Darboux Transformations

Andrew Osborne

Introduction and Background

There is a great deal of ongoing research in the symmetry of differential equations in physics and mathematics. In some respects, it makes a great deal of sense to think of differential equations as matrix equations over infinite–dimensional vector spaces. That is, in basic linear algebra courses, a great deal of time is spent considering matrix equations of the form $A\hat{x} = \hat{b}$ where $A \in \mathbb{R}^{n \times n}$, $\hat{x} \in \mathbb{R}^n$, and $\hat{b} \in \mathbb{R}^n$ for some $n \in \mathbb{N}$. Following this illustration, we say that $\hat{x}$ is an eigenvector of A if $A\hat{x} = \lambda\hat{x}$ for some $\lambda \in \mathbb{C}$ and $\hat{x} \neq \hat{0}$. The finite dimensionality of A imposes that there can be only finitely–many distinct eigenvalues. On the other hand, when we are considering differential equations, matrices generalize to operators and our eigenspace is no longer limited to a finite dimensionality. It is with these spaces and their beautiful symmetries that I concern myself.

Jean Gaston Darboux was a French contemporary of Joseph Fourier and Henri Poincare. Darboux's most famous work was in geometry, but he also contributed to the theory of linear differential equations (Darboux, 1882). Darboux transformations, as I shall call them, were first theorized by Darboux, but they have been rediscovered and generalized in a variety of contexts. In particular, Nobel Laureate Subrahmanyan Chandrasekhar rediscovered them as they apply to perturbation equations which govern the motion of particles around black holes (Chandrasekhar, 1983). Chandrasekhar lamented that the physics community at large did not appreciate the "mysterious" and strangely fundamental properties of the Darboux transformation. Darboux transformations are also fundamental to super–symmetry and they present simple solution methods to a special class of problems, though the uses of Darboux transformations are not limited to the cases explored here.

Darboux transformations are, in fact, linear transformations on the solution spaces of differential equations, but it is not, at this time, clear what is meant by "transformation". In the case where $\frac{d^2y}{dx^2} + qy = 0$, with $y : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$, a Darboux transformation creates a solution, Y, to the equation $\frac{d^2Y}{dx^2} + QY = 0$ for some other function $Y : \mathbb{R} \to \mathbb{R}$ and some other function $Q : \mathbb{R} \to \mathbb{R}$. We could refer to the Darboux transformation as a relationship between y and Y, but the existence of such a transformation and its properties depend only on the form of q and Q. In this way, it is clear that, when we refer to a "transformation", we are talking about a differential operator which turns solutions corresponding to q into solutions corresponding to Q. If such a Darboux transformation exists, we say that q and Q are *Darboux related*. The purpose of this writing is to explore some of the properties of Darboux related potentials and to demonstrate a simple soltion method.

Definitions and Derivations

In this exposition, we shall limit ourselves to particularly amicable cases.

Definition. *$\hat{H}_V = \frac{d^2}{dx^2} + V$ is called the Q–form with potential $V : \mathbb{R} \to \mathbb{R}$. We say that $y : \mathbb{R} \to \mathbb{R}$ solves $\hat{H}_V$ if $\frac{d^2y}{dx^2} + Vy = 0$ and we write $\hat{H}_V y = 0$.*

Notice that Q–forms govern a great portion of interesting physical phenomena (e.g. simple pendulum, mass on a spring). In some applications, it is not the case that the equation of interest is explicitly a Q–form, but rather that it is easily manipulated into one. Using the AC circuit with an inductor and a capacitor as an example, our equation is

$$L\frac{d^2Q}{dt^2} + C\,Q = 0$$

In order to facilitate manipulation as a Q–form with Darboux transformations, we simply write an equivalent equation

$$\frac{d^2Q}{dt^2} + \frac{C}{L}\,Q$$

which implies that this equation is a Q–form with potential $\frac{C}{L}$ and is solved by $Q(t)$.

The chief subject of this paper is the mechanism by which we may construct and use differential operators to elucidate and exploit the rich symmetry of Q–forms. In pursuit of this goal, we have a definition.

Definition. *We say that $\frac{d}{dx} + f$ is a Darboux transformation with generator $f : \mathbb{R} \to \mathbb{R}$ and we write such a transformation as $\hat{D}_f$, and we use the notation $\hat{D}_f y$ to denote $\frac{dy}{dx} + fy$.*

Now, referring to my earlier clarificaition of the word "transformation", we say that a potential $q : \mathbb{R} \to \mathbb{R}$ is Darboux related to a potential $Q : \mathbb{R} \to \mathbb{R}$ if there exists $f : \mathbb{R} \to \mathbb{R}$ such that $\hat{H}_Q \hat{D}_f y = 0$ whenever $y : \mathbb{R} \to \mathbb{R}$ and $\hat{H}_q y = 0$. In this case, it is clear that $\hat{D}_f y$ solves $\hat{H}_Q$.

Existence of Darboux Transformation

Lemma. *A potential, $q : \mathbb{R} \to \mathbb{R}$ is Darboux related to $Q : \mathbb{R} \to \mathbb{R}$ by $\hat{D}_f$ where $f : \mathbb{R} \to \mathbb{R}$ if $\frac{df}{dx} - f^2 - q = c \in \mathbb{R}$ and $Q = q - 2\frac{df}{dx}$.*

Proof. Suppose that there is exist functions $y : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ such that $\hat{H}_q y = 0$ and $\hat{D}_f y = \frac{dy}{dx} + fy$. Then, differentiating this equation, we have

$$\begin{aligned}
\frac{d}{dx}\hat{D}_f y &= \frac{d^2y}{dx^2} + \frac{df}{dx}y + f\frac{dy}{dx} \text{ and substituting} \\
\frac{d^2y}{dx^2} &= -qy, \text{ we have} \\
\frac{d}{dx}\hat{D}_f y &= f\frac{dy}{dx} + (\frac{df}{dx} - q)y, \text{ and differentiating again, we have} \\
\frac{d^2}{dx^2}\hat{D}_f y &= f\frac{d^2y}{dx^2} + (2\frac{df}{dx} - q)\frac{dy}{dx} + (\frac{d^2f}{dx^2} - \frac{dq}{dx})y \\
&= (2\frac{df}{dx} - q)\frac{dy}{dx} + (\frac{d^2f}{dx^2} - \frac{dq}{dx} - fq)y + 2f\frac{df}{dx}y - 2f\frac{df}{dx}y \\
&= (2\frac{df}{dx} - q)\frac{dy}{dx} + y\frac{d}{dx}(\frac{df}{dx} - q - f^2) + (2\frac{df}{dx} - q)fy
\end{aligned}$$

but $\frac{df}{dx} - f^2 - q = c \in \mathbb{R}$ so $\frac{d}{dx}(\frac{df}{dx} - f^2 - q) = 0$, and we have

$$\begin{aligned}
\frac{d^2}{dx^2}\hat{D}_f y &= (2\frac{df}{dx} - q)(\frac{dy}{dx} + fy) \\
\frac{d^2}{dx^2}\hat{D}_f y &= (2\frac{df}{dx} - q)\hat{D}_f y
\end{aligned}$$

Finally, we have that $\frac{d^2}{dx^2}\hat{D}_f y + (q - 2\frac{df}{dx})\hat{D}_f y = 0$, so then, letting $Q = q - 2\frac{df}{dx}$ we have $\hat{H}_Q \hat{D}_f y = 0$ and therefore q is Darboux related to Q. $\square$

This result gives a sufficient condition for Darboux relation between two potentials, but the statement seems to suggest that, in some sense, Q was produced by f and q rather than f being produced by Q and q as I have promised earlier. As it turns out, the conditions

$$\frac{df}{dx} - f^2 - q = c \in \mathbb{R} \qquad \textit{and} \qquad Q = q - 2\frac{df}{dx} \tag{1}$$

are exactly equivalent to the agreement of the latter two conditions. (Glampedakis,2017)

$$\frac{df}{dx} = -\frac{1}{2}(Q - q) \qquad \textit{and} \qquad 2(Q - q)f = \frac{d}{dx}(Q + q) \tag{2}$$

It should be noted that the (1) is a Ricatti equation and there is no known method by which to find solutions without *a priori* knowledge of the system at hand. Interestingly, we can find some soltion by assuming that f takes the form $-\frac{1}{u}\frac{du}{dx}$ for some $u : \mathbb{R} \to \mathbb{R}$. In this case, the Ricatti equation in (1) becomes

$$-(\frac{du}{dx})^2 \frac{1}{u^2} + \frac{d^2u}{dx^2}\frac{1}{u} + (\frac{du}{dx})^2 \frac{1}{u^2} - q = c \in \mathbb{R}$$

This is equivalently $\frac{d^2u}{dx^2} - (q + c)u = 0$. Apparently, the solution of a Q–form is sufficient to determine a generator between Q–forms.

Then, the natural question arises: Is Darboux relation an equivalence relation on the set of Q–form differential operators, or on the set of real–valued functions? The answer is a resounding no. If you let $Q = q$ then the second condition in (1) implies that $\frac{df}{dx} = 0$ and therefore that f is a constant. Then the first condition implies that $-q = c + f^2$ for some $c \in \mathbb{R}$. This restriction that q be a real number is the statement that the Darboux relation is reflexive only on constant potentials, and in any other case, we are forbidden from reflexive Darboux relations. Indeed, the indeterminate division by zero indicates an infinite number of Darboux transformations, rather than a unique Darboux transformation. Take $f : \mathbb{R} \to \mathbb{R}$ to be a constant, and see that $\frac{d^2y}{dx^2} + c_0 y = 0$ implies that $\frac{d^3y}{dx^3} + c_0\frac{dy}{dx} = \frac{d}{dx}0 = 0$ and $f(\frac{d^2y}{dx^2} + c_0 y) = f * 0 = 0$ for all $c_0 \in \mathbb{R}$. Then any constant function on $\mathbb{R}$ generates a Darboux transformation from a Q–form with a constant potential to itself, and in no other situations is the Darboux relation reflexive, so Darboux relation is not an equivalence relation. It also lacks transitivity, but that is a more involved demonstration which is not worth the space. On the other hand, the Darboux relation is symmetric.

Symmetry of Darboux Relation

Theorem. *If a potential, $q : \mathbb{R} \to \mathbb{R}$ is Darboux related to a potential $Q : \mathbb{R} \to \mathbb{R}$ by $\hat{D}_f$, then Q is Darboux related to q by $\hat{D}_{-f}$.*

$$\begin{tikzcd} \ker \hat{H}_q \arrow[r, bend right, "\hat{D}_f"'] & \ker \hat{H}_Q \arrow[l, bend right, "\hat{D}_{-f}"'] \end{tikzcd}$$

Proof. Suppose that $\hat{D}_f$ relates q to Q. Then, we have that $\frac{df}{dx} - f^2 - q = c$ where $c \in \mathbb{R}$ is an arbitrary but fixed constant. We also must have that $Q = q - 2\frac{df}{dx}$. Then, $-\frac{df}{dx} - (-f)^2 - Q = -\frac{df}{dx} - f^2 - q + 2\frac{df}{dx} = \frac{df}{dx} - f^2 - q = c$ by hypothesis. On the other hand, if we accept that $Q = q - 2\frac{df}{dx}$, then this is $q = Q - 2\frac{d}{dx}(-f)$ $\square$

Theorem. *$\hat{H}_q$ has the same solutions as $\hat{D}_{-f}\hat{D}_f - c$ where c is the constant imposed by the Ricatti equation in* (1).

Proof. Suppose $y : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ generates a Darboux Transformation between q and $Q : \mathbb{R} \to \mathbb{R}$. Then suppose y is a subjected to $(\hat{D}_{-f}\hat{D}_f - c)$. Then this is

$$\frac{d^2y}{dx^2} + (\frac{df}{dx} - f^2 - q)y - cy + qy = \hat{H}_q y$$

Evidently, $(\hat{D}_{-f}\hat{D}_f - c)y = \hat{H}_q y$ in general and therefore $(\hat{D}_{-f}\hat{D}_f - c)y = 0$ if and only if $\hat{H}_q y = 0$. The result above does not depend on the form of the target potential, and can be achieved with only one of the conditions in (1). That is to say that, if we can solve the Ricatti equation, then, as we will see, we may trivially ascertain a solution of $\hat{H}_q$. Note that if we apply $\hat{D}_f$ to the left of this identity, then we obtain

$$(\hat{D}_f\hat{D}_{-f} - c)\hat{D}_f y = 0 \iff \hat{H}_Q\hat{D}_f y = 0$$

This result depends explicitly on the form of Q. It is this property which will later allow us to use Darboux transformations as "ladder" operators. $\square$

Corollary. *If $f : \mathbb{R} \to \mathbb{R}$ generates a transformation between $q : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}$, and $y : \mathbb{R} \to \mathbb{R}$, then $\hat{H}_q\, y = 0$ if and only if $\hat{D}_{-f}\hat{D}_f\, y = c\, y$ where $c \in \mathbb{R}$ is the constant imposed by the Ricatti equation in* (1)

Proof. Using the last theorem, it is clear that $(\hat{D}_{-f}\hat{D}_f - c)\, y = 0$ if and only if $\hat{H}_q = 0$, and therefore $\hat{D}_{-f}\hat{D}_f\, y = c\, y$ if and only if $\hat{H}_q\, y = 0$. $\square$

From this, it follows very clearly that $\hat{D}_f : \ker \hat{H}_q \to \ker \hat{H}_Q$ is surjective onto $\ker \hat{H}_Q$, and in turn $\hat{D}_{-f}$ is surjective onto $\ker \hat{H}_q$. From these facts follows very clearly a method of showing that two potentials are not Darboux related.

Applications

Then, in the interest of further applications, I shall define the set of valid quantum mechanical wave functions.

Definition. *The set of quantum mechanical wave functions of one variable, $\mathcal{Q}$ is defined in the following way. A function, $f : \mathbb{R} \to \mathbb{R} \in \mathcal{Q}$ iff $f \in L^2$ and* $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$.

Simple Harmonic Oscillator

To illustrate the usefulness of the Darboux transformation and to apply it more carefully to an example, I will now define the equation governing a quantum mechanical simple harmonic oscillator as follows.

$$\hat{H}_\lambda \equiv \frac{d^2}{dx^2} + q_\lambda$$

Where $q_\lambda = -x^2 - \lambda$ and $\lambda \in \mathbb{R}$. The simple harmonic oscillator has a countable set of eigenvalues, which can be labeled in a convenient way. I have not as of yet given any reason to think that a single $f : \mathbb{R} \to \mathbb{R}$ can be found to generate multiple eigenvectors of a general q–form but, in some cases, it can be done. That is, if $y : \mathbb{R} \to \mathbb{R}$, and $\hat{H}_\lambda y = 0$, and there exists some $f : \mathbb{R} \to \mathbb{R}$ so that $\hat{H}_\mu \hat{D}_f y = 0$, it is not true in general that there is some κ so that $\hat{H}_\kappa (\hat{D}_f)^2\, y = 0$, but, as we will see, the Darboux transformation is useful with and without this condition.

On the other hand, if there does exist λ, μ, κ and $f : \mathbb{R} \to \mathbb{R}$ so that $\hat{H}_{q_\lambda}\, y = 0$, $\hat{H}_{q_\mu} \hat{D}_f y = 0$, and $\hat{H}_{q_\kappa} (\hat{D}_f)^2\, y = 0$, then it is not necessarily the case that there is a Darboux transformation exists between the kernels of $\hat{H}_{q_\lambda}$ and $\hat{H}_{q_\kappa}$. Certainly, if such a Darboux transformation exists, it is not $\hat{D}_f$. Notice also that $\hat{D}_f^2$ is not necessarily a Darboux transformation at all! This raises some questions about labeling: if we are to label eigenvalues, such labeling must in some sense arise naturally from the Darboux transformation itself.We seek to find all solutions of this equation which are members of $\mathcal{Q}$. Then we must Darboux relate $\hat{H}_\lambda$ to $\hat{H}_\mu$. Now using (2), we know that $q_\mu - q_\lambda = -\mu + \lambda$ and therefore that $\frac{df}{dx} = \frac{1}{2}(\mu - \lambda) \implies f = \frac{1}{2}(\mu - \lambda)x$. We must also have that $q_\lambda + q_\mu = -x^2 - \mu - \lambda$ and therefore that $\frac{d}{dx}(q_\lambda + q_\mu) = -4x$ and then that $f = 2\frac{x}{\mu - \lambda}$. Then we have

$$f = \frac{1}{2}(\mu - \lambda)x = \frac{2}{\mu - \lambda}x$$

. And if we are to have agreement between these two equations, and Darboux relation, we must also have

$$(\frac{1}{2}(\mu - \lambda))^2 = 1 \implies \frac{1}{2}(\mu - \lambda) = \pm 1$$

In the interest of verification, we see that, with $f = x$, and $q_\lambda = -x^2 - \lambda$, we have that

$$\frac{df}{dx} - f^2 - q = 1 - x^2 + x^2 + \lambda = c = 1 + \lambda \in \mathbb{R}$$

Which is to say that if $\hat{H}_\lambda$ is Darboux related to $\hat{H}_\mu$, then μ and λ are separated by 2. Note that one could also have worked out the value of c by using $\hat{D}_{-f}$ which would yield

$c = \lambda - 1$. This is a mathematical manifestation of the equal spacing of the energy levels of the simple harmonic oscillator, which is a computationally nontrivial result acquired very easily with Darboux transformations. We also note that, while the transformations between related eigenvectors are unaffected by the eigenvalues themselves, the separation constant, c, is.

This is also an interesting result because we have shown that we can use a single Darboux transformation to transform between any two eigenvectors whose eigenvalues differ by two. Now, to solve this differential equation we must only observe the fact that this relation is easily soluble in the case where $c = 0$ (when $\lambda = -1$):

$$\hat{D}_{-f} \hat{D}_f \, y = c \, y$$

Notice that $\hat{D}_f y = 0 = \frac{dy}{dx} + x \, y$ has a unique solution, which is $y = e^{-\frac{1}{2}x^2}$. Then, by our previous theorem, $\hat{H}_{-1} e^{-\frac{1}{2}x^2} = 0$. In general $\hat{H}_{(-2n+1)} (\hat{D}_{-f})^n e^{-\frac{1}{2}x^2} = 0$, or more clearly

$$\frac{d^2 y_n}{dx^2} - x^2 \, y_n = -(2n+1) \, y_n \; : \; y_n = (\hat{D}_{-f})^n e^{-\frac{1}{2}x^2}$$

Moreover, notice that $y_{n+1} = \frac{dy_n}{dx} - x y_n$, which is precisely the recursion relation satisfied by the Hermite polynomials. Then evidently, the nth solution of our simple harmonic oscillator is a constant multiple of the nth Hermite polynomial multiplied by a Gaussian distribution. We could also repeat this process by using $\hat{D}_{-f}$ to yield another set of solutions which can be naturally indexed, but which are not elements of $\mathcal{Q}$.

Using this knowledge, we can find a general solution to the time–dependent Schrodinger equation which governs the Harmonic Oscillator.

$$-2i \frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2} - x^2 \varphi \tag{3}$$

With the above analysis in mind, and noticing that this equation is simply separated, we can clearly see that

$$\varphi = \sum_{j=0}^{\infty} c_j \, e^{i\lambda_j t/2} (\hat{D}_{-f})^j \varphi_0$$

is a general solution of (3) where $\hat{D}_{-f} \, \varphi_0 = 0$ with $f = -x$ and $\lambda_j = 1 - 2j$, and $c_j \in \mathbb{C} \; \forall \; j \in \mathbb{N}$. Though I will not demonstrate it here, these solutions span the solution space of the simple harmonic oscillator in $\mathcal{Q}$, and this mathematical construction is the foundation of quantum field theory. Interestingly, one can also apply a Fourier transform to the nth solution of the simple harmonic oscillator, and the Darboux relation is preserved. In the language of physics, Darboux relation is preserved under change of representation.

Radial Oscillator

We shall extend the previous example slightly. We define $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in the typical Cartesian coordinates. Then we consider

$$\hat{H}_\lambda \equiv \nabla^2 - (x^2 + y^2 + z^2) - \lambda$$

which can very clearly be separated into

$$\hat{X}_\mu \equiv \frac{\partial^2}{\partial x^2} - x^2 - \mu \quad , \quad \hat{Y}_\nu \equiv \frac{\partial^2}{\partial y^2} - y^2 - \nu \quad \text{and} \quad \hat{Z}_\gamma \equiv \frac{\partial^2}{\partial z^2} - z^2 - \gamma$$

where $\mu + \nu + \gamma = \lambda$ and we can clearly, by previous results, find $f(x) : \mathbb{R} \to \mathbb{R}\,,\, g(y) : \mathbb{R} \to \mathbb{R}\, and\, h(z) : \mathbb{R} \to \mathbb{R}$ such that

$$\hat{D}^x_{f(x)} \equiv \frac{\partial}{\partial x} + f(x) \quad , \quad \hat{D}^y_{g(y)} \equiv \frac{\partial}{\partial y} + g(y) \quad \text{and} \quad \hat{D}^z_{h(z)} \equiv \frac{\partial}{\partial z} + h(z)$$

are valid Darboux transformations on $\hat{X}_\mu$, $\hat{Y}_\nu$ and $\hat{Z}_\gamma$ respectively with eigenvectors given by $(\hat{D}^x_{f(x)})^l\, e^{-\frac{1}{2}x^2}$, $(\hat{D}^y_{g(y)})^m\, e^{-\frac{1}{2}y^2}$, and $(\hat{D}^z_{h(z)})^n\, e^{-\frac{1}{2}z^2}$ with eigenvalues given by $-(2l+1)$, $-(2m+1)$, and $-(2n+1)$. Once again, clearly,

$$\varphi_{lmn}(x,y,z) = (\hat{D}^x_{f(x)})^l (\hat{D}^y_{g(y)})^m (\hat{D}^z_{h(z)})^n e^{-\frac{1}{2}(x^2+y^2+z^2)}$$

is a solution to $\hat{H}_\lambda \;:\; \lambda = -2(l+m+n) - 3$. The same argument as above can be repeated to demonstrate orthogonality and a general solution to the time–dependent Schrodinger equation. This is an interesting example because it demonstrates that Darboux transformations can be useful beyond 1–D quantum mechanics, and that, while I have imposed significant restrictions (e.g. nonrepeating eigenvalues), these restrictions are not strict requirements for the applicability of Darboux transformations. That is, so long as the sum of l,m, and n is constant, λ is constant, and the multiplicity of λ as l, m, and n grow is unbounded. This "degeneracy" is most elegantly explained by a two–fold rotation symmetry in our potential, which depends only on the distance from a point to the origin and not to its polar or azimuthal orientation. In general, multi–variate potentials with purely radial potentials. This problem is also analytically solvable in spherical coordinates and the solutions involves the spherical harmonics and the Laguerre polynomials, rather than Gaussians and Hermite polynomials, but strictly speaking, neither of these solutions admit treatment with Darboux transformations before a separability argument. In general, an increase in dimensionality obscures the symmetry that Darboux transformations exploit. In the context of quantum mechanics, derivatives are replaced by momentum operators which are Hermitan and complex, and we, in practice, do not require a purely real values in any of the places which I have so far. This allows a more elegant notation, where $\hat{D}_f$ and $\hat{D}_{-f}$ are represented instead by $\hat{D}_f$ and $\hat{D}_f^\dagger$ and the differentiation changes sign rather than f. A more thorough set of examples can be found in §2.6 of (Sasaki, 2014).

Conclusions

I have here written about Darboux transformations, but in the present examples, the distinction between a Darboux transformation and a Backlund transformation is essentially a change of sign. I have, in some sense, done this for convenience's sake, and in the context of physics, the distinction between a Backlund transformation and a Darboux transformation is seldom made, and Backlund transformations are much more frequently used.

In the examples which I have used, and, indeed in the majority of mathematical and physical literature, the utility of Darboux transformations is in generating solutions to seemingly complicated differential equations through the use of a single trivial solution as in (Sasaki, 2014).

In general relativity, on the other hand, Darboux transformations are used to illucidate symmetries. In (Chandrasekhar, 1982), Chandrasekhar used a Darboux transformation to relate the radial and axial perturbation equations of an orbit about a Swarzchild black hole. This is essentially the statement that the deviation of a particle from its unperturbed orbit has many properties which are a consequence of the geometry imposed by the black hole rather than of the nature of any perturbation. The transformation which demonstrates this fact is called the Chandrasekhar transformation. Chandrasekhar, however, was seemingly unaware of the work of Darboux, and he lamented in (Chandrasekhar, 1982) that his transformation and its implications for symmetry had not been thorougly explored. In (Glampedakis, 2017) it was shown that the Chandrasekhar transformation is indeed a Darboux transformation. Indeed, the derivation of the Chandrasekhar transformation is much simpler with Darboux transformations in mind than with Chandrasekhar's original calculation.

In general relativity, there are a great number of quantities for which calculation requires an integration of the potential of a Q–form which is often a rational function of polynomials. These integrations sometimes admit closed–form solutions but regularly require numerical integration. That is, we must repeatedly calculate

$$\int_{\mathbb{R}^+} V(r)\, dr \quad \text{where} \quad V(r) = \sum_{i=m}^{n} \frac{c_i}{r^i} \quad \text{and} \quad c_i \in \mathbb{C}\, \forall\, i \in \mathbb{Z}$$

This is no complicated task, but some $V(r)$ are difficult to integrate because of the order of the leading term of $\frac{1}{r}$. Thus it would be beneficial if one could find a way to change the order of the leading term of $V(r)$ in a way which also preserves the nature of the solutions of the corresponding Q–form. It is for this problem that Darboux transformations have been used in (Glampedakis, 2017). In fact, a generalization of the Darboux transformation (which is admittedly less accessible) can be used to accomplish this task and is not thoroughly understood by the physics and mathematics communities.

Darboux transformations are also a current topic of research interest in mathematical physics, though interest in them has declined with the interest in string theory. From the perspective of an applied mathematician or theoretical physicist, Darboux transformations can be used to solve some complicated linear and non–linear differential equations, and also to provide food for thought into the nature of differential equations and their algebraic structure.

References(APA)

Darboux, J. G., & Rosu, H. C. (1882). On a Proposition Relative to Linear Equations. Acad. Sci., 94, pp. 1456-1459

Chandrasekhar, S. (1982). The Mathematical Theory of Black Holes. Oxford, UK: Oxford University Press.

Sasaki, R. (2014). Exactly Solvable Quantum Mechanics. The Universe, 2, pp. 2–32

Glampedakis, K., (2017). The Darboux transformation in black hole perturbation theory. Phys. Rev. D, 96, pp. 1 – 13