

DIFFERENTIAL GEOMETRY HOMEWORK 2

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1. DETERMINE THE CURVATURE AND THE TORSION OF THE CURVE GIVEN BY THE INTERSECTION OF THE SURFACES $x^3 = 3a^2y$ AND $2xz = a^2$

Firstly if $a = 0$, then the resulting intersection is the y, z plane and hence not a curve. So let $a \neq 0$ which means $x, y, z \neq 0$ so we can divide by them. Hence,

$$\begin{aligned} 8x^3z^3 &= a^6 & x^3 &= 3a^2y \\ 24a^2yz^3 &= a^6 & x\frac{a^4}{4z^2} &= 3a^2\frac{a^4}{24z^3} \\ 24yz^3 &= a^4 & x &= \frac{a^2}{2z} \\ y &= \frac{a^4}{24z^3} \end{aligned}$$

So, we can let $z = t$, and we have that the curve is, $c(t) = \left(\frac{a^2}{2t}, \frac{a^4}{24t^3}, t\right)$ (so formally this would be two curves, one for the positive t and one for negative, and note that the denominators throughout this problem cannot be zero because t has to be on one side of zero). $\dot{c} = \left(\frac{-a^2}{2t^2}, \frac{-a^4}{8t^4}, 1\right)$, and $||\dot{c}|| = \sqrt{\frac{a^4}{4t^4} + \frac{a^8}{64t^8} + 1} = \frac{\sqrt{64t^8 + 16a^4t^4 + a^8}}{8t^4} = \frac{\sqrt{(a^4 + 8t^4)^2}}{8t^4} = \frac{8t^4 + a^4}{8t^4}$. Hence, $T = \frac{8t^4}{8t^4 + a^4} \left(\frac{-a^2}{2t^2}, \frac{-a^4}{8t^4}, 1\right) = \frac{1}{8t^4 + a^4} (-4a^2t^2, -a^4, 8t^4)$.

$$\begin{aligned} \dot{T} &= \frac{1}{8t^4 + a^4} (-8a^2t, 0, 32t^3) - \frac{32t^3}{(8t^4 + a^4)^2} (-4a^2t^2, -a^4, 8t^4) \\ &= \frac{8t^4 + a^4}{(8t^4 + a^4)^2} (-8a^2t, 0, 32t^3) - \frac{32t^3}{(8t^4 + a^4)^2} (-4a^2t^2, -a^4, 8t^4) \\ &= \frac{1}{(8t^4 + a^4)^2} (-(8t^4 + a^4)8a^2t + 128a^2t^5, 32a^4t^3, 32t^3(8t^4 + a^4 - 8t^4)) \\ &= \frac{1}{(8t^4 + a^4)^2} (-64a^2t^5 - 8a^6t + 128a^2t^5, 32a^4t^3, 32a^4t^3) \\ &= \frac{8a^2t}{(8t^4 + a^4)^2} (8t^4 - a^4, 4a^2t^2, 4a^2t^2) \end{aligned}$$

Since $T' = \dot{T}t' = \frac{\dot{T}}{\dot{s}}$, where s is the arc-length parameter, it suffices to find what $\frac{ds}{dt} (\dot{s})$ is. Note, $s(t) = \int_{t_0}^t ||\dot{c}(t)|| dt = F(t) - F(t_0)$, where $\frac{dF}{dt} = ||\dot{c}||$ (by the fundamental theorem of Calculus). Hence,

$$\dot{s} = \frac{ds}{dt} = \frac{d}{dt}(F(t) - F(t_0)) = \frac{dF}{dt}(t) = \|\dot{c}\|. \text{ So,}$$

$$\begin{aligned} T' &= \frac{8t^4}{8t^4 + a^4} \frac{8a^2t}{(8t^4 + a^4)^2} (8t^4 - a^4, 4a^2t^2, 4a^2t^2) \\ &= \frac{64a^2t^5}{(8t^4 + a^4)^3} (8t^4 - a^4, 4a^2t^2, 4a^2t^2) \end{aligned}$$

So,

$$\begin{aligned} \kappa &= \|c''\| = \|T'\| \\ &= \left\| \frac{64a^2t^5}{(8t^4 + a^4)^3} (8t^4 - a^4, 4a^2t^2, 4a^2t^2) \right\| \\ &= \frac{64a^2t^5}{(8t^4 + a^4)^3} \|(8t^4 - a^4, 4a^2t^2, 4a^2t^2)\| \\ &= \frac{64a^2t^5}{(8t^4 + a^4)^3} \sqrt{(8t^4 - a^4)^2 + 16a^4t^4 + 16a^4t^4} \\ &= \frac{64a^2t^5}{(8t^4 + a^4)^3} \sqrt{64t^8 - 16a^4t^4 + a^8 + 16a^4t^4 + 16a^4t^4} \\ &= \frac{64a^2t^5}{(8t^4 + a^4)^3} \sqrt{64t^8 + 16a^4t^4 + a^8} \\ &= \frac{64a^2t^5}{(8t^4 + a^4)^3} \sqrt{(8t^4 + a^4)^2} \\ &= \frac{64a^2t^5}{(8t^4 + a^4)^2} \end{aligned}$$

$$T = \frac{1}{8t^4 + a^4} (-4a^2t^2, -a^4, 8t^4)$$

$$N = \frac{1}{8t^4 + a^4} (8t^4 - a^4, 4a^2t^2, 4a^2t^2).$$

$$\begin{aligned} \text{So, } B &= \frac{1}{(8t^4 + a^4)^2} (-4a^6t^2 - 32a^2t^6, 64t^8 - 8a^4t^4 + 16a^4t^4, -16a^4t^4 + 8a^4t^4 - a^8) \\ &= \frac{1}{(8t^4 + a^4)^2} (-4a^2t^2(8t^4 + a^4), 8t^4(8t^4 + a^4), -a^4(8t^4 + a^4)) = \frac{1}{8t^4 + a^4} (-4a^2t^2, 8t^4, -a^4) \end{aligned}$$

$$\tau B = N' + \kappa T$$

$$= \left(\frac{T'}{\kappa} \right)' + \kappa T$$

$$= \left(\frac{1}{8t^4 + a^4} (8t^4 - a^4, 4a^2t^2, 4a^2t^2) \right)' + \kappa T$$

$$= \frac{-256t^7}{(8t^4 + a^4)^3} (8t^4 - a^4, 4a^2t^2, 4a^2t^2) + \frac{8t^4}{(8t^4 + a^4)^2} (32t^3, 8a^2t, 8a^2t) + \frac{64a^2t^5}{(8t^4 + a^4)^3} (-4a^2t^2, -a^4, 8t^4)$$

$$= \frac{8t^5}{(8t^4 + a^4)^3} (-8a^2(-4a^2t^2), -8a^2(8t^4), -8a^2(-a^4))$$

$$= \frac{-64a^2t^5}{(8t^4 + a^4)^2} \left(\frac{1}{8t^4 + a^4} (-4a^2t^2, 8t^4, -a^4) \right) = \frac{-64a^2t^5}{(8t^4 + a^4)^2} B \Rightarrow \tau = \frac{-64a^2t^5}{(8t^4 + a^4)^2}$$

2. If c is a closed curve of length L on the unit sphere, show that:

a. $\int_0^L \tau(s) ds = 0$. Firstly we note that if the curve is planar then $\tau = 0$ and the proof is done. So let us assume that τ is not constantly 0.

Next, we shall prove that $\tau = \frac{J'}{1+J^2}$, where $J = \det[c, c', c'']$, which would lead us to (by the fundamental theorem of calculus)

$$\begin{aligned} \int_0^L \tau ds &= \int_0^L \frac{J'}{1+J^2} ds = \tan^{-1}(J(L)) - \tan^{-1}(J(0)) \\ &= 0 \text{ because the three vectors must be the same at the endpoints (closed curve)} \end{aligned}$$

Now to prove our claim:

Note that since the curve is on the unit sphere, the vectors $c, c', c \times c'$ form an orthonormal frame along the curve. Hence, $c'' = \langle c'', c \rangle c + \langle c'', c' \rangle c' + \langle c'', c \times c' \rangle c \times c'$. But since $\langle c, c' \rangle = 0$ (again, because it's on a sphere), we have that $0 = \langle c, c' \rangle' = \langle c', c' \rangle + \langle c, c'' \rangle \Rightarrow \langle c, c'' \rangle = -\langle c', c' \rangle = -1$. Furthermore, $J = \det[c, c', c''] = \langle c'', c \times c' \rangle$. This last part can be seen because the determinant is the unique alternating multi-linear map that sends e_1, e_2, e_3 (orthonormal basis) to 1. So that $c'' = -c + Jc \times c'$ and $\kappa^2 = \|c''\|^2 = 1 + J^2$. Kühnel uses this to show that $\kappa^2 > 0$, but we can already see that $\kappa > 1$ by considering the oscillating circle at that point (which has radius $1/\kappa$) that must have radius ≤ 1 if it's to be on S^1 . Which means, $\kappa \geq 1$. Either way, we have $\kappa > 0$.

Furthermore, $T = c'$, $N = \frac{1}{\kappa}c''$, and $B = T \times N$. But $\langle T', c \rangle = \langle c'', c \rangle = -1 \Rightarrow 0 = \langle T', c \rangle' = \langle T'', c \rangle + \langle T', c' \rangle = \langle T'', c \rangle + 0 \Rightarrow \langle T'', c \rangle = 0$

$$\begin{aligned} \tau &= \langle B, N' \rangle = -\langle B', N \rangle \\ &= -\frac{1}{\kappa} \langle B', T' \rangle \\ &= -\frac{1}{\kappa} \langle (T \times N)', T' \rangle \\ &= -\frac{1}{\kappa} \langle \left(\frac{1}{\kappa} T \times T' \right)', T' \rangle \\ &= -\frac{1}{\kappa} \left\langle \frac{1}{\kappa} T' \times T' + \frac{1}{\kappa} T \times T'' + \frac{-\kappa'}{\kappa^2} T \times T', T' \right\rangle \\ &= -\frac{1}{\kappa} \left\langle \frac{1}{\kappa} T \times T'' + \frac{-\kappa'}{\kappa} T \times T', T' \right\rangle \\ &= -\frac{1}{\kappa} \left(\left\langle \frac{1}{\kappa} T \times T'', T' \right\rangle + \left\langle \frac{-\kappa'}{\kappa^2} T \times T', T' \right\rangle \right) \\ &= -\frac{1}{\kappa^2} \langle T \times T'', T' \rangle \\ &= -\frac{1}{\kappa^2} \langle T \times T'', -c + Jc \times T \rangle \\ &= -\frac{1}{\kappa^2} \langle T \times T'', -c - JT \times c \rangle \\ &= \frac{1}{\kappa^2} (\langle T \times T'', c \rangle + \langle T \times T'', JT \times c \rangle) \\ &= \frac{1}{\kappa^2} \langle T \times T'', c \rangle \text{ because } \langle T'', c \rangle = 0 \text{ } T'' \text{ is in the } T, T \times c \text{ plane which makes the above 0} \\ &= \frac{1}{\kappa^2} \det[c, T, T''] = \frac{-1}{\kappa^2} \det[T'', T, c] = \frac{1}{\kappa^2} \det[T'', c, T] = \frac{1}{\kappa^2} \langle c''', c \times c' \rangle = \frac{J'}{1+J^2} \end{aligned}$$

Since $J' = \langle c'', c \times c' \rangle' = \langle c''', c \times c' \rangle + \langle c'', c' \times c' \rangle + \langle c'', c \times c'' \rangle = \langle c''', c \times c' \rangle$

b. $\int_0^L \frac{\tau}{\kappa} ds = 0$. Firstly we note that if τ is constantly zero then the integral is obviously 0. Next we will prove that $\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$, so by the fundamental theorem of calculus,

$$\int_0^L \frac{\tau}{\kappa} ds = \frac{\kappa'(L)}{\tau(L)\kappa^2(L)} - \frac{\kappa'(0)}{\tau(0)\kappa^2(0)} = 0 \text{ by periodicity.}$$

If we consider the oscillating sphere (which must be the same sphere), we find that the center of the sphere is given by $m = c + \frac{1}{\kappa}N - \frac{\kappa'}{\tau\kappa^2}B$ So that

$$\begin{aligned} 0 = m' &= \left(c + \frac{1}{\kappa}N - \frac{\kappa'}{\tau\kappa^2}B\right)' \\ &= T + \frac{-\kappa'}{\kappa^2}N + \frac{1}{\kappa}(-\kappa T + \tau B) - \left(\frac{\kappa'}{\tau\kappa^2}\right)'B + \frac{\kappa'}{\kappa^2}N \\ &= \left(\frac{\tau}{\kappa} - \left(\frac{\kappa'}{\tau\kappa^2}\right)'\right)B \Rightarrow \frac{\tau}{\kappa} - \left(\frac{\kappa'}{\tau\kappa^2}\right)' = 0. \end{aligned}$$

Note: this works even if τ is zero on intervals or discrete points. Since we're assuming $\frac{\tau}{\kappa}$ is integrable (otherwise the question is nonsense), we know the set of discontinuities of $\frac{\tau}{\kappa}$ must have measure zero and therefore the zeros of τ cannot be dense unless τ is constantly zero. For simplicity of the proof, let us assume that τ is only zero on the interval (a, b) for some $0 \leq a < b \leq L$. Then,

$$\begin{aligned} \int_0^L \frac{\tau}{\kappa} ds &= \int_0^a \frac{\tau}{\kappa} ds + \int_a^b \frac{\tau}{\kappa} ds + \int_b^L \frac{\tau}{\kappa} ds \\ &= \lim_{x \rightarrow a} \int_0^x \frac{\tau}{\kappa} ds + \int_a^b 0 ds + \lim_{x \rightarrow b} \int_x^L \frac{\tau}{\kappa} ds \\ &= \lim_{x \rightarrow a} \frac{\kappa'(x)}{\tau(x)\kappa^2(x)} - \frac{\kappa'(0)}{\tau(0)\kappa^2(0)} + \frac{\kappa'(L)}{\tau(L)\kappa^2(L)} - \lim_{x \rightarrow b} \frac{\kappa'(x)}{\tau(x)\kappa^2(x)} \\ &= \lim_{x \rightarrow a} \frac{\kappa'(x)}{\tau(x)\kappa^2(x)} - \lim_{x \rightarrow b} \frac{\kappa'(x)}{\tau(x)\kappa^2(x)} \end{aligned}$$

But $\lim_{x \rightarrow a} \frac{\kappa'(x)}{\tau(x)\kappa^2(x)} - \lim_{x \rightarrow b} \frac{\kappa'(x)}{\tau(x)\kappa^2(x)} = 0$ because $\tau = 0$ for all $t \in (a, b)$, and this curve is on a sphere. That means the curve is an arc of a circle on the sphere on the interval (a, b) . But then $\lim_{x \rightarrow a} \kappa = \kappa_0 = \lim_{x \rightarrow b} \kappa$, and $\lim_{x \rightarrow a} \kappa' = \lim_{x \rightarrow b} \kappa'$ and also, $\lim_{x \rightarrow a} \tau = \lim_{x \rightarrow b} \tau$.

This argument immediately generalizes to finitely many intervals and discrete points.

3. PROVE THAT FOR ANY REAL NUMBER c THERE EXISTS A CLOSED \mathcal{C}^3 CURVE r OF LENGTH L , SUCH THAT $\int_0^L \tau ds = c$.

Let $L, c \in \mathbb{R}_+$ ($\neq 0$ because that case was done in question 2 and for negatives, just consider the reverse of our construction). We will define a curve, r (parameterized by arc-length), in 4 parts such that r has length L with total torsion c . Part 1 will be $[0, \alpha]$ for $\alpha = L/10$, part two will be a curve on a sphere with $s \in [\alpha, \beta]$, for some $\beta \leq 3L/10$, then part three will be a plane curve for $s \in [\beta, \gamma]$ for some $\gamma < L$, and part four will be another curve on a sphere with $s \in [\gamma, L]$.

Part 1 ($s \in [0, \alpha]$): $r(t) = (a \cos(t), a \sin(t), bt)$, for some $a \in \mathbb{R}$ (to be figured out later and assumed positive for our purposes).

Then, $\dot{r} = (-a \sin(t), a \cos(t), b) \rightarrow \|\dot{r}\| = \sqrt{a^2 + b^2}$, so that $s(t) = \int_0^t \sqrt{a^2 + b^2} dx = \sqrt{a^2 + b^2}t$, and

$$r = r(s) = \left(a \cos \left(\frac{s}{\sqrt{a^2 + b^2}} \right), a \sin \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{bs}{\sqrt{a^2 + b^2}} \right).$$

$$\text{So, } T = \left(\frac{-a}{\sqrt{a^2 + b^2}} \sin \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{a}{\sqrt{a^2 + b^2}} \cos \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{b}{\sqrt{a^2 + b^2}} \right),$$

$$\text{and } r'' = T' = \left(\frac{-a}{a^2 + b^2} \cos \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{-a}{a^2 + b^2} \sin \left(\frac{s}{\sqrt{a^2 + b^2}} \right), 0 \right) \Rightarrow \kappa = \frac{|a|}{a^2 + b^2} = \frac{a}{a^2 + b^2}.$$

$$N = \left(-\cos \left(\frac{s}{\sqrt{a^2 + b^2}} \right), -\sin \left(\frac{s}{\sqrt{a^2 + b^2}} \right), 0 \right).$$

$$\text{Next, } B = T \times N = \left(\frac{b}{\sqrt{a^2 + b^2}} \sin \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{-b}{\sqrt{a^2 + b^2}} \cos \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{a}{\sqrt{a^2 + b^2}} \right).$$

$$\tau B = N' + \kappa T$$

$$= \left(\frac{1}{\sqrt{a^2 + b^2}} \sin \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{-1}{\sqrt{a^2 + b^2}} \cos \left(\frac{s}{\sqrt{a^2 + b^2}} \right), 0 \right) + \kappa T$$

$$= \left(\frac{a^2 + b^2 - a^2}{\sqrt{a^2 + b^2}^3} \sin \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{-a^2 - b^2 + a^2}{\sqrt{a^2 + b^2}^3} \cos \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{ab}{\sqrt{a^2 + b^2}^3} \right)$$

$$= \frac{b}{a^2 + b^2} \left(\frac{b}{\sqrt{a^2 + b^2}} \sin \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{-b}{\sqrt{a^2 + b^2}} \cos \left(\frac{s}{\sqrt{a^2 + b^2}} \right), \frac{a}{\sqrt{a^2 + b^2}} \right) \Rightarrow$$

$$\tau = \frac{b}{a^2 + b^2}$$

Then, $\int_0^\alpha \tau ds = \frac{b\alpha}{a^2 + b^2}$. But we want $\alpha = 2n\pi\sqrt{a^2 + b^2}$, so that things will play out nicely in parts 2,3, and 4. So $\int_0^\alpha \tau ds = \frac{2n\pi b}{\sqrt{a^2 + b^2}}$. By letting $a = 0$, $n \rightarrow \infty$ (hence $b \rightarrow 0$) we see $\int_0^\alpha \tau ds \rightarrow \infty$. On the other hand, by setting $b = 0$ (hence $2n\pi a = \alpha = L/10$) we see that $\int_0^\alpha \tau ds = 0$. So, for any positive $c \in \mathbb{R}$, we can find a, b, n such that $\int_0^\alpha \tau ds = c$. Next we will connect the two endpoints of this helix in a way such that the total torsion is 0.

Part 2 ($[\alpha, \beta]$ on the sphere): In particular, this portion will be on the oscillating sphere at the point $s = \alpha$. This will be any regular curve of length $\leq L/5$ on said sphere that ends on the great circle parallel to the plane defined by $T(\alpha)$ and the line segment connecting $r(0)$ with $r(\alpha)$ (the two ends of the helix). Let us call this endpoint of part 2 of this curve, d . The idea is that part 3 will be a plane curve connecting to the same point on the oscillating sphere at the bottom of the helix, thereby making that into a pseudo-closed curve on a sphere and by question 2, the total torsion 0.

Part 3 ($[\beta, \gamma]$) is just any plane curve that smoothly connects the point d from above to the corresponding point d' on the oscillating sphere at $r(0)$ in such a way that it is a continuation of part 2 on the sphere.

Part 4 ($[\gamma, L]$): Essentially a continuation of part two but on the oscillating sphere at $r(0)$. Note: $\int_\alpha^\beta \tau ds + \int_\gamma^L \tau ds = 0$ because it is the total torsion of a closed curve on a sphere.

$$\text{Then, } \int_0^L \tau ds = \int_0^\alpha \tau ds + \int_\alpha^\beta \tau ds + \int_\beta^\gamma \tau ds + \int_\gamma^L \tau ds = c + \left(\int_\alpha^\beta \tau ds + \int_\gamma^L \tau ds \right) = c.$$

4. PROVIDE A DEFINITION OF CONVEX CURVE IN A PLANE AND A PROOF OF THE FOUR VERTEX THEOREM (THEOREM 2.33).

“A simply closed plane curve is called *convex*, if the image set of the boundary is a convex subset $C \subseteq \mathbb{R}^2$. The convexity of a subset C is defined in the usual way, namely, for any two points contained in C , also the segment joining these two points is completely contained in C .”

In other words, a plane curve c is called *convex* if it is the boundary of a convex set in \mathbb{R}^2 .

Next up... The Four Vertex Theorem!

Claim: A simply closed, regular and convex plane curve which is of class \mathcal{C}^3 has at least four local extremal points for its curvature κ (such a point is referred to as a *vertex*).

Proof. Let c be a simply closed, regular and convex plane curve parameterized by arc length and from the interval $[0, L]$ to the x, y plane. Firstly, if κ is constant on any interval, then every point is a vertex. So from now on, let us assume that κ is not constant anywhere. Furthermore, since κ is a local extremal point, $\kappa' = 0$ and κ changes sign.

Note: Since $[0, L]$ is compact and κ is continuous, by the extreme value theorem, κ has an absolute minimum and maximum value in $[0, L]$. So two vertexes are given already.

Since the curve is a closed curve, we can assume that $\kappa(0)$ is an absolute minimum and $\kappa(s_0)$ is an absolute maximum. Next, we can choose the coordinate system to be such that $c(0)$ and $c(s_0)$ are both on the x axis. Since c is convex, if $c(s_1)$ is also on the x axis, then every point between 0 and s_1 , and s_0 and s_1 , should be on that line too. But that contradicts the curve being regular (second derivative cannot be 0).

Now, assume toward a contradiction that these are the only two points where κ' changes sign. Then, if we write $c(s)$ as $(x(s), y(s))$, then the function $\kappa'y$ never changes sign (because of our choice of axis). By the Frenet equations we have,

$$T = (x', y') \quad T' = (x'', y'') = \kappa N \quad N = (-y', x') \text{ because } T \cdot N = 0$$

By matching coordinates, we find, $x'' = -y'\kappa$. Furthermore,

$$\begin{aligned} \int_0^L \kappa'y ds &= \kappa y|_0^L - \int_0^L \kappa y' ds \\ &= \int_0^L x'' ds = x'(L) - x'(0) = 0 \end{aligned}$$

But since $\kappa'y$ never changes sign, we know $\kappa'y$ must be constant. Since y cannot be constant (in particular it cannot be constantly 0) on any interval (because the curve is regular), we find that κ' is constantly zero. But that means κ is constant, which is a contradiction to our criteria that κ not be constant on any interval.

Hence, there must be at least three vertexes. Since κ' changes sign at each vertex and the curve is L periodic, we know there must be an even number (if finite) of vertexes. Hence, there must be at least 4.

Hence, the 4 vertex theorem. □

5. SUPPOSE THAT A FRENET CURVE IS AN INTERSECTION OF TWO REGULAR (PARAMETERIZED) SURFACE ELEMENTS. SHOW THAT IF IT IS A LINE OF CURVATURE FOR BOTH SURFACES, THEN THE SURFACES INTERSECT AT A CONSTANT ANGLE.