Wave Equation:

$$u_{tt} = c^2 u_{xx} \quad ; \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

$$u(0,t) = \alpha(t), \quad u(L,t) = \beta(t).$$

$\Delta t = k, \quad \Delta x = h$

Explicit method : Algorithm

$$u_i^{J+1} = \frac{k^2 c^2}{h^2}\left[u_{i+1}^{J} + u_{i-1}^{J}\right] - u_i^{J-1} + 2\left[1 - \frac{k^2c^2}{h^2}\right]u_i^{J} \quad -(1)$$

Let $\frac{k^2c^2}{h^2} = \lambda^2$ ie $\frac{c\Delta t}{\Delta x} = \lambda$.

In order to start computations, we need the data on two lines $t=0$, $t=k$. The information required on the line $t=k$ (first time step) is obtained by using a suitable approximation to the initial condition. $\frac{\partial u}{\partial t}(x,0) = g(x)$. *

If we use the central difference approximation

$$u_t(x_i,0) = \frac{u_i^1 - u_i^{-1}}{2k} = g(x_i)$$

$$\therefore \boxed{u_i^{-1} = u_i^1 - 2k\, g(x_i)} \quad -(2)$$

Corresponding to $J=0$, ie for the first time step, eqn. (1) becomes *

$=0 \quad u_i^1 = \lambda^2\left(u_{i+1}^0 + u_{i-1}^0\right) - \left(u_i^1 - 2k\, g(x_i)\right) + 2(1-\lambda^2)u_i^0 \quad (3)$

$1, n \quad u_i^{J+1} = \lambda^2\left(u_{i+1}^J + u_{i-1}^J\right) - u_i^{J-1} + 2(1-\lambda^2)u_i^J \quad (4)$

We use equations (3) and (4) to numerically solve wave equation by the explicit method.

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* We may use forward difference approximation $u_t(x,0) = \frac{u_i^1 - u_i^0}{k} = g(x_i)$ to approximate $u(x,t)$ on the first time step. But, accuracy will suffer.) * $u_i^1 = \frac{\lambda^2}{2}\left(u_{i+1}^0 + u_{i-1}^0\right) + k\, g(x_i) + (1-\lambda^2)u_i^0$

This explicit scheme is stable if $\lambda \leq 1$. If $\lambda > 1$, we cannot be sure of convergence.

<u>Problem:</u>

$$u_{tt} = u_{xx} \quad , \quad 0 \leq x \leq 1$$

subject to $\quad u(x,0) = \sin \pi x \qquad 0 \leq x \leq 1$

$\qquad\qquad u_t(x,0) = 0 \qquad 0 \leq x \leq 1$

and the boundary conditions $\quad u(0,t) = 0 = u(1,t), \ t > 0.$

Let $h = \frac{1}{4}$, $\lambda = \frac{3}{4}$, we have $\frac{\Delta t}{\Delta x} = \frac{3}{4}$.

$$\therefore \ \underline{\Delta t} = k = \frac{3}{4} \times \frac{1}{4} = \underline{\underline{\frac{3}{16}}}.$$

The explicit algorithm equations become.

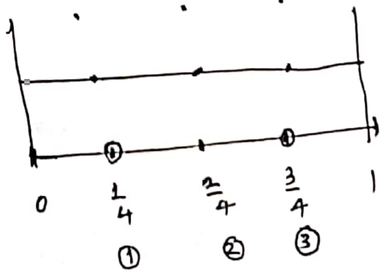
$$u_i^1 = \frac{9}{16}\left(u_{i+1}^0 + u_{i-1}^0\right) - u_i^1 + 2\left(1 - \frac{9}{16}\right) u_i^0$$

i.e. $$u_i^1 = \frac{9}{32}\left(u_{i+1}^0 + u_{i-1}^0\right) + \frac{7}{16} u_i^0$$

$u(\frac{1}{4}, 0) = 0.7071$

$u(\frac{1}{2}, 0) = 1.0000$

$u(\frac{3}{4}, 0) = 0.7071$



$$u_1^1 = \frac{9}{32}(1) + \frac{7}{16} \times 0.7071$$

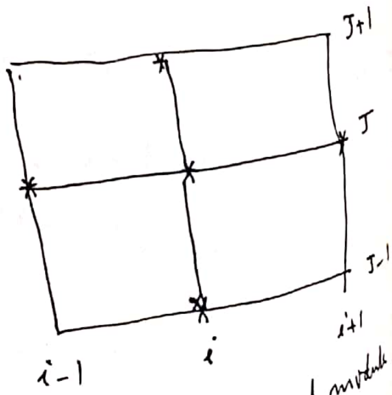
$$= 0.59061$$

$$u_2^1 = \frac{9}{32}\left(0.7071 + 0.7071\right) + \frac{7}{16} \times 1$$

$$= 0.83525$$

$$u_3^1 = 0.59061$$

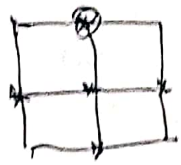
Note $u_1^J = u_3^J$.



computational module for wave equation

For the other time steps, the algorithm (4) becomes

$$u_i^{j+1} = \frac{9}{16}\left(u_{i+1}^{j} + u_{i-1}^{j} \right) - u_i^{j-1} + \frac{7}{8} u_i^{j} \qquad j \geq 1, n.$$



$j=1$

$$u_1^2 = \frac{9}{16}\left(\cancel{0.59061+} \; 0 + 0.83525 \right) - 0.7071 + \frac{7}{8} \times 0.59061$$

$$= 0.27951 = u_3^2$$

$i=2$

$$u_2^2 = \frac{9}{16}\left(0.59061 + 0.59061 \right) - 1.0000 + \frac{7}{8} \times 0.83525$$

$$= 0.39528$$

Similarly for the other time steps.

$u_1^3 = u_3^3 = -0.12369, \quad u_2^3 = -0.17493$

$u_1^4 = u_3^4 = -0.48614, \quad u_2^4 = -0.68750$

$u_1^5 = u_3^5 = -0.68844, \quad u_2^5 = -0.97354$

Exact soln: $\boxed{u(x,t) = \sin(\pi x) \cos \pi t.}$

for $x = \frac{1}{4}$, $t = \frac{3}{8}$ (Two time steps).

$$u\left(\frac{1}{4}, \frac{3}{8}\right) = \sin\left(\frac{\pi}{4}\right) \cdot \cos\left(\frac{3\pi}{8}\right) = 0.2706$$

$$u\left(\frac{1}{2}, \frac{3}{8}\right) = \sin\frac{\pi}{2} \cdot \cos\frac{3\pi}{8} = 0.3827.$$

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