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Implementation details for cause-specific ATE

Anders Munch

We assume given the functions,

$$\begin{aligned}
&\pi(w)\\
&\Lambda_j(t \mid a, w), \quad j \in \{1,2\},\\
&\Gamma(t \mid a, w),
\end{aligned}$$

as defined in the main paper.

Define

$$\begin{aligned}
\omega_a(A, W; \pi) &= \frac{(-1)^{a+1}\mathbb{1}\{A = a\}}{\pi(W)^a(1-\pi(W))^{1-a}},\\
g(t, A, W; \Lambda_1, \Lambda_2) &= \int_0^t e^{-\Lambda_1(s- \mid W, A) - \Lambda_2(s- \mid W, A)}\Lambda_1(\mathrm{d}s \mid W, A),\\
M_j(\mathrm{d}t \mid A, W; \Lambda_j) &= N_j(\mathrm{d}t) - \mathbb{1}\{\tilde{T} \geq t\}\Lambda_j(\mathrm{d}t \mid W, A), \quad j \in \{1,2\},\\
M(\mathrm{d}t \mid A, W; \Lambda_1, \Lambda_2) &= M_1(\mathrm{d}t \mid A, W; \Lambda_1) + M_2(\mathrm{d}t \mid A, W; \Lambda_2).
\end{aligned}$$

The efficient influence function is

$$\begin{aligned}
&\psi_\tau(O; \Lambda_1, \Lambda_2, \Gamma, \pi)\\
&= \sum_{a=0}^{1} \omega_a(A, W; \pi) \int_0^\tau e^{\Gamma(t- \mid A, W)} M_1(\mathrm{d}t \mid A, W; \Lambda_1)\\
&\quad - \sum_{a=0}^{1} \omega_a(A, W; \pi) g(\tau, A, W; \Lambda_1, \Lambda_2) \int_0^\tau e^{[\Gamma + \Lambda_1 + \Lambda_2](t- \mid A, W)} M(\mathrm{d}t \mid A, W; \Lambda_1, \Lambda_2)\\
&\quad + \sum_{a=0}^{1} \omega_a(A, W; \pi) \int_0^\tau g(t, A, W; \Lambda_1, \Lambda_2) e^{[\Gamma + \Lambda_1 + \Lambda_2](t- \mid A, W)} M(\mathrm{d}t \mid A, W; \Lambda_1, \Lambda_2)\\
&\quad + g(\tau, 1, W; \Lambda_1, \Lambda_2) - g(\tau, 0, W; \Lambda_1, \Lambda_2)\\
&\quad - \tilde{\Psi}^0_t(\Lambda_1, \Lambda_2, \mu).
\end{aligned}$$

The one-step estimator is

$$\Psi_{\mathrm{OS}}(\Lambda_1, \Lambda_2, \Gamma, \pi, \mu) = \tilde{\Psi}^0_t(\Lambda_1, \Lambda_2, \mu) + \mathbb{P}_n[\psi_\tau(\cdot; \Lambda_1, \Lambda_2, \Gamma, \pi)]$$

Define

$$
\begin{aligned}
\texttt{termW}_i(\pi) &= \frac{(-1)^{A_i+1}}{\pi(W_i)^{A_i}(1-\pi(W_i))^{1-A_i}}, \\
\texttt{termA\_N}_i(\Gamma) &= \int_0^\tau e^{\Gamma(t-|A_i,W_i)} N_{1,i}(\mathrm{d}t) \\
&= \mathbb{1}\{\tilde{D}_i = 1\}\mathbb{1}\{\tilde{T}_i \le \tau\} e^{\Gamma(\tilde{T}_i - |A_i,W_i)} \\
\texttt{termA\_L}_i(\Gamma,\Lambda_1) &= \int_0^\tau e^{\Gamma(t-|A_i,W_i)} \mathbb{1}\{\tilde{T}_i \ge t\}\Lambda_1(\mathrm{d}t \mid A_i, W_i) \\
\texttt{termB\_N}_i(\Gamma,\Lambda_1,\Lambda_2) &= g(\tau, A_i, W_i; \Lambda_1, \Lambda_2) \int_0^\tau e^{[\Gamma+\Lambda_1+\Lambda_2](t-|A_i,W_i)} [N_{1,i} + N_{2,i}](\mathrm{d}t) \\
&= g(\tau, A_i, W_i; \Lambda_1, \Lambda_2)\mathbb{1}\{\tilde{D}_i \ne 0\}\mathbb{1}\{\tilde{T}_i \le \tau\} e^{[\Gamma+\Lambda_1+\Lambda_2](\tilde{T}_i - |A_i,W_i)}, \\
\texttt{termB\_L}_i(\Gamma,\Lambda_1,\Lambda_2) &= g(\tau, A_i, W_i; \Lambda_1, \Lambda_2) \int_0^\tau e^{[\Gamma+\Lambda_1+\Lambda_2](t-|A_i,W_i)} \mathbb{1}\{\tilde{T}_i \ge t\}[\Lambda_1+\Lambda_2](\mathrm{d}t \mid A_i, W_i), \\
\texttt{termC\_N}_i(\Gamma,\Lambda_1,\Lambda_2) &= \int_0^\tau g(t, A_i, W_i; \Lambda_1, \Lambda_2) e^{[\Gamma+\Lambda_1+\Lambda_2](t-|A_i,W_i)} [N_{1,i} + N_{2,i}](\mathrm{d}t) \\
&= g(\tilde{T}_i \wedge \tau, A_i, W_i; \Lambda_1, \Lambda_2)\mathbb{1}\{\tilde{D}_i \ne 0\}\mathbb{1}\{\tilde{T}_i \le \tau\} e^{[\Gamma+\Lambda_1+\Lambda_2](\tilde{T}_i - |A_i,W_i)}, \\
&= \int_0^\tau e^{-\Lambda_1(s-|W_i,A_i)-\Lambda_2(s-|W_i,A_i)} \mathbb{1}\{\tilde{T}_i \ge t\}\Lambda_1(\mathrm{d}s \mid W_i, A_i) \\
&\quad \times \mathbb{1}\{\tilde{D}_i \ne 0\}\mathbb{1}\{\tilde{T}_i \le \tau\} e^{[\Gamma+\Lambda_1+\Lambda_2](\tilde{T}_i - |A_i,W_i)}, \\
\texttt{termC\_L}_i(\Gamma,\Lambda_1,\Lambda_2) &= \int_0^\tau \int_0^t e^{-\Lambda_1(s-|W_i,A_i)-\Lambda_2(s-|W_i,A_i)} \Lambda_1(\mathrm{d}s \mid W_i, A_i) \\
&\quad \times e^{[\Gamma+\Lambda_1+\Lambda_2](t-|A_i,W_i)} \mathbb{1}\{\tilde{T}_i \ge t\}[\Lambda_1+\Lambda_2](\mathrm{d}t \mid A_i, W_i), \\
\texttt{naiv}_i(\Lambda_1,\Lambda_2) &= g(\tau, 1, W_i; \Lambda_1, \Lambda_2) - g(\tau, 0, W_i; \Lambda_1, \Lambda_2)
\end{aligned}
$$

Then we can write the one-step estimator as

$$
\begin{aligned}
\Psi_{\mathrm{OS}}(\Lambda_1,\Lambda_2,\Gamma,\pi,\mu) = \frac{1}{n}\sum_{i=1}^n \Bigg\{ & \texttt{term}_i(\pi)\Big([\texttt{termA\_N}_i(\Gamma) - \texttt{termA\_L}_i(\Gamma,\Lambda_1)] \\
& \qquad - [\texttt{termB\_N}_i(\Gamma,\Lambda_1,\Lambda_2) - \texttt{termB\_L}_i(\Gamma,\Lambda_1,\Lambda_2)] \\
& \qquad + [\texttt{termC\_N}_i(\Gamma,\Lambda_1,\Lambda_2) - \texttt{termC\_L}_i(\Gamma,\Lambda_1,\Lambda_2)] \Big) \\
& + \texttt{naiv}_i(\Lambda_1,\Lambda_2) \Bigg\}
\end{aligned}
$$