The negative log-likelihood loss and cross-validation with censored data

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Outline

Model and hyper-parameter selection for survival models

The least false model in the presence of censoring

Hold-out samples and survival model estimators

Selecting a model from a collection of candidate models

Consider estimation of the parameter

$$\theta(P) := \operatorname*{argmin}_{f \in \mathcal{F}} P[L(f, \cdot)], \quad \text{where} \quad P[g] := \int_{\mathcal{O}} g(o) P(\mathrm{d}o),$$

for some loss function $L \colon \mathcal{F} \times \mathcal{O} \to \mathbb{R}_+$.

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Hyper-parameter selection

For estimation in high-dimensional settings we often introduce a regularization parameter λ (e.g., LASSO, kernel smoothing). To select a value for λ we would typically split the data $\mathcal{D}_n = \{O_1,\dots,O_n\}$ randomly in two, $\mathcal{D}_n^1$ and $\mathcal{D}_n^2$, and calculate

$$\operatorname*{argmin}_{\lambda\in\Lambda} \hat{\mathbb{P}}_n^2[L(\hat{f}_\lambda^1,\cdot)],$$

where $\hat{\mathbb{P}}_n^2$ denotes the empirical measure based on the sample $\mathcal{D}_n^2$, and $\hat{f}_\lambda^1$ denotes an estimator calculated on $\mathcal{D}_n^1$ with regularization parameter λ.

A loss function for survival data

$O = (\tilde{T}, \Delta, X) \sim P \in \mathcal{P}$ Oberved data with $\mathcal{O} = \mathbb{R}_+ \times \{0, 1\} \times \mathbb{R}^p$.

$(T, X) \sim Q \in \mathcal{Q}$ The distribution Q (or a feature of it) is of interest.

Assuming coarsening at random [Gill et al., 1997] we can write

$$\mathcal{P} = \{P_{Q,G} : Q \in \mathcal{Q}, G \in \mathcal{G}\},$$

where $\mathcal{G}$ denotes a collection of conditional distributions for the censoring mechanism, and the likelihood factorizes as

$$\ell(P_{Q,G}, O) = \ell_F(Q, O) \cdot \ell_{\mathcal{C}}(G, O),$$

with

$$\ell_F(Q, O) := q(\tilde{T} \mid X)^{\Delta} \bar{Q}(\tilde{T} \mid X)^{1-\Delta} m(X),$$

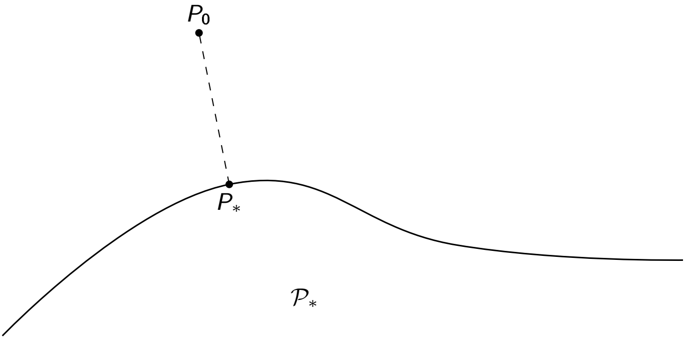
where q and $\bar{Q}$ are the conditional density and survivor function, respectively, and m the marginal distribution of X.

Natural to use the negative partial log-likelihood $-\log \ell_F$ as loss function, or even only the first part concerning the conditional distribution of T given X.

Kullback-Leibler divergence and partial likelihoods

Maximum likelihood estimation is connected to minimizing the Kullback-Leibler divergence and gives an interpretation of the MLE under mis-specified models.

$$D_{\mathrm{KL}}(P_0 \,||\, P) := P_0\left[\log \frac{p_0}{p}\right], \quad \text{where} \quad P_0 = p_0 \cdot \mu, P = p \cdot \mu.$$



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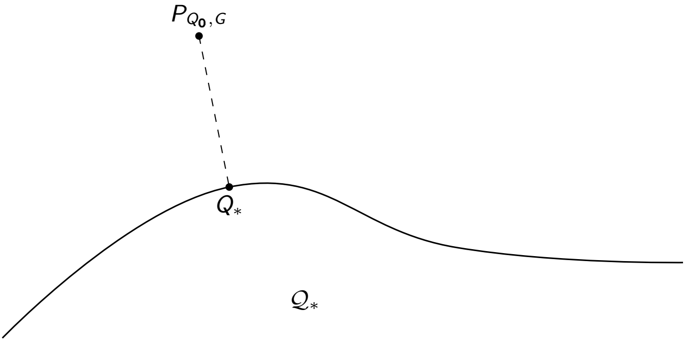
$$Q \longmapsto D_{\mathrm{KL}}(P_{Q_0,G} \,||\, P_{Q,G}), \quad \text{with} \quad Q \in \mathcal{Q}_*.$$



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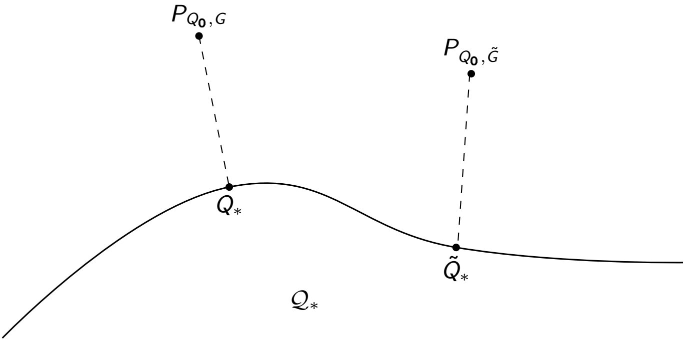

Least false model depends on the censoring distribution

For any value $G \in \mathcal{G}$ we have that $D_{\mathrm{KL}}(P_{Q_0,G} \,||\, P_{Q_0,G}) = 0$, so the correct model Q_0 is ranked better than any other model independently of $G \in \mathcal{G}$. However, if $Q_0 \notin \mathcal{Q}_*$ the minimizer might depend on the value of G.[1]

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For the simple survival case with no baseline covariates, we have the following result stating that for a mis-specified model Q we can alway find an alternative model $\tilde{Q}$ that is ranked better under one censoring regime but worse under another.

Let Q_0 and G be given together with some $Q \neq Q_0$. Then (under regularity conditions) we can find $\tilde{Q}$ and $\tilde{G}$ such that

$$D_{\mathrm{KL}}(P_{Q_0,G} \,||\, P_{Q,G}) < D_{\mathrm{KL}}(P_{Q_0,G} \,||\, P_{\tilde{Q},G}),$$

and

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Sketch of proof

- Divide $(0, \tau)$ into $(0, \tau_0)$ and $[\tau_0, \tau)$, where $(0, \tau)$ is the support of T.
- Construct $\tilde{Q}$ such that it performs better than Q on $(0, \tau_0)$ but worse on (τ_0, τ) under the censoring regime G.
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Whether the alternative model $\tilde{Q}$ can be constructed such that $\tilde{Q} \in \mathcal{Q}_*$ for some model class $\mathcal{Q}_*$ will depend on the model class and on Q_0 and $\mathcal{G}$.

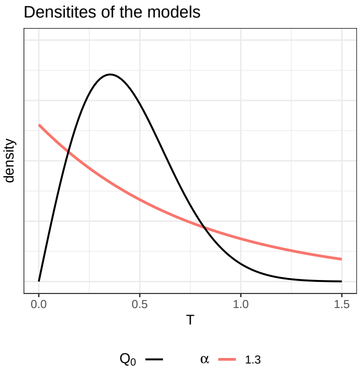
A simple example with mis-specified survival models

Assume the data generating distribution given by

$$Q_0 = \text{Weibull}(2, 0.5), \quad \text{and} \quad G_\gamma = \text{Weibull}(2, \gamma),$$

and consider the four candidate models indexed by α,

$$Q_\alpha = \text{Exp}(\alpha), \quad \text{with} \quad \alpha \in \{1.3,\ 1.5,\ 1.8,\ 2\}.$$



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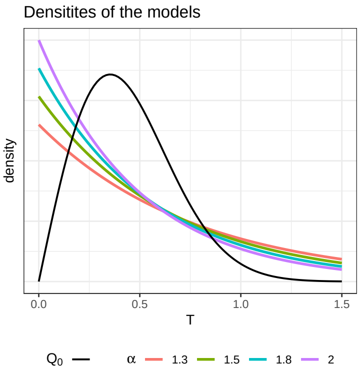

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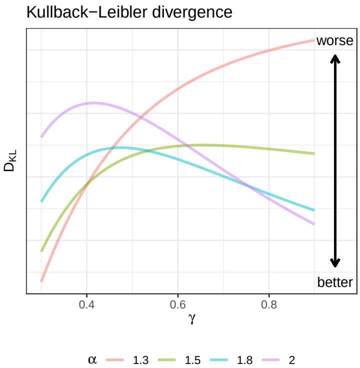

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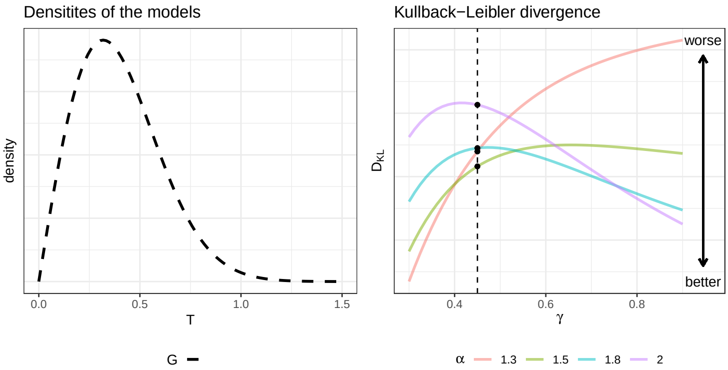

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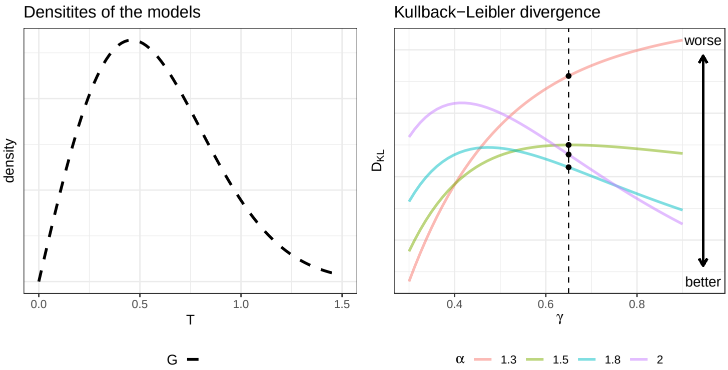

Survival curve estimators evaluated on hold-out samples

Consider the problem of selecting a hyper-parameter or model using cross-validation. We split the data $\mathcal{D}_n = \{O_1, \ldots, O_n\}$ in two, $\mathcal{D}_n^1$ and $\mathcal{D}_n^2$.

On split $\mathcal{D}_n^1$ Fit models $\{\hat{f}_\lambda : \lambda \in \Lambda\}$ or $\{\hat{f}_1, \hat{f}_2, \ldots \hat{f}_k\}$.

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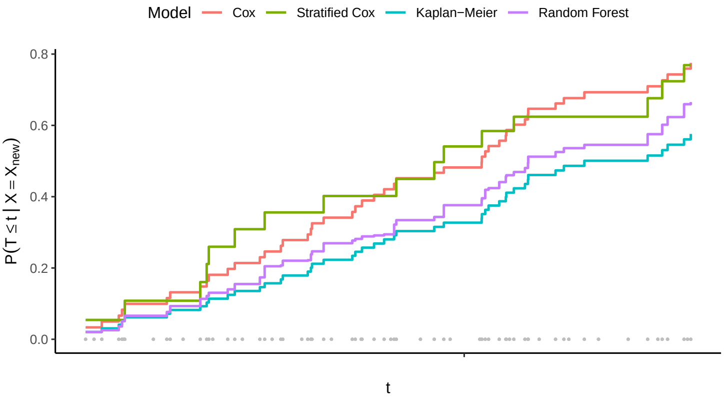

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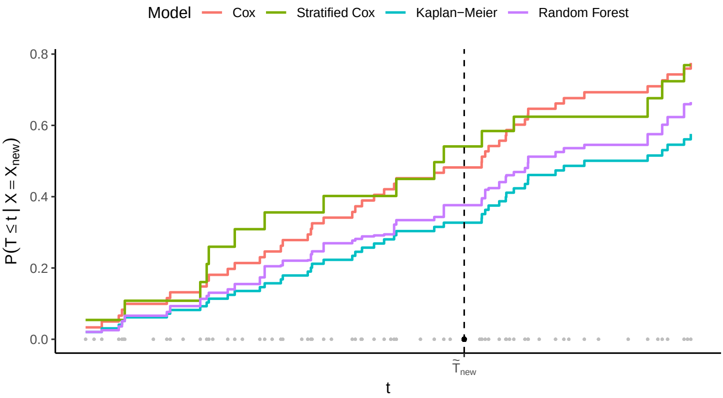

Taking the censoring distribution into account

To alliviate these problems problems we can reweight the observed outcome or the loss function to account/adjust for the censoring:

- Inverse probability of censoring weighted loss functions [van der Laan and Dudoit, 2003]. For instance, weighted negative log-likelihood or (integrated) Brier score.
- Pseudo-values [Andersen et al., 2003, Mogensen and Gerds, 2013].
- Censoring unbiased transformations [Fan and Gijbels, 1996, Steingrimsson et al., 2019].

These approaches are particularly attractive when we are willing to assume that the censoring does not depend on the baseline covariates.

Modeling the censoring distribution

If we are not sure that the censoring is independent we need to model the dependence on the covariates.

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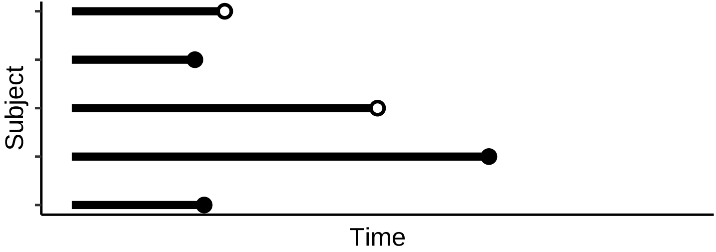

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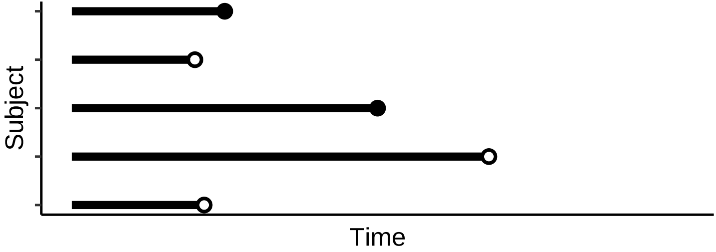

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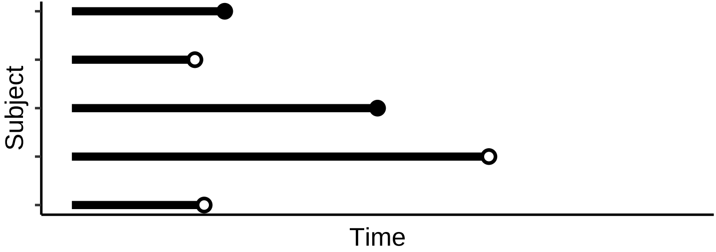

Inverse-weighted survival games [Han et al., 2021]: Iterate the estimation until convergence (hopefully).

Conclusion

How should we do cross-validation for general survival models?

- Using the negative partial log-likelihood is problematic
  - $\rightarrow$ The least false model is not well-defined (without reference to the censoring regime)
  - $\rightarrow$ For many standard survival estimators, we cannot use it on hold-out samples
- Using loss functions designed to measure the loss for the model of interest is challenging in the presence of complicated censoring
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Questions, comments, suggestions?

Thank you!

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