Algorithms Homework Assignment 2

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Conventions

When I refer to $\mathbb{N}$, I speak of

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

And, when I label a variable to be m or n, I am indicating that these variables take values only in $\mathbb{N}$.

Problem 4.5-1

Part a.

Our recursion equation is

$$T(n) = 2T(n/4) + 1$$

Then, in the context of the master theorem, we have $a = 2, \quad b = 4, \quad f(n) = 1$. Then, we can see clearly that $\log_4(2) = \frac{1}{2}$. If $\epsilon = \frac{1}{4}$, we see that $n^{\log_4 2 - \epsilon} = n^{\frac{1}{2} - \frac{1}{4}} = n^{\frac{1}{4}}$ and $0 \leq 1 \leq n^{\frac{1}{4}} \quad \forall n \geq 1$ so clearly $f(n) = O(n^{\frac{1}{4}})$ and

$$T(n) = \Theta(n^{\frac{1}{2}})$$

Part b.

Our recursion equation is

$$T(n) = 2T(n/4) + \sqrt{n}$$

Then, from the last problem, we know that $n^{\log_b(a)} = \sqrt{n}$ and, in this case, $f(n) = \sqrt{n}$ so clearly

$$\sqrt{n} = \Theta(\sqrt{n})$$

and therefore, by case 2 of the master theorem,

$$T(n) = \Theta(\sqrt{n} \log_2(n))$$

Part c.

Our recursion euation is

$$T(n) = 2T(n/4) + n$$

Once again, from our previous work, we know that $n^{\log_b(a)} = \sqrt{n}$. Then take $\epsilon = \frac{1}{4}$ and

$$n^{\frac{1}{2} + \epsilon} = n^{\frac{3}{4}}$$

Then, for $c = 1$ and $n_0 = 1$,

$$0 \leq n^{\frac{3}{4}} \leq n \quad \forall\, n \geq n_0$$

so $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ for some constant $\epsilon > 0$ and $a\, f(n/b) = \frac{a}{b}\, n = \frac{n}{2}$. and clearly, if $c = \frac{3}{4}$, then

$$\frac{1}{2}n \leq \frac{3}{4}n \quad \forall\, n \in \mathbb{N}$$

Therefore, by the third case of the master theorem,

$$T(n) = \Theta(n)$$

Part d.

Our recursion relation is the same as the above three but with $f(n) = n^2$. And, again, $n^{\log_b\, a} = \sqrt{n}$ and clearly with $\epsilon = 1$, $n^{\log_b(a)\,+\epsilon} = n^{\frac{3}{2}}$. Furthermore, for all $n \geq 2$,

$$0 \leq n^{\frac{3}{2}} \leq n^2$$

So $f(n) = \Omega(n^{\log_b(a)\,+\epsilon})$. Additionally, we can see that $a\, f(n/b) = a\,\frac{n^2}{b^2} = \frac{1}{8}n^2 \leq \frac{n^2}{2} \quad \forall\, n \in \mathbb{N}$. Then, by the third case of the master theorem,

$$T(n) = \Theta(n^2)$$

Problem 4-1

Part a.

Our recursion equation is

$$T(n) = 2T(n/2) + n^4$$

We see that $n^{\log_b(a)} = n$. Then quite clearly, if we take $\epsilon = 1$ then $n^4 = \Omega(n^{\log_2(2)\,+1}) = \Omega(n^2)$. Moreover, $a\, f(n/b) = \frac{1}{8}n^4 \leq \frac{1}{2}n^4 \quad \forall n \in \mathbb{N}$. Therefore by the master theorem,

$$T(n) = \Theta(n^4)$$

Part b.

Our recursion relation is

$$T(n) = T(7n/10) + n$$

Then $n^{\log_b(a)} = n^0 = 1$. And if we let $\epsilon = \frac{1}{2}$, then $n^{\log_b(a)\,+\epsilon} = \sqrt{n}$ and

$$\forall\, n \in \mathbb{N}\,,\, 0 \leq \sqrt{n} \leq n$$

So then $n = \Omega(n^{\log_b(a)\,+\epsilon}) = \Omega(\sqrt{n})$. Then $a\, f(n/b) = \frac{7}{10}n \leq \frac{8}{10}n \quad \forall\, n \in \mathbb{N}$. Therefore, by the master theorem,

$$T(n) = \Theta(n)$$

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Part c.

Our recursion relation is

$$T(n) = 16T(n/4) + n^2$$

Then $n^{\log_b(a)} = n^2$ and it is obvious that $n^2 = \Theta(n^2)$ so by the master theorem,

$$T(n) = \Theta(n^2 \log_2 n)$$

Part d.

Our recursion relation is

$$T(n) = 7T(n/3) + n^2$$

Then $n^{\log_b(a)} = n^{\log_3(7)}$. Note that $1 < \log_3(7) < 2$. In fact $\log_3(7) = 1.77124\ldots$. Then, for $\epsilon = 0.1$, $n^{\log_3(7)+0.1} = n^{1.87124\ldots}$ and for all $n \in \mathbb{N}$, $0 \leq n^{1.87124\ldots} \leq n^2$ so $n^2 = \Omega(n^{1.87124\ldots})$ and $a\,f(n/b) = \frac{7}{9}n^2 \leq \frac{8}{9}n^2 \quad \forall\, n \in \mathbb{N}$, so, by the master theorem,

$$T(n) = \Theta(n^2)$$

Part e.

Our recursion relation is

$$T(n) = 7T(n/2) + n^2$$

Then, $n^{\log_b(a)} = n^{\log_2(7)} \in (2,3)$. In fact, $n^{\log_2(7)} = n^{2.807\ldots}$. Then if $\epsilon = 0.1$, then $n^{\log_2(7)-0.1} = n^{2.707\ldots}$. Finally, using $c = 1$ and noticing that $0 \leq n^2 \leq n^{2.707\ldots} \quad \forall\, n \in \mathbb{N}$, we see that $n^2 = O(n^{\log_b(a)-\epsilon})$ and therefore that

$$T(n) = \Theta(n^{2.807\ldots})$$

Part f.

Our recursion relation is

$$T(n) = 2T(n/4) + \sqrt{n}$$

Then $n^{\log_b(a)} = \sqrt{n}$. Obviously $\sqrt{n} = \Theta(\sqrt{n})$ and therefore by the master theorem

$$T(n) = \Theta(\sqrt{n}\; lg\, n)$$

I solved this problem exactly earlier in the homework.

Part g.

Our recursion relation is

$$T(n) = T(n-2) + n^2$$

We cannot apply the master theorem to this situation because of the form of the recursion equation. Assuming that n is even for the moment, then $T(n) = n^2 + (n-2)^2 + (n-4)^2 + \cdots + (4)^2 + T(2)$. This is precisely

$$T(2) + \sum_{i=2}^{n/2} (2\,i)^2 = T(2) - 4 + \sum_{i=0}^{n/2} (2i)^2$$

but, noting our knowledge of series', this is

$$T(n) = T(2) - 4 + \frac{1}{6}(2(n/2)^3 + 3(n/2)^2 + n/2)$$

We note that, from this expression, we may only claim formally that $T(n) = \Omega(n)$ & $T(n) = O(n^3)$. If we tolerate some sloppiness, however, and consider only the term of the greatest asymptotic growth, we can state that

$$T(n) = \Theta(n^3)$$

We have here assumed n to be even, but the same argument may be applied directly to

$$T(1) - \sum_{i=2}^{(n-1)/2} (2i+1)^2$$

to acquire exactly the same result. Furthermore, we may have guessed correctly that a sum of squares is cubic in leading order, just as the integral of a quadratic is cubic in growth. My tolerance for sloppy analysis is justified in section 4.6 in the text.

References

I used this website to look–up some series identities that were used in solving recursion equations

https://en.wikipedia.org/wiki/List_of_mathematical_series