Influence functions and functional derivatives

Anders Munch

May 11, 2021

Outline

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- Do NOT write like this in your report!

A statistical problem

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Example (Average treatment effect)

We are given n iid. sample of $O \sim \mathrm{P}$, with $\mathrm{P} \in \mathcal{P}$ and where $O = (X, A, Y)$, with $X \in \mathbb{R}^d$, $A \in \{0, 1\}$, and $Y \in \{0, 1\}$. We want to estimate the average treatment effect

$$\mathbb{E}_{\mathrm{P}}\left[f(1, X) - f(0, X)\right],$$

with $f(a, x) := \mathbb{E}_{\mathrm{P}}\left[Y \mid A = a, X = x\right]$. The target parameter is

$$\Psi(\mathrm{P}) = \mathbb{E}_{\mathrm{P}}\left[f_{\mathrm{P}}(1, X) - f_{\mathrm{P}}(0, X)\right].$$

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Example (ATE)

The ATE can be written as $\Psi(\mathrm{P}) = \mathrm{P}[\varphi_1] = \mathrm{P}[\varphi_2] = \mathrm{P}[\varphi_3]$, for

$$\varphi_1(o; f) := f(1, x) - f(0, x),$$

$$\varphi_2(o; \pi) := \frac{a\,y}{\pi(x)} - \frac{(1-a)\,y}{1-\pi(x)},$$

$$\varphi_3(o; f, \pi) := \varphi_1(o; f) + \varphi_2(o; \pi) - \frac{a\,f(1, x)}{\pi(x)} + \frac{(1-a)\,f(0, x)}{1-\pi(x)}$$

$\mathrm{P}[\varphi]$ means

$$\mathrm{P}[\varphi] = \mathbb{E}_{\mathrm{P}}\left[\varphi(O)\right] = \int \varphi(o)\,\mathrm{dP}(o).$$

High-/infinite-dimensional nuisance parameters

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Trying to control for confounding $\implies$ nice to have:

- flexible model
- many covariates

Toy example: Integrated kernel density

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$$\Psi(\mathrm{P}) = \Psi_0(f) := \int_{-\infty}^{x} f(z)\,\mathrm{d}z, \quad \text{for} \quad \mathrm{P} = f \cdot \lambda,$$

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$$\hat{f}_n(x) = \hat{\mathbb{P}}_n[k_h(X, x)] = \frac{1}{n}\sum_{i=1}^{n} k_h(X_i, x),$$

where k_h is, e.g, $k_h(x, y) = g\left(\frac{x-y}{h}\right)$, with g the density for the standard Gaussian distribution, and the bandwidth h is chosen using cross-validation.

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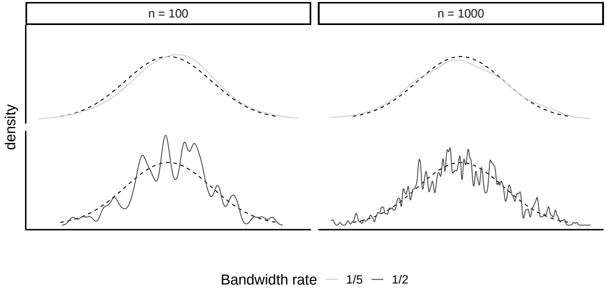

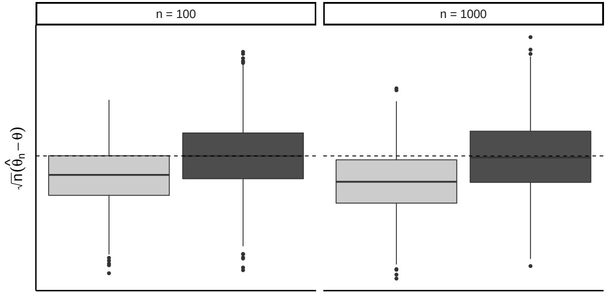

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with $\mathbb{G}_n := \sqrt{n}(\hat{\mathbb{P}}_n - \mathrm{P})$ the empirical process.

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$\mathbb{G}_n[\varphi(O, \hat{\nu}_n)]$ determines the (main) variance
$\Psi_0(\mathrm{P}, \hat{\nu}_n) - \Psi_0(\mathrm{P}, \nu)$ is bias!

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The decomposition then becomes

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(3) is specific to the nuisance estimator (and the functional Ψ). Importantly, the rate $\sqrt{n}\|\hat{\nu}_n - \nu\|_{\mathcal{V}} = o_{\mathrm{P}}(n^{-1/4})$ is sufficient.

Defining a functional derivative

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A linear approximation $\dot{\Psi}_x$ to the map Ψ at $x \in \mathcal{M}$, i.e.,

$$\left\| \Psi(x + \varepsilon_n h_n) - \Psi(x) - \dot{\Psi}_x(\varepsilon_n h_n) \right\| = \mathcal{O}(\varepsilon_n),$$

when $\varepsilon_n \to 0$.

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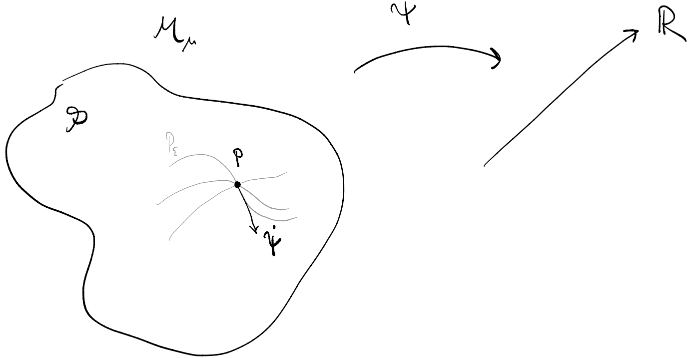
- Which norm on $\mathcal{M}$ should we use?
- In which space should we represent $\mathcal{P}$?

Pathwise Hadamard differentiability

Think of the gradient of a function defined on a manifold (surface).



Canonical gradient

Definition (Canonical gradient)

Let $(\mathcal{P}, \Psi)$ be a statistical problem, with $\mathcal{P} \subset \mathcal{M}_\mu$, and $\dot{\mathcal{P}}_{\mathrm{P}}$ the tangent space of $\mathcal{P}$ at $\mathrm{P} \in \mathcal{P}$. If $\Psi \colon \mathcal{P} \to \mathbb{R}$ is Hadamard differentiable at P tangential to $\dot{\mathcal{P}}_{\mathrm{P}}$, we refer to the Hadamard derivative $\dot{\Psi}_{\mathrm{P}}$ as the *canonical gradient of the statistical problem*.

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Characterizing property

With $\Gamma_{\mathrm{P}} := \overline{\mathrm{span}}\{\dot{\ell}_0\} \subset \mathcal{L}^2_{\mathrm{P}}$, where $\dot{\ell}_0 = \partial_0 \log(p_\varepsilon)$ is the score function of the sub-model P_ε, there exists a unique element $\varphi_{\mathrm{P}} \in \Gamma_{\mathrm{P}}$ such that

$$\partial_0 \Psi(\mathrm{P}_\varepsilon) = \langle \varphi_{\mathrm{P}}, \dot{\ell}_0 \rangle_{\mathrm{P}}$$

holds for any differentiable submodel P_ε with score function $\dot{\ell}_0$.

Canonical gradient for the ATE

Example (ATE)

When we make no assumptions about $\mathcal{P}$, the canonical gradient for the ATE problem

$$\begin{aligned}\varphi_{\mathrm{P}}(o; f, \pi) := f(1, x) - f(0, x) \\ + \frac{a\, y}{\pi(x)} - \frac{(1-a)\, y}{1-\pi(x)} \\ - \frac{a\, f(1, x)}{\pi(x)} + \frac{(1-a)\, f(0, x)}{1-\pi(x)} \\ - \Psi(\mathrm{P})\end{aligned}$$

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One way to show this is to first show that the tangent space Γ_{P} is the full subset $\mathbb{H}_0 \subset \mathcal{L}^2_{\mathrm{P}}$ of zero-mean functions, and then show that $\partial_0 \Psi(\mathrm{P}_\varepsilon) = \langle \varphi_{\mathrm{P}}, \dot{\ell}_0 \rangle_{\mathrm{P}}$ for all P_ε (see for instance Kennedy [2016]).

Neyman orthogonality

Theorem (Neyman orthogonality)

If $\Psi(\mathrm{P}) = \Psi_0(\mathrm{P}, \nu) = \mathrm{P}[\varphi(O, \nu(\mathrm{P}))]$ *and* $\varphi(\cdot, \nu) - \mathrm{P}[\varphi(O, \nu)]$ *is the canonical gradient of* $(\mathcal{P}, \Psi)$ *then* $\mathrm{D}_\nu \Psi_0 = 0$.

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Debiasing

The *first order* bias, coming from $\Psi_0(\mathrm{P}, \hat{\nu}_n) - \Psi_0(\mathrm{P}, \nu)$, is removed.

Efficiency

Definition (RAL estimators)

An estimator $\hat{\theta}_n$ of the parameter $\theta = \Psi(\mathrm{P})$ under the model $\mathcal{P}$, is called *asymptotically linear* with *influence function* $\mathrm{IF}(\cdot, \mathrm{P}) \in \mathcal{L}^2_{\mathrm{P}}$, if $\mathrm{P}[\mathrm{IF}(O, \mathrm{P})] = 0$ for all $\mathrm{P} \in \mathcal{P}$, and

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Theorem (Efficient influence function)

The RAL estimator with lowest possible asymptotic variance has the canonical gradient as its influence function.

Constructing estimators: Solve the efficient score equation

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&= \mathbb{G}_n[\varphi(O, \nu)] + \mathcal{o}_{\mathrm{P}}(1).
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Find a parametrization $\Psi(\mathrm{P}) = \mathrm{P}[\varphi(O, \nu)]$ such that φ is the (canonical) gradient. Then by Neyman orthogonality and assumptions we can write

$$\begin{aligned}
\sqrt{n}\left(\hat{\theta}_n - \theta\right) &= \mathbb{G}_n[\varphi(O, \hat{\nu}_n)] && = \mathbb{G}_n[\varphi(O, \nu)] \\
&\quad + \mathrm{D}_\nu \Psi_0\left[\sqrt{n}(\hat{\nu}_n - \nu)\right] && = 0 \\
&\quad + \mathcal{O}_{\mathrm{P}}(\sqrt{n}\|\hat{\nu}_n - \nu\|^2_{\mathcal{V}}) && = o_{\mathrm{P}}(1) \\
&= \mathbb{G}_n[\varphi(O, \nu)] + o_{\mathrm{P}}(1).
\end{aligned}$$

Hence $\hat{\theta}_n$ is a RAL estimator, and if $\varphi - \mathrm{P}[\varphi]$ is the canonical gradient it will be *asymptotically efficient*.

Constructing estimators: Solve the efficient score equation

Find a parametrization $\Psi(\mathrm{P}) = \mathrm{P}[\varphi(O,\nu)]$ such that φ is the (canonical) gradient. Then by Neyman orthogonality and assumptions we can write

$$
\begin{aligned}
\sqrt{n}\left(\hat{\theta}_n - \theta\right) &= \mathbb{G}_n[\varphi(O,\hat{\nu}_n)] && = \mathbb{G}_n[\varphi(O,\nu)] \\
&\quad + \mathrm{D}_\nu \Psi_0\left[\sqrt{n}(\hat{\nu}_n - \nu)\right] && = 0 \\
&\quad + \mathcal{O}_{\mathrm{P}}(\sqrt{n}\|\hat{\nu}_n - \nu\|_{\mathcal{V}}^2) && = o_{\mathrm{P}}(1) \\
&= \mathbb{G}_n[\varphi(O,\nu)] + o_{\mathrm{P}}(1).
\end{aligned}
$$

Hence $\hat{\theta}_n$ is a RAL estimator, and if $\varphi - \mathrm{P}[\varphi]$ is the canonical gradient it will be *asymptotically efficient*.

This is the approach taken in Chernozhukov et al. [2018]. See also Example 4.1 of the note.

TMLE

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