CENG 789 – Digital Geometry Processing

02- Polygons and Triangulations

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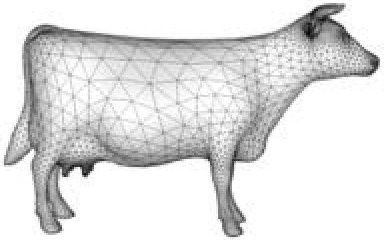
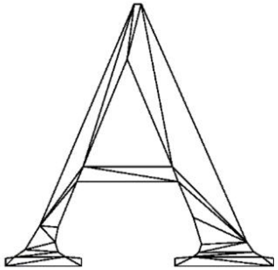
Polygons

- We'll deal with thin-shell surfaces, represented by polygon meshes.
  - set of polygons, e.g., triangles, representing a 2D *surface* embedded in 3D.



  - set of polygons representing a 2D *surface* embedded in 2D //not our interest.



- Let's analyze polygons in detail.

Polygons

- ✓ Analogy

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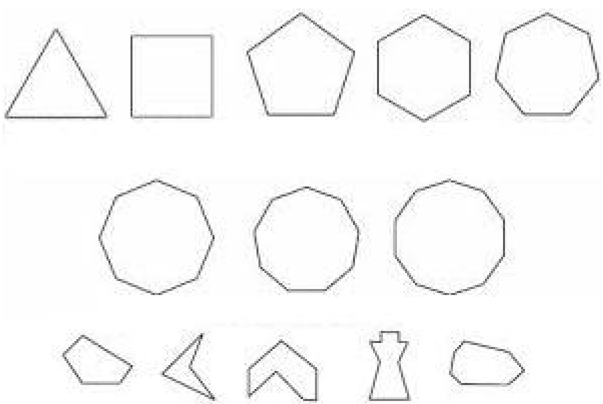

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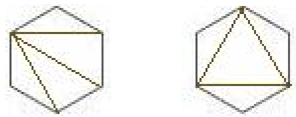
- ✓ Polygons are to planar geometry as integers are to numerical math.

Polygons

✓ Analogy

$100 = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$ ~ Prime factorization (unique)



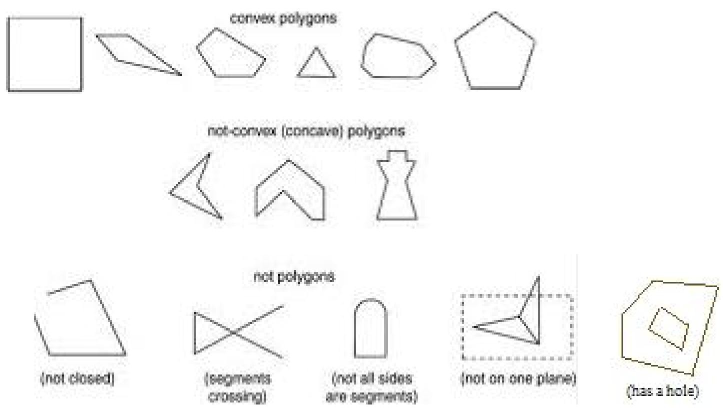
. . ~ Prime factorization (not unique*)

* no benefit of Fundamental Theorem of Arithmetic

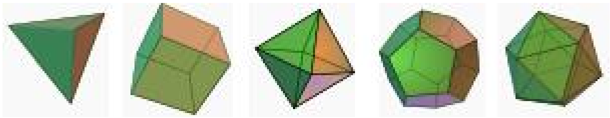
✓ Triangulations are the prime factorization of polygons (come to that later).

Polygons

- ✓ Polygon: closed region of the plane bounded by a finite collection of line segments forming a closed curve that does not intersect itself.
- ✓ Line segments are called edges.
- ✓ Points where 2 edges meet are called vertices.
- ✓ Vertices are ordered in a polygon.



- ✓ 3D generalization: polyhedron. //some cool theorems proved in the first class: 01-intro.ppt



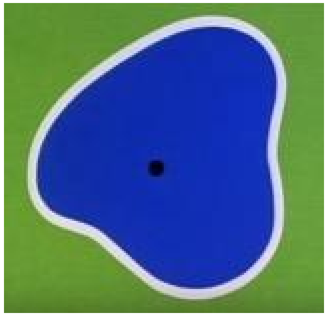
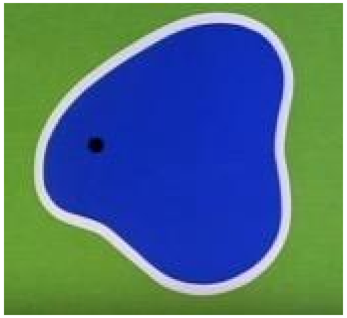
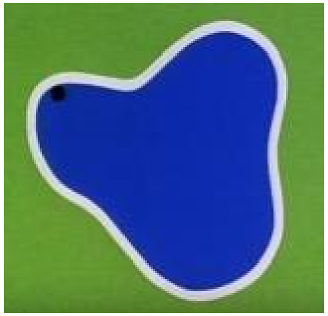
Polygonal Jordan Curve Thm

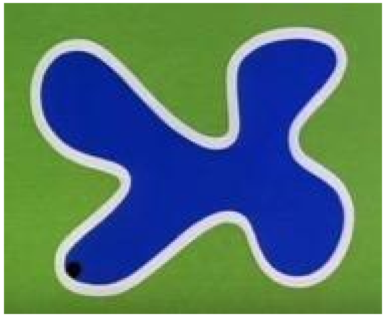
- ✓ Theorem: boundary B (white) of a polygon P partitions the plane into 2 parts, namely bounded interior (b) and unbounded exterior (g).
- ✓ Intuition: Shows the Fundamental similarity between all polygons!

- ✓ Each form above divides the paper into an inside part (blue) and an outside part (green).

Polygonal Jordan Curve Thm

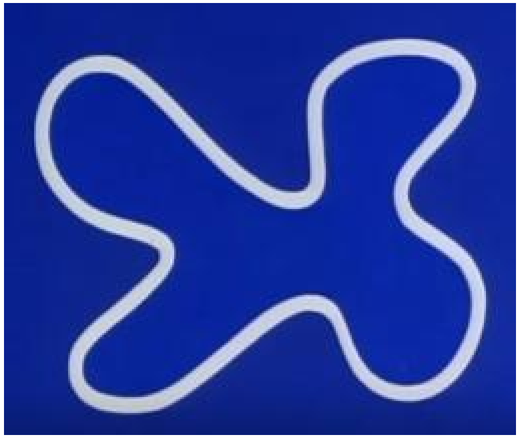
- ✓ Theorem: boundary B (white) of a polygon P partitions the plane into 2 parts, namely bounded interior (b) and unbounded exterior (g).
- ✓ A point that lies in 1 part cannot get to the other no matter how the curve/boundary is stretched.

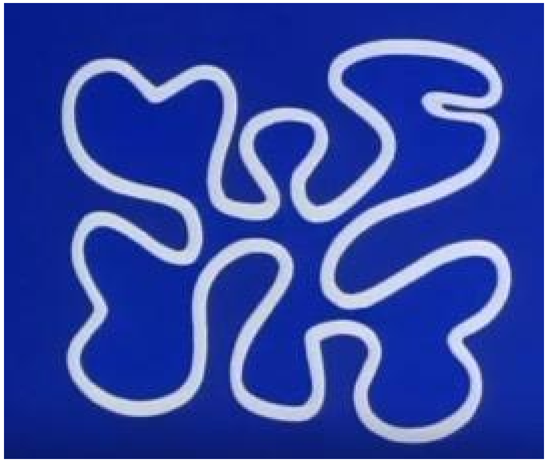






Polygonal Jordan Curve Thm

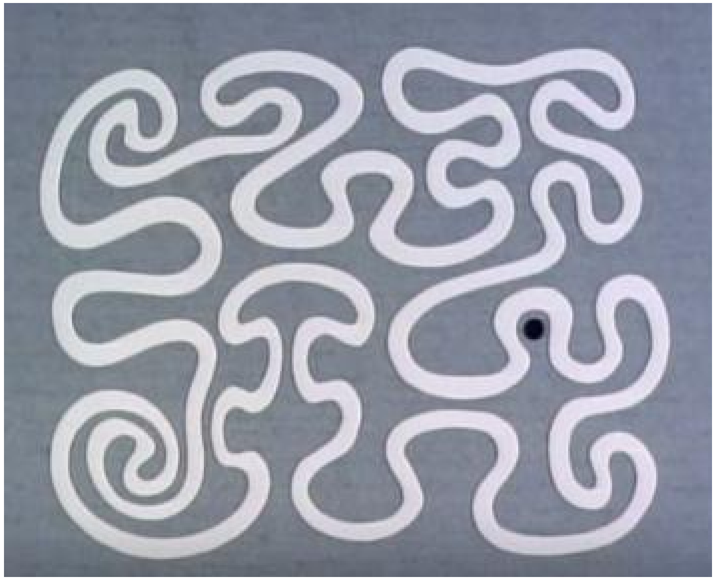
- ✓ Theorem: boundary B (white) of a polygon P partitions the plane into 2 parts, namely bounded interior (b) and unbounded exterior (g).
- ✓ A point that lies in 1 part cannot get to the other no matter how the curve/boundary is stretched.
- ✓ It will behave the same no matter how the boundary is distorted.





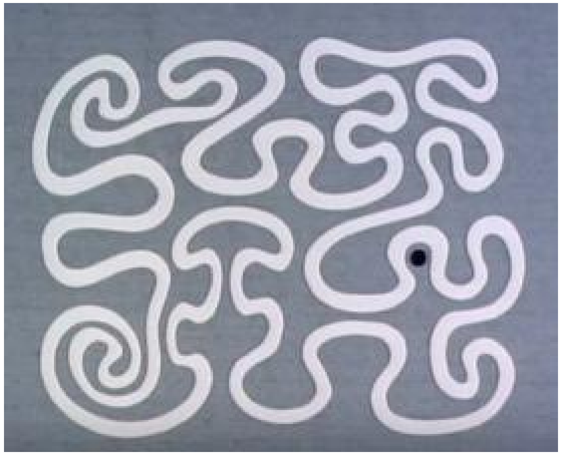
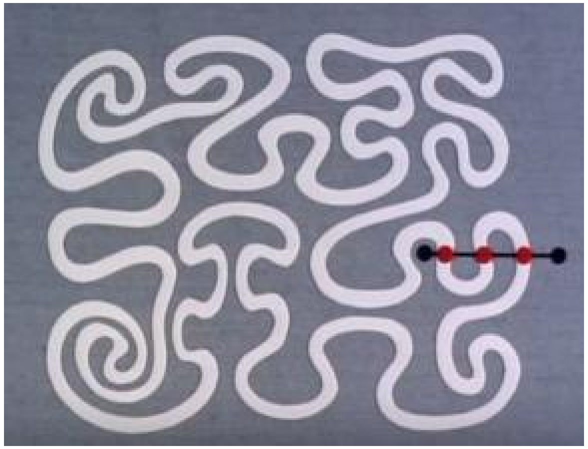
Polygonal Jordan Curve Thm

- ✓ Theorem: boundary B (white) of a polygon P partitions the plane into 2 parts, namely bounded interior (b) and unbounded exterior (g).
- ✓ A point that lies in 1 part cannot get to the other no matter how the curve/boundary is stretched.
- ✓ Now it is more difficult to tell whether point x is inside or outside. But wait there is an easy way!



Polygonal Jordan Curve Thm

- ✓ Theorem: boundary B (white) of a polygon P partitions the plane into 2 parts, namely bounded interior (b) and unbounded exterior (g).
- ✓ In/Out: draw a line from x to infinity(this line not parallel to any edge of P; ok ‘cos P has finite # edges).
- ✓ If line intersects B odd # of times: x is inside.
- ✓ Else: x is outside.
- ✓ Holds for arbitrary x.

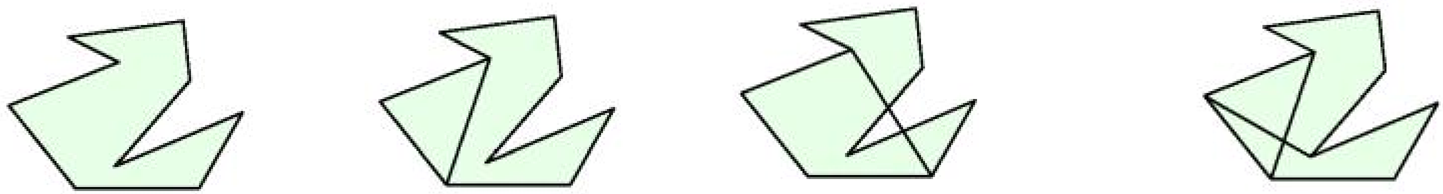

Polygonal Jordan Curve Thm

- ✓ Theorem: boundary B (white) of a polygon P partitions the plane into 2 parts, namely bounded interior (b) and unbounded exterior (g).

- ✓ Observation: the trivial odd-even algorithm is triggered to decided whether a user clicks inside a region in a game or a button in a GUI.

Diagonal

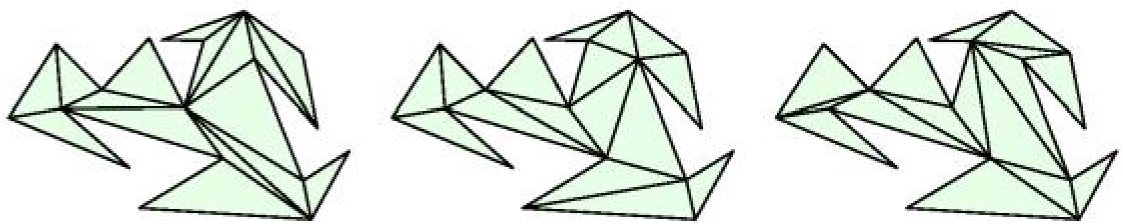
- ✓ Diagonal of polygon P is a line segment connecting 2 vertices of P and lying in the interior of P.



P with a *diagonal* a *line segment* *crossing diagonals*

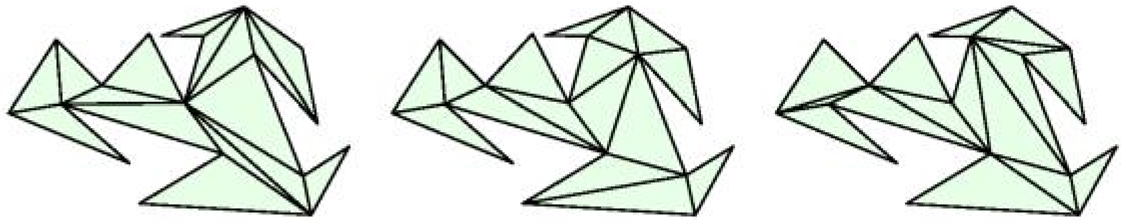
Triangulation

- ✓ Triangulation of polygon P is a decomposition of P into triangles by a maximal set of noncrossing diagonals.
- ✓ Maximal: no new diagonal may be added without crossing existings.
- ✓ Easier than triangulation of a structureless point set, e.g., Delaunay.
  - ✓ We will see this general case later; now just assume we have the boundary as an ordered set of vertices, i.e., as polygon P.



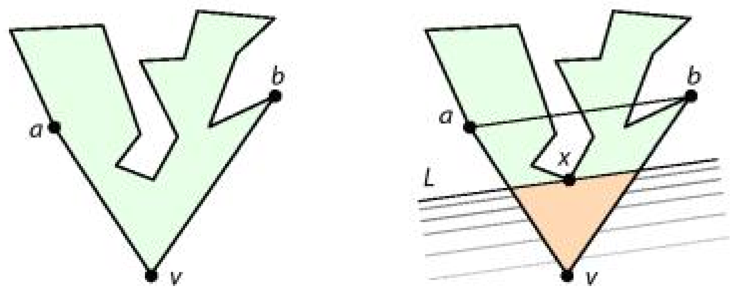
Triangulation

- ✓ How many different triangulations does polygon P have?
- ✓ How many triangles are in each triangulation of P?
- ✓ Does every polygon have a triangulation?
- ✓ Must every polygon have at least 1 diagonal?



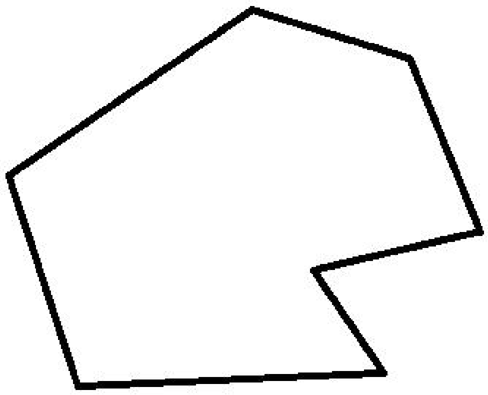
Triangulation

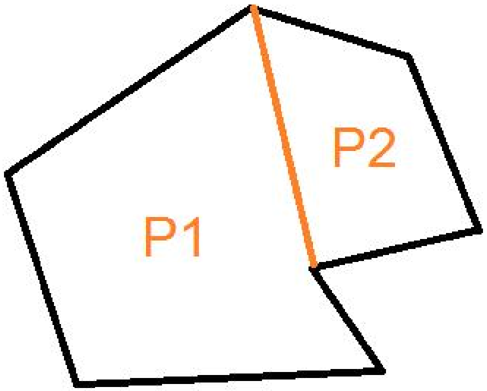
- ✓ Must every polygon w/ 4+ vertices have at least 1 diagonal?
  - ✓ YES.
- ✓ Proof: v bottom vertex. a & b adjacent to v. If line segment ab lies in P, it is the diagonal. Else sweep line L from v towards ab, parallel to ab. x first vertex L touches. vx is the desired diagonal.



Triangulation

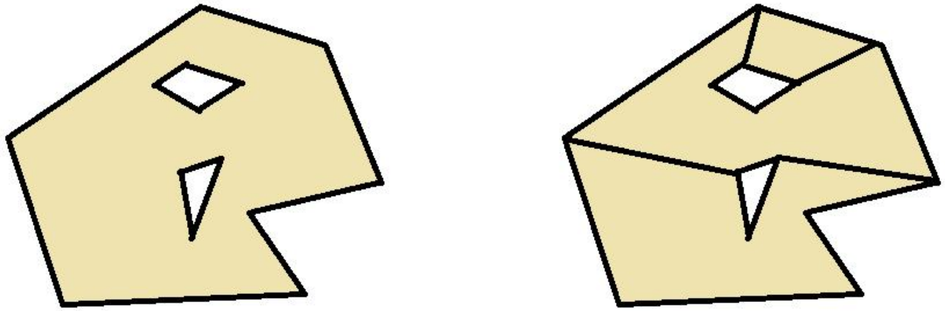
- ✓ Does every polygon have a triangulation?
  - ✓ YES.
- ✓ Proof: Induct on n = # of vertices of P. If n=3, P=triangle; done. For n>3, use prev theorem to find a diagonal cutting P into P1 and P2, which are triangulatable by induction hypothesis. By Jordan theorem, interior of P1 is exterior of P2 and vice versa; hence no triangle overlap.





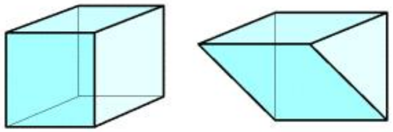
Triangulation

- ✓ Does every polygonal region w/ holes have a triangulation?
  - ✓ YES.
- ✓ Proof: Induct on h = # of holes.



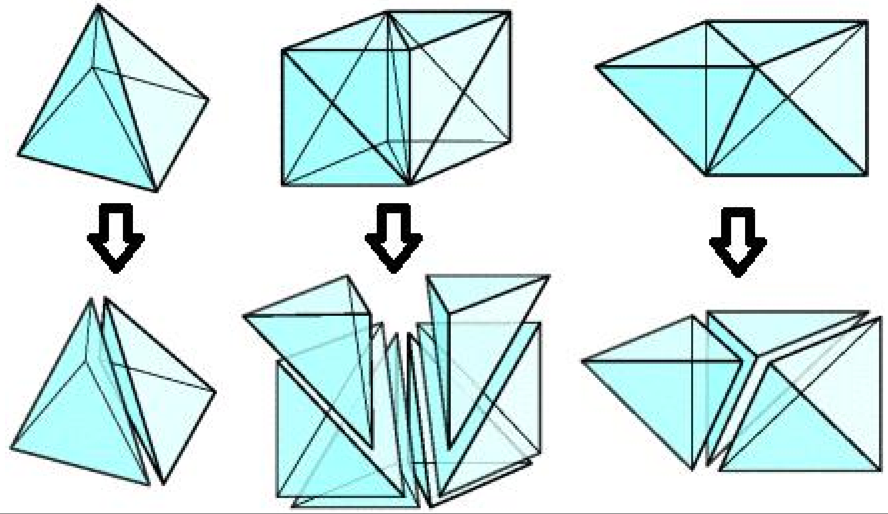
Triangulation

- Does every polyhedron have a tetrahedralization?
  - NO.
  - Polyhedron: 3D version of a polygon, a 3D solid bounded by finitely many polygons.



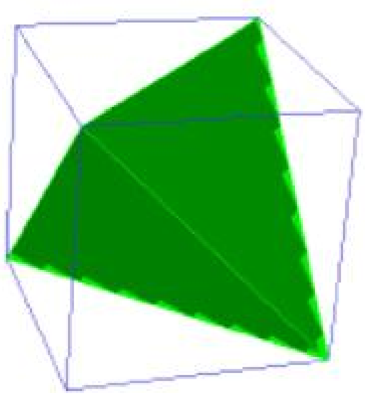
  - Tetrahedron: simplest polyhedron; pyramid with a triangular base.

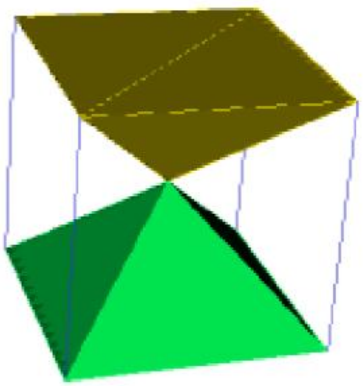
  - Tetrahedralization: partition of a polyhedron into tets.



Triangulation

- Does every polyhedron have a tetrahedralization?
  - NO.
  - Tetrahedralization of a cube into 5 and 12 tetrahedra.

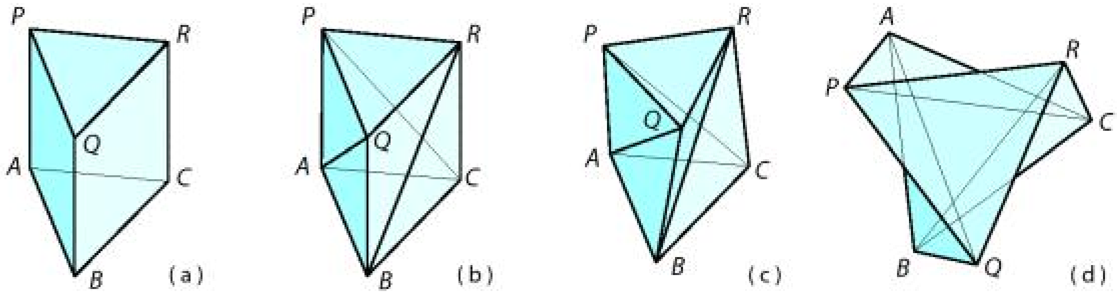




  - Related read: There are 174 Subdivisions of the Hexahedron into Tetrahedra, SIGGRAPH Asia 2018.

Triangulation

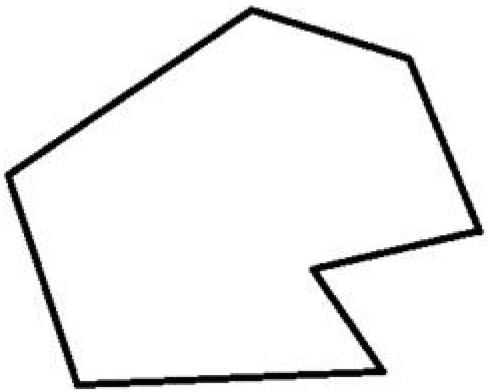
- ✓ Does every polyhedron have a tetrahedralization?
  - ✓ NO. A counter example exists: Schonhardt polyhedron.

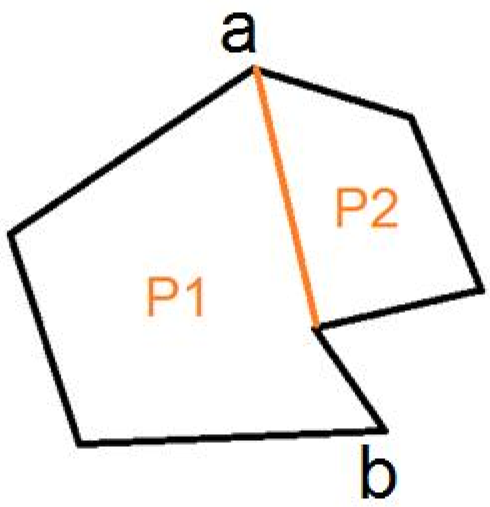


  - ✓ Notice that all possible diagonals are external.

Triangulation

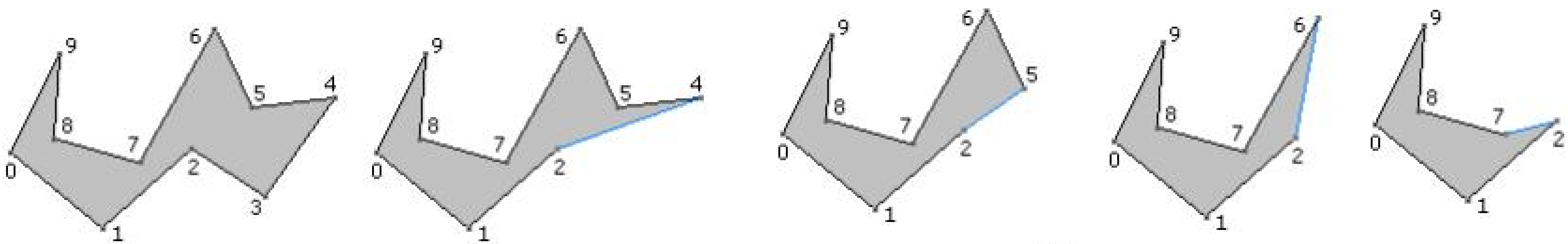
- ✓ Number of triangles in any triangulation of a polygon P is the same and equal to n-2 if P has n vertices.
- ✓ Proof: Induct on n = # of vertices of P. If n=3, trivially true. Diagonal ab exists. P1 and P2 have n1 and n2 vertices. n1 + n2 = n + 2 'cos a and b appear in both P1 and P2. By induction hypo, there are n1-2 and n2-2 triangles in P1 and P2. Hence P has
  - ✓ (n1-2) + (n2-2) = n1+n2 – 4 = n+2 – 4 = n - 2 triangles.

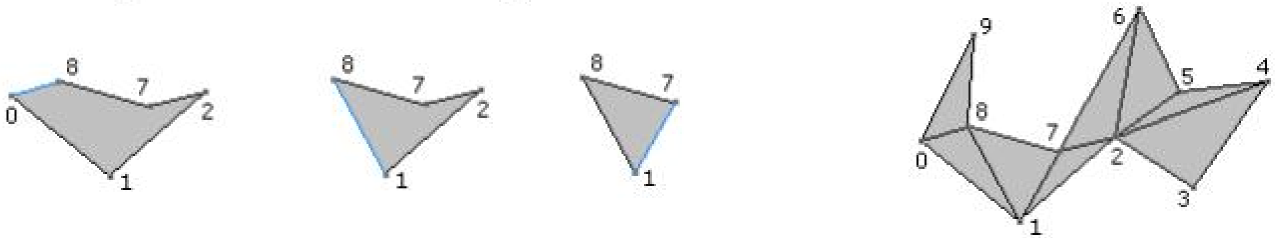




Triangulation

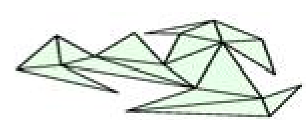
- ✓ Ears: 3 consecutive vertices a b c form ear of P if ac is a diagonal.





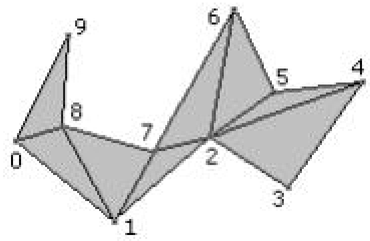
(our 1st triangulation algo aka *ear clipping method*)

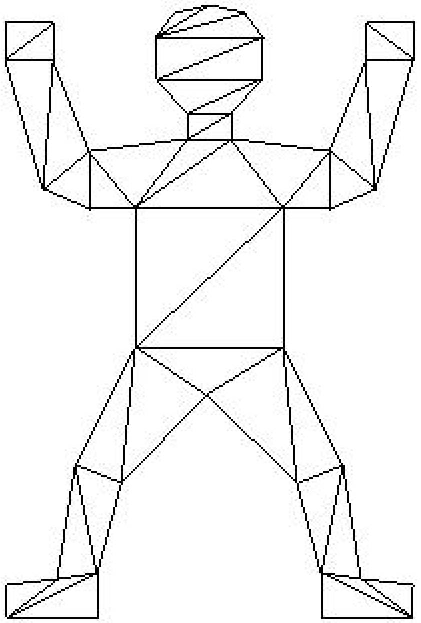
- ✓ Every polygon w/ 4+ vertices has at least 2 ears.
- ✓ Proof: By prev thm P has n-2 triangles. Each tri covers at most 2 edges of boundary. Because there're n edges on boundary but only n-2 tris, at least 2 tris must contain 2 edges of boundary (assume each tri covers 1; n-2 covered; need +2). These are the ears.



Triangulation

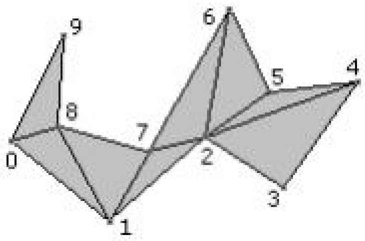
- ✓ Sum of interior angles of any polygon with n vertices is pi(n-2).
- ✓ Proof: every triangulation has n-2 triangles and sum of each tri is pi.

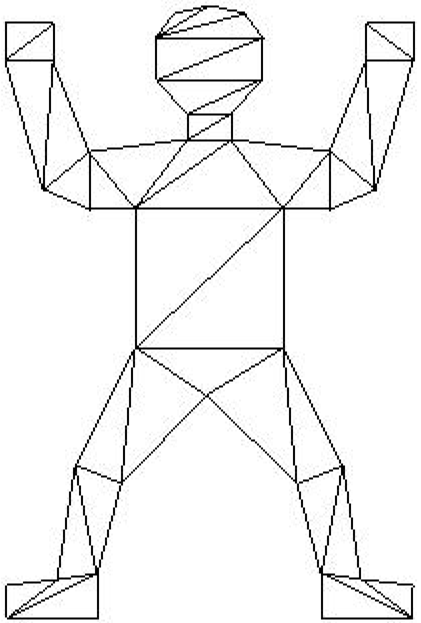


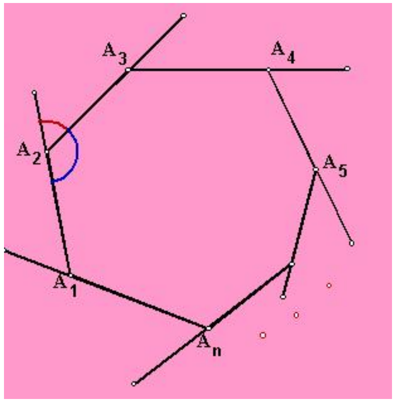


Triangulation

- Sum of turn angles of any polygon with n vertices is 2pi. Turn angle at a vertex is pi – internal angle at v.
- Proof: Follows from npi – (n-2)pi = 2pi.

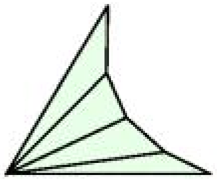






Triangulation

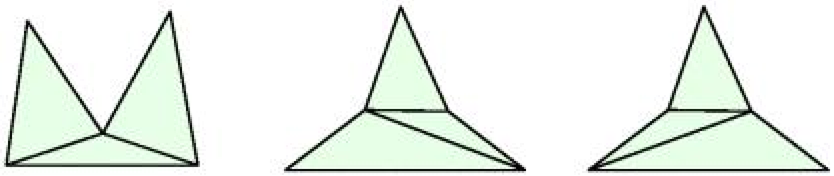
- ✓ How many different triangulations does polygon P have?
  - ✓ No closed formula ‘cos small position changes can lead to different triangulatns.
- ✓ Find a polygon with n vertices that has a unique triangulation.
  - ✓ 3 convex vertices and a reflex chain of n-3 vertices, generalizing:
  - ✓ A vertex is convex/reflex if its angle is less than or equal to/greater than pi.



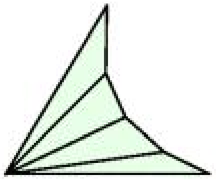
- ✓ A polygon is convex if all its vertices are convex. Or equivalently, if every pair of nonadjacent vertices determines a diagonal.
- ✓ How many different triangulations does a convex polygon P have?
  - ✓ n+2 vertices P has C_n (Catalan number) triangulations.

Triangulation

- ✓ How many different triangulations do these polygons have?

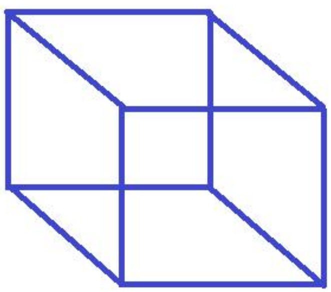


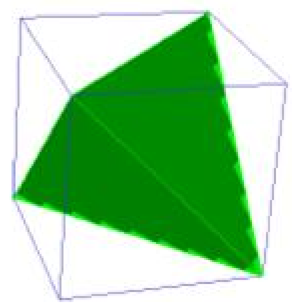
1 2 2

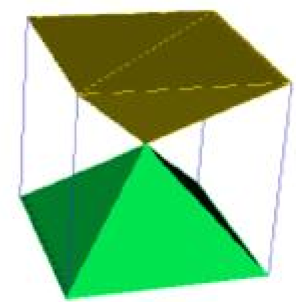
- ✓ Find a polygon with n vertices that has exactly 2 triangulations.
  - ✓ Hint:  and a rectangle.

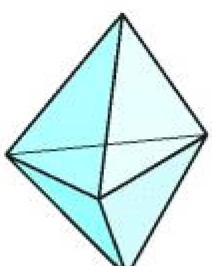
Triangulation

- ✓ Recall in 2D every polygon with n vertices has the same # of triangles in any triangulations: n-2 triangles.
- ✓ How about polyhedra in 3D?
  - ✓ Not valid ☹ Two different tetrahedralizations of the same polyhedron have different # of tets:

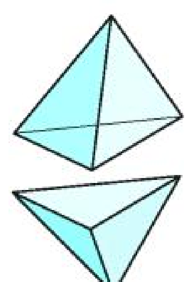


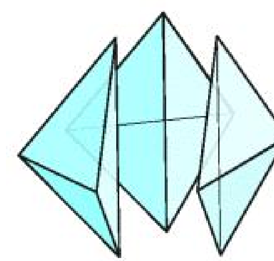






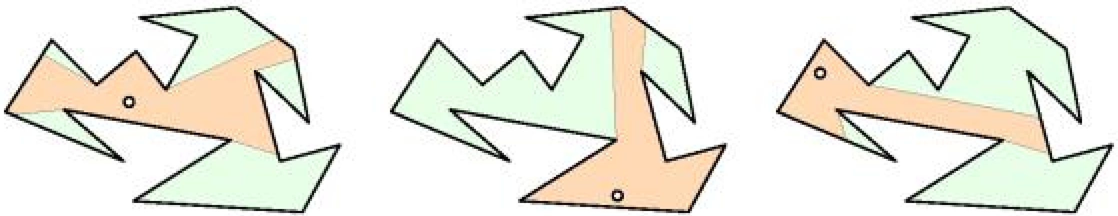
OR





Art Gallery Problem

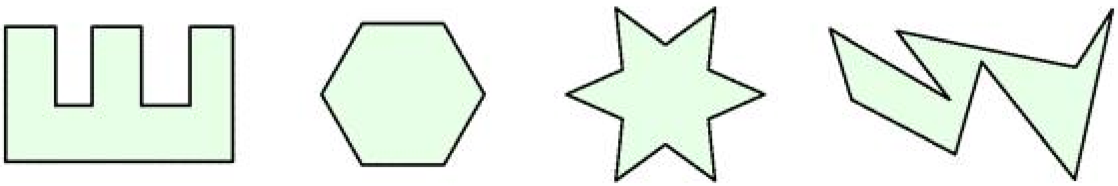
- ✓ Min # of stationary guards needed to protect the gallery.
- ✓ Polygonal floor plan of the gallery.
- ✓ Each guard has 360 degree visibility.
- ✓ Protect: every point in P is visible to some guard.



- ✓ x in P is visible to y in P if line segment xy lies in P.

Art Gallery Problem

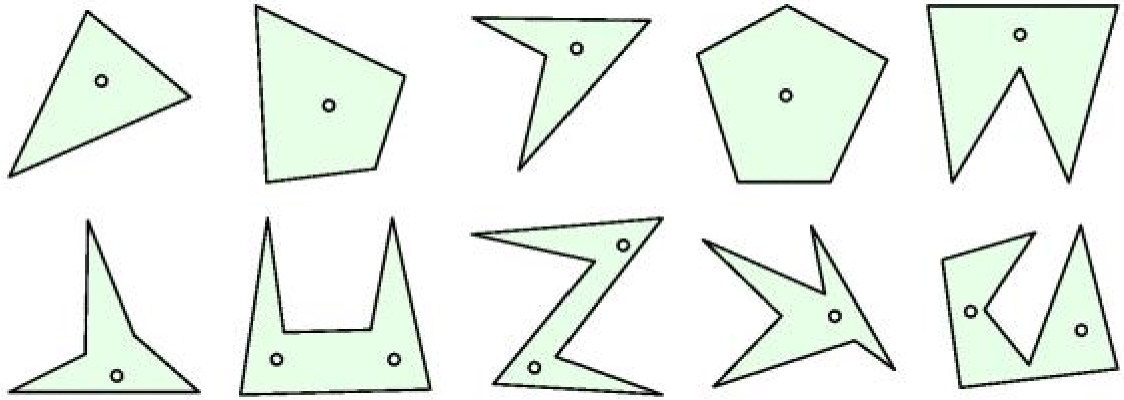
- ✓ Min # of guards to protect/cover these polygons.



3 1 1 2

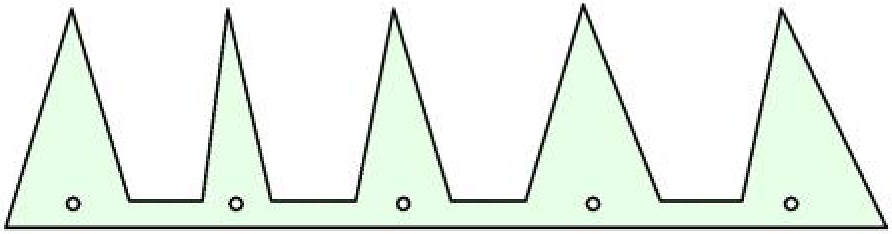
Art Gallery Problem

✓ First time 2 guides are needed is for some hexagons.



Art Gallery Problem

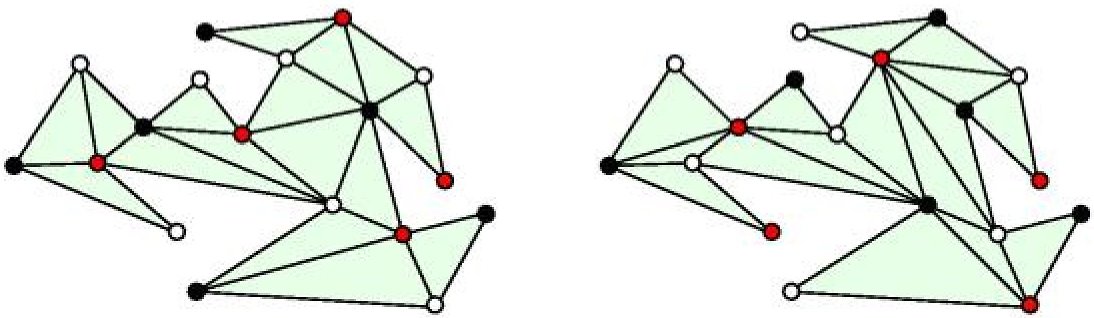
- ✓ Bound that is good for any polygon with n vertices?
- ✓ Reflex vertices will cause problems w/ large # of vertices.
- ✓ Since there can exist so many reflexes construct a useful example based on prongs: a comb-shaped example.



- ✓ A comb of m prongs has 3m vertices.
- ✓ Since each prong needs its own gard, at least floor(n/3) guards needed.
- ✓ Lower bound: sometimes floor(n/3) guards necessary.
- ✓ Is this also sufficient? (yes)

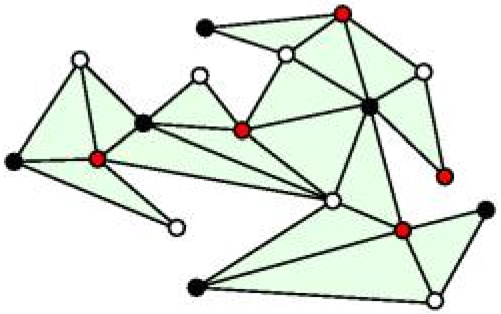
Art Gallery Problem

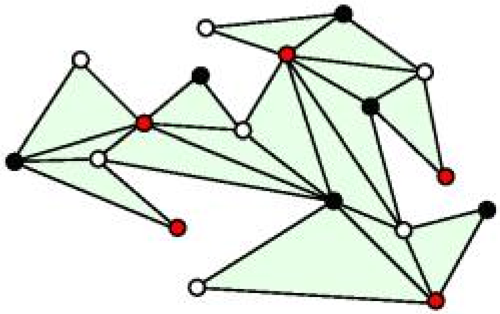
- ✓ Is floor(n/3) guards also sufficient to cover n-vertex polygon?
- ✓ = Is this also an upper bound?
- ✓ Trick: triangulation as 1 guard clearly covers a triangle.
- ✓ We have n-2 triangles so is n-2 the upper bound?
  - ✓ Not too tight ☹.
- ✓ Put guards not inside the triangles but on the vertices.



Art Gallery Problem

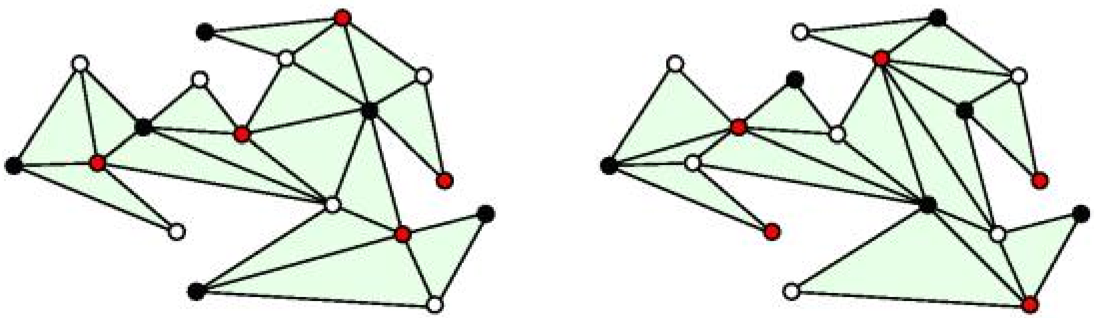
- ✓ Use induction to prove any triangulation of P is 3-colorable: each vert of P can be assigned a color so that any connected pair has different color.
- ✓ Induct on n = # of vertices. For n=3, trivially true. For n>3, we know P has an ear abc with b as tip. Removing the ear produces a P' w/ n-1 vertices where b is removed. By induction hypo, P' is 3-colorable. Replacing the ear, coloring b with the color not used by a or c, provides the desired coloring.





Art Gallery Problem

- ✓ Since there're n vertices, the least frequently used color appears at most floor(n/3) times (pigeonhole principle). Put guards at these verts.
- ✓ Since every triangle has 1 vertex of this color, and this guard covers the triangle, gallery is completely covered (with at most floor(n/3) guards).
  - ✓ If I had used 4 colors, then a 3-vertex triangle might be missing that least-freq color; but I guarantee to use 3 colors so all triangles are definitely covered.



n=18; least-freq color red is used 5 times.

Potential Project Topic

- ✓ Implement the polygon interpolation paper: Shape Blending Using the Star-Skeleton Representation.
- ✓ Implement the 3D Art Gallery paper: Computing 3D Shape Guarding and Star Decomposition.