The negative log-likelihood loss and cross-validation with censored data

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Outline

Model and hyper-parameter selection for survival models

The least false model in the presence of censoring

Hold-out samples and survival model estimators

Selecting a model from a collection of candidate models

Consider estimation of the parameter

$$\theta(P) := \operatorname*{argmin}_{f \in \mathcal{F}} P[L(f, \cdot)], \quad \text{where} \quad P[g] := \int_{\mathcal{O}} g(o) P(\mathrm{d}o),$$

for some loss function $L \colon \mathcal{F} \times \mathcal{O} \to \mathbb{R}_+$.

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Maximum likelihood estimator (MLE)

If $\mathcal{F}$ is a collection of densities on $\mathcal{O}$ and $L(f, O) := -\log(f(O))$, then $\theta(\hat{\mathbb{P}}_n)$ is the MLE for the model $\mathcal{F}$, where $\hat{\mathbb{P}}_n$ denotes the empirical measure.

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Hyper-parameter selection

For estimation in high-dimensional settings we often introduce a regularization parameter λ (e.g., LASSO, kernel smoothing). To select a value for λ we would typically split the data $\mathcal{D}_n = \{O_1, \ldots, O_n\}$ randomly in two, $\mathcal{D}_n^{\text{train}}$ and $\mathcal{D}_n^{\text{test}}$, and calculate

$$\underset{\lambda \in \Lambda}{\operatorname{argmin}}\, \hat{\mathbb{P}}_n^{\text{test}}[L(\hat{f}_\lambda^{\text{train}}, \cdot)],$$

where $\hat{\mathbb{P}}_n^{\text{test}}$ is the empirical measure based on the sample $\mathcal{D}_n^{\text{test}}$, and $\hat{f}_\lambda^{\text{train}}$ denotes an estimator calculated on $\mathcal{D}_n^{\text{train}}$ with regularization parameter λ.

A loss function for survival data

$O = (\tilde{T}, \Delta, X) \sim P \in \mathcal{P}$ Oberved data with $\mathcal{O} = \mathbb{R}_+ \times \{0, 1\} \times \mathbb{R}^p$.

$(T, X) \sim Q \in \mathcal{Q}$ The distribution Q (or a feature of it) is of interest.

Assuming coarsening at random [Gill et al., 1997] we can write

$$\mathcal{P} = \{P_{Q,G} : Q \in \mathcal{Q}, G \in \mathcal{G}\},$$

where $\mathcal{G}$ denotes a collection of conditional distributions for the censoring mechanism, and the likelihood factorizes as

$$\ell(P_{Q,G}, O) = \ell_F(Q, O) \cdot \ell_C(G, O),$$

with

$$\ell_F(Q, O) := q(\tilde{T} \mid X)^{\Delta} \bar{Q}(\tilde{T} \mid X)^{1-\Delta} m(X),$$

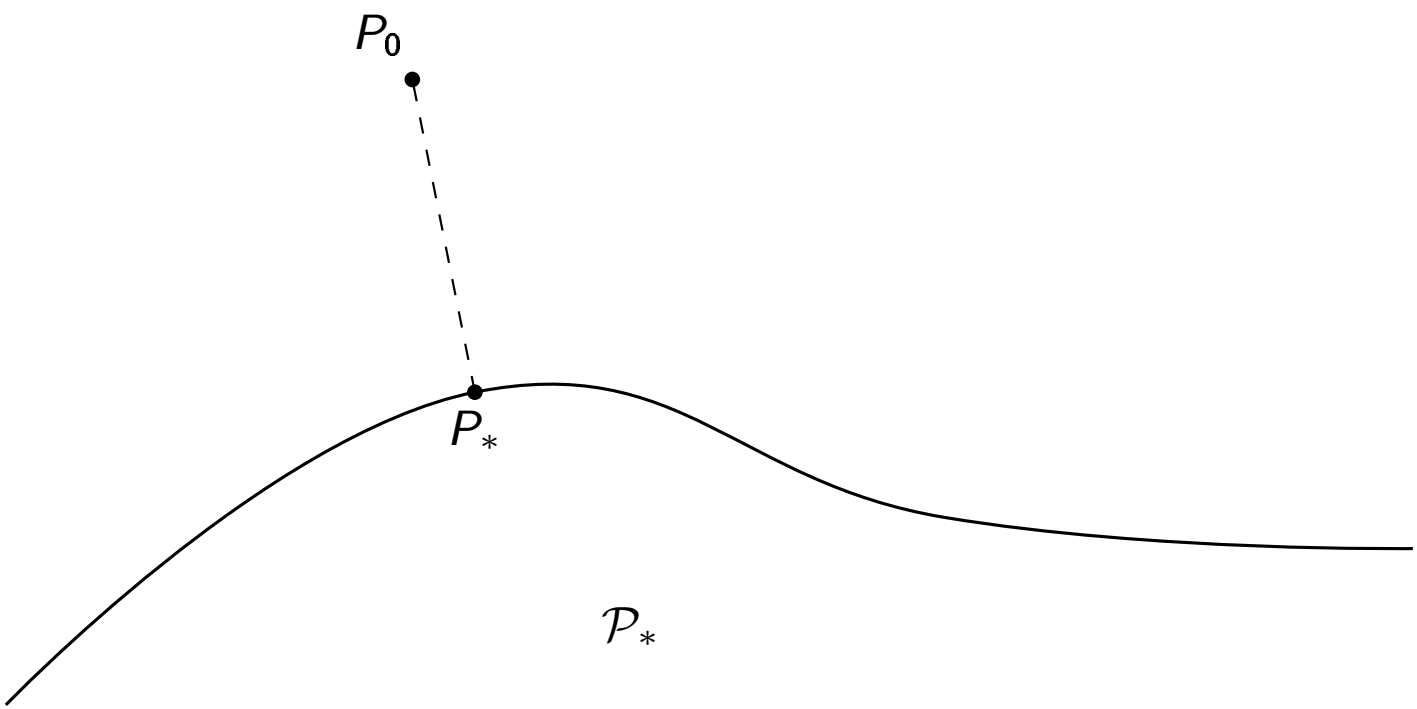
where q and $\bar{Q}$ are the conditional density and survivor function, respectively, and m the marginal distribution of X.

Natural to use the negative partial log-likelihood $-\log \ell_F$ as loss function, or even only the first part concerning the conditional distribution of T given X.

Kullback-Leibler divergence and partial likelihoods

Maximum likelihood estimation is connected to minimizing the Kullback-Leibler divergence and gives an interpretation of the MLE under misspecified models.

$$D_{\mathrm{KL}}(P_0 \,||\, P) := P_0\left[\log \frac{p_0}{p}\right], \quad \text{where} \quad P_0 = p_0 \cdot \mu, P = p \cdot \mu.$$



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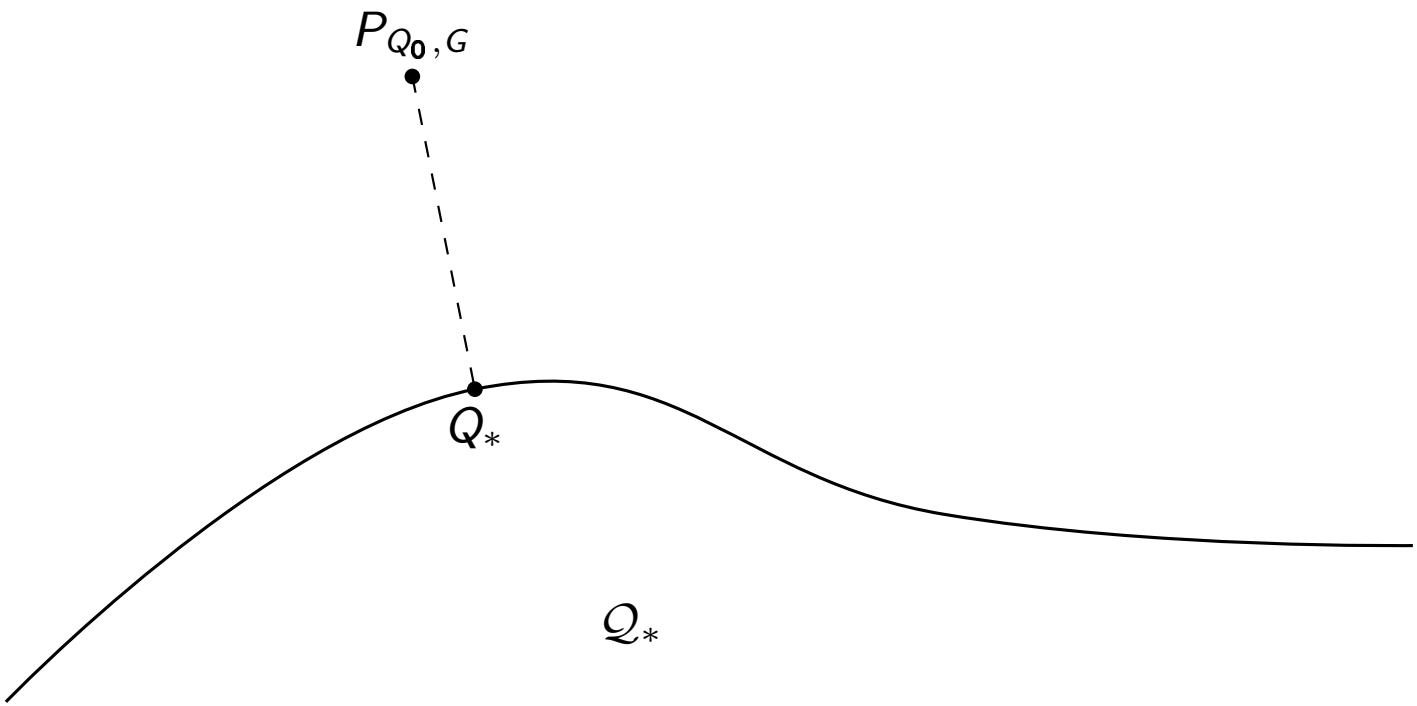

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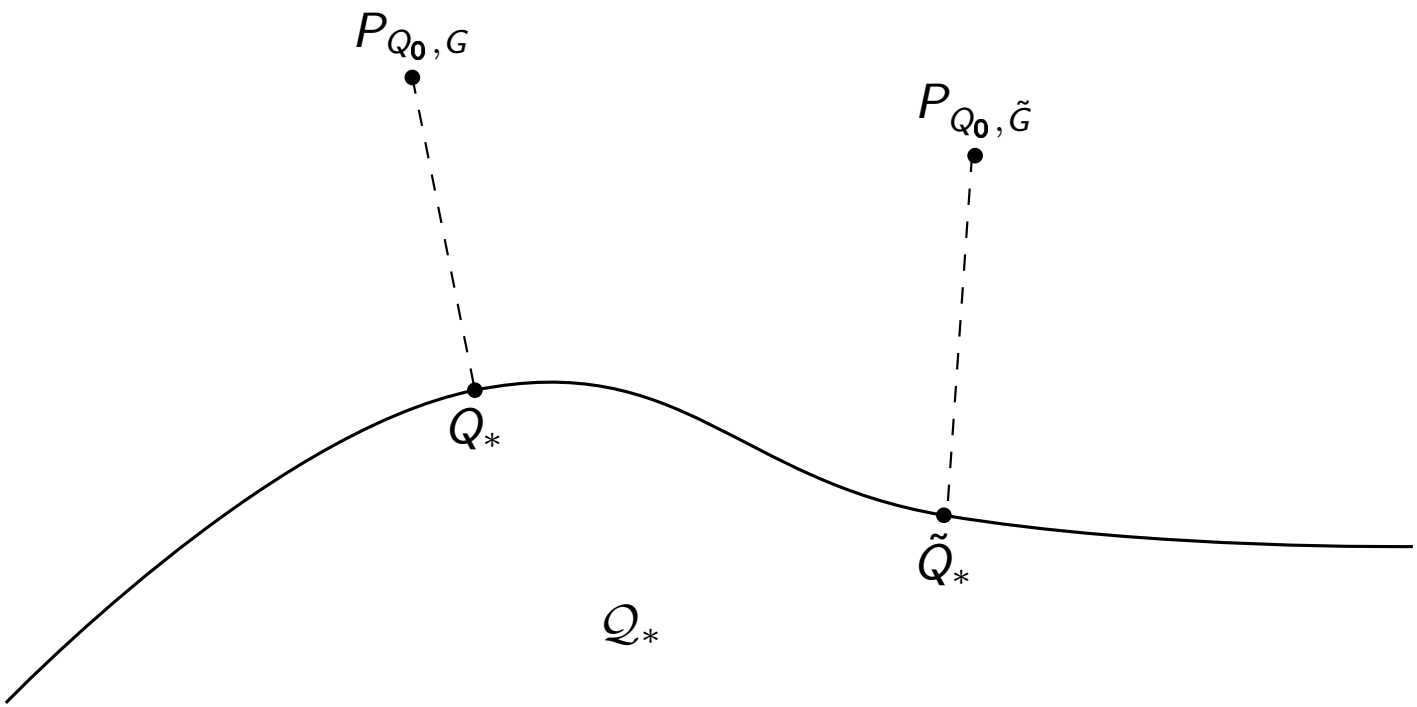

Least false model depends on the censoring distribution

For any value $G \in \mathcal{G}$ we have that $D_{\mathrm{KL}}(P_{Q_0,G} \,||\, P_{Q_0,G}) = 0$, so the correct model Q_0 is ranked better than any other model independently of $G \in \mathcal{G}$. However, if $Q_0 \notin \mathcal{Q}_*$ the minimizer might depend on the value of G.[1]

[1] This is mentioned in Whitney et al. [2019] and van der Laan and Dudoit [2003], and a similar phenomenon is well studied for the Cox model [Struthers and Kalbfleisch, 1986, Hjort, 1992, Fine, 2002].

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For the simple survival case with no baseline covariates, we have the following result stating that for a misspecified model Q we can alway find an alternative model $\tilde{Q}$ that is ranked better under one censoring regime but worse under another.

Let Q_0 and G be given together with some $Q \neq Q_0$. Then (under regularity conditions) we can find $\tilde{Q}$ and $\tilde{G}$ such that

$$D_{\mathrm{KL}}(P_{Q_0,G} \,||\, P_{Q,G}) < D_{\mathrm{KL}}(P_{Q_0,G} \,||\, P_{\tilde{Q},G}),$$

and

$$D_{\mathrm{KL}}(P_{Q_0,\tilde{G}} \,||\, P_{Q,\tilde{G}}) > D_{\mathrm{KL}}(P_{Q_0,\tilde{G}} \,||\, P_{\tilde{Q},\tilde{G}}).$$

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Sketch of proof

- Divide $(0, \tau)$ into $(0, \tau_0)$ and $[\tau_0, \tau)$, where $(0, \tau)$ is the support of T.
- Construct $\tilde{Q}$ such that it performs better than Q on $(0, \tau_0)$ but worse on (τ_0, τ) under the censoring regime G.
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Whether the alternative model $\tilde{Q}$ can be constructed such that $\tilde{Q} \in \mathcal{Q}_*$ for some model class $\mathcal{Q}_*$ will depend on the model class and on Q_0 and $\mathcal{G}$.

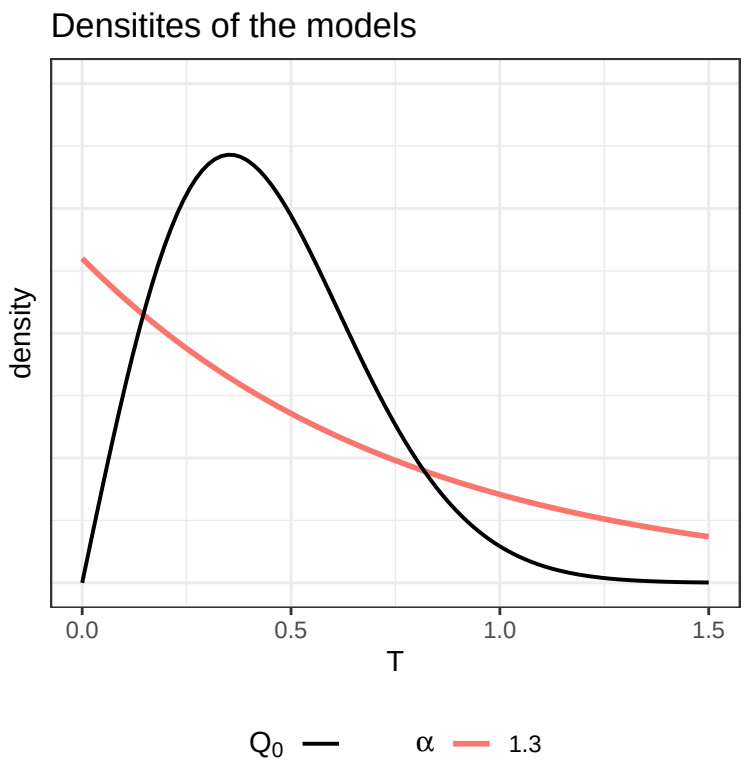
A simple example with misspecified survival models

Assume the data generating distribution given by

$$Q_0 = \text{Weibull}(2, 0.5), \quad \text{and} \quad G_\gamma = \text{Weibull}(2, \gamma),$$

and consider the four candidate models indexed by α,

$$Q_\alpha = \text{Exp}(\alpha), \quad \text{with} \quad \alpha \in \{1.3, 1.5, 1.8, 2\}.$$



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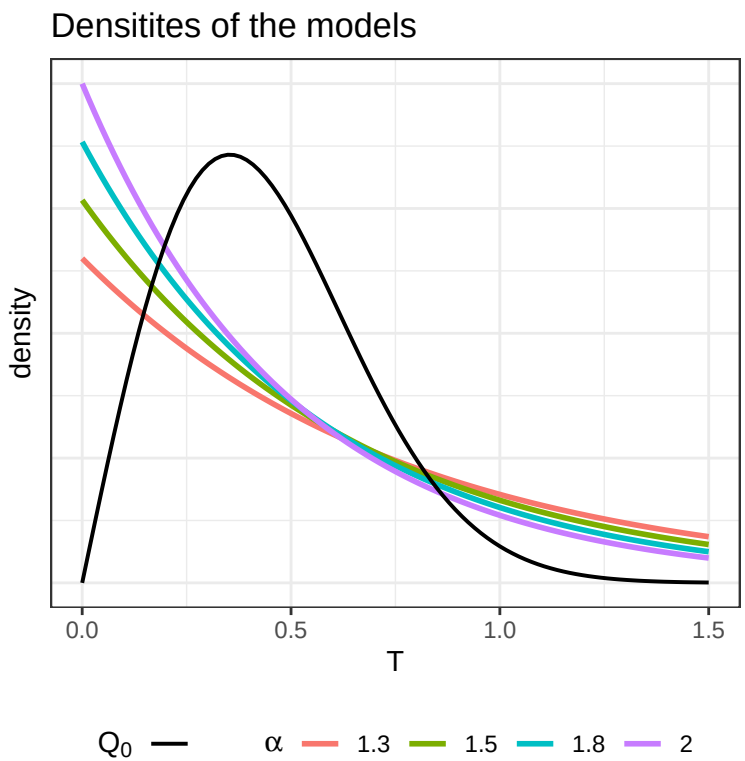

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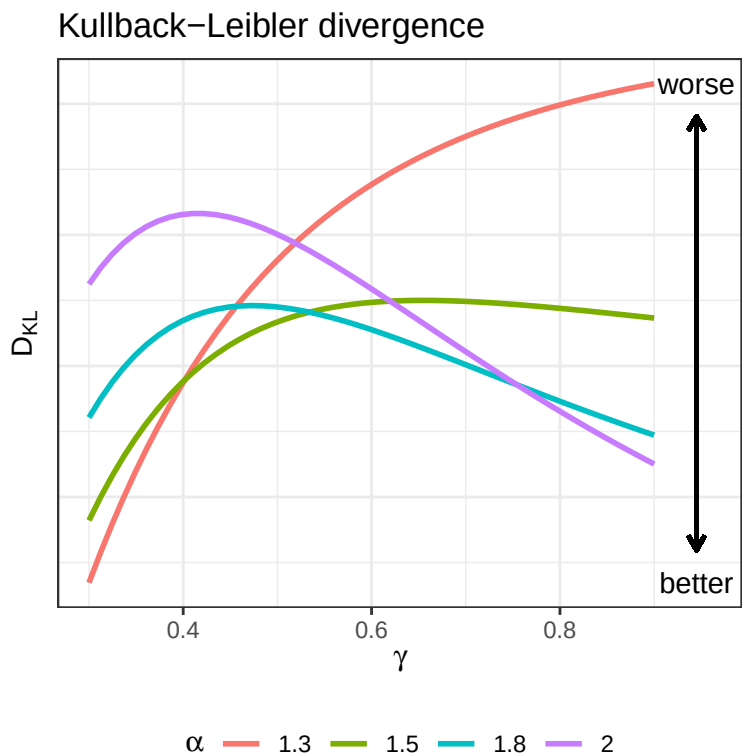

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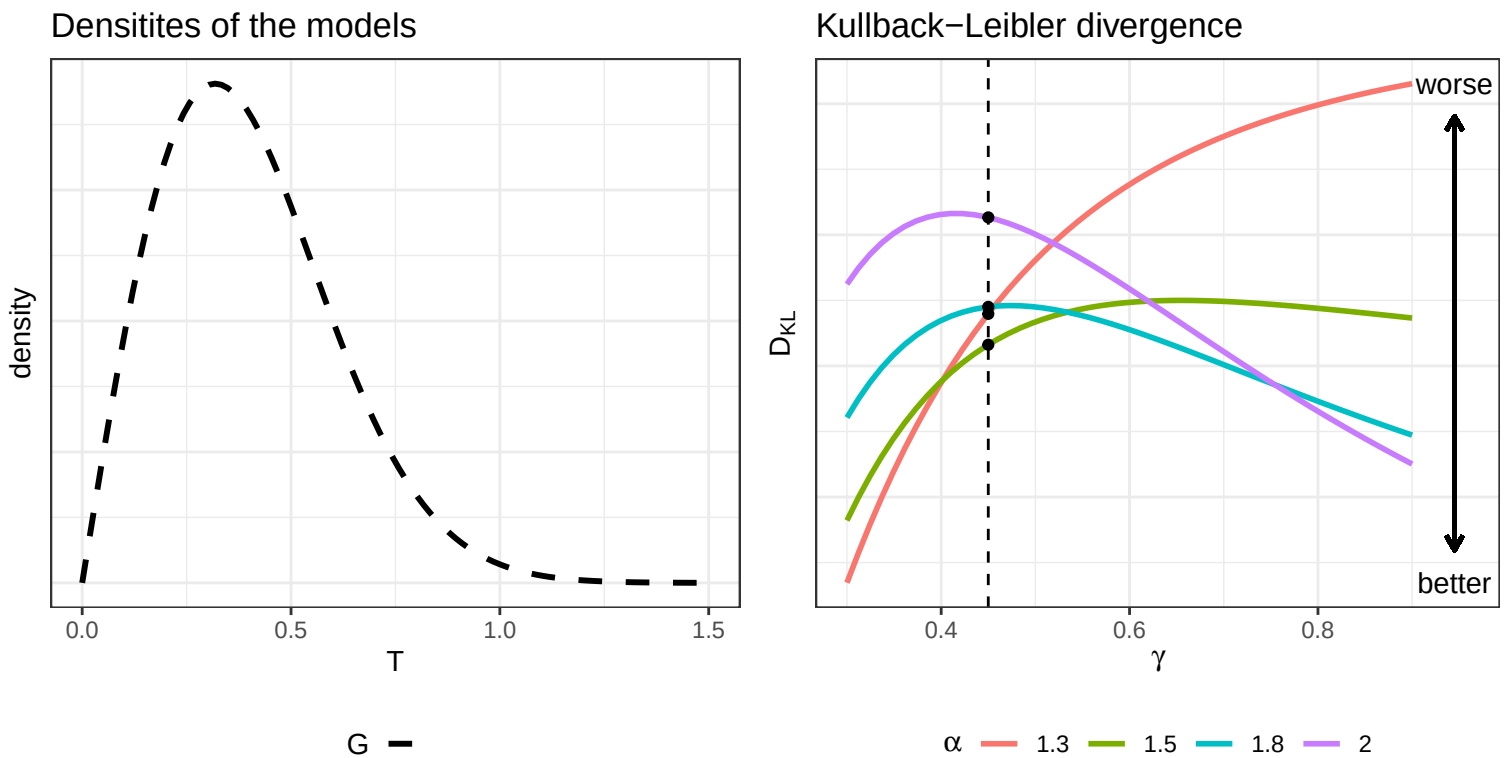

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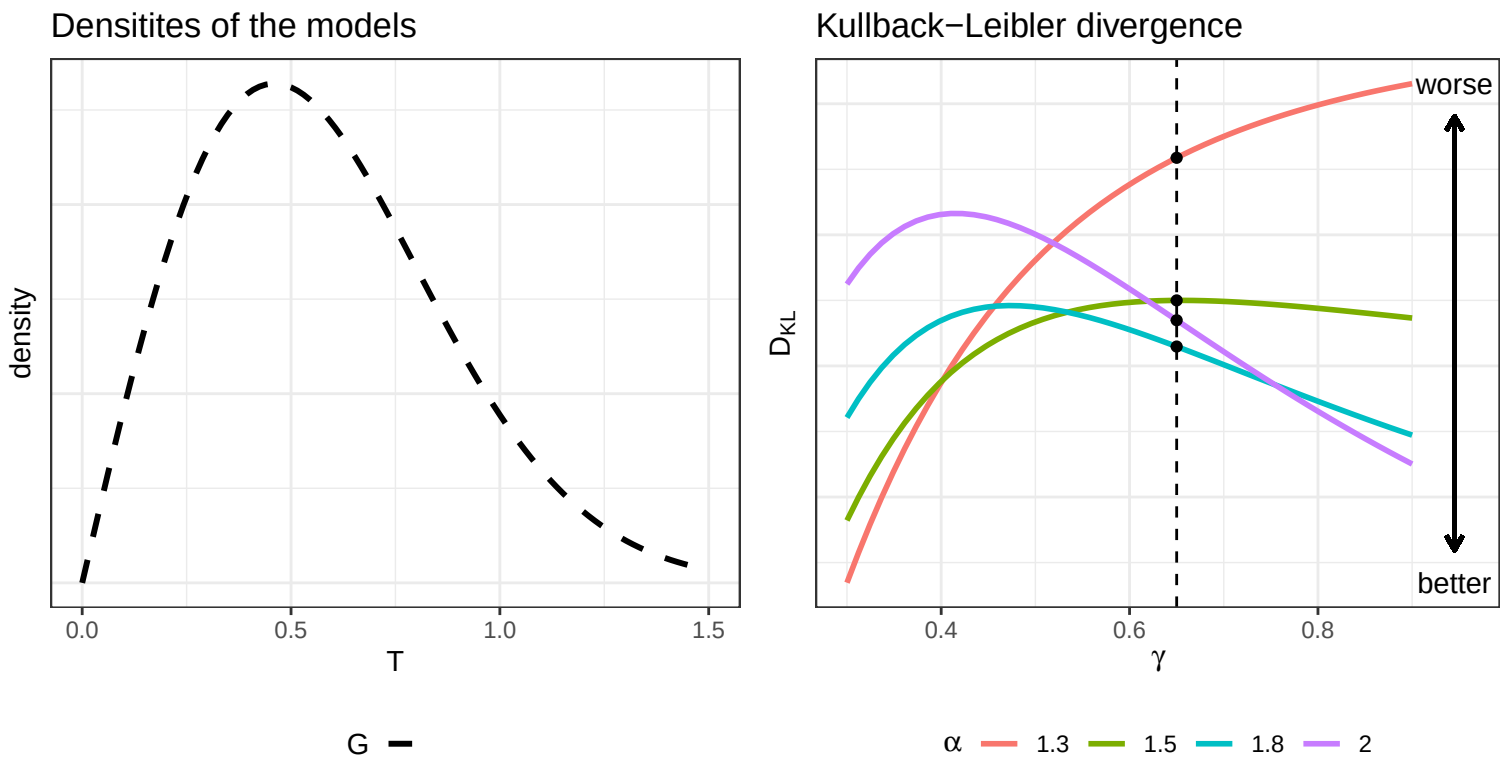

Survival curve estimators evaluated on hold-out samples

Consider the problem of selecting a hyper-parameter or model using cross-validation. We split the data $\mathcal{D}_n$ in two, $\mathcal{D}_n^{\text{train}}$ and $\mathcal{D}_n^{\text{test}}$.

On split $\mathcal{D}_n^{\text{train}}$ Fit models $\{\hat{f}_\lambda : \lambda \in \Lambda\}$ or $\{\hat{f}_1, \hat{f}_2, \ldots \hat{f}_k\}$.

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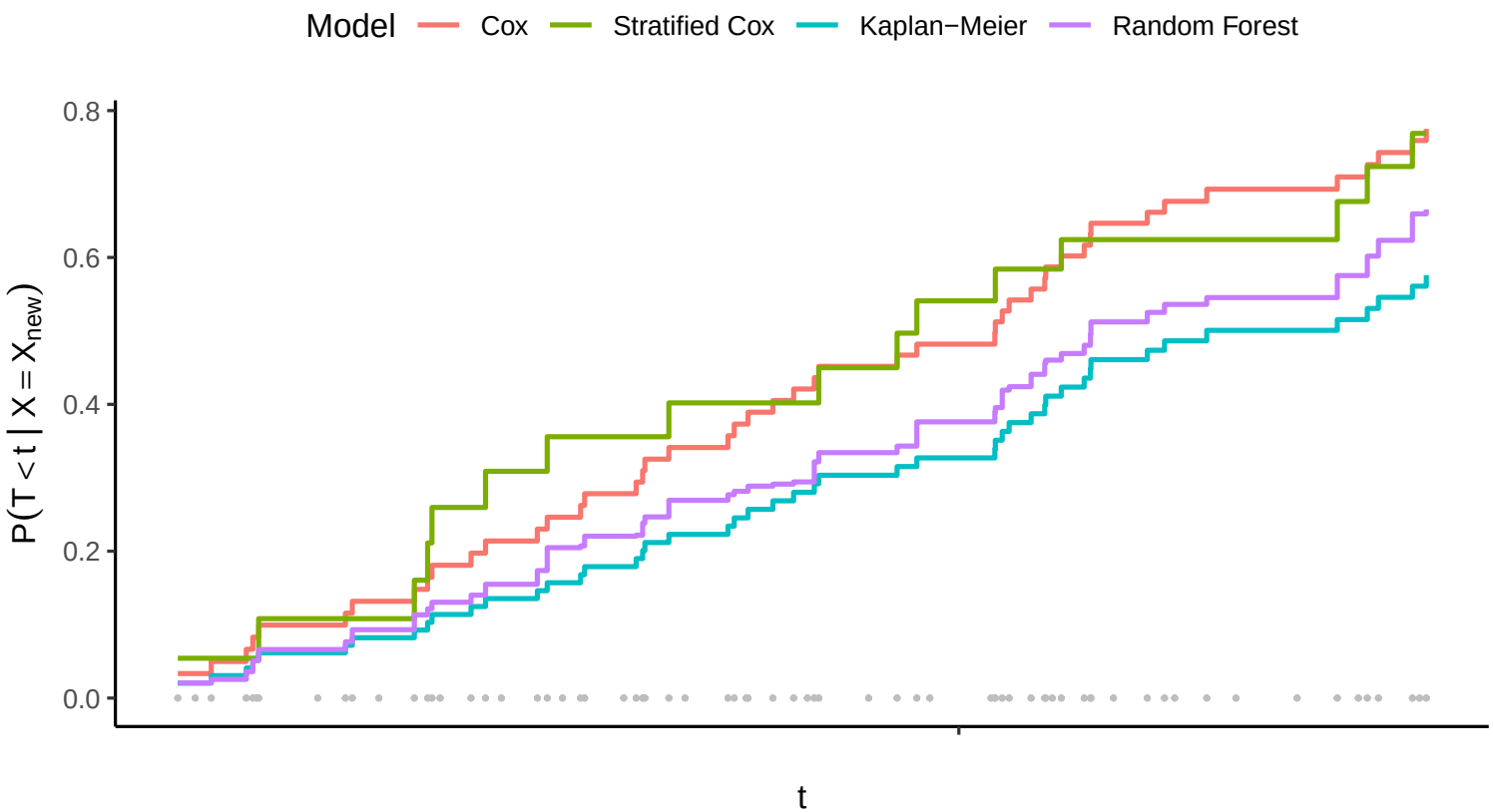

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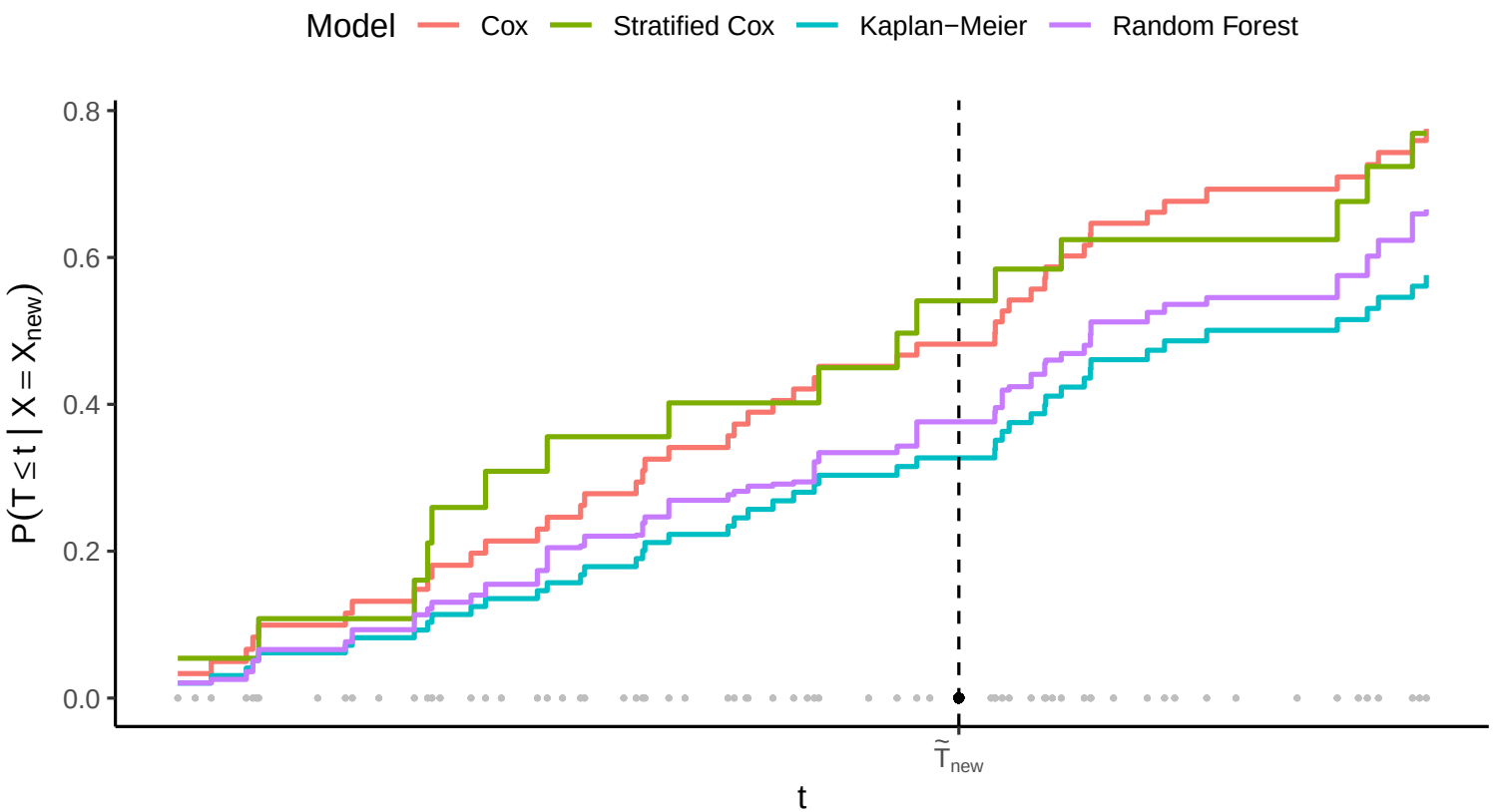

Taking the censoring distribution into account

To alliviate these problems problems we can reweight the observed outcome or the loss function to account/adjust for the censoring:

- Inverse probability of censoring weighted loss functions [Graf et al., 1999, Gerds and Schumacher, 2006, van der Laan and Dudoit, 2003]. For instance, weighted negative log-likelihood or (integrated) Brier score.
- Pseudo-values [Andersen et al., 2003, Mogensen and Gerds, 2013].
- Censoring unbiased transformations [Fan and Gijbels, 1996, Steingrimsson et al., 2019].

These approaches are particularly attractive when we are willing to assume that the censoring does not depend on the baseline covariates.

Modeling the censoring distribution

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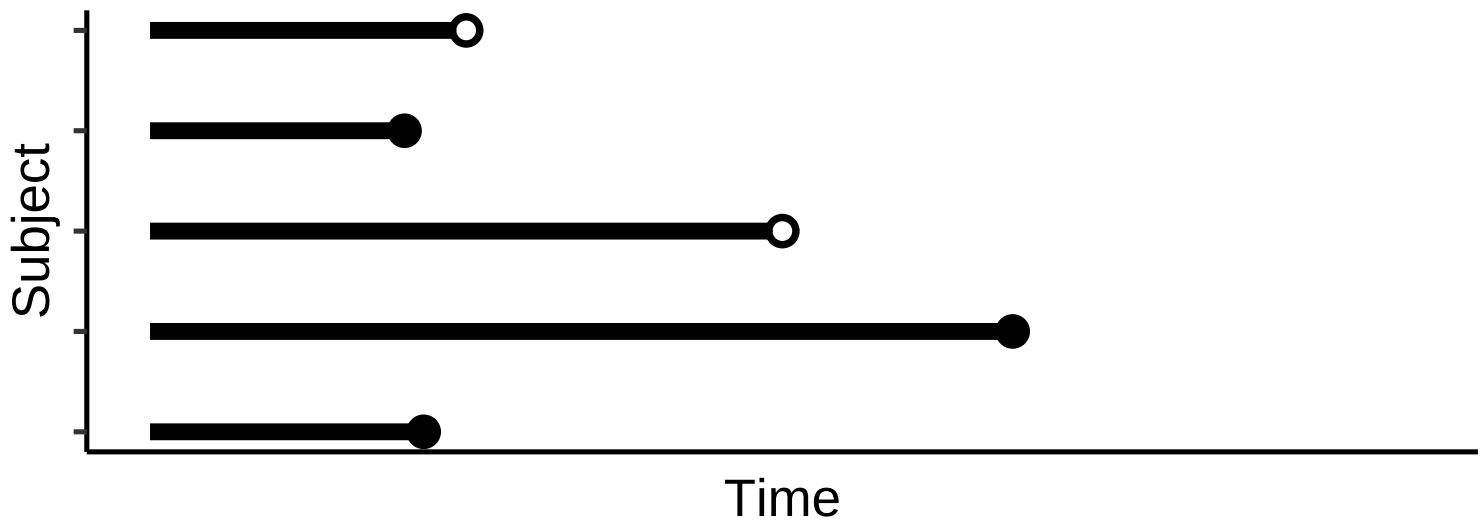
An (infinite?) loop



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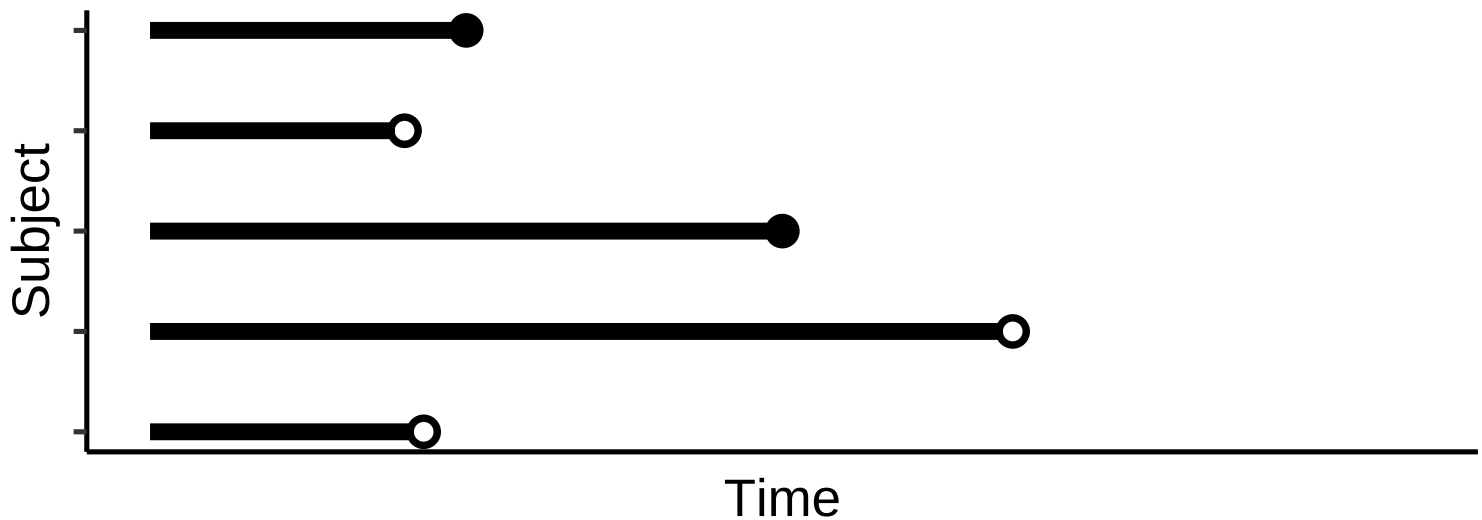

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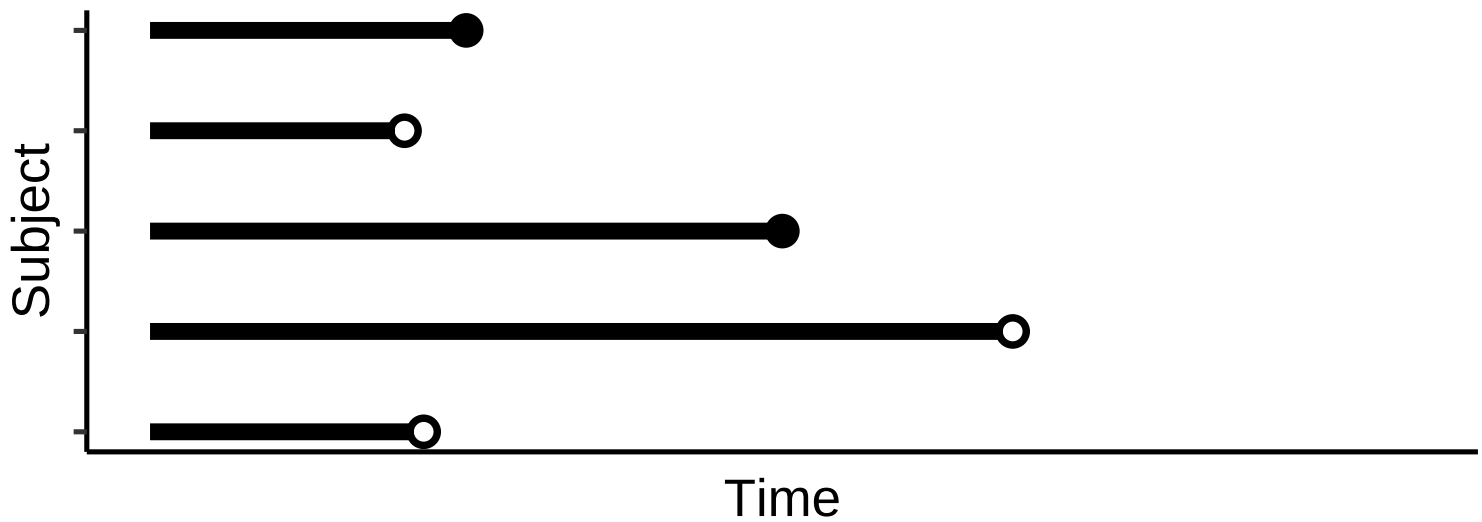

Iterate the estimation and hope for convergence [Han et al., 2021, Westling et al., 2021].

Conclusion

How should we do cross-validation for general survival models?

- Using the negative partial log-likelihood is problematic
  - → The least false model is not well-defined (without reference to the censoring regime)
  - → For many standard survival estimators, we cannot use it on hold-out samples
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Questions, comments, suggestions?

Thank you!

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