Notes on the Darboux Transformation

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Definitions

A **Q–form** is a second order differential operator of the form

$$\frac{d^2}{dx^2} + Q \textit{ where } Q : \mathbb{R} \to \mathbb{R}$$

We shall denote such an operator as $\hat{H}_Q$

Definitions

A **Darboux Transformation** is a differential operator of the form

$$\frac{d}{dx} + f \ \textit{where} \ f : \mathbb{R} \to \mathbb{R}$$

When we are referring to some Darboux transformation, we shall refer to f as a **generating function** and we shall denote this operator as $\hat{D}_f$.
We shall consider the properties of Darboux Transformations as maps from the kernel of one Q–form to another.

Definitons

A Q–form $\hat{H}_q$ is said to be **Darboux related** to another Q–form $\hat{H}_Q$ if there exists some $f : \mathbb{R} \to \mathbb{R}$ so that

$$\frac{df}{dx} - f^2 - q = c \in \mathbb{R} \text{ and } Q = q - 2\frac{df}{dx}$$

If this is the case, we also say that $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$ This is exactly equivalent to the case that

$$\frac{df}{dx} = -\frac{1}{2}(Q - q) \text{ and } 2(Q - q)f = \frac{d}{dx}(Q + q)$$

Note that either of these sets of conditions may be used to show the Darboux relation of two Q–forms. The former set of conditions provides a simple technique to determine c, the **separation constant** of some Darboux relation once f is known, but the latter provides a more tractable method to discover f provided that it exists.

Basic Theorems

Theorem: If $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$ then $\hat{H}_Q$ is Darboux related to $\hat{H}_q$ by $\hat{D}_{-f}$

$$\begin{tikzcd} \ker \hat{H}_q \arrow[r, bend right, "\hat{D}_f"'] & \ker \hat{H}_Q \arrow[l, bend right, "\hat{D}_{-f}"'] \end{tikzcd}$$

Theorem: if $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$, then $\hat{H}_Q \hat{D}_f y = 0$ whenever $\hat{H}_q y = 0$

(Big) Theorem: if $\hat{H}_q$ is Darboux related to $\hat{H}_Q$ by $\hat{D}_f$ then $\hat{H}_q y = 0$ if and only if $\hat{D}_{-f}\hat{D}_f y = c\, y$ where $c = \frac{df}{dx} - f^2 - q$ which is fixed by the Darboux relation

This result is the crux of my solution method, and it's derivation is somewhat subtle. For the physicists out there, this is the foundation of supersymmetry and Darboux transformations preserve transmission and reflection in quantum mechanical systems. Also, the power of this statement is obvious; if we could force $c = 0$ somehow, then we could solve

$$\hat{D}_f\, y = 0$$

to generate a solution of $\hat{H}_q$.

Example

Let λ be some yet undetermined fixed real number, and

$$\hat{H}_\lambda = \frac{d^2}{dx^2} - \frac{1}{2}x^2 - \lambda$$

and let us define $\hat{H}_\mu$ similarly. Let us find a Darboux transformation which will relate $\hat{H}_\lambda$ to $\hat{H}_\mu$/ To this end, let us determine f.

Example

$$Q = -\tfrac{1}{2}x^2 - \mu \text{ and } q = -\tfrac{1}{2}x^2 - \lambda$$

Recall that f is determined by

$$\frac{df}{dx} = -\frac{1}{2}(Q - q) \ and \ 2(Q - q)f = \frac{d}{dx}(Q + q)$$

Beginning on the first equation, we have that

$$\frac{df}{dx} = \frac{1}{2}(\mu - \lambda) \implies f = \frac{1}{2}(\mu - \lambda)x$$

Then, taking our attention to the second equation, we have

$$2(\lambda - \mu)\,f = \frac{d}{dx}(-x^2 - \mu - \lambda) = -2\,x$$

$$f = \frac{2}{\mu - \lambda}x$$

Example

Evidently, then, if we are to achieve Darboux Relation

$$f = \frac{1}{2}(\mu - \lambda)x = \frac{2}{\mu - \lambda}x$$

and, if this is to be the case, then we must have that $\mu - \lambda = \pm 2$. Then $f = x$. Look! It's the equal spacing of the energy levels of the simple harmonic oscillator! Now we must calculate c from

$$\frac{df}{dx} - f^2 - q = c$$

$$\frac{df}{dx} - f^2 - q = 1 - x^2 + x^2 + \lambda = 1 + \lambda$$

Example

Apparently $c = 1 + \lambda$. Then, if $\lambda = -1$, $c = 0$ and we may find a solution of $\hat{H}_{-1}$ by solving

$$\hat{D}_f y == \frac{d\,y}{dx} + f\;y = \frac{d\,y}{dx} + x\,y = 0$$

Which has the trivial solution

$$y = e^{-\frac{x^2}{2}}$$

Then, we have a solution of $\hat{H}_{2n-1}$ which is $\hat{D}^n_{-f}\;y$.