Boundary value problems: Practical applications give rise to many such problems and we here discuss two-point linear boundary value problems of the following types:

(i) $\frac{d^2y}{dx^2}+\lambda(x)\frac{dy}{dx}+\mu(x)y=\gamma(x)$ with the conditions $y(x_0)=a, y(x_n)=b$

(ii) $\frac{d^4y}{dx^4}+\lambda(x)y=\mu(x)$ with the conditions $y(x_0)=y'(x_0)=a, y(x_n)=y'(x_n)=b$

As a matter of fact, there exist many numerical methods for solving such boundary value problems but the method of finite differences is most commonly used. We shall explain this method in the next article.

Finite-difference method: In this method, the derivative s existing in the differential equation and the boundary conditions are replaced by finite difference approximations and the resulting linear system of equations are solved by any standard method. These roots are the values of the required solution at the pivotal points.
The finite difference approximations to the various derivatives are obtained as follows:
Let $y(x)$ and its derivatives be single valued continuous functions of x, then Taylor's expansion gives:

$$y(x+h)=y(x)+hy'(x)+\frac{h^2}{2!}y''(x)+\frac{h^3}{3!}y'''(x)+\ldots \tag{1}$$

$$y(x-h)=y(x)-hy'(x)-\frac{h^2}{2!}y''(x)-\frac{h^3}{3!}y'''(x)+\ldots \tag{2}$$

Equation (1) then gives

$$y'(x)=\frac{1}{h}[y(x+h)-y(x)]-\frac{h}{2}y''(x)-\ldots$$

$$\Rightarrow \quad y'(x)=\frac{1}{h}[y(x+h)-y(x)]+O(h)$$

This is the forward difference approximation of $y'(x)$ with an error of the order h.
Similarly (2) gives

$$y'(x)=\frac{1}{h}[y(x)-y(x-h)]+O(h)$$

This is the backward difference approximation of $y'(x)$ with an error of the order h^2. Clearly central difference approximation of $y'(x)$ is better than the forward or difference approximations and hence must be preffered.
Further adding (1) and (2), we readily obtain

$$y''(x)=\frac{1}{2h}[y(x+h)-2y(x)+y(x-h)]+O(h^2)$$

Which is the central difference approximation of $y''(x)$. Similarly, we can derive central difference approximations to higher derivatives.
To sum up, the working expressions for the central difference approximations to the four derivatives of y_1 are given below:

$$y_i'=\frac{1}{2h}(y_{i+1}-y_{i-1}) \tag{a}$$

$$y_i'' = \frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) \tag{b}$$

$$y_i''' = \frac{1}{2h^3}\left(y_{i+2} - 2y_{i+1} + 2y_{i-1} + y_{i-2}\right) \tag{c}$$

$$y^{iv}{}_i = \frac{1}{h^4}\left(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right) \tag{d}$$

Problem 1: Determine value of y at the pivotal points of the interval (0,1), if y satisfies the boundary value problem $y^{iv} + 81y = 81x^2$, $y(0) = y(1) = y''(0) = y''(1) = 0$ (take n=3)

Sol: Let $h = 1/3$ and the pivotal points are $x_0 = 0,\ x_1 = (1/3),\ x_2 = (2/3),\ x_3 = 1$.

The corresponding y-values are $y_0(=0), y_1, y_2, y_3(=0)$.

Then replacing y^{iv} by its central difference approximation, the differential equation reduces to

$$\frac{1}{h^4}\left(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right) + 81y_i = 81x_i^2$$

$$\Rightarrow y_{i+2} - 4y_{i+1} + 7y_i - 4y_{i-1} + y_{i-2} = x_i^2,\ i = 1,2$$

Putting $i = 1$, $i = 2$; $y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = (1/9)$ and $y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = (4/9)$

Then using $y_0 = y_3 = 0$, we readily get

$$-4y_2 + 7y_1 + y_{-1} = (1/9) \tag{a}$$

And $y_4 + 7y_2 - 4y_1 = (4/9)$ (b)

Then knowing the conditions $y_0'' = y_3'' = 0$, we have $y'' = \dfrac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right)$

Putting $i = 0$, $y_0'' = 9\left(y_1 - 2y_0 + y_{-1}\right) \Rightarrow y_{-1} = -y_1 \quad (\because y_0 = y_0'' = 0)$ (c)

Putting $i = 3$, $y_3'' = 9\left(y_4 - 2y_3 + y_2\right) \Rightarrow y_4 = -y_2 \quad (\because y_3 = y_3'' = 0)$ (d)

Then using (c), the equation (a) reduces to

$$-4y_2 + 6y_1 = (1/9) \tag{e}$$

While using (d), the equation (b) reduces to

$$6y_2 - 4y_1 = (4/9) \tag{f}$$

Solving (e) and (f), we get

$y_1 = (11/90)$ and $y_2 = 7/45 \Rightarrow y(1/3) = 0.12222$ and $y(2/3) = 0.1556$.

Problem 2: Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$

Sol: We divide the interval (0,1) into four sub-intervals so that $h = (1/4)$ and the pivotal points are $x_0 = 0,\ x_1 = (1/4),\ x_2 = (1/2),\ x_3 = (3/4),\ x_4 = 1$

Now the differential equation is approximated as $\dfrac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) = x_i + y_i$

$\Rightarrow 16y_{i+1} - 33y_i + 16y_{i-1} = x_i;\ i = 1,2,3$. But $y_0 = y_4 = 0$

Hence we get $16y_2 - 33y_1 = (1/4);\quad 16y_3 - 33y_2 + 16y_1 = (1/2);\quad 33y_3 + 16y_2 = (3/4)$

$\Rightarrow y_1 = -.03488,\quad y_2 = -.05632,\quad y_3 = -.05003.$

Problem 3: Using finite difference method, solve $y'' - 64y + 10 = 0$ given $y(0) = y(1) = 0$, subdividing the interval into (i) 4 equal parts (ii) 2 equal parts.

Solution: Since n = 4 and $nh = 1$, h = ¼

Converting the differential equation into difference equation we have,

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} - 64y_i + 10 = 0$$

i.e. $$y_{i+1} + y_{i-1} - \left(2 + 64h^2\right) y_i + 10h^2 = 0$$

Putting h = ¼, this becomes, $y_{i+1} - 6y_i + y_{i-1} = \frac{-5}{8}$. (1)

where i = 1, 2, 3, $y(0) = y(1) = 0$

Hence using, $y_0 = 0$, $y_4 = 0$, we get

$$y_2 - 6y_1 = -5/8 \quad (2)$$

$$y_3 - 6y_2 + y_1 = -5/8 \quad (3)$$

$$-6y_3 + y_2 = -5/8 \quad (4)$$

(2)-(4) gives, $6(y_3 - y_1) = 0$

i.e. $y_1 = y_3$

(3) becomes, $2y_3 - 6y_2 = -5/8$ (5)

$$-6y_3 + y_2 = -5/8$$

Eliminating y_3 we have, $-17y_2 = -5/2$

$$y(0.5) = y_2 = 5/34 = 0.1471$$

Hence (2) reduces to $6y_1 = \frac{5}{34} + \frac{5}{8} = \frac{105}{36}$

Therefore $y_3 = y_1 = \frac{35}{272} = 0.1287$. Exact value of y_1 is 0.1505

When n = 2, y_2 is 0.1389.

Problem 4: Solve $y'' - xy = 0$ given $y(0) = -1$, $y(1) = 2$ by finite difference method using n = 2.

Solution: If n = 2, then $h = 1/2$ since range is (0, 1).

The nodal points are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

The differential equation reduces to, $\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} - xy_i = 0$

i.e. $$y_{i+1} + -\left(2 + h^2 x_i\right) y_i + y_{i-1} = 0$$

where $i = 1$, $h = 1/2$, $x_i = 0.5$, $y_0 = -1$, $y_2 = 2$

$$y_2 - \left(2 + \frac{1}{8}\right) y_1 + y_0 = 0$$

$$2 - \frac{17}{8} y_1 - 1 = 0$$

$$y_1 = \frac{8}{17} = 0.4706.$$