DIFFERENTIAL GEOMETRY HOMEWORK 1

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a. **Suppose that a particle in 3-space is moving under a central force F. That is equivalent to the following condition: there is a fixed point A such that the acceleration vector points in direction of A at all time. Prove that its trajectory lies in a fixed plane.** Let $r(s)$ be a Frenet curve in $\mathbb{R}^3$ parameterized by arc length (s is the arc length parameter) such that $r''(s) = \alpha(s)\,(A - r(s))$ for some $\mathcal{C}^2$ function, α that is non-zero on I. We can define $c(s) := A - r(s)$ so that $T(s), N(s), B(s)$ are the same for both curves. Furthermore, $c''(s) = -r''(s) = -\alpha(s)c(s)$.
Claim: $B(s)$ is constant, and hence $c(s)$ must lie in the plane normal to $B(s_0)$.

Proof.

$$T(s) = c'(s) \tag{1}$$

$$N(s) = \frac{c''(s)}{\kappa(s)} = \frac{-\alpha(s)c(s)}{\kappa(s)} \tag{2}$$

$$B(s) = T(s) \times N(s) \tag{3}$$

Then,

$$T'(s) = c''(s) = -\alpha(s)c(s) \tag{4}$$

$$-\kappa(s)T(s) + \tau(s)B(s) = N'(s) = \left(\frac{-\alpha(s)c(s)}{\kappa(s)}\right)' \tag{5}$$

$$B'(s) = -\tau(s)N(s) \tag{6}$$

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Note:

$$\begin{aligned}\left(\frac{-\alpha(s)c(s)}{\kappa(s)}\right)' &= \frac{-\alpha'(s)\kappa(s)c(s)+\alpha(s)\kappa'(s)c(s)-\alpha(s)\kappa(s)c'(s)}{\kappa(s)^2}\\ &= -\left(\frac{\alpha(s)}{\kappa(s)}\right)' c(s) - \frac{\alpha(s)c'(s)}{\kappa(s)}\\ &= \left(\frac{\alpha(s)}{\kappa(s)}\right)' \frac{\kappa(s)}{\alpha(s)}N(s) - \frac{\alpha(s)}{\kappa(s)}T(s)\end{aligned}$$

Since T, N, B are the Frenet basis (hence ortho-normal) by equation (5) we know $-\kappa(s) = -\frac{\alpha(s)}{\kappa(s)} \to \kappa^2(s) = \alpha(s)$, and $\tau(s) = 0$ for all s which, given equation (6), concludes the proof. But we also see some other interesting results. Such as $\left(\frac{\alpha(s)}{\kappa(s)}\right)' \frac{\kappa(s)}{\alpha(s)} = 0$. But since $\frac{\kappa(s)}{\alpha(s)} = \frac{1}{\kappa(s)} \neq 0$, $\left(\frac{\alpha(s)}{\kappa(s)}\right)' = \kappa'(s) = 0 \to \kappa(s) = \beta$, for some $\beta \in \mathbb{R}$. But why should there being a central force mean constant curvature? $\square$

b. **Fix the plane to be the xy-plane and write the equation for the trajectory of $\boldsymbol{A}$ in polar coordinates.** Fix the x, y plane to be the one that the curve is in (normal to B). Let us further let x be in the direction of $T(s_0)$, and y in the direction of $N(s_0)$. $c(s) = (||c(s)||,)$

c. **If the force F is given by F=$\frac{c(r)}{||r||^3}$, show that the trajectory is part of an ellipse, hyperbola or parabola (second order or quadratic curve).**

2. A Frenet curve in $\mathbb{R}^3$ is called a *Bertrand curve*, if there is a second curve such that the principle normal vectors to these two curves (at corresponding points) are identical, viewed as lines in space. One speaks in the case of a *Bertrand pair of curves*. Show that non-planar Bertrand curves are characterized by the existence of a linear relation $a\kappa + b\tau = 1$ with constants a, b, where $a \neq 0$.

Let c_1, c_2 be a pair of Bertrand curves in $\mathbb{R}^3$ and let N_1, N_2 be their corresponding Principal Normal vectors at t. Then $\forall t \in I$, $N_1 = N_2$. Hence, $\forall t \in I$, $\dot{N}_1 = \dot{N}_2$ (so we can drop the subscripts on N and $\dot{N}$). Now, $\frac{\ddot{c}_1}{||\ddot{c}_1||} = N = \frac{\ddot{c}_2}{||\ddot{c}_2||}$

3. Suppose $r = r(t)$ is a regular curve satisfying $r'' = r' \times H$ for a constant vector H (according to one source this is the equation of an electron moving under a magnetic field force). Prove that τ and κ are constants.

Note: $r''(s) =$

4. Let A_x and A_y be the operators corresponding to the cross product with the vectors x and y respectively. Show that $A_x A_y - A_y A_x = A_{x \times y}$

This is just a computation. There's definitely a theory way to go here (involving commutators, or lie groups or whatever), but I can't see that way right now, so I'll just do this computationally.

$$\begin{aligned}(A_X A_Y - A_Y A_X)(Z) =& X \times (Y \times Z) - Y \times (X \times Z)\\ =& X \times (Y \times Z) + Y \times (Z \times X)\\ =& - Z \times (X \times Y)\\ =& (X \times Y) \times Z = A_{X \times Y}(Z)\end{aligned}$$