Course code: CSE3008

Course Title: Introduction to Machine Learning

Module 4: Expectation Maximization

Expectation Maximization

- EM algorithm provides a general approach to learning in presence of unobserved variables.
- In many practical learning settings, only a subset of relevant features or variables might be observable. – Eg: Hidden Markov, Bayesian Belief Networks
- Estimation: Estimate the expectation from some random data
- Maximization: Whatever is estimated should be maximized to find the best result.
- From given data EM learn a theory which tells that how each example to be classified and how to predict the feature value of each class.

Expectation Maximization

Suppose you have 2 coins, A and B, each with a certain bias of landing heads, θ_A , θ_B.

Given data sets $X_A = \{x_{1,A}, \ldots, x_{m_A,A}\}$ and $X_B = \{x_{1,B}, \ldots, x_{m_B,B}\}$

Where $x_{i,j} = \{ \begin{matrix} 1\,; if\ heads \\ 0\,; otherwise \end{matrix}$

No hidden variables – easy solution. $\theta_j = \frac{1}{m_j} \sum_{i=1}^{m_j} x_{i,j}$; sample mean

Example

- Assume that we have two coins, C1 and C2
- Assume the bias of C1 is θ_1

  (i.e., probability of getting heads with C1)
- Assume the bias of C2 is θ_2

  (i.e., probability of getting heads with C2)

- We want to find θ_1, θ_2 by performing a number of trials

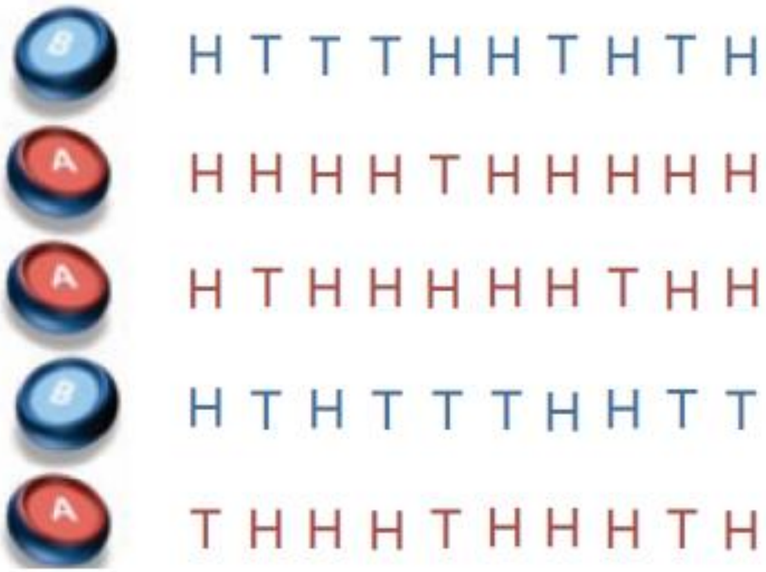
  (i.e., coin tosses)

Example

First experiment

- We choose 5 times one of the coins.

- We toss the chosen coin 10 times



$$\theta_1 = \frac{number\ of\ heads\ using\ C1}{total\ number\ of\ flips\ using\ C1}$$

$$\theta_2 = \frac{number\ of\ heads\ using\ C2}{total\ number\ of\ flips\ using\ C2}$$

Example


H T T T H H T H T H

H H H H T H H H H H

H T H H H H H T H H


H T H T T T H H T T

T H H H T H H H T H

Coin A	Coin B
	5 H, 5 T
9 H, 1 T	
8 H, 2 T	
	4 H, 6 T
7 H, 3 T	
24 H, 6 T	9 H, 11 T

$$\theta_1 = \frac{24}{24+6} = 0.8$$

$$\theta_2 = \frac{9}{9+11} = 0.45$$

Example with Hidden Variable

- What if you were given the same dataset of coin flip results, but no coin identities defining the datasets?

Here: $X = \{x_1, \dots x_m\}$; the observed variable

$$Z = \begin{Bmatrix} z_{1,1} & \cdots & z_{m,1} \\ \cdots & z_{i,j} & \cdots \\ z_{1,k} & \cdots & z_{m,k} \end{Bmatrix} \quad \text{where } z_{i,j} = \begin{Bmatrix} 1\,; if\ x_i\ is\ from\ j^{th}\ coin \\ 0; otherwise \end{Bmatrix}$$

But Z is not known. (Ie: ‘hidden’ / ‘latent’ variable)

Example with Hidden Variable

Assume a more challenging problem

H T T T H H T H T H

H H H H T H H H H H

H T H H H H H T H H

H T H T T T H H T T

T H H H T H H H T H

- We do not know the identities of the coins used for each set of tosses (we treat them as hidden variables).

Example with Hidden Variable

0) Initialize some arbitrary hypothesis of parameter values (θ):

$\theta = \{ \theta_1, \dots, \theta_k\}$ coin flip example: $\theta = \{\theta_A, \theta_B\} = \{0.6, 0.5\}$

1) Expectation (E-step)

$$E[z_{i,j}] = \frac{p(x = x_i | \theta = \theta_j)}{\sum_{n=1}^{k} p(x = x_i | \theta = \theta_n)}$$

2) Maximization (M-step)

$$\theta_j = \frac{\sum_{i=1}^{m} E[z_{i,j}] x_i}{\sum_{i=1}^{m} E[z_{i,j}]}$$

If $z_{i,j}$ is known:

$$\theta_j = \frac{\sum_{i=1}^{m_j} x_i}{m_j}$$

Example with Hidden Variable

θ = P(up),   1-θ = P(down)

Observe:

Likelihood of the observation sequence depends on θ:

$$
\begin{aligned}
l(\theta) &= \theta(1-\theta)\theta(1-\theta)\theta\theta\theta\theta\theta\theta\theta\theta \\
&= \theta^8(1-\theta)^2
\end{aligned}
$$

Example with Hidden Variable

$L(C)=\Theta^k (1-\Theta)^{n-k}$

Likelihood For first coin Flips

$L(A)= 0.6^5 (1-0.6)^{10-5} = 0.0007963$

$L(B)= 0.5^5 (1-0.5)^{10-5} = 0.0009766$

$P(A)=L(A)/L(A)+L(B) = 0.0007963/(0.0007963+0.0009766)=0.45$

$P(B)=L(B)/L(A)+L(B) = 0.0009766/(0.0007963+0.0009766) = 0.55$

Estimate **Likely No of Heads and Tails for First Toss**

For A: H = 0.45*5 = 2.2, T = 0.45*5 = 2.2

For B: H = 0.55*5 = 2.8, T = 0.55*5 = 2.8

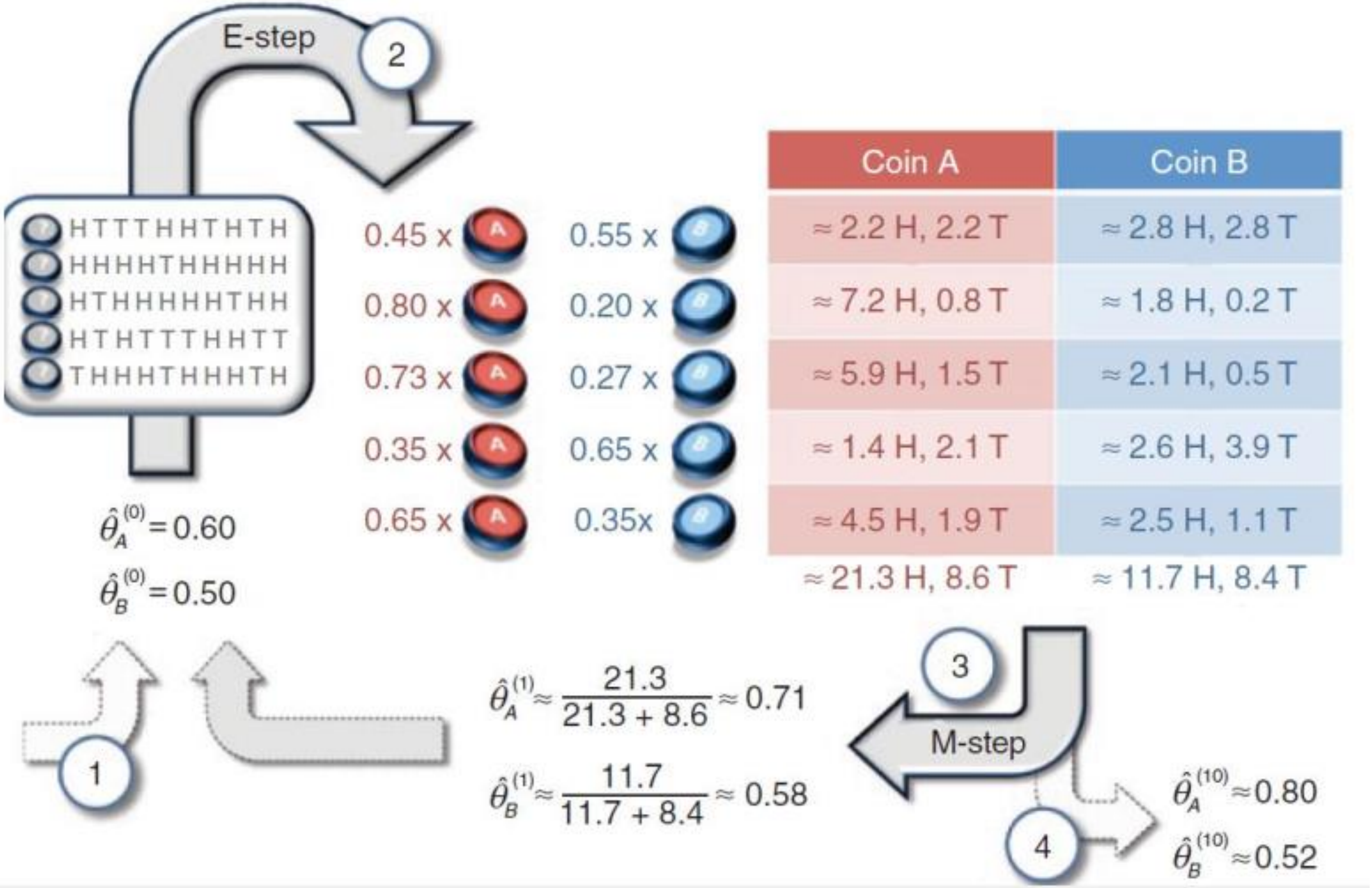
Example with Hidden Variable

In similar fashion find probability of all coins with all flips. It will be as follows:

L(H): Likely no of heads   L(T): Likely no of tails

											Iteration 1->:		Coin A		Coin B	
											P(A)	P(B)	L(H)	L(T)	L(H)	L(T)
B	H	T	T	T	H	H	T	H	T	H	0.45	0.55	2.2	2.2	2.8	2.8
A	H	H	H	H	T	H	H	H	H	H	0.80	0.20	7.2	0.8	1.8	0.2
A	H	T	H	H	H	H	H	T	H	H	0.73	0.27	5.9	1.5	2.1	0.5
B	H	T	H	T	T	T	H	H	T	T	0.35	0.65	1.4	2.1	2.6	3.9
A	T	H	H	H	T	H	H	H	T	H	0.65	0.35	4.5	1.9	2.5	1.1

Example with Hidden Variable



Expectation Maximization

1. Choose starting parameters
2. Estimate probability using these parameters that each data set (x_i) came from j^{th} coin ($E[z_{i,j}]$)
3. Use these probability values ($E[z_{i,j}]$) as weights on each data point when computing a new θ_j to describe each distribution
4. Summate these expected values, use maximum likelihood estimation to derive new parameter values to repeat process