DIFFERENTIAL GEOMETRY HOMEWORK 1

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a. **Suppose that a particle in 3-space is moving under a central force F. That is equivalent to the following condition: there is a fixed point A such that the acceleration vector points in direction of A at all time. Prove that its trajectory lies in a fixed plane.** Let r be a Frenet curve in $\mathbb{R}^3$ such that $\ddot{r} = \alpha\,(A - r)$ for some smooth real valued function, α that is non-zero on I and some fixed point A. We can define $c := A - r$ so that T, N, B are the same for both curves (up to negation). Furthermore, $\ddot{c} = -\ddot{r} = -\alpha c$.
Claim: B is constant, and hence c must lie in the plane normal to B.

Proof. Firstly,

$$-\alpha c = \ddot{c} = \frac{d^2c}{dt^2} = \frac{d}{dt}\left(\frac{dc}{ds}\frac{ds}{dt}\right) = \frac{d}{dt}\left(c'\dot{s}\right) = c''\dot{s}^2 + c'\ddot{s} \tag{1}$$

$$N = \frac{c''}{\kappa} = -\frac{\alpha c + \ddot{s}c'}{\dot{s}^2\kappa} \tag{2}$$

$$-\kappa T + \tau B = N' = \left(-\frac{\alpha c + \ddot{s}c'}{\dot{s}^2\kappa}\right)' \tag{3}$$

$$B' = -\tau N \tag{4}$$

By taking equation (3), simplifying and replacing every occurrence of c with $-\frac{\dot{s}^2c''+\ddot{s}c'}{\alpha} = -\frac{\dot{s}^2\kappa N+\ddot{s}T}{\alpha}$ from (1), we find,

$$\begin{aligned}
-\kappa T + \tau B &= \left(-\frac{\alpha c + \ddot{s}c'}{\dot{s}^2\kappa}\right)' \\
&= \frac{-(\alpha' c + c'\alpha + \ddot{s}c'')\dot{s}^2\kappa + \dot{s}^2\kappa'(\alpha c + \ddot{s}c')}{\dot{s}^4\kappa^2} \\
&= \frac{-\alpha'(\dot{s}^2\kappa N + \ddot{s}T)\kappa/\alpha - \kappa\alpha T - \ddot{s}\kappa^2 N - \kappa'\dot{s}^2\kappa N}{\dot{s}^2\kappa^2} \\
&= \left(\frac{-\alpha'\ddot{s} - \alpha^2}{\dot{s}^2\kappa\alpha}\right)T - \left(\frac{\alpha'\dot{s}^2\kappa + \ddot{s}\kappa\alpha + \kappa'\alpha\dot{s}^2}{\dot{s}^2\kappa\alpha}\right)N
\end{aligned}$$

is a linear combination of T, N. Since T, N, B are orthonormal, we see that $\tau = 0$. ☐

Note: If $k = 0$ at a single point (hence finitely many), all we would have to show is that before and after said point, the plane in which the curve lies is not shifted. To do this, we simply note that any shift in said plane means a non-zero τ (because $B' \neq 0$). If we fix an epsilon, then we find that there needs to be some interval with $\tau \neq 0$, and hence comes our contradiction.
Next, if the plane before and after are the same, then the point in question must lie in that plane for

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the same reason (fix epsilon).
Furthermore, $\kappa^2\dot{s}^2\alpha = \alpha'\ddot{s} + \alpha^2$, and $\alpha'\dot{s}^2\kappa + \ddot{s}\kappa\alpha + \kappa'\alpha\dot{s}^2 = 0$

$$\begin{aligned} 0 =& \alpha'\dot{s}^2\ddot{s}\kappa + \ddot{s}^2\kappa\alpha + \ddot{s}\kappa'\alpha\dot{s}^2 \\ =& (\kappa^2\dot{s}^2\alpha - \alpha^2)\dot{s}^2\kappa + \ddot{s}^2\kappa\alpha + \ddot{s}\kappa'\alpha\dot{s}^2 \\ =& (\kappa^2\dot{s}^2 - \alpha)\dot{s}^2\kappa + \ddot{s}^2\kappa + \ddot{s}\kappa'\dot{s}^2 \end{aligned}$$

b. **Fix the plane to be the xy-plane and write the equation for the trajectory of A in polar coordinates.** Let x be in the direction of $T(s_0)$, y in the direction of $N(s_0)$ and z in the direction of B. Then $r(s) = (||c(s)||, \theta, 0) + A$, where θ is the angle given by $\theta = \int_0^t \kappa(t)dt$.

c. **If the force F is given by F=$\frac{Cr}{||r||^3}$, show that the trajectory is part of an ellipse, hyperbola or parabola (second order or quadratic curve).** For ease of notation, let $d = ||r||$. If we assume that $\ddot{r} = F$, then $\ddot{r} = F = \frac{C}{d^2}\frac{r}{||r||}$, then $A = \vec{0}$, and we have $\ddot{r} = -\alpha r$, hence, $-\alpha r = \frac{C}{d^2}\frac{r}{||r||}$. But then, $\alpha = \alpha(d) = \frac{-C}{d^3}$. Hence, either $C > 0$ (and $\alpha < 0$), $C < 0$ (and $\alpha > 0$), or $\alpha = C = 0$ for all t. From this we can conclude that $r = r(\theta)$ (or r is a straight line).

Proof. Firstly, if $C = \alpha = 0$, then $\ddot{r} = 0$, and $r'' = N = 0$. So from now on we will assume that $C \neq 0$. Just for ease of notation, let $s_0 = 0$ (if not we can reparameterize). Let $r(0) = p$, and $r'(0)$ not be in the direction of the origin (if it is in the direction of the origin, again we have a straight line). Then if we let l be the ray starting at the origin and passing through p, our claim is that r passes through l only at p.

$$\dot{r} = r'\dot{s} \qquad\qquad \frac{Cr}{d^3} = \ddot{r} = r''(\dot{s})^2 + r'\ddot{s} = (\dot{s})^2\kappa N + \ddot{s}T$$

$\square$

Furthermore, $\dot{r}(t) - \dot{r}(t_0) = \int_{t_0}^t \ddot{r}(t)dt = \int_{t_0}^t \frac{C}{||r||^2}\frac{r}{||r||}dt =$
Case 1: $\alpha = C = 0$. Then, $\ddot{c} = 0 \to c'' = 0 \to c$ is a straight line.
Case 2: $\alpha < 0$, then $C > 0$ and we have a hyperbola...
Case 3: $\alpha > 0$, then $C < 0$ and we have either a parabola or an ellipse...

2. A Frenet curve in $\mathbb{R}^3$ is called a *Bertrand curve*, if there is a second curve such that the principle normal vectors to these two curves (at corresponding points) are identical, viewed as lines in space. One speaks in the case of a *Bertrand pair of curves*. Show that non-planar Bertrand curves are characterized by the existence of a linear relation $a\kappa + b\tau = 1$ with constants a, b, where $a \neq 0$.

Let c_1, c_2 be a pair of distinct Bertrand curves in $\mathbb{R}^3$. WLOG: we can assume that c_1 is parameterized by arc-length already (if not, then re-parameterize it, and whatever the re-parameterization function is, apply it to the c_2 parameter too, and we have what we want).
First we notice that $d(c_1(t), c_2(t))$ is constant. To see this,

$$c_2(t) = c_1(t) + f(t)N(t) \text{ because the two } \vec{N}\text{s define the same line in space}$$

If we prove that $f(t)$ is constant, we've done what we need. But note:

$$\begin{aligned}
\frac{T_2}{t'} &= \dot{c}_2 \\
&= c_1' + f'N + fN' \\
&= T_1 + f'N + f(-\kappa_1 T_1 + \tau_1 B_1) \\
&= (1 - f\kappa_1)T_1 + f'N + f\tau_1 B_1 \Rightarrow \\
f't' &= \langle c_2', N\rangle = \langle T_2, N\rangle = 0
\end{aligned}$$

So, f is constant (because t is never constant wrt the arc-length parameter, or differently stated, $t' \neq 0$), and hence $d(c_1(t), c_2(t))$ is constant (let's say $= \alpha$) as well.
Next we claim that $\theta = \Theta(t) = \frac{\langle T_1, T_2\rangle}{||T_1|| ||T_2||} = \langle T_1, T_2\rangle$ is constant. To see this,

$$\begin{aligned}
T_2 &= Proj_{T_2} T_1 + Proj_{T_2} N + Proj_{T_2} B_1 \\
&= T_1 \cos\theta + B_1 \sin\theta \text{ because } T_2 \perp N \\
\frac{N}{t'} &= \dot{T}_2 \\
&= T_1' \cos\theta - T_1 \sin\theta\theta' + B_1' \sin\theta + B_1 \cos\theta\theta' \\
&= \kappa_1 N \cos\theta - T_1 \sin\theta\theta' - \tau_1 N \sin\theta + B_1 \cos\theta\theta' \\
&= -\sin\theta\theta' T_1 + (\kappa_1 \cos\theta - \tau_1 \sin\theta)N + \cos\theta\theta' B_1 \Rightarrow \\
-\sin\theta\theta' &= 0 = \cos\theta\theta' \Rightarrow \\
\theta' &= 0
\end{aligned}$$

So θ is constant. So, by matching coefficients we find,

$$\cos\theta = t' - f\kappa_1 t' \qquad\qquad \sin\theta = f\tau_1 t'$$

Since the curves were distinct and the distance is constant, we know $f \neq 0$, and since the curves are non-planar, we know $\tau_1 \neq 0$. So,

$$\begin{aligned}
\cot\theta = \frac{\cos\theta}{\sin\theta} &= \frac{t' - f\kappa_1 t'}{f\tau_1 t'} = \frac{1 - f\kappa_1}{f\tau_1} \rightarrow \\
f\kappa_1 + \cot\theta f\tau_1 &= 1 \rightarrow \text{ let } a = f\kappa_1, \text{ and } b = \cot\theta f.
\end{aligned}$$

Now, let us assume there is a non-planar Frenet curve c_1 in $\mathbb{R}^3$, and constants a, b with $a \neq 0$, such that for all s (I'm assuming c_1 is parameterized by arc length) $a\kappa + b\tau = 1$. We need to find a Bertrand pair for this curve. To do that, let $d = a$, and define a new curve, $c_2 = c_1 + dN_1$. The only thing we need to do is show that c_2 is a Frenet curve and that the two N's define the same line in space.

3. Suppose $r = r(t)$ is a regular curve satisfying $r'' = r' \times H$ for a constant vector H (according to one source this is the equation of an electron moving under a magnetic field force). Prove that τ and κ are constants.

Proof. Let s be the arc length parameter for r. Then $r' = \dot{r}\frac{dt}{ds}$. Since $1 = ||r'|| = ||\dot{r}|| \cdot ||\frac{dt}{ds}|| \to \frac{1}{||dt/ds||} = ||\frac{ds}{dt}|| = ||\dot{s}|| = ||\dot{r}||$. Hence

$$\begin{aligned}(\dot{s})^2 =& \dot{r} \cdot \dot{r} \to \\ 2\dot{s}\ddot{s} =& 2\ddot{r} \cdot \dot{r} \\ =& (\dot{r} \times H) \cdot \dot{r} = 0\end{aligned}$$

Hence, either $\dot{s} = 0$ or, $\ddot{s} = 0$. Either way, $\ddot{s} = 0$, and $s = at + b$ with $a, b \in \mathbb{R}$. Next we have,

$$\begin{aligned}T =& r'(s) \\ =& \dot{r}\frac{dt}{ds} \\ =& \frac{1}{a}\dot{r} \to ||\dot{r}|| = a\end{aligned}$$

Furthermore,

$$\begin{aligned}T' =& \frac{d^2r}{ds^2} \\ =& \frac{d}{dt}\left(\frac{1}{a}\dot{r}\right)\frac{dt}{ds} \\ =& \frac{1}{a^2}\ddot{r} \\ =& \frac{1}{a^2}\dot{r} \times H \\ =& \frac{1}{a}r' \times H\end{aligned}$$

and $T' = \kappa N$, so

$$\begin{aligned}\kappa' N + \kappa N' = T'' =& \frac{1}{a}r'' \times H \\ =& \frac{1}{a^2}(r' \times H) \times H\end{aligned}$$

so that

$$\begin{aligned}\kappa' =& \langle \kappa' N + \kappa N', N\rangle \\ =& \langle \frac{1}{a^2}(r' \times H) \times H, N\rangle \\ =& \frac{1}{a}\langle T' \times H, N\rangle \\ =& \frac{1}{a}\langle \kappa N \times H, N\rangle = 0\end{aligned}$$

Hence, κ is constant.
Next, let us first note: Since $T' = \frac{1}{a}T \times H$, then $N = \frac{T \times H}{||T \times H||} = CT \times H$, where $C = \frac{1}{||T \times H||} = \frac{a}{\kappa}$ which

is constant by part 1 of this question, and, $B = T \times N$.

$$\begin{aligned}\tau =&\langle B, N'\rangle \text{ by definition of } \tau\\ =&\langle B, (CT \times H)'\rangle\\ =&\langle B, CT' \times H\rangle\\ =&\langle B, C\frac{1}{a}(T \times H) \times H\rangle\\ =&\frac{1}{a}\langle B, N \times H\rangle\end{aligned}$$

Hence, first noting that $\langle B, N\rangle = 0 = \langle B, N\rangle' = \langle B', N\rangle + \langle B, N'\rangle = 0 \to \langle B', N\rangle = -\tau$

$$\begin{aligned}a\tau' =&\langle B, N \times H\rangle'\\ =&\langle B', N \times H\rangle + \langle B, N' \times H\rangle\\ =&\langle -\tau N, N \times H\rangle + \langle B, (-\kappa T + \tau B) \times H\rangle\\ =&0 - \kappa\langle B, T \times H\rangle\\ =&-\frac{\kappa}{C}\langle B, CT \times H\rangle\\ =&-\frac{\kappa}{C}\langle B, N\rangle = 0\end{aligned}$$

So, τ is constant too. □

4. Let A_X and A_Y be the operators corresponding to the cross product with the vectors X and Y respectively. Show that $A_X A_Y - A_Y A_X = A_{X \times Y}$

This is just a computation. There's definitely a theory way to go here (involving commutators, or lie groups or whatever), but I can't see that way right now, so I'll just do this computationally.

$$\begin{aligned}(A_X A_Y - A_Y A_X)(Z) =&X \times (Y \times Z) - Y \times (X \times Z)\\ =&X \times (Y \times Z) + Y \times (Z \times X)\\ =&-Z \times (X \times Y)\\ =&(X \times Y) \times Z = A_{X \times Y}(Z)\end{aligned}$$