

DIFFERENTIAL GEOMETRY HOMEWORK 1

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a. Suppose that a particle in 3-space is moving under a central force \mathbf{F} . That is equivalent to the following condition: there is a fixed point A such that the acceleration vector points in direction of A at all time. Prove that its trajectory lies in a fixed plane. Let r be a Frenet curve in \mathbb{R}^3 such that $\ddot{r} = \alpha(A - r)$ for some \mathcal{C}^2 function, α that is non-zero on I and some fixed point A . We can define $c := A - r$ so that T, N, B are the same for both curves (up to negation). Furthermore, $\ddot{c} = -\ddot{r} = -\alpha c$.

Claim: B is constant, and hence c must lie in the plane normal to B .

Proof. Firstly,

$$(1) \quad -\alpha c = \ddot{c} = \frac{d^2 c}{dt^2} = \frac{d}{dt} \left(\frac{dc}{ds} \frac{ds}{dt} \right) = \frac{d}{dt} (c' \dot{s}) = c'' \dot{s}^2 + c' \ddot{s}$$

$$(2) \quad T = c'$$

$$(3) \quad N = \frac{c''}{\kappa} = -\frac{\alpha c + \ddot{s} c'}{\dot{s}^2 \kappa}$$

$$(4) \quad B = T \times N$$

Then,

$$(5) \quad T' = c'' = \kappa N$$

$$(6) \quad -\kappa T + \tau B = N' = \left(-\frac{\alpha c + \ddot{s} c'}{\dot{s}^2 \kappa} \right)'$$

$$(7) \quad B' = -\tau N$$

Note:

$$(8) \quad -\kappa T + \tau B = \left(-\frac{\alpha c + \ddot{s} c'}{\dot{s}^2 \kappa} \right)' = \frac{-(\alpha' c + c' \alpha + \ddot{s} c'') \dot{s}^2 \kappa + \dot{s}^2 \kappa' (\alpha c + \ddot{s} c')}{\dot{s}^4 \kappa^2}$$

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By taking the right hand side of equation (8), simplifying and replacing every occurrence of c with $-\frac{\dot{s}^2 c'' + \ddot{s} c'}{\alpha} = -\frac{\dot{s}^2 \kappa N + \ddot{s} T}{\alpha}$ from (1), we find,

$$\begin{aligned} -\kappa T + \tau B &= \frac{-(\alpha' c + c' \alpha + \ddot{s} c'') \kappa + \dot{\kappa}' (\alpha c + \ddot{s} c')}{\dot{s}^2 \kappa^2} \\ &= \frac{-\alpha' (\dot{s}^2 \kappa N + \ddot{s} T) \kappa / \alpha - \kappa \alpha T - \ddot{s} \kappa^2 N - \kappa' \dot{s}^2 \kappa N}{\dot{s}^2 \kappa^2} \\ &= \left(\frac{-\alpha' \ddot{s} - \alpha^2}{\dot{s}^2 \kappa \alpha} \right) T - \left(\frac{\alpha' \dot{s}^2 \kappa + \ddot{s} \kappa \alpha + \kappa' \alpha \dot{s}^2}{\dot{s}^2 \kappa \alpha} \right) N \end{aligned}$$

is a linear combination of T, N . Since T, N, B are orthonormal, we see that $\tau = 0$. \square

Furthermore, $\kappa^2 \dot{s}^2 \alpha = \alpha' \ddot{s} + \alpha^2$, and $\alpha' \dot{s}^2 \kappa + \ddot{s} \kappa \alpha + \kappa' \alpha \dot{s}^2 = 0$

$$\begin{aligned} 0 &= \alpha' \dot{s}^2 \ddot{s} \kappa + \ddot{s}^2 \kappa \alpha + \ddot{s} \kappa' \alpha \dot{s}^2 \\ &= (\kappa^2 \dot{s}^2 \alpha - \alpha^2) \dot{s}^2 \kappa + \ddot{s}^2 \kappa \alpha + \ddot{s} \kappa' \alpha \dot{s}^2 \\ &= (\kappa^2 \dot{s}^2 - \alpha) \dot{s}^2 \kappa + \ddot{s}^2 \kappa + \ddot{s} \kappa' \dot{s}^2 \end{aligned}$$

b. Fix the plane to be the xy -plane and write the equation for the trajectory of A in polar coordinates. Let x be in the direction of $T(s_0)$, y in the direction of $N(s_0)$ and z in the direction of B . Then $r(s) = (||c(s)||, \theta, 0) + A$, where θ is the angle.

c. If the force F is given by $F = \frac{cr}{||r||^3}$, show that the trajectory is part of an ellipse, hyperbola or parabola (second order or quadratic curve). If we assume that $\ddot{c} = F$, then $\ddot{c}(t) = F = \frac{c}{||r||^2} \frac{r}{||r||}$, then $A = \vec{0}$

2. A FRENET CURVE IN \mathbb{R}^3 IS CALLED A *Bertrand curve*, IF THERE IS A SECOND CURVE SUCH THAT THE PRINCIPLE NORMAL VECTORS TO THESE TWO CURVES (AT CORRESPONDING POINTS) ARE IDENTICAL, VIEWED AS LINES IN SPACE. ONE SPEAKS IN THE CASE OF A *Bertrand pair of curves*. SHOW THAT NON-PLANAR BERTRAND CURVES ARE CHARACTERIZED BY THE EXISTENCE OF A LINEAR RELATION $a\kappa + b\tau = 1$ WITH CONSTANTS a, b , WHERE $a \neq 0$.

Let c_1, c_2 be a pair of Bertrand curves in \mathbb{R}^3 and let N_1, N_2 be their corresponding Principal Normal vectors at t . Then $\forall t \in I, N_1 = N_2$. Hence, $\forall t \in I, \dot{N}_1 = \dot{N}_2$ (so we can drop the subscripts on N and \dot{N}). Now, $\frac{\ddot{c}_1}{||\ddot{c}_1||} = N = \frac{\ddot{c}_2}{||\ddot{c}_2||}$

3. SUPPOSE $r = r(t)$ IS A REGULAR CURVE SATISFYING $r'' = r' \times H$ FOR A CONSTANT VECTOR H (ACCORDING TO ONE SOURCE THIS IS THE EQUATION OF AN ELECTRON MOVING UNDER A MAGNETIC FIELD FORCE). PROVE THAT τ AND κ ARE CONSTANTS.

Note: $r''(s) =$

4. LET A_x AND A_y BE THE OPERATORS CORRESPONDING TO THE CROSS PRODUCT WITH THE VECTORS x AND y RESPECTIVELY. SHOW THAT $A_x A_y - A_y A_x = A_{x \times y}$

This is just a computation. There's definitely a theory way to go here (involving commutators, or lie groups or whatever), but I can't see that way right now, so I'll just do this computationally.

$$\begin{aligned}
 (A_X A_Y - A_Y A_X)(Z) &= X \times (Y \times Z) - Y \times (X \times Z) \\
 &= X \times (Y \times Z) + Y \times (Z \times X) \\
 &= -Z \times (X \times Y) \\
 &= (X \times Y) \times Z = A_{X \times Y}(Z)
 \end{aligned}$$