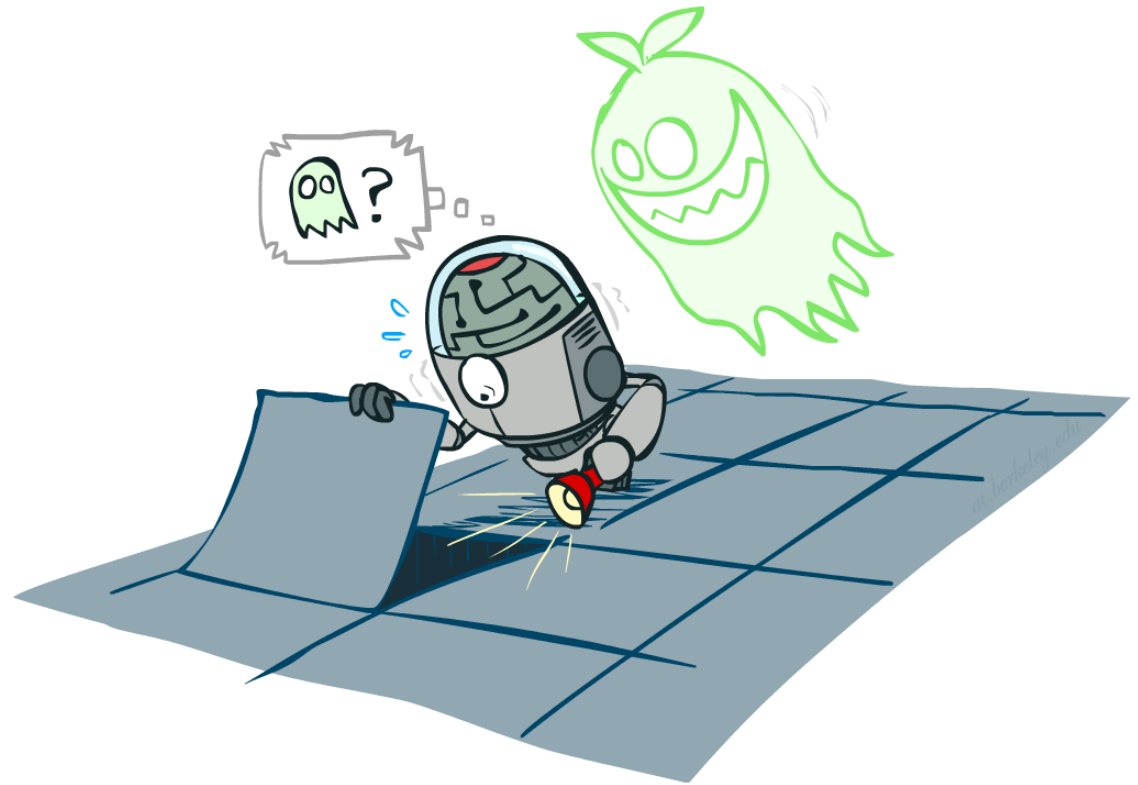


# Hidden Markov Models

# Probabilistic Reasoning

- Probabilistic Reasoning

- Diagnosis
- Speech recognition
- Tracking objects
- Robot mapping
- Genetics
- Error correcting codes
- ... lots more!

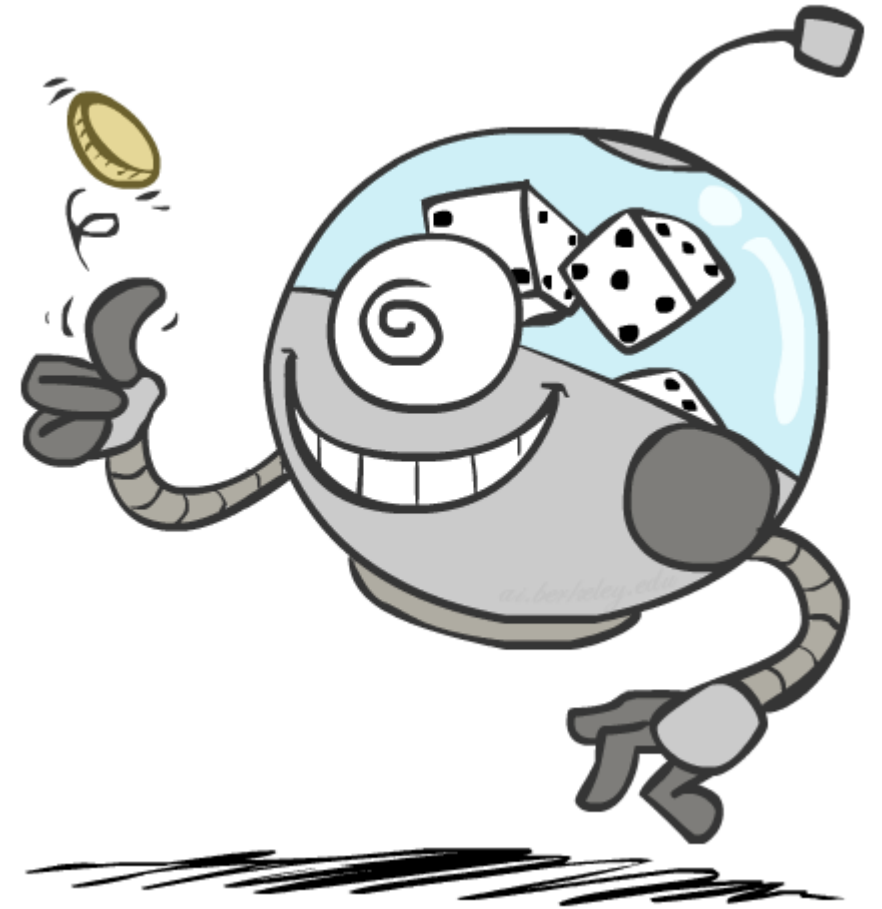


# Probability



# Today

- Probability
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distribution
  - Product Rule, Chain Rule, Bayes' Rule
  - Inference
  - Independence



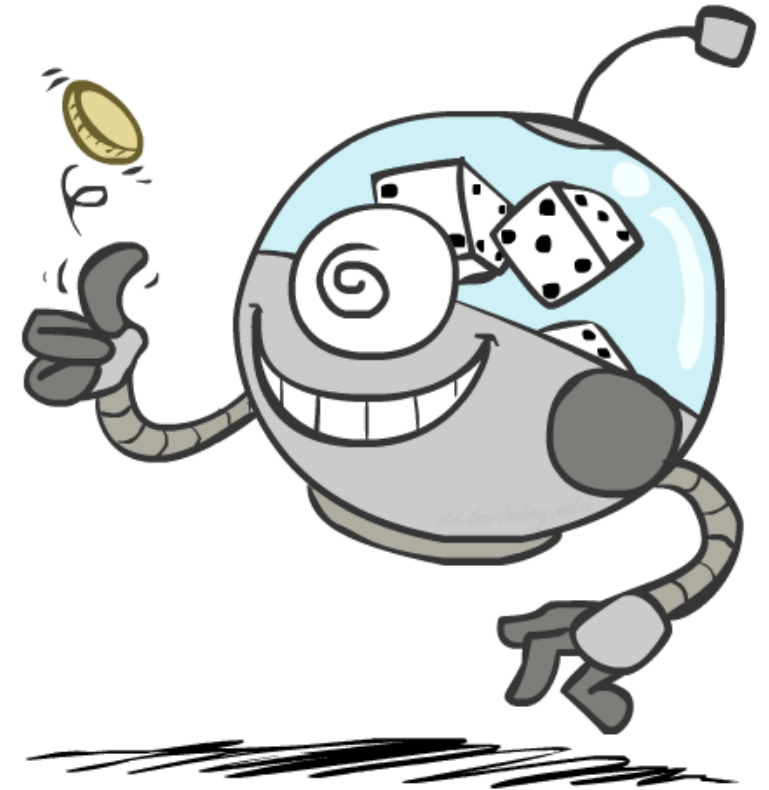
# Uncertainty

---

- General situation:
  - **Observed variables (evidence):** Agent knows certain things about the state of the world (e.g., sensor readings or symptoms or lab tests)
  - **Unobserved variables:** Agent needs to reason about other aspects (e.g. where an object is or what disease is present)
  - **Model:** Agent knows something about how the known variables relate to the unknown variables
- Probabilistic reasoning gives us a framework for managing our beliefs and knowledge

# Random Variables

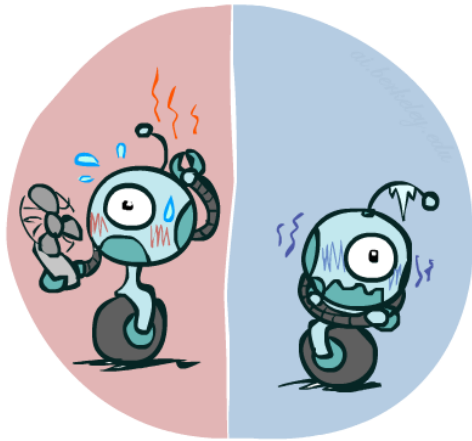
- A random variable is some aspect of the world about which we (may) have uncertainty
  - $R$  = Is it raining?
  - $T$  = Is it hot or cold?
  - $D$  = How long will it take to drive to work?
  - $L$  = Where is the ghost?
- We denote random variables with capital letters
- Random variables have domains
  - $R$  in  $\{\text{true}, \text{false}\}$  (often write as  $\{+r, -r\}$ )
  - $T$  in  $\{\text{hot}, \text{cold}\}$
  - $D$  in  $[0, \infty)$
  - $L$  in possible locations, maybe  $\{(0,0), (0,1), \dots\}$



# Probability Distributions

- Associate a probability with each value

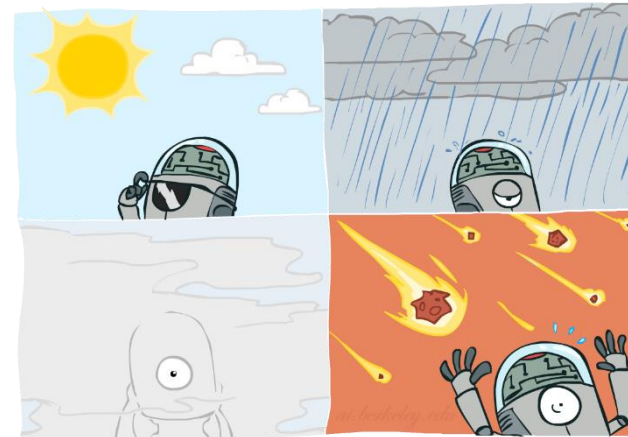
- Temperature:



$$P(T)$$

T	P
hot	0.5
cold	0.5

- Weather:



$$P(W)$$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0001

# Probability Distributions

- Unobserved random variables have distributions

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

Shorthand notation:

$$P(\textit{hot}) = P(T = \textit{hot}),$$

$$P(\textit{cold}) = P(T = \textit{cold}),$$

$$P(\textit{rain}) = P(W = \textit{rain}),$$

...

OK if all domain entries are unique

- A distribution is a TABLE of probabilities of values
- A probability (lower case value) is a single number

$$P(W = \textit{rain}) = 0.1$$

- Must have:  $\forall x \ P(X = x) \geq 0$  and  $\sum_x P(X = x) = 1$



# Joint Distributions

- A *joint distribution* over a set of random variables:  $X_1, X_2, \dots, X_n$  specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

$$P(x_1, x_2, \dots, x_n)$$

- Must obey:  $P(x_1, x_2, \dots, x_n) \geq 0$

$$\sum_{(x_1, x_2, \dots, x_n)} P(x_1, x_2, \dots, x_n) = 1$$

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- Size of distribution if  $n$  variables with domain sizes  $d$ ?
  - For all but the smallest distributions, impractical to write out!

# Probabilistic Models

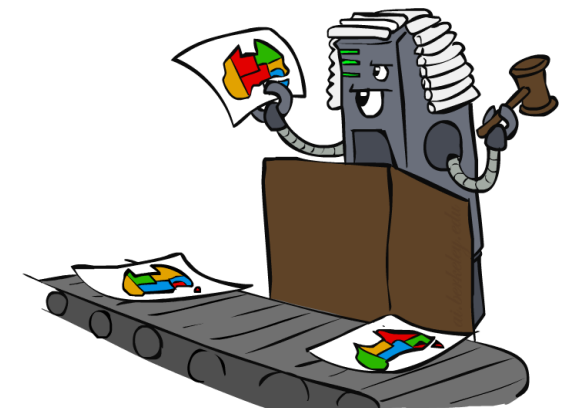
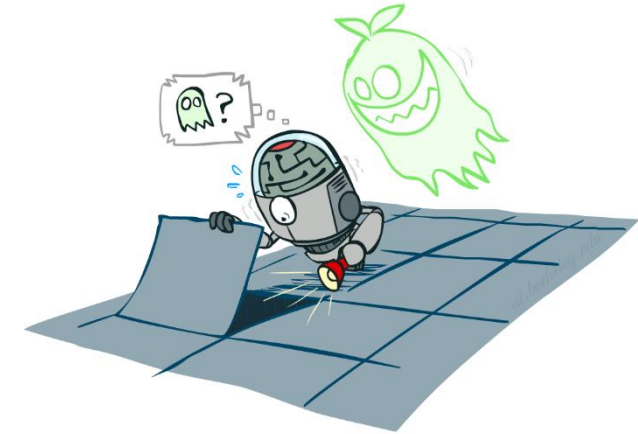
- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
  - (Random) variables with domains
  - Assignments are called *outcomes*
  - Joint distributions: say whether assignments (outcomes) are likely
  - *Normalized*: sum to 1.0
  - Ideally: only certain variables directly interact
- Constraint satisfaction problems:
  - Variables with domains
  - Constraints: state whether assignments are possible
  - Ideally: only certain variables directly interact

Distribution over T,W

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Constraint over T,W

T	W	P
hot	sun	T
hot	rain	F
cold	sun	F
cold	rain	T



# Events

- An *event* is a set  $E$  of outcomes

$$P(E) = \sum_{(x_1 \dots x_n) \in E} P(x_1 \dots x_n)$$

- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny? 0.4
  - Probability that it's hot? 0.5
  - Probability that it's hot OR sunny? 0.7
- Typically, the events we care about are *partial assignments*, like  $P(T=\text{hot})$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

# Quiz: Events

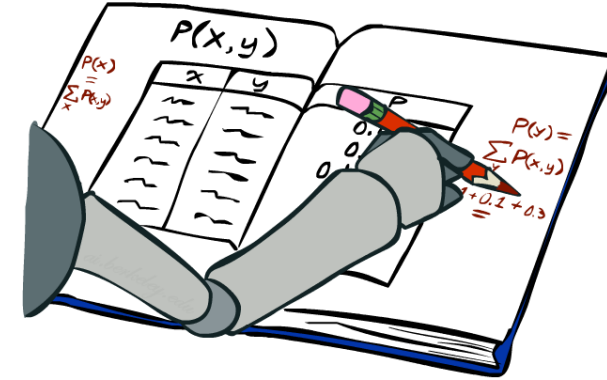
- $P(+x, +y)$  ?
- $P(+x)$  ?
- $P(-y \text{ OR } +x)$  ?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

# Marginal Distributions

- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding



$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3



$$P(t) = \sum_s P(t, s)$$

$P(T)$

T	P
hot	0.5
cold	0.5



$$P(s) = \sum_t P(t, s)$$

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

# Quiz: Marginal Distributions

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1



$$P(x) = \sum_y P(x, y)$$

$P(X)$

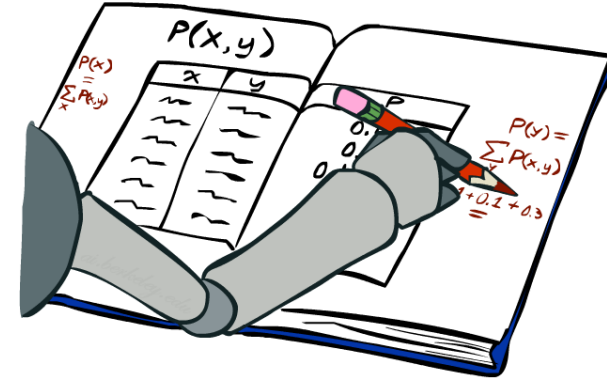
X	P
+x	
-x	

$P(Y)$

Y	P
+y	
-y	



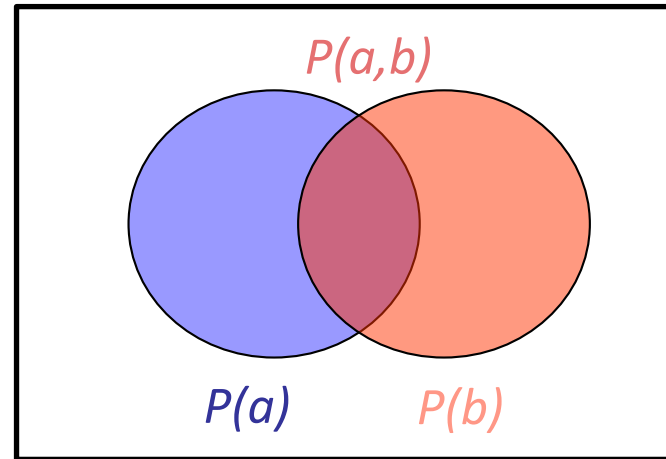
$$P(y) = \sum_x P(x, y)$$



# Conditional Probabilities

- A simple relation between joint and conditional probabilities
  - In fact, this is taken as the *definition* of a conditional probability

$$P(a|b) = \frac{P(a, b)}{P(b)}$$



$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W = s|T = c) = \frac{P(W = s, T = c)}{P(T = c)} = \frac{0.2}{0.5} = 0.4$$

$$\begin{aligned} &= P(W = s, T = c) + P(W = r, T = c) \\ &= 0.2 + 0.3 = 0.5 \end{aligned}$$

# Quiz: Conditional Probabilities

---

■  $P(+x \mid +y)$  ?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

■  $P(-x \mid +y)$  ?

■  $P(-y \mid +x)$  ?



# Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$P(W T)$	$P(W T = \text{hot})$	
	W	P
	sun	0.8
	rain	0.2
	$P(W T = \text{cold})$	
	W	P
	sun	0.4
	rain	0.6

Joint Distribution

$P(T, W)$		
T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

# Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$\begin{aligned}P(W = s|T = c) &= \frac{P(W = s, T = c)}{P(T = c)} \\&= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\&= \frac{0.2}{0.2 + 0.3} = 0.4\end{aligned}$$



$P(W|T = c)$

W	P
sun	0.4
rain	0.6

$$\begin{aligned}P(W = r|T = c) &= \frac{P(W = r, T = c)}{P(T = c)} \\&= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\&= \frac{0.3}{0.2 + 0.3} = 0.6\end{aligned}$$

# Normalization Trick

$$\begin{aligned} P(W = s|T = c) &= \frac{P(W = s, T = c)}{P(T = c)} \\ &= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.2}{0.2 + 0.3} = 0.4 \end{aligned}$$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Joint distribution  
to

**SELECT** the joint  
probabilities  
matching the  
evidence



$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

**NORMALIZE** the  
selection  
(make it sum to one)



$P(W|T = c)$

W	P
sun	0.4
rain	0.6

Conditional  
distribution

$$\begin{aligned} P(W = r|T = c) &= \frac{P(W = r, T = c)}{P(T = c)} \\ &= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.3}{0.2 + 0.3} = 0.6 \end{aligned}$$

# Normalization Trick

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

**SELECT** the joint probabilities matching the evidence



$P(c, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

**NORMALIZE** the selection (make it sum to one)



$P(W|T = c)$

W	P
sun	0.4
rain	0.6

- Why does this work? Sum of selection is  $P(\text{evidence})$ ! ( $P(T=c)$ , here)

$$P(x_1|x_2) = \frac{P(x_1, x_2)}{P(x_2)} = \frac{P(x_1, x_2)}{\sum_{x_1} P(x_1, x_2)}$$

# Quiz: Normalization Trick

■  $P(X \mid Y=-y)$  ?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

**SELECT** the joint probabilities matching the evidence



**NORMALIZE** the selection  
(make it sum to one)



# To Normalize

- (Dictionary) To bring or restore to a normal condition

All entries sum to ONE

- Procedure:

- Step 1: Compute  $Z = \text{sum over all entries}$
- Step 2: Divide every entry by  $Z$

- Example 1

W	P
sun	0.2
rain	0.3

Normalize  
Z = 0.5

W	P
sun	0.4
rain	0.6

- Example 2

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

Normalize  
Z = 50

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

# Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)
- We generally compute conditional probabilities
  - $P(\text{on time} \mid \text{no reported accidents}) = 0.90$
  - These represent the agent's *beliefs* given the evidence
- Probabilities change with new evidence:
  - $P(\text{on time} \mid \text{no accidents, 5 a.m.}) = 0.95$
  - $P(\text{on time} \mid \text{no accidents, 5 a.m., raining}) = 0.80$
  - Observing new evidence causes *beliefs to be updated*



# Inference by Enumeration

- General case:


- Evidence variables:  $E_1 \dots E_k = e_1 \dots e_k$
  - Query\* variable:  $Q$
  - Hidden variables:  $H_1 \dots H_r$
- $\left. \begin{array}{l} X_1, X_2, \dots, X_n \\ \text{All variables} \end{array} \right\}$

- We want:

*\* Works fine with multiple query variables, too*

$$P(Q|e_1 \dots e_k)$$

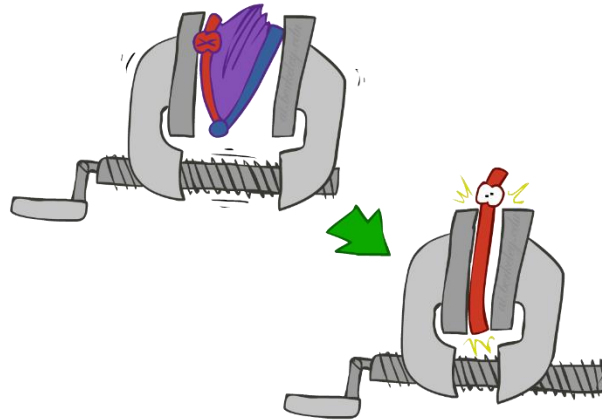
- Step 1: Select the entries consistent with the evidence



x	P(x)
-3	0.05
-1	0.25
0	0.07
1	0.2
5	0.01

2      0.15

- Step 2: Sum out H to get joint of Query and evidence



- Step 3: Normalize

$$\times \frac{1}{Z}$$

Note: We get a conditional distribution normalization which is the actual result

$$Z = \sum_q P(Q, e_1 \dots e_k)$$

$$P(Q|e_1 \dots e_k) = \frac{1}{Z} P(Q, e_1 \dots e_k)$$

$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots, X_n}$$



# Inference by Enumeration

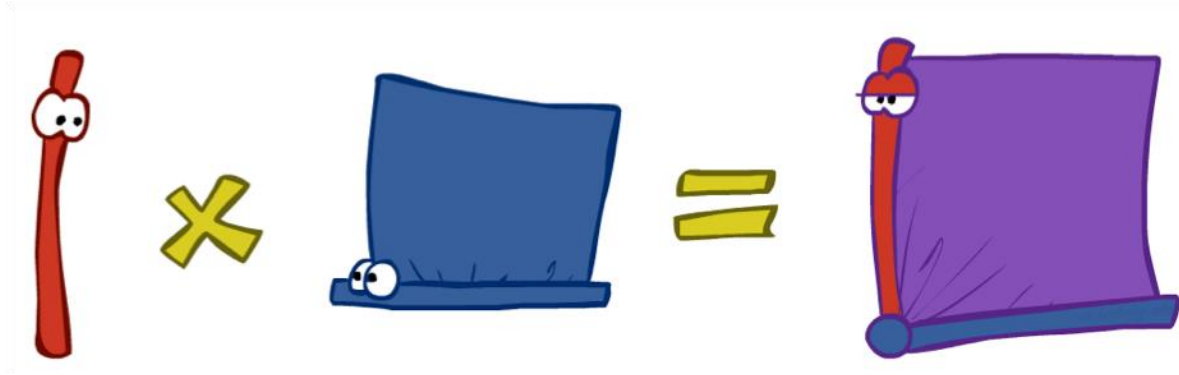
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- Obvious problems:
  - Worst-case time complexity  $O(d^n)$
  - Space complexity  $O(d^n)$  to store the joint distribution

# The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(y)P(x|y) = P(x, y) \quad \longleftrightarrow \quad P(x|y) = \frac{P(x, y)}{P(y)}$$



# The Product Rule

$$P(y)P(x|y) = P(x, y)$$

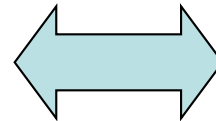
- Example:

$P(W)$

R	P
sun	0.8
rain	0.2


$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3



$P(D, W)$

D	W	P
wet	sun	0.08
dry	sun	0.72
wet	rain	0.14
dry	rain	0.06



# The Chain Rule

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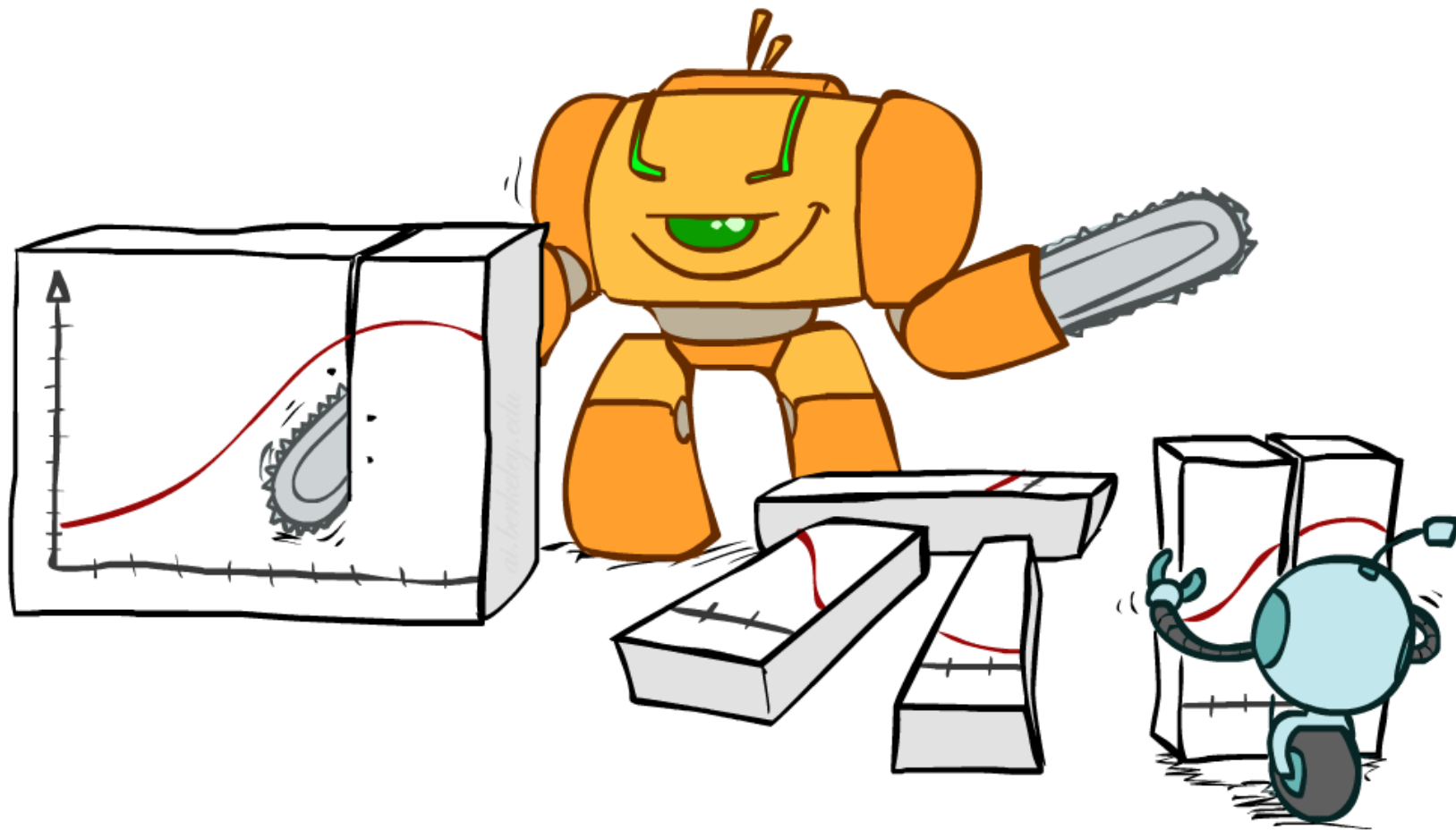
- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \dots, x_n) = \prod_i P(x_i|x_1 \dots x_{i-1})$$

- Why is this always true?

# Bayes Rule



# Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

That's my rule!

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
- In the running for most important AI equation!



# Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- Example:

- M: meningitis, S: stiff neck

$$\left. \begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned} \right\} \text{Example gives}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.999}$$

- Note: posterior probability of meningitis still very small
- Note: you should still get stiff necks checked out! Why?

# Quiz: Bayes' Rule

- Given:

$$P(W)$$

R	P
sun	0.8
rain	0.2

$$P(D|W)$$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is  $P(W \mid \text{dry})$  ?