
ME 536

— Weeks 4: Some more math —
2 remember

Matrices

Collection of vectors, where a vector generally represents a data point.

Vectors can be in the rows or columns of the **matrix**.

Convention for the rest of the course:

***Columns** of a data matrix represent **data points** in the rest of the class unless otherwise mentioned*

*Let's refresh our memories on **Matrices** and **Vectors** in action*

Matrix multiplied by a Vector

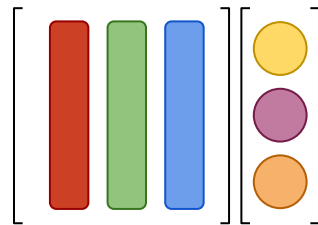
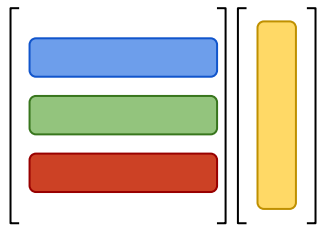
$$\mathbf{M}\mathbf{x} = \mathbf{a}$$

Vector multiplies the matrix on the right.

Recall: \mathbf{M} is the matrix and \mathbf{x} , \mathbf{a} are vectors.

Matrix multiplied by a Vector: 2 Main Interpretations

$$\mathbf{M}\mathbf{x} = \mathbf{a}$$



$$\mathbf{a}^T = \left[\begin{array}{c} \text{blue bar} \cdot \text{yellow bar} \\ \text{green bar} \cdot \text{yellow bar} \\ \text{red bar} \cdot \text{yellow bar} \end{array} \right]$$

Dot (or inner) product version

$$\mathbf{a} = \text{yellow circle} \cdot \text{red bar} + \text{purple circle} \cdot \text{green bar} + \text{orange circle} \cdot \text{blue bar}$$

Linear Combination version

Generally we are used to: square \mathbf{M} with no rank deficiency, i.e. \mathbf{M} is full rank

Row and Column Space of a Matrix

$$\begin{bmatrix} 4 & 2 & -1 & -8 & 12 & 1 \\ 2 & 6 & -3 & -4 & 6 & 3 \\ 5 & 4 & -2 & -10 & 15 & 2 \end{bmatrix}$$

Given \mathbf{M}

$$\begin{bmatrix} -4 & -2 & 1 & 3 & -4 & -2 \\ 0 & 2 & 1 & 3 & 4 & -2 \end{bmatrix}$$

Column space of \mathbf{M} : $\mathbf{C}(\mathbf{M})$

Spanned by columns of \mathbf{M}

Row space of \mathbf{M} : $\mathbf{R}(\mathbf{M})$

Spanned by rows of \mathbf{M}

Row and Column Space of a Matrix

Column space of \mathbf{M} : $\mathbf{C}(\mathbf{M}) \rightarrow$ Spanned by columns of \mathbf{M}

Row space of \mathbf{M} : $\mathbf{R}(\mathbf{M}) \rightarrow$ Spanned by rows of \mathbf{M}

$$\mathbf{C}(\mathbf{M}) \stackrel{?}{=} \mathbf{R}(\mathbf{M})$$

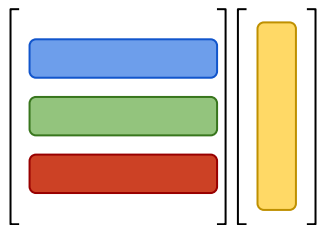
1- Fix for general case if it is not always True

2- Special case when this is correct?

Matrix multiplied by a Vector: 2 Main Interpretations

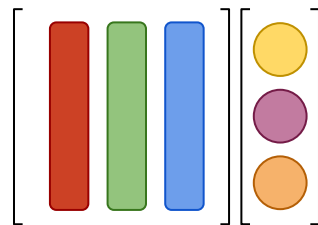
$$\mathbf{M}\mathbf{x} = \mathbf{0}$$

What if the trivial is not the only alternative: $\mathbf{x} \neq \mathbf{0}$



Dot (or inner) product version

$$0 = \begin{array}{|c|} \hline \text{blue} \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \text{yellow} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{green} \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \text{yellow} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{red} \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \text{yellow} \\ \hline \end{array}$$



Linear Combination version

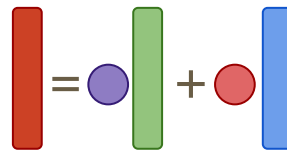
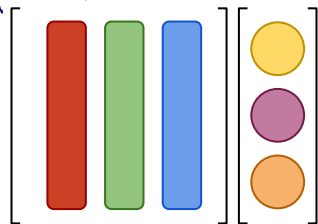
$$0 = \begin{array}{|c|} \hline \text{yellow} \\ \hline \end{array} \begin{array}{|c|} \hline \text{red} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{purple} \\ \hline \end{array} \begin{array}{|c|} \hline \text{green} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{orange} \\ \hline \end{array} \begin{array}{|c|} \hline \text{blue} \\ \hline \end{array} - \begin{array}{|c|} \hline \text{red} \\ \hline \end{array} \begin{array}{|c|} \hline \text{purple} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{yellow} \\ \hline \end{array} \begin{array}{|c|} \hline \text{green} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{orange} \\ \hline \end{array} \begin{array}{|c|} \hline \text{blue} \\ \hline \end{array}$$

$\mathbf{M}\mathbf{x} = \mathbf{0}$: Different Cases

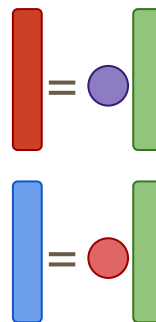
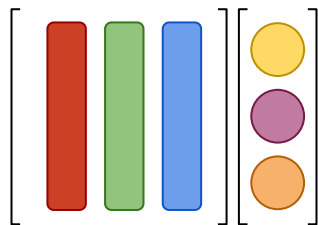
for null space finding

Let:

$$\text{rank}(\mathbf{M}_{3 \times 3}) = 2$$



$$\text{rank}(\mathbf{M}_{3 \times 3}) = 1$$



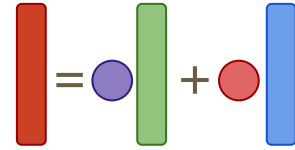
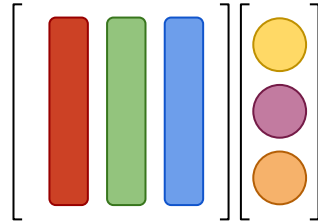
What does $C(\mathbf{M})$ represent in each case?

What about the basis of \mathbf{M} ?

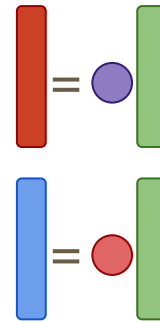
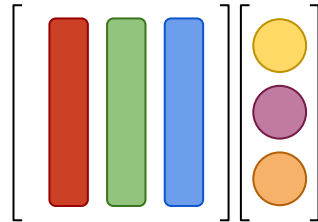
$\mathbf{M}\mathbf{x} = \mathbf{0}$: NULL Space - $N(\mathbf{M})$

Let:

$$\text{rank}(\mathbf{M}_{3 \times 3}) = 2$$



$$\text{rank}(\mathbf{M}_{3 \times 3}) = 1$$

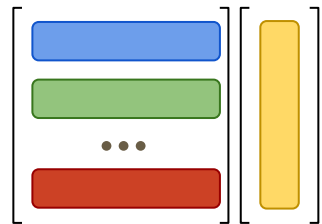


Where does \mathbf{x} come from? Where do all \mathbf{x} live?

Dimension of the $N(\mathbf{M})$? Number of free Parameters?

WRAP UP for $N(\mathbf{M}_{m \times n})$

Let: $\mathbf{M}_{m \times n}$



$$N(\mathbf{M}) \perp R(\mathbf{M})$$

similarly

$$N(\mathbf{M}) \perp C(\mathbf{M}^T)$$

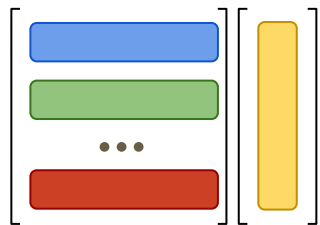
where

$$\dim(R(\mathbf{M})) = \dim(C(\mathbf{M}^T)) = \text{rank}(\mathbf{M}) = r$$

$$\dim(N(\mathbf{M})) = ?$$

$$0 = \begin{bmatrix} \text{blue} \\ \text{green} \\ \vdots \\ \text{red} \end{bmatrix} \cdot \begin{bmatrix} \text{yellow} \end{bmatrix} = \begin{bmatrix} \text{green} \end{bmatrix} \cdot \begin{bmatrix} \text{yellow} \end{bmatrix} \dots = \begin{bmatrix} \text{red} \end{bmatrix} \cdot \begin{bmatrix} \text{yellow} \end{bmatrix}$$

WRAP UP for $N(\mathbf{M}_{m \times n})$



Similarly for $\mathbf{M}_{n \times m}^T$

$$N(\mathbf{M}^T) \perp R(\mathbf{M}^T)$$

hence

$$N(\mathbf{M}^T) \perp C(\mathbf{M})$$

where

$$\dim(R(\mathbf{M}^T)) = \dim(C(\mathbf{M})) = \text{rank}(\mathbf{M}) = r$$

$$\dim(N(\mathbf{M}^T)) = ?$$

$$0 = \begin{bmatrix} \text{blue} \\ \text{green} \\ \vdots \\ \text{red} \end{bmatrix} \cdot \begin{bmatrix} \text{yellow} \end{bmatrix} = \begin{bmatrix} \text{green} \end{bmatrix} \cdot \begin{bmatrix} \text{yellow} \end{bmatrix} \dots = \begin{bmatrix} \text{red} \end{bmatrix} \cdot \begin{bmatrix} \text{yellow} \end{bmatrix}$$

Time Capsule: note to the Future

Recall that choosing a basis **from the data** matrix \mathbf{M} is:

- Practical
- Yet subject to *ill-conditioned* cases!
 - For a square matrix, you can check for very small $|\cdot|$
 - But **what if the Basis vectors do not form a square matrix?**

Time Capsule: note to the Future

Given a full rank square matrix $\mathbf{A}_{n \times n}$

We can consider this as a mapping from \mathbb{R}^n to \mathbb{R}^n , i.e. $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$

For $\mathbf{Ax} = \mathbf{a}$, we have $\mathbf{x}, \mathbf{a} \in \mathbb{R}^n$

What if $\mathbf{A}_{n \times m}$ is not square?

Then we have a mapping from \mathbb{R}^m to \mathbb{R}^n , i.e. $\mathbf{A} : \mathbb{R}^m \rightarrow \mathbb{R}^n$

For $\mathbf{Ax} = \mathbf{a}$, we have $\mathbf{x} \in \mathbb{R}^m$, and $\mathbf{a} \in \mathbb{R}^n$

Consider cases:

1. $\mathbf{A}_{3 \times 2}$
2. $\mathbf{A}_{200 \times 10}$
3. $\mathbf{A}_{2 \times 3}$

Can you move back and forth?

Try to think about it, drawing might help

A famous Space: \mathbb{R}^n

- ◆ All of us know a very well-known vector space: \mathbb{R}^n

For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we know that the length of vector x

$$\|x\| = \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2}$$

- ◆ For general vector spaces, we need a concept that corresponds to length in \mathbb{R}^n
- ◆ We use “norm” instead of “length”

Good news:

Our data will almost exclusively live in \mathbb{R}^n

So no weird brain freezing spaces

A famous Norm: length

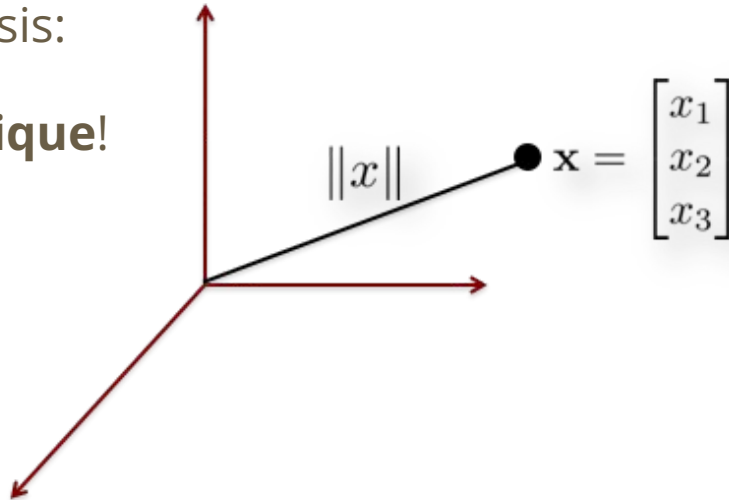
Euclidean Vector Norm

For a vector $x \in \mathbb{R}^n$, the euclidean norm is defined by:

$$\|x\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

Just like the standard basis:

- *Famous*, but **not unique**!



In general: Norm

A norm for a real or complex vector space \mathcal{V} is a function

$$\|\cdot\| : \mathcal{V} \rightarrow \mathbb{R}$$

with three properties:

1. $\|x\| \geq 0$ for all $x \in V$ and $\|x\| = 0 \iff x = 0$
2. $\|\alpha x\| = |\alpha| \|x\|$ for all scalars α and for all $x \in \mathcal{V}$
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{V}$

p-Norms

p-Norms

For $p \geq 1$, the p-norm of a vector $x \in \mathbb{R}^n$ is defined by:

$$\|x\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{1/p}$$

- l_1 norm \rightarrow • $\|x\|_1$ - a.k.a Taxicab or Manhattan norm
- l_2 norm \rightarrow • $\|x\|_2$ - Euclidean norm
- ... • ...
- l_∞ norm \rightarrow • $\|x\|_\infty$ - $\max(|x_1|, \dots, |x_n|)$

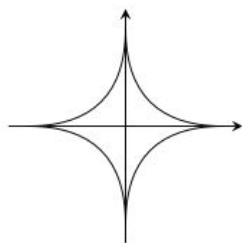
Even though value of p determines the norm, *in general* they are referred to as l_p norm. (with a *lowercase L*)

NOTE: shape of l_p norms:

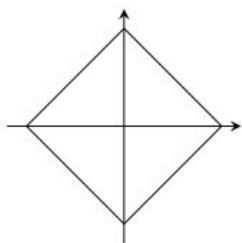
What are the set of **points with unit norm** look like:

$$l_p = 1$$

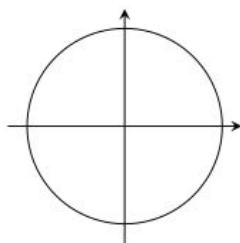
for different values of $p = \{ \frac{1}{2}, 1, 2, \infty \}$ in 2D.



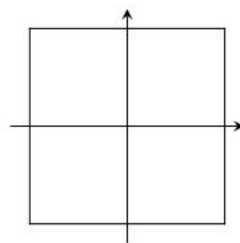
$$p = \frac{1}{2}$$



$$p = 1$$

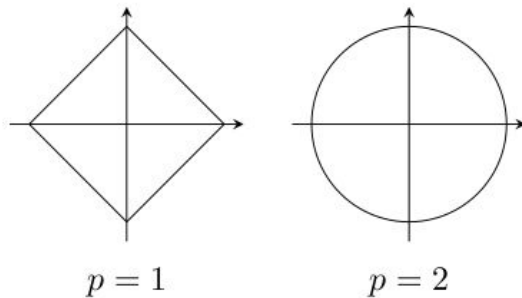


$$p = 2$$



$$p = \infty$$

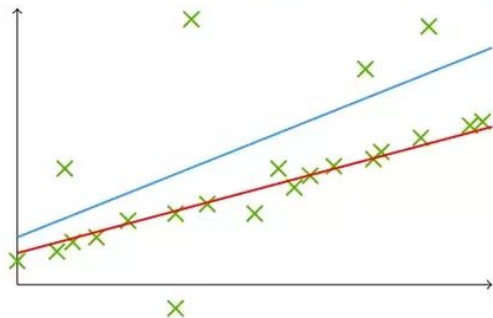
NOTE: l_1 over l_2 ?



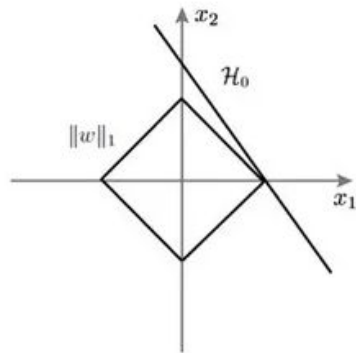
Given : A set of **points** in 2-dimension

Goal : Find a line to fit those points

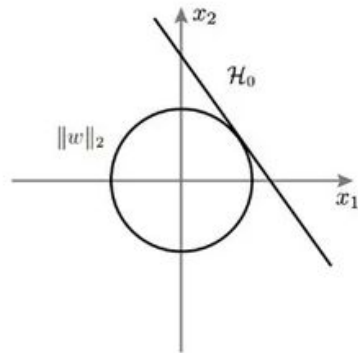
Output : l_2 minimizer **line**, l_1 minimizer **line**



A l_1 regularization



B l_2 regularization



NOTE: l_0 norm

Despite the fact that l_0 norm is **not a norm by definition** but it is very often *used in practice*.

l_0 norm is defined by the **number of non-zero elements** in the vector.

→ i.e. minimizing l_0 results in sparse solution

NOTE: yet another norm

Note that if \mathbf{D} is a symmetric matrix with positive diagonal terms, then for the vector \mathbf{v} :

$$\mathbf{v}^T \mathbf{D} \mathbf{v}$$

defines a norm in the form of a weighted sum of squares.

Example: a 2x2 case

$$\mathbf{D} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\mathbf{v}^T \mathbf{D} \mathbf{v} = av_1^2 + bv_2^2$$

Good news:

Every finite dimensional real or complex topological vector space has a norm

So no hunt for finding norms, well mostly...

A famous Inner Product in: \mathbb{R}^n a.k.a *DOT Product*

\mathbb{R}^n : Collection of all finite sequences

$$x = (x_1, x_2, \dots, x_n), \quad x_k \in \mathbb{R} \text{ for } k = 1, 2, \dots, n$$

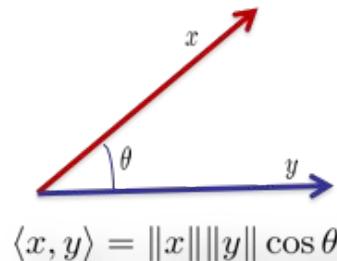
We can define norm for $\|x\| = (\sum_{k=1}^n x_k^2)^{1/2}$

Recall: inner product of $x, y \in \mathbb{R}^n$, $\langle x, y \rangle = \sum_{k=1}^n x_k y_k$

Clearly: $\langle x, x \rangle = \|x\|^2$

Inner product is a very important tool for analysis in \mathbb{R}^n .

It is a measure of angle between vectors



In general: Inner Product

Definition: An *inner product* on a real (or complex) vector space \mathcal{V} is a function that maps each ordered pairs of vectors v and w to a real (or complex) scalar $\langle v, w \rangle$ such that the following properties hold:

1. $\langle v, v \rangle$ is real with $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle = 0 \iff v = 0$
2. $\langle v, \alpha w \rangle = \overline{\alpha} \langle v, w \rangle$ for all scalars α and for all $v, w \in \mathcal{V}$
3. $\langle v, w + u \rangle = \langle v, w \rangle + \langle v, u \rangle$ for all $v, u, w \in \mathcal{V}$
4. $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in \mathcal{V}$ (note that $\langle v, w \rangle = \langle w, v \rangle$ for real spaces)

Brain teaser:

Try to come up with *your custom*:

- norm for vectors in \mathbb{R}^n

with three properties:

1. $\|x\| \geq 0$ for all $x \in V$ and $\|x\| = 0 \iff x = 0$
2. $\|\alpha x\| = |\alpha| \|x\|$ for all scalars α and for all $x \in \mathcal{V}$
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{V}$

- inner product for vector pairs in \mathbb{R}^n

1. $\langle v, v \rangle$ is real with $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle = 0 \iff v = 0$
2. $\langle v, \alpha w \rangle = \overline{\alpha} \langle v, w \rangle$ for all scalars α and for all $v, w \in \mathcal{V}$
3. $\langle v, w + u \rangle = \langle v, w \rangle + \langle v, u \rangle$ for all $v, u, w \in \mathcal{V}$
4. $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in \mathcal{V}$ (note that $\langle v, w \rangle = \langle w, v \rangle$ for real spaces)

Inner Product Spaces

Simple Recipe: *A vector space with an Inner Product*

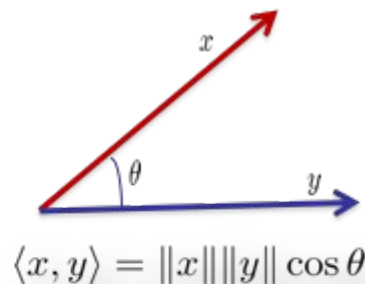
Examples: \mathbb{R}^n with $\langle x, y \rangle = \sum_{k=1}^n x_k y_k$

$$\mathbb{C}^n \text{ with } \langle x, y \rangle = \sum_{k=1}^n x_k \overline{y_k}$$

Cauchy-Schwarz Inequality

Inner product for \mathbb{R}^2

$$\langle x, y \rangle = \|x\| \|y\| \cos(\theta)$$



$$|\langle x, y \rangle| \leq \|x\| \|y\| = \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}$$

For a general normed vector space \mathcal{V}

Theorem. $|\langle v, w \rangle| \leq \langle v, v \rangle^{1/2} \langle w, w \rangle^{1/2}, \quad \forall v, w \in \mathcal{V}$

Good to know

- ◆ In any inner product vector space, regardless of the inner product we can always define a norm

Lemma. *If \mathcal{V} has the inner product $\langle \cdot, \cdot \rangle$, then*

$$\|v\| = |\langle v, v \rangle|^{1/2} \text{ is a norm}$$

- ◆ But opposite is not true. We may not always define an inner product from a given norm

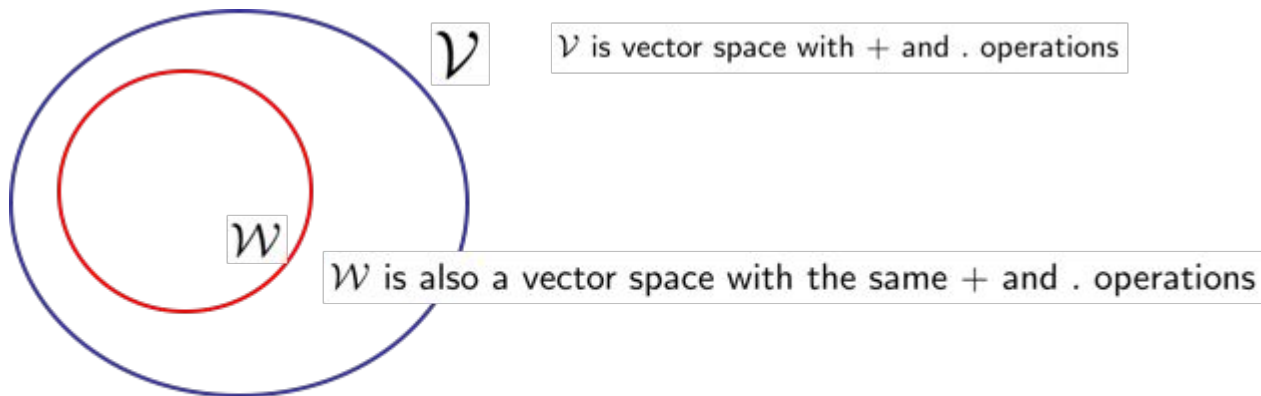
But...

- ◆ We may define an inner product from a given norm if the parallelogram law holds for the norm

$$\|v + w\|^2 + \|v - w\|^2 = 2 (\|v\|^2 + \|w\|^2)$$

for all $v, w \in \mathcal{V}$

Subspaces

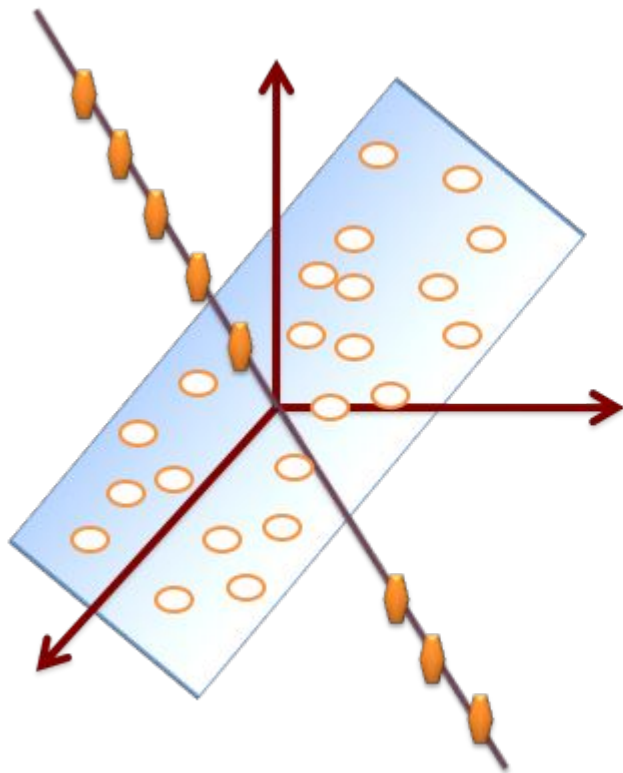


Lemma. *A subset $\mathcal{W} \subset \mathcal{V}$ is a subspace if $\alpha, \beta \in \mathcal{F}$ and $v, w \in \mathcal{W} \Rightarrow \alpha v + \beta w \in \mathcal{W}$*

Practical consequence:

Any *line, plane, etc* are **subspaces** if they **include the origin**

Subspaces: Examples



A line through origin in \mathbb{R}^3 is a 1-dimensional subspace of \mathbb{R}^3

A plane through origin in \mathbb{R}^3 is a 2-dimensional subspace of \mathbb{R}^3

Sum of Subspaces

If \mathcal{X} and \mathcal{Y} are subspaces of a vector space \mathcal{V} , then the *sum* of \mathcal{X} and \mathcal{Y} is defined as all possible sums of vectors from \mathcal{X} and \mathcal{Y} :

$$\mathcal{X} + \mathcal{Y} = \{x + y : x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}$$

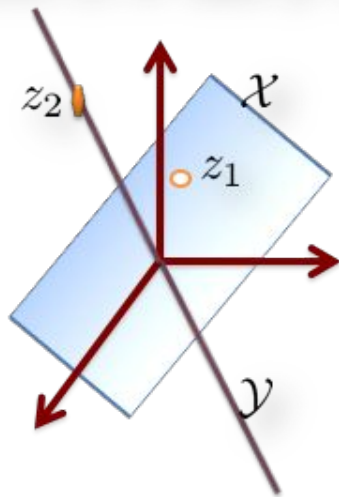
The sum of \mathcal{X} and \mathcal{Y} is a subspace of \mathcal{V}

Union of Subspaces

If \mathcal{X} and \mathcal{Y} are subspaces of a vector space \mathcal{V} , then the *union* of \mathcal{X} and \mathcal{Y} is defined:

$$\mathcal{X} \cup \mathcal{Y} = \{z : z \in \mathcal{X} \text{ or } z \in \mathcal{Y}\}$$

The union of \mathcal{X} and \mathcal{Y} may not be a subspace of \mathcal{V}



$$z_1 \in \mathcal{X} \cup \mathcal{Y}, z_2 \in \mathcal{X} \cup \mathcal{Y}$$

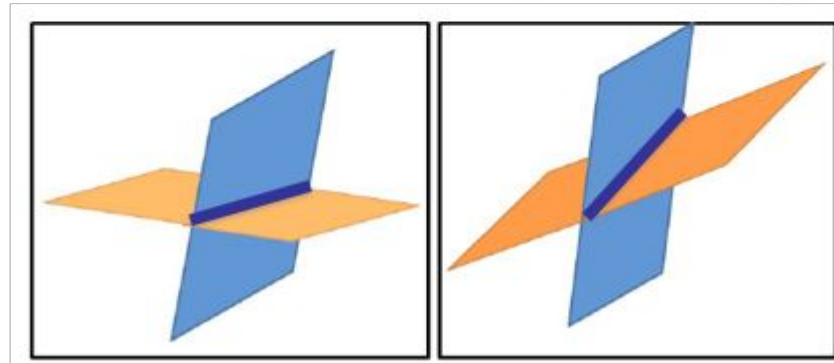
$$\text{but } z_1 + z_2 \notin \mathcal{X} \cup \mathcal{Y}$$

Intersection of Subspaces

If \mathcal{X} and \mathcal{Y} are subspaces of a vector space \mathcal{V} , then the *intersection* of \mathcal{X} and \mathcal{Y} is defined:

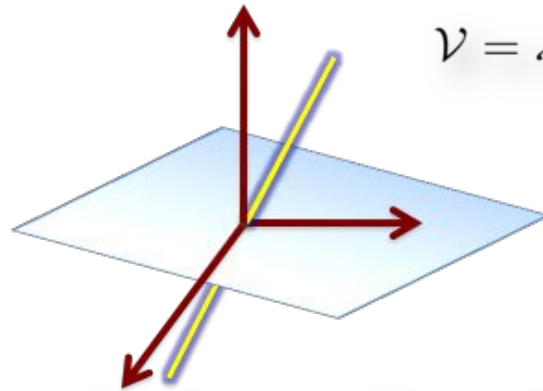
$$\mathcal{X} \cap \mathcal{Y} = \{z : z \in \mathcal{X} \text{ and } z \in \mathcal{Y}\}$$

The intersection of \mathcal{X} and \mathcal{Y} is a subspace of \mathcal{V}



Complementary Subspaces

Subspaces \mathcal{X} and \mathcal{Y} of a space \mathcal{V} are *complementary* if



$$\mathcal{V} = \mathcal{X} + \mathcal{Y} \text{ and } \mathcal{X} \cap \mathcal{Y} = \emptyset$$

$$(v_1, v_2) + (u_1, u_2) = (v_1 + u_1, v_2 + u_2)$$

matrix $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$

In this case, \mathcal{V} is said to be the direct sum of \mathcal{X} and \mathcal{Y}

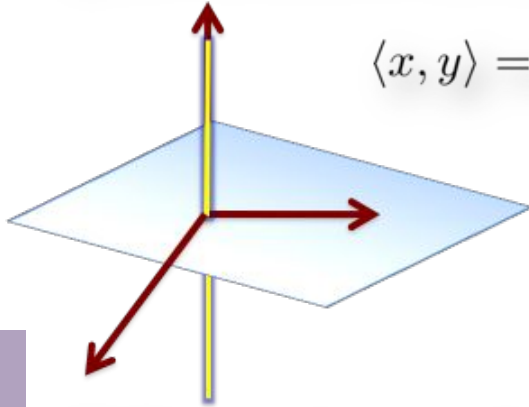
$$\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$$

For each $v \in \mathcal{V}$, there are *unique* vectors $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $v = x + y$

Orthogonal Subspaces

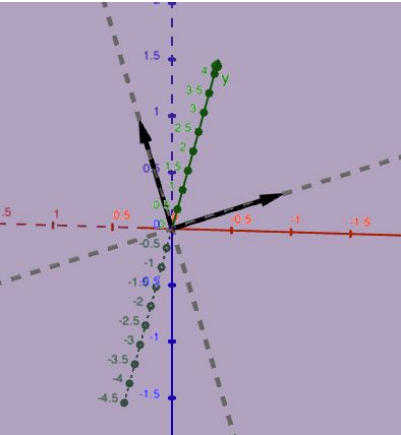
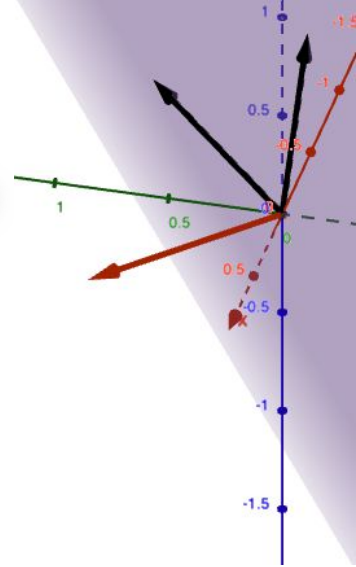
If \mathcal{X} and \mathcal{Y} are subspaces of an *inner-product* vector space \mathcal{V} , then \mathcal{X} and \mathcal{Y} are orthogonal $\mathcal{X} \perp \mathcal{Y}$ if

$$\langle x, y \rangle = 0 \text{ for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}$$



If \mathcal{X} is a subspace of a finite dimensional inner-product space \mathcal{V} , then \mathcal{X}^\perp is its orthogonal complement if

$$\mathcal{V} = \mathcal{X} \oplus \mathcal{X}^\perp$$



Independent vs Disjoint

Linear subspaces $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_k$ of \mathbb{R}^n are independent

if and only if

$$\dim(\mathcal{S}_1 + \mathcal{S}_2 + \dots + \mathcal{S}_k) = \dim(\mathcal{S}_1) + \dim(\mathcal{S}_2) + \dots + \dim(\mathcal{S}_k)$$

Linear subspaces $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_k$ of \mathbb{R}^n are disjoint

if they intersect only at the origin

Independence is stronger than disjointness

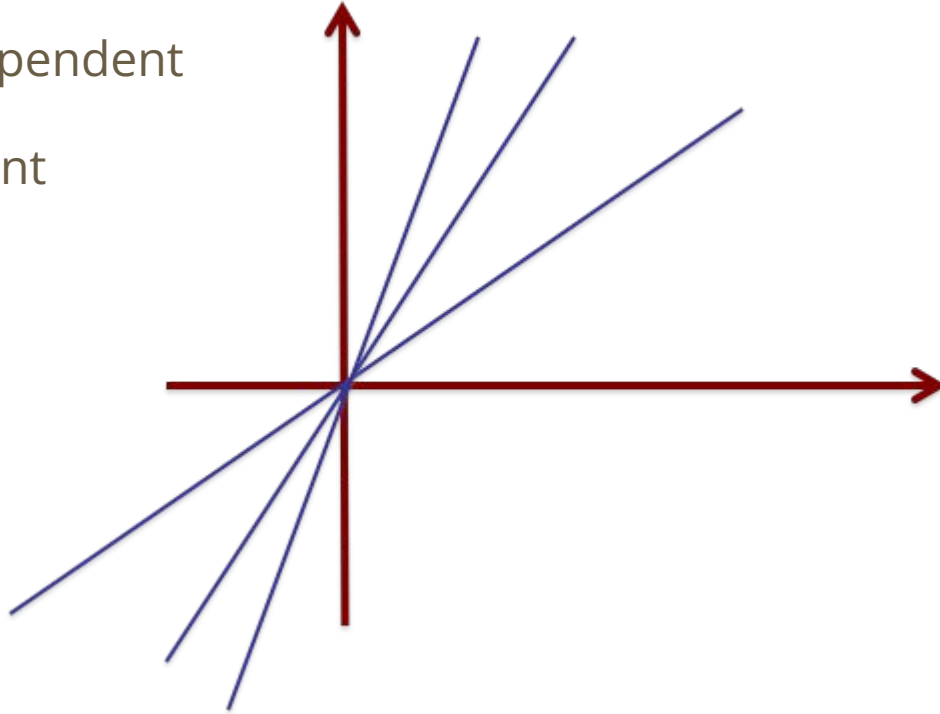
Three lines in \mathbb{R}^2 intersecting at 0 are disjoint

But they are not independent

Independent vs Disjoint

In \mathbb{R}^2 any 2 lines are independent

3 or more lines are disjoint

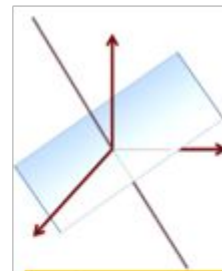


Minimal Angle

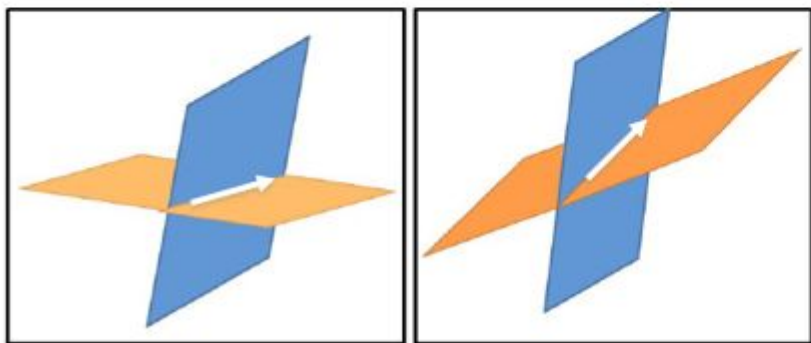
Minimum Angle

Let \mathcal{F} and \mathcal{G} be subspaces of \mathbb{R}^D . The minimal angle between \mathcal{F} and \mathcal{G} is defined as

$$\theta_{\min} = \arccos \left[\max_{\substack{f \in \mathcal{F} \\ g \in \mathcal{G} \\ \|f\|_2 = \|g\|_2 = 1}} f^T g \right]$$



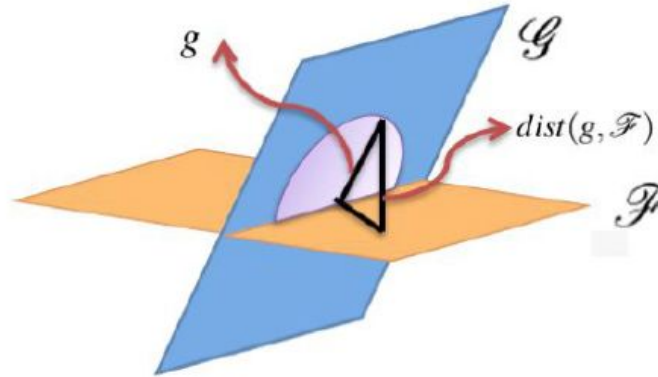
It is a good measure for complementary subspaces



It is not a good measure for non-complementary subspaces

Gap between Subspaces

$$d(\mathcal{F}, \mathcal{G}) = \max_{\substack{g \in \mathcal{G} \\ \|g\|_2=1}} \text{dist}(g, \mathcal{F}) = \max_{\substack{g \in \mathcal{G} \\ \|g\|_2=1}} \|(I - P_{\mathcal{F}})g\|_2$$



$$\text{gap}(\mathcal{F}, \mathcal{G}) = \min(d(\mathcal{F}, \mathcal{G}), d(\mathcal{G}, \mathcal{F}))$$

Maximal Angle

Maximum Angle

The maximal angle between \mathcal{F} and \mathcal{G} is defined as

$$\theta_{\max} = \arcsin(\text{gap}(\mathcal{F}, \mathcal{G})),$$

where $0 \leq \theta_{\max} \leq \pi/2$.

It is useful for subspaces
of equal dimension

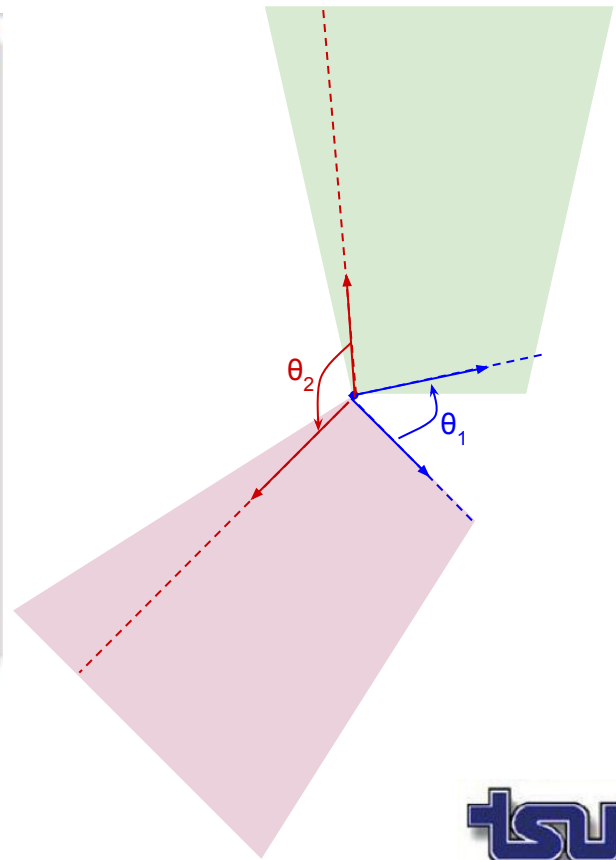
Principal Angles: An iterative process

Principle Angles

Let \mathcal{F} and \mathcal{G} be subspaces of \mathbb{R}^D . Let $k = \min(\dim \mathcal{F}, \dim \mathcal{G})$. Then, the principle angles $\theta_1, \theta_2, \dots, \theta_k$ are the numbers $0 \leq \theta_i \leq \pi/2$ and they are defined as

$$\cos \theta_i = \max_{\substack{f \in \mathcal{F}_i \\ g \in \mathcal{G}_i}} f^t g = f_i^t g_i \quad i = 1, \dots, k$$
$$f_{||_2} = g_{||_2} = 1$$

where $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{G}_1 = \mathcal{G}$, $\|f_i\|_2 = 1$, $\|g_i\|_2 = 1$, $\mathcal{F}_i = f_{i-1}^\perp \cap \mathcal{F}_{i-1}$, and $\mathcal{G}_i = g_{i-1}^\perp \cap \mathcal{G}_{i-1}$. Note that $\theta_1 \leq \theta_2 \leq \dots \leq \theta_k$.



Principal Angles

Principle Angles

Let \mathcal{F} and \mathcal{G} be subspaces of \mathbb{R}^D . Let $k = \min(\dim \mathcal{F}, \dim \mathcal{G})$. Then, the principle angles $\theta_1, \theta_2, \dots, \theta_k$ are the numbers $0 \leq \theta_i \leq \pi/2$ and they are defined as

$$\cos \theta_i = \max_{\substack{f \in \mathcal{F}_i \\ g \in \mathcal{G}_i \\ \|f\|_2 = \|g\|_2 = 1}} f^T g = f_i^T g_i \quad i = 1, \dots, k$$

where $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{G}_1 = \mathcal{G}$, $\|f_i\|_2 = 1$, $\|g_i\|_2 = 1$, $\mathcal{F}_i = \mathcal{F}_{i-1}^\perp \cap \mathcal{F}$, and $\mathcal{G}_i = \mathcal{G}_{i-1}^\perp \cap \mathcal{G}$. Note that $\theta_1 \leq \theta_2 \leq \dots \leq \theta_k$.

Time Capsule: note to the Future

Given two subspaces \mathcal{F} and \mathcal{G} let \mathbf{F} , \mathbf{G} form orthogonal bases for subspaces \mathcal{F} and \mathcal{G} respectively.

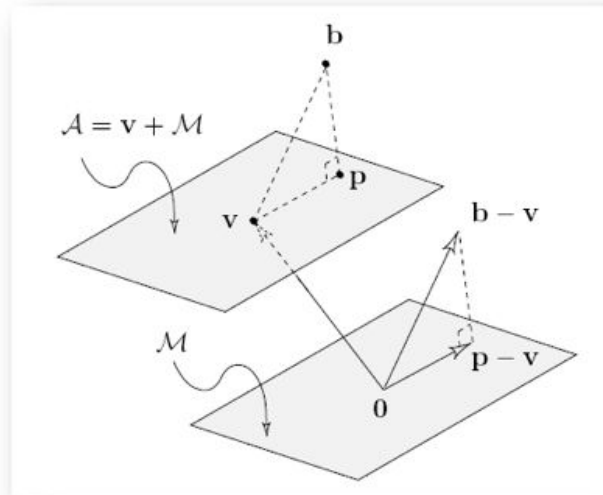
$$\cos \theta_i = S(\mathbf{F}^T \mathbf{G})$$

where $S(\cdot)$ correspond to the *singular values* of \cdot

$\sigma = \cos \theta$

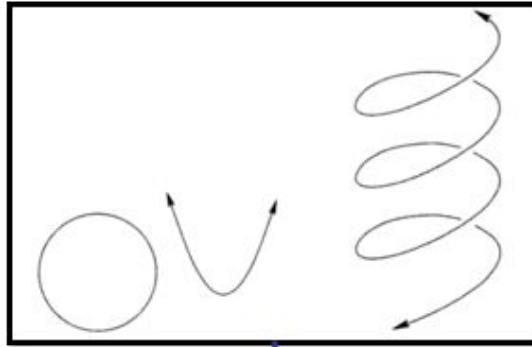
Affine Space

Affine Projections. If $\mathbf{v} \neq \mathbf{0}$ is a vector in a space \mathcal{V} , and if \mathcal{M} is a subspace of \mathcal{V} , then the set of points $\mathcal{A} = \mathbf{v} + \mathcal{M}$ is called an *affine space* in \mathcal{V} .

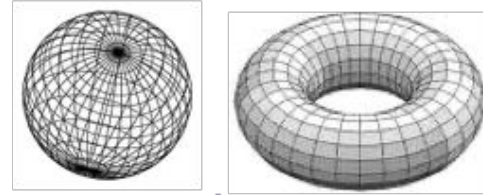


Manifolds

- ◆ A manifold is a mathematical space that (on a sufficiently small scale) resembles to the Euclidean space of a specific dimension



1-dimensional



2-dimensional

Manifolds

- ◆ Manifolds are like curves and surfaces, except that they might be of higher dimension
- ◆ Every manifold has a dimension
 - ◆ The number of independent parameters to specify a point
- ◆ n dimensional manifold is an object modeled *locally* on
 - ◆ It takes exactly n numbers to specify a point

Recall ME210:
Parametric definition
of curves and surfaces

to be continued...

With matrix decomposition methods