DIFFERENTIAL GEOMETRY HOMEWORK 3

AARON NISKIN
PID: 3337729

1. Topics from the theory:

(1) Definitions of regular and Frenet curve. Fundamental theorem of the local theory of curves - proof. (It could be from the book for $\mathbb{R}^n$ or from the lectures for $\mathbb{R}^3$).

Date: 20ish FEB 2015.

(2) Definitions of closed curve, simply closed curve, and convex curve. Proof of the four vertex theorem.

(3) Definition of closed curve. Proof of Fenchel's theorem about the total curvature of space curve.

A closed curve is a regular curve $c : [a, b] \to \mathbb{R}^n$ such that there is a second regular curve $\tilde{c} : \mathbb{R} \to \mathbb{R}^n$ with the property that $\forall m \in \mathbb{Z}, \tilde{c}(x + m(b-a)) = c(x)$ and $\forall x \in [a, b], \tilde{c}(x) = c(x)$.

Fenchel's theorem states: For every closed and regular space curve $c : [a, b] \to \mathbb{R}^3$ of total length l on has the equality

$$\int_0^l \kappa ds = \int_a^b \kappa(t)||\dot{c}(t)||dt \geq 2\pi,$$

with equality if and only if the curve is a convex, simple plane curve.

It's theorem 2.34 on page 46.

(4) Definition of regular surface element, first and second fundamental forms, Gauss map. Proof of Theorem 3.14 on p. 73 from the book.

(5) Definition of minimal surface and proof of Theorem 3.28. Definition of conformal parametrization and proofs of Consequence 3.30 and Corollary 3.31 from the book.

(6) Definition of directional and covariant derivative. Proofs of lemma 4.4, Theorem 4.6, Corollary 4.8 from the book. Directional derivative, $D_X Y|_p = DY(p)(X) = \lim_{h\to 0} \frac{Y(p+hX)-Y(p)}{h}$.
Covariant - $\nabla_X Y(p) = (D_X Y(p))^{tang} = D_X Y(p) - \langle D_X Y(p), \nu\rangle \nu$
Lemma 4.4 follows from standard derivatives.
THM 4.6 - The covariant derivative depends only on the first fundamental form.

Proof. We set $X = \sum \xi^i \frac{\partial f}{\partial u^i}$ and $Y = \sum \eta^j \frac{\partial f}{\partial u^j}$. In order to determine $\nabla_X Y$, it is sufficient to know the quantities $\left\langle \nabla_X Y, \frac{\partial f}{\partial u^k} \right\rangle$ for all k. From the calculus rules 4.4 we get the equation,

$$\begin{aligned}
\nabla_X Y &= \sum \xi^i \nabla_{\frac{\partial f}{\partial u^i}} Y \\
&= \sum \xi^i \sum \nabla_{\frac{\partial f}{\partial u^i}} \left(\eta^j \frac{\partial f}{\partial u^j} \right) \\
&= \sum \xi^i \left(\frac{\partial eta^j}{\partial u^i} \frac{\partial f}{\partial u^j} + \eta^j \nabla_{\frac{\partial f}{\partial u^i}} \frac{\partial f}{\partial u^j} \right)
\end{aligned}$$

□

(7) Definition of geodesic line, line of curvature and asymptotic line. Definition of parallel vector field on a surface element and along a curve, parallel displacement. Proofs of Corollary 4.11, Theorem 4.12, Theorem 4.13. Parallel vector field - 4.9

(8) Fundamental theorem of the local theory of surfaces. Proofs of Lemma 4.23 and Theorem 4.24

(9) Proof of Theorema egregium in the two variants- Theorem 4.15, Corollary 4.16, Lemma 4.17, Corollary 4.20.

(10) Definition of 1-form and 2-form, properties and exterior differential. Proofs of Theorem 4.34, Theorem 4.35, Local Gauss-Bonnet formula (Theorem 4.38).

(11) Proof of Theorem 4.39, Corollary 4.40, 4.43 (global Gauss-Bonnet formula).

2. Problems:

(1) Prove that a curve satisfies $\ddot{c}c^{(3)}c^{(4)} = 0$ iff its principal normals are parallel to a fixed plane. Since $c^{(4)}$ exists, we know $c \in \mathcal{C}^4$ at least (and $c^{(4)}$ must have at most countably many discontinuities). Hence, if $c^{(4)} \neq 0$ on some interval, then we know $c^{(3)}$ is changing (and hence not constantly 0) on that interval. But then by similar logic we see $\ddot{c} \neq 0$. But then $\ddot{c}c^{(3)}c^{(4)} \neq 0$ for at least some points in there. So, then $c^{(4)} \neq 0$ doesn't hold for any intervals. But then, either $c^{(4)} \equiv 0$, or there are some points of discontinuities in $c^{(4)}$, but at those points one of the others must be 0. But since the

(2) Solve problem 25 on p. 52 from the book.
A Frenet curve in $\mathbb{R}^3$ is called a *Bertrand curve*, if there is a second curve such that the principle normal vectors to these two curves (at corresponding points) are identical, viewed as lines in space. One speaks in the case of a *Bertrand pair of curves.* Show that non-planar Bertrand curves are characterized by the existence of a linear relation $a\kappa + b\tau = 1$ with constants a, b, where $a \neq 0$. Let c_1, c_2 be a pair of distinct Bertrand curves in $\mathbb{R}^3$. WLOG: we can assume that c_1 is parameterized by arc-length already (if not, then re-parameterize it, and whatever the re-parameterization function is, apply it to the c_2 parameter too, and we have what we want).
First we notice that $d(c_1(t), c_2(t))$ is constant. To see this,

$$c_2(t) = c_1(t) + f(t)N(t) \text{ because the two } \vec{N}\text{s define the same line in space}$$

If we prove that $f(t)$ is constant, we've done what we need. But note:

$$\begin{aligned}
\frac{T_2}{t'} &= \dot{c}_2 \\
&= c_1' + f'N + fN' \\
&= T_1 + f'N + f(-\kappa_1 T_1 + \tau_1 B_1) \\
&= (1 - f\kappa_1)T_1 + f'N + f\tau_1 B_1 \Rightarrow \\
f't' &= \langle c_2', N\rangle = \langle T_2, N\rangle = 0
\end{aligned}$$

So, f is constant (because t is never constant wrt the arc-length parameter, or differently stated, $t' \neq 0$), and hence $d(c_1(t), c_2(t))$ is constant (let's say $= \alpha$) as well.
Next we claim that $\theta = \Theta(t) = \frac{\langle T_1, T_2\rangle}{||T_1||||T_2||} = \langle T_1, T_2\rangle$ is constant. To see this,

$$\begin{aligned}
T_2 &= Proj_{T_2}T_1 + Proj_{T_2}N + Proj_{T_2}B_1 \\
&= T_1\cos\theta + B_1\sin\theta \text{ because } T_2 \perp N \\
\frac{N}{t'} &= \dot{T}_2 \\
&= T_1'\cos\theta - T_1\sin\theta\theta' + B_1'\sin\theta + B_1\cos\theta\theta' \\
&= \kappa_1 N\cos\theta - T_1\sin\theta\theta' - \tau_1 N\sin\theta + B_1\cos\theta\theta' \\
&= -\sin\theta\theta' T_1 + (\kappa_1\cos\theta - \tau_1\sin\theta)N + \cos\theta\theta' B_1 \Rightarrow \\
-\sin\theta\theta' &= 0 = \cos\theta\theta' \Rightarrow \\
\theta' &= 0
\end{aligned}$$

So θ is constant. So, by matching coefficients we find,

$$\cos\theta = t' - f\kappa_1 t' \qquad\qquad \sin\theta = f\tau_1 t'$$

Since the curves were distinct and the distance is constant, we know $f \neq 0$, and since the curves are non-planar, we know $\tau_1 \neq 0$. So,

$$\begin{gathered}
\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{t' - f\kappa_1 t'}{f\tau_1 t'} = \frac{1 - f\kappa_1}{f\tau_1} \rightarrow \\
f\kappa_1 + \cot\theta f\tau_1 = 1 \rightarrow \text{ let } a = f\kappa_1, \text{ and } b = \cot\theta f.
\end{gathered}$$

Now, let us assume there is a non-planar Frenet curve c_1 in $\mathbb{R}^3$, and constants a, b with $a \neq 0$, such that for all s (I'm assuming c_1 is parameterized by arc length) $a\kappa + b\tau = 1$. We need to find a Bertrand pair for this curve. To do that, let $d = a$, and define a new curve, $c_2 = c_1 + dN_1$. The only thing we need to do is show that c_2 is a Frenet curve and that the two N's define the same line in space.

(3) Determine the curvature and torsion of the curve given by:

(a) The intersection of the surfaces $x^3 = 3a^2y$ and $2xz = a^2$.
Firstly if $a = 0$, then the resulting intersection is the y, z plane and hence not a curve. So let $a \neq 0$ which means $x, y, z \neq 0$ so we can divide by them. Hence,

$$\begin{aligned} 8x^3z^3 &= a^6 \\ 24a^2yz^3 &= a^6 \\ 24yz^3 &= a^4 \\ y &= \frac{a^4}{24z^3} \end{aligned} \qquad\qquad \begin{aligned} x^3 &= 3a^2y \\ x\frac{a^4}{4z^2} &= 3a^2\frac{a^4}{24z^3} \\ x &= \frac{a^2}{2z} \end{aligned}$$

So, we can let $z = t$, and we have that the curve is, $c(t) = \left(\frac{a^2}{2t}, \frac{a^4}{24t^3}, t\right)$ (so formally this would be two curves, one for the positive t and one for negative, and note that the denominators throughout this problem cannot be zero because t has to be on one side of zero). $\dot{c} = \left(\frac{-a^2}{2t^2}, \frac{-a^4}{8t^4}, 1\right)$, and $||\dot{c}|| = \sqrt{\frac{a^4}{4t^4} + \frac{a^8}{64t^8} + 1} = \frac{\sqrt{64t^8+16a^4t^4+a^8}}{8t^4} = \frac{\sqrt{(a^4+8t^4)^2}}{8t^4} = \frac{8t^4+a^4}{8t^4}$. Hence, $T = \frac{8t^4}{8t^4+a^4}\left(\frac{-a^2}{2t^2}, \frac{-a^4}{8t^4}, 1\right) = \frac{1}{8t^4+a^4}\left(-4a^2t^2, -a^4, 8t^4\right)$.

$$\begin{aligned} \dot{T} &= \frac{1}{8t^4+a^4}\left(-8a^2t, 0, 32t^3\right) - \frac{32t^3}{(8t^4+a^4)^2}\left(-4a^2t^2, -a^4, 8t^4\right) \\ &= \frac{8t^4+a^4}{(8t^4+a^4)^2}\left(-8a^2t, 0, 32t^3\right) - \frac{32t^3}{(8t^4+a^4)^2}\left(-4a^2t^2, -a^4, 8t^4\right) \\ &= \frac{1}{(8t^4+a^4)^2}\left(-(8t^4+a^4)8a^2t + 128a^2t^5, 32a^4t^3, 32t^3(8t^4+a^4-8t^4)\right) \\ &= \frac{1}{(8t^4+a^4)^2}\left(-64a^2t^5 - 8a^6t + 128a^2t^5, 32a^4t^3, 32a^4t^3\right) \\ &= \frac{8a^2t}{(8t^4+a^4)^2}\left(8t^4 - a^4, 4a^2t^2, 4a^2t^2\right) \end{aligned}$$

Since $T' = \dot{T}t' = \frac{\dot{T}}{\dot{s}}$, where s is the arc-length parameter, it suffices to find what $\frac{ds}{dt}$ ($\dot{s}$) is. Note, $s(t) = \int_{t_0}^{t} ||\dot{c}(t)||dt = F(t) - F(t_0)$, where $\frac{dF}{dt} = ||\dot{c}||$ (by the fundamental theorem of Calculus). Hence, $\dot{s} = \frac{ds}{dt} = \frac{d}{dt}(F(t) - F(t_0)) = \frac{dF}{dt}(t) = ||\dot{c}||$. So,

$$\begin{aligned} T' &= \frac{8t^4}{8t^4+a^4}\frac{8a^2t}{(8t^4+a^4)^2}\left(8t^4 - a^4, 4a^2t^2, 4a^2t^2\right) \\ &= \frac{64a^2t^5}{(8t^4+a^4)^3}\left(8t^4 - a^4, 4a^2t^2, 4a^2t^2\right) \end{aligned}$$

So,

$$
\begin{aligned}
\kappa =& ||c''|| = ||T'|| \\
=& \left|\left| \frac{64a^2t^5}{(8t^4+a^4)^3}\left(8t^4-a^4, 4a^2t^2, 4a^2t^2\right)\right|\right| \\
=& \frac{64a^2t^5}{(8t^4+a^4)^3}\left|\left|\left(8t^4-a^4, 4a^2t^2, 4a^2t^2\right)\right|\right| \\
=& \frac{64a^2t^5}{(8t^4+a^4)^3}\sqrt{(8t^4-a^4)^2+16a^4t^4+16a^4t^4} \\
=& \frac{64a^2t^5}{(8t^4+a^4)^3}\sqrt{64t^8-16a^4t^4+a^8+16a^4t^4+16a^4t^4} \\
=& \frac{64a^2t^5}{(8t^4+a^4)^3}\sqrt{64t^8+16a^4t^4+a^8} \\
=& \frac{64a^2t^5}{(8t^4+a^4)^3}\sqrt{(8t^4+a^4)^2} \\
=& \frac{64a^2t^5}{(8t^4+a^4)^2}
\end{aligned}
$$

$T = \frac{1}{8t^4+a^4}\left(-4a^2t^2, -a^4, 8t^4\right)$
$N = \frac{1}{8t^4+a^4}\left(8t^4-a^4, 4a^2t^2, 4a^2t^2\right)$.
So, $B = \frac{1}{(8t^4+a^4)^2}\left(-4a^6t^2-32a^2t^6, 64t^8-8a^4t^4+16a^4t^4, -16a^4t^4+8a^4t^4-a^8\right)$
$= \frac{1}{(8t^4+a^4)^2}\left(-4a^2t^2(8t^4+a^4), 8t^4(8t^4+a^4), -a^4(8t^4+a^4)\right) = \frac{1}{8t^4+a^4}\left(-4a^2t^2, 8t^4, -a^4\right)$

$$
\begin{aligned}
\tau B =& N' + \kappa T \\
=& \left(\frac{T'}{\kappa}\right)' + \kappa T \\
=& \left(\frac{1}{8t^4+a^4}\left(8t^4-a^4, 4a^2t^2, 4a^2t^2\right)\right)' + \kappa T \\
=& \frac{-256t^7}{(8t^4+a^4)^3}\left(8t^4-a^4, 4a^2t^2, 4a^2t^2\right) + \frac{8t^4}{(8t^4+a^4)^2}\left(32t^3, 8a^2t, 8a^2t\right) + \frac{64a^2t^5}{(8t^4+a^4)^3}\left(-4a^2t^2, -a^4, 8t^4\right) \\
=& \frac{8t^5}{(8t^4+a^4)^3}\left(-8a^2(-4a^2t^2), -8a^2(8t^4), -8a^2(-a^4)\right) \\
=& \frac{-64a^2t^5}{(8t^4+a^4)^2}\left(\frac{1}{8t^4+a^4}\left(-4a^2t^2, 8t^4, -a^4\right)\right) = \frac{-64a^2t^5}{(8t^4+a^4)^2}B \Rightarrow \tau = \frac{-64a^2t^5}{(8t^4+a^4)^2}
\end{aligned}
$$

(b) $c = (\cos^3 t, \sin^3 t, \cos 2t)$.

$$
\begin{aligned}
\dot{c} =& (-\cos^2 t\sin t, \sin^2 t\cos t, -2\sin 2t) \quad \Rightarrow \quad ||\dot{c}||^2 = \cos^4 t\sin^2 t + \sin^4 t\cos^2 t + 4\sin^2 2t \\
&= \cos^2 t\sin^2 t(\cos^2 t + \sin^2 t) + 4(2\sin t\cos t)^2 \\
&= 17\cos^2 t\sin^2 t = \frac{17}{4}\sin^2 2t
\end{aligned}
$$

So that, $||\dot{c}|| = \sqrt{17}\cos t\sin t = \frac{\sqrt{17}}{2}\sin 2t$ and $c' = \frac{1}{\sqrt{17}}(-\cos t, \sin t, -4)$. Then,

(4) if c is a closed curve on the unit sphere of length L, show that:

a. $\int_0^L \tau(s)ds = 0$.

b. $\int_0^L \frac{\tau}{\kappa}ds = 0$.

(5) Suppose that a Frenet curve is an intersection of two regular (parameterized) surface elements. Show that if it is a line of curvature for both surfaces than the surfaces intersect at a constant angle.

(6) Find asymptotic lines of the surface $z = a(\frac{x}{y} + \frac{y}{x})$ Firstly, this surface is only defined when $x, y \neq 0$, so let's let $U = \{v \in \mathbb{R}^2 | \pi_1(v) \neq 0 \neq \pi_2(v)\}$ and let $f : U \to \mathbb{R}^3$ given by, $f(x, y) = (x, y, a(\frac{x}{y} + \frac{y}{x}))$

(7) Prove that a line of curvature on a surface is planar if its osculating plane forms a constant angle with the tangent plane to the surface.

(8) Show that the catenoid is the only surface of rotation for which $H \equiv 0, K \not\equiv 0$.

(9) The *rotational torus* is given by

$$f(u,v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)\,,$$

$0 \leq u, v \leq 2\pi$, cf. Figure 3.3. Here $a > b > 0$ are arbitrary (but fixed) parameters. Calculate the *total mean curvature* of this torus as the surface integral of the function $(H(u,v))^2, 0 \leq u, v \leq 2\pi$, explicitly as a function of a and b. What is the smallest possible value of the total mean curvature?

Hint: The minimum occurs at $a = \sqrt{2}b$. Note that the integral is invariant under homotheties $x \to \lambda x$ of space with a fixed number λ.

(10) For a surface element $f : U \to \mathbb{R}^3$ we define the *parallel surface* at distance ϵ by
$$f_\epsilon(u_1, u_2) := f(u_1, u_2) + \epsilon \cdot \nu(u_1, u_2),$$
cf. Section 3D. ν is the unit normal of the surface f. Decide for which ϵ this defines a regular surface, and show the following.

(a) The principal curvatures of f_ϵ and f have a ratio of $\kappa_i^{(\epsilon)} = \kappa_i/(1 - \epsilon\kappa_i)$.

(b) In case f has constant mean curvature $H \neq 0$, f_ϵ has constant Gaussian curvature for $\epsilon = \frac{1}{2H}$.

(11) Problems 11-20 on p. 187 (CH 4) from the book.

(12) Show that on a parameterized surface element with $K \leq 0$ there is no closed geodesic.