Recursive State Estimation & Bayes Filters

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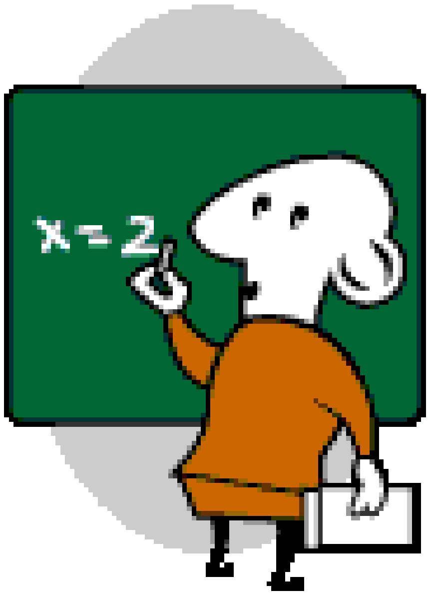
Lecture slides heavily use material from the textbook and
Sebastian Thrun, Lecture Slides; http://www.probabilistic-robotics.org/
Andrew Moore, Data mining tutorials; http://www.autonlab.org/tutorials/

What we will discuss



- Recall our motivation for using Probability for robotics,
- Review probability theory, its axioms and important results,
- Take another look in robot environment interaction,
- Discuss recursive state estimation through the formalism of Bayes Filters,

Why are we doing this?

- Robotics would have been so easy <u>if only we knew</u> certain quantities,
- We could make decisions, apply control etc easily.
- **What are those “quantities”?**

The World “State”
(The state of robot and its environment)

- ***Unfortunately, we do not know directly and often cannot directly measure the “state”.***

Why are we doing this?

- *Probabilistic Robotics* is about ***estimating "the state" based on noisy models and noisy measurement data***.
- **In fact:** We will attempt to compute *belief distributions over all possible world states.*
- **These will be in the form:**
  - *Probability Mass Functions (discrete case)*
  - *Probability Density Functions (continuous case)*

Basic Concepts in Probability Theory

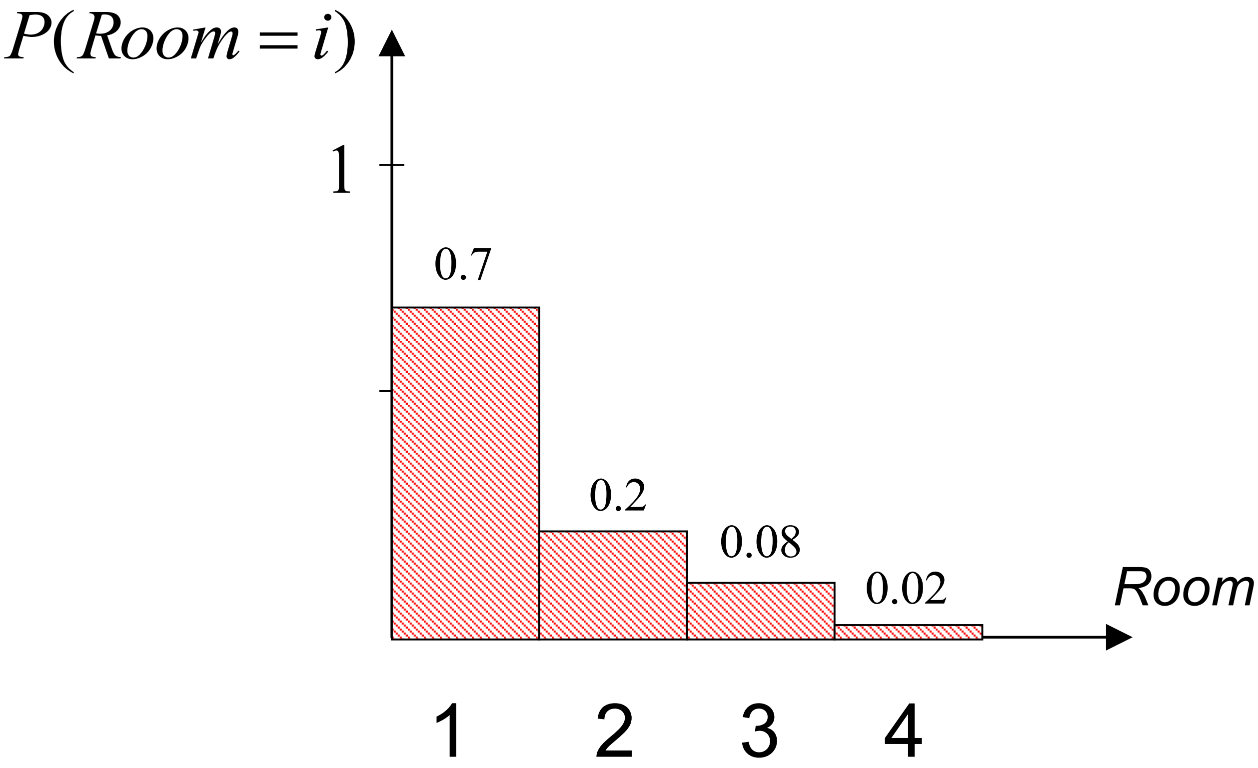
- *A Random Variable (rv) is*
  "a mathematical variable that takes on values based on the outcome of an uncertain event E"
- ***Examples of Events:***
  - E1: The US president in the 2028 will be Trump again!
  - E2: You will wake up tomorrow with a headache.
  - E3: You have Corona Virus!
- Each possible event is mapped into a value of the rv: For example $\boldsymbol{x}$ = 1,2,3 for $E1$, $E2$, $E3$
- Can be generalized to represent "continuous events" (e.g., battery level of a robot)

Discrete Random Variables

- X denotes a random variable.
- X can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $\Pr(X=x_i)$, or $P(x_i)$, is the *probability* that the random variable X takes on value x_i.
- $P(.)$ is called *probability mass function*.
- E.g.: $P(Room = i) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$

Discrete Random Variables

$$\Pr(Room = i) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$$

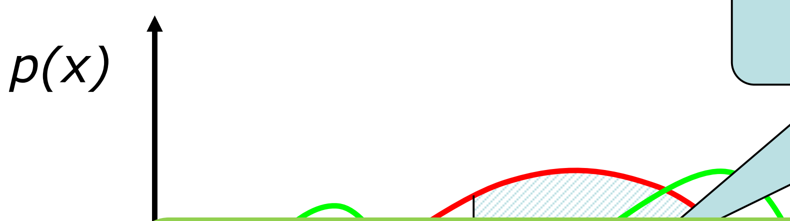


Continuous Random Variables

- X takes on values in the continuum.
- $p(X=x)$, or $p(x)$, is called a *probability density function*.

$$P(x \in (a,b)) = \int_a^b p(x)dx$$

- E.g.



$$P(x \in (a,b))$$

*However individual values of $P(X=x)$ have a name: called the “**likelihood**” of $X=x$.*

We cannot talk ab
for individual values of $X=x$ but of <u>intervals!!</u>

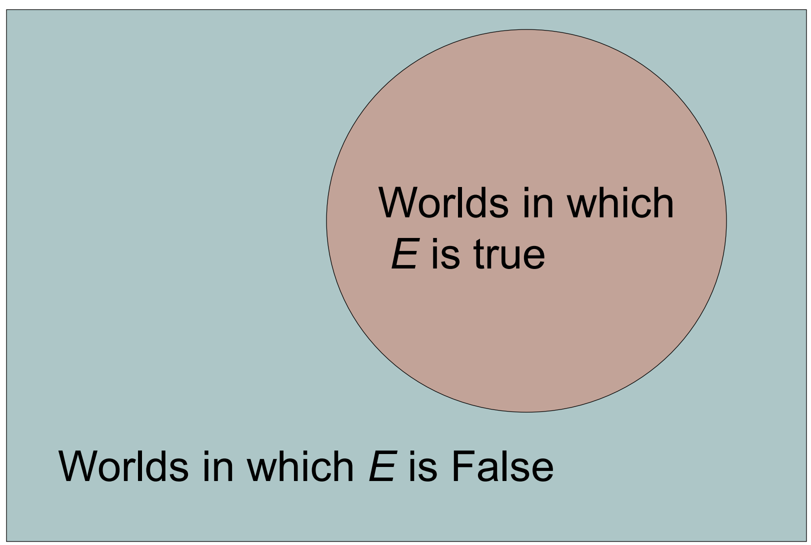
Meaning of Probabilities

- For the continuous case, $P(x \in (a,b))$ is the fraction of worlds in which event *E* causing $x \in (a,b)$ is true.

$$P(E) = P(x \in (a,b))$$

"<u>Event space</u>" of all possible worlds

Its area is 1



$P(x \in (a,b)) =$ P(*E*)= Area of reddish oval

Axioms of Probability Theory

Assume events *A and B*

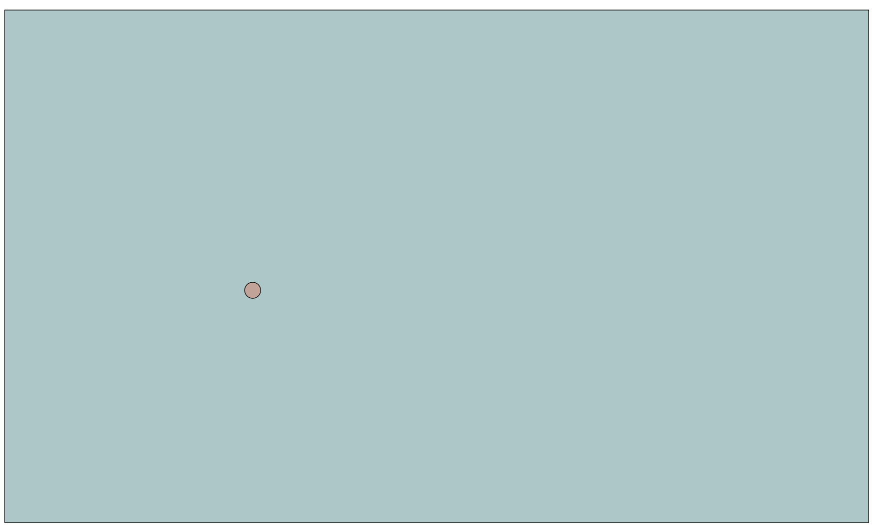
- **0 <= P(A) <= 1**
- **P(True) = P("*Sure Event*") = 1**
- **P(False) = P("*Impossible Event*") = 0**
- **P(A or B) = P(A) + P(B) - P(A and B)**

- *More popularly used Symbols* **"Or": V, "And": Λ**

Interpreting the Axioms

- <= 1
- P(True) = 1
-
- P(A or B) = P(A) + P(B) - P(A and B)

Assume events *A and B*



The area of A can't get any smaller than 0

And a zero area would mean "*no world could ever have A true*" – A is an "impossible event"

Interpreting the Axioms

- 0 <=
-
- P(False) = 0
- P(A or B) = P(A) + P(B) - P(A and B)

Assume events *A and B*

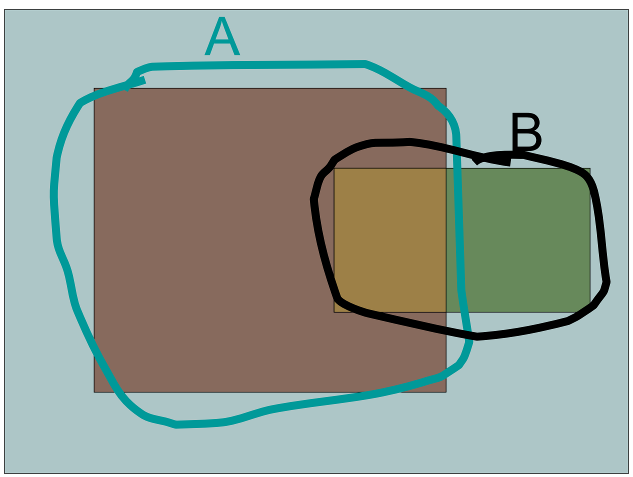
The area of A can't get any bigger than 1

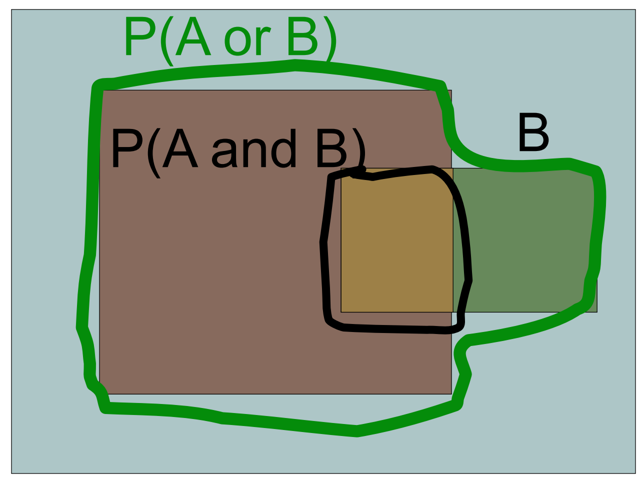
And an area of 1 would mean "*all worlds will have A true*" – A is a "sure event"

Interpreting the Axioms

- 0 <= P(A) <= 1
- P(True) = 1
- P(False) = 0
-

Assume events *A and B*





Simple addition and subtraction for *P*(.) values

These Axioms are Not to be Trifled With

- There have been attempts to propose different methodologies for uncertainty
  - Fuzzy Logic
  - Three-valued logic
  - Dempster-Shafer "Theory of Evidence"
  - Non-monotonic reasoning
- But the axioms of probability are the only system with this property:
- *If you gamble using them you can't be unfairly exploited by an opponent using some other system* [di Finetti 1931]
  http://plato.stanford.edu/archives/sum2003/entries/probability-interpret/#3.5

Theorems from the Axioms

- **Anything other than the axioms needs *proof* based on the axioms!**
- **Simple example: From the axioms,**

  0 <= P(A) <= 1, P(True) = 1, P(False) = 0
  P(A or B) = P(A) + P(B) - P(A and B)

- **we can prove:**

$$\boldsymbol{P(\text{not } A) = P(\neg A) = 1\text{-}P(A)}$$

$$\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(True) &= \Pr(A) + \Pr(\neg A) - \Pr(False) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}$$

Another important Theorem

- **Again from the Axioms:**

  0 <= P(A) <= 1, P(True) = 1, P(False) = 0
  P(A or B) = P(A) + P(B) - P(A and B)

- **We can prove:**

$$P(A) = P(A \wedge B) + P(A \wedge \neg B)$$

Do it for next week!!

- **How?**

Joint Probabilities and Independence

- **Probability that two or more events happen at the same time or "jointly".**

$$P(A \text{ and } B) = P(A \wedge B) = P(A, B)$$

$$P(X = x \text{ and } Y = y) = P(x, y)$$

- ***X* and *Y* are <u>independent r.v.s</u>, if and only if:**

$$p(x, y) = p(x)\,p(y)$$

A reminder about notation

- **Events are sets:**

$$P(A) = \Pr(A = true) = \Pr(\text{event A happens})$$

- **Continuous event spaces: We can only talk about actual probabilities of <u>intervals.</u>**

$$\Pr(x \in (a,b)) = \int_a^b p(x)dx$$

- **Otherwise in general a prob. density function:**

$$p(x) \qquad p(x,y)$$

Density function

Joint density function

Conditional Probability

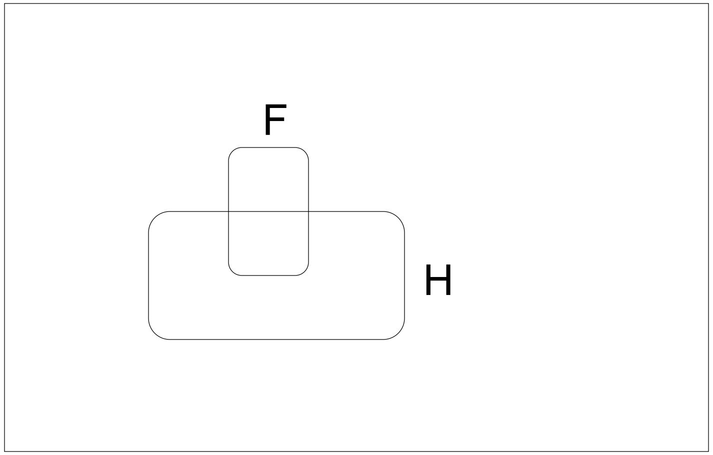
- Conditional probability is about an event A, <u>given that we know exatly</u> what happened with event B
- *Notation: P("A given B") =* $P(A|B)$

Conditional Probability

- $P(A|B)$ = Fraction of worlds in which B is true that also have A true

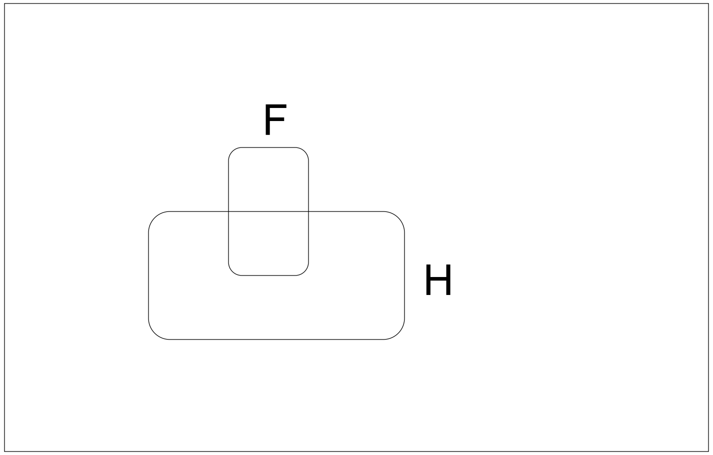
H = “Having a headache”
F = “Coming down with Flu”

“*Headaches are rare and flu is rarer, but if you’re coming down with flu what is the chance that you also have a headache?*



Conditional Probability

- *P*(*A*|*B*) = Fraction of worlds in which *B* is true that also have *A* true

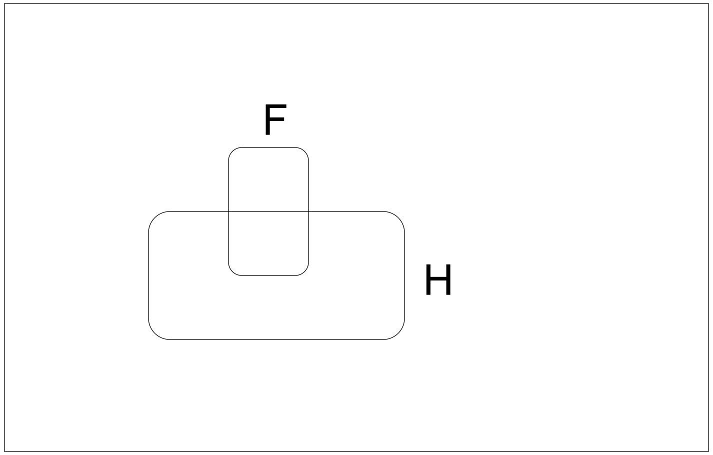


H = "Having a headache"
F = "Coming down with Flu"

"*Headaches are rare and flu is rarer, but if you're coming down with flu there's a 50-50 chance you'll have a headache.*"

P(H) = 1/10
P(F) = 1/40
P(H|F) = 1/2

Conditional Probability



H = "Have a headache"
F = "Coming down with Flu"

P(H) = 1/10
P(F) = 1/40
P(H|F) = 1/2

P(H|F) = Fraction of flu-infected worlds in which you have a headache

$$= \frac{\text{\#worlds with flu and headache}}{\text{\#worlds with flu}}$$

$$= \frac{\text{Area of "H and F" region}}{\text{Area of "F" region}}$$

Hence:

$$P(H|F) = \frac{P(H \wedge F)}{P(F)}$$

Definition of Conditional Probability

$$P(A \mid B) \triangleq \frac{P(A \wedge B)}{P(B)}$$

$$p(x \mid y) \triangleq \frac{p(x, y)}{p(y)}$$

Event set notation

pdf notation

Most convenient when we talk about discrete events & discrete random variables

Definition of Conditional Probability

$$P(A \mid B) \triangleq \frac{P(A \wedge B)}{P(B)}$$

Event set notation

$$p(x \mid y) \triangleq \frac{p(x, y)}{p(y)}$$

pdf notation

Most convenient when we talk about continuous random variables

Definition of Conditional Probability

$$P(A \mid B) \triangleq \frac{P(A \wedge B)}{P(B)}$$

Event set notation

$$p(x \mid y) \triangleq \frac{p(x, y)}{p(y)}$$

pdf notation

- **Corollary: The Chain Rule**

$$\Pr(A \wedge B) = \Pr(A \mid B) \Pr(B)$$

$$p(x, y) = p(x \mid y) p(y)$$

Multi-valued random variables

- Binary random variables have only “true” and “false” values,
- Represent whether an event *A* has happened or not,
- In general, discrete random variables also take multiple (but finite number of) values,
- Continuous r.v.s take *infinitely many values* (hence why we are dealing with intervals)

Total Probability and Marginals

- For *multi-valued discrete* and *continuous* random variables, we can show:

Discrete case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous case

$$\int p(x)\, dx = 1$$

$$p(x) = \int p(x, y)\, dy$$

$$p(x) = \int p(x \mid y) p(y)\, dy$$

"Total Probability Theorem"

OK. Now…

$$p(x, y) = p(y, x)$$

$$p(x \mid y)\, p(y) = p(y \mid x)\, p(x)$$

The "Bayes Rule"

$$p(x,y) = p(y,x)$$

$$\Rightarrow p(x \mid y)\,p(y) = p(y \mid x)\,p(x)$$

and hence

$$\Rightarrow p(x \mid y) = \frac{p(y \mid x)\,p(x)}{p(y)}$$

Bayes, Thomas (1763) "*An essay towards solving a problem in the doctrine of chances.*" *Philosophical Transactions of the Royal Society of London,* **53:370-418**

The “Bayes Rule”

“Posterior” belief about the state, having made the observation

would

sensor data”

“Prior” or “a-priori knowledge”

$p($

$\Rightarrow p(x$

and he ce

$$\Rightarrow p(x \mid y) = \frac{p(y \mid x)\, p(x)}{p(y)}$$

“Generative model” or “sensor model”

A normalization factor independent of *x*

“An essay

doctrine

actions of

370-418

How to compute *p*(*y*)?

$$p(x \mid y) = \frac{p(y \mid x)\,p(x)}{p(y)} = \frac{p(y \mid x)\,p(x)}{\sum_{x'} p(y \mid x')\,p(x')}$$ Discrete

$$p(x \mid y) = \frac{p(y \mid x)\,p(x)}{p(y)} = \frac{p(y \mid x)\,p(x)}{\int p(y \mid x')\,p(x')\,dx'}$$ Continuous

η

η is a *normalization factor* independent of *x*.
Can always be found using the fact $\sum_{x} P(x \mid y) = 1$

With the normalization factor

$$p(x \mid y) = \eta \cdot p(y \mid x) p(x)$$

- Mathematical convenience,
- Keep propagating η in all equations.
- It is always a different number!
- Reminds us simply that “the final result has to be normalized to 1”

Conditioning on a third variable

- Law of total probability:

$$P(x) = \int P(x, z) dz$$

$$P(x) = \int P(x \mid z) P(z) dz$$

$$P(x \mid \quad = \int P(x \mid \quad z) \, P(z \mid \quad dz$$

- "Bayes Rule with background knowledge":

$$P(x \mid y, \quad = \frac{p(y \mid x, \quad p(x \mid \quad}{p(y \mid \quad}$$

Conditional Independence

- *Conditional Independence* of *x* and *y*, *given z*, *iff:*

$$P(x, y \mid z) = P(x \mid z) P(y \mid z)$$

- Which is equivalent to:

$$P(x \mid z, y) = P(x \mid z)$$

$$P(y \mid z, x) = P(y \mid z)$$

Conditional Independence is extremely important and makes almost all algorithms in this book computationally feasible.

Conditional Independence

- Important note:

Conditional Independence **does not** imply *(absolute) independence*

- More interestingly:

(absolute) independence **does not** imply *conditional independence*

Expected Value of a R.V.

- "Expected Value" or "Expectation" of a r.v. *X* is given by

$$E[X] = \sum_x x \cdot p(x) \qquad \text{(discrete)}$$

$$E[X] = \int x \cdot p(x) dx \quad \text{(continuous)}$$

- *E*[.] is a linear operator:

$$E[aX + b] = aE[X] + b$$

- *E*[*x*] is also often called "the mean of *x*"

Variance of a R.V.

- Defined using the “Expectation” operator:

$$\mathrm{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

- Mean and Variance are the *first* and *second* “moments” of the r.v., *X.*
- Measures: “*the expected value of the squared deviation from the mean*”
- If we have a (column) vector r.v. ***X***, then we have the “Covariance Matrix” instead

The Covariance Matrix

- Defined again using the "Expectation" operator:

$$\mathrm{Cov}(\mathbf{X}) = E\left[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T\right]$$

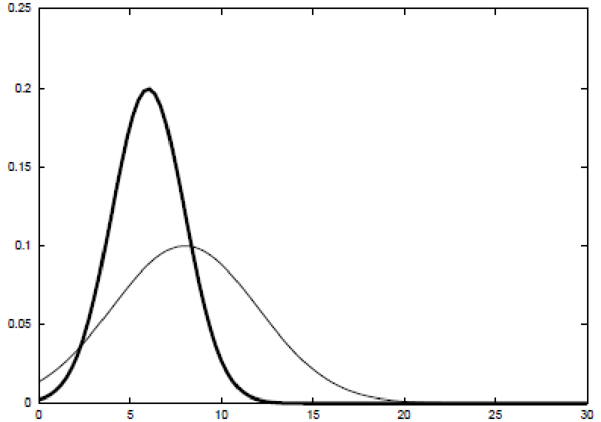
where ***X*** is column vector.

Example: Gaussian pdf

- Very common (why?) and useful probability distribution function,
- Described **for a scalar** r.v. *X* by:

$$p(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}$$

- which has a mean μ and variance σ^2

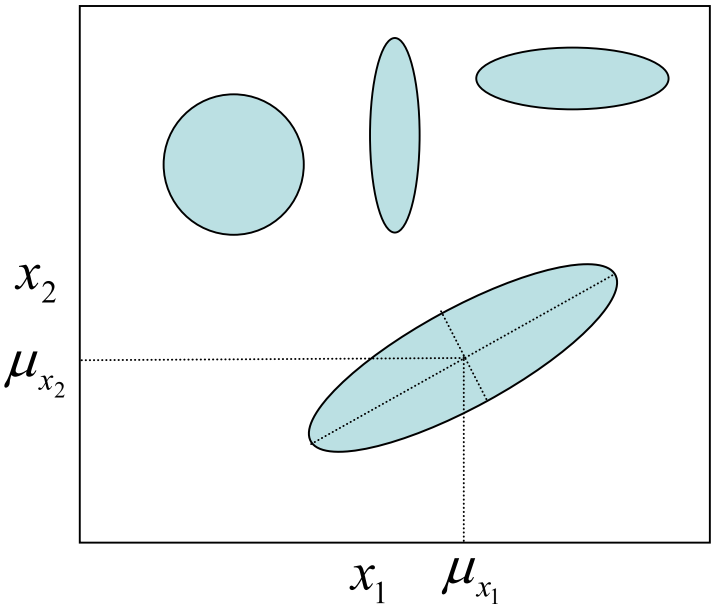


Multivariate Gaussian pdf

- For a multivariate (vector) r.v. ***X*** :

$$p(\mathbf{x}) = \det(2\pi\mathbf{\Sigma})^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

- which has a mean vector ***μ*** and Covariance matrix ***Σ***
- Notation: $N\{x; \mu, \Sigma\}$



Enthropy

- Originates in *Information Theory*,
- Represents the *expected information* (*in bits*) that a random variable carries

$$H_p(x) = E[-\log_2 p(x)]$$

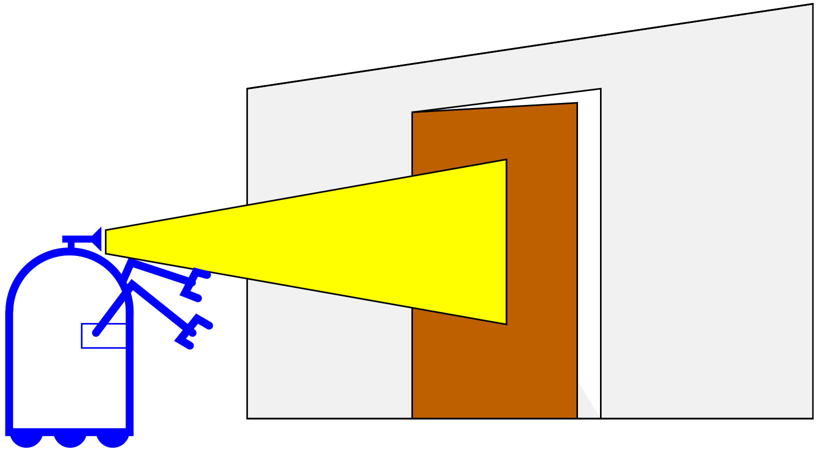
$$H_p(x) = -\sum_x p(x) \log_2 p(x) \qquad \text{(discrete)}$$

$$H_p(x) = -\int p(x) \log_2 p(x) dx \qquad \text{(continuous)}$$

- Used for robotic exploration in this context

A Simple example of state estimation

- Suppose a robot obtains a measurement *z* indicating that a *door is open.* (observation)
- What is the probability that the door is in fact open (x=open)
- Namely: What is $P(x=open|z=open)$?



A Simple example of state estimation

- *P(x=open|z)* is diagnostic. (A question)
- *P(z|x=open)* is causal.
- Often causal knowledge is easier to obtain (often by repeated experimentation).
- Bayes rule allows us to use causal knowledge to obtain diagnostic answers:

$$P(x = open \mid z) = \frac{P(z \mid x = open)P(x = open)}{P(z)}$$

How can we compute these terms?

A Simple example of state estimation

- **A priori knowledge:** *P(x=open) = P(x=¬open) = 0.5*
- **Sensor Behavior ("Model"):** *P(z=op|x=op)=0.6; P(z=op|x=¬op)=0.3*

$$P(x=op\mid z=op)=\frac{P(z=op\mid x=op)P(x=open)}{P(z=op\mid x=op)\,p(x=open)+P(z=op\mid x=\neg op)\,p(x=\neg open)}$$

$$P(x=op\mid z=op)=\frac{0.6\cdot 0.5}{0.6\cdot 0.5+0.3\cdot 0.5}=\frac{2}{3}=0.67$$

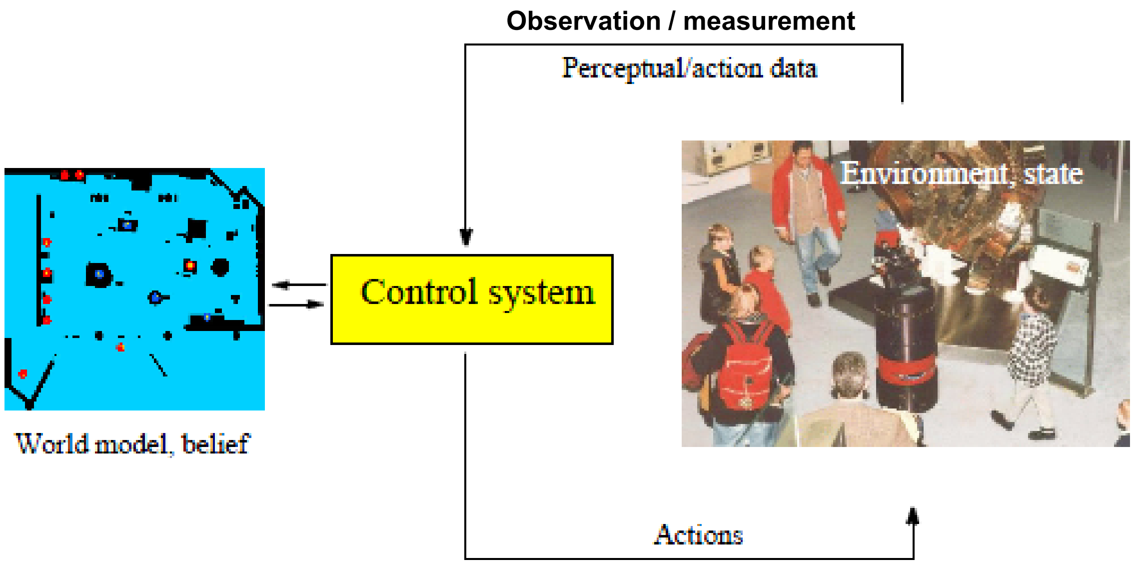
- Observation *z=open* brings our "belief" on the door state closer to the "open" state!!
- A similar calculation can be done for when *z*=closed is our observation!!

Recursively combining new data

- Suppose our robot obtains another observation z_2.
- How can we *recursively integrate* this new information to our present "*belief*" ?
- More generally, how can we estimate *the "belief"* $p(x \mid z_n, z_{n-1}, \ldots z_1)$?

Robot and Its Environment

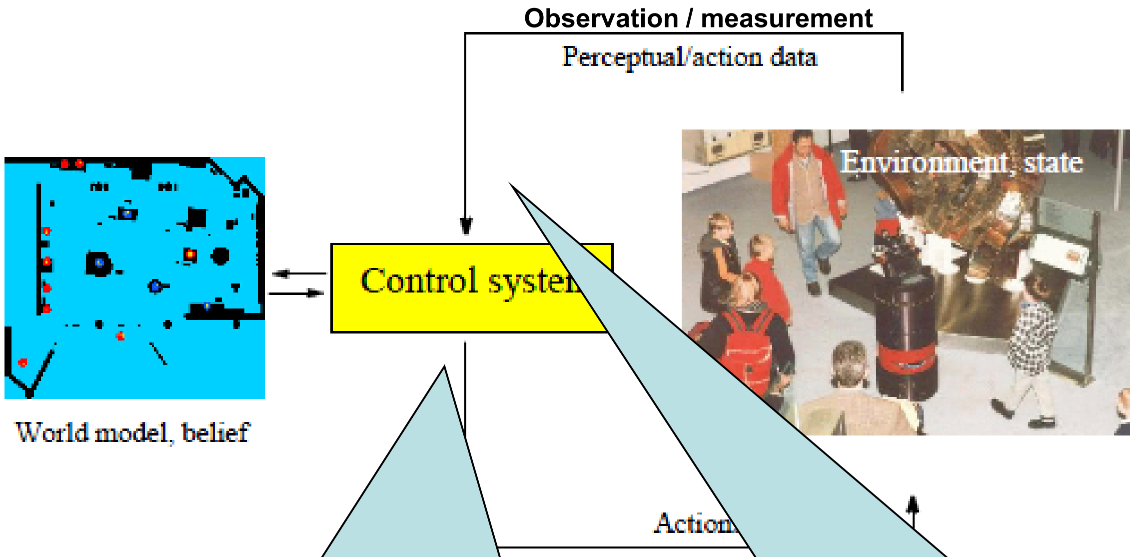
- Some terminology first,



- State (Static/Dynamic) – Pose – Landmark – Complete / Incomplete State – Markov Chain (Markov Assumption) -

Robot / Environment Interaction

- Measurement and actuation is represented by time-sequences of data...



Control Data

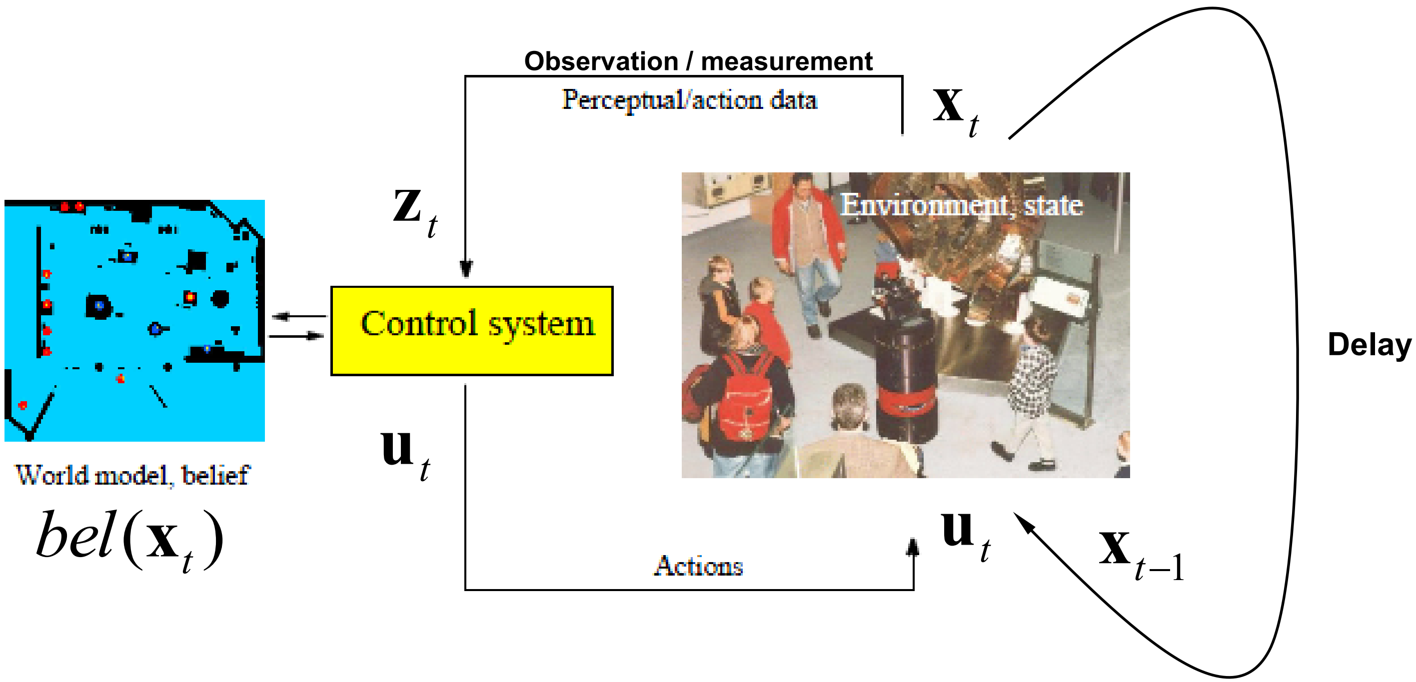
$$\mathbf{u}_{t_1:t_2} = \mathbf{u}_{t_1}, \mathbf{u}_{t_1+1}, \ldots, \mathbf{u}_{t_2}$$

Measurement Data

$$\mathbf{z}_{t_1:t_2} = \mathbf{z}_{t_1}, \mathbf{z}_{t_1+1}, \ldots, \mathbf{z}_{t_2}$$

Probabilistic Generative Laws

- All interaction (measurement and actuation) is uncertain,



Probabilistic Generative Laws

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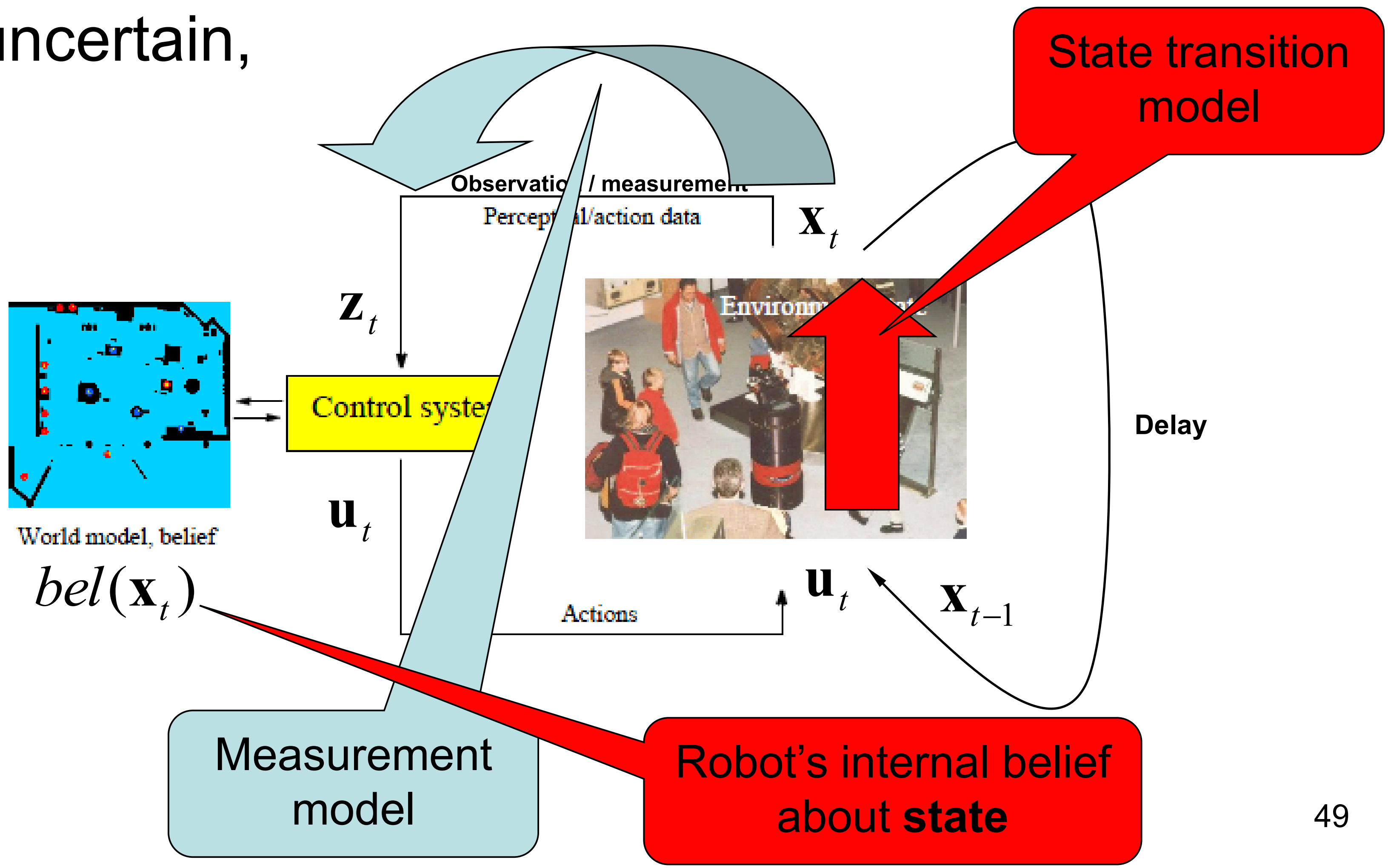


Probabilistic Generative Laws

- Variables influence each other through probabilistic models: ***conditional density functions***
- ***Complete State Assumption*** and ***Markov Property*** *allows us to write*

State-transition Model

$$p(\mathbf{x}_t \mid \mathbf{x}_{0:t-1}, \mathbf{z}_{1:t-1}, \mathbf{u}_{1:t}) = p(\mathbf{x}_t \mid \mathbf{x}_{t-1}, \mathbf{u}_t)$$

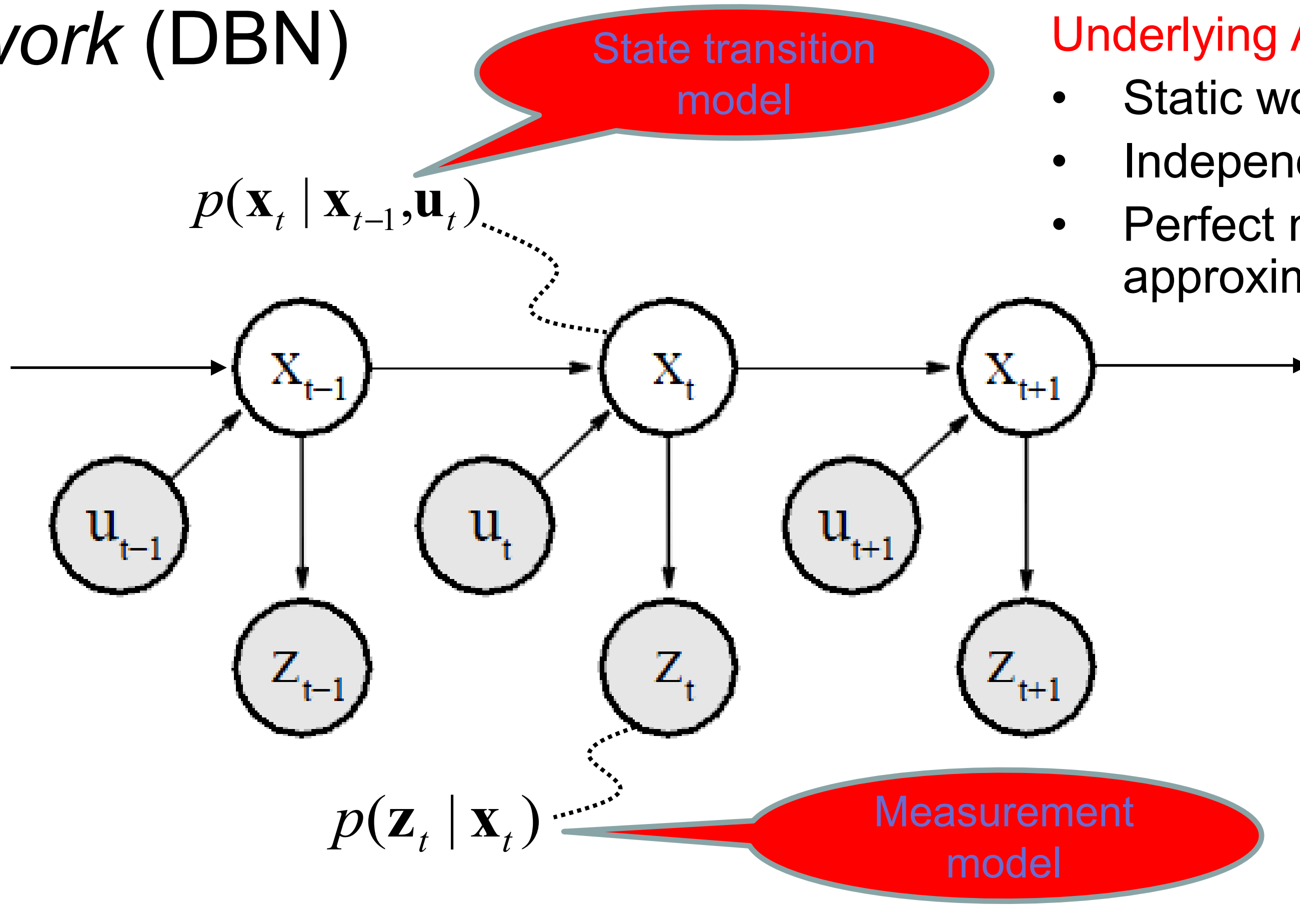
Measurement Model

$$p(\mathbf{z}_t \mid \mathbf{x}_{0:t}, \mathbf{z}_{1:t-1}, \mathbf{u}_{1:t}) = p(\mathbf{z}_t \mid \mathbf{x}_t)$$

Conditional Independence

Probabilistic Generative Laws

- The probabilistic relation forms a graph called *Hidden-Markov-Model* (HMM) or *Dynamic Bayes Network* (DBN)



Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Bayes Filter: The framework

- **Given:**
  - Stream of observations $\{z_t\}$ and action data $\{u_t\}$ *:*

$$d_t = \{u_1, z_1 \ldots, u_t, z_t\}$$

  - Sensor model $p(\mathbf{z}_t \mid \mathbf{x}_t)$
  - Action (state transition) model $p(\mathbf{x}_t \mid \mathbf{x}_{t-1}, \mathbf{u}_t)$
  - Prior density of the system state $p(x_0)$.
- **Wanted:**
  - Estimate of the state x_t of the dynamical system at *t.*
  - This "posterior" of the state is also called **Belief**:

$$Bel(x_t) = P(x_t \mid u_{1:t}, z_{1:t})$$

  - We would like to do this *recursively* for all time *t.*

Bayes Filter: Recursive estimation

z = observation
u = action
x = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Bayes
$$= \eta\ P(z_t \mid x_t, u_1, z_1, \ldots, u_t)\ P(x_t \mid u_1, z_1, \ldots, z_{t-1}, u_t)$$

Markov
$$= \eta\ P(z_t \mid x_t)\ P(x_t \mid u_1, z_1, \ldots, z_{t-1}, u_t)$$

Total prob. + chain rule
$$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, z_{t-1}, u_t, x_{t-1})\ P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}, u_t)\ dx_{t-1}$$

Markov
$$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}, u_t)\ dx_{t-1}$$

Markov
$$= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\, P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1})\ dx_{t-1}$$

$$\boxed{= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ Bel(x_{t-1})\ dx_{t-1}}$$

The Bayes Filter Algorithm in Short

$$bel(x_t) = \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; Bel(x_{t-1}) \; dx_{t-1}$$

“*prediction*” stage without the measurement z_t at final time t

Uses the “Motion Model”

Uses the “Measurement Model”

“*measurement update*” stage intcorporating measurement z_t

```
Algorithm Bayes_filter(bel(x_t ...
    for all x_t do
        bel‾(x_t) = ... bel(x_{t-1}) dx_{t-1}
        bel(x_t) = η ... bel(x_t)
    endfor
    return bel(x_t)
```

Table 2.1 The general algorithm for Bayes filtering.

The Bayes Filter Algorithm

$$bel(x_t) = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ Bel(x_{t-1}) \ dx_{t-1}$$

1. Algorithm **Bayes_filter**(*Bel(x),d*):
2. η=0
3. If *d* is a perceptual data item *z* then
4. For all *x* do
5. $Bel'(x) = P(z \mid x) Bel(x)$
6. $\eta = \eta + Bel'(x)$
7. For all *x* do
8. $Bel'(x) = \eta^{-1} Bel'(x)$
9. Else if *d* is an action data item *u* then
10. For all *x* do
11. $Bel'(x) = \int P(x \mid u, x') \ Bel(x') \ dx'$
12. Return *Bel'(x)*

Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- A Bayes filter is a probabilistic tool for recursively estimating the state of a dynamic system.

Exercises

- *Homework is coming (will be in metuonline soon! You will be notified.)*
- *Read Chapter 1 & 2 from the textbook*
- *Study the full example at the end of Chapter 2.*