MOCK EXAM MM562 AND MM853

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- The mock exam consists of 10 problems each worth 10 points. The format and difficulty is approximately the same as the final exam.
- In Problem 10 there is a version for students on MM562 and one for MM853. You should answer the one for the course you are following.
- You can answer the exam in Danish or in English.
- All exercises can be solved by hand. If you want to use Maple or other tools for computations, that is fine, as long as you demonstrate that you understand the theory that the problem relates to.

Problem 1. (10 points) Let $A\colon \mathbb{C} \to \mathbb{C}$ be the map given by $A(\alpha) = \overline{\alpha}$ where $\overline{\alpha}$ denotes the complex conjugate. Show that if we consider $\mathbb{C}$ as complex vector space, then A is not linear, and that if we consider $\mathbb{C}$ as a real vector space, then A is linear.

Problem 2. (10 points) Prove that $\mathcal{B} = \{\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right), \left(\begin{smallmatrix}3\\2\end{smallmatrix}\right)\}$ is a basis for $\mathbb{C}^2$.

Problem 3. (10 points) Show that the map $A\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by $A\left(\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right) = \left(\begin{smallmatrix}3\alpha\\2\alpha\end{smallmatrix}\right)$ is linear, and write down the matrix ${}_{\mathcal{B}}[A]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis from Problem 2.

Problem 4. (10 points) Consider the $\mathbb{Q}$-vector space V of polynomials of degree at most three with rational coefficients, i.e.

$$V := \{p \in \mathbb{Q}[x] \mid p \text{ has degree at most } 3\}.$$

Show that the map $V \to \mathbb{Q}$ given by $y(p) = p(0)$ is a linear functional, and compute the dimension of the quotient space $V/\ker y$.

Problem 5. (10 points) Let $n \in \mathbb{N}$ and let V be an n-dimensional vector space over a field $\mathcal{F}$. Let $\{x_1, \ldots, x_n\}$ be a basis for V. Show that

$$U := \{x_n \otimes v \mid v \in V\}$$

is a subspace of $V \otimes V$.

Problem 6. (10 points) Consider $\mathbb{C}^3$ as an inner product space with the standard inner product. Show that $A\colon \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$A\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \begin{pmatrix}i\beta+3\gamma\\-i\alpha+2\beta\\3\alpha-2\gamma\end{pmatrix}$$

is a self-adjoint endomorphism (you may assume without proof that A is linear). As usual, $i \in \mathbb{C}$ denotes the imaginary constant.

Problem 7. (10 points) Consider $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$A\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right) = \begin{pmatrix}\alpha+2\beta\\ \beta\end{pmatrix}$$

which is linear (you do not need to prove that A is linear). Show that 1 is the only eigenvalue for A and determine the generalised eigenspace M_1 for the eigenvalue 1.

Problem 8. (10 points) Let V be an inner product space over $\mathbb{R}$ and let $x, y \in V$. Show that $\langle x, y\rangle = 0$ if and only if $\|x\| \leq \|x + \alpha y\|$ for all $\alpha \in \mathbb{R}$.

Problem 9. (10 points) Consider $\mathbb{C}^3$ as an inner product space with the standard inner product. Let $A \in \operatorname{End}(\mathbb{C}^3)$ be given by

$$A\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}\alpha+\beta\\ \beta+\gamma\\ \alpha+\gamma\end{pmatrix}.$$

Show that A is orthogonally diagonalisable.

Problem 10. (10 points) For $n \in \mathbb{N}$, let V_n denote the $\mathbb{R}$-vector space of polynomials of degree at n with real coefficients, i.e.

$$V_n := \{p \in \mathbb{R}[x] : p \text{ has degree at most } n\}.$$

Let $D_n \in \operatorname{End}(V_n)$ be the endomorphism given by differentiation, i.e. $D_n(p)(x) = \frac{d}{dx}p(x)$. You do not have to show that D_n is linear.

- *MM562* Show that D_2 is nilpotent.
- *MM853* Show that D_n is nilpotent for all $n \in \mathbb{N}$.

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