Explicit Solution of the One-Dimensional Heat-Conduction Equation:

Problem: Use the explicit method to solve for the temperature distribution of the long, thin rod with a length of 10 cm and the following values: $k' = 0.49$ cal/(s.cm. °C). $\Delta x = 2$ cm, and $\Delta t = 0.1$s. At t = 0, the temperature of the rod is zero and the boundary conditions are fixed for all times at T(0) = 100 °C and T(10) =50 °C. Note that the rod is aluminum with C = 0.2174 cal/(g . °C) and ρ = 2.7g/cm^3. Therefore, $k_* = 0.49/(2.7 \bullet 0.2174) = 0.835$ cm^2/s and $\lambda = 0.835(0.1)/(2)^2 = 0.020875$.

($\frac{\partial u}{\partial t} = k_* \frac{\partial^2 u}{\partial x^2}$; $k_* = \frac{k}{c\rho}$; k_* is the diffusivity of the substance; k=Coeff of thermal conductivity, C is the heat capacity of the material, ρ is the density of the material))

Applying Eq. $\boxed{T_i^{l+1} = T_i^l + \lambda (T_{i+1}^l - 2T_i^l + T_{i-1}^l)}$, where $\lambda = k\Delta t / (\Delta x)^2$ gives the following value at t = 0.1 s for the node at x = 2cm:

$T_1^1 = 0 + 0.020875[0 - 2(0) + 100] = 2.0875$

At the other interior points, x = 4,6, and 8 cm, the results are

$T_2^1 = 0 + 0.020875[0 - 2(0) + 0] = 0$

$T_3^1 = 0 + 0.020875[0 - 2(0) + 0] = 0$

$T_4^1 = 0 + 0.020875[50 - 2(0) + 0] = 1.0438$

At t = 0.2 s, the values at the four interior nodes are computed as

$T_1^2 = 2.0875 + 0.020875[0 - 2(2.0875) + 100] = 4.0878$

$T_2^2 = 0 + 0.020875[0 - 2(0) + 2.0875] = 0.043577$

$T_3^2 = 0 + 0.020875[1.0438 - 2(0) + 0] = 0.021788$

$T_4^2 = 1.0438 + 0.020875[50 - 2(1.0438) + 0] = 2.0439$

The computation is continued, and the results at 3-s intervals are depicted in Figure-1. The general rise in temperature with time indicates that the computation captures the diffusion of heat from the boundaries into the bar.

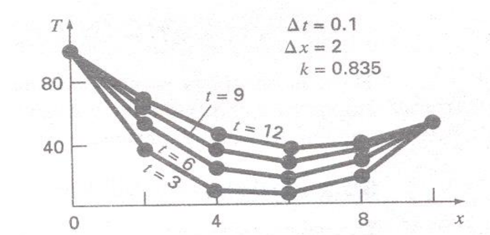

Figure 1

(Temperature distribution in a long rod as computed With the explicit method described)

Convergence and Stability:

Convergence means that as Δx and Δt approach zero, the results of the finite-difference technique approach the true solution. *Stability* means that errors at any stage of the computation are not amplified but are attenuated as the computation progresses. It can be shown that the explicit method is both convergent and stable if $\lambda \le 1/2$, or

$$\Delta t \le \frac{1}{2}\frac{\Delta x^2}{k} \qquad \text{... (1)}$$

In addition, it should be noted that setting $\lambda \le 1/2$ could result in a solution in which errors do not grow, but oscillate. Setting $\lambda \le 1/4$ ensures that the solution will not oscillate. It is also known that setting $\lambda = 1/6$ tends to minimize truncation error

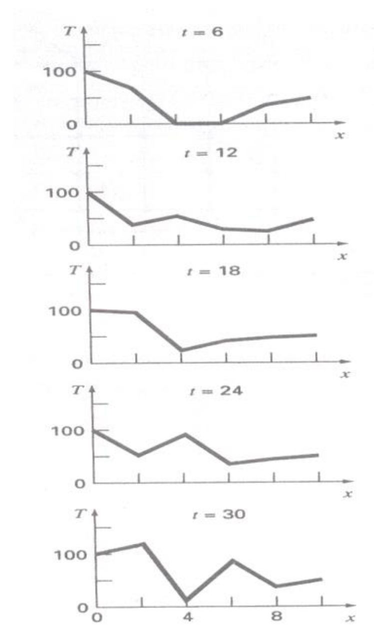

Figure 2

An illustration of instability with λ =0.735

Fig.2 is an example of instability caused by violating Eq.(1). This plot is for the same problem discussed above but with $\lambda = 0.735$, which is considerably greater than 0.5. As in Fig.2, the solution undergoes progressively increasing oscillations. This situation will continue to deteriorate as the computation continues.

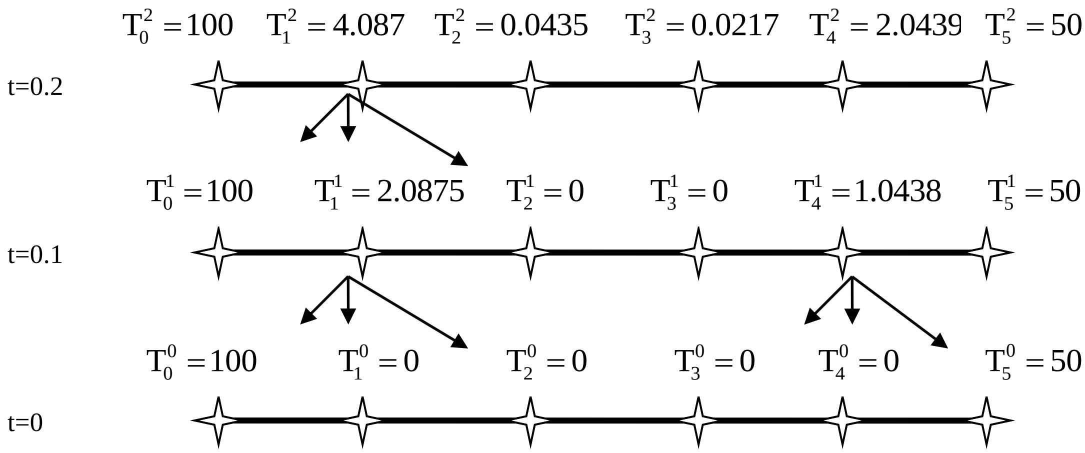
$T_0^2 = 100$ $T_1^2 = 4.087$ $T_2^2 = 0.0435$ $T_3^2 = 0.0217$ $T_4^2 = 2.0439$ $T_5^2 = 50$
t=0.2
$T_0^1 = 100$ $T_1^1 = 2.0875$ $T_2^1 = 0$ $T_3^1 = 0$ $T_4^1 = 1.0438$ $T_5^1 = 50$
t=0.1
$T_0^0 = 100$ $T_1^0 = 0$ $T_2^0 = 0$ $T_3^0 = 0$ $T_4^0 = 0$ $T_5^0 = 50$
t=0

Figure 1 (b)

SAQ

Problem1: Use Crank-Nicolson method to solve the same problem (discussed above)

Hint / Intermediate steps: $\begin{bmatrix} 2.04175 & -0.020875 & 0 & 0 \\ -0.020875 & 2.04175 & -0.020875 & 0 \\ 0 & -0.020875 & 2.04175 & -0.020875 \\ 0 & 0 & -0.020875 & 2.04175 \end{bmatrix} \begin{Bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{Bmatrix} = \begin{Bmatrix} 4.175 \\ 0 \\ 0 \\ 2.0875 \end{Bmatrix}$

Which can be solved for the temperatures at t=0.1 s: $\begin{Bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{Bmatrix} = \begin{Bmatrix} 2..0450 \\ 0.0210 \\ 0.0107 \\ 1.0225 \end{Bmatrix}$

To solve for the temperatures at t=0.2 s, the right-hand-side vector must be changed to: $\begin{Bmatrix} 8.1801 \\ 0.0841 \\ 0.0427 \\ 4.0901 \end{Bmatrix}$

The simultaneous equations can then be solved for $\begin{Bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{Bmatrix} = \begin{Bmatrix} 4.0073 \\ 0.0826 \\ 0.0422 \\ 2.0036 \end{Bmatrix}$

Problem 2: (a) Find the region in the xy-plane in which the following equation is Hyperbolic:
$[(x-y)^2-1]U_{xx} + 2U_{xy} + [(x-y)^2-1]U_{yy}=0.$

(b) Classify the following equation: $x^2 \dfrac{\partial^2 u}{\partial x^2} + (1-y^2)\dfrac{\partial^2 u}{\partial y^2} = 0$; $-\infty < x < \infty$; $-1<y<1$.

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* The teacher if he is indeed wise does not teach bid you to enter the house of wisdom but leads you to the threshold of your own mind." ***- Kahlil Gilbran***
* "The mediocre teacher tells. The good teacher explains. The superior teacher demonstrates. The great teacher inspires." ***- William Arthur Ward***.
* "What I hear, I forget.
  What I see, I remember.
  What I do, I understand." ***-- Confucius***