ASSIGNMENT - 3

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18MIS7250

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Q. Find the value of $y(1.1)$ by using the fourth-order Runge-Kutta method from the differential eq? $\frac{dy}{dx} = x - y$ with the initial condition $y(1) = 1$. Assume $h = 0.1$.

Sol: $h = 0.1$

given $f(x, y) = x - y$ $\qquad$ $y(1) = 1$

$$y_2 = y(1.1) = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

Using Runge-Kutta 4th order Method.

$$k_1 = h f(x_i, y_i)$$
$$= 0.1 \times f(1.1, 1)$$
$$= 0.1 \times [1.1 - 1] = 0.1 \times 0.1 = 0.01$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1 f\left(1.1 + \frac{0.1}{2}, 1 + \frac{0.01}{2}\right) = 0.1 f[1.15, 1.005]$$
$$= 0.1 [1.15 - 1.005]$$
$$= 0.0145$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1 f\left(1.1 + \frac{0.1}{2}, 1 + \frac{0.0145}{2}\right)$$
$$= 0.1 f(1.15, 1.00725) = 0.1 (1.15 - 1.00725)$$
$$= 0.014275$$

$k_4 = hf(x_1 + \frac{h}{2}, y_1 + k_3)$

$= 0.1\, f(1.1 + 0.1, 1 + 0.014275)$

$= 0.1\, f(1.2, 1.014275)$

$= 0.1\, (1.2 - 1.014275)$

$= 0.185725 \times 0.1$

$= 0.0185725$

Putting k_1, k_2, k_3 and k_4 in the formula

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[0.01 + 2(0.0145) + 2(0.014275) + (0.018572$

$= 1 + \frac{0.0861225}{6}$

$= 1 + 0.01435375$

$= 1.01435375$

2. Solve the steady state temperature distribution equation numerically with a square grid of size $h = \frac{2}{3}$ as shown in fig. $\nabla^2 T = 0$ in the region

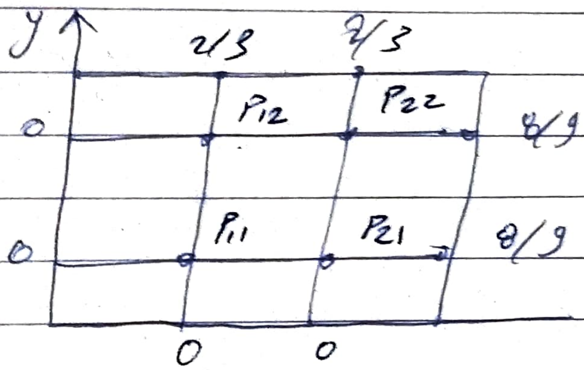
$$0 \le x \le 2 \quad \text{and} \quad 0 \le x \le 2$$

with condition:

$T(0,y) = 0$

$T(2,y) = y(2-y)$ for $0 \le y < 2$ $\qquad T(x,0) = 0$

$$T(x,2) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \end{cases}$$



sol: $h = \frac{2}{3}$

Using Crank Nicolson Formula when $\lambda = 1$

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right]$$

Let $x = \frac{1}{8}$ $\lambda = \frac{k}{h^2} = \frac{1/8}{1 \cdot (2/3)^2} = \frac{7/8}{4/9} = \frac{1}{8} \times \frac{9}{4} = \frac{9}{32}$

Crank Nicolson Formula.

$$\lambda(u_{i+1,j+1} + u_{i-1,j+1}) - 2(\lambda+1)u_{i,j+1} = 2(\lambda-1)u_{i,j} - \lambda(u_{i+1,j} + u_{i-1,j})$$

$$\frac{9}{32}(u_{i+1,j+1} + u_{i-1,j+1}) - 2\left(\frac{9}{32}+1\right)u_{i,j+1} = 2\left(\frac{9}{32}-1\right)u_{i,j} - \frac{9}{32}(u_{i+1,j} + u_{i-1,j})$$

$$\frac{9}{32}(u_{i+1,j+1} + u_{i-1,j+1}) - \frac{41}{16}u_{i,j+1} = -\frac{23}{16}u_{i,j} - \frac{9}{32}(u_{i+1,j} + u_{i-1,j})$$

$$23u_{i,j} + \frac{9}{2}(u_{i+1,j+1} + u_{i-1,j+1}) + \frac{9}{2}(u_{i+1,j} + u_{i-1,j}) = 41u_{i,j+1}$$

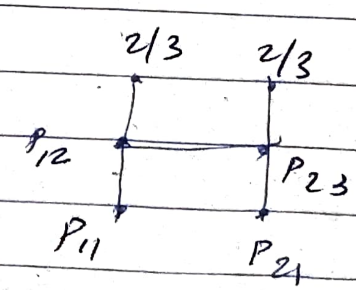
$$\lambda(L+R) - 2(\lambda+1)M = 2(\lambda-1)TM - \lambda(T_L + T_R)$$

Since $T(0,y) = 0$ $\quad 0 < y < 2$

$T(2,y) = y(2-y)$

$T(x,0) = 0$

$$T(x,2) = \begin{cases} x & 0 < x < 1 \\ 2-x & 0 \le x < 2 \end{cases}$$



~~$t = 0$~~ & $x = 0$ at point P_{12}

so the eq^n become

$\cancel{\frac{9}{32} \cdot \frac{8}{9}} \; \cancel{\left(-2\left(\frac{9}{32}+1\right)\right)}$

$$\frac{9}{32}\left(\frac{2}{3}+P_{11}\right) - 2\left(\frac{9}{32}+1\right)P_{12} = 2\left(\frac{9}{32}-1\right)P_{22} - \frac{9}{32}\left(\frac{2}{3}+P_{21}\right)$$

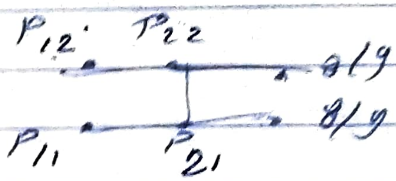
$$6 + 3P_{11} - 82P_{12} = -46P_{22} - 5 + 3P_{21}$$

$$11 + 3P_{11} - 82P_{12} = 3P_{21} - 46P_{22}$$

— (1)

Similarly

$$\frac{9}{32}\left(P_{11}+\frac{8}{9}\right) - 2\left(\frac{9}{32}+1\right)P_{21} = 2\left(\frac{9}{32}-1\right)P_{22} - \frac{9}{32}\left(P_{12}+\frac{8}{9}\right)$$



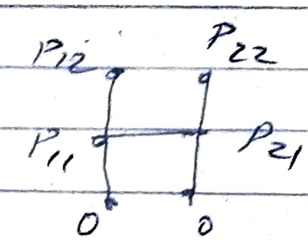
$$9P_{11} + \cancel{8} - 82P_{21} = -46P_{22} - 9P_{12} + \cancel{8}$$

$$9P_{11} - 82P_{21} = -46P_{22} - 9P_{12}$$

— (2)

So now

$$\frac{9}{32}(P_{22}) - 2\left(\frac{9}{32}+1\right)P_{21} = 2\left(\frac{9}{32}-1\right)P_{11} - \frac{9}{32}(P_{12})$$

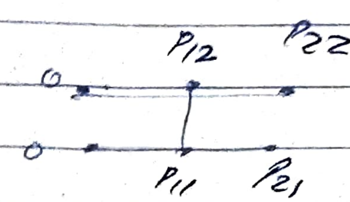


$$9P_{22} - 82P_{21} = -46P_{11} - 9P_{12}$$

— (3)

For this

$$\frac{9}{32}P_{21} - 2\left(\frac{9}{32}+1\right)P_{11} = 2\left(\frac{9}{32}-1\right)P_{12} - \frac{9}{32}(P_{22})$$



$$9P_{21} - 82P_{11} = -46P_{12} - 9P_{22}$$

— (4)

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solving these 4 equations

$$11 + 3P_{11} - 82P_{12} = 3P_{21} - 46P_{22} \quad -①$$

$$9P_{11} - 82P_{21} = -46P_{22} - 9P_{12} \quad -②$$

$$9P_{22} - 82P_{21} = -46P_{11} - 9P_{12} \quad -③$$

$$9P_{21} - 82P_{11} = -46P_{12} - 9P_{22} \quad -④$$

solving.

② & ③

$$9P_{11} - 9P_{22} = -46P_{22} + 46P_{11}$$

$$37P_{22} = 55P_{11}$$

$$\boxed{P_{22} = \frac{55P_{11}}{37}}$$

① & ②

$$11 + 3P_{11} - 82P_{12} = 3P_{21} - 46P_{22}$$

$$9P_{11} - 82P_{21} = -46P_{22} - 9P_{12}$$

$$11 - 6P_{11} = 3P_{21} + 9P_{12} + 82P_{12} - 82P_{21}$$

$$11 - 6P_{11} = -79P_{21} + 91P_{12}$$

$$6P_{11} = 11 - 79P_{21} + 91P_{12}$$

keeping in eq 1 ①

$$11 + 3\left(\frac{11 - 79P_{21} + 91P_{12}}{6}\right) - 82P_{12} = 3P_{21} - 46P_{22}$$

$$22 + 11 - 79P_{21} + 91P_{12} - 164P_{12} = 6P_{21} - 92P_{22}$$

$$33 - 73P_{21} - 64P_{12} = -92P_{22}$$

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P_{22} value $P_{22} = \frac{55}{37} P_{11}$ in eqn (2)

$$9P_{11} - 82P_{21} = -46\left(\frac{55}{37}P_{11}\right) - 9P_{12}$$

$$37(9P_{11} - 82P_{21}) = -46 \times 55 P_{11} - (9 \times 37)P_{12}$$

$$333P_{11} - 3034P_{21} = 2530P_{11} - 333P_{12}$$

$$2197P_{11} = 333P_{12} - 3034P_{21}$$

solving again

$$33 - 73P_{21} - 67P_{12} = -92\left(\frac{55}{37}\right)P_{11}$$

$$2197P_{11} = 333P_{12} - 3034P_{21}$$

} solving

$$1221 - 2701P_{21} - 2479P_{12} = -5060P_{11}$$

$$2863P_{11} + 1221 - 333P_{21} - 2146P_{12} = 0$$

we get

$P_{11} = 1.0896$
$P_{22} = 1.6197$
$P_{12} = 1.1704$
$P_{22} = 1.3341$