09-Classical Assumptions

Monday, November 4, 2019 11:53 AM

Prove that $E(\hat{\beta}_2) = \beta_2$.

$$\hat{\beta}_2 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

we'll use the fact that

$$\sum (x_i - \bar{x})(y_i - \bar{y}) = \sum (x_i - \bar{x}) y_i$$

$$\Rightarrow \hat{\beta}_2 = \frac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2}$$

Pop. model: $Y_i = \beta_1 + \beta_2 X_i + u_i$

$\hookrightarrow$ plug in to slope estimator:

$$\text{Then} \quad \hat{\beta}_2 = \frac{\sum (x_i - \bar{x})(\beta_1 + \beta_2 X_i + u_i)}{\sum (x_i - \bar{x})^2}$$

$$= \frac{\sum (x_i - \bar{x})\beta_1 + \sum (x_i - \bar{x}) x_i \beta_2 + \sum (x_i - \bar{x}) u_i}{\sum (x_i - \bar{x})^2}$$

$$= \frac{\cancel{\beta_1 \sum (x_i - \bar{x})} + \beta_2 \sum (x_i - \bar{x}) x_i + \sum (x_i - \bar{x}) u_i}{\sum (x_i - \bar{x})^2}$$

Showed $\sum (x_i - \bar{x}) = 0$

$$\sum (x_i - \bar{x}) x_i = \sum (x_i - \bar{x})^2 \longrightarrow$$

$$\sum (x_i - \bar{x}) x_i = \sum x_i^2 - \bar{x} \sum (x_i)$$

$$= \sum x_i^2 - \bar{x} \sum x_i - \bar{x} \underbrace{\sum (x_i - \bar{x})}_{=0}$$

$$= \sum x_i^2 - 2\bar{x} \sum x_i + \sum \bar{x}^2$$

$$= \sum (x_i - \bar{x})^2$$

$$\text{Then} \quad \hat{\beta}_2 = \beta_2 + \frac{\sum (x_i - \bar{x}) u_i}{\sum (x_i - \bar{x})^2}$$

Take expectations:

$$E(\hat{\beta}_2) = \beta_2 + E\left(\frac{\sum (x_i - \bar{x}) u_i}{\sum (x_i - \bar{x})^2}\right)$$

$$= E\left(\frac{1}{\sum (x_i - \bar{x})^2} \sum (x_i - \bar{x}) u_i\right)$$

$$= \frac{1}{\sum (x_i - \bar{x})^2} \sum E (x_i - \bar{x}) u_i$$

$$E(u|X) = 0 \Rightarrow E(u) = 0$$

$E(u|X)=0 \Rightarrow E(u)=0$

$$E(\beta_2) = \beta_2 + \frac{1}{\sum(X_i-\bar{X})^2}\sum(X_i-\bar{X})(0)$$

$$= \beta_2$$

Q.E.D.

Derive $var(\hat{\beta}_2)$

$$\hat{\beta}_2 = \beta_2 + \frac{1}{\sum(X_i-\bar{X})^2}\sum(X_i-\bar{X})u_i$$

$$var(\hat{\beta}_2) = var\left(\beta_2 + \frac{1}{\sum(X_i-\bar{X})^2}\sum(X_i-\bar{X})u_i\right)$$

$$= var\left(\frac{1}{\sum(X_i-\bar{X})^2}\sum(X_i-\bar{X})u_i\right)$$

$$= \left(\frac{1}{\sum(X_i-\bar{X})^2}\right)^2 var\left(\sum(X_i-\bar{X})u_i\right)$$

$$= \left(\frac{1}{\sum(X_i-\bar{X})^2}\right)^2 \sum(X_i-\bar{X})^2 \underline{var(u_i)}$$

Random sampling

$$= \left(\frac{1}{\sum(X_i-\bar{X})^2}\right)^2 \sum(X_i-\bar{X})^2 \sigma^2$$

$$var(\hat{\beta}_2) = \frac{\sigma^2}{\sum(X_i-\bar{X})^2}$$