DIFFERENTIAL GEOMETRY HOMEWORK 1

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1. Determine the curvature and the torsion of the curve given by the intersection of the surfaces $x^3 = 3a^2y$ and $2xz = a^2$

Firstly let us assume that $a \neq 0$ which means $x, y, z \neq 0$ so we can divide by them. Hence,

$$
\begin{aligned}
8x^3z^3 =& a^6 & & \\
24a^2yz^3 =& a^6 & x =& 3a^2y \\
24yz^3 =& a^4 & x =& 3a^2\frac{a^4}{24z^3} \\
y =& \frac{a^4}{24z^3} & x =& \frac{a^6}{8z^3}
\end{aligned}
$$

So, we can let $z = t$, and we have that the curve is, $c(t) = (\frac{a^6}{8t^3}, \frac{a^4}{24t^3}, t)$. So, next we find the arc-length parameter. To do that, we first find $||\dot{c}(t)|| = ||(\frac{-3a^6}{8t^4}, \frac{-a^4}{8t^4}, 1)|| = \sqrt{\frac{9a^{12}}{64t^8} + \frac{a^8}{64t^8} + 1} \neq 0$. So, let $s = \frac{t}{\sqrt{\frac{9a^{12}}{64t^8} + \frac{a^8}{64t^8} + 1}} = \frac{8t^5}{\sqrt{9a^{12}+a^8+64t^8}}$ and s is the arc-length parameter. Furthermore, $t' = \frac{1}{ds/dt} = \frac{1}{}$

$$
\begin{aligned}
T =& \dot{c}t' \\
=&
\end{aligned}
$$

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2. If c is a closed curve of length L on the unit sphere, show that:

a. $\int_0^L \tau(s)ds = 0$. First we can write c in spherical coordinates as, $c(s) = (1, \varphi(s), \psi(s))$, where φ is the polar angle, ψ is the azimuthal angle, 1 is the radius and s is the arc-length parameter. Note: since c is on the unit sphere, we know $\kappa \neq 0$ everywhere. Furthermore, Then, $T = (0, \varphi'(s), \psi'(s)) \rightarrow (\varphi')^2 + (\psi')^2 = 1$. Next, $N = \frac{T'}{||T'||} = \frac{(0,\varphi''(s),\psi''(s))}{\sqrt{(\varphi''(s))^2+(\psi''(s))^2}} = \frac{(0,\varphi''(s),\psi''(s))}{\kappa}$. Note, $N' = -\kappa T + \tau B$

b. $\int_0^L \frac{\tau}{\kappa} ds = 0$.

3. Prove that for any real number r there exists a closed curve c of length L, such that $\int_0^L \tau ds = r$.

4. Provide a definition of convex curve in a plane and a proof of the Four Vertex Theorem (Theorem 2.33).

"A simply closed plane curve is called *convex*, if the image set of the boundary is a convex subset $C \subseteq \mathbb{R}^2$. The convexity of a subset C is defined in the usual way, namely, for any two points contained in C, also the segment joining these two points is completely contained in C."
In other words, a plane curve c is called *convex* if it is the boundary of a convex set in $\mathbb{R}^2$.

Next, to prove the Four Vertex Theorem.

5. Suppose that a Frenet curve is an intersection of two regular (parameterized) surface elements. Show that if it is a line of curvature for both surfaces, then the surfaces intersect at a constant angle.