

MAT2001 – Numerical Methods for Engineers
Numerical Integration – Gaussian Quadrature
Prof. S. Srinivas

Department of Mathematics, School of Advanced Sciences, VIT-AP University, Amaravati - 522 237.

1 Gaussian-quadrature two-point formula

Consider the numerical evaluation of the integral

$$\int_{-1}^1 f(x) dx.$$

The numerical integration techniques described so far involve equally spaced values of the interval. However, for a fixed number of points the accuracy may be increased, if we do not insist that the points are equidistant. Gauss derived a formula which uses the same number of function values but with different spacing and gives better accuracy.

Gauss' formula is expressed in the form

$$\int_{-1}^1 F(x) dx = \sum_{i=1}^n s_i F(x_i).$$

Thus, assuming that we can compute a specified number of values of the integrand (at arbitrary points), we shall construct a formula by selecting arguments (or abscissas) within the range of integration in order to arrive at a most accurate integration rule. An obvious advantage of this formula is that the 'abscissae and weights' are symmetrical with respect to the middle point of the interval.

2 Gauss-Legendre Two-Point Formula

Let

$$y = f(x), \quad -1 \leq x \leq 1$$

and let

$$\int_{-1}^1 f(x) dx \approx w_1 f(x_1) + w_2 f(x_2).$$

We need to find the weights w_1, w_2 and abscissas x_1, x_2 . We do this by method of undetermined coefficients such that $\int_{-1}^1 f(x) dx$ is exact for cubic polynomials i.e., $f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$. Since four coefficients w_1, w_2, x_1, x_2 are need to be determined we require four conditions. Now,

$$f(x) = 1 \Rightarrow \int_{-1}^1 1 \, dx = 2 = w_1 + w_2$$

$$f(x) = x \Rightarrow \int_{-1}^1 x \, dx = 0 = w_1 x_1 + w_2 x_2$$

$$f(x) = x^2 \Rightarrow \int_{-1}^1 x^2 \, dx = 2/3 = w_1 x_1^2 + w_2 x_2^2$$

$$f(x) = x^3 \Rightarrow \int_{-1}^1 x^3 \, dx = 0 = w_1 x_1^3 + w_2 x_2^3$$

Solving the above four equations, we obtain

$$w_1 = w_2 = 1; \quad x_1 = -\frac{1}{\sqrt{3}} = -0.5773502692, \quad x_2 = \frac{1}{\sqrt{3}} = 0.5773502692$$

So,

$$\int_{-1}^1 f(x) \, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \approx f(-0.57735) + f(0.57735) = G_2(f)$$

Note: Above formula has degree of precession $n = 3$.

If $f \in C^4[-1, 1]$ then

$$\int_{-1}^1 f(x) \, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) + E_2(f),$$

where $E_2(f) = \frac{f^{(4)}(c)}{135}$, here $f^{(4)}(c) = \max_{-1 \leq x \leq 1} |f^{(4)}(x)|$

3 Gauss-Legendre Three-Point Formula

Let

$$y = f(x), \quad -1 \leq x \leq 1$$

and let

$$\int_{-1}^1 f(x) \, dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3).$$

Following the same procedure as above, we obtain

$$\begin{aligned} \int_{-1}^1 f(x) \, dx &\approx \frac{1}{9} \left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) \right] \\ &\approx \frac{1}{9} [5f(-0.774597) + 8f(0) + 5f(0.774597)] = G_3(f). \end{aligned}$$

$G_3(f)$ has a degree of precession $n = 5$. For $f \in C^6[-1, 1]$ error is $E_3(f) = \frac{f^{(6)}(c)}{15750}$ where $f^{(6)}(c) = \max_{-1 \leq x \leq 1} |f^{(6)}(x)|$.

4 Gauss-Legendre Translation

Let

$$\int_a^b f(t) dt$$

be the integral to be evaluated. We can apply Gauss-Legendre formula by transforming the interval $[a, b]$ to $[-1, 1]$. This can be done by the change of variable

$$t = \frac{b+a}{2} + \frac{b-a}{2}x \quad \text{and} \quad dt = \frac{b-a}{2} dx.$$

So,

$$\int_a^b f(t) dt = \int_{-1}^1 f\left(\frac{b+a}{2} + \frac{b-a}{2}x\right) \frac{b-a}{2} dx = \frac{b-a}{2} \int_{-1}^1 g(x) dx$$

where $g(x) = f\left(\frac{b+a}{2} + \frac{b-a}{2}x\right)$.

Problem. Evaluate $\int_{-1}^1 \frac{1}{x+2} dx$ by Gauss-Legendre two-point formula

Solution. Given integral value by Gauss-Legendre two-point formula is:

$$G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.70291 + 0.38800 = 1.09091$$

Now, evaluate the same integral by Trapezoidal as well as Simpons's 1/3-rule, we obtain

$$T_2(f) = \frac{2}{2} [f(-1) + f(1)] = 1 + 0.33333 = 1.33333$$

$$S_2(f) = \frac{1}{3} [f(-1) + 4f(0) + f(1)] = \frac{1}{3} \left[1 + 2 + \frac{1}{3}\right] = 1.11111$$

Exact answer is: $\ln(3) - \ln(2) \approx 1.09861$. Observe that the Gauss-Legendre two-point formula is very close the exact answer compared to the values obtained by Trapezoidal and Simpons's rules.

Problem. Evaluate $\int_{-1}^1 5x^4 dx$ using Gauss-Legendre three-point formula

Solution. The given integral value using Gauss-Legendre three-point formula is:

$$G_3(f) = \frac{1}{9} \left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) \right] = \frac{1}{9} [5(5)(0.6)^2 + 0 + 5(5)(0.6)^2] = \frac{18}{9} = 2$$

Problem. Evaluate $\int_1^5 \frac{1}{t} dt$ by Gauss-Legendre three-point formula.

Solution. The first step is to change the limits. Using the change of variable $t = \frac{1}{2}[6 + 4x] = 3 + 2x$, we have $dt = 2dx$ and that,

$$\int_1^5 \frac{1}{t} dt = \int_{-1}^1 \frac{1}{3+2x} (2dx) = 2 \int_{-1}^1 f(x) dx$$

Now, applying Gauss-Legendre three-point formula, we have

$$\begin{aligned} 2 \int_{-1}^1 f(x) dx &= \frac{2}{9} \left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) \right] \\ &= \frac{2}{9} [3.4464 + 2.66671 + 1.0991] = 1.6027 \end{aligned}$$

Problem. Evaluate $\int_0^{\pi/2} \sin t dt$ by Gaussian quadrature of two-point formula.

Solution. Using the change of variables, we have $t = \frac{1}{2}[\frac{\pi}{2} + \frac{\pi}{2}x] = \frac{\pi}{4}(x+1)$; $dt = \frac{\pi}{4} dx$. So,

$$\begin{aligned} \int_0^{\pi/2} \sin t dt &= \frac{\pi}{4} \int_{-1}^1 \sin\left(\frac{\pi}{4}(x+1)\right) dx \\ &= \frac{\pi}{4} \int_{-1}^1 g(x) dx \quad \text{where } g(x) = \sin\left(\frac{\pi}{4}(x+1)\right) \\ &= \frac{\pi}{4} \left[g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) \right] \approx 0.998473 \end{aligned}$$

Problem. Evaluate $I = \int_0^1 x dx$ by two-point and three-point

Solution. Let $x = \frac{1}{2}(1+t)$ and $dx = \frac{dt}{2}$.

Two-point formula:

$$\begin{aligned} I = \int_0^1 x dx &= \frac{1}{2} \int_{-1}^1 \frac{1}{2}(1+t) dt = \frac{1}{4} \int_{-1}^1 f(t) dt, \quad \text{where } f(t) = 1+t \\ &= \frac{1}{4} \left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \right] = \frac{1}{4} [0.4226 + 1.5774] = 0.5 \end{aligned}$$

Three-point formula:

$$\begin{aligned} I = \int_0^1 x dx &= \frac{1}{4} \int_{-1}^1 (1+t) dt \\ &= \frac{1}{4} \cdot \frac{1}{9} [5(1 - 0.774597) + 8(1+0) + 5(1 + 0.774597)] \\ &= \frac{1}{4} \cdot \frac{1}{9} [5(0.2254) + 8(1) + 5(1.7746)] = 0.5. \end{aligned}$$