Middle East Technical University
Department of Electrical and Electronics Engineering
EE230: Probability and Random Variables

Homework 1
Sets, Probability Spaces, Conditional Probability
Due: February 28, 2011

1. Out of 60 people attending a party, half are over the age of 21. Thirty-five of all attendees have high school diplomas, 15 have driver's licenses, and 15 have neither a high school diploma nor a driver's license. Only 2 people without diplomas are over 21. Furthermore, we know that five people are in all three sets (A= {Over 21}, B= {has license}, C= {has diploma}) and 14 people are in none of the three.
   (a) Draw a Venn Diagram illustrating the sets A, B C, and on it mark the region {People over 21 with diploma but no license}.
   (b) Express this region in terms of the sets A, B, C, and the operations of complement and intersection.
   (c) How many people over 21 have a diploma but no license?
   (d) You enter the party blindfolded and pick a person uniformly at random. What is the likelihood that this person has both a driver's license and a high school diploma?
   (e) Express in terms of A, B, C, and the operations of union, intersection and complement, the following event: The person I pick is either under 21, or, if not, then this person has a diploma and a license.
   (f) What is the probability of the event in part (e)?

2. Eylem, who lives in the Dormitories, is a student in EE230. On a normal day, he wakes up at some time between $8:00$ to $9:00$ (he is equally likely to wake up at any point in this interval.) When he studies until late at night he wakes up, again uniformly, at some time between $8:30$ to $9:30$. We are interested in his wake-up time.
   (a) Define a sample space S for this experiment.
   (b) Define two events that are disjoint.
   (c) Define two events that have a nonempty intersection.
   (d) Considering the uniform probability law, what is the probability that he wakes up no later than $8:30$ on a normal day?
   (e) If Eylem showed up at EA207 at $8:29$, what is the probability that he studied late the previous night?

3. My keys got lost either at home or at the office, but I know that they are twice as likely to be at home than at the office. If they are at home, I will find them the next day with probability 50%. If they are in the office, I will find them the next day with probability 80%.
   (a) What is the probability that the next day, I find my keys at home?
   (b) What is the probability that after a day, I have not found my keys?
   (c) **Given that I found my keys**, what is the probability that they were at the office? (Hint: simply combine your answers to the above two questions.) Your answer should be above 1/3. Explain why.

Homework 1 Solutions
February 28, 2011

1.

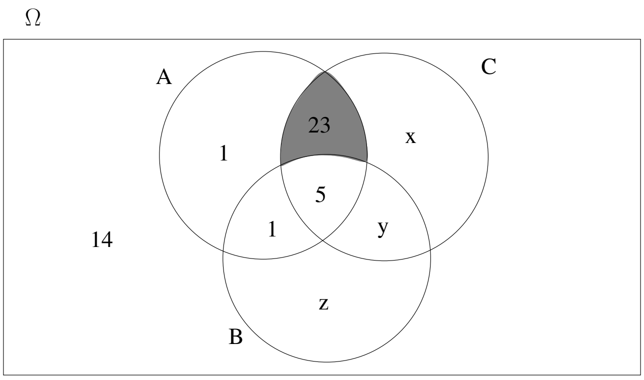
Let A : {over 21}, B : {has driver's license}, and C : {has high-school diploma}.

(a)

We are given that |A| = 60/2 = 30, |B| = 15, |C| = 35, $|(B\cup C)^c| = 15$, $|A\cap C^c| = 2$, $|(A\cap B\cap C)| = 5$, and $|(A\cup B\cup C)^c| = 14$. Therefore, we have

$$\begin{aligned}
|A \cap (B \cup C)^c| &= |(B \cup C)^c| - |(A \cup B \cup C)^c| = 15 - 14 = 1, \\
|(A \cap B) \cap C^c| &= |A \cap C^c| - |A \cap (B \cup C)^c| = 2 - 1 = 1, \\
|(A \cap C) \cap B^c| &= |A| - (|A \cap C^c| + |(A \cap B \cap C)|) = 30 - (2 + 5) = 23.
\end{aligned}$$

The following Venn Diagram illustrates sets A, B, and C together with the universal set of people attending the party, Ω. Some cardinalities are also shown on it in addition to the shaded area of {People over 21 with diploma but no license}.



(b)

$(A\cap C)\cap B^c$

(c)

$|(A\cap C)\cap B^c|=23$.

(d)

Using the Venn Diagram of part (a), we obtain,

$$\begin{aligned}
x + y &= 8, \\
x + y + z &= 16, \\
y + z &= 9.
\end{aligned}$$

Then, $x = 7$, $y = 0$, and $z = 9$. According to the uniform probability law,

$$\text{P(Selected person has both a license and a diploma)} = \frac{|B \cap C|}{|P|} = \frac{5}{60} = \frac{1}{12}$$

(e)

$A^c \cup [A \cap (B \cap C)] = A^c \cup (B \cap C)$

(f)

$$\text{P(Selected person is either under 21, or, if not, has a diploma and a license)} = \frac{|A^c \cup (B \cap C)|}{|P|} = \frac{35}{60} = \simeq 0.583$$

2.

(a)

A sample space would be defined as $\Omega = [8{:}00,\ 9{:}30] = \{t \in \Re : 8.0 < t < 9.5\}$, which is an uncountable set of possible wake-up times shown by t in hours.

(b)

We may define two subsets of Ω as E_1=[8:00, 8:30] and E_2=[9:00, 9:30] that denote two disjoint events (i.e., $E_1 \cap E_2 = \emptyset$). E_1 corresponds to waking up of Eylem between 8:00 and 8:30. Similarly, E_1 corresponds to observing the wake-up time to be within the interval 9:00 and 9:30.

(c)

E_3=[8:15, 8:45] and E_4=[8:30, 9:30] correspond to two events that may occur simultaneously for a single observation (trial) of wake-up time.

(d)

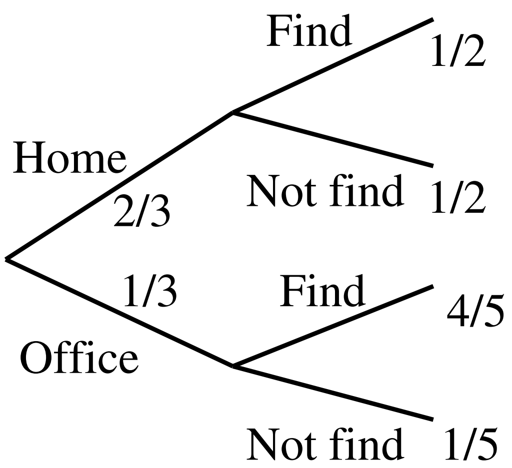
P(wake-up time $\leq$ 8:30 | Normal day) = P(E_1 | Normal day) $= \frac{30}{60} = \frac{1}{2}$.

(e)

P(Late at previous night | wake-up time $\leq$ 8:29) = P(E_4 | wake-up time $\leq$ 8:29) = 0.

3.

A sequential sample-space description is useful for this example as shown in the following figure.



(a)

Defining the events (sets) of losing the keys at home and at the office as H = {Keys at home} and O = {Keys at the office} respectively, and the event of finding the keys as F = {find keys next day}, we have

$$\begin{aligned} P_1 &= \mathrm{P}(\mathrm{H} \cap \mathrm{F}) \\ &= \mathrm{P}(\mathrm{H})\ \mathrm{P}(\mathrm{F} \mid \mathrm{H}) \\ &= \frac{2}{3} \cdot \frac{1}{2} \\ &= \frac{1}{3} \end{aligned}$$

(b)

Losing keys at home and at the office are two disjoint events, $\mathrm{O} = \mathrm{H}^c$. Hence, using total probability theorem,

$$\begin{aligned} P_2 &= \mathrm{P}(\mathrm{F}^c) \\ &= \mathrm{P}(\mathrm{H})\ \mathrm{P}(\mathrm{F}^c \mid \mathrm{H}) + \mathrm{P}(\mathrm{H}^c)\ \mathrm{P}(\mathrm{F}^c \mid \mathrm{H}^c) \\ &= \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{5} \\ &= \frac{6}{15} = \frac{2}{5} \end{aligned}$$

(c)

$$\begin{aligned} P_3 &= \mathrm{P}(\mathrm{H}^c \mid \mathrm{F}) \\ &= \frac{\mathrm{P}(\mathrm{H}^c \cap \mathrm{F})}{\mathrm{P}(\mathrm{F})} \\ &= \frac{1 - (P_1 + P_2)}{1 - P_2} \\ &= \frac{1 - \frac{11}{15}}{1 - \frac{6}{15}} \\ &= \frac{\frac{4}{15}}{\frac{9}{15}} \\ &= \frac{4}{9} \simeq 0.444. \end{aligned}$$

We would like to obtain $\mathrm{P}(\mathrm{H}^c \mid \mathrm{F})$, which is equal to

$$\mathrm{P}(\mathrm{H}^c \mid \mathrm{F}) = \frac{\mathrm{P}(\mathrm{F} \mid \mathrm{H}^c)\ \mathrm{P}(\mathrm{H}^c)}{\mathrm{P}(\mathrm{F})}$$

We know that $\frac{1}{2} = \mathrm{P}(\mathrm{F} \mid \mathrm{H}) < \mathrm{P}(\mathrm{F}) < \mathrm{P}(\mathrm{F} \mid \mathrm{H}^c) = \frac{4}{5}$ so that

$$\frac{\mathrm{P}(\mathrm{F} \mid \mathrm{H}^c)}{\mathrm{P}(\mathrm{F})} > 1.$$

Therefore, we should have

$$\mathrm{P}(\mathrm{H}^c \mid \mathrm{F}) > \mathrm{P}(\mathrm{H}^c) = \mathrm{P}(\mathrm{O}) = \frac{1}{3}$$

The keys are more likely to be found at the office (if they were lost there). Given that the keys are found, the *a posteriori* probability of them being lost at the office, 44.4%, is thus higher than the *a priori* probability, which is 33.3%. Remember the folk tale where "Nasreddin Hoca" has lost something, most probably at home, but looks for it under the streetlamp outside. Under the lamp where it's well-lit, the *conditional probability* of finding it is higher!

Homework 2
Probability Spaces, Conditional Probability, Law of Total Probability
Due: March 7, 2011

1. In a psychology-related experiment, a monkey sits in front of a computer with a keyboard of 4 characters $(1, 2, 3, 4)$ and presses the keys in a random manner with equal likelihood. The researchers stop the monkey after it types two characters.

   (a) What is the sample space of this random experiment?

   (b) Write down the event A that the sum of the two numbers equals five and find its probability.

   (c) The event B is defined as the case that the product of the two numbers equals 4. Evaluate $\mathrm{P}(B)$.

   (d) Find the conditional probabilities $\mathrm{P}(B|A)$ and $\mathrm{P}(A|B)$.

   (e) Define another event C that corresponds to the sum being equal to 9. Find $\mathrm{P}(A|C)$.

2. A person is randomly selected from the whole population to check whether this person has a certain rare disease through a blood test. The random experiment is related to person's actual sickness status and the test result.

   (a) Define a sample space S for this experiment.

   (b) Pick a probability law that satisfies all the axioms. Pay attention to the actual scenario given here when assigning probability to events. Show that all the axioms are satisfied.

   (c) Based on your probability law, what is the probability that the person is sick given that s/he is tested positive for the disease?

   (d) How should the event probabilities be set to have the probability in part c as large as possible?

3. A binary-valued source generates a random sequence of numbers from the alphabet $\{0, 1\}$. The source generates numbers with probabilities based on the previously generated number. The probability that the generated number is the same as the previous one is 0.8. The probability of the first number being 0 is 0.6. The random experiment is stopped after three numbers are generated.

   (a) What is the probability of the event that the second number is 1?

   (b) What is the probability of the event that the third number is 0?

   (c) Find the probability of the event that the third number is 0 given that the second number is 1.

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Homework 3
Probability, Independence, Counting
Due: 17:30 on March 14, 2011

1. (a) Prove that the following property holds for any two events A and B belonging to the same probability space. Clearly indicate all the steps in your proof.

$$P(A \cap B) \geq P(A) + P(B) - 1$$

   (b) Consider three events A, B and C from the same experiment, and suppose P(A)= 0.70, P ($B|A$) = 0.5. While A and C are independent, the events C and $(A \cap B)$ are *disjoint*. Designate each statement below as "True", "False", or "Cannot be concluded", clearly explaining your answer.

      i. B and C are mutually exclusive.
      ii. C and B are independent.
      iii. P (C) = P ($A|C$).
      iv. P($A|B$)<P($B|A$).
      v. P($B \cap A^c$) $\leq$ P(A).

2. In a casino, there are two groups of die-pairs: fair pairs and biased pairs. Each die in a biased pair has the probabilities $(3/12, 3/12, 2/12, 2/12, 1/12, 1/12)$, for each of its six faces numbered 1 to 6, respectively (i.e. face-1 and -2 are more, and face-5 and -6 are less likely). It is known that the number of biased pairs constitute 5% of the total.

   (a) After a single roll of a biased pair, what is the probability for observing the same number come up on the faces of both dice? Compare this probability with that of a fair die pair.

   (b) Unfortunately, all of these die pairs are mixed in such a way that any selected pair could be either fair or biased (no pair with one fair and one biased die). We would like to discriminate a biased pair from a fair one, after rolling a randomly selected pair 20 times. Find the conditional probability that a biased pair has been selected, given that equal faces are obtained 6 times out of 20 rolls. Find the probability of a fair pair having been selected under the same condition.

3. A deck of 52 cards is divided into two (equal) halves. An Ace is drawn from one of the halves and placed on the second half-deck. Then, the second half-deck is shuffled and another card is drawn. What is the probability that this selected card is an Ace?

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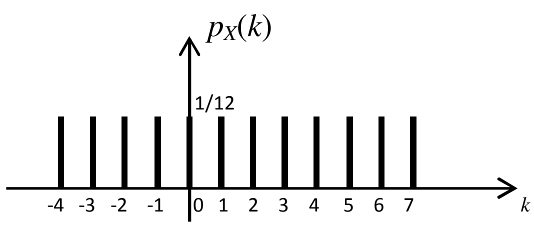
Homework 4

Discrete Random Variables-I
Due: 17:30 on March 21, 2011

1.

(a) Consider the single tossing of an unfair coin with P(H)= 0.3. Define two random variables *X* and *Y* for this experiment. Plot the probability mass functions (PMF) of *X* and *Y*.
(b) Repeat part-(a) for the double toss of the same coin.
(c) Write the mathematical expression and plot the PMFof the random variable *X* defined as the number of heads out of 6 tosses of this unfair coin. What is the name of this distribution?

2. The PMF, $p_X(k)$, of a discrete random variable *X* is given.



Let the the events *A* and *B* be defined as $X \in \{-3,-2,5,7\}$ and $X \in \{-4,-2,4\}$, respectively.

(a) What are the probabilities of *A* and *B*?
(b) Are *A* and *B* independent?
(c) How many events with a single common element with *A* and independent of it can be found?

3.

(a) Plot the PMF of the random variable *Y* defined as $Y = \frac{3}{2}X$ where *X* is the random variable in problem-2.
(b) Repeat part-(a) for $Y = \frac{1}{3}X$.
(c) Repeat part-(a) for $Y = X^2$.