DIFFERENTIAL GEOMETRY HOMEWORK 1

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a. **Suppose that a particle in 3-space is moving under a central force F. That is equivalent to the following condition: there is a fixed point *A* such that the acceleration vector points in direction of *A* at all time. Prove that its trajectory lies in a fixed plane.** Let r be a Frenet curve in $\mathbb{R}^3$ such that $\ddot{r} = \alpha\,(A - r)$ for some $\mathcal{C}^2$ function, α that is non-zero on I and some fixed point A. We can define $c := A - r$ so that T, N, B are the same for both curves (up to negation). Furthermore, $\ddot{c} = -\ddot{r} = -\alpha c$.
Claim: B is constant, and hence c must lie in the plane normal to B.

Proof. Firstly,

$$-\alpha c = \ddot{c} = \frac{d^2 c}{dt^2} = \frac{d}{dt}\left(\frac{dc}{ds}\frac{ds}{dt}\right) = \frac{d}{dt}\left(c'\dot{s}\right) = c''\dot{s}^2 + c'\ddot{s} \tag{1}$$

$$N = \frac{c''}{\kappa} = -\frac{\alpha c + \ddot{s}c'}{\dot{s}^2\kappa} \tag{2}$$

$$-\kappa T + \tau B = N' = \left(-\frac{\alpha c + \ddot{s}c'}{\dot{s}^2\kappa}\right)' \tag{3}$$

$$B' = -\tau N \tag{4}$$

By taking equation (3), simplifying and replacing every occurrence of c with $-\frac{\dot{s}^2 c'' + \ddot{s}c'}{\alpha} = -\frac{\dot{s}^2\kappa N + \ddot{s}T}{\alpha}$ from (1), we find,

$$\begin{aligned}
-\kappa T + \tau B = &\left(-\frac{\alpha c + \ddot{s}c'}{\dot{s}^2\kappa}\right)' \\
=&\frac{-(\alpha' c + c'\alpha + \ddot{s}c'')\dot{s}^2\kappa + \dot{s}^2\kappa'(\alpha c + \ddot{s}c')}{\dot{s}^4\kappa^2} \\
=&\frac{-\alpha'(\dot{s}^2\kappa N + \ddot{s}T)\kappa/\alpha - \kappa\alpha T - \ddot{s}\kappa^2 N - \kappa'\dot{s}^2\kappa N}{\dot{s}^2\kappa^2} \\
=&\left(\frac{-\alpha'\ddot{s} - \alpha^2}{\dot{s}^2\kappa\alpha}\right) T - \left(\frac{\alpha'\dot{s}^2\kappa + \ddot{s}\kappa\alpha + \kappa'\alpha\dot{s}^2}{\dot{s}^2\kappa\alpha}\right) N
\end{aligned}$$

is a linear combination of T, N. Since T, N, B are orthonormal, we see that $\tau = 0$. ☐

Furthermore, $\kappa^2\dot{s}^2\alpha = \alpha'\ddot{s} + \alpha^2$, and $\alpha'\dot{s}^2\kappa + \ddot{s}\kappa\alpha + \kappa'\alpha\dot{s}^2 = 0$

$$\begin{aligned}
0 =&\alpha'\dot{s}^2\ddot{s}\kappa + \ddot{s}^2\kappa\alpha + \ddot{s}\kappa'\alpha\dot{s}^2 \\
=&(\kappa^2\dot{s}^2\alpha - \alpha^2)\dot{s}^2\kappa + \ddot{s}^2\kappa\alpha + \ddot{s}\kappa'\alpha\dot{s}^2 \\
=&(\kappa^2\dot{s}^2 - \alpha)\dot{s}^2\kappa + \ddot{s}^2\kappa + \ddot{s}\kappa'\dot{s}^2
\end{aligned}$$

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b. **Fix the plane to be the xy-plane and write the equation for the trajectory of $\boldsymbol{A}$ in polar coordinates.** Let x be in the direction of $T(s_0)$, y in the direction of $N(s_0)$ and z in the direction of B. Then $r(s) = (||c(s)||, \theta, 0) + A$, where θ is the angle given by $\theta = \int_0^t \kappa(t)dt$.

c. **If the force F is given by F=$\frac{Cr}{||r||^3}$, show that the trajectory is part of an ellipse, hyperbola or parabola (second order or quadratic curve).** If we assume that $\ddot{r} = F$, then $\ddot{r} = F = \frac{C}{||r||^2}\frac{r}{||r||}$, then $A = \vec{0}$, and we have $\ddot{r} = -\alpha r$, hence, $-\alpha r = \frac{C}{||r||^2}\frac{r}{||r||}$. But then, $\left|\frac{C}{\alpha}\right| = ||r||^3$. Hence, either $\alpha < 0$, $C < 0$, or $\alpha = C = 0$. Furthermore, $\dot{r}(t) = \dot{r}(t) - \dot{r}(t_0) = \int_{t_0}^t \ddot{r}(t)dt = \int_{t_0}^t \frac{C}{||r||^2}\frac{r}{||r||}dt =$
Case 1: $\alpha = C = 0$. Then, $\ddot{C} = 0 \to c'' = 0 \to C$ is a straight line.
Case 2: $\alpha < 0$, then $C > 0$ and we have a hyperbola...
Case 3: $\alpha > 0$, then $C < 0$ and we have either a parabola or an ellipse...

2. A Frenet curve in $\mathbb{R}^3$ is called a *Bertrand curve*, if there is a second curve such that the principle normal vectors to these two curves (at corresponding points) are identical, viewed as lines in space. One speaks in the case of a *Bertrand pair of curves*. Show that non-planar Bertrand curves are characterized by the existence of a linear relation $a\kappa + b\tau = 1$ with constants a, b, where $a \neq 0$.

Let c_1, c_2 be a pair of Bertrand curves in $\mathbb{R}^3$ and let N_1, N_2 be their corresponding Principal Normal vectors at t. Then $\forall t \in I$, $N_1 = N_2$. Hence, $\forall t \in I$, $\dot{N}_1 = \dot{N}_2$ (so we can drop the subscripts on N and $\dot{N}$). Now, $\frac{\ddot{c}_1}{\kappa_1} = N = \frac{\ddot{c}_2}{\kappa_2}$. Since the N vectors are identical at every point, then $\frac{dN}{dt}$ must have the same property. Hence, if we let s_1, s_2 be the arc-length parameters then $(-\kappa_1 T_1 + \tau_1 B_1)\frac{ds_1}{dt} = (-\kappa_2 T_2 + \tau_2 B_2)\frac{ds_2}{dt}$.

3. Suppose $r = r(t)$ is a regular curve satisfying $r'' = r' \times H$ for a constant vector H (according to one source this is the equation of an electron moving under a magnetic field force). Prove that τ and κ are constants.

Proof. Let s be the arc length parameter for r, let $C = ||H||$ and $V = \frac{H}{||H||}$. Then $r' = \dot{r}\frac{dt}{ds}$. Since $1 = ||r'|| = ||\dot{r}|| \cdot ||\frac{dt}{ds}|| \to \frac{1}{||dt/ds||} = ||\frac{ds}{dt}|| = ||\dot{s}|| = ||\dot{r}||$. Hence

$$\begin{aligned}
(\dot{s})^2 =& \dot{r} \cdot \dot{r} \to \\
2\dot{s}\ddot{s} =& 2\ddot{r} \cdot \dot{r} \\
=& (\dot{r} \times H) \cdot \dot{r} = 0
\end{aligned}$$

Hence, either $\dot{s} = 0$ or, $\ddot{s} = 0$. Either way, $\ddot{s} = 0$, and $s = at + b$ with $a, b \in \mathbb{R}$. Next we have,

$$\begin{aligned}
T =& r'(s) \\
=& \dot{r}\frac{dt}{ds} \\
=& \frac{1}{a}\dot{r} \to ||\dot{r}|| = a
\end{aligned}$$

Furthermore,

$$\begin{aligned}
T' =& \frac{d^2 r}{ds^2} \\
=& \frac{d}{dt}\left(\frac{1}{a}\dot{r}\right)\frac{dt}{ds} \\
=& \frac{1}{a^2}\ddot{r} \\
=& \frac{1}{a^2}\dot{r} \times H \\
=& \frac{1}{a}r' \times H
\end{aligned}$$

and $T' = \kappa N$, so

$$\begin{aligned}
\kappa' N + \kappa N' = T'' =& \frac{1}{a}r'' \times H \\
=& \frac{1}{a^2}(r' \times H) \times H
\end{aligned}$$

so that

$$\begin{aligned}\kappa' =&\langle \kappa' N + \kappa N', N\rangle \\ =&\langle \frac{1}{a^2}(r' \times H) \times H, N\rangle \\ =&\frac{1}{a}\langle T' \times H, N\rangle \\ =&\frac{1}{a}\langle \kappa N \times H, N\rangle = 0\end{aligned}$$

Hence, κ is constant.
Next, let us first note: Since $T' = \frac{1}{a} T \times H$, then $N = \frac{T\times H}{||T\times H||} = CT \times H$, where $C = \frac{1}{||T\times H||}$, and, $B = T \times N$.

$$\begin{aligned}\tau =& -\langle B, N'\rangle \\ =& -\langle T \times N, (CT \times H)'\rangle \\ =& -\langle T \times N, C'T \times H + CT' \times H\rangle \\ =& -\langle T \times N, CT' \times H\rangle \text{ because } T \times H \text{ and } N \text{ are lin dep} \\ =& -\langle T \times N, C\frac{1}{a}(T \times H) \times H\rangle \\ =&\frac{-1}{a}\langle B, N \times H\rangle\end{aligned}$$

Hence,

$$\begin{aligned}-a\tau' =&\langle B, N \times H\rangle' \\ =&\langle B', N\rangle + \langle B, N'\rangle \\ =&\langle -\tau N, N\rangle + \langle B, -\kappa T + \tau B\rangle \\ =& -\tau + \tau = 0\end{aligned}$$

So, τ is constant too. $\square$

4. Let A_X and A_Y be the operators corresponding to the cross product with the vectors X and Y respectively. Show that $A_X A_Y - A_Y A_X = A_{X\times Y}$

This is just a computation. There's definitely a theory way to go here (involving commutators, or lie groups or whatever), but I can't see that way right now, so I'll just do this computationally.

$$\begin{aligned}(A_X A_Y - A_Y A_X)(Z) =&X \times (Y \times Z) - Y \times (X \times Z) \\ =&X \times (Y \times Z) + Y \times (Z \times X) \\ =& -Z \times (X \times Y) \\ =&(X \times Y) \times Z = A_{X\times Y}(Z)\end{aligned}$$