac{1}{16}\right)\cos\left(\frac{\sqrt{3}}{2}\right) + 1\right] = 1.13200$$

Using the 3- point Gauss- Legendre formula, we get (with $f(x) = (1-x^2)^{3/2} \cos x$)

$$I = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right] = \frac{1}{9}\left[10\left(\frac{2}{5}\right)^{3/2} \cos\left(\sqrt{\frac{3}{5}}\right) + 8\right] = 1.08979.$$