Notes on the Darboux Transformation

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A **Q–form** is a second order differential operator of the form

$$\frac{d^2}{dx^2} + Q \ \mathit{where}\ Q : \mathbb{R} \to \mathbb{R}$$

We shall denote such an operator as $\hat{H}_Q$

Definitions

A **Darboux Transformation** is a differential operator of the form

$$\frac{d}{dx} + f \ \textit{where} \ f : \mathbb{R} \to \mathbb{R}$$

When we are referring to some Darboux transformation, we shall refer to f as a **generating function** and we shall denote this operator as $\hat{D}_f$.

We shall consider the properties of Darboux Transformations as maps from the kernel of one Q–form to another.

Definitons

A Q–form $\hat{H}_q$ is said to be **Darboux related** to another Q–form $\hat{H}_Q$ if there exists some $f : \mathbb{R} \to \mathbb{R}$ so that

$$\frac{df}{dx} - f^2 - q = c \in \mathbb{R} \; and \; Q = q - 2\frac{df}{dx}$$

If this is the case, we also say that $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$ This is exactly equivalent to the case that

$$\frac{df}{dx} = -\frac{1}{2}(Q - q) \; and \; 2(Q - q)f = \frac{d}{dx}(Q + q)$$

Basic Theorems

Theorem: If $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$ then $\hat{H}_Q$ is Darboux related to $\hat{H}_q$ by $\hat{D}_{-f}$

Theorem: if $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$, then $\hat{H}_Q\hat{D}_f y = 0$ whenever $\hat{H}_q y = 0$

$$\begin{tikzcd} \ker \hat{H}_q \arrow[r, bend right, "\hat{D}_f"'] & \ker \hat{H}_Q \arrow[l, bend right, "\hat{D}_{-f}"'] \end{tikzcd}$$

(Big) Theorem: if $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$ then $\hat{H}_q y = 0$ if and only if $\hat{D}_{-f}\hat{D}_f y = c\, y$ where $c = \frac{df}{dx} - f^2 - q$ which is fixed by the Darboux relation

Also, for the physicists out there, this is the foundation of supersymmetry and Darboux transformations preserve transmission and reflection in quantum mechanical systems.

Clearly, if we could force $c = 0$, then our problem is greatly simplified!

Example

Let λ be some yet undetermined fixed real number, and

$$\hat{H}_\lambda = \frac{d^2}{dx^2} - \frac{1}{2}x^2 - \lambda$$

. Define $\hat{H}_\mu$ similarly. We seek a Darboux transformation which will relate $\hat{H}_\lambda$ to $\hat{H}_\mu$ To this end, let us determine f.

Example

$$Q = -\tfrac{1}{2}x^2 - \mu \text{ and } q = -\tfrac{1}{2}x^2 - \lambda$$

Recall that f is determined by

$$\frac{df}{dx} = -\frac{1}{2}(Q - q) \; and \; 2(Q - q)f = \frac{d}{dx}(Q + q)$$

Beginning on the first equation, we have that

$$\frac{df}{dx} = \frac{1}{2}(\mu - \lambda) \implies f = \frac{1}{2}(\mu - \lambda)x$$

Then, taking our attention to the second equation, we have

$$2(\lambda - \mu)\,f = \frac{d}{dx}(-x^2 - \mu - \lambda) = -2\,x$$

$$f = \frac{2}{\mu - \lambda}x$$

Example

Evidently, then, if we are to achieve Darboux Relation

$$f = \frac{1}{2}(\mu - \lambda)x = \frac{2}{\mu - \lambda}x \implies (\mu - \lambda)^2 = 4$$

Then $f = x$. Now we must calculate c from

$$\frac{df}{dx} - f^2 - q = c$$

$$\frac{df}{dx} - f^2 - q = 1 - x^2 + x^2 + \lambda = 1 + \lambda$$

Example

Apparently $c = 1 + \lambda$. Then, if $\lambda = -1$, $c = 0$ and we may find a solution of $\hat{H}_{-1}$ by solving

$$\hat{D}_f y = \frac{dy}{dx} + f\, y = \frac{dy}{dx} + x\, y = 0$$

Which has the trivial solution

$$y = e^{-\frac{x^2}{2}}$$

Then, we have a solution of $\hat{H}_{2n-1}$ which is $\hat{D}^n_{-f}\ y$.

Questions?

If you would like to read my paper or see my references, it can be found at

https://github.com/amo004/Math-Thesis