Course Summary

Advanced Geometry 1 (Algebraic Topology Introduction)

Andrew Moyer

Contents

Lecture 0 – Preliminary Definitions from Munkres Topology Chapter 2

Definition (Topology). *Let X be a set. A* topology *on X is a collection $\mathcal{T}$ of subsets of X (called* open sets*) satisfying:*

1. *$\varnothing, X \in \mathcal{T}$,*
2. *the union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$,*
3. *the intersection of any finite collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.*

The pair $(X, \mathcal{T})$ is called a topological space*.*

Definition (Basis of a Topology). *A collection $\mathcal{B}$ of subsets of X is a* basis *for a topology on X if:*

1. *for each $x \in X$, there exists $B \in \mathcal{B}$ with $x \in B$,*
2. *if $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

The topology generated *by $\mathcal{B}$ is the collection of all unions of elements of $\mathcal{B}$.*

Definition (Subspace Topology). *Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. The* subspace topology *on Y is*

$$\mathcal{T}_Y = \{\, U \cap Y : U \in \mathcal{T} \,\}.$$

Definition (Closed Set). *A subset $A \subseteq X$ is called* closed *if its complement $X \setminus A$ is open.*

Definition (Topology via Closed Sets). *A collection $\mathcal{C}$ of subsets of X is the collection of* closed sets *of a topology on X if:*

1. *$\varnothing, X \in \mathcal{C}$,*
2. *the intersection of any collection of sets in $\mathcal{C}$ is in $\mathcal{C}$,*
3. *the union of any finite collection of sets in $\mathcal{C}$ is in $\mathcal{C}$.*

Definition (Closure and Interior). *Let $A \subseteq X$.*

- *The* closure *of A, denoted $\overline{A}$, is the intersection of all closed sets containing A.*
- *The* interior *of A, denoted A°, is the union of all open sets contained in A.*

Definition (Hausdorff Space). *A topological space $(X, \mathcal{T})$ is called* Hausdorff *(or T_2) if for every pair of distinct points $x, y \in X$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.*

Definition (Continuous Function). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is* continuous *if any of the following equivalent conditions hold:*

1. *for every open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X;*
2. *for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X;*
3. *for every $x \in X$ and every neighborhood V of $f(x)$ in Y, there exists a neighborhood U of x in X such that $f(U) \subseteq V$.*

Definition (Homeomorphism). *A function $f : X \to Y$ between topological spaces is a* homeomorphism *if it is bijective, continuous, and its inverse f^{-1} is also continuous. Two spaces are* homeomorphic *if there exists a homeomorphism between them; they are then considered topologically equivalent.*

Definition (Metric). *A* metric *on a set X is a function $d : X \times X \to [0, \infty)$ satisfying, for all $x, y, z \in X$:*

1. $d(x, y) = 0$ *iff* $x = y$*;*
2. $d(x, y) = d(y, x)$ *(symmetry);*
3. $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*

The pair (X, d) is called a metric space*.*

Definition (Metric Topology). *Given a metric space (X, d), the* metric topology *$\mathcal{T}_d$ is the collection of all subsets $U \subseteq X$ such that for every $x \in U$, there exists $\varepsilon > 0$ with the open ball*

$$B_d(x, \varepsilon) = \{\, y \in X : d(x, y) < \varepsilon \,\} \subseteq U.$$

Definition (Quotient Map and Quotient Topology). *Let X and Y be topological spaces and $q : X \to Y$ a surjective map.*

1. *The map q is a* quotient map *if a subset $U \subseteq Y$ is open in Y iff $q^{-1}(U)$ is open in X.*
2. *If q is surjective, the* quotient topology *on Y induced by q is defined by*

$$\mathcal{T}_Y = \{\, U \subseteq Y : q^{-1}(U) \text{ is open in } X \,\}.$$

Lecture 1 – Introduction to Algebraic Topology

The big question. Given two topological spaces X, Y (likely manifolds), are they homeomorpic? Does there exists $f : X \to Y$ a homeomorphism? To prove this, one must construct a homeomorphism. But to disprove it, we can use topological invariants and show that X and Y are different. We must be able to construct and calculate them. Given a topological space, we can construct maps to number systems, polynomials, groups, vector spaces and so on. If we can prove that these properties are preserved under homeomorphism, we have a topological invariant

In the course we will cover simplicial homolgy for polyhedra, singular homology for all topological spaces and cellular homology for CW complexes. At the end we will study cohomology, which is the geometric dual of homology. To begin simplicial homology, we must start with the building blocks called simplices.

Definition. *Given $X = \{a_0, ..., a_n\} \subseteq \mathbb{R}^n$, X is said to be geometrically independent if the following equations hold true:*

- $\sum_{i=0}^{n} t_i = 0$
- $\sum_{i=0}^{n} t_i a_i = 0$

with $t_i \in \mathbb{R}$ imply that $t_1 + t_2 + ... + t_n = 0$

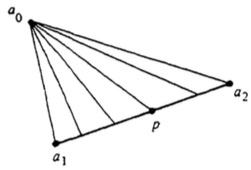
The set X is just a bunch of points in $\mathbb{R}^n$. If we subtract the set from a fixed point then we get a set of linearly independent vectors.

Definition. *Let $\{a_0, ..., a_n\}$ be a set and consider*

$$\{x = \sum_{i=0}^{n} t_i a_i | \sum_{i=0}^{n} t_1 = 1 \text{ and } t_i \geq 0 \forall i\}$$

This is called an n simplex generated by $a_0, ..., a_n$

One can do a little algebra with a triangle and show that this sweeps out the edges and interior.



In general it will sweep out the polyhedra and interior of with the geometrically independent set as the vertices.

Definition. *Let σ be an n-simplex generated by $a_0, ..., a_n$*

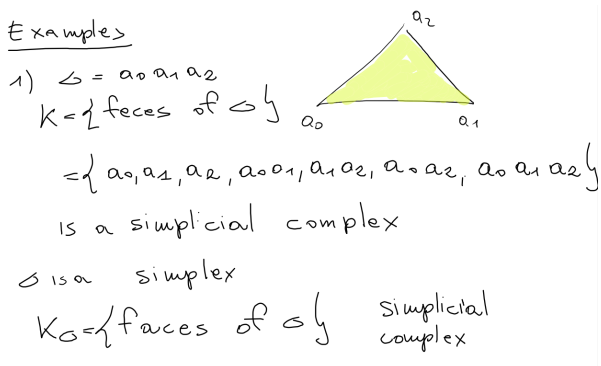
- *n is the dimension of the simplex*
- *The simplex $\{a_{i_0}, ...a_{i_k}\} \subseteq \{a_0, ..., a_k\}$ is a k dimensional face of σ*

Finally, we define a simplicial complex so it glues together in a way that makes sense.

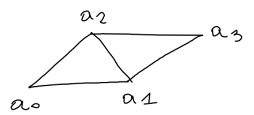
Definition. *A simplicial complex $K \subset \mathbb{R}^n$ is a set of simplices in $\mathbb{R}^n$ such that*

- *$\sigma \in K$, σ' a face of $K \Rightarrow \sigma' \in K$*
- *$\sigma, \sigma' \in K$ and $\sigma \cap \sigma' \neq \emptyset \Rightarrow \sigma \cap \sigma'$ is a face common to both σ and σ'*

A few examples:



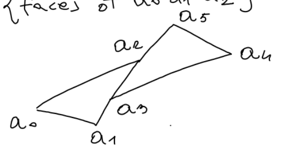
2)



Yes

$K = \{\text{faces of } a_0 a_1 a_2\} \cup \{\text{faces of } a_1 a_2 a_3\}$

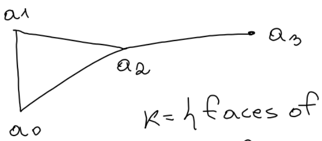
3)



No

$K = \{\text{faces of } a_0 a_1 a_2\} \cup \{\text{faces of } a_3 a_4 a_5\}$

4)



Yes

$K = \{\text{faces of } a_0 a_1 a_2\}$
$\cup \{\text{faces of } a_2 a_3\}$

Lecture 2 – Topologies of Simplicial Complexes, Free Abelian Groups and Homology Group Introduction

In previous lecture we defined as simplical complex K. Now we define a topology on K

Definition. *Let* $\bigcup_{\sigma \in K} \sigma$

- *The topology of each σ is induced by the standard topology of $\mathbb{R}^n$*
- *$A \subseteq |K|$ is closed iff $A \cap \sigma$ is closed for all $\sigma \in K$*

So each individual σ just has the topology from little ϵ-balls in $\mathbb{R}^n$ restricted to the subspace σ. The professor proves that the subset topology of K is coarser than the topology when the number of vertices are infinite. But they are equivalent when the number of vertices are finite ($\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{T}_2 \subseteq \mathcal{T}_1$) Now we review abelian groups.

Definition. *Let G be an abelian group. The set $\{g_\alpha\}$ such that every $g \in G$ is*

$$g = \sum n_\alpha g_\alpha, \text{ finitely many nonzero } n_\alpha$$

is a generator system of G. If the coeffiecents are unique then $\{g_\alpha\}$ basis.

If G has a basis it is called a free abelian group. We can have abelian groups that do not have a basis. The cardinality of the basis is the rank of σ. These groups are convienent to construct homomorphisms with. We prove a key fact about homomorphsims on these groups in the exercises.

Theorem. *Let G be a finitely generated free Abelian Group. We define $T = \{$Elements of finite order$\}$ (T is a subroup of G). The following properties hold:*

- *There exists H a free abelian subroup of G such that $G = T \oplus H$*
- *There exist some finite cyclic groups $T_1, ..., T_k$ with orders $t_1, ..., t_k$ respectively such that $T = T_1 \oplus ... \oplus T_k$ and t_1 divides t_2, t_2 divdes t_3,..., and t_{k-1} divides t_k*
- *rank H and $t_1, ..., t_k$ are uniquely determined by G*

T is the torsion subgroup, $t_1, ..., t_k$ are the torsion coefficients of G and rank H is the *Betti Number* of G In summary, a finitely generated abelian group always decomposes nicely into finite and infinite parts

$$G \cong \mathbb{Z}_{t_2} \oplus ... \oplus \mathbb{Z}_{t_k} \oplus \mathbb{Z}^r$$

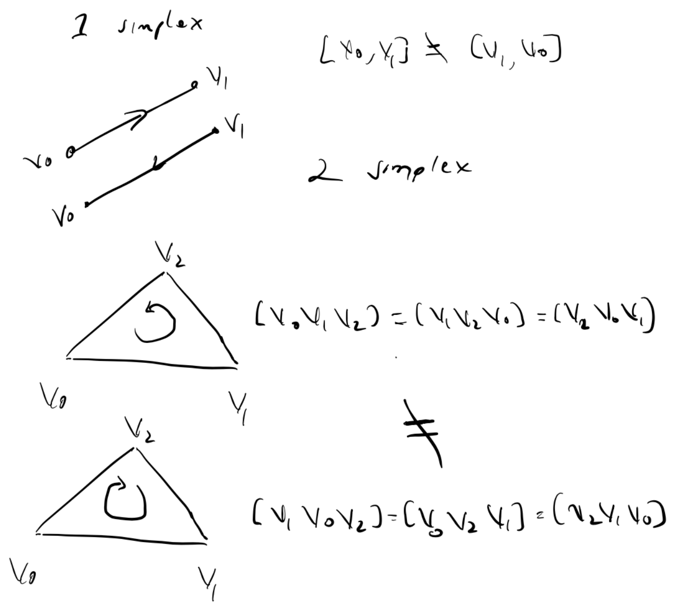
Moving on, now that we have defined a topology on our simplex, we are ready to endow it with a group structure.

Let σ be a simplex, consider the set $\{$Orderings of Vertices$\}$.

- We define an equivalence class and identify two orderings if they differ by an even permutation (even permutations decompose into an even number of transpositions)
- This partitions the set into two equivalence classes
- Each of these classes is called an *orientation* of σ
- 0-simplexes only have one orientation

Definition. *A simplex with a chosen orientation is called a oriented simplex*

$[v_0, ..., v_p]$ denotes the simplex $v_0, ..., v_p$ with the orientation given by $v_0 < ... < v_p$



Exercise. *Let K be a simplicial complex, prove that the topology of $|K|$ which we defined during the lesson is actually a topology.*

Solution

Recall the toplogy we defined in class for $|K|$, which was finer than the topology it inherits as a subset of $\mathbb{R}^n$. $|K| = \bigcup\limits_{\sigma \in K} \sigma$

1) The topology of each simplex σ is induced by $\mathbb{R}^n$
2)$A \subseteq |K|$ is closed iff $A \cap \sigma$ is closed for all $\sigma \in K$

Let A = $\emptyset$. Then $\{\emptyset\} \cap \sigma = \emptyset \ \forall \sigma \in K$. But σ inherits a subspace topology from $\mathbb{R}^n$ so $\emptyset$ is closed for all $\sigma \in K$. Now let $A = |K|$. We have $|K| \cap \sigma = \sigma \forall \ \sigma \in K$. Because σ inherits the subspace topology, σ is closed in σ for all sigma.

Now we check closure under finite unions. Let $A = \bigcup_{i \in I} A_i$ be a union of closed sets. $(|I| \in \mathbb{N})$. Then we have

$$\big(\bigcup_{i \in I} A_i\big) \cap \sigma = (A_1 \cap \sigma) \cup ... \cup (A_k \cap \sigma)$$

Because each A_i is closed, all $(A_i \cap \sigma)$ must be closed by (2). And a finite union of closed sets is closed.

Now let $A = \bigcap_{i \in I} A_i$ be an intersection of closed sets. Then we have

$$\big(\bigcap_{i \in I} A_i\big) \cap \sigma = (A_1 \cap \sigma) \cap ... \cap (A_k \cap \sigma) \cap ...$$

Each $(A_1 \cap \sigma)$is closed by (2). And intersections of closed sets are closed.

Exercise. *Prove that the symmetric group S_n is generated by transpositions of type $(j, j+1)$.*

Solution

Recall that S_n is the set of all permutations of the set $\{1, ..., n\}$. So $\pi \in S_n$ maps $\pi : \{1, ..., n\} \to \{1, ..., n\}$ where the group operation is composition. Let π be an arbitrary element of S_n. We would like to show that

$$\pi \ = \ \Pi_j (j, j+1)^{n_j}, \quad \text{where } (j, j+1)^{n_j} \ = \ \underbrace{(j, j+1) \circ (j, j+1) \circ \cdots \circ (j, j+1)}_{n_j \text{ times}}$$

But we also know that any π can be decomposed into a composition of transpositions (2-cycles)

$$\pi = (a_1\, a_2\, \ldots\, a_k) \ = \ (a_1\, a_k) \circ (a_1\, a_{k-1}) \circ \cdots \circ (a_1\, a_2)$$

But each transposition can be decomposed into compositions of our generators $\{(j, j+1)\}$.

$$(a_m a_n) = (j, k) = (j, j+1) \circ (j+1, j+2) \circ ...(k-1, k) \circ (k-2, k-1) \circ ... \circ (j+1, j+2) \circ (j, j+1)$$

Putting this into the previous decomposition and using the fact that the symmetric group is abelian we conclude that every element of the permuation group can be expressed in the form

$$\pi \ = \ \Pi_j (j, j+1)^{n_j}$$

Exercise. *Let F be a free abelian group with basis $(e_\alpha)_{\alpha \in J}$ and A be an abelian group. Prove that for each system of vectors $(g_\alpha)_{\alpha \in J}$ of A there exists exactly one homomorphism $h : F \to A$ such that $h(e_\alpha) = g_\alpha$ for all $\alpha \in J$.*

Solution

F is a free abelian group, so every element $f \in F$ can be written as

$$f = \sum n_\alpha e_\alpha \text{with finitely many } n_\alpha \neq 0$$

Before showing uniqueness we construct the homomorphism h. Define $h : F \to A$ such that

$$h(f) = h(\sum n_\alpha e_\alpha) = \sum n_\alpha h(e_\alpha) = \sum n_\alpha g_\alpha$$

It is the same n_α from before so the sum is well defined. Let us choose $x, y \in F$ so that

$$h(x) + h(y) = h(\sum m_\alpha e_\alpha) + h(\sum n_\alpha e_\alpha) = \sum(n_\alpha + m_\alpha)g_\alpha = h(\sum(m_\alpha + n_\alpha)) = h(x + y)$$

So we see it is a homomorphism. Let us assume $h'(e_\alpha) = g_\alpha$is another homomorphism and let x be an arbitrary element in F Thus we have

$$h(x) = h(\sum m_\alpha e_\alpha) = \sum m_\alpha g_\alpha = h'(\sum m_\alpha e_\alpha) = h'(x)$$

So our homomorphism f is unique.

Exercise. *Let G be isomorphic to $\mathbb{Z}_{28} \oplus \mathbb{Z}_{42} \oplus \mathbb{Z}_{100} \oplus \mathbb{Z} \oplus \mathbb{Z}_{154} \oplus \mathbb{Z} \oplus \mathbb{Z}_{99}$. Compute the torsion coefficieients and the Betti number of G.*

Solution

We see that the group has two copies of $\mathbb{Z}$ so the betty number is 2. The strategy to get the torsion coefficients is described in the next problem. They are $2, 14, 462, 13860$. We also find a factor of one in our decomposition but that gives the trivial group so we ignore it.

Exercise. *(Extra) Compute the torsion coefficients of $\mathbb{Z}_{30} \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}_{75}$.*

Solution

We do a prime factorization, $30 = 2 \cdot 3 \cdot 5$, $18 = 2 \cdot 3^2$ and $75 = 3 \cdot 5^2$. If we make vectors of the powers each of the prime factors in the original numbers (for example $18 \equiv [1, 2, 0]$), we can then stack all the vectors and rearrange them in ascending order. Then we read the new numbers powers of the exponents down the columns. So we get the torsion coefficients $2^0 \cdot 3^1 \cdot 5^0 = 3$, $2^1 \cdot 3^1 \cdot 5^1 = 30$ and $2^1 \cdot 3^2 \cdot 5^2 = 450$. As a check one can see that 3 divides 30 and 30 divedes 450. Also $3 \cdot 30 \cdot 450 = 30 \cdot 18 \cdot 75$

Exercise. *(Extra) Compute the torsion coefficients of $\mathbb{Z}_{50} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$.*

Solution

The coefficients are $2, 10, 900$.

Lecture 3 – Group of Oriented P Chains

Definition. *Let $X = \{x_\alpha\}$ be a set. We define*

$$G = \{\sum n_\alpha x_\alpha | n_\alpha \in \mathbb{Z}, n_\alpha \text{ almost all 0}\}$$

We define the group operation to be addition where

$$\sum n_\alpha x_\alpha + \sum m_\alpha x_\alpha = \sum(n_\alpha + m_\alpha)x_\alpha$$

This is a free abelian group with basis $X \subseteq G$

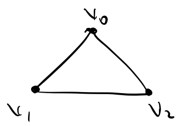
Definition. *Let K be a simplicial complex.*

$$C_p(K) = \frac{\text{Free Abelian Group generatd by oriented p-simplices of } K}{\text{the subgroup generated by } \sigma + \sigma' \text{ such that } \sigma \text{ and } \sigma' \text{ are the same simplex with opposite orientation}} \tag{1}$$

$C_p(K)$ is the group of oriented p-chains.

In plain english we take a simplicial complex and pickout all the simplices with codimension k. Then we treat the simplices that are the same with opposite orientations as one.

$\sigma = v_0 v_1 v_2$



$K_\sigma = \{\text{faces of } \sigma\}$

$C_0(K_\sigma) = \{ n_0 v_0 + n_1 v_1 + n_2 v_2 \mid n_i \in \mathbb{Z} \}$

free abelian group generated by 0 simplices

$\cong \mathbb{Z}_3$

$$C_1(K_\sigma) = \frac{\left\{ \begin{array}{l} n_0[v_0, v_1] + n_1[v_1 v_0] + n_2[v_0 v_2] \\ + n_3[v_2 v_0] + n_4[v_1 v_2] + n_5[v_2 v_1] \\ \text{st } n_i \in \mathbb{Z} \end{array} \right.}{\left\langle \begin{array}{c} [v_0 v_1] + [v_1 v_0], [v_1 v_2] + [v_2 v_1], \\ [v_0 v_2] + [v_2 v_0] \end{array} \right\rangle}$$

in $C_2(K_\sigma)$ $\quad [v_0 v_1] = -[v_1 v_0]$

$C_1(K_\sigma) \cong \mathbb{Z}^3$ a basis

$\{[v_0, v_1], [v_0, v_2], [v_1 v_2]\}$ $\quad C_2(K_\sigma) \cong \mathbb{Z}$

If we just choose an orientation for each p simplex then we have a basis for $C_p(K)$. There are arguments from general group theory that imply the existence of a homomorphism. We skip these details but note that a homomorphism f would map our free abelian group like

$$f\left(\sum n_\alpha x_\alpha\right) = \sum n_\alpha f(x\alpha)$$

on the basis elements. We use this to define the boundary map.

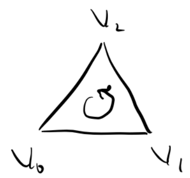
Definition. *Let K be a simplicial complex and we define the following function.*

$$\partial_p : \{\textit{Oriented p-simplices of K}\} \to C_{p-1}(K)$$

$$\partial_p([v_0, ..., v_p]) = \sum_{i=0}^{p} (-1)^i [v_0, ..., \hat{v}_i, ...v_p]$$

where $\hat{v}_i$ indicates we remove that vertex from the simplex. This is called the boundary operator

Notice that $(-1)^i$ ensures that the orientations of the faces are consistent. This will be necessary to ensure $\partial_{p-1} \circ \partial_p = 0$. In the notes it is shown that this is in fact a homomorphism and preserves group structure. Now time for an example.



$$\partial[v_0 v_1 v_2] = (v_1 v_2) \underbrace{- [v_0 v_2]}_{= +[v_2 v_0]} + (v_0 v_1)$$

Exercise. *Show that $[\partial_p, \pi] = 0$ with $\pi \in S_p$. This is, show that the boundary operator commutes with permutation when acting on elements of $C_p(K)$*

Lecture 4 – Boundary Operator Properties and Homology Group

Now that we have defined the boundary operator, let us prove the property $\partial_{p-1} \circ \partial_p = 0$.

$$\partial_{p-1} \circ \partial_p[v_0, ..., v_p] = \partial_{p-1} \sum_{i=0}^{p} (-1)^i [v_0, ..., \hat{v}_i, ..., v_p] = \sum_{i=0}^{p} (-1)^i \partial_{p-1}[v_0, ..., \hat{v}_i, ..., v_p] =$$

ask professor about the proof and how to break up the sums ...

$$= \sum_{i>j} (-1)^{i+j} [v_0, ..., \hat{v}_j, ..., \hat{v}_i, ..., v_p] - \sum_{j>i} (-1)^{i+j} [v_0, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_p] = 0$$

The geometric idea is that the boundary of a boundary is nothing.

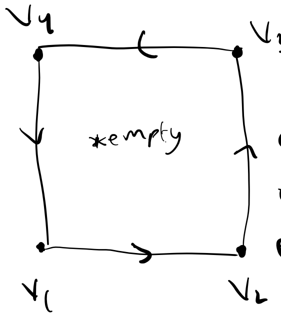
Definition. *We have two preliminary definitions.*

- *The kernel of $\partial_p : C_p(K) \to C_{p-1}(K)$ is called the group of p-cycles and is denoted $Z_p(K)$*
- *The image of $\partial_{p+1} : C_{p+1}(K) \mapsto C_p(K)$ is called the group of p-boundaries and is denoted $B_p(K)$*

Not that $\partial_{p-1} \circ \partial_p = 0 \Rightarrow B_p(K) \subseteq Z_p(K)$. We have

$$H_p(K) \equiv Z_p(K)/B_p(K) \text{ is called the } p^{th} \text{ homology group.}$$

Now let us do some example calculations.



$e_1 = [v_1, v_2]$
$e_2 = [v_3, v_4]$
$e_3 = [v_3, v_4]$
$e_4 = [v_4, v_1]$

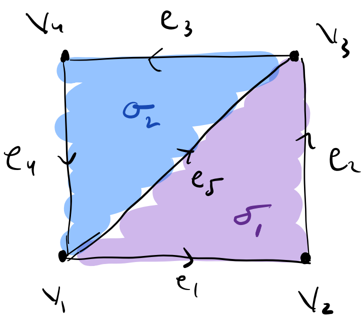
$|K|$ = union of edges of square

$C_0(K) = \{n_1 v_1 + n_2 v_2 + n_3 v_3 + n_4 v_4 \mid n_i \in \mathbb{Z}\}$

$C_1(K) = \{n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4 e_4 \mid n_i \in \mathbb{Z}\}$

$C_p(K) = 0 \quad \forall p > 1$

(our polytope only has 0 and 1 simplexes)



$\sigma_1 = [v_1, v_2, v_3]$
$\sigma_2 = [v_1, v_3, v_4]$

$L = \{v_1, v_2, v_3, v_4, v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3, v_1v_2v_3, v_1v_3v_4\}$

$Z_1(L)$ generated by $\{e_1 + e_2 - e_5, e_3 + e_4 + e_5\}$
$\cong \mathbb{Z} \oplus \mathbb{Z}$

$B_1(L) = \langle e_1 + e_2 - e_5, e_3 + e_4 + e_5 \rangle = Z_1(L)$

So $H_1(L) = 0$

$$Z_1(k): \quad 0 = \partial_p(n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4 e_4)$$

$$= (n_4 - n_1)V_1 + (n_1 - n_2)V_2 + (n_2 - n_3)V_3 + (n_3 - n_4)V_4$$

$$\iff$$

$$n_4 - n_1 = 0$$
$$n_1 - n_2 = 0$$
$$n_2 - n_3 = 0$$
$$n_3 - n_4 = 0$$

$$\iff$$

$$n_1 = n_4 = n_2 = n_3$$

$$Z_1(k) = \{n(e_1 + e_2 + e_3 + e_4) \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$$

$$B_1(k): \quad \partial_2 C_2(k) = 0$$

$$\Rightarrow H_1(k) = \frac{Z_1(k)}{B_p(k)} = \mathbb{Z}$$

$$H_2(L) = \frac{Z_2(L)}{B_2(L)}$$

$$Z_2(L) = ?$$

$$0 = \partial_2(n_1\sigma_1 + n_2\sigma_2)$$

$$= n_1 \partial_2 \sigma_1 + n_2 \partial_2 \sigma_2 =$$

$$n_1([V_2, V_3] - [V_1, V_3] + [V_1, V_2]) + n_2([V_3, V_4] - [V_1, V_4] + [V_1, V_3]) =$$

$$n_1(e_2 - e_5 + e_1) + n_2(e_3 - e_4 + e_5)$$

$$=$$

$$n_1 e_2 + (n_2 - n_1)e_5 + n_1 e_1 + n_2 e_3 - n_2 e_4 = 0$$

$$\Rightarrow n_1 = n_2 = 0$$

$$H_2(L) = 0/B_2(L) = 0$$

The Homology group $H_p(K)$ detects holes p dimensional holes in your space from the triangulation of the manifold. We can see that in the two examples we just did

Intermezzo 2 – Review of Some Relevant Definitions from Munkes Topology Chapter 3

Definition (Connected Space). *A topological space X is said to be* connected *if it cannot be written as the union of two disjoint nonempty open subsets. Equivalently, there do not exist disjoint nonempty open sets $U, V \subseteq X$ with $X = U \cup V$.*

Definition (Path Connected Space). *A space X is* path connected *if for every pair of points $x, y \in X$ there exists a continuous map*

$$\gamma : [0,1] \to X$$

such that $\gamma(0) = x$ and $\gamma(1) = y$. The map γ is called a path *from x to y.*

Definition (Components). *Let X be a topological space.*

- *A subset $C \subseteq X$ is* connected *if it is connected as a subspace.*
- *A* component *(or* connected component*) of X is a maximal connected subset of X, i.e. a connected subset not properly contained in any larger connected subset.*

The components of X form a partition of X, and each component is closed in X.

Definition (Compact Space). *A topological space X is* compact *if every open cover of X admits a finite subcover; that is, whenever $\{\, U_\alpha \,\}_{\alpha \in A}$ is a collection of open subsets with $X = \bigcup_{\alpha \in A} U_\alpha$, there exist finitely many indices $\alpha_1, \dots, \alpha_n$ such that*

$$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

Example (Connected but not Path Connected: The Topologist's Sine Curve). *Consider the subset*

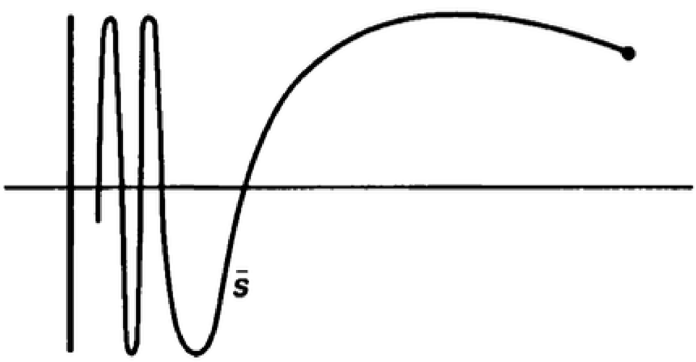
$$S = \big\{(x, \mathsf{sin}(1/x)) : 0 < x \leq 1\big\} \cup \big(\{0\} \times [-1,1]\big) \subset \mathbb{R}^2.$$

1. *The set $\{(x, \mathsf{sin}(1/x)) : 0 < x \leq 1\}$ is the graph of* $\mathsf{sin}(1/x)$*, which oscillates infinitely often as $x \to 0^+$.*
2. *The vertical segment $\{0\} \times [-1,1]$ is added to make the set closed in $\mathbb{R}^2$.*

Then:

- *S is* connected — *any attempt to separate S into disjoint nonempty open subsets fails, because the oscillations of* $\mathsf{sin}(1/x)$ *accumulate densely along the vertical segment.*
- *S is* not path connected — *there is no continuous path in S joining a point on the oscillating part (where $x > 0$) to a point on the segment $\{0\} \times [-1,1]$. Any such path would force a limit of* $\mathsf{sin}(1/x)$ *as $x \to 0^+$, which does not exist.*

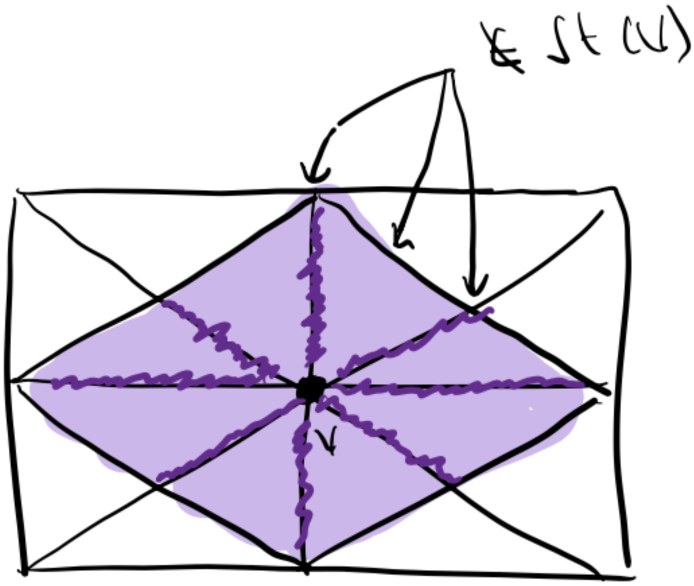
Thus, S provides a standard example of a space that is connected but not path connected.



Lecture 5 – Homology Group Structure and Equivalent Homology Group

Definition. *Let K be a simplicial complex with $v \in K^{(0)}$ We define the star of v*

$$\text{st}(v) = \cup_{\sigma \in K} \text{Int}(\sigma) \text{ such that } v \text{ is a vertex of } \sigma$$



We can take its closure. We can also see that st(v) is an open set of $|K|$ so that $|K|/\text{st}(v)$ is closed in the topology.

Theorem. *Let K be a simplicial complex and $H_0(K)$ is a free abelian group. If $\{v_\alpha\} \subseteq K^{(0)}$ such that each connected component of $|K|$ contains exactly one element of $\{v_\alpha\}$, then the equivalence classes of $\{v_\alpha\}$ give a basis of H_0(K)*

This is an extremely confusing definition. The point is that $H_0(K)$ just counts the number of connected components. It would make more sense to consider a space of multiple components, then show that each component is generated by one vertex using properties of connectedness. But I omit the proof from these notes. Using this idea we can create a new definition from our star definition.

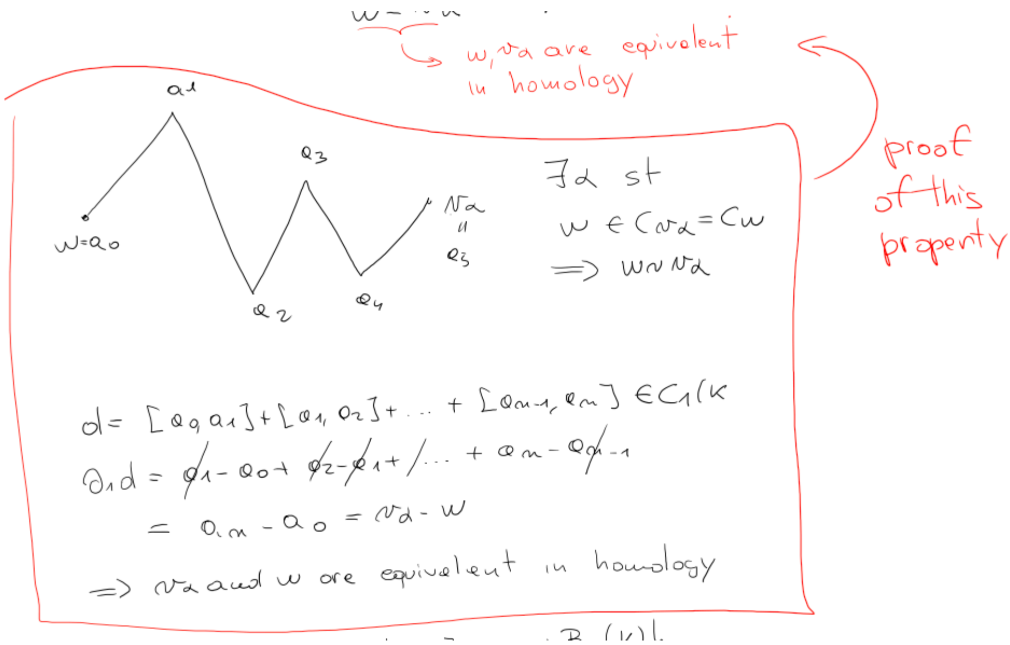
Definition.

$$C_v = \cup_{w \sim v} \text{st}(w)$$

are the connected components of $|K|$

- *C_v are open in the topology on $|K|$*
- *C_v are (path) connected*

I dont put the proof here but it should be clear that if there is an edge between v and w That if we take the union of all these stars it gives us the connected components of $|K|$. Now let $\{v_\alpha\}$ be a set of vertices such there is one vertex from each connected component. The boundary map is trivially 0, $\partial_0 C_0 \to C_{-1} = 0$. It follows that $C_0(K) = Z_0(K)$. Now let $w \in K^{(0)}$. Then there exists α and $d \in C_1(K)$ such that $w - v_\alpha = \partial d$. In plain english, if w is a vertex then there is a one cycle d such that $w - \alpha$ is the boundary of d. The proof is a calculation



So now we can undrestand $H_0(K)$ a little better. If two vertices are connected by edges then they are in the same $H_0(K)$ equivalent class. But that is only the case if they are connected. So the $H_0(K)$ counts the number of connected components of the manifold. The $\{v_\alpha\}$ form a basis for $H_0(K)$. So all elements of this group are linear combinations. The proof uses definiton of a basis and the boundary map. I omit it but we can use this idea to define a new function.

Definition. *Define a homomorphism $\varepsilon : C_0(K) \to \mathbb{Z}$ such that $\varepsilon(v) = 1\ \forall v \in C_0(K)$. That is $\varepsilon(\sum n_\alpha v_\alpha) = \sum n_\alpha$ We call this function the augmentation map.*

From this definition it is clear that $\varepsilon(\partial[v, w]) = 1 - 1 = 0 \Rightarrow \varepsilon \circ \partial_1 = 0$. Now we can define the reduced homology groups. First recall $H_0(K) = C_0(K)/\operatorname{Im}\{\partial_1\}$

Definition. *Attach the augmentation map $\varepsilon : C_0(K) \to \mathbb{Z}$ to the chain complex to form the* augmented *complex*

$$\cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

The reduced homology groups *$\widetilde{H}_n(K)$ are the homology groups of this augmented complex; equivalently,*

$$\widetilde{H}_n(K) = \begin{cases} \ker(\partial_n)/\operatorname{im}(\partial_{n+1}), & n \geq 1, \\ \ker(\varepsilon)/\operatorname{im}(\partial_1), & n = 0. \end{cases}$$

In particular, for $n \geq 1$ we have $\widetilde{H}_n(K) \cong H_n(K)$, while $\widetilde{H}_0(K) = \ker(\varepsilon)/\operatorname{im}(\partial_1)$.

We can contrast this to Homology groups:

$$\text{Homology Groups:} \quad \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{0} 0 \xrightarrow{0} 0$$

This lecture concluded with the following theorem

Theorem. *$\tilde{H}_0(K)$ is free and $H_0(K) \simeq \tilde{H}_0(K) \oplus \mathbb{Z}$. If K is connected than $\tilde{H}_0(K) = 0$*

There is a proof in the notes but we already know $H_0(K) = \mathbb{Z}$ if it is connected. So of course this reduced homolgy group must be 0.

Exercise. *Let K be a simplicial complex and Y be a topological space. Prove that $f : |K| \to Y$ is a continuous function if and only if $f|_\sigma$ is continuous for all $\sigma \in K$.*

Solution

($\Rightarrow$) We use theorem 18.2 (d) of Munkres Topology for constructing continuous functions. If $f : |K| \to Y$ is continuous and $\sigma \subset |K|$, then $f|_\sigma : \sigma \to Y$ is also continuous. This is true for all the σ in $|K|$, so the forwards statement is true.

($\Leftarrow$) Now we assume that for all $\sigma_i \in |K|$ that $f|_{\sigma_i} : \sigma_i \to Y$ are continuous functions. So for each closed set $C \subseteq Y$ we have that $(f|_{\sigma_i})^{-1}(C)$ is closed in σ_i. Each σ_i has the toplogy it inherits from $\mathbb{R}^n$ so $(f|_{\sigma_i})^{-1}(C)$ is closed in this topology. But because $(f|_{\sigma_i})^{-1}(C)$ is closed and σ_i is closed in the toplogy induced by $\mathbb{R}^n$, we have that $(f|_{\sigma_i})^{-1}(C) \cap \sigma_i = (f|_{\sigma_i})^{-1}(C)$. Because this holds for all the possible $\sigma_i \in K$, $(f|_{\sigma_i})^{-1}(C)$ will always be closed in the topology of $|K|$. Because $C \subseteq Y$ was a closed set, it follows that $f : |K| \to Y$ is continuous.

Exercise.

1. *Let x be a point of the simplicial complex $v_0v_1 \cdots v_p$, then we have that*
$$x = \sum_{i=0}^{p} t_i v_i$$
with $t_i \geq 0$ and $\sum_{i=0}^{p} t_i = 1$. Prove that the coefficients t_i are uniquely determined by x. The real numbers t_i are called barycentric coordinates of x.

2. *Let K be a simplicial complex. Prove that each $x \in |K|$ is contained in the interior of exactly one simplex of K. Then fix $v \in K^{(0)}$ and define*
$$t_v : |K| \to \mathbb{R}$$
as the function associating $x \in |K|$ with the barycentric coordinate of x with respect to v if x is contained in the interior of a simplex of vertex v, otherwise we set $t_v(x) = 0$. Prove that t_v is continuous.

Solution

1. *Consider two points in $|K|$, x and x'. Let us assume $x = x'$. Thus we have*
$$\sum_{i=0}^{p} t_i v_i = \sum_{i=0}^{p} t'_i v_i, \quad \sum_{i=0}^{p} t_i = \sum_{i=0}^{p} t'_i = 1, \quad t_i, t'_i \geq 0$$
We can subtract a vertex v_j from each point to get a set of linearly independent vectors. Now we have
$$\vec{x} = \sum_{i=0}^{p} t_i(v_i - v_j), \quad \vec{x}' = \sum_{i=0}^{p} t'_i(v_i - v_j)$$
Because $\vec{x} = \vec{x}'$ we have
$$\vec{0} = \vec{x} - \vec{x}' = \sum_{i=0}^{p} (t_i - t'_i)\vec{v}_{ij}$$
The basis is linearly independent so $t_i = t'_i \ \forall i$. So the coefficients t_i uniquely determine the point x.

2. *Now let x belong to the interior of two simplexes of σ and τ of the simplicial complex K. $\sigma \cap \tau \neq \emptyset \Rightarrow \sigma \cap \tau$ is a face common to both σ and τ. But this cannot be a proper face because this would mean $x \in \partial\sigma$ or $\partial\tau$. So we can conclude*
$$\sigma \cap \tau = \sigma \text{ and } \sigma \cap \tau = \tau \Rightarrow \sigma = \tau$$

So x can only belong to the interior of one simplex of K

Now we fix $v \in K^{(0)}$ and define $t_v : |K| \to \mathbb{R}$ by

$$t_v(x) = \begin{cases} t_i, & \text{if } v = v_i \in \sigma \text{ and } x \in \operatorname{int}\sigma, \\ 0, & \text{if } v \notin \sigma. \end{cases}$$

Choose $\sigma = \langle v_0, v_1, \ldots, v_p \rangle$. If $v \notin \sigma$, then $(t_v|_\sigma)(x) = 0$ for all x in the interior of σ. This is a constant map, hence continuous.

Now fix $v_0 \in \sigma$ and consider $x, x' \in \sigma$. Define the vectors

$$\vec{x} = x - v_0 = \sum_{i=1}^{p} t_i(v_i - v_0) = \tilde{B}\,\vec{t}, \qquad \vec{x}' = x' - v_0 = \sum_{i=1}^{p} t_i'(v_i - v_0) = \tilde{B}\,\vec{t}',$$

where

$$\tilde{B} = [\, v_1 - v_0 \;\; v_2 - v_0 \;\; \cdots \;\; v_p - v_0 \,], \quad \vec{t} = (t_1, \ldots, t_p)^{\mathsf{T}}.$$

Because the vertices of σ are affinely independent, the columns of $\tilde{B}$ are linearly independent. Thus $\tilde{B}$ is invertible on its span, and we have

$$\vec{t} - \vec{t}' = \tilde{B}^{-1}(\vec{x} - \vec{x}').$$

Taking norms gives

$$\|\vec{t} - \vec{t}'\| \leq \|\tilde{B}^{-1}\| \, \|\vec{x} - \vec{x}'\|.$$

Now consider

$$|t_0 - t_0'| = \Big| \sum_{i=1}^{p} (t_i' - t_i) \Big| \leq \sum_{i=1}^{p} |t_i' - t_i| = (1, \ldots, 1) \cdot (|t_1 - t_1'|, \ldots, |t_p - t_p'|) \leq \sqrt{p}\, \|\vec{t} - \vec{t}'\| \leq \sqrt{p}\, \|\tilde{B}^{-1}\| \, \|\vec{x} - \vec{x}'\|.$$

Finally, given $\varepsilon > 0$, choose

$$\delta = \frac{\varepsilon}{\sqrt{p}\, \|\tilde{B}^{-1}\|}.$$

Then, whenever $\|\vec{x} - \vec{x}'\| < \delta$, we have

$$|t_{v_0}(x) - t_{v_0}(x')| = |t_0 - t_0'| \leq \sqrt{p}\, \|\tilde{B}^{-1}\| \, \|\vec{x} - \vec{x}'\| < \sqrt{p}\, \|\tilde{B}^{-1}\| \cdot \frac{\varepsilon}{\sqrt{p}\, \|\tilde{B}^{-1}\|} = \varepsilon.$$

This holds for all v_i so t_v is continuous on σ. Because the function restricted to σ is continuous for all $\sigma \in K$ we use the result of the first exercise to conclude that the function is continuous on the whole space.

Exercise. *Let K be a simplicial complex. Prove that:*

1. *$|K|$ is Hausdorff;*
2. *in $|K|$ path-connected components and connected components coincide;*
3. *if K is finite, $|K|$ is compact.*

Solution

1. Let σ and σ' be disctinct simplexes in $|K|$. First, let x, y be distinct points in σ. Because σ inherits the topology of $\mathbb{R}^n$ and $\mathbb{R}^n$ is Hausdorff, there exists U and V disjoint sets in σ such that $x \in U$ and $y \in V$. Now let $x \in \sigma$ and $y \in \sigma'$. If $\sigma \cap \sigma' = \emptyset$ then the Hausdorff criteria automatically follows. But if $\sigma \cap \sigma' \neq \emptyset$ then either x or y or both are in $\sigma \cap \sigma'$. In that case we invoke the Hausdorff property of $\mathbb{R}^n$ inherited by $\sigma \cap \sigma'$. In all cases $|K|$ is Hausdorff.

2. Path connected always imply connected. So we only need to prove that the connected components of $|K|$ are also path connected. Each simplex $\sigma \subset |K|$ is convex in $\mathbb{R}^n$, hence path-connected. If two simplices σ and τ intersect, then $\sigma \cap \tau$ is a common face, which is also convex and therefore path-connected. Thus, $\sigma \cup \tau$ is path-connected.

   Now note that any two simplices of K that intersect can be joined by a path through their common face. If we can move from one simplex to another through a chain of intersecting simplices, then any two points in their union can be connected by a continuous path in $|K|$. Each connected component of $|K|$ is exactly such a union, and therefore is path-connected. Hence the connected components and path-connected components of $|K|$ coincide.

3. Because the polytope $|K|$ is embedded in $\mathbb{R}^n$ we use the Heine-Borel theorem which states that compactness is equivalent to boundedness and closedness. Because the vertices $\{v\}$ in $|K|$ are geometrically independent, we subtract all vertices from $\vec{0}$ to get $\{\vec{v}\}$. Let $R = \max ||\vec{v}||$ such that $\vec{v} \in \{\vec{v}\}$. Then we can bound the Polytope in a ball $B_R(0)$. So $|K|$ is bounded. Because each σ is closed in $\mathbb{R}^n$, $|K| = \cup_{\sigma \in K} \sigma$ is also closed. It follows that $|K|$ is compact.

Lecture 6 – Homology Group of A Cone and more Reduced Homology Group Properties

Given a simplicial complex $|K|$, we can define an operation to add another vertex and create a new simplicial complex.

Definition. *We define the cone on K with vertex w as a point $w \in \mathbb{R}^n$ that intersects $|K|$ in at most one point.*

$$w * K = K \cup \{wa_1...a_p | a_1...a_p \in K\} \cup \{w\}$$

One can verify that this new structure is also a simplicial complex. The point w cannot be collinear with any of the 1 faces of $|K|$.

Definition. *Let $w * K$ be a cone. We define a homomorphism*

$$C_p(K) \to C_{p+1}(w * K)$$

$$[a_0, ..., a_p] \to [w, a_0, ..., a_p]$$

Now we have a nice interesting theorem.

Theorem. $\tilde{H}_p(w * K) = 0 \; \forall \; p$. *We call this property acyclic.*

Adding a cone to even a nonconnected subspace makes in contractible to a point. And removes all holes. Like a torus, adding a cone will make the entire space connected and remove the holes. The proof is a calculation again. $p = 0$ trivial because $w * K$ is connected. Then you do a calculation for $p > 1$. The proof is carried out in the lecture notes.

Theorem. *Let σ be an n-simplex. We have K_σ, the simplicial complex built from σ (filled in polyhedron) is acyclic. For $n > 0$ we define*

$$\Sigma^{n-1} = \{\textit{proper faces of } \sigma\}$$

We have that

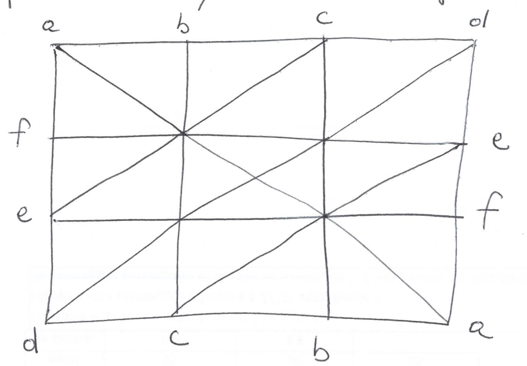
- $\tilde{H}_{n-1}(\Sigma^{n-1}) \simeq \mathbb{Z}$
- $\tilde{H}_i(\Sigma^{n-1}) = 0 \; \forall i \neq n-1$

If we give an orientation to σ then $\partial\sigma$ generates $\tilde{H}_{n-1}(\Sigma^{n-1})$

Homework 3

Exercise 1

Let K be the simplicial complex represented by the following scheme:



where vertices and 1-simplices in the boundary are identified if they are labelled by the same letters.

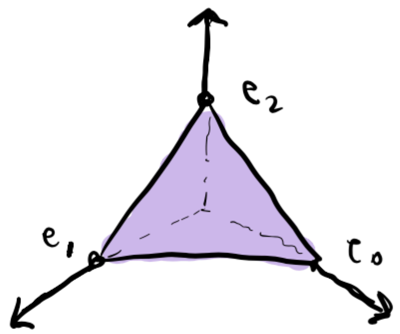
1. Prove that $H_1(K) \cong \mathbb{Z}_2$ and $H_2(K) = 0$
2. What surface is homeomorphic to $|K|$?

Exercise. *Do the above exercise*

Lecture 7 – Beginning of Singular Homology and its Topological Invariance

We use the notation $e_0, ..., e_p$ is the standard basis for $\mathbb{R}^{p+1}$. We define $\Delta_p \equiv e_0...e_p$ as the standard p-simplex

$$\Delta_p = \{(t_0, \ldots, t_p) \in \mathbb{R}^{p+1} \mid t_i \geq 0, \sum t_i = 1\}.$$



Definition. *Let X be a topological space. A singular $p-$simplex is a continuous map*

$$\sigma : \Delta_p \to X$$

The only condition on σ is continuity. In simplicial homology we focused on the combinatorial structure of the building blocks of the polytope. Now we are focusing on the whole space, studying all possible continuous maps from the polytope to a topological space X.

Definition. *The free abelian group generated by the singular p-simplices is called the singular p-chain group $S_p(X)$.*

The members of this group are the maps σ_i and X is the target space, the space we wish to study. Let $a_0, ..., a_p \subseteq \mathbb{R}^n$, not necessarily geometricall independent. We define

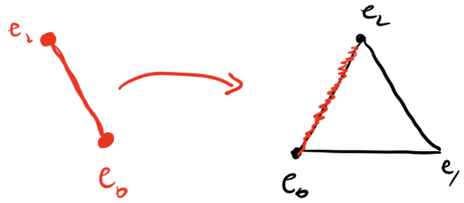
$$\varphi(a_0, ..., a_p) : \Delta_P \to \mathbb{R}^n$$

$$\sum t_i e_i \to \sum t_i a_i$$

$\varphi(a_0, ..., a_p)$ is a p simplex of R^n (continuous). We have $\varphi(e_0, ..., e_P) \equiv \text{Id}_{\Delta_P}$. We also have $\varphi(e_0, .., \hat{e}_i, ., e_P) : \Delta_{p-1} \to \Delta_P$. As an example

example e_0, e_1, e_2

$\varphi(e_0, e_2) : \Delta_2 \longrightarrow \Delta_2$



We have a basis for $S_P(X)$, the set $\{\sigma|$ singular p simplices$\} = \mathcal{S}$ Any function $\mathcal{S} \to G$ (Abelian Group) can be uniquely extend to homomorphism $S_P(X) \to G$ We can now define another boundary operator in this Homology setting.

Definition. *Let $\partial_P : S_P(X) \to S_{P-1}(X)$ by the homomorphism*

$$\partial_P(\sigma) = \sum_{i=0}^{p}(-1)^i \sigma \circ \varphi(e_0, ..., \hat{e}_i, ..., e_p)$$

where $\sigma \circ \varphi(e_0, ..., \hat{e}_i, ...e_p)$

This operator works like the boundary in simplicial homology, Im$\{\partial_p\} \subseteq$ Ker∂_{p-1}. The proof is analagous to the simplical homology proof. With this definition we define the chain complex.

Definition. *We define the chain complex $\{S_P(X), \partial_P\}$*

$$\cdots \xrightarrow{\partial_{p+2}} S_{p+1}(X) \xrightarrow{\partial_{p+1}} S_p(X) \xrightarrow{\partial_p} S_{p-1}(X) \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_1} S_0(X) \longrightarrow 0$$

- $Z_P(X) \equiv$ *ker∂_P group of (singular) p-cycles*
- $B_P(X) \equiv \partial_{P+1}(S_{P+1}(X))$ *group of (singular) p-boundaries*

We have $\partial_{P-1}\partial_P = 0 \Rightarrow B_P(X) \subseteq Z_P(X)$. We have that

$$H_P(X) = Z_P(X)/B_P(X) \equiv \text{ the singular pth homology group}$$

We would eventually like to prove that these groups are topological invariants, invariants under homeomorphism. To prove this we will need to build up a bit of the algebra of these groups before we can make this claim that singular homology groups are topological invariants.

Definition. *Let $\{A_p\},\ p \in \mathbb{Z}$ be a family of abelian groups and $\{\partial_P : A_p \to A_{p-1}\}$ be homomorphisms such that $\partial_p \circ \partial_{p+1} = 0$. Then in general we have what is called a chain complex*

$$\cdots \xrightarrow{\partial_{p+2}} A_{p+1} \xrightarrow{\partial_{p+1}} A_p \xrightarrow{\partial_p} A_{p-1} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_1} A_0 \longrightarrow 0$$

In this more general setting where we dont care about the topological space X, we still have that ker $\partial_P \subseteq$ Im$\{\partial_{p+1}\}$. From this we have a general homology group for $\{A_p, \partial_P\}$

$$H_P(A) = \text{Im}\{\partial_{p+1}\}/\ker\partial_P$$

We can actually define homomorphisms between two chain maps.

Definition. *Let $\varphi = \{C_p, \partial_P\}$ and $\varphi' = \{C'_p, \partial'_P\}$ be chain complexes. A family of homomorphisms $\{\phi_P : C_P \to C'_P\}$ such that $\phi_{P-1} \circ \phi_P = \partial'_P \circ \phi_P$ is called a chain map from φ to φ'*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{p+2}} & C_{p+1} & \xrightarrow{\partial_{p+1}} & C_p & \xrightarrow{\partial_p} & C_{p-1} & \xrightarrow{\partial_{p-1}} & \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0 \\
 & & \downarrow \phi_{p+1} & & \downarrow \phi_p & & \downarrow \phi_{p-1} & & \\
\cdots & \xrightarrow{\partial'_{p+2}} & C'_{p+1} & \xrightarrow{\partial'_{p+1}} & C'_p & \xrightarrow{\partial'_p} & C'_{p-1} & \xrightarrow{\partial'_{p-1}} & \cdots \xrightarrow{\partial'_1} C'_0 \longrightarrow 0
\end{array}$$

With our chain map definition we can study maps between singular homology groups. Now let $\phi : \varepsilon \to \varepsilon'$ be chain map. We can define a new function.

Definition.

$$(\phi_*)_p : H_p(\epsilon) \to H_p(\epsilon')$$

$$(\phi_*)_p([c]) = [\phi([c])]$$

and we have a set for all the simplices $\phi_* = \{(\phi_*)_p\}$

This map now maps between the equivalence classes, but ϕ just mapped between teh chain groups. ϕ_* is well defined. Let $c, c' \in \ker \partial_p$ such that$[c] = [c']$.

$$[c] = [c'] \Rightarrow c - c' = \partial d$$

$$\phi_*([c]) = \phi_*(\partial d + c') = [\phi(\partial d + c')] = [\phi(\partial d) + \phi(c')] = [\partial\phi(d) + \phi(c')] = 0 + [\phi(c')] = \phi_*([c'])$$

A few remarks are in order.

- $(\phi_*)_p$ is a homomorphism
- $\phi : \varepsilon \to \epsilon'$ and $\Phi : \varepsilon' \to \varepsilon''$ then $\Phi \circ \phi : \varepsilon \to \varepsilon''$
- $(\Phi \circ \phi)_* = \Phi_* \circ \phi_*$
- Id $: \varepsilon \to \varepsilon$ is a chain map then $(\text{Id}_*)_p = \text{Id}_{H_p}$

  prove these properties. Third and fourth properties are called functorial properties

- proof 1
- proof 2
- proof 3
- proof 4

We can use all this now to prove topological invariance of the singular homology groups.

Definition. *Let $f : X \to Y$ be a continuous function. We define*

$$(f)_p : S_p(X) \to S_p(Y)$$

$$(f)_p(\sigma) = f \circ \sigma$$

We have that $(f)_p$ is a chain map, but I dont prove it. At this point we are just defining new functions in terms of compositions of functions, then defining functions on top of that. The reason that most people do not like algebra. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. They satisfy the functorial properties:

- $f_\# \circ g_\# = (f \circ g)_\#$
- For $\text{Id}_X : X \to X$ we have that $(\text{Id}_X)_\# = \text{Id}_{\text{Sp}(X)}$

Now we define a final function in terms of the other functions to get our prized topological invariance.

Definition. *Let $(f)_* : H_p(X) \to H_p(Y)$ be a homomorphism. It has functorial properties like the other ones.*

Corollary. *If $h : X \to Y$ is a homeomorphism then $(h)_* : H_p(X) \to H_p(Y)$ is an isomorphism for all p.*

Exercise. *Prove the following functorial properties. Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then*

1. $g_\# \circ f_\# = (g \circ f)_\#$
2. *For Id$_X : X \to X$ we have that $(\text{Id}_X)_\# = \text{Id}_{Sp(X)}$*

Solution

1. Let σ_i be a generator of $S_p(X)$. Thus we have
$$g_\# \circ f_\#(\sigma_i) = g_\#(f_\#(\sigma_i)) = g_\#(f \circ \sigma_i) = g \circ f \circ \sigma_i = (g \circ f)_\#(\sigma_i)$$
2. Let σ_i be a generator of $S_p(X)$. Thus we have
$$(\text{Id}_X)_\#(\sigma_i) = \text{Id}_X \circ \sigma_i = \sigma_i$$
so $(\text{Id}_X)_\# = \text{Id}_{S_p(X)}$

Lecture 8 – Reduced Homology, Properties of Singular Homology, Relative Homology and Homological Algebra Part II

Consider the chain complex
$$... \to S_2(X) \to S_1(X) \to S_0(X) \to 0 \to 0...$$

Definition. *We define $\epsilon : S_0(X) \to \mathbb{Z}$ by*
$$\epsilon(\sigma) = 1 \ \forall \text{ 0 simplices } \sigma$$
We can prove $\epsilon \circ \sigma = 0$. Then we get the augmented chain complex
$$... \to S_2(X) \to S_1(X) \to S_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0...$$
With this chain complex we define reduced homology groups
$$\tilde{H}_0(X) = \frac{\ker \epsilon}{\partial_1 S_1}$$
For the rest of the p they are equivalent to rest of homology groups.

It seems pedantic to create this definition but it will help us in our calculations. It also gets rid of a factor of $\mathbb{Z}$ so $\tilde{H}_0(point) = 0$. Now we move on to properties of the singular homology group.

Proposition. *Let $X = \cup_\alpha X_\alpha$ be a topological space where each X_α are the path connected components. Then*
$$H_p(X) \cong \oplus_\alpha H_p(X_\alpha)$$

This is a nice property and I don't prove it. We will use it in a later proof though.

Proposition. *Let X be a topological space. Then $H_0(X)$ is free abelian and if $\{\sigma_\alpha\}$ is a family of 0 simplices such that for $\forall$ path connected components X_α there existes exactly one σ_α such that $\sigma_\alpha(\Delta_0) \in X_\alpha$ then $\{\sigma_\alpha\}$ is a basis for $H_0(X)$.*

I don't prove this one either but we use this to make another proposition.

Proposition. *Let X, X_α and $\{\sigma_\alpha\}$ be as in the previous proposition. Then $\tilde{H}_0(X)$ is a free abelian group. If we fix α_0 then $\{[\sigma_\alpha - \sigma_0]|\alpha \neq \alpha_0\}$ is a basis of $\tilde{H}_0(X)$.*

Now we put all these propositions to use.

Theorem. *If $X = \{x_0\}$ as single point then X is acyclic. $\tilde{H}_p(X) = 0$ for all p*

This makes sense because the p^{th} homology group measures the number of p dimensional holes of a space. Lets see how easy or difficult it is to get this trivial result.

Proof. There exists a unique p-simplex (for each p)

$$\sigma_p : \Delta_p \to \{x_0\}$$

which is clearly a constant function. Let us consider the chain map

$$... \xrightarrow{\partial_{2k+1}} S_{2k+1}(X) \xrightarrow{\partial_{2k+1}} S_{2k}(X) \xrightarrow{\partial_{2k}} S_{2k-1}(X) \xrightarrow{\partial_{2k-1}} S_{2k-2}(X) \xrightarrow{\partial_{2k-2}} ...$$

Given that there is only one σ_p for each p we have that this chain becomes

$$... \xrightarrow{\partial_{2k+2}} \mathbb{Z} \xrightarrow{\partial_{2k+1}} \mathbb{Z} \xrightarrow{\partial_{2k}} \mathbb{Z} \xrightarrow{\partial_{2k-1}} \mathbb{Z} \xrightarrow{\partial_{2k-2}} ...$$

If we apply the boundary operator to any σ_p we have

$$\partial_p(\sigma_p) = \sum_{i=0}^{p}(-1)^i \sigma_p \circ \varphi_i = (\sum_{i}^{p}(-1)^i)\sigma_{p-1}$$

which is just adding a constant map over and over. We have

$$\partial_p(\sigma_p) = \begin{cases} 0, & \text{p is odd} \\ \sigma_{p-1}, & \text{p is even} \end{cases}$$

So the chain complex is now

$$... \xrightarrow{Id} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{Id} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{Id} ...$$

Let us look at the Homology groups $H_p(X) = \ker \partial_p / \operatorname{Im}\{\partial_{p+1}\}$. We have that $\ker Id = 0$, $\operatorname{Im}\{Id\} = \mathbb{Z}$, $\ker 0 = \mathbb{Z}$ and $\operatorname{Im}\{0\} = 0$. This gives us that for p odd we have

$$\tilde{H}_p(X) = \mathbb{Z}/\mathbb{Z} = 0$$

and for p even we have

$$\tilde{H}_p(X) = 0/0 = 0$$

. Because X is connected $\tilde{H}_0(X) = 0$. Thus it is acyclic. □

Alot of work for a trivial point. We will move towards doing calculations with exact sequences, which are a better computational tool. Before we do that we define relative homology but I skip over most of the details.

Definition. *Let $A \subseteq X$. The group of relative p-chains is defined as the quotient*

$$S_p(X, A) = S_p(X)/S_p(A),$$

where $S_p(X)$ and $S_p(A)$ are the singular p-chain groups of X and A.

The boundary map is induced from the usual one:

$$\partial_p[\sigma] = [\partial_p \sigma],$$

and satisfies $\partial_p^2 = 0$.

The relative homology groups of the pair (X, A) are then

$$H_p(X, A) = \ker(\partial_p)/\operatorname{im}(\partial_{p+1}),$$

that is, the homology of the quotient chain complex $S_*(X, A)$.

Most results carry over from regular homology. Before we get to the main computational tool of exact sequences we return to Homological Algebra to build up some results we will need.

Definition. *Consider a sequence of abelian groups and homomorphisms*

$$\dots \xrightarrow{\phi_{i+2}} A_{i+1} \xrightarrow{\phi_{i+1}} A_i \xrightarrow{\phi_i} \dots$$

The sequence is exact if and only if $\text{Im}\{\phi_{i+1}\} = \ker \phi_i$ *An exact sequence is a chain complex with trivial homology. We have a specific type of exact sequence called a short exact sequence with the following form*

$$0 \to A \xrightarrow{\Phi} B \xrightarrow{\Psi} C \to 0$$

A sequence being short exact is equivalent to the following

- Φ *is injective*
- $\ker \Psi = \text{Im}\{\Phi\}$
- Ψ *is surjective*

We can now specialize this definition to chain complexes.

Definition (Short exact sequence of chain complexes)**.** *A sequence of chain complexes*

$$0 \longrightarrow (A_*, \partial_A) \xrightarrow{\Phi_*} (B_*, \partial_B) \xrightarrow{\Psi_*} (C_*, \partial_C) \longrightarrow 0$$

is short exact *if for every* $p \in \mathbb{Z}$ *the sequence*

$$0 \longrightarrow A_p \xrightarrow{\Phi_p} B_p \xrightarrow{\Psi_p} C_p \longrightarrow 0$$

is exact, and the boundary maps satisfy

$$\partial_B \Phi_p = \Phi_{p-1}\partial_A, \qquad \partial_C \Psi_p = \Psi_{p-1}\partial_B.$$

This can be visualized as the following "ladder" of exact rows:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & A_{p+1} & \xrightarrow{\Phi_{p+1}} & B_{p+1} & \xrightarrow{\Psi_{p+1}} & C_{p+1} & \longrightarrow & 0 \\
 & & \downarrow \partial_A & & \downarrow \partial_B & & \downarrow \partial_C & & \\
0 & \longrightarrow & A_p & \xrightarrow{\Phi_p} & B_p & \xrightarrow{\Psi_p} & C_p & \longrightarrow & 0 \\
 & & \downarrow \partial_A & & \downarrow \partial_B & & \downarrow \partial_C & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Lecture 9 – Snake Lemma, more Homological Algebra and Applications to Algebraic Topology

Lemma. *Let*

$$0 \to \alpha \xrightarrow{\Phi} \beta \xrightarrow{\Psi} \gamma \to 0$$

Be a short exact sequence of chain complexes. Then there exists a family of homomorphisms

$$H_p(\gamma) \xrightarrow{\partial_*} H_{p-1}(\alpha)$$

such that the following is a long exact sequence

$$\dots \to H_p(\alpha) \xrightarrow{\Phi_*} H_p(\beta) \xrightarrow{\Psi_*} H_p(\gamma) \xrightarrow{\partial_*} H_{p-1}(\alpha) \xrightarrow{\Phi_*} \dots$$

where Φ_* *and* Ψ_* *are the homomorphisms induced by* Φ *and* Ψ*. This is called the snake lemma or zig-zag lemma.*

Proof. We define $\partial_* : H_p(\gamma) \to H_{p-1}(\delta)$ and consider $[e_p] \in H_p(\gamma)$ where e_p is a cycle. Because Ψ_p is surjective there exists d_p such that $\Psi_p(d_p) = e_p$. Now consider $\partial_D(d_p)$. By commutativity we have

$$\Psi_{p-1}(\partial_D(d_p)) = \partial_E(\Psi_p(d_p)) = \partial_E e_p = 0$$

So $\partial_D(d_p) \in \ker \Psi_{p-1} = \text{Im}\{\Phi_{p-1}\}$. Therefore there exists c_{p-1} such that $\Phi_{p-1}(c_{p-1}) = \partial_D(d_P)$. We define

$$\partial_*([e_p]) = [c_{p-1}]$$

Consider the following diagram for reference

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & C_p & \xrightarrow{\boldsymbol{\Phi_p}} & \mathbf{D_p} & \xrightarrow{\boldsymbol{\Psi_p}} & \mathbf{E_p} & \longrightarrow & 0 \\
 & & \downarrow \partial_C & & \downarrow \partial_{\mathbf{D}} & & \downarrow \partial_{\mathbf{E}} & & \\
0 & \longrightarrow & \mathbf{C_{p-1}} & \xrightarrow{\boldsymbol{\Phi_{p-1}}} & \mathbf{D_{p-1}} & \xrightarrow{\Psi_{p-1}} & E_{p-1} & \longrightarrow & 0 \\
 & & \downarrow \partial_C & & \downarrow \partial_D & & \downarrow \partial_E & & \\
0 & \longrightarrow & C_{p-2} & \xrightarrow{\Phi_{p-2}} & D_{p-2} & \xrightarrow{\Psi_{p-2}} & E_{p-2} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

In the diagram, the connecting morphism is visualized as a path that moves *left, down, and left* across the rows and columns:

$$E_p \xleftarrow{\Psi_p} D_p \xrightarrow{\partial_D} D_{p-1} \xleftarrow{\Phi_{p-1}} C_{p-1}.$$

Starting with a cycle in the top right (E_p), one moves left to D_p using the surjectivity of Ψ_p, then downward via the boundary ∂_D, and finally left again to C_{p-1} using the exactness of the lower row. This "zig–zag" path through the commutative diagram visually represents the construction of the connecting homomorphism $\partial_* : H_p(E) \to H_{p-1}(C)$. We still need to prove $\partial_*([e_p]) = [c_{p-1}]$ is well defined but I skip that part of the proof. Its a few pages of work and not interesting. $\square$

Now we consider the naturality of the long exact sequence in homology.

Theorem. *Consider the following commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\Phi} & B & \xrightarrow{\Psi} & C & \longrightarrow & 0 \\
 & & \downarrow \alpha & \circlearrowright & \downarrow \beta & \circlearrowright & \downarrow \gamma & & \\
0 & \longrightarrow & D & \xrightarrow{\Phi'} & E & \xrightarrow{\Psi'} & F & \longrightarrow & 0
\end{array}$$

where the horizontal rows are short exact sequences of chain complexes and α, β, γ are chain maps. Then the following ladder of homology groups commutes.

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & H_p(A) & \xrightarrow{\Phi_*} & H_p(B) & \xrightarrow{\Psi_*} & H_p(C) & \xrightarrow{\partial_*} & H_{p-1}(A) & \xrightarrow{\Phi_*} & H_{p-1}(B) & \longrightarrow & \cdots \\
 & & \downarrow \alpha_* & \circlearrowleft & \downarrow \beta_* & \circlearrowleft & \downarrow \gamma_* & \circlearrowleft & \downarrow \alpha_* & \circlearrowleft & \downarrow \beta_* & & \\
\cdots & \longrightarrow & H_p(D) & \xrightarrow{\Phi'_*} & H_p(E) & \xrightarrow{\Psi'_*} & H_p(F) & \xrightarrow{\partial'_*} & H_{p-1}(D) & \xrightarrow{\Phi'_*} & H_{p-1}(E) & \longrightarrow & \cdots
\end{array}$$

We can now apply this to algebraic topology.

Theorem. *Let X be a topological space and $A \subseteq X$. Then there exists a homomorphism $\partial_* : H_p(X, A) \to H_{p-1}(A)$ such that*

$$... \to H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{\pi_*} H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A) \to ...$$

is a long exact sequence where i_* is induced by $i : A \to X$. The map π_* is induced by

$$\pi_\# : S_p(X) \to \frac{S_p(X)}{S_p(A)} = S_p(X, A)$$

π_* is a chain map. Moreover if $A \neq \emptyset$ this holds for the reduced homology.

$$... \to \tilde{H}_p(A) \to \tilde{H}_p(X) \to H_p(X, A) \to \tilde{H}_{p-1}(A) \to \tilde{H}_{p-1}(X) \to ...$$

($\tilde{H}_p(X, A)$ does not exist). If $f(X, A) \to f(Y, B)$ is a continuous map then f_* induces a chain map between the long exact sequence in homology

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_p(A) & \to & H_p(X) & \to & H_p(X,A) & \to & H_{p-1}(A) & \longrightarrow & \cdots \\
& & \downarrow (f|_A)_* & \circlearrowleft & \downarrow f_* & \circlearrowleft & \downarrow f_* & \circlearrowleft & \downarrow (f|_A)_* & & \\
\cdots & \longrightarrow & H_p(B) & \to & H_p(Y) & \to & H_p(Y,B) & \to & H_{p-1}(B) & \longrightarrow & \cdots
\end{array}$$

The groups $H_p(A)$, $H_p(X)$, and $H_p(X, A)$ measure related but distinct topological information. The absolute group $H_p(A)$ records p-dimensional holes contained entirely in A, while $H_p(X)$ records those of the whole space X. The relative group $H_p(X, A)$ measures the new p-dimensional features of X that disappear when restricted to A, that is, the topology of X *relative* to A. The long exact sequence

$$\cdots \to H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{\pi_*} H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A) \to \cdots$$

links these groups together and allows one to compute the homology of a pair from the homology of its parts. If $f : (X, A) \to (Y, B)$ is a continuous map, the induced maps on homology make the entire sequence commute, expressing the *naturality* of the long exact sequence. This could be used, for example, when studying the circle as a subspace of the disc.

Exercise. *Prove the Snake Lemma.*

Solution

insert proof here

Exercise. *Let X be a subset of $\mathbb{R}^n$ containing w such that for all $x \in X$ the segment connecting x and w is contained in X (the set X is said to be* stair-convex *with center w). Prove that X is acyclic with respect to the singular homology.*

Hint: recall that $z \in \Delta_{p+1}$ can be expressed as $z = te_{p+1} + (1-t)z'$ where $z' \in \Delta_p$; given a singular p-simplex $\sigma : \Delta_p \to X$, one can construct a singular $(p+1)$-simplex $[w, \sigma] : \Delta_{p+1} \to X$ where $z = te_{p+1} + (1-t)z' \in \Delta_{p+1}$ is mapped to $tw + (1-t)\sigma(z')$.

Exercise. *Consider the following commutative diagram of Abelian groups:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\Phi_1} & B & \xrightarrow{\Psi_1} & C & \longrightarrow & 0 \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \longrightarrow & D & \xrightarrow{\Phi_2} & E & \xrightarrow{\Psi_2} & F & \longrightarrow & 0
\end{array}$$

Suppose that the two horizontal sequences are exact and prove that there exists an exact sequence of the following type:

$$0 \longrightarrow \ker \alpha \xrightarrow{\Omega_1} \ker \beta \xrightarrow{\Omega_2} \ker \gamma \xrightarrow{\Omega_3} D/\alpha(A) \xrightarrow{\Omega_4} E/\beta(B) \xrightarrow{\Omega_5} F/\gamma(C) \longrightarrow 0.$$

Hint. *This can be done either directly or by using the Snake Lemma; both proofs are interesting!*

Solution

Insert Proof Here

Lecture 10 – Homotopy Review

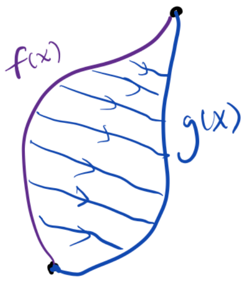
Definition. *Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps. We say that f and g are* homotopic, *written $f \simeq g$, if there exists a continuous map*

$$H : X \times [0,1] \to Y$$

such that

$$H(x,0) = f(x) \quad \text{and} \quad H(x,1) = g(x) \quad \text{for all } x \in X.$$

The map H is called a homotopy *between f and g.*



In other words, they map into the same space and can be deformed in to eachother without ever leaving the space. This homotopy which defines the fundamental group also helps us in homology.

Lemma (Homotopy Lemma). *Let $f, g : X \to Y$ be continuous maps between topological spaces. If f and g are homotopic, then they induce the same homomorphism on homology:*

$$f_* = g_* : H_n(X) \to H_n(Y) \quad \text{for all } n \geq 0.$$

This holds for the reduced and unreduced homology groups.

Corollary (Homotopy Invariance of Homology). *If two topological spaces X and Y are homotopy equivalent, then they have isomorphic homology groups:*

$$H_p(X) \cong H_p(Y) \quad \text{for all } p \geq 0.$$

The simplest spaces are described as contractible.

Definition (Contractible Space). *A topological space X is said to be* contractible *if the identity map*

$$\text{id}_X : X \to X$$

is homotopic to a constant map $c_{x_0} : X \to X$, where $c_{x_0}(x) = x_0$ for some fixed point $x_0 \in X$.

With the following definition we can connext contractible spaces to singular homology.

Remark. *A space X is contractible if and only if the identity map $\text{id}_X : X \to X$ is homotopic to a constant map $c_{x_0} : X \to X$, where $c_{x_0}(x) = x_0$ for some $x_0 \in X$.*

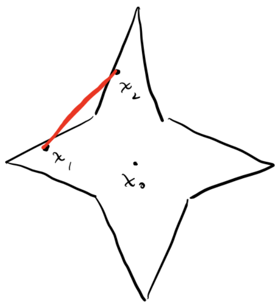
We conclude with the following corollary that connects homotopy and singular homology.

Corollary. *A contractible space is acyclic.*

Lecture 11 – Star Convexity, Excision Theorem and Homology of Spheres

Definition. *$X \subseteq \mathbb{R}^n$ is called a star convex set if $\exists x_0 \in X$ such that every x has a straight line that connects to it that is also contained in X*

Remark. *Every convex set in $\mathbb{R}^n$ is star convex, but not every star convex set is convex.*



We are now ready to discuss the excision theorem.

Theorem. *Let X be a topological space and $A \subseteq X$. If $U \subseteq X$ and $\overline{U} \subseteq$ Int$A$ then the inclusion map*

$$j : (X, A) \to (X \setminus U, A \setminus U)$$

induces and isomorphism in homology

$$j_* : H_p(X \setminus U, A \setminus U) \to H_p(X, A) \;\forall p$$

The excision theorem means that if a subset U lies entirely inside the interior of A, then removing U from both X and A does not change the relative homology. In other words, the homology groups $H_p(X, A)$ depend only on how A meets the boundary of X, not on what happens deep inside A. Thus, any region U completely contained in Int(A) can be "excised" without affecting the homology of the pair.

Definition. *Let X be a topological space and $\mu = \{U_j\}$ be a family of subsets of X such that $X = \cup_j$IntU_j. A singular $\sigma : \Delta_p \to X$ is called $\mu-$small if there exists a j_0 such that $\sigma(\Delta_p) \subseteq U_{j_0}$ We define $S_p^\mu(X)$ as*

$$S_p^\mu(X) = \langle \sigma \text{is a p simplex} | \sigma \text{ is } \mu\text{−small} \rangle$$

. $S_p^\mu(X) \subseteq S_p(X)$ is the group of $\mu-$small p-chains.

The idea of μ-small simplices is to control where each simplex lies with respect to an open cover $\mu = \{U_j\}$ of X. A simplex is called μ-small if its entire image is contained in one element of the cover. Intuitively, this means the simplex is "small enough" not to cross between different sets of the cover. Such μ-small chains form a subcomplex of the singular chain complex and are useful in the proof of the excision theorem.

Remark. *The boundary operator $\partial_p : S_p(X) \to S_{p-1}(X)$ preserves μ-smallness. Indeed, if σ is μ-small with $\sigma(\Delta_p) \subseteq U_{j_0}$, then each face of σ also lies in U_{j_0}, so $\partial_p\sigma$ is a μ-small $(p-1)$-chain. Hence, the subgroups $S_p^\mu(X)$ form a subchain complex of the singular chain complex:*

$$\cdots \xrightarrow{\partial_{p+1}} S_p^\mu(X) \xrightarrow{\partial_p} S_{p-1}^\mu(X) \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_1} S_0^\mu(X) \to 0.$$

We can define $H_p^\mu(X)$ as the homology groups of this chain complex.

Theorem. *Let X be a topological space and $\mu = \{U_j\}$ an open cover of X such that $X = \bigcup_j \operatorname{Int} U_j$. Then the inclusion of chain complexes*

$$i : S_*^\mu(X) \hookrightarrow S_*(X)$$

induces an isomorphism in homology:

$$i_* : H_p(S_*^\mu(X)) \xrightarrow{\cong} H_p(X) \quad \text{for all } p \geq 0.$$

We are now ready to prove the excision theorem, which we will use to calculate our first not trivial homology of the spheres S^n.

Proof. Let X be a topological space, $A \subseteq X$, and $U \subseteq X$ an open subset such that $\overline{U} \subseteq \operatorname{Int}(A)$.

We define the family

$$\mu = \{A,\, X \setminus \overline{U}\}.$$

We claim that μ satisfies the condition

$$X = \operatorname{Int}(A) \cup \operatorname{Int}(X \setminus \overline{U}),$$

which allows us to apply the μ-small theorem.

Indeed, since $\overline{U} \subseteq \operatorname{Int}(A)$, for any $x \in X$ we have two possibilities:

- If $x \in \overline{U}$, then $x \in \operatorname{Int}(A)$.
- If $x \notin \overline{U}$, then $x \in X \setminus \overline{U} \subseteq \operatorname{Int}(X \setminus U) \subseteq \operatorname{Int}(X \setminus \overline{U})$.

Thus every $x \in X$ belongs to $\operatorname{Int}(A) \cup \operatorname{Int}(X \setminus \overline{U})$, and therefore

$$X = \operatorname{Int}(A) \cup \operatorname{Int}(X \setminus \overline{U}).$$

This verifies that μ is an open cover of X in the sense required by the μ-small simplices theorem.

Consequently, by the μ-small theorem, the inclusion map

$$i : S^\mu(X, A) \hookrightarrow S(X, A)$$

induces an isomorphism in homology. Recall the definition of relative homology

$$S^\mu(X, A) = \frac{S^\mu(X)}{S^\mu(A)}$$

. Now the claim is that this inclusion map is an isomorphsim

$$\frac{S_p(X \setminus U)}{S_p(A \setminus U)} \to \frac{S_p(X)}{S_p(A)}$$

is an isomporphism. To prove this we first define the auxilary map

$$\phi : S_p(X \setminus U) \longrightarrow \frac{S_p^\mu(X)}{S_p^\mu(A)}$$

as the composition of the inclusion and projection maps:

$$S_p(X \setminus U) \xrightarrow{\text{inclusion}} S_p^\mu(X) \xrightarrow{\text{projection}} \frac{S_p^\mu(X)}{S_p^\mu(A)}.$$

That is,

$$\phi = \text{projection} \circ \text{inclusion}.$$

Now let

$$c_p + S_p^\mu(A) \ \in \ \frac{S_p^\mu(X)}{S_p^\mu(A)}.$$

Then c_p is a μ-small chain. Hence c_p is a sum of simplices whose images are contained either in A or in $X \setminus U$.

If $\sigma_i : \Delta_p \to X$ is one such simplex and $\sigma_i(\Delta_p) \subseteq A$, then

$$\sigma_i + S_p^\mu(A) = S_p^\mu(A) \quad \text{in} \quad \frac{S_p^\mu(X)}{S_p^\mu(A)}.$$

Therefore, there exists a chain

$$c_p' = \sum \text{(simplices with image contained in } X \setminus U\text{)}$$

such that

$$c_p' + S_p^\mu(A) = c_p + S_p^\mu(A).$$

We note that $c_p' \in S_p(X \setminus U)$, and

$$\phi(c_p') = c_p' + S_p^\mu(A) = c_p + S_p^\mu(A).$$

Hence ϕ is surjective. We now claim that

$$\ker \phi = S_p(A \setminus U).$$

($\subseteq$ direction). Let $c_p \in S_p(A \setminus U)$. Then by definition,

$$\phi(c_p) = c_p + S_p^\mu(A) = S_p^\mu(A) = 0 \quad \text{in} \ \frac{S_p^\mu(X)}{S_p^\mu(A)}.$$

Hence $c_p \in \ker \phi$.

($\supseteq$ direction). Let $c_p \in S_p(X \setminus U)$ such that

$$\phi(c_p) = c_p + S_p^\mu(A) = 0.$$

This means $c_p \in S_p^\mu(A)$.

Therefore, every simplex of c_p has image contained in A. Since $c_p \in S_p(X \setminus U)$ as well, the images of its simplices are disjoint from U. Combining these two facts, we conclude that

$$c_p \in S_p(A \setminus U).$$

Thus both inclusions hold, and therefore

$$\ker \phi = S_p(A \setminus U).$$

By the First Isomorphism Theorem, since $\ker \phi = S_p(A \setminus U)$ and ϕ is surjective, we have

$$\frac{S_p(X \setminus U)}{S_p(A \setminus U)} \cong \frac{S_p(X)}{S_p(A)} = S_p(X, A)$$

Passing to homology, this chain-level isomorphism induces

$$\overline{\phi}_* : H_p(X \setminus U,\, A \setminus U) \ \xrightarrow{\cong}\ H_p^\mu(X, A).$$

Since the inclusion $i : S_*^\mu(X, A) \hookrightarrow S_*(X, A)$ induces an isomorphism in homology by the μ-small theorem, we obtain that

$$j_* = i_* \circ \phi_* : H_p(X \setminus U,\, A \setminus U) \ \xrightarrow{\cong}\ H_p(X, A)$$

for all p. □

So if we are studying the relative homology of a space, we can remove a piece of the contained entirely withint the subset without affecting the homology groups. This will allow to get homology groups for the spheres.

Theorem.

$$\tilde{H}_i(S^n) = 0 \text{ if } i \neq n$$

When $i = n$

$$\tilde{H}_i(S^n) = \mathbb{Z}$$

Proof. We proceed by induction. For $n = 0$, we have the space $S^0 = \{-1, 1\}$. This is a disjoint space so

$$H_0(S^0) \cong H_0(-1) \oplus H_0(1) \cong \mathbb{Z} \oplus \mathbb{Z} \Rightarrow \tilde{H}_0(S^0) \cong \mathbb{Z}$$

All the higher homology groups are zero because the points are 0 dimensional. Now we proceed to the inductive step. For $n > 0$, we have

$$S^n = \left\{ x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i^2 = 1 \right\}.$$

We also define the upper hemisphere:

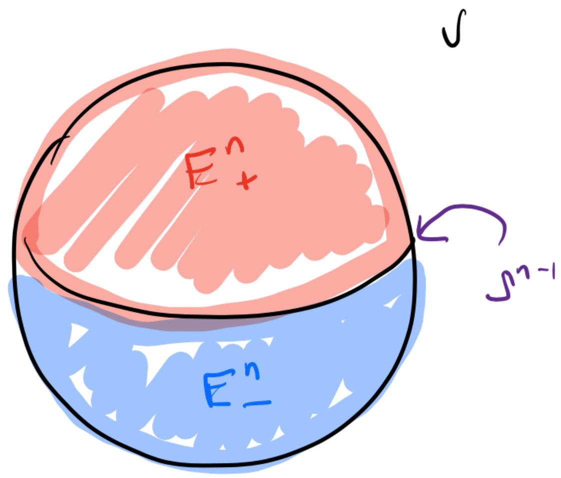
$$E^n_+ = \{(x_0, x_1, \ldots, x_n) \in S^n \mid x_n \geq 0\}.$$

and thelower hemisphere:

$$E^n_- = \{(x_0, x_1, \ldots, x_n) \in S^n \mid x_n \leq 0\}.$$

with their common boundary (the "equator"):

$$S^{n-1} = \{(x_0, x_1, \ldots, x_n) \in S^n \mid x_n = 0\} = E^n_+ \cap E^n_-$$



We also define the south pole $q = \{-1, 0, ..., 0\}$. The next step requires a couple of definitions which we put here in the middle of this proof.

Definition (Retraction)**.** *Let X be a topological space and let $A \subseteq X$ be a subspace. A continuous map*

$$r : X \to A$$

is called a retraction *if*

$$r|_A = \mathrm{Id}_A.$$

In this case, A is called a retract *of X.*

Remark. *Let $i : A \hookrightarrow X$ denote the inclusion map. Then $r : X \to A$ is a retraction if and only if*

$$r \circ i = \mathrm{Id}_A .$$

Definition (Deformation Retraction)**.** *A retraction $r : X \to A$ is called a* deformation retraction *if the composition*

$$i \circ r : X \to X$$

is homotopic to the identity map Id_X, i.e.

$$i \circ r \simeq \mathrm{Id}_X .$$

In this case, we say that A is a deformation retract *of X.*

Remark. *A deformation retraction is a special kind of homotopy equivalence.*

Our next step is to establish

$$H_i(E^n_+, S^{n-1}) \xrightarrow[\text{isom.}]{\cong} H_{i-1}(S^{n-1}), \quad \forall i.$$

by finding a long exact sequence in Homology for the pair (E^n_+, S^{n-1}). Since S^{n-1} is a subspace of the closed upper hemisphere E^n_+, we can consider the homology of the pair (E^n_+, S^{n-1}). From the definition of relative homology, there is a long exact sequence connecting the homology of the subspace, the total space, and the pair:

$$\cdots \longrightarrow H_i(S^{n-1}) \xrightarrow{i_*} H_i(E^n_+) \xrightarrow{j_*} H_i(E^n_+, S^{n-1}) \xrightarrow{\partial} H_{i-1}(S^{n-1}) \xrightarrow{i_*} H_{i-1}(E^n_+) \longrightarrow \cdots$$

Because E^n_+ is homeomorphic to a closed n-disk, it is contractible and therefore has

$$H_i(E^n_+) = \begin{cases} \mathbb{Z}, & i = 0, \\ 0, & i > 0. \end{cases}$$

Substituting this into the sequence gives, for all $i \geq 2$,

$$0 \longrightarrow H_i(E^n_+, S^{n-1}) \xrightarrow{\partial} H_{i-1}(S^{n-1}) \longrightarrow 0,$$

so that

$$H_i(E^n_+, S^{n-1}) \cong H_{i-1}(S^{n-1}).$$

The remaining low-degree terms can be checked similarly, confirming that the same relationship holds for all i. Hence we obtain

$$H_i(E^n_+, S^{n-1}) \xrightarrow[\text{isom.}]{\cong} H_{i-1}(S^{n-1}), \quad \forall i.$$

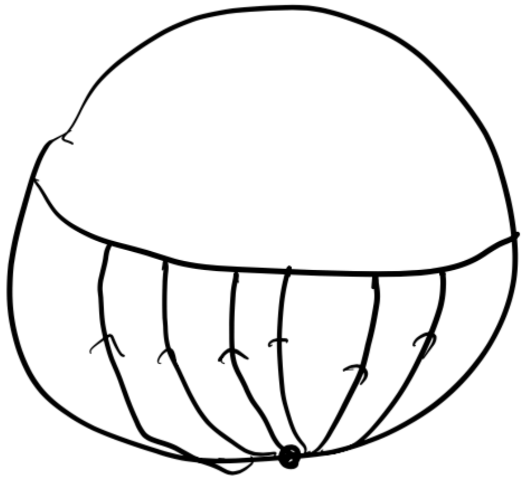
Next, we relate the pair (E^n_+, S^{n-1}) to the sphere S^n itself. Consider the inclusion

$$i : E^n_+ \hookrightarrow S^n \setminus \{q\}$$

and define a map

$$r : S^n \setminus \{q\} \longrightarrow E^n_+$$

by projecting each point of the lower hemisphere along its great circle through q to the upper hemisphere.



Geometrically, r fixes every point of E_+^n and sends the lower hemisphere continuously onto it. This map satisfies

$$r \circ i = \mathsf{Id}_{E_+^n}, \qquad i \circ r \simeq \mathsf{Id}_{S^n \setminus \{q\}},$$

so r is a deformation retraction of $S^n \setminus \{q\}$ onto E_+^n. Because r restricts to the identity on the common boundary S^{n-1}, it induces a homotopy equivalence of pairs:

$$r : (S^n \setminus \{q\}, E_+^n \setminus \{q\}) \;\simeq\; (E_+^n, S^{n-1}).$$

By the homotopy invariance of homology, this gives

$$H_i(E_+^n, S^{n-1}) \xrightarrow[\text{isom.}]{r_*} H_i(S^n \setminus \{q\}, E_+^n \setminus \{q\}), \quad \forall i.$$

We also consider the pair (E_+^n, S^{n-1}) to (S^n, E_-^n) and the long exact sequence in reduced homology

$$\cdots \longrightarrow \tilde{H}_i(E_-^n) \longrightarrow \tilde{H}_i(S^n) \longrightarrow H_i(S^n, E_-^n) \longrightarrow \tilde{H}_{i-1}(E_-^n) \longrightarrow \cdots$$

Since E_-^n is also contractible, we have $\tilde{H}_i(E_-^n) = 0$ for all i, and the sequence reduces to

$$\tilde{H}_i(S^n) \xrightarrow[\text{isom.}]{} H_i(S^n, E_-^n), \quad \forall i.$$

Next, we apply the Excision Theorem Taking $X = S^n$, $A = E_-^n$, and U to be a small neighborhood of the south pole q, we have $\overline{U} \subset \mathsf{Int}(E_-^n)$, so excision yields

$$H_i(S^n, E_-^n) \;\cong\; H_i(S^n \setminus \{q\}, E_+^n \setminus \{q\}).$$

Combining this with the previous isomorphism gives

$$H_i(S^n, E_-^n) \;\cong\; H_i(E_+^n, S^{n-1}).$$

From the inductive step, we already know that

$$H_i(E_+^n, S^{n-1}) \cong H_{i-1}(S^{n-1}).$$

Therefore,

$$\tilde{H}_i(S^n) \;\cong\; H_i(S^n, E_-^n) \;\cong\; H_i(E_+^n, S^{n-1}) \;\cong\; H_{i-1}(S^{n-1}),$$

which completes the induction.

$$H_{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z}, & i-1 = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z}, & i = n, \\ 0, & i \neq n. \end{cases}$$

□

Exercise. *Let X be a topological space and let $x_0 \in X$. Prove that for all $i \in \mathbb{Z}$ we have*

$$\widetilde{H}_i(X) \cong H_i(X, x_0).$$

Solution

Since we aim to find an isomorphism between homology groups, it is enough to construct an isomorphism between the corresponding chain complexes. Because reduced homology only differs from ordinary homology in degree $n = 0$, we start with the simpler case $n > 0$. For each $n \geq 0$, the group $S_n(X)$ consists of all finite integer combinations of singular n-simplices in X. Concretely, a typical element of $S_n(X)$ is

$$c = a_1\sigma_1 + a_2\sigma_2 + \cdots + a_k\sigma_k,$$

where each $\sigma_i : \Delta^n \to X$ is a continuous map (called a singular simplex) and $a_i \in \mathbb{Z}$ are integer coefficients. We can think of c as a formal sum of oriented n–dimensional pieces drawn in X. The boundary operator $\partial_n : S_n(X) \to S_{n-1}(X)$ is defined on each simplex by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]},$$

that is, the alternating sum of its $(n-1)$–dimensional faces. Extending this linearly gives the full chain complex

$$\cdots \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} S_0(X) \to 0.$$

Now we fix a basepoint $x_0 \in X$. Among all singular simplices $\sigma : \Delta^n \to X$, there are the constant ones that send every point of Δ^n to x_0. Each such map σ_{x_0} is "entirely collapsed" inside x_0 and has no geometric extent. We denote by $S_n(x_0)$ the set of all integer combinations of these constant simplices:

$$S_n(x_0) = \{\, a_1\sigma_{x_0}^{(1)} + \cdots + a_m\sigma_{x_0}^{(m)} \mid a_i \in \mathbb{Z} \,\}.$$

Since all constant simplices have the same image point x_0, this group is essentially a copy of $\mathbb{Z}$ sitting inside $S_n(X)$. The relative chain group is the quotient

$$S_n(X, x_0) = S_n(X)/S_n(x_0).$$

This means that in $S_n(X, x_0)$ we identify any two chains that differ by an element of $S_n(x_0)$. Equivalently, we declare all chains lying completely inside the point x_0 to be zero. Formally, for a chain $c \in S_n(X)$, its equivalence class is

$$[c] = c + S_n(x_0) = \{\, c + d \mid d \in S_n(x_0) \,\}.$$

Two chains $c_1, c_2 \in S_n(X)$ represent the same element of $S_n(X, x_0)$ if and only if $c_1 - c_2$ consists entirely of constant simplices at x_0. For $n > 0$, we now define

$$\Phi_n : \widetilde{S}_n(X) = S_n(X) \longrightarrow S_n(X, x_0)$$

by sending each chain to its equivalence class modulo the constant simplices:

$$\Phi_n(c) = [c] = c + S_n(x_0).$$

Explicitly, on a single simplex σ,

$$\Phi_n(\sigma) = \begin{cases} [\sigma], & \text{if } \sigma \text{ is not constant,} \\ 0, & \text{if } \sigma(t) = x_0 \text{ for all } t. \end{cases}$$

Linearly extending this rule to sums of simplices gives $\Phi_n(c) = \sum_i a_i[\sigma_i]$. Geometrically, Φ_n simply "forgets" any collapsed simplices sitting at the basepoint. Now we check that Φ_n is an isomorphism. Every element of $S_n(X, x_0)$ is by definition an equivalence class $[c]$ with $c \in S_n(X)$. But $\Phi_n(c) = [c]$ for all c, so every class is hit. Hence Φ_n is surjective. Now suppose $\Phi_n(c) = 0$. Then $[c] = 0$ in the quotient, which means $c \in S_n(x_0)$. But $S_n(x_0)$ consists precisely of chains supported entirely at x_0, which are declared to be zero in both theories for $n > 0$. Thus Φ_n has trivial kernel, and it is injective. So Φ is an isomorphism. But we need it to comput with the boundary opearator to have a chain complex. For any $c \in S_n(X)$ we have

$$\Phi_{n-1}(\partial_n c) = [\, \partial_n c\,] = \partial_n^{\text{rel}}([c]) = \partial_n^{\text{rel}}(\Phi_n(c)).$$

Hence Φ commutes with the boundary operator, and therefore is a chain map. Now we must adress $n = 0$. Here, reduced chains differ from ordinary ones:

$$\widetilde{S}_0(X) = \ker(\varepsilon : S_0(X) \to \mathbb{Z}), \qquad \varepsilon\left(\sum_i a_i[x_i]\right) = \sum_i a_i.$$

This means $\widetilde{S}_0(X)$ consists of integer combinations of points whose total coefficient is zero—in other words, formal differences of points in X. We define

$$\Phi_0 : \widetilde{S}_0(X) \longrightarrow S_0(X, x_0)$$

by the same rule $\Phi_0(c) = [c] = c + S_0(x_0)$. To see that this is an isomorphism, note that every class $[c] \in S_0(X, x_0)$ has a representative $c' = c - m[x_0]$ with $\varepsilon(c') = 0$, so Φ_0 is surjective. If $c \in \ker \varepsilon$ and $\Phi_0(c) = 0$, then $c \in S_0(x_0)$, hence $c = k[x_0]$ for some $k \in \mathbb{Z}$. But $\varepsilon(c) = k = 0$, so $c = 0$. Thus Φ_0 is injective. Therefore Φ_0 is also an isomorphism.

Since we have an isomorphism between the chain complexes

$$\Phi_* : \widetilde{S}_*(X) \xrightarrow{\cong} S_*(X, x_0),$$

we also have isomorphisms between the corresponding homology groups

$$\widetilde{H}_i(X) \cong H_i(X, x_0) \qquad \text{for all } i \in \mathbb{Z}.$$

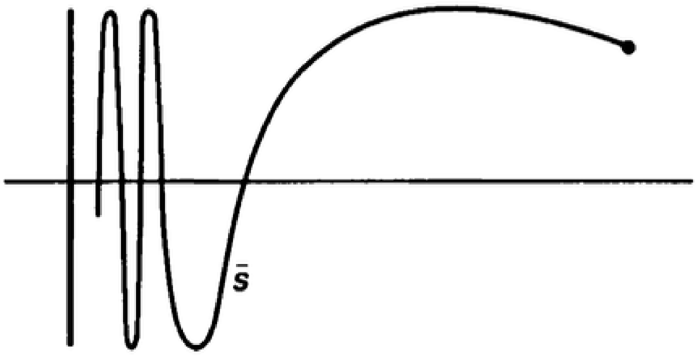
This means that reduced Intuitively, the reduced homology groups $\widetilde{H}_n(X)$ remove the redundant information that ordinary homology $H_n(X)$ carries about the mere existence of a point. In standard homology, each connected component contributes a copy of $\mathbb{Z}$ in degree 0, coming from constant simplices that collapse the whole space to a single point. Reduced homology "forgets" this trivial contribution, keeping only the part that reflects the actual shape of X. Thus both $\widetilde{H}_n(X)$ and $H_n(X, x_0)$ describe the same essential topological information: they ignore the simplices supported entirely at the basepoint and record only how X differs from a single point.

Exercise. *Compute the homology groups of the following subspace of $\mathbb{R}^2$:*

$$A = \{(0, y) \mid -1 \le y \le 1\} \cup \{(x, \sin(1/x)) \mid 0 < x \le 1\}.$$

Solution

This is the topologist's sine curve mentioned the Intermezzo 2.



It is connected but not path connected and has two components.

$$C_1 = \{(0, y) \mid -1 \leq y \leq 1\} \qquad C_2 = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}.$$

Each path component is contractible to a point. So

$$H_0 = \mathbb{Z} \oplus \mathbb{Z}$$

We see that there are no closed loops to make a cycles that are not boundaries, so

$$H_1 = 0$$

Since it is a 1 dimenstional surface embedded in $\mathbb{R}^2$ all higher homology groups are zero. This emphasizes the fact that H_0 is measuring path components, not components. We must be careful when these don't coincide.

Exercise. *Compute the homology groups of the following subspace of $\mathbb{R}^2$:*

$$\{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\} \,\cup\, \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid 0 < x \leq 1\}.$$

Exercise. *Find explicitly a generator of $H_1(S^1)$.*

Lecture 12 – ???

Lecture X – Beginning of Simplicial Homology

Exercise. *Let Δ^3 be the standard 3-simplex, and consider the topological space X obtained by identifying the edge $[v_0, v_1]$ with the edge $[v_0, v_2]$ and $[v_1, v_3]$ with $[v_2, v_3]$. Prove that S^2 is a deformation retract of X. What identification of the edges of Δ^3 gives a topological space having the torus T^2 as a deformation retract? And what identification gives the projective plane $\mathbb{RP}^2$?*

Exercise. *Compute the homology groups of $\mathbb{RP}^2$.*