Notes on the Darboux Transformation

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Definitions

A **Q–form** is a second order differential operator of the form

$$\frac{d^2}{dx^2} + Q \; where \; Q : \mathbb{R} \to \mathbb{R}$$

We shall denote such an operator as $\hat{H}_Q$

Definitions

A **Darboux Transformation** is a differential operator of the form

$$\frac{d}{dx} + f \ \ where\ f : \mathbb{R} \to \mathbb{R}$$

When we are referring to some Darboux transformation, we shall refer to f as a **generating function** and we shall denote this operator as $\hat{D}_f$.

We shall consider the properties of Darboux Transformations as maps from the kernel of one Q–form to another.

Definitons

A Q–form $\hat{H}_q$ is said to be **Darboux related** to another Q–form $\hat{H}_Q$ if there exists some $f : \mathbb{R} \to \mathbb{R}$ so that

$$\frac{df}{dx} - f^2 - q = c \in \mathbb{R} \; and \; Q = q - 2\frac{df}{dx}$$

If this is the case, we also say that $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$ This is exactly equivalent to the case that

$$\frac{df}{dx} = -\frac{1}{2}(Q - q) \; and \; 2(Q - q)f = \frac{d}{dx}(Q + q)$$

Theorem: If $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$ then $\hat{H}_Q$ is Darboux related to $\hat{H}_q$ by $\hat{D}_{-f}$

Theorem: if $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$, then $\hat{H}_Q\hat{D}_f y = 0$ whenever $\hat{H}_q y = 0$

$$\begin{tikzcd} \ker \hat{H}_q \arrow[r, bend right, "\hat{D}_f"'] & \ker \hat{H}_Q \arrow[l, bend right, "\hat{D}_{-f}"'] \end{tikzcd}$$

(Big) Theorem: if $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$ then $\hat{H}_q y = 0$ if and only if

$$\hat{D}_{-f}\hat{D}_f y = c\, y$$

where $c = \frac{df}{dx} - f^2 - q$ which is fixed by the Darboux relation

Clearly, if we could force $c = 0$, then our problem is greatly simplified!

Example

Let λ be some yet undetermined fixed real number, and

$$\hat{H}_\lambda = \frac{d^2}{dx^2} - \frac{1}{2}x^2 - \lambda$$

. Define $\hat{H}_\mu$ similarly. We seek a Darboux transformation which will relate $\hat{H}_\lambda$ to $\hat{H}_\mu$ To this end, let us determine f.

Example

$$Q = -\tfrac{1}{2}x^2 - \mu \text{ and } q = -\tfrac{1}{2}x^2 - \lambda$$

Recall that f is determined by

$$\frac{df}{dx} = -\frac{1}{2}(Q - q) \; and \; 2(Q - q)f = \frac{d}{dx}(Q + q)$$

Beginning on the first equation, we have that

$$\frac{df}{dx} = \frac{1}{2}(\mu - \lambda) \implies f = \frac{1}{2}(\mu - \lambda)x$$

Then, taking our attention to the second equation, we have

$$2(\lambda - \mu)\,f = \frac{d}{dx}(-x^2 - \mu - \lambda) = -2\,x$$

$$f = \frac{2}{\mu - \lambda}x$$

Example

Evidently, then, if we are to achieve Darboux Relation

$$f = \frac{1}{2}(\mu - \lambda)x = \frac{2}{\mu - \lambda}x \implies (\mu - \lambda)^2 = 4$$

Then $f = x$. Now we must calculate c from

$$\frac{df}{dx} - f^2 - q = c$$

$$\frac{df}{dx} - f^2 - q = 1 - x^2 + x^2 + \lambda = 1 + \lambda$$

Example

Apparently $c = 1 + \lambda$. Then, if $\lambda = -1$, $c = 0$ and we may find a solution of $\hat{H}_{-1}$ by solving

$$\hat{D}_f y = \frac{dy}{dx} + f\,y = \frac{dy}{dx} + x\,y = 0$$

Which has the trivial solution

$$y = e^{-\frac{x^2}{2}}$$

Then, we have a solution of $\hat{H}_{2n-1}$ which is $\hat{D}^n_{-f}\ y$.

Questions?

If you would like to read my paper or see my references, it can be found at

`https://github.com/amo004/Math-Thesis`