Week 14 & 15 - Exercises

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Advanced Linear Algebra

Week 14 - Exercise 1

1. Consider $\mathbb{R}^2$ endowed with the usual dot-product as inner product. Give a geometric interpretation of what it means for two vectors to be orthogonal. Also, what does Lemma 8.3 have to do with the usual Pythagoras theorem about right angled triangles?

Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Orthogonality means that

$$0 = x \cdot y = (x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 y_2$$

Geometrically, this means that the two vectors are perpendicular. Lemma 8.3 says that

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

when x and y are orthogonal. Geometrically, $x + y$, x and y yield a right angled triangle with the bases $\|x\|$, $\|y\|$ and hypotenuse $\|x + y\|$, and thus Lemma 8.3 reflects the usual Pythagoras theorem about right angled triangles.

Week 14 - Exercise 2

2. If you have not done so already, try out the Gram-Schmidt orthonormalization algorithm on a set of vectors. For instance $\{(2,0,0),(1,1,0),(1,1,1)\} \subset \mathbb{R}^3$.

We set $y_1 = (2,0,0)$, $y_2 = (1,1,0)$, $y_3 = (1,1,1)$. Then the Gram-Schmidt process constructs three orthonormal vectors x_1, x_2, x_3 in the following way:

$$x_1 = \frac{y_1}{\|y_1\|} = \frac{1}{2}(2,0,0) = (1,0,0)$$

$$\begin{aligned} x_2 &= y_2 - \langle y_2, x_1 \rangle x_1 \\ &= (1,1,0) - \langle (1,1,0),(1,0,0) \rangle (1,0,0) \\ &= (1,1,0) - (1,0,0) \\ &= (0,1,0) \end{aligned}$$

$$\begin{aligned} x_3 &= y_3 - \langle y_3, x_1 \rangle x_1 - \langle y_3, x_2 \rangle x_2 \\ &= (1,1,1) - (1,0,0) - (0,1,0) \\ &= (0,0,1) \end{aligned}$$

Notice that in general we need to normalize x_2 and x_3 in the end, but in this case $(0,1,0)$ and $(0,0,1)$ are already normalized.

Week 14 - Exercise 3

3. Let V be an inner product space and let $X \subseteq V$ be a subset. Show that $(\mathrm{span}(X))^{\perp} = X^{\perp}$

Let $y \in X^{\perp}$. Take $\sum_{x \in X}^{f} \alpha_x x \in \mathrm{span}(X)$. Then

$$\langle \sum_{x \in X}^{f} \alpha_x x, y \rangle = \sum_{x \in X}^{f} \alpha_x \langle x, y \rangle = 0$$

Hence, $y \in \mathrm{span}(X)^{\perp}$. It follows that $X^{\perp} \subseteq \mathrm{span}(X)^{\perp}$.

Conversely, let $y \in \mathrm{span}(X)^{\perp}$. Take $x \in X$. Then in particular $x \in \mathrm{span}(X)$ and consequently $\langle x, y \rangle = 0$. Hence $y \in X^{\perp}$, and we conclude that $\mathrm{span}(X)^{\perp} \subseteq X^{\perp}$.

Week 14 - Exercise 4

4. Let V be an inner product space. Show that $d(x, y) := \|x - y\|$ defines a metric on V.

Let $x, y \in V$. Then

1. $d(x, y) = 0 \Leftrightarrow \|x - y\| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$
2. $d(x, y) = \|x - y\| = |-1| \cdot \|x - y\| = \|(-1)(x - y)\| = \|y - x\| = d(y, x)$
3. $d(x, y) = \|x - y\| = \|(x - z) + (z - y)\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$.

Week 14 - Exercise 5

5. If time permits, understand how Theorem 8.11 says that the orthogonal projection of $y \in V$ onto U is the vector in U closest to y with respect to the metric from the previous question.

Theorem 8.11. *Let $v \in V$ and $u \in U$. Then $v - u \perp U$ if and only if*

$$\|v - u\| = \min_{x \in U} \|v - x\|. \tag{8.5}$$

Take $y \in V$. We write $E(y)$ for the orthogonal projection onto U. Since $y - E(y) \perp U$, we have by Theorem 8.11 that

$$d(y, E(y)) = \|y - E(y)\| = \min_{x \in U} \|y - x\| = \min_{x \in U} d(y, x)$$

Week 14 - Exercise 6

6. Consider the space $C([0,1],\mathbb{R})$ of continuous functions on $[0,1]$. Show that

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx,$$

defines an inner product on $C([0,1],\mathbb{R})$.

Let $f, g, h \in C([0,1],\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$. Then

1. $\langle \alpha f + \beta g, h\rangle = \int_0^1 (\alpha f(x) + \beta g(x))h(x)\,dx = \int_0^1 (\alpha f(x)h(x) + \beta g(x)h(x))\,dx = \alpha \int_0^1 f(x)h(x)\,dx + \beta \int_0^1 g(x)h(x)\,dx = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$
2. $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx = \langle g, f\rangle$
3. $\langle f, f\rangle = \int_0^1 f(x)^2\,dx \geq 0$ and $\langle f, f\rangle = \int_0^1 f(x)^2\,dx = 0$ if and only if $f = 0$ since f is continuous.

Week 14 - Exercise 7

7. Let $1_{[0,1]}$ denote the function on $[0,1]$ which is constantly equal to one. Show that $E(f) = (\int_0^1 f(x)dx) \cdot 1_{[0,1]}$ is the orthogonal projection onto $\operatorname{span}(1_{[0,1]})$.

Take $f \in C([0,1], \mathbb{R})$. Then

$$f = \left(\int_0^1 f(x)\,dx\right) \cdot 1_{[0,1]} + f - \left(\int_0^1 f(x)\,dx\right) \cdot 1_{[0,1]}.$$

and

$$f - \left(\int_0^1 f(x)\,dx\right) \cdot 1_{[0,1]} \in \operatorname{span}(1_{[0,1]})^{\perp}$$

Hence,

$$E(f) = \left(\int_0^1 f(x)\,dx\right) \cdot 1_{[0,1]}$$

is the orthogonal projection onto $\operatorname{span}(1_{[0,1]})$.

Week 15 - Exercise 1

1. Consider the map $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $A(x_1, x_2) = (2x_1, x_1 - x_2)$. Determine the adjoint A^* with respect to the ordinary inner product on $\mathbb{R}^2$.

Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. The adjoint A^* is defined by the criterion

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

We have

$$\begin{aligned} A(x_1, x_2) \cdot (y_1, y_2) &= (2x_1, x_1 - x_2) \cdot (y_1, y_2) \\ &= 2x_1 y_1 + x_1 y_2 - x_2 y_2 \\ &= x_1(2y_1 + y_2) + x_2(-y_2) \\ &= (x_1, x_2) \cdot (2y_1 + y_2, -y_2) \end{aligned}$$

Thus,

$$A^*(y_1, y_2) = (2y_1 + y_2, -y_2)$$

Week 15 - Exercise 2

2. Let V be an inner product space and consider the map $\Phi\colon V \to V'$ given by $\Phi(v) = \langle -\,, v\rangle$ that we studied in the lecture. Show that this is indeed an antilinear map as I claimed.

Let $v, w \in V$, $\alpha, \beta \in \mathbb{F}$, $x \in V$. Then

$$\Phi(\alpha v + \beta w)(x) = \langle x, \alpha v + \beta w\rangle = \bar{\alpha}\langle x, v\rangle + \bar{\beta}\langle x, w\rangle$$

and consequently

$$\Phi(\alpha v + \beta w) = \bar{\alpha}\langle -, v\rangle + \bar{\beta}\langle -, w\rangle = \bar{\alpha}\Phi(v) + \bar{\beta}\Phi(w)$$

Week 15 - Exercise 3

3. Let U and V be finite dimensional inner product spaces and let $A \in \mathrm{Hom}(U, V)$ be given. Show that the map A^* that we constructed in the lecture is actually linear, so that have defined an element in $\mathrm{Hom}(V, U)$.

Let $v, w \in V$, $\alpha, \beta \in \mathbb{F}$ and $u \in U$. We have

$$\begin{aligned}
\langle A(u), \alpha v + \beta w \rangle &= \langle A(u), \alpha v \rangle + \langle A(u), \beta w \rangle \\
&= \bar{\alpha} \langle A(u), v \rangle + \bar{\beta} \langle A(u), w \rangle \\
&= \bar{\alpha} \langle u, A^*(v) \rangle + \bar{\beta} \langle u, A^*(w) \rangle \\
&= \langle u, \alpha A^\star(v) \rangle + \langle u, \beta A^*(w) \rangle \\
&= \langle u, \alpha A^\star(v) + \beta A^\star(w) \rangle
\end{aligned}$$

Thus,

$$A^*(\alpha v + \beta w) = \alpha A^\star(v) + \beta A^\star(w)$$

Week 15 - Exercise 4

4. Consider an inner product space V over $\mathbb{C}$ and let $\lambda \in \mathbb{C}$ have $|\lambda| = 1$. Show that the map $U : V \to V$ given by $U(v) = \lambda v$ is a unitary and determine its adjoint.

Linearity of U is clear. Let $v \in V$ and assume $U(v) = 0$. Then $\lambda v = 0$ and consequently $v = \lambda^{-1}\lambda v = 0$. Thus, U is injective. Next, we show surjectivity. Let $v \in V$. Then $\lambda^{-1} v \in V$ and $U(\lambda^{-1} v) = \lambda^{-1}\lambda v = v$. Thus, U is an isomorphism.

It remains to be shown that U preserves inner products. Let $v, w \in V$. Then

$$\langle U(v), U(w)\rangle = \langle \lambda v, \lambda w\rangle = \bar{\lambda}\lambda\langle v, w\rangle = |\lambda|^2\langle v, w\rangle = \langle v, w\rangle$$

Thus, U is a unitary.

To determine the adjoint, we let $v, w \in V$. We have

$$\langle U(v), w\rangle = \langle \lambda v, w\rangle = \lambda\langle v, w\rangle = \langle v, \bar{\lambda} w\rangle$$

Thus,

$$U^*(w) = \bar{\lambda} w$$

Week 15 - Exercise 5

5. Let V be an inner product space and $U : V \to V$ a unitary. Show that U is an isometry with respect to the metric d studied in the exercise above (i.e. prove that $d(Ux, Uy) = d(x, y)$ for all $x, y \in V$).

Let $x, y \in V$. We have

$$\begin{aligned} d(U(x), U(y)) &= \|U(x) - U(y)\| \\ &= \sqrt{\langle U(x) - U(y), U(x) - U(y)\rangle} \\ &= \sqrt{\langle U(x - y), U(x - y)\rangle} \\ &= \sqrt{\langle x - y, x - y\rangle} \\ &= \|x - y\| \\ &= d(x, y) \end{aligned}$$

Week 15 - Exercise 6

6. Consider the linear map $A\colon \mathbb{C}^2 \to \mathbb{C}^3$ given by $A(z_1, z_2) = (z_1, z_2, 0)$. Determine A^* and show that A satisfies the relation $A^*A = I_{\mathbb{C}^2}$ but not the relation $AA^* = I_{\mathbb{C}^3}$, thus showing the the assumption in Lemma 8.23 is necessary.

Let $(z_1, z_2) \in \mathbb{C}^2$ and $(x_1, x_2, x_3) \in \mathbb{C}^3$. Then

$$A(z_1, z_2) \cdot (x_1, x_2, x_3) = (z_1, z_2, 0) \cdot (x_1, x_2, x_3) = z_1x_1 + z_2x_2 = (z_1, z_2) \cdot (x_1, x_2)$$

Thus,

$$A^\star(x_1, x_2, x_3) = (x_1, x_2)$$

Then

$$A^*A(z_1, z_2) = A^*(z_1, z_2, 0) = (z_1, z_2) \Longrightarrow A^*A = I_{\mathbb{C}^2}$$

However,

$$AA^\star(x_1, x_2, x_3) = A(x_1, x_2) = (x_1, x_2, 0) \Longrightarrow AA^* \neq I_{\mathbb{C}^3}$$

Week 15 - Exercise 7

Lemma 8.23. *Assume* $\dim U = \dim V < \infty$*. The following are equivalent*

(i) *A preserves inner products*

(ii) *A carries orthonormal bases to orthonormal bases*

(iii) $A^*A = I$

(iv) *A is a unitary isomorphism.*

We prove the implications $(i) \implies (iii) \implies (iv) \implies (ii) \implies (i)$.

$(i) \implies (iii)$: Assume that A preserves inner products. Let $x, y \in U$. Then

$$\langle A^* Ax, y\rangle = \langle Ax, Ay\rangle = \langle x, y\rangle$$

and consequently $A^*A = I_U$.

$(iii) \implies (iv)$: Assume that $A^\star A = I_U$. First, we show that A is an isomorphism. Since the dimensions agree, it suffices to show that A is injective. So let $u \in U$ and assume that $Au = 0$. Then $u = A^*Au = A^*0 = 0$ and consequently A is injective. It remains to check that A preserves inner products. Let $x, y \in U$. Then

$$\langle Ax, Ay\rangle = \langle x, A^*Ay\rangle = \langle x, y\rangle$$

Thus, A is a unitary isomorphism.

Week 15 - Exercise 7 - continued

Lemma 8.23. *Assume* $\dim U = \dim V < \infty$. *The following are equivalent*

(i) *A preserves inner products*

(ii) *A carries orthonormal bases to orthonormal bases*

(iii) $A^*A = I$

(iv) *A is a unitary isomorphism.*

$(iv) \implies (ii)$: Assume that A is a unitary isomorphism. Let $B = \{u_1, \ldots, u_n\}$ be an orthonormal basis for U. We must show that $C = \{Au_1, \ldots, Au_n\}$ is an orthonormal basis for V. Since A is an isomorphism and B is basis for U, we know that C is a basis for V. Further, C is orthonormal since

$$\langle Au_j, Au_k \rangle = \langle u_j, u_k \rangle = \left\{ \begin{array}{l} 1 \text{ for } j = k \\ 0 \text{ for } j \neq k \end{array} \right\}$$

$(ii) \implies (i)$: Assume that A carries orthonormal bases to orthonormal bases. Since U is finite-dimensional, it has an orthonormal basis $\{u_1, \ldots, u_n\}$ (see Theorem 8.7). By assumption $\{Au_1, \ldots, Au_n\}$ is then an orthonormal basis for V. Thus,

$$\langle Au_j, Au_k \rangle = \left\{ \begin{array}{l} 1 \text{ for } j = k \\ 0 \text{ for } j \neq k \end{array} \right\} = \langle u_j, u_k \rangle$$

Now, let $x = \sum_{j=1}^n \alpha_j u_j, y = \sum_{k=1}^n \beta_k u_k \in U$. Then

$$\langle Ax, Ay \rangle = \sum_{j=1}^n \sum_{k=1}^n \alpha_j \bar{\beta}_k \langle Au_j, Au_k \rangle = \sum_{j=1}^n \sum_{k=1}^n \alpha_j \bar{\beta}_k \langle u_j, u_k \rangle = \langle x, y \rangle$$

which shows that A preserves inner products.