Numerical Solution of Ordinary Differential Equations

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Picard's Method:

Consider the first order equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

It is required to find that particular solution of (1)

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx \;\; \text{(or)} \;\; y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \tag{2}$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign. As a first approximation y_1 to the solution, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2),giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx.$$

Similarly, the third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx.$$

Continuing this process, a sequence of functions of x, *i.e.*, $y_1, y_2, y_3, \ldots$ is obtained each giving a better approximation of the desired solution than the preceding one.

NOTE

Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily. The method can be extended to simultaneous equations and equations of higher order.

Example

Using Picard's process of successive approximation, obtain a solution up to the fifth approximation of the equation $dy/dx = y + x$, such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.

Solution:

(a) We have

$$y = 1 + \int_0^x (y + x)\, dx.$$

First approximation. Put $y = 1$, in $y + x$, giving

$$y_1 = 1 + \int_0^x (1 + x)\, dx = 1 + x + x^2/2.$$

Second approximation. Put $y = y_1$ in $y + x$, giving

$$y_2 = 1 + \int_0^x \left(1 + 2x + x^2/2\right) dx = 1 + x + x^2 + x^3/6.$$

Third approximation. Put $y = y_2$ in $y + x$, giving

$$y_3 = 1 + \int_0^x \left(1 + 2x + x^2 + x^3/6\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Fourth approximation. Put $y = y_3$ in $y + x$, giving

$$y_4 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

Fifth approximation. Put $y = y^4$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx$$

$$y_5 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}. \tag{3}$$

(b)Given equation:

$$\frac{dy}{dx} - y = x \text{ is a Leibnitz's linear in } x.$$

Its Integrating Factor being e^{-x}, the solution is

$$ye^{-x} = \int xe^{-x}\, dx + c = -xe^{-x} - \int \left(-e^{-x}\right) dx + c = -xe^{-x} - e^{-x} + c$$

$\therefore\ y = ce^x - x - 1.$
Since $y = 1$, when $x = 0$,
$\therefore\quad c = 2.$

Thus the desired particular solution is

$$y = 2e^x - x - 1 \tag{4}$$

Or using the series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\infty,$$

We get

$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \ldots\infty \tag{5}$$

Comparing (3) and (5), it is clear that (3) approximates to the exact particular solution (4) up to the term in x^5.

Obs. At $x = 1$, the fourth approximation $y_4 = 3.433$ and the fifth approximation $y_5 = 3.433$ whereas exact value is 3.44.

Consider the first order equation

$$dy/dx = f(x, y) \tag{6}$$

Differentiating (6), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \text{ i.e., } y'' = f_x + f_y f' \tag{7}$$

Differentiating this successfully, we can get y''', y^{iv} etc. Putting $x = x_0$ and $y = 0$, the above values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series

$$y(x) = y_0 + (x - x_0)\left(y'\right)_0 + \frac{(x - x_0)^2}{2!}\left(y''\right)_0 + \frac{(x - x_0)^3}{3!}\left(y'''\right)_0 + \cdots \tag{8}$$

gives the values of y for every value of x for which (8) converges.

On finding the value y_1 for $x = x_1$ form (8), y', y'' can be evaluated at $x = x_1$ by means of (6), (7) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (8).

Example

Find by Taylor's series method, the value of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from $dy/dx = x^2y - 1$, $y(0) = 1$.

Solution:

Here $(y)_0 = 1$, $y' = x^2y - 1$, $(y')_0 = -1$

$\therefore$ Differentiating successfully and substituting, we get

$$\begin{aligned} y'' &= 2xy + x^2y', & (y'')_0 &= 0 \\ y''' &= 2y + 4xy' + x^2y'', & (y''')_0 &= 2 \\ y^{iv} &= 6y' + 6xy'' + x^2y''', & \left(y^{iv}\right)_0 &= -6 \text{ etc.} \end{aligned}$$

Putting these values in the Taylor's series,

$$y(x) = y_0 + (x)\left(y'\right)_0 + \frac{(x)^2}{2!}\left(y''\right)_0 + \frac{(x)^3}{3!}\left(y'''\right)_0 + \frac{(x)^4}{4!}\left(y^{iv}\right)_0 + \cdots$$

we have

$$y(x) = 1 + (x)(-1) + \frac{(x)^2}{2!}(0) + \frac{(x)^3}{3!}(2) + \frac{(x)^4}{4!}(-6) + \cdots = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Hence, $y(0.1) = 0.90033$ and $y(0.2) = 0.80227$.

The fourth-order Runge-Kutta method is most commonly used and is often referred to as 'Runge-Kutta Method' only.
Working rule for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from $\dfrac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ is as follows:
Calculate successively

$$\begin{aligned}
k_1 &= \quad hf(x_0, y_0) \\
k_2 &= \quad hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right) \\
k_3 &= \quad hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2\right) \\
k_4 &= \quad hf(x_0 + h,\ y_0 + k_3)
\end{aligned}$$

Finally compute $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
which gives the approximate value $y_1 = y_0 + k$.

(Note that k is the weighted mean of $k_1,\ k_2,\ k_3$ and k_4)

Obs. One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example

Apply R-K fourth order method, to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

Here, $x_0 = 0,\ y_0 = 1,\ h = 0.2,\ f(x_0, y_0) = 1$

$$\therefore \begin{aligned} k_1 &= hf(x_0, y_0) = 0.2 \times 1 && = 0.2000 \\ k_2 &= hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) && = 0.2400 \\ k_3 &= hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) && = 0.2440 \\ k_4 &= hf(x_0 + h,\ y_0 + k_3) = 0.2 \times f(0.2, 1.244) && = 0.2888 \end{aligned}$$

$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2428.$

Hence the required approximate value of $y = y_0 + k$ is 1.2428.

Consider the second order differential equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

By writing $dy/dx = z$, it can be reduced to two first order simultaneous differential equations $\frac{dy}{dx} = z, \quad \frac{dz}{dx} = \phi(x, y, z)$.

These equations can be solved as follows:

Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-Kutta method gives,

$$\begin{aligned}
k_1 &= hf(x_0, y_0, z_0) & l_1 &= h\phi(x_0, y_0, z_0)\\
k_2 &= hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1,\ z_0 + \tfrac{1}{2}l_1\right) & l_2 &= h\phi\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1,\ z_0 + \tfrac{1}{2}l_1\right)\\
k_3 &= hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2,\ z_0 + \tfrac{1}{2}l_2\right) & l_3 &= h\phi\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2,\ z_0 + \tfrac{1}{2}l_2\right)\\
k_4 &= hf(x_0 + h,\ y_0 + k_3,\ z_0 + l_3) & l_4 &= h\phi(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)
\end{aligned}$$

Hence $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ and $z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$. To compute y_2 and z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

Example

Using Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0,\ y = 1,\ y' = 0$.

Solution:

Let $dy/dx = z = f(x, y, z.)$ Then $dz/dx = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0,\ y_0 = 1,\ z_0 = 0,\ h = 0.2$.

Using $k_1,\ k_2, \cdots$ for $f(x, y, z)$ and $l_1,\ l_2, \cdots$ for $\phi(x, y, z)$, R-K formulae become

$$k_1 = hf\left(x_0, y_0, z_0\right) = 0 \qquad l_1 = h\phi\left(x_0, y_0, z_0\right) = -0.2$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1,\ z_0 + \tfrac{1}{2}l_1\right) = -0.02 \qquad l_2 = h\phi\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1,\ z_0 + \tfrac{1}{2}l_1\right) = -0.1998$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2,\ z_0 + \tfrac{1}{2}l_2\right) = -0.02 \qquad l_3 = h\phi\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2,\ z_0 + \tfrac{1}{2}l_2\right) = -0.1958$$

$$k_4 = hf\left(x_0 + h,\ y_0 + k_3,\ z_0 + l_3\right) = -0.0392 \qquad l_4 = h\phi\left(x_0 + h,\ y_0 + k_3,\ z_0 + l_3\right) = -0.1905$$

Hence

$y = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.9801$ and

$y' = z = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = -0.1970.$