Advanced Linear Algebra

Week 18

Jamie Gabe

SDU
DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE

Recall that $A \in \mathrm{End}(V)$ is nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$.

Theorem (7.4 (slightly different than lecture notes))

Suppose $\dim V < \infty$ *and let* $A \in \mathrm{End}(V)$. *TFAE:*

(a) *A is nilpotent;*

(b) *there exists a basis* $\{x_1, \ldots, x_n\}$ *for* V *such that* $Ax_1 = 0$ *and* $Ax_j \in \mathrm{Span}\{x_1, \ldots, x_{j-1}\}$ *for* $j = 2, \ldots, n$;

(c) *there exists an ordered basis* $\mathcal{B}$ *for* V *such that* ${}_{\mathcal{B}}[A]_{\mathcal{B}}$ *is strictly upper triangular.*

In particular, if A is nilpotent, then its index of nilpotency is at most $\dim V$.

Moreover, (which we saw in the proof) $N = \bigcup_{k \in \mathbb{N}} N(A^k)$ and $R = \bigcap_{k \in \mathbb{N}} R(A^k)$.

Consider the following complex matrices:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 4 & i \\ 0 & 0 & -2+i \\ 0 & 0 & 0 \end{pmatrix}$$

Question: Which of the above matrices are nilpotent? (Multiple answers)

Theorem (7.5)

Let V be a finite-dimensional vector space and $A \in \mathrm{End}(V)$.

(a) *There exist (unique) A-invariant subspaces $N, R \subseteq V$ such that $V = N \oplus R$, $A|_N \in \mathrm{End}(N)$ is nilpotent, and $A|_R \in \mathrm{End}(R)$ is invertible;*

(b) *If $M \subseteq V$ is an A-invariant subspace for which $A|_M \in \mathrm{End}(M)$ is nilpotent, then $M \subseteq N$;*

(c) *If $S \subseteq V$ is an A-invariant subspace for which $A|_S \in \mathrm{End}(S)$ is invertible, then $S \subseteq R$.*

Recall: $E_1 \in \mathrm{End}(V)$ is an idempotent if $E_1^2 = E_1$. When this is the case, $V = R(E_1) \oplus R(I - E_1)$ and E_1 is the projection onto $R(E_1)$ along $R(I - E_1)$.
Note that $E_2 := (I - E_1)$ is also an idempotent,
$E_1 E_2 = E_1(I - E_1) = E_1 - E_1^2 = 0$, and
$E_1 + E_2 = E_1 + (I - E_1) = I$.

Consider $E_1, E_2, E_3 \in \mathrm{End}(\mathbb{C}^3)$ given by

$$E_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$R(E_1) = \{ \begin{pmatrix} \alpha \\ \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{C}\}, \quad R(E_2) = \{ \begin{pmatrix} \alpha \\ -\alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{C}\}$$

$$R(E_3) = \{ \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} : \alpha \in \mathbb{C}\}.$$

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Lemma (6.20 (weak version))

Let $E_1, \ldots, E_k \in \mathrm{End}(V)$ *be idempotents and let* $U_i := R(E_i)$*. Suppose that*

$$E_i E_j = 0 \textit{ for } i \neq j, \qquad E_1 + E_2 + \cdots + E_k = I.$$

Then

$$V = U_1 \oplus \cdots \oplus U_k.$$

Theorem (7.7)

Assume $E_1, \ldots, E_k \in \mathrm{End}(V)$ *are idempotents such that* $E_i E_j = 0$ *for* $i \neq j$*. Let* $U_i = R(E_i)$ *and* $W_i = N(E_i)$ *for each* i*. Then*

$$V = U_1 \oplus \cdots \oplus U_k \oplus (W_1 \cap \cdots \cap W_k)$$

with projections $E_1, \ldots, E_k$ *and* $I - (E_1 + \cdots + E_k)$*.*

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Recall that for $A \in \mathrm{End}(V)$ and $\lambda \in \mathcal{F}$, the generalised eigenspace is

$$M_\lambda = \{x \in V : \exists k \in \mathbb{N} \text{ s.t. } (A-\lambda I)^k x = 0\} = \bigcup_{k\in\mathbb{N}} N((A-\lambda I)^k).$$

Note that when $\dim V < \infty$, then M_λ is the space N in Theorem 7.5 for the endomorphism $A - \lambda I$. So there is a unique $A - \lambda I$-invariant subspace R so that $V = M_\lambda \oplus R$, $(A - \lambda I)|_R$ is invertible and $(A - \lambda I)|_{M_\lambda}$ is nilpotent.

Theorem ((Main part of) 7.8 - Jordan decomposition)

Assume $\dim V < \infty$ *and let* $A \in \mathrm{End}(V)$. *Let* $\lambda_1, \dots, \lambda_m$ *be the eigenvalues of* A. *There exists a unique* A*-invariant subspace* $R \subseteq V$ *such that*

$$V = M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m} \oplus R.$$

Moreover, $A|_R$ *has no eigenvectors.*

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