DIFFERENTIAL GEOMETRY HOMEWORK 2

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1. Determine the curvature and the torsion of the curve given by the intersection of the surfaces $x^3 = 3a^2y$ and $2xz = a^2$

Firstly if $a = 0$, then the resulting intersection is the y, z plane and hence not a curve. So let $a \neq 0$ which means $x, y, z \neq 0$ so we can divide by them. Hence,

$$\begin{aligned} 8x^3z^3 &= a^6 & & \\ 24a^2yz^3 &= a^6 & x &= 3a^2y \\ 24yz^3 &= a^4 & x &= 3a^2\frac{a^4}{24z^3} \\ y &= \frac{a^4}{24z^3} & x &= \frac{a^6}{8z^3} \end{aligned}$$

So, we can let $z = t$, and we have that the curve is, $c(t) = \left(\frac{a^6}{8t^3}, \frac{a^4}{24t^3}, t\right)$ (so formally this would be two curves, one for the positive t and one for negative, and note that the denominators throughout this problem cannot be zero because t has to be on one side of zero). $\dot{c} = \left(\frac{-3a^6}{8t^4}, \frac{a^4}{8t^4}, 1\right)$, and $||\dot{c}|| = \sqrt{\frac{9a^{12}}{64t^8} + \frac{a^8}{64t^8} + 1}$. Hence, $T = \frac{8t^4}{\sqrt{9a^{12}+a^8+64t^8}}\left(\frac{-3a^6}{8t^4}, \frac{a^4}{8t^4}, 1\right) = \frac{1}{\sqrt{9a^{12}+a^8+64t^8}}\left(-3a^6, a^4, 8t^4\right)$.
Next

$$\begin{aligned} \dot{T} &= \frac{1}{\sqrt{9a^{12}+a^8+64t^8}}\left(0,0,32t^3\right) + \frac{-512t^7}{2\sqrt{9a^{12}+a^8+64t^8}^3}\left(-3a^6, a^4, 8t^4\right) \\ &= \frac{2(9a^{12}+a^8+64t^8)}{2\sqrt{9a^{12}+a^8+64t^8}^3}\left(0,0,32t^3\right) - \frac{512t^7}{2\sqrt{9a^{12}+a^8+64t^8}^3}\left(-3a^6, a^4, 8t^4\right) \\ &= \frac{1}{2\sqrt{9a^{12}+a^8+64t^8}^3}\left(-3a^6, a^4, 64t^3(9a^{12}+a^8+64t^8) - 4096t^{11}\right) \\ &= \frac{a^4}{2\sqrt{9a^{12}+a^8+64t^8}^3}\left(-3a^2, 1, 64a^4(9a^4+1)t^3\right) \end{aligned}$$

Since $T' = \dot{T}t' = \frac{\dot{T}}{\dot{s}}$, where s is the arc-length parameter, it suffices to find what $\frac{ds}{dt}$ ($\dot{s}$) is. Note, $s(t) = \int_{t_0}^{t} ||\dot{c}(t)||dt = \int_{t_0}^{t}\sqrt{\frac{9a^{12}}{64t^8} + \frac{a^8}{64t^8} + 1}dt = F(t) - F(t_0)$, where $\frac{dF}{dt} = ||\dot{c}|| = \sqrt{\frac{9a^{12}}{64t^8} + \frac{a^8}{64t^8} + 1}$ (by the fundamental theorem of Calculus). Hence, $\frac{ds}{dt} = \frac{d}{dt}(F(t) - F(t_0)) = \frac{dF}{dt}(t) = ||\dot{c}|| = \sqrt{\frac{9a^{12}}{64t^8} + \frac{a^8}{64t^8} + 1}$.

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So,

$$\begin{aligned}
T' =& \frac{a^4}{2\sqrt{9a^{12}+a^8+64t^8}^3}\left(-3a^2,1,64a^4(9a^4+1)t^3\right)/\sqrt{\frac{9a^{12}}{64t^8}+\frac{a^8}{64t^8}+1} \\
=& \frac{8t^4a^4}{2\sqrt{9a^{12}+a^8+64t^8}^3}\left(-3a^2,1,64a^4(9a^4+1)t^3\right)/\sqrt{9a^{12}+a^8+64t^8} \\
=& \frac{4t^4a^4}{\left(9a^{12}+a^8+64t^8\right)^2}\left(-3a^2,1,64a^4(9a^4+1)t^3\right)
\end{aligned}$$

So,

$$\begin{aligned}
\kappa =& ||c''|| \\
=& \left|\left|\frac{4t^4a^4}{\left(9a^{12}+a^8+64t^8\right)^2}\left(-3a^2,1,64a^4(9a^4+1)t^3\right)\right|\right| \\
=& \frac{4t^4a^4}{\left(9a^{12}+a^8+64t^8\right)^2}\left|\left|\left(-3a^2,1,64a^4(9a^4+1)t^3\right)\right|\right| \text{ b/c the constant is} \geq 0 \\
=& \frac{4t^4a^4}{\left(9a^{12}+a^8+64t^8\right)^2}\left|\left|\left(-3a^2,1,64a^4(9a^4+1)t^3\right)\right|\right| \\
=& \frac{4t^4a^4}{\left(9a^{12}+a^8+64t^8\right)^2}\sqrt{9a^4+1+4096a^8(9a^4+1)^2t^6}
\end{aligned}$$

$$\begin{aligned}
\tau B =& N' + \kappa T \\
=& \left(\frac{T'}{\kappa}\right)' + \kappa T \\
=& \left(\frac{1}{\sqrt{9a^4+1+4096a^8(9a^4+1)^2t^6}}\left(-3a^2,1,64a^4(9a^4+1)t^3\right)\right)' + \kappa T \\
=& \left(\frac{12288a^8(9a^4+1)^2t^5}{\dot{s}(9a^4+1+4096(9a^4+1)^2a^8t^6)^{3/2}}\right)\left(-3a^2,1,64a^4(9a^4+1)t^3\right) \\
&+ \frac{1}{\dot{s}\sqrt{9a^4+1+4096a^8(9a^4+1)^2t^6}}\left(0,0,192a^4(9a^4+1)t^2\right) \\
&+ \kappa\frac{1}{\sqrt{9a^{12}+a^8+64t^8}}\left(-3a^6,a^4,8t^4\right)
\end{aligned}$$

2. If c is a closed curve of length L on the unit sphere, show that:

a. $\int_0^L \tau(s)ds = 0$**.** Firstly we note that if the curve is planar then $\tau = 0$ and the proof is done. So let us assume that τ is not constantly 0.
Next, we shall prove that $\tau = \frac{J'}{1+J^2}$, where $J = \det[c, c', c'']$, which would lead us to (by the fundamental theorem of calculus)

$$\int_0^L \tau ds = \int_0^L \frac{J'}{1+J^2} ds = \tan^{-1}(J(L)) - \tan^{-1}(J(0))$$

=0 because the three vectors must be the same at the endpoints (closed curve)

Now to prove our claim:
Note that since the curve is on the unit sphere, the vectors $c, c', c \times c'$ form an orthonormal frame along the curve. Hence, $c'' = \langle c'', c\rangle c + \langle c'', c'\rangle c' + \langle c'', c \times c'\rangle c \times c'$. But since $\langle c, c'\rangle = 0$ (again, because it's on a sphere), we have that $0 = \langle c, c'\rangle' = \langle c', c'\rangle + \langle c, c''\rangle \Rightarrow \langle c, c''\rangle = -\langle c', c'\rangle = -1$. Furthermore, $J = \det[c, c', c''] = \langle c'', c \times c'\rangle$. This last part can be seen because the determinant is the unique multi-linear map that sends e_1, e_2, e_3 (orthonormal basis) to 1. So that $c'' = -c + Jc \times c'$ and $\kappa^2 = ||c''||^2 = 1 + J^2$. Kühnel uses this to show that $\kappa^2 > 0$, but we can already see that $\kappa > 1$ by considering the oscillating circle at that point (which has radius $1/\kappa$) that must have radius ≤ 1 if it's to be on S^1. Which means, $\kappa \geq 1$. Either way, we have $\kappa > 0$.
Furthermore, $T = c'$, $N = \frac{1}{\kappa}c''$, and $B = T \times N$. But $\langle T', c\rangle = \langle c'', c\rangle = -1 \Rightarrow 0 = \langle T', c\rangle' = \langle T'', c\rangle + \langle T', c'\rangle = \langle T'', c\rangle + 0 \Rightarrow \langle T'', c\rangle = 0$

$$\begin{aligned}
\tau =& \langle B, N'\rangle = -\langle B', N\rangle \\
=& \frac{-1}{\kappa}\langle B', T'\rangle \\
=& \frac{-1}{\kappa}\langle (T \times N)', T'\rangle \\
=& \frac{-1}{\kappa}\langle (\frac{1}{\kappa}T \times T')', T'\rangle \\
=& \frac{-1}{\kappa^2}\langle \frac{1}{\kappa}T' \times T' + \frac{1}{\kappa}T \times T'' + \frac{-\kappa'}{\kappa^2}T \times T', T'\rangle \\
=& \frac{-1}{\kappa^2}\langle \frac{1}{\kappa}T \times T'' + \frac{-\kappa'}{\kappa}T \times T', T'\rangle \\
=& \frac{-1}{\kappa}\left(\langle \frac{1}{\kappa}T \times T'', T'\rangle + \langle \frac{-\kappa'}{\kappa^2}T \times T', T'\rangle\right) \\
=& \frac{-1}{\kappa^2}\langle T \times T'', T'\rangle \\
=& \frac{-1}{\kappa^2}\langle T \times T'', -c + Jc \times T\rangle \\
=&
\end{aligned}$$

b. $\int_0^L \frac{\tau}{\kappa} ds = 0$**.** Firstly we note that if τ is constantly zero then the integral is obviously 0. Next we will prove that $\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$, so by the fundamental theorem of calculus,

$$\int_0^L \frac{\tau}{\kappa} ds = \frac{\kappa'(L)}{\tau(L)\kappa^2(L)} - \frac{\kappa'(0)}{\tau(0)\kappa^2(0)} = 0$$

3. Prove that for any real number c there exists a closed curve r of length L, such that $\int_0^L \tau ds = c$.

Let $L, c \in \mathbb{R}_+$ ($\neq 0$ because that case was done in question 2 and for negatives, just consider the reverse of our construction). We will define a curve, r (parameterized by arc-length), in 4 parts such that r has length L with total torsion c. Part 1 will be $[0, \alpha]$ for $\alpha = L/10$, part two will be a curve on a sphere with $s \in [\alpha, \beta]$, for some $\beta \leq 3L/10$, then part three will be a plane curve for $s \in [\beta, \gamma]$ for some $\gamma < L$, and part four will be another curve on a sphere with $s \in [\gamma, L]$.

Part 1 ($s \in [0, \beta]$): $r(t) = (a\cos(t), a\sin(t), bt)$, for some $a \in \mathbb{R}$ (to be figured out later and assumed positive for our purposes).

Then, $\dot{r} = (-a\sin(t), a\cos(t), b) \to ||\dot{r}|| = \sqrt{a^2+b^2}$, so that $s(t) = \int_0^t \sqrt{a^2+b^2}dx = \sqrt{a^2+b^2}t$, and $r = r(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right)$.

So, $T = \left(\frac{-a}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{a}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{b}{\sqrt{a^2+b^2}}\right)$,

and $r'' = T' = \left(\frac{-a}{a^2+b^2}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{-a}{a^2+b^2}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right) \Rightarrow \kappa = \frac{|a|}{a^2+b^2} = \frac{a}{a^2+b^2}$.

$N = \left(-\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right)$.

Next, $B = T \times N = \left(\frac{b}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{-b}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{a}{\sqrt{a^2+b^2}}\right)$.

$$\begin{aligned}
\tau B =& N' + \kappa T \\
=& \left(\frac{1}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{-1}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right) + \kappa T \\
=& \left(\frac{a^2+b^2-a^2}{\sqrt{a^2+b^2}^3}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{-a^2-b^2+a^2}{\sqrt{a^2+b^2}^3}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{ab}{\sqrt{a^2+b^2}^3}\right) \\
=& \frac{b}{a^2+b^2}\left(\frac{b}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{-b}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{a}{\sqrt{a^2+b^2}}\right) \implies \\
\tau =& \frac{b}{\sqrt{a^2+b^2}}
\end{aligned}$$

Then, $\int_0^\alpha \tau ds = \frac{b\alpha}{\sqrt{a^2+b^2}}$. But we want $\alpha = 2n\pi\sqrt{a^2+b^2}$, so that things will play out nicely in parts 2,3, and 4. So $\int_0^\alpha \tau ds = \frac{2n\pi\sqrt{a^2+b^2}b}{\sqrt{a^2+b^2}} = 2n\pi b$. By keeping b constant and letting $n \to \infty$ (hence $a \to 0$) we see $\int_0^\alpha \tau ds \to \infty$. On the other hand, by setting $b = 0$ (hence $2n\pi a = \alpha = L/10$) we see that $\int_0^\alpha \tau ds = 0$. So, for any positive $c \in \mathbb{R}$, we can find a, b, n such that $\int_0^\alpha \tau ds = c$. Next we will connect the two endpoints of this helix in a way such that the total torsion is 0.

Part 2 ($[\alpha, \beta]$ on the sphere): In particular, this portion will be on the oscillating sphere at the point $s = \alpha$. This will be any regular curve of length $\leq L/5$ on said sphere that ends on the great circle parallel to the plane defined by $T(\alpha)$ and the line segment connecting $r(0)$ with $r(\alpha)$ (the two ends of the helix). Let us call this endpoint of part 2 of this curve, d. The idea is that part 3 will be a plane curve connecting to the same point on the oscillating sphere at the bottom of the helix, thereby making that into a pseudo-closed curve on a sphere and by question 2, the total torsion 0.

Part 3 ($[\beta, \gamma]$) is just any plane curve that smoothly connects the point d from above to the corresponding point d' on the oscillating sphere at $r(0)$ in such a way that it is a continuation of part 2 on the sphere.

Part 4 ($[\gamma, L]$): Essentially a continuation of part two but on the oscillating sphere at $r(0)$. Note: $\int_\alpha^\beta \tau ds + \int_\gamma^L \tau ds = 0$ because it is the total torsion of a closed curve on a sphere.

Then, $\int_0^L \tau ds = \int_0^\alpha \tau ds + \int_\alpha^\beta \tau ds + \int_\beta^\gamma \tau ds + \int_\gamma^L \tau ds = c + \left(\int_\alpha^\beta \tau ds + \int_\gamma^L \tau ds\right) = c$.

4. Provide a definition of convex curve in a plane and a proof of the Four Vertex Theorem (Theorem 2.33).

"A simply closed plane curve is called *convex*, if the image set of the boundary is a convex subset $C \subseteq \mathbb{R}^2$. The convexity of a subset C is defined in the usual way, namely, for any two points contained in C, also the segment joining these two points is completely contained in C."
In other words, a plane curve c is called *convex* if it is the boundary of a convex set in $\mathbb{R}^2$.

Next up... The Four Vertex Theorem!
Claim: A simply closed, regular and convex plane curve which is of class $\mathcal{C}^3$ has at least four local extremal points for its curvature κ (such a point is referred to as a *vertex*).

Proof. Let c be a simply closed, regular and convex plane curve parameterized by arc length and from the interval $[0, L]$ to the x, y plane. Firstly, if κ is constant on any interval, then every point is a vertex. So from now on, let us assume that κ is not constant anywhere. Furthermore, since κ is a local extremal point, $\kappa' = 0$ and κ changes sign.
Note: Since $[0, L]$ is compact and κ is continuous, by the extreme value theorem, κ has an absolute minimum and maximum value in $[0, L]$. So two vertexes are given already.
Since the curve is a closed curve, we can assume that $\kappa(0)$ is an absolute minimum and $\kappa(s_0)$ is an absolute maximum. Next, we can choose the coordinate system to be such that $c(0)$ and $c(s_0)$ are both on the x axis. Since c is convex, if $c(s_1)$ is also on the x axis, then every point between 0 and s_1, and s_0 and s_1, should be on that line too. But that contradicts the curve being regular (second derivative cannot be 0).
Now, assume toward a contradiction that these are the only two points where κ' changes sign. Then, if we write $c(s)$ as $(x(s), y(s))$, then the function $\kappa' y$ never changes sign (because of our choice of axis). By the Frenet equations we have,

$$T = (x', y') \qquad T' = (x'', y'') = \kappa N \qquad N = (-y', x') \text{ because } T \cdot N = 0$$

By matching coordinates, we find, $x'' = -y'\kappa$. Furthermore,

$$\begin{aligned} \int_0^L \kappa' y ds &= \kappa y|_0^L - \int_0^L \kappa y' ds \\ &= \int_0^L x'' ds = x'(L) - x'(0) = 0 \end{aligned}$$

But since $\kappa' y$ never changes sign, we know $\kappa' y$ must be constant. Since y cannot be constant (in particular it cannot be constantly 0) on any interval (because the curve is regular), we find that κ' is constantly zero. But that means κ is constant, which is a contradiction to our criteria that κ not be constant on any interval.
Hence, there must be at least three vertexes. Since κ' changes sign at each vertex and the curve is L periodic, we know there must be an even number (if finite) of vertexes. Hence, there must be at least 4.
Hence, the 4 vertex theorem. □

5. Suppose that a Frenet curve is an intersection of two regular (parameterized) surface elements. Show that if it is a line of curvature for both surfaces, then the surfaces intersect at a constant angle.

Check page 266 of the other diff geom book.