Course Summary

Advanced Geometry 1 (Algebraic Topology Introduction)
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Contents

Lecture 0 – Preliminary Definitions from Munkres Topology Chapter 2

Definition (Topology). Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets of X (called open sets) satisfying:

1. $\varnothing, X \in \mathcal{T}$,

2. the union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$,

3. the intersection of any finite collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space.

Definition (Basis of a Topology). A collection $\mathcal{B}$ of subsets of X is a basis for a topology on X if:

1. for each $x \in X$, there exists $B \in \mathcal{B}$ with $x \in B$,

2. if $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The topology generated by $\mathcal{B}$ is the collection of all unions of elements of $\mathcal{B}$.

Definition (Subspace Topology). Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. The subspace topology on Y is
$$\mathcal{T}_Y = \{ U \cap Y : U \in \mathcal{T} \}.$$

Definition (Closed Set). A subset $A \subseteq X$ is called closed if its complement $X \setminus A$ is open.

Definition (Topology via Closed Sets). A collection $\mathcal{C}$ of subsets of X is the collection of closed sets of a topology on X if:

1. $\varnothing, X \in \mathcal{C}$,

2. the intersection of any collection of sets in $\mathcal{C}$ is in $\mathcal{C}$,

3. the union of any finite collection of sets in $\mathcal{C}$ is in $\mathcal{C}$.

Definition (Closure and Interior). Let $A \subseteq X$.

· The closure of A, denoted $\overline{A}$, is the intersection of all closed sets containing A.

· The interior of A, denoted A°, is the union of all open sets contained in A.

Definition (Hausdorff Space). A topological space $(X, \mathcal{T})$ is called Hausdorff (or T_2) if for every pair of distinct points $x, y \in X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.

Definition (Continuous Function). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is continuous if any of the following equivalent conditions hold:

1. for every open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X;

2. for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X;

3. for every $x \in X$ and every neighborhood V of $f(x)$ in Y, there exists a neighborhood U of x in X such that $f(U) \subseteq V$.

Definition (Homeomorphism). A function $f : X \to Y$ between topological spaces is a homeomorphism if it is bijective, continuous, and its inverse f^{-1} is also continuous. Two spaces are homeomorphic if there exists a homeomorphism between them; they are then considered topologically equivalent.

Definition (Metric). *A metric on a set X is a function $d : X \times X \to [0, \infty)$ satisfying, for all $x, y, z \in X$:*

   1. $d(x, y) = 0$ iff $x = y$;

   2. $d(x, y) = d(y, x)$ (symmetry);

   3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

The pair (X, d) is called a metric space.

Definition (Metric Topology). *Given a metric space (X, d), the metric topology $\mathcal{T}_d$ is the collection of all subsets $U \subseteq X$ such that for every $x \in U$, there exists $\varepsilon > 0$ with the open ball*

$$B_d(x, \varepsilon) = \{\, y \in X : d(x, y) < \varepsilon \,\} \subseteq U.$$

Definition (Quotient Map and Quotient Topology). *Let X and Y be topological spaces and $q : X \to Y$ a surjective map.*

   1. The map q is a quotient map if a subset $U \subseteq Y$ is open in Y iff $q^{-1}(U)$ is open in X.

   2. If q is surjective, the quotient topology on Y induced by q is defined by

$$\mathcal{T}_Y = \{\, U \subseteq Y : q^{-1}(U) \text{ is open in } X \,\}.$$

Lecture 1 – Introduction to Algebraic Topology

The big question. Given two topological spaces X, Y (likely manifolds), are they homeomorpic? Does there exists $f : X \to Y$ a homeomorphism? To prove this, one must construct a homeomorphism. But to disprove it, we can use topological invariants and show that X and Y are different. We must be able to construct and calculate them. Given a topological space, we can construct maps to number systems, polynomials, groups, vector spaces and so on. If we can prove that these properties are preserved under homeomorphism, we have a topological invariant

In the course we will cover simplicial homolgy for polyhedra, singular homology for all topological spaces and cellular homology for CW complexes. At the end we will study cohomology, which is the geometric dual of homology. To begin simplicial homology, we must start with the building blocks called simplices.

Definition. *Given $X = \{a_0, ..., a_n\} \subseteq \mathbb{R}^n$, X is said to be geometrically independent if the following equations hold true:*

   · $\sum_{i=0}^{n} t_i = 0$

   · $\sum_{i=0}^{n} t_i a_i = 0$

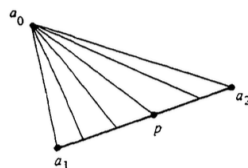*with $t_i \in \mathbb{R}$ imply that $t_1 + t_2 + ... + t_n = 0$*

The set X is just a bunch of points in $\mathbb{R}^n$. If we subtract the set from a fixed point then we get a set of linearly independent vectors.

Definition. *Let $\{a_0, ..., a_n\}$ be a set and consider*

$$\{x = \sum_{i=0}^{n} t_i a_i \,|\, \sum_{i=0}^{n} t_1 = 1 \text{ and } t_i \geq 0 \forall i\}$$

This is called an n simplex generated by $a_0, ..., a_n$

One can do a little algebra with a triangle and show that this sweeps out the edges and interior.

In general it will sweep out the polyhedra and interior of with the geometrically independent set as the vertices.

Definition. Let σ be an n-simplex generated by $a_0, ..., a_n$

· n is the dimension of the simplex

· The simplex $\{a_{i_0}, ... a_{i_k}\} \subseteq \{a_0, ..., a_k\}$ is a k dimensional face of σ

Finally, we define a simplicial complex so it glues together in a way that makes sense.

Definition. A simplicial complex $K \subset \mathbb{R}^n$ is a set of simplices in $\mathbb{R}^n$ such that

· $\sigma \in K$, σ' a face of $K \Rightarrow \sigma' \in K$

· $\sigma, \sigma' \in K$ and $\sigma \cap \sigma' \neq \emptyset \Rightarrow \sigma \cap \sigma'$ is a face common to both σ and σ'

A few examples:
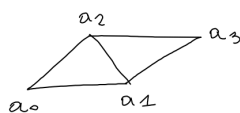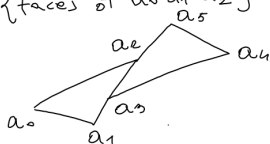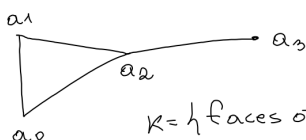
Lecture 2 – Topologies of Simplicial Complexes, Free Abelian Groups and Homology Group Introduction

In previous lecture we defined as simplical complex K. Now we define a topology on K

Definition. Let $\bigcup_{\sigma \in K} \sigma$

- The topology of each σ is induced by the standard topology of $\mathbb{R}^n$

- $A \subseteq |K|$ is closed iff $A \cap \sigma$ is closed for all $\sigma \in K$

So each individual σ just has the topology from little ϵ-balls in $\mathbb{R}^n$ restricted to the subspace σ. The professor proves that the subset topology of K is coarser than the topology when the number of vertices are infinite. But they are equivalent when the number of vertices are finite ($\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{T}_2 \subseteq \mathcal{T}_1$) Now we review abelian groups.

Definition. Let G be an abelian group. The set $\{g_\alpha\}$ such that every $g \in G$ is

$$g = \sum n_\alpha g_\alpha, \text{ finitely many nonzero } n_\alpha$$

is a generator system of G. If the coeffiecents are unique then $\{g_\alpha\}$ basis.

If G has a basis it is called a free abelian group. We can have abelian groups that do not have a basis. The cardinality of the basis is the rank of σ. These groups are convenent to construct homomorphisms with. We prove a key fact about homomorphsims on these groups in the exercises.

Theorem. Let G be a finitely generated free Abelian Group. We define $T = \{Elements\ of\ finite\ order\}$ (T is a subroup of G). The following properties hold:

- There exists H a free abelian subgroup of G such that $G = T \oplus H$

- There exist some finite cyclic groups $T_1, ..., T_k$ with orders $t_1, ..., t_k$ respectively such that $T = T_1 \oplus ... \oplus T_k$ and t_1 divides t_2, t_2 divdes t_3,..., and t_{k-1} divides t_k

- rank H and $t_1, ..., t_k$ are uniquely determined by G

T is the torsion subgroup, $t_1, ..., t_k$ are the torsion coefficients of G and rank H is the Betti Number of G In summary, a finitely generated abelian group always decomposes nicely into finite and infinite parts

$$G \cong \mathbb{Z}_{t_2} \oplus ... \oplus \mathbb{Z}_{t_k} \oplus \mathbb{Z}^r$$

Moving on, now that we have defined a topology on our simplex, we are ready to endow it with a group structure.

Let σ be a simplex, consider the set $\{Orderings\ of\ Vertices\}$.

- We define an equivalence class and identify two orderings if they differ by an even permutation (even permutations decompose into an even number of transpositions)

- This partitions the set into two equivalence classes

- Each of these classes is called an orientation of σ

- 0-simplexes only have one orientation

Definition. A simplex with a chosen orientation is called a oriented simplex

$[v_0, ..., v_p]$ *denotes the simplex* $v_0, ..., v_p$ *with the orientation given by* $v_0 < ... < v_p$
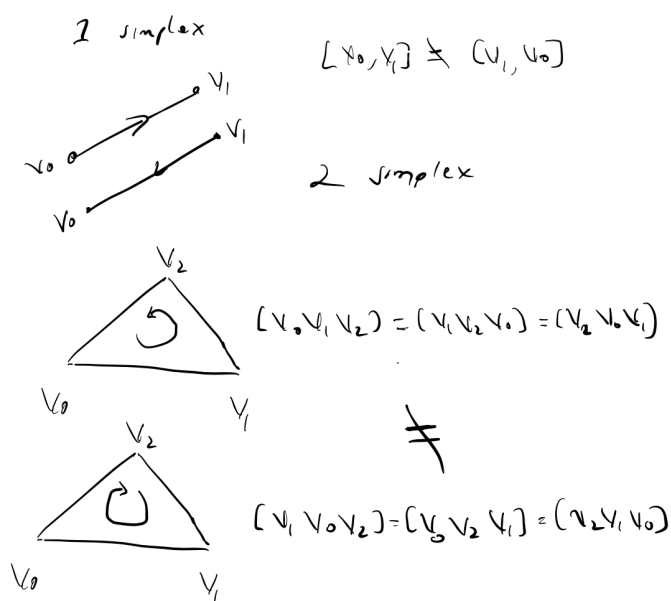
Exercise. *Let* K *be a simplicial complex, prove that the topology of* $|K|$ *which we defined during the lesson is actually a topology.*

Solution

Recall the toplogy we defined in class for $|K|$, *which was finer than the topology it inherits as a subset of* $\mathbb{R}^n$. $|K| = \bigcup\limits_{\sigma \in K} \sigma$

1) The topology of each simplex σ *is induced by* $\mathbb{R}^n$
2) $A \subseteq |K|$ *is closed iff* $A \cap \sigma$ *is closed for all* $\sigma \in K$

Let $A = \emptyset$. *Then* $\{\emptyset\} \cap \sigma = \emptyset \ \forall \sigma \in K$. *But* σ *inherits a subspace topology from* $\mathbb{R}^n$ *so* $\emptyset$ *is closed for all* $\sigma \in K$. *Now let* $A = |K|$. *We have* $|K| \cap \sigma = \sigma \forall \ \sigma \in K$. *Because* σ *inherits the subspace topology,* σ *is closed in* σ *for all sigma.*

Now we check closure under finite unions. Let $A = \bigcup\limits_{i \in I} A_i$ *be a union of closed sets.* $(|I| \in \mathbb{N})$.
Then we have

$$\left(\bigcup\limits_{i \in I} A_i \right) \cap \sigma = (A_1 \cap \sigma) \cup ... \cup (A_k \cap \sigma)$$

Because each A_i *is closed, all* $(A_i \cap \sigma)$ *must be closed by (2). And a finite union of closed sets is closed.*

Now let $A = \bigcap\limits_{i \in I} A_i$ *be an intersection of closed sets. Then we have*

$$\left(\bigcap\limits_{i \in I} A_i \right) \cap \sigma = (A_1 \cap \sigma) \cap ... \cap (A_k \cap \sigma) \cap ...$$

Each $(A_1 \cap \sigma)$ *is closed by (2). And intersections of closed sets are closed.*

Exercise. *Prove that the symmetric group* S_n *is generated by transpositions of type* $(j, j+1)$.

Solution

Recall that S_n is the set of all permutations of the set $\{1,...,n\}$. So $\pi \in S_n$ maps $\pi : \{1,...,n\} \to \{1,...,n\}$ where the group operation is composition. Let π be an arbitrary element of S_n. We would like to show that

$$\pi = \Pi_j(j,j+1)^{n_j}, \quad \text{where } (j,j+1)^{n_j} = \underbrace{(j,j+1) \circ (j,j+1) \circ \cdots \circ (j,j+1)}_{n_j \text{ times}}$$

But we also know that any π can be decomposed into a composition of transpositions (2-cycles)

$$\pi = (a_1 \, a_2 \, \ldots \, a_k) = (a_1 \, a_k) \circ (a_1 \, a_{k-1}) \circ \cdots \circ (a_1 \, a_2)$$

But each transposition can be decomposed into compositions of our generators $\{(j,j+1)\}$.

$$(a_m a_n) = (j,k) = (j,j+1) \circ (j+1,j+2) \circ ...(k-1,k) \circ (k-2,k-1) \circ ... \circ (j+1,j+2) \circ (j,j+1)$$

Putting this into the previous decomposition and using the fact that the symmetric group is abelian we conclude that every element of the permuation group can be expressed in the form

$$\pi = \Pi_j(j,j+1)^{n_j}$$

Exercise. Let F be a free abelian group with basis $(e_\alpha)_{\alpha \in J}$ and A be an abelian group. Prove that for each system of vectors $(g_\alpha)_{\alpha \in J}$ of A there exists exactly one homomorphism $h : F \to A$ such that $h(e_\alpha) = g_\alpha$ for all $\alpha \in J$.

Solution

F is a free abelian group, so every element $f \in F$ can be written as

$$f = \sum n_\alpha e_\alpha \text{with finitely many } n_\alpha \neq 0$$

Before showing uniqueness we construct the homomorphism h. Define $h : F \to A$ such that

$$h(f) = h(\sum n_\alpha e_\alpha) = \sum n_\alpha h(e_\alpha) = \sum n_\alpha g_\alpha$$

It is the same n_α from before so the sum is well defined. Let us choose $x, y \in F$ so that

$$h(x) + h(y) = h(\sum m_\alpha e_\alpha) + h(\sum n_\alpha e_\alpha) = \sum(n_\alpha + m_\alpha)g_\alpha = h(\sum(m_\alpha + n_\alpha)) = h(x+y)$$

So we see it is a homomorphism. Let us assume $h'(e_\alpha) = g_\alpha$is another homomorphism and let x be an arbitrary element in F Thus we have

$$h(x) = h(\sum m_\alpha e_\alpha) = \sum m_\alpha g_\alpha = h'(\sum m_\alpha e_\alpha) = h'(x)$$

So our homomorphism f is unique.

Exercise. Let G be isomorphic to $\mathbb{Z}_{28} \oplus \mathbb{Z}_{42} \oplus \mathbb{Z}_{100} \oplus \mathbb{Z} \oplus \mathbb{Z}_{154} \oplus \mathbb{Z} \oplus \mathbb{Z}_{99}$. Compute the torsion coefficieients and the Betti number of G.

Solution

We see that the group has two copies of $\mathbb{Z}$ so the betty number is 2. The strategy to get the torsion coefficients is described in the next problem. They are $2, 14, 462, 13860$. We also find a factor of one in our decomposition but that gives the trivial group so we ignore it.

Exercise. (Extra) Compute the torsion coefficients of $\mathbb{Z}_{30} \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}_{75}$.

Solution

We do a prime factorization, $30 = 2 \cdot 3 \cdot 5$, $18 = 2 \cdot 3^2$ and $75 = 3 \cdot 5^2$. If we make vectors of the powers each of the prime factors in the original numbers (for example $18 \equiv [1, 2, 0]$), we can then stack all the vectors and rearrange them in ascending order. Then we read the new numbers powers of the exponents down the columns. So we get the torsion coefficients $2^0 \cdot 3^1 \cdot 5^0 = 3$, $2^1 \cdot 3^1 \cdot 5^1 = 30$ and $2^1 \cdot 3^2 \cdot 5^2 = 450$. As a check one can see that 3 divides 30 and 30 divedes 450. Also $3 \cdot 30 \cdot 450 = 30 \cdot 18 \cdot 75$

Exercise. (Extra) Compute the torsion coefficients of $\mathbb{Z}_{50} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$.

Solution

The coefficients are $2, 10, 900$.

Lecture 3 – Group of Oriented P Chains

Definition. Let $X = \{x_\alpha\}$ be a set. We define

$$G = \{\sum n_\alpha x_\alpha | n_\alpha \in \mathbb{Z}, n_\alpha \text{ almost all } 0\}$$

We define the group operation to be addition where

$$\sum n_\alpha x_\alpha + \sum m_\alpha x_\alpha = \sum (n_\alpha + m_\alpha) x_\alpha$$

This is a free abelian group with basis $X \subseteq G$

Definition. Let K be a simplicial complex.

$$C_p(K) = \frac{\text{Free Abelian Group generatd by oriented p-simplices of } K}{\text{the subgroup generated by } \sigma + \sigma' \text{ such that } \sigma \text{ and } \sigma' \text{ are the same simplex with opposite orientation}} \tag{1}$$

$C_p(K)$ is the group of oriented p-chains.

In plain english we take a simplicial complex and pickout all the simplices with codimension k. Then we treat the simplices that are the same with opposite orientations as one.

$\sigma = V_0 V_1 V_2$

$K_\sigma = \{$ faces of $\sigma\}$

$C_0(K_\sigma) = \{ n_0 V_0 + n_1 V_1 + n_2 V_2 \mid n_i \in \mathbb{Z} \}$

free abelian group generated by
0 simplices

$\cong \mathbb{Z}_3$

$C_1(K_\sigma) = \left\{ \begin{array}{l} n_0 [V_0, V_1] + n_1 [V_1, V_0] + n_2 [V_0 V_2] \\ + n_3 [V_2 V_0] + n_4 [V_1 V_2] + n_5 [V_2 V_1] \end{array} \right\}$ s.t. $n_i \in \mathbb{Z}$

$\Big\langle [V_0 V_1] + [V_1 V_0], [V_1 V_2] + [V_2 V_1], [V_0 V_2] + [V_2 V_0] \Big\rangle$

in $C_2(K_\sigma)$ $[V_0 V_1] = -[V_1 V_0]$

$C_1(K_\sigma) \cong \mathbb{Z}^3$ a basis

$\{[V_0, V_1], [V_0, V_2], [V_1 V_2]\}$ $C_2(K_\sigma) \cong \mathbb{Z}$

If we just choose an orientation for each p simplex then we have a basis for $C_p(K)$. There are arguments from general group theory that imply the existence of a homomorphism. We skip these details but note that a homomorphism f would map our free abelian group like

$$f\Big(\sum n_\alpha x_\alpha\Big) = \sum n_\alpha f(x\alpha)$$

on the basis elements. We use this to define the boundary map.

Definition. Let K be a simplicial complex and we define the following function.

$$\partial_p : \{Oriented\ p\text{-simplices of } K\} \to C_{p-1}(K)$$

$$\partial_p([v_0, ..., v_p]) = \sum_{i=0}^{p}(-1)^i[v_0, ..., \hat{v}_i, ...v_p]$$

where $\hat{v}_i$ indicates we remove that vertex from the simplex. This is called the boundary operator

Notice that $(-1)^i$ ensures that the orientations of the faces are consistent. This will be necessary to ensure $\partial_{p-1} \circ \partial_p = 0$. In the notes it is shown that this is in fact a homomorphism and preserves group structure. Now time for an example.

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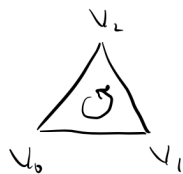
$$\partial [v_0, v_1, v_2] = (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$$
$$\mapsto (v_2, v_0)$$

Exercise. *Show that* $[\partial_p, \pi] = 0$ *with* $\pi \in S_p$. *This is, show that the boundary operator commutes with permutation when acting on elements of* $C_p(K)$

Lecture 4 – Boundary Operator Properties and Homology Group

Now that we have defined the boundary operator, let us prove the property $\partial_{p-1} \circ \partial_p = 0$.

$$\partial_{p-1} \circ \partial_p [v_0, ..., v_p] = \partial_{p-1} \sum_{i=0}^{p} (-1)^i [v_0, ..., \hat{v}_i, ..., v_p] = \sum_{i=0}^{p} (-1)^i \partial_{p-1} [v_0, ..., \hat{v}_i, ..., v_p] =$$

ask professor about the proof and how to break up the sums ...

$$= \sum_{i>j} (-1)^{i+j} [v_0, ..., \hat{v}_j, ..., \hat{v}_i, ..., v_p] - \sum_{j>i} (-1)^{i+j} [v_0, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_p] = 0$$

*The geometric idea is that the boundary of a boundary is nothing.*

Definition. *We have two preliminary definitions.*

- *The kernel of* $\partial_p : C_p(K) \to C_{p-1}(K)$ *is called the group of p-cycles and is denoted* $Z_p(K)$

- *The image of* $\partial_{p+1} : C_{p+1}(K) \mapsto C_p(K)$ *is called the group of p-boundaries and is denoted* $B_p(K)$

Not that $\partial_{p-1} \circ \partial_p = 0 \Rightarrow B_p(K) \subseteq Z_p(K)$. *We have*

$$H_p(K) \equiv Z_p(K)/B_p(K) \text{ is called the } p^{th} \text{ homology group.}$$

Now let us do some example calculations.
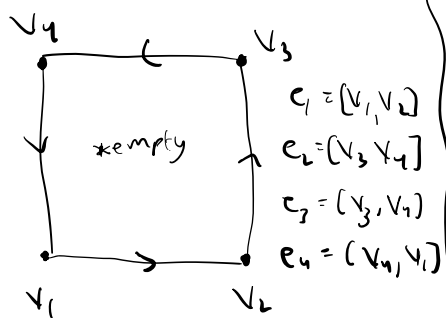
$e_1 = (V_1, V_2)$
$e_2 = (V_3, V_4]$
$e_3 = (V_3, V_4)$
$e_4 = (V_4, V_1)$
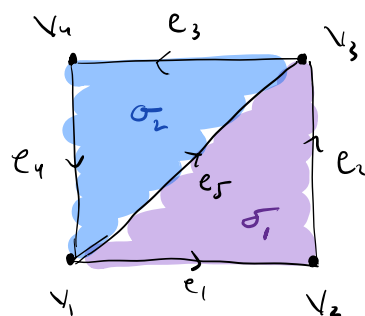
$|K| =$ union of edges of square

$$C_0(K) = \{ n_1 V_1 + n_2 V_2 + n_3 V_3 + n_4 V_4 \mid n_i \in \mathbb{Z} \}$$

$$C_1(K) = \{ n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4 e_4 \mid n_i \in \mathbb{Z}$$

$$C_p(K) = 0 \quad \forall p > 1$$
(our polytope only has 0 and 1 simplexes

$\sigma_1 = (V_1, V_2, V_3]$
$\sigma_2 = (V_1, V_3, V_4)$

$$L = \{ V_1, V_2, V_3, V_4, V_1 V_2, V_2 V_3, V_3 V_4, V_4 V_1, V_1 V_3, V_1 V_2 V_3, V_1 V_3 V_4 \}$$

$Z_1(L)$ generated
by $\{ e_1 + e_2 - e_5, e_3 + e_4 + e_5 \}$
$\cong \mathbb{Z} \oplus \mathbb{Z}$

$B_1(L) = \langle e_1 + e_2 - e_5,$
$e_3 + e_4 + e_5 \rangle = Z_1(L)$

So $H_1(L) = 0$

$Z_1(k):$ $0 =$

$\partial_p \left(n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4 e_4 \right)$

$= (n_4 - n_1) V_1 + (n_1 - n_2) V_2$

$+ (n_2 - n_3) V_3 + (n_3 - n_4) V_4$

$\Longleftrightarrow$

$n_4 - n_1 = 0$

$n_1 - n_2 = 0$

$n_2 - n_3 = 0$

$n_3 - n_4 = 0$

$\Longleftarrow$

$n_1 = n_4 = n_2 = n_3$

$Z_1(k) = \{ n (e_1 + e_2 + e_3 + e_4) \mid n \in \mathbb{Z} \} \cong \mathbb{Z}$

$B_1(k): \quad \partial_2 C_2(k) = 0$

$\Longrightarrow H_1(k) = \dfrac{Z_1(k)}{B_p(k)} = \mathbb{Z}$

$H_2(L) = \dfrac{Z_2(L)}{B_2(L)}$

$Z_2(L) = ?$

$0 = \partial_2 (n_1 \sigma_1 + n_2 \sigma_2)$

$= n_1 \partial_2 \sigma_1 + n_2 \partial_2 \sigma_2 =$

$n_1 ([V_2, V_3] - [V_1, V_3]$

$+ [V_1, V_2]) + n_2 ([V_3, V_4]$

$- [V_1, V_4] + [V_1, V_3]) =$

$n_1 (e_2 - e_5 + e_1) +$

$n_2 (e_3 - e_4 + e_5)$

$=$

$n_1 e_2 + (n_2 - n_1) e_5$

$+ n_1 e_1 + + n_2 e_3 +$

$- n_2 e_4 = 0$

$\Longrightarrow n_1 = n_2 = 0$

$H_2(L) = {}^{0}\!/\!_{B_2(L)} = 0$

The Homology group $H_p(K)$ detects holes p dimensional holes in your space from the triangulation of the manifold. We can see that in the two examples we just did

Intermezzo 2 – Review of Some Relevant Definitions from Munkes Topology Chapter 3

Definition (*Connected Space*). *A topological space X is said to be connected if it cannot be written as the union of two disjoint nonempty open subsets. Equivalently, there do not exist disjoint nonempty open sets $U, V \subseteq X$ with $X = U \cup V$.*

Definition (*Path Connected Space*). *A space X is path connected if for every pair of points $x, y \in X$ there exists a continuous map*

$$\gamma : [0, 1] \to X$$

such that $\gamma(0) = x$ and $\gamma(1) = y$. The map γ is called a path from x to y.

Definition (*Components*). *Let X be a topological space.*

- *A subset $C \subseteq X$ is connected if it is connected as a subspace.*

- *A component (or connected component) of X is a maximal connected subset of X, i.e. a connected subset not properly contained in any larger connected subset.*

The components of X form a partition of X, and each component is closed in X.

Definition (*Compact Space*). *A topological space X is compact if every open cover of X admits a finite subcover; that is, whenever $\{U_\alpha\}_{\alpha \in A}$ is a collection of open subsets with $X = \bigcup_{\alpha \in A} U_\alpha$, there exist finitely many indices $\alpha_1, \ldots, \alpha_n$ such that*

$$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

Example (*Connected but not Path Connected: The Topologist's Sine Curve*). *Consider the subset*

$$S = \left\{ (x, sin(1/x)) : 0 < x \leq 1 \right\} \cup \left(\{0\} \times [-1, 1] \right) \subset \mathbb{R}^2.$$

1. *The set $\{(x, sin(1/x)) : 0 < x \leq 1\}$ is the graph of $sin(1/x)$, which oscillates infinitely often as $x \to 0^+$.*

2. *The vertical segment $\{0\} \times [-1, 1]$ is added to make the set closed in $\mathbb{R}^2$.*

Then:

- *S is connected — any attempt to separate S into disjoint nonempty open subsets fails, because the oscillations of $sin(1/x)$ accumulate densely along the vertical segment.*

- *S is not path connected — there is no continuous path in S joining a point on the oscillating part (where $x > 0$) to a point on the segment $\{0\} \times [-1, 1]$. Any such path would force a limit of $sin(1/x)$ as $x \to 0^+$, which does not exist.*
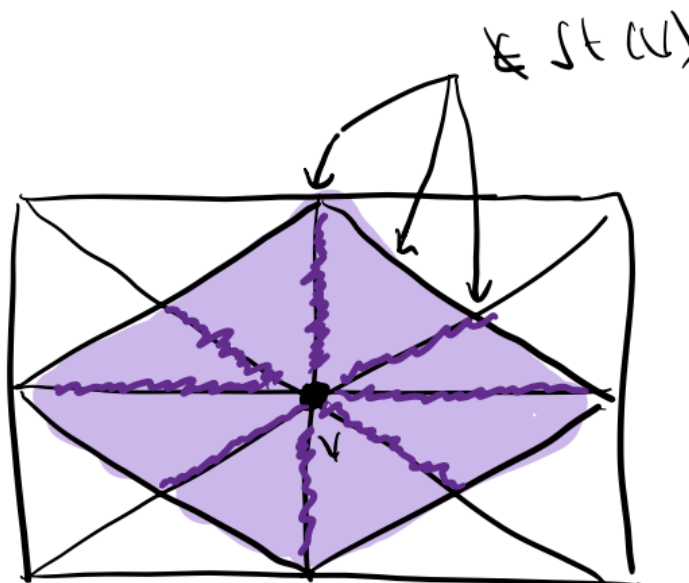
Thus, S provides a standard example of a space that is connected but not path connected.
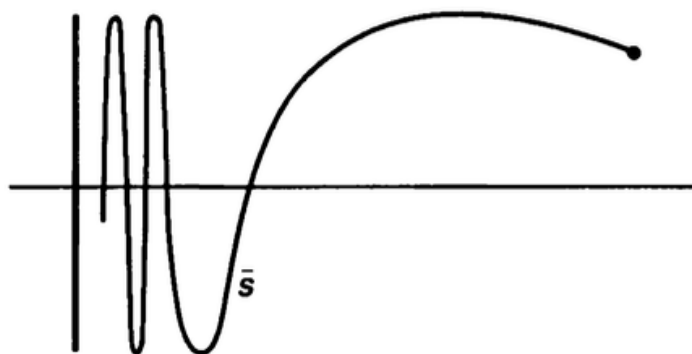
Lecture 5 – Homology Group Structure and Equivalent Homology Group

Definition. Let K be a simplicial complex with $v \in K^{(0)}$ We define the star of v

$$st(v) = \cup_{\sigma \in K} Int(\sigma) \text{ such that } v \text{ is a vertex of } \sigma$$

We can take its closure. We can also see that $st(v)$ is an open set of $|K|$ so that $|K|/st(v)$ is closed in the topology.

Theorem. Let K be a simplicial complex and $H_0(K)$ is a free abelian group. If $\{v_\alpha\} \subseteq K^{(0)}$ such that each connected component of $|K|$ contains exactly one element of $\{v_\alpha\}$, then the equivalence classes of $\{v_\alpha\}$ give a basis of $H_0(K)$

This is an extremely confusing definition. The point is that $H_0(K)$ just counts the number of connected components. It would make more sense to consider a space of multiple components, then show that each component is generated by one vertex using properties of connectedness. But I omit the proof from these notes. Using this idea we can create a new definition from our star definition.

Definition.
$$C_v = \cup_{w \sim v} st(w)$$

are the connected components of $|K|$

  · C_v are open in the topology on $|K|$

  · C_v are (path) connected

Exercise. Let K be a simplicial complex and Y be a topological space. Prove that $f : |K| \to Y$ is a continuous function if and only if $f|_\sigma$ is continuous for all $\sigma \in K$.

Solution

($\Rightarrow$) We use theorem 18.2 (d) of Munkres Topology for constructing continuous functions. If $f : |K| \to Y$ is continuous and $\sigma \subset |K|$, then $f|_\sigma : \sigma \to Y$ is also continuous. This is true for all the σ in $|K|$, so the forwards statement is true.

($\Leftarrow$) Now we assume that for all $\sigma_i \in |K|$ that $f|_{\sigma_i} : \sigma_i \to Y$ are continuous functions. So for each closed set $C \subseteq Y$ we have that $(f|_{\sigma_i})^{-1}(C)$ is closed in σ_i. Each σ_i has the toplogy it inherits from $\mathbb{R}^n$ so $(f|_{\sigma_i})^{-1}(C)$ is closed in this topology. But because $(f|_{\sigma_i})^{-1}(C)$ is closed and σ_i is closed in the topology induced by $\mathbb{R}^n$, we have that $(f|_{\sigma_i})^{-1}(C) \cap \sigma_i = (f|_{\sigma_i})^{-1}(C)$. Because this holds for all the possible $\sigma_i \in K$, $(f|_{\sigma_i})^{-1}(C)$ will always be closed in the topology of $|K|$. Because $C \subseteq Y$ was a closed set, it follows that $f : |K| \to Y$ is continuous.

Exercise. 1. Let x be a point of the simplicial complex $v_0 v_1 \cdots v_p$, then we have that

$$x = \sum_{i=0}^{p} t_i v_i$$

with $t_i \geq 0$ and $\sum_{i=0}^{p} t_i = 1$. Prove that the coefficients t_i are uniquely determined by x. The real numbers t_i are called barycentric coordinates of x.

2. Let K be a simplicial complex. Prove that each $x \in |K|$ is contained in the interior of exactly one simplex of K. Then fix $v \in K^{(0)}$ and define

$$t_v : |K| \to \mathbb{R}$$

as the function associating $x \in |K|$ with the barycentric coordinate of x with respect to v if x is contained in the interior of a simplex of vertex v, otherwise we set $t_v(x) = 0$. Prove that t_v is continuous.

Exercise. Let K be a simplicial complex. Prove that:

1. $|K|$ is Hausdorff;

2. in $|K|$ path-connected components and connected components coincide;

3. if K is finite, $|K|$ is compact.

Solution

1. Let σ and σ' be disctinct simplexes in $|K|$. First, let x, y be distinct points in σ. Because σ inherits the topology of $\mathbb{R}^n$ and $\mathbb{R}^n$ is Hausdorff, there exists U and V disjoint sets in σ such that $x \in U$ and $y \in V$. Now let $x \in \sigma$ and $y \in \sigma'$. If $\sigma \cap \sigma' = \emptyset$ then the Hausdorff criteria automatically follows. But if $\sigma \cap \sigma' \neq \emptyset$ then either x or y or both are in $\sigma \cap \sigma'$. In that case we invoke the Hausdorff property of $\mathbb{R}^n$ inherited by $\sigma \cap \sigma'$. In all cases $|K|$ is Hausdorff.

2. Path connected always imply connected. So we only need to prove that the connected components of $|K|$ are also path connected.

3. Because the polytope $|K|$ is embedded in $\mathbb{R}^n$ we use the Heine-Borel theorem which states that compactness is equivalent to boundedness and closedness. Because the vertices $\{v\}$ in $|K|$ are geometrically independent, we subtract all vertices from $\vec{0}$ to get $\{\vec{v}\}$. Let $R = max \, ||\vec{v}||$ such that $\vec{v} \in \{\vec{v}\}$. Then we can bound the Polytope in a ball $B_R(0)$. So $|K|$ is bounded. Because each σ is closed in $\mathbb{R}^n$, $|K| = \cup_{\sigma \in K} \sigma$ is also closed. It follows that $|K|$ is compact.

Homework 3

Exercise 1

Let k be the simplicial complex represented by the following scheme:
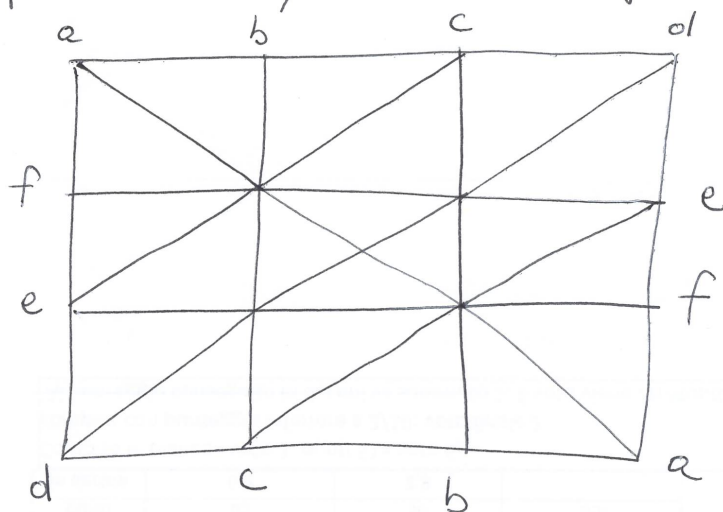
where vertices and 1-simplices in the boundary are identified if they are labelled by the same letters.

1. Prove that $H_1(k) \cong \mathbb{Z}_2$ and $H_2(k) = 0$

2. What surface is homeomorphic to $|k|$?