DIFFERENTIAL GEOMETRY TEST (CONTINUED)

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(1) Curvature tensor: Given a connection ∇, the associated curvature tensor, is a (0,3) tensor defined by,

$$\mathrm{R}^{\nabla}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla[X,Y]Z$$

If T is a (0,p) tensor field, then $\nabla_X T$ is a (0,p) tensor defined by,

$$(\nabla_X T)(X_1, ..., X_p) = X(T(X_1, ..., X_p)) - T(\nabla_X X_1, X_2, ..., X_p) - ... - T(X_1, ..., X_{p-1}, \nabla_X X_p)$$

The sectional curvature is defined by

$$K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - (g(X,Y))^2}$$

(2) Prove the symmetries of the curvature tensor.
   (a) $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = -(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z) = -R(Y,X)Z$
   (b) For this, it is sufficient to prove that this property holds at every point. So pick a point $p \in M$, then every tangent vector can be expressed as $\sum \alpha^i \frac{\partial}{\partial x^i}$, where α^i are constants. Then, since $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$, we have

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(3) Schur's Theorem (6.5,6.6,6.7)

(4) The cross product makes $\mathbb{R}^3$ into a Lie Algebra. What Lie Group has a Lie Algebra isomorphic to it?