

Some comments about the Highly-Adaptive LASSO with focus on survival data

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joint work with Thomas G., Helene, and Mark van der Laan

February 15, 2023

Outline

Setting and motivation

Multivariate càdlàg functions of bounded sectional variation norm

Some challenges with the exact definition of the estimator

Approximate minimization is sufficient

Motivation

Reasons for using HAL [van der Laan, 2017]

- Theoretically important
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Theoretical discussion

- In practice we will have to approximate the HAL estimator for computational reasons
- Still nice to know that the estimator we approximate has nice properties

Definition of the estimator

$\mathcal{D}_M([0, 1]^d)$ the space of càdlàg functions $f: [0, 1]^d \rightarrow \mathbb{R}$ with **sectional variation norm** bounded by M .

\mathcal{O} the sample space

L a loss function, $L(f, O) \in \mathbb{R}_+$

The parameter of interest is the function minimizing the expected loss (risk)

$$f_0 = \operatorname{argmin}_{f \in \mathcal{D}_M([0, 1]^d)} P[L(f, \cdot)] = \operatorname{argmin}_{f \in \mathcal{D}_M([0, 1]^d)} \int_{\mathcal{O}} L(f, o) P(\mathrm{d}o).$$

We estimate f_0 with the function minimizing the empirical risk

$$\hat{f}_n = \operatorname{argmin}_{f \in \mathcal{D}_M([0, 1]^d)} \hat{\mathbb{P}}_n[L(f, \cdot)] = \operatorname{argmin}_{f \in \mathcal{D}_M([0, 1]^d)} \frac{1}{n} \sum_{i=1}^n L(f, O_i).$$

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Definition (Neuhaus [1971])

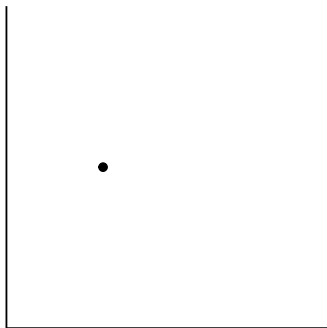
For a point $u \in [0, 1]^d$ and a vertex $\mathbf{a} \in \{0, 1\}^d$ look at quadrants $Q_{\mathbf{a}}(u)$ spanned by u and \mathbf{a} . The limit of $f(u_n)$ for $\{u_n\} \subset Q_{\mathbf{a}}(u)$, $u_n \rightarrow u$ should exist, and if $\mathbf{a} = (1, 1, \dots, 1)$ then $\lim_{n \rightarrow \infty} f(u_n) = f(u)$.

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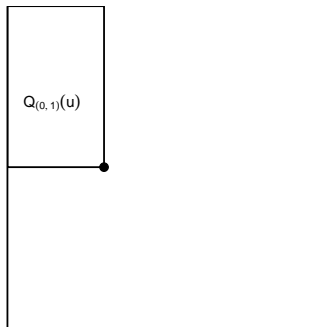


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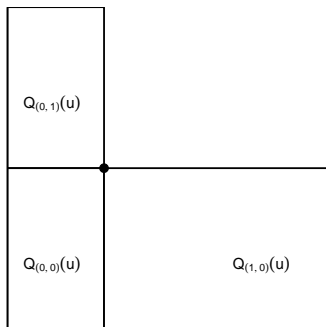


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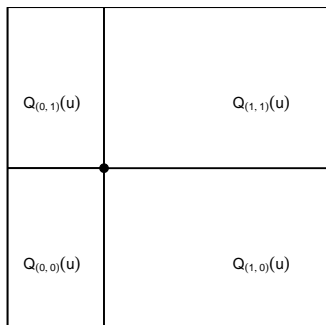


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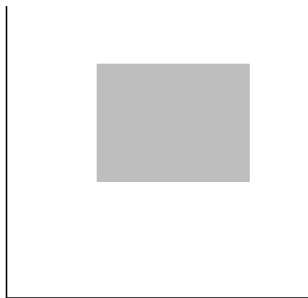
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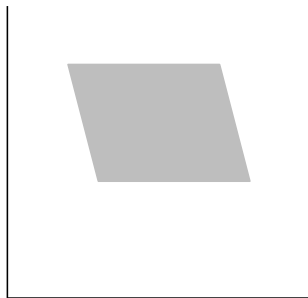
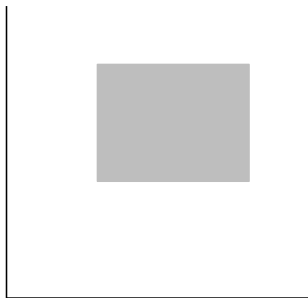
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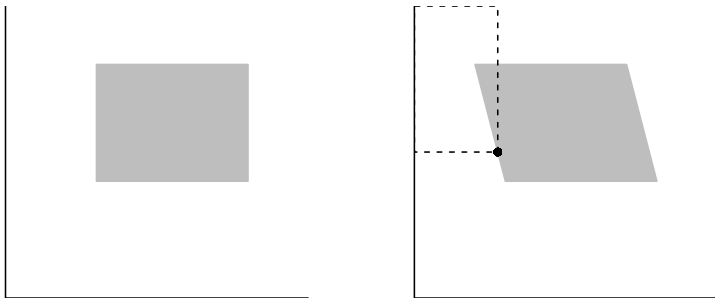
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In dimension $d = 1$, the variation norm of a function is

$$\|f\|_v = \sup_{\pi} \sum_{i=1}^{|\pi|} |f(t_i) - f(t_{i-1})|,$$

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At first sight, a natural generalization seems to be the *Vitali variation*:

$$V^{(d)}(f) = \sup_{\pi} \sum_{A \in \pi} |\Delta(f; A)|,$$

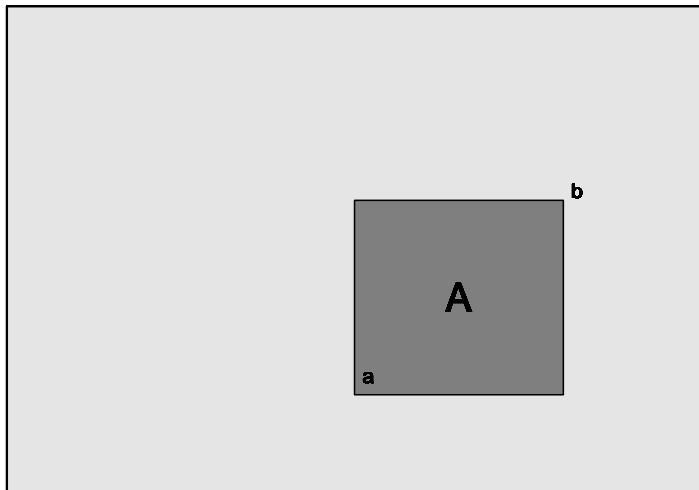
where the supremum is taken over all “grid partitions” and $\Delta(f; A)$ is the *quasi-volume* that f assigns the rectangle A .

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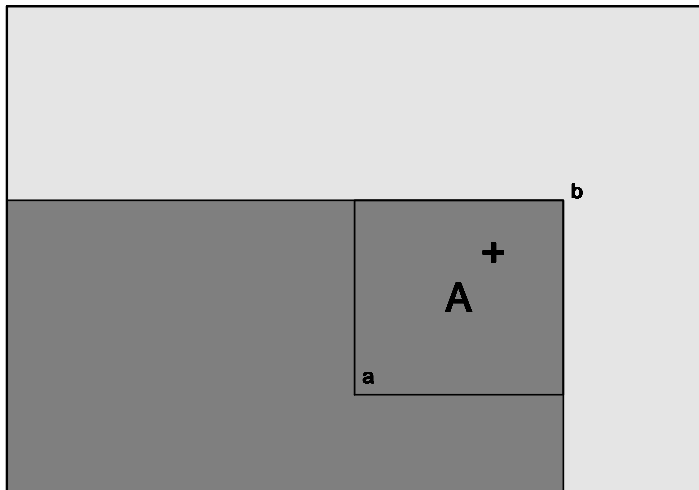
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When $d = 2$, $\Delta(f; A) = f(b_1, b_2) - f(b_1, a_2) - f(a_1, b_2) + f(a_1, a_2)$.



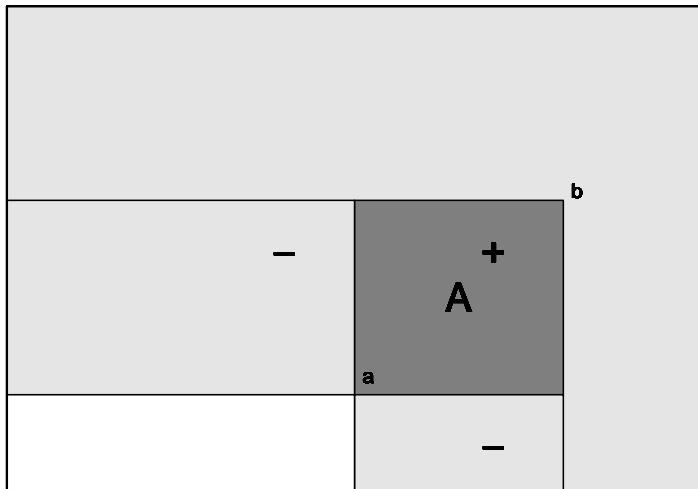
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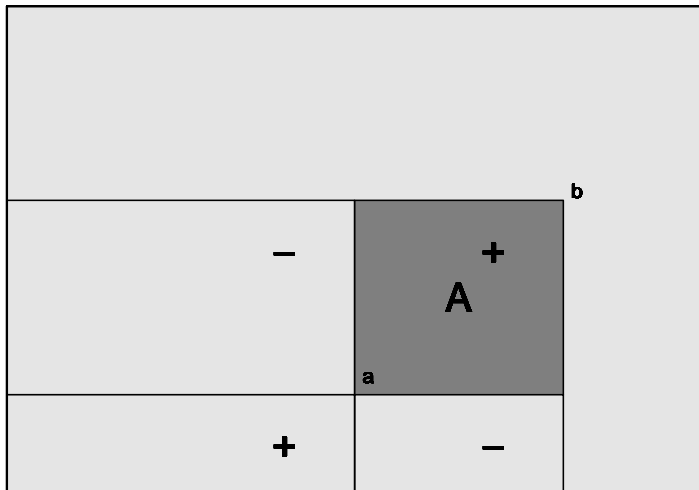
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We should also look at the function on the *lower dimensional faces*

$$U_s = \{(u_1, \dots, u_d) \in [0, 1]^d : u_j = 0, j \notin s\},$$

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We denote by $f_s: [0, 1]^{|s|} \rightarrow \mathbb{R}$ the restriction of f to U_s and define the norm

$$\|f\|_v = \sum_s V^{(|s|)}(f_s),$$

where the sum is taken over all non-empty subsets $s \subset \{1, \dots, d\}$.

This is referred to as the *Hardy-Krause variation* by Fang et al. [2021] and the *sectional variation norm* by van der Laan [2017].

The sectional variation norm of smooth functions

In $d = 1$, if f is differentiable then

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→ The sum contains $\sum_{k=1}^n \binom{n}{k} = (2^d - 1)$ terms.

Implementation of the estimator

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Gill et al. [1995] and van der Laan [2017] give the following representation of any $f \in \mathcal{D}_M([0, 1]^d)$:

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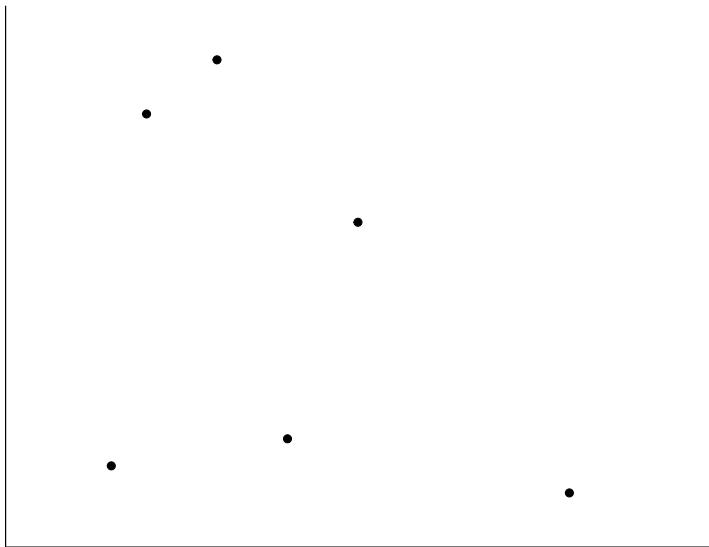
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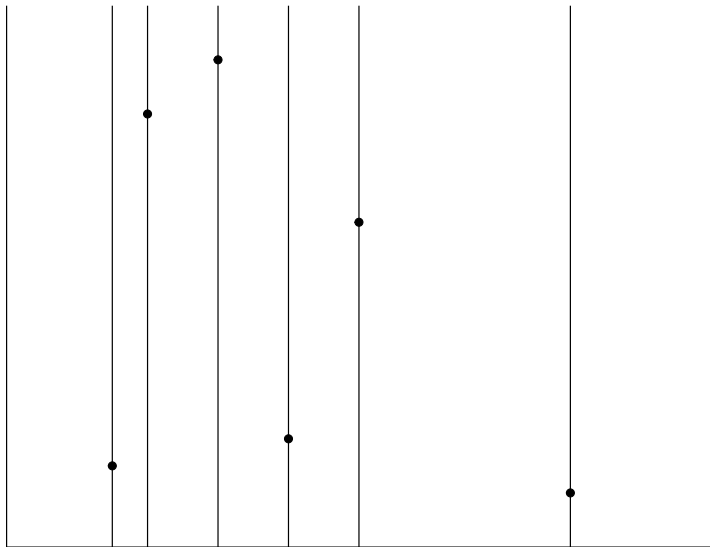
This can be phrased as the LASSO problem

$$\underset{\beta}{\operatorname{argmin}} \hat{\mathbb{P}}_n[L(f_\beta, \cdot)], \quad \text{such that} \quad \|\beta\|_1 \leq M.$$

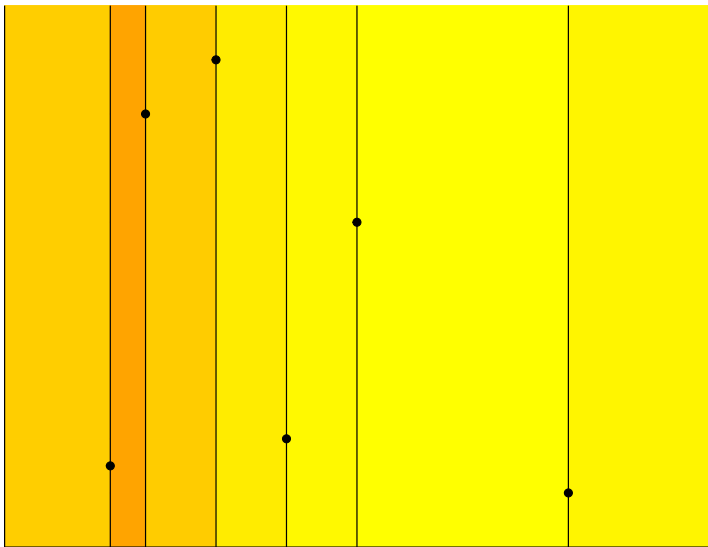
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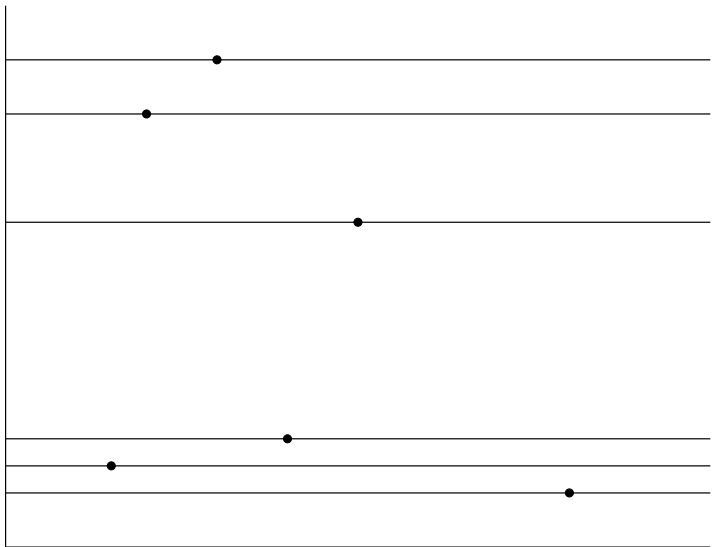
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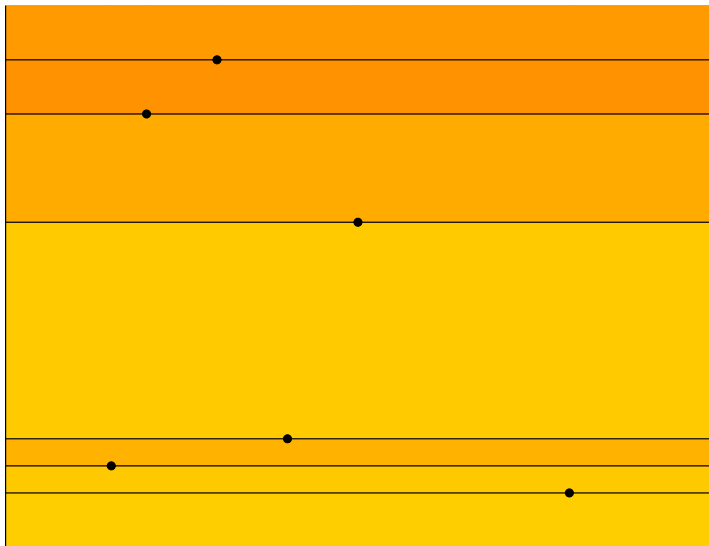
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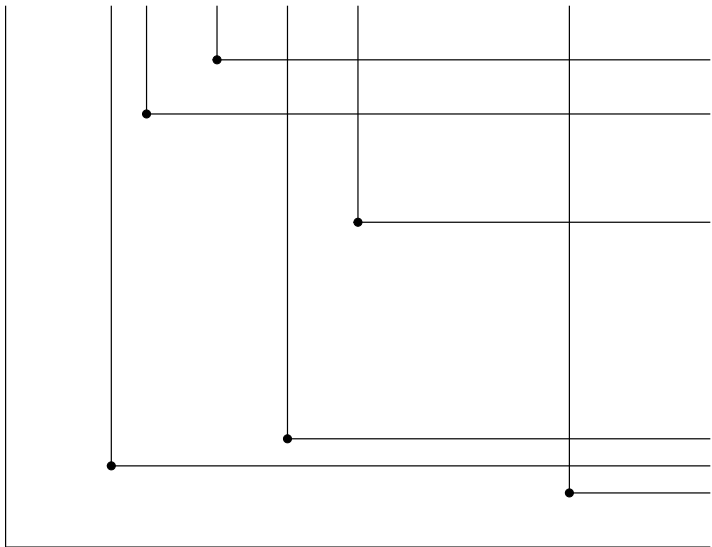
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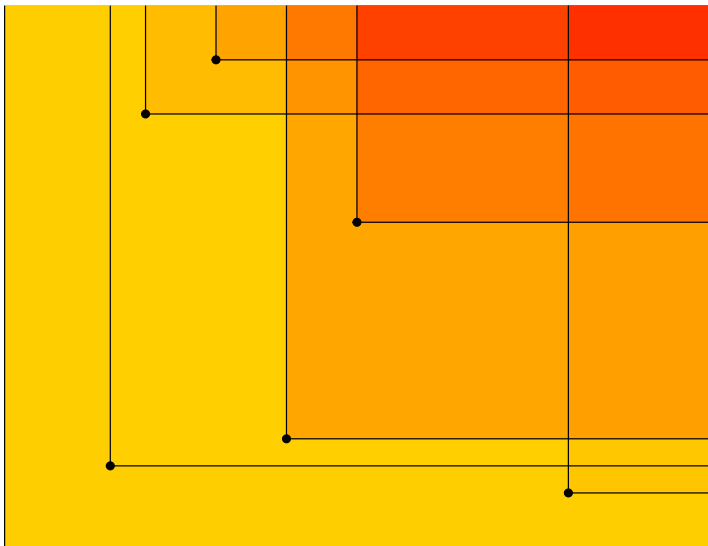
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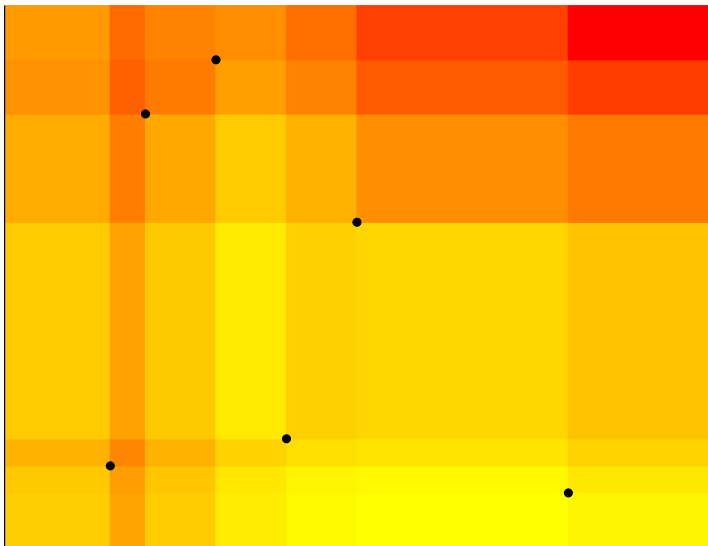
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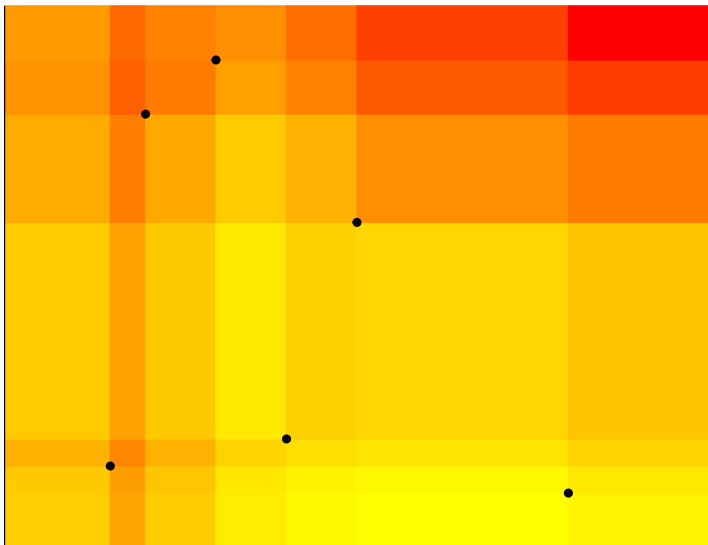
and, with $0 = X_{(0)} \leq X_{(1)} \leq \dots \leq X_{(n)}$,

$$\begin{aligned} \|\bar{f}\|_v &= \sum_{i=1}^n |\bar{f}(X_{(i)}) - \bar{f}(X_{(i-1)})| = \sum_{i=1}^n |f(X_{(i)}) - f(X_{(i-1)})| \\ &\leq \sup_{\pi} \sum_{i=1}^{|\pi|} |f(t_i) - f(t_{i-1})| = \|f\|_v. \end{aligned}$$

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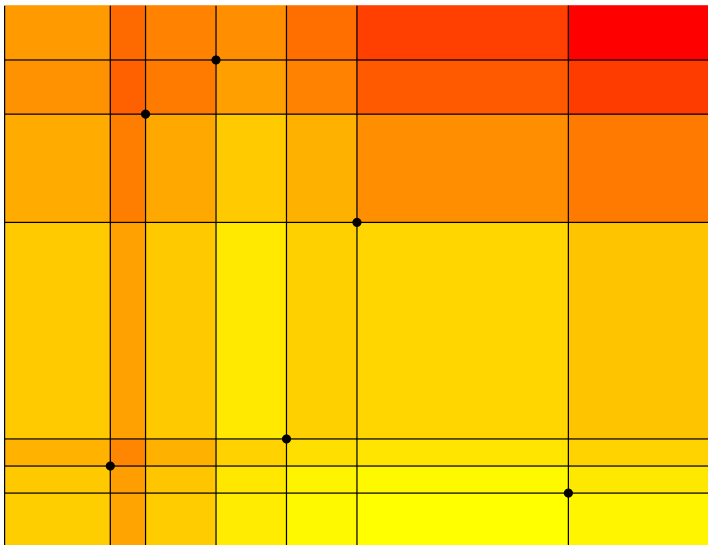
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We have $6 \times 3 + 1 = 17$ basis functions but $(6 + 1)^2 = 49$ rectangles.



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Fang et al. [2021] formally show that when L is the squared error loss, the minimizer

$$\hat{f}_n = \operatorname{argmin}_{f \in \mathcal{D}_M([0,1]^d)} \hat{\mathbb{P}}_n[L(f, \cdot)]$$

can be taken to be the solution to a LASSO problem using indicator functions as basis functions.

However, they need up to $\asymp n^d$ basis functions whereas the HAL estimator is made up of only $n \times (2^d - 1) + 1 \asymp n$ basis functions.

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Consider estimation of the hazard for the survival time T .

Data $O = (\tilde{T}, \Delta)$, $\tilde{T} = T \wedge C$, $\Delta \in \{0, 1\}$

Hazard $h = e^f$, $f \in \mathcal{D}_M([0, 1])$

Loss $L(f, O) = \int_0^{\tilde{T}} e^{f(s)} - \Delta f(\tilde{T})$

The estimator \hat{f}_n in the survival setting

Consider estimation of the hazard for the survival time T .

Data $O = (\tilde{T}, \Delta)$, $\tilde{T} = T \wedge C$, $\Delta \in \{0, 1\}$

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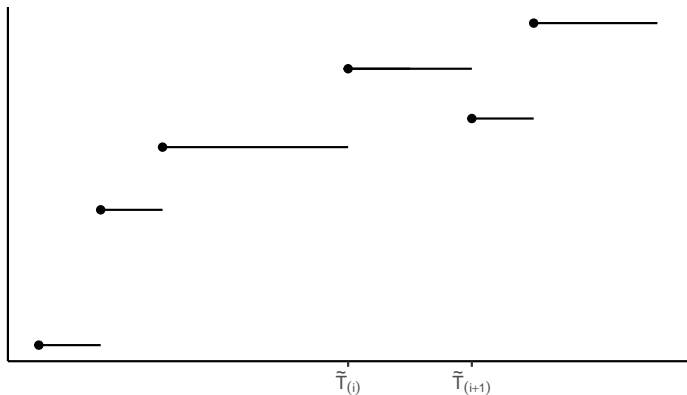
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\Rightarrow The empirical risk minimizer \hat{f}_n is in general either not well-defined or a very bad estimator.

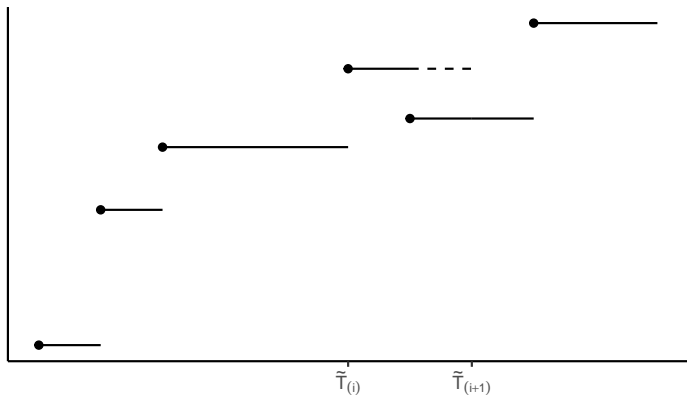
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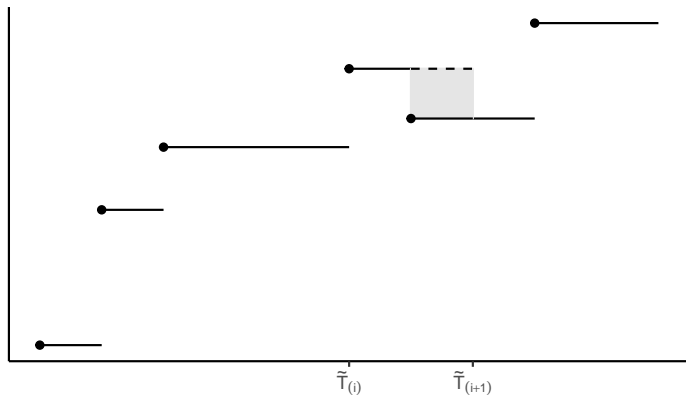
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(Note that $f_{\hat{\beta}_n}$ is well-defined in the survival setting.)

Deriving convergence rates using empirical processes theory

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The convergence rate for an empirical loss minimizer over a function space \mathcal{F} can be read off from the *modulus of continuity* of the empirical process $\mathbb{G}_n = \sqrt{n}(\hat{\mathbb{P}}_n - P)$ over the space

$$\mathcal{L} = \{L(f, \cdot) - L(f_0, \cdot) : f \in \mathcal{F}\},$$

which is defined as

$$\varphi_n(\delta) = \mathbb{E} [\|\mathbb{G}_n\|_{\mathcal{L}(\delta)}], \quad \text{where} \quad \|\mathbb{G}_n\|_{\mathcal{L}(\delta)} = \sup_{h \in \mathcal{L}(\delta)} |\mathbb{G}_n[h]|,$$

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The modulus φ_n can be controlled by the covering or bracketing entropy for \mathcal{F} . When $\mathcal{F} = \mathcal{D}_M([0, 1]^d)$ this leads to the convergence rate

$$\|\hat{f}_n - f_0\| = \mathcal{O}_P(r_n), \quad \text{for} \quad r_n = n^{-1/3} \log(n)^{2(d-1)/3}.$$

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Exact minimization is not needed – we just need the estimator f_n^* to fulfill

$$\hat{\mathbb{P}}_n[L(f_n^*, \cdot)] \leq \hat{\mathbb{P}}_n[L(f_0, \cdot)] + \mathcal{O}_P(r_n^2).$$

This holds for $f_{\hat{\beta}_n}$: $\hat{\mathbb{P}}_n[L(f_{\hat{\beta}_n}, \cdot)] \leq \hat{\mathbb{P}}_n[L(f_0, \cdot)] + \mathcal{O}_P(r_n^2)$

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Combine this with a bound on $\varphi_n(\delta)$ for $\mathcal{D}_M([0, 1]^d)$ to obtain

$$\hat{\mathbb{P}}_n[L(\tilde{f}_n, \cdot)] - \hat{\mathbb{P}}_n[L(f_0, \cdot)] = \mathcal{O}_P(r_n^2).$$

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- What kind of constraints are put on functions in $\mathcal{D}_M([0,1]^d)$ that are continuous but not much smoother?

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