Lecture notes MM852 “Computational option pricing part I”

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Learning aims (from course description)
In this course we introduce modelling of problems from financial mathematics, especially option pricing, by stochastic and partial differential equations. The models are solved by computational methods based on a sound mathematical analysis.
Expected learning outcome

1. Deal with stochastic differential equation models in finance
2. Analyse and simulate stochastic differential equations using appropriate, advanced methods and modern software
3. Design and perform reliable simulations of stochastic differential equation models

Practicalities

- Lectures and exercises mixed
- Feedback until noon before next lecture
- Work in study groups strongly recommended
- After every lecture updated study phase assignments
- Expected work load per week: 4 hours (intro + trainingsphase) plus 10 hours studyphase
- Project (5 ECTS, pass / fail)

Contents

1 Introduction

Financial options

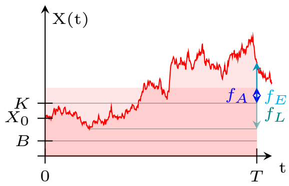
buyer's option (call): contract, which gives the buyer ("holder") the right (not the obligation) to purchase from the "writer" a prescribed amount of a prescribed asset ("underlying", e. g. currencies, commodities, shares in a company) during the contract period (for an American option) or only at the end of the period ("expiration time", for European options) to a price fixed in advance ("strike price", "exercise price").

Opposite: seller's option (put)

Example 1.1. *A bank writes a European call option that gives the holder the right to buy US \$ 100 for 120 € in six months from now.*
What is the value of the European call option at the expiry date?

Question: How to determine a fair price of the option?

Example: Payoff functions



Consider the value $f(X)$ of an option at maturity ($t = T$) based on an underlying asset X.

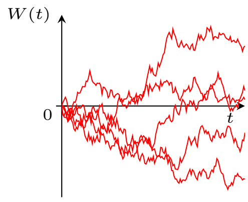
- European call: $f_E(X) = \max\{0, X(T) - K\}$,
- Digital call: $f_D(X) = \chi(X(T) > K)$,
- Barrier call: $f_B(X) = f_E(X)\chi(\inf\{t > 0 : X(t) < B\} > T)$,
- Lookback: $f_L(X) = \Big(X(T) - \min\limits_{0\le t\le T} X(t)\Big)$.
- Asian call: $f_A(X) = \max\Big\{0, \frac{1}{T}\int_0^T X(t)\mathrm{d}t - K\Big\}$,

Generalizations: American style options, basket options
Recap: Stochastic differential equations

Stochastic differential equations (SDEs)

$\mathrm{d}X(t) = g_0\big(X(t)\big)\mathrm{d}t + \sum\limits_{l=1}^{m} g_l\big(X(t)\big) \star \mathrm{d}W_l(t), \quad X(t_0) = X_0$

$$X(t) = X_0 + \int_{t_0}^{t} g_0\big(X(s)\big)\mathrm{d}s + \sum_{l=1}^{m} \underbrace{\int_{t_0}^{t} g_l\big(X(s)\big) \star \mathrm{d}W_l(s)}, \; t_0 \le t \le T$$



$$\lim_{\Delta s \to 0} \sum_j g_l\Big(X(\xi_j)\Big)\Big(W_l(s_{j+1}) - W_l(s_j)\Big)$$

$\xi_j = s_j$: Itô-integral,

$$\int_{t_0}^{t} g_l\Big(X(s)\Big)\ dW_l(s)$$

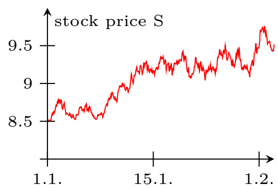
$\xi_j = \frac{1}{2}(s_j + s_{j+1})$: Stratonovich-integral,

$$\int_{t_0}^{t} g_l\Big(X(s)\Big)\ \circ dW_l(s)$$

$W(t)$: standard Wiener-process

- $W(0) = 0$ a. s.
- $W(t_2) - W(t_1) \sim N(0, t_2 - t_1)$ for $0 \le t_1 < t_2 \le T$
- $W(t_2) - W(t_1)$ and $W(t_4) - W(t_3)$ are independent for $0 \le t_1 < t_2 \le t_3 < t_4 \le T$

Example model: stock price development



- Black[1]–Scholes[2] model:

$$dS(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), \quad S(t_0) = S_0$$

(μ - expected annual rate of return, σ - volatility)

- Heston[3] model:

$$\begin{aligned} dS(t) &= \mu S(t)\mathrm{d}t + \sqrt{\nu(t)}S(t)\mathrm{d}W_1(t), & S(t_0) &= S_0 \\ \mathrm{d}\nu(t) &= \kappa\Big(\vartheta - \nu(t)\Big)\mathrm{d}t + \xi\sqrt{\nu(t)}\mathrm{d}W_2(t), & \nu(t_0) &= \nu_0 \end{aligned}$$

- Further generalizations, e. g. Lévy[4] driven assets, ...

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[1] Fischer Sheffey Black, *Washington 11.1.1938, †New York 30.8.1995, US-American economist, professor at University of Chicago and MIT

[2] Myron Samuel Scholes, * Timmins (Ontario) 1.7.1941, Canadian financial economist, Nobel memorial Prize in Economic Sciences 1997, professor in Princeton, at MIT and in Stanford

[3] Steven L. Heston, US-American mathematician, economist and financier, professor of finance at the Washington Unversity in St. Louis and University of Maryland

[4] Paul Pierre Lévy, *Paris 15.9.1886, †Paris 15.12.1971, French mathematician, professor at École des Mines and École Polytechnique

2 Numerical methods for SDEs (Repetition from MM547/MM802)

Definition 2.1 (Wiener[5] process)**.** *A Wiener process (or Brown[6]ian motion) is a stochastic process* $(W(t))_{t\geq 0}$ *fulfilling the following properties:*

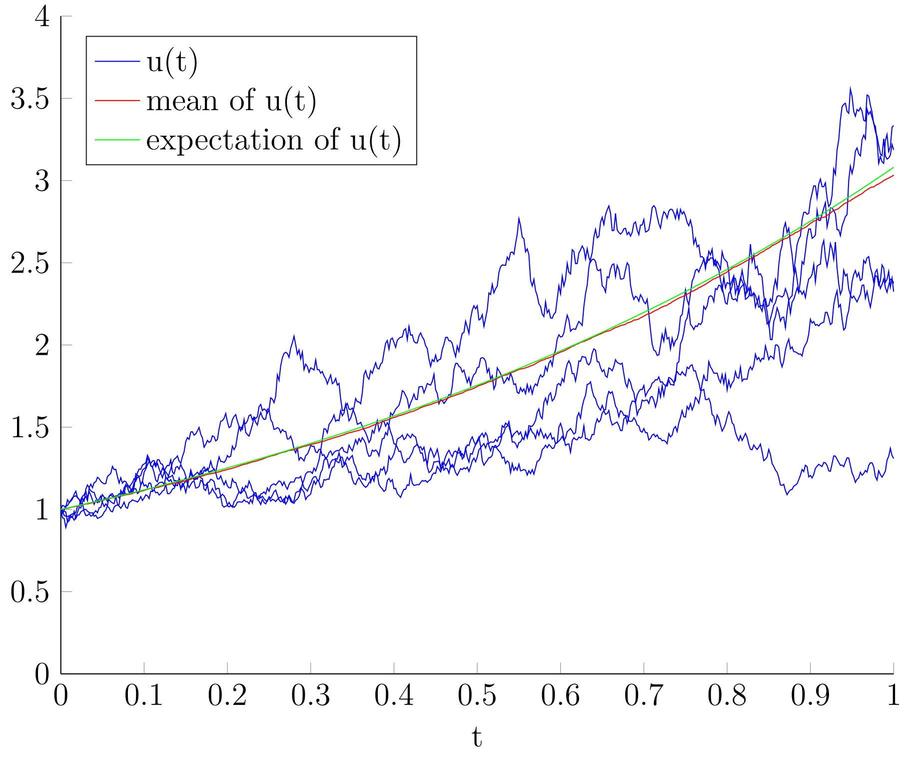
1.) $W(0) = 0$ *a.s.,*

2.) $(W(t))_{t\geq 0}$ *has a.s. continuous paths* $t \mapsto W(t)$*,*

3.) $W(t) - W(s) \sim N(0, t-s)$ *for* $0 < s < t$ *(stationary increments)*

4.) $W(t) - W(s)$ *is stochastically independent of* $W(u) - W(v)$ *for* $0 \leq v < u \leq s < t$ *(independent increments)*

Exercise 1 (Discretized Brownian Paths)**.** *Simulate 1000 paths of a one-dimensional Wiener process* $W(t)$ *on the time interval* $[0, 1]$ *with time step size* $1/500$*. Evaluate the function* $u(t, W(t)) = e^{t+\frac{1}{2}W(t)}$ *along the paths, and plot the result for five individual paths as well as the mean value in one diagram.*



[5]Norbert Wiener, *Columbia (Missouri) 26.11.1894, †Stockholm, 18.3.1964, US-American mathematician and philosopher, professor at MIT

[6]Robert Brown, * Montrose 21.12.1773, †London 10.6.1858, Scottish botanist

We consider the Itō SDE

$$\mathrm{d}X(t) = f(t, X(t))\mathrm{d}t + g(t, X(t))\mathrm{d}W(t)$$

on the interval $[t_0, T]$ with initial value $X(t_0)$ and an m dimensional Wiener process W, $m \geq 1$.
The solution $X(t)$ fulfills the Itō integral equation

$$X(t) = X(t_0) + \int\limits_{t_0}^{t} f(s, X(s))\mathrm{d}s + \int\limits_{t_0}^{t} g(s, X(s))\mathrm{d}W(s)$$

on the interval $[t_0, T]$ as well as the integral equations

$$X(t_{n+1}) = x(t_n) + \int\limits_{t_n}^{t_{n+1}} f(s, x(s))\mathrm{d}s + \int\limits_{t_n}^{t_{n+1}} g(s, x(s))\mathrm{d}W(s)$$

on each subinterval $[t_n, t_{n+1}]$ of $[t_0, T]$. As in the deterministic case we “freeze” the integrand functions f and g at $s = t_n$. We obtain the approximation

$$\begin{aligned}
X(t_{n+1}) &\approx X(t_n) + \int\limits_{t_n}^{t_{n+1}} f(t_n, X(t_n))\mathrm{d}s + \int\limits_{t_n}^{t_{n+1}} g(t_n, X(t_n))\mathrm{d}W(s) \\
&= X(t_n) + f(t_n, X(t_n)) \int\limits_{t_n}^{t_{n+1}} 1\mathrm{d}s + g(t_n, X(t_n)) \int\limits_{t_n}^{t_{n+1}} \mathrm{d}W(s) \\
&= x(t_n) + f(t_n, X(t_n))\,\Delta_n t + g(t_n, X(t_n))\Delta_n W
\end{aligned}$$

where

$$\begin{array}{lcl cl}
\Delta_n t & = & t_{n+1} - t_n & = & \int_{t_n}^{t_{n+1}} ds \\
\Delta_n W & = & W(t_{n+1}) - W(t_n) & = & \int_{t_n}^{t_{n+1}} \mathrm{d}W(s)
\end{array}$$

This yields the stochastic Euler[7] method

$$X_{n+1} = X_n + f(t_n, X_n)\,\triangle_n + g(t_n, X_n)\,\Delta_n W\ ,\ n = 0, \ldots, N-1,$$

also called Euler–Maruyama[8] method.

$$\begin{array}{ccccccc}
 & & \Delta_0 W(\omega) & & \Delta_1 W(\omega) & & \searrow \\
 & & \searrow & & \searrow & & \\
x_0(\omega) & \longrightarrow & x_1(\omega) & \longrightarrow & x_2(\omega) & \longrightarrow & \ldots
\end{array}$$

[7]Leonhard Euler, *Basel (Schweiz) 15.4.1707, †Sankt Petersburg 18.9.1783, mathematician and physicist, professor at the University of Sankt Petersburg and the Academy of Berlin

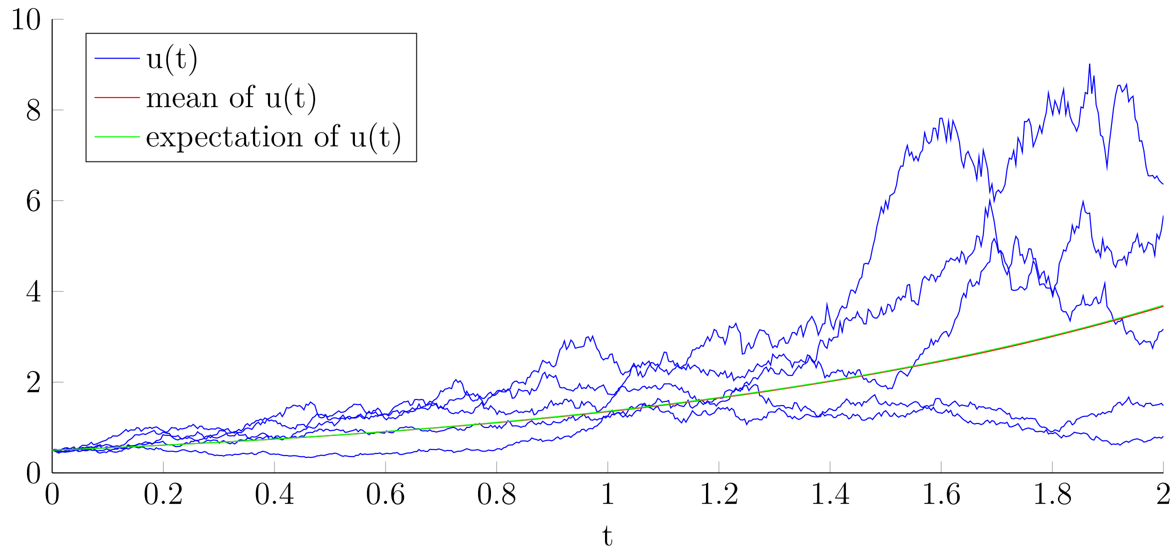
[8]Maruyama Gisirō, *4.4.1916, †5.7.1986, Japanese mathematician, professor at Ochanomizu university, Kyushu University, Tokyo university, Tokyo University of Education, the University of Tokyo, the University of Electro-Communications, Toky Denko University

$$W(t_{n+1}) = W(t_0) + \sum_{j=0}^{n} \triangle W_j$$

Exercise 2 (Euler-Maruyama method)**.** *Apply the Euler-Maruyama method for solving the linear SDE*

$$dX(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \quad X(0) = X_0,$$

with parameters $\lambda = 1$*,* $\mu = 0.6$ *and initial value* $X_0 = 0.5$ *on the interval* $[0, 2]$*. Simulate 100000 trajectories of the solution with each 512 equidistant steps and use them to approximate its expectation. Plot the result for five individual paths as well as the mean value in one diagram.*



More generally, we approximate the solution of an SDE by one step methods as follows:

One step approximations

A general one step approximation is given by

$$Y^{t,x}(t + h) = A(t, x, h; \xi),$$

where ξ is a vector of random variables, with moments of sufficiently high order, and A is a vector function of dimension d.
Let a discretization $I^h = \{t_0, t_1, \ldots, t_N\}$ with $t_0 < t_1 < \cdots < t_N = T$ of the time interval $I = [t_0, T]$ with step sizes $h_n = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N - 1$ and maximal step size

$$h = \max_{n=0}^{N-1} h_n$$

be given.
We write $Y_{n+1} = Y^{t_n, Y_n}(t_{n+1})$ and we construct the sequence

$$\begin{aligned} Y_0 &= X_0, \\ Y_{n+1} &= A(t_n, Y_n, h_n; \xi_n), \quad n = 0, 1, \ldots, N-1, \end{aligned}$$

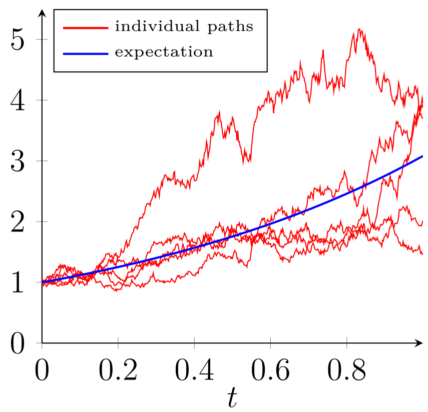
where ξ_0 is independent of Y_0, while ξ_n for $n \geq 1$ is independent of $Y_0, \ldots, Y_n$ and $\xi_0, \ldots, \xi_{n-1}$. What can one say about the accuracy of one step methods?

Strong and weak convergence

Classically, we say that an approximation method is convergent of order p, if the difference of the numerical approximation $Y^h(t)$ and the exact solution $X(t)$ is uniformly bounded by a constant times the maximal step size to the power p. But for SDEs, the solution, the numerical approximation and their difference are random variables. So we have to use the expectation of this difference. This concept of convergence is called strong convergence. The strong order of convergence measures the rate at which the mean of the error decays as h tends to zero. Although the definition of strong convergence involves an expected value, it has also implications for individual simulations. By applying a Markov[9] inequality (If $EX < \infty$, then $P(|X| \geq a) \leq E|X|/a$) one can show that the probability that the error is large goes to zero as the maximal step size h goes to zero. A similar concept of convergence is mean-square convergence. Here, in contrast to strong convergence we consider the square root of the expectation of the square of the error. By Jensen[10]'s inequality it follows that mean-square convergence implies strong convergence of the same order.

Strong convergence: $\forall h \in (0, H]$
$\max\limits_{t \in I^h} \mathbb{E}(\|X(t) - Y^h(t)\|) \leq Ch^p$

Mean-square convergence:
$\max\limits_{t \in I^h} \sqrt{\mathbb{E}(\|X(t) - Y^h(t)\|^2)} \leq Ch^p$
implies strong convergence



A completely different concept of convergence for stochastic approximation methods is weak convergence. The main motivation for considering weak approximations is

[9]Andrej Andreevič Markov, *Rjasan 2.6.1856, †Petrograd 20.7.1922, Russian mathematician, professor in Sankt Petersburg

[10]Johan Ludwig William Valdemar Jensen, *Nakskov 8.5.1859, †Kopenhagen 5.3.1925, Danish mathematician

the computation of the expectation of functionals of the solution. Often, we are not interested in approximating individual solution paths, but in approximating, e. g., the expectation of the solution with good accuracy. This problem arises for example in stochastic finance for the fair pricing of options.
[0.2cm] Weak convergence: $\forall f \in C_P^{2(p+1)}(\mathbb{R}^d, \mathbb{R})$

$$\max_{t \in I^h} \left| \mathbb{E}\left(f\big(X(t)\big) - f\big(Y^h(t)\big) \right) \right| \leq C_f h^p$$

Definition 2.2. *$C_P^l(\mathbb{R}^d, \mathbb{R}^{\hat{d}})$ denotes the space of all $g \in C^l(\mathbb{R}^d, \mathbb{R}^{\hat{d}})$ fulfilling a polynomial growth condition, i. e. there exist a constant $C > 0$ and $r \in \mathbb{N}$, such that $\|\partial_x^i g(x)\| \leq C(1 + \|x\|^{2r})$ holds for all $x \in \mathbb{R}^d$ and any partial derivative of order $i \leq l$. Further, let $g \in C_P^{k,l}(I \times \mathbb{R}^d, \mathbb{R})$ if $g(\cdot, x) \in C^k(I, \mathbb{R})$ and $g(t, \cdot) \in C_P^l(\mathbb{R}^d, \mathbb{R})$ for all $t \in I$ and $x \in \mathbb{R}^d$.*

Definition 2.3. *A time discrete one-step approximation $Y^h = (Y(t))_{t \in I^h}$ converges weakly / strongly / in the mean square with order p to X as $h \to 0$ at time $t \in I^h$ if (in the case of weak convergence for all $f \in C_P^{2(p+1)}(\mathbb{R}^d, \mathbb{R})$) there exist constants $C_f, \delta_0 \in \mathbb{R}$, such that for all $h \in (0, \delta_0)$*

$$\left| \mathbb{E}\left(f\big(X(t)\big) - f\big(Y(t)\big) \right) \right| \leq C_f\, h^p,$$
$$\mathbb{E}\, \|X(t) - Y(t)\| \leq C\, h^p,$$

respectively

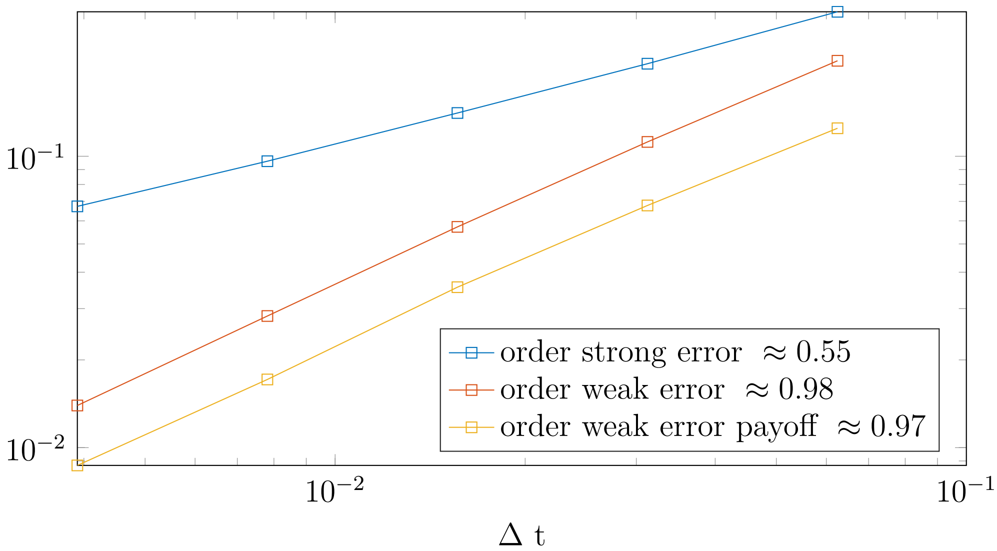
$$\sqrt{\mathbb{E}(\|X(t) - Y(t)\|^2)} \leq C\, h^p.$$

The Euler-Maruyama method is of strong order $\frac{1}{2}$ and weak order 1.

Exercise 3 (Convergence of the Euler-Maruyama method)**.** *Examine the strong and weak convergence of the Euler-Maruyama method for the linear SDE*

$$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \quad X(0) = X_0,$$

with parameters $\lambda = 1$, $\mu = 0.6$ and initial value $X_0 = 0.5$ on the interval $[0, 2]$. Simulate 100000 trajectories of the solution and use them to approximate the expectation of the absolute value of the error and the absolute value of the expected error of the solution at the end point in time for five different step sizes $\Delta t = 2^{p-10}$, $1 \leq p \leq 5$. Plot these in a doubly logarithmic diagram and calculate the slope of the respective regression lines. Repeat the exercise, now for the error of the payoff function $f(X) = (X - K)^+ := \max(X - K, 0)$ with $K = 8$.

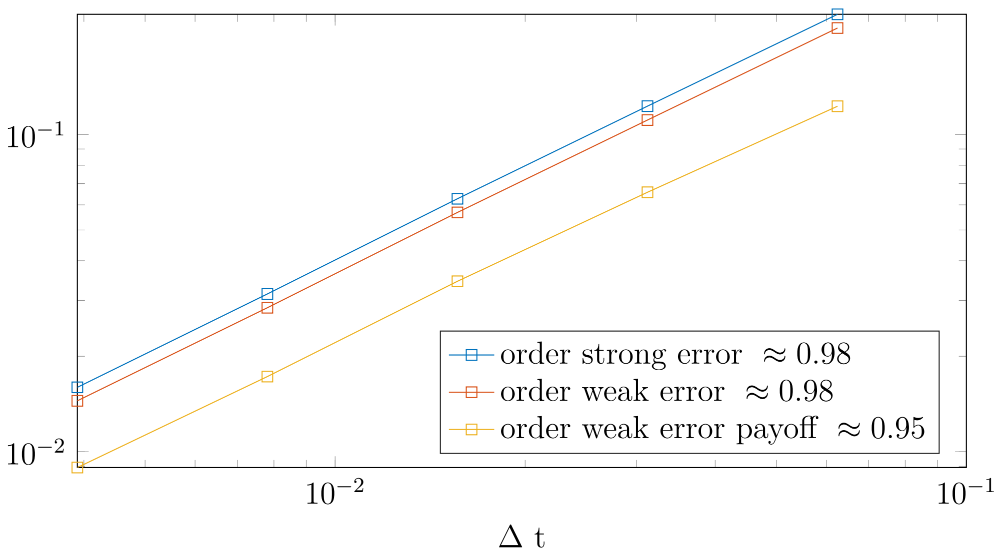


The Euler-Maruyama method is a (stochastic) Taylor[11] method taking into account terms up to order $\frac{1}{2}$. Taking also the terms of order 1 into account, we obtain for $m = 1$ the Milstein[12] method

$$Y_{n+1} = Y_n + g_1(Y_n)\Delta_n W + g_0(Y_n)\Delta_n t + \frac{(\Delta_n W)^2 - \Delta_n t}{2} \frac{\partial g_1}{\partial y}(Y_n) g_1(Y_n).$$

Though of different strong order, both methods are of weak order 1.

Exercise 4. *Repeat exercise 3 for the Milstein method.*



[11]Brook Taylor, *Edmonton (Middlesex) 18.8.1685, †Somerset House (London) 29.12.1731, British mathematician

[12]Grigori Noichowitsch Milstein, Russian mathematician, professor at Ural State University

Remark. *For weak approximation, we do not really need to exactly simulate the stochastic terms occurring in the Taylor expansion of the exact solution; random variables with accurate moments up to sufficiently high order suffice. E. g. in the Euler-Maruyama method, the Brownian increments can be replaced by e. g. two-point distributed random variables* ξ *with*

$$P(\xi = \sqrt{h}) = P(\xi = -\sqrt{h}) = \frac{1}{2}.$$

Example (Platen[13]'s explicit order 1.0 strong scheme)**.**

$$\begin{aligned}Y_{n+1} = Y_n + g_1(Y_n)\Delta_n W + g_0(Y_n)\Delta_n t \\ + \frac{(\Delta_n W)^2 - \Delta_n t}{2\sqrt{\Delta_n t}} \left(g_1(Y_n + \sqrt{\Delta_n t} g_1(Y_n)) - g_1(Y_n) \right).\end{aligned}$$

Example (Platen's explicit order 2.0 weak scheme)**.**

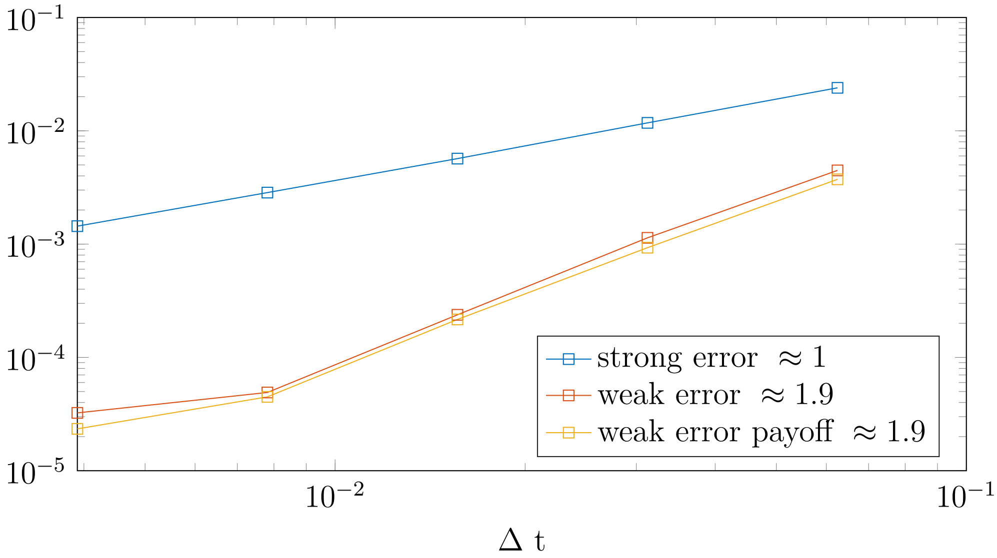
$$\begin{aligned}Y_{n+1} = Y_n + \frac{1}{2} \left(g_0(H_2^{(0)}) + g_0(Y_n) \right) \Delta_n t \\ + \frac{1}{4} \left(g_1(H_2^{(1)}) + g_1(H_3^{(1)}) + 2g_1(Y_n) \right) \Delta_n W \\ + \frac{1}{4} \left(g_1(H_2^{(1)}) - g_1(H_3^{(1)}) \right) \frac{(\Delta_n W)^2 - \Delta_n t}{\sqrt{\Delta_n t}}\end{aligned}$$

with

$$\begin{aligned}H_2^{(0)} &= Y_n + g_0(Y_n)\Delta_n t + g_1(Y_n)\Delta_n W, \\ H_2^{(1)} &= Y_n + g_0(Y_n)\Delta_n t + g_1(Y_n)\sqrt{\Delta_n t}, \\ H_3^{(1)} &= Y_n + g_0(Y_n)\Delta_n t - g_1(Y_n)\sqrt{\Delta_n t}.\end{aligned}$$

Exercise 5. *Repeat exercise 3 for Platen's explicit order 2.0 weak scheme.*

[13] Eckhard Platen, German mathematician, professor at University of Technology Sydney



Exercise 6 (Itō[14] formula)**.** *Approximate both*

$$\mathrm{d}X(t) = (2 - X(t))\mathrm{d}t + X(t)\mathrm{d}W(t), \qquad X(0) = 1$$

and the corresponding SDE for $V(t) = X^2$ by the Euler-Maruyama method and compare graphically the numerical approximations you obtain for $X(t)^2$.

[14]Itō Kiyoshi, *Hokusei-chō 7.9.1915, †Kyōto 10.11.2008, Japanese mathematician, professor at the University of Kyoto

3 Black–Scholes model (Repetition from MM547/ MM802)

We consider an economy with d assets whose prices $S_i(t)$, $i = 1, \dots, d$, are described by

$$\mathrm{d}S_i(t) = \mu_i(t, S(t))S_i(t)\mathrm{d}t + \sigma_i(t, S(t))S_i(t)\mathrm{d}W(t) \tag{3.1}$$

with $\mu_i : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $\sigma_i : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^m$
trading strategy: $\vartheta(t) \in \mathbb{R}^d$
initial portfolio value: $\vartheta(0)^\top S(0)$
portfolio value at t: $\vartheta(t)^\top S(t)$
gains from trading over $[0, t]$: $\int_0^t \vartheta(u)^\top \mathrm{d}S(u)$
self-financing trading strategy:

$$\vartheta(t)^\top S(t) - \vartheta(0)^\top S(0) = \int\limits_0^t \vartheta(u)^\top \mathrm{d}S(u)$$

Consider a derivative security with a payoff of $f(S(T))$ at time T
Suppose that its value at time t is given by $V(t, S(t))$
Itô's formula $\Longrightarrow$

$$\begin{aligned}\mathrm{d}V(t, S(t)) = \frac{\partial V(t, S(t))}{\partial t}\mathrm{d}t + \sum_{i=1}^{d} \frac{\partial V(t, S(t))}{\partial S_i}\mathrm{d}S_i(t) \\ + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 V(t, S(t))}{\partial S_i \partial S_j} S_i S_j \sigma_i^\top \sigma_j \mathrm{d}t.\end{aligned}$$

If $V(t, S(t))$ can be achieved from $V(0, S(0))$ by a self-financing trading strategy ϑ, it holds

$$V(t, S(t) = \vartheta(t)^\top S(t),$$
$$V(t, S(t)) = V(0, S(0)) + \sum_{i=1}^{d} \int\limits_0^t \vartheta_i(u)\mathrm{d}S_i(u).$$

Comparing term by term yields

$$\vartheta_i(t) = \frac{\partial V(t, S(t))}{\partial S_i}, \qquad i = 1, \dots, d,$$
$$\frac{\partial V(t, S)}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 V(t, S)}{\partial S_i S_j} S_i S_j \sigma_i^\top \sigma_j = 0.$$

Further,

$$V(t,S) = \sum_{i=1}^{d} \frac{\partial V(t,S)}{\partial S_i} S_i,$$

and $V(T,S) = f(S)$.
$\Longrightarrow V(0,S(0))$ is the fair price

Remark. *The PDE for V does not depend on μ_i's, thus the effect of the drifts on the price of a derivative is already reflected in the underlying asset prices S_i themselves.*

Assume now that there is one asset which is risk-free in the sense that its diffusion coefficient is 0. W. l. o. g. assume that it is S_d with $S_d(0) = 1$, thus $\sigma_d = 0_{\mathbb{R}^m}$. Let $\tilde{S} = (S_1, \dots, S_{d-1})^\top$, $r = \mu_d$ and $\beta = S_d$. Then $\beta(t) = e^{\int_0^t r(s)\mathrm{d}s}$. Let $\tilde{V}(t,\tilde{S}) = V(t,\tilde{S},\beta(t))$.

Exercise 7. *Show that it holds*

$$\frac{\partial \tilde{V}(t,\tilde{S})}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{d-1} \frac{\partial^2 \tilde{V}(t,\tilde{S})}{\partial \tilde{S}_i \tilde{S}_j} \tilde{S}_i \tilde{S}_j \sigma_i^\top \sigma_j + r(t) \sum_{i=1}^{d-1} \frac{\partial \tilde{V}(t,\tilde{S})}{\partial \tilde{S}_i} \tilde{S}_i - r(t)\tilde{V}(t,\tilde{S}) = 0 \tag{3.2}$$

Now, applying Itô's lemma to the process

$$Y(t) = e^{-\int_0^t r(\tau)\mathrm{d}\tau} \tilde{V}\Big(t, X(t)\Big)$$

yields

$$\begin{aligned}\mathrm{d}Y(t) = e^{-\int_0^t r(s)\mathrm{d}s} \Bigg(-r(t)\tilde{V}\Big(t,X(t)\Big)\mathrm{d}t + \frac{\partial \tilde{V}\Big(t,X(t)\Big)}{\partial t}\mathrm{d}t + \sum_{i=1}^{d-1} \frac{\partial \tilde{V}\Big(t,X(t)\Big)}{\partial X_i}\mathrm{d}X_i \\ + \frac{1}{2}\sum_{i,j=1}^{d-1} \frac{\partial^2 \tilde{V}\Big(t,X(t)\Big)}{\partial X_i X_j}\mathrm{d}X_i \mathrm{d}X_j \Bigg)\end{aligned}$$

Comparing with (3.2), we see that choosing

$$\mathrm{d}X_i(t) = r(t)X_i(t)\mathrm{d}t + \tilde{\sigma}_i\Big(t,X(t)\Big)X_i(t)\mathrm{d}W, \;\; X_i(0) = \tilde{S}(0) \tag{3.3}$$

with $\tilde{\sigma}_i(t,x) = \sigma_i(t,x,\beta(t))$ it follows that

$$\mathrm{d}Y(t) = e^{-\int_0^t r(s)\mathrm{d}s} \sum_{i=1}^{d-1} \frac{\partial \tilde{V}\Big(t,X(t)\Big)}{\partial X_i} \tilde{\sigma}_i X_i(t)\mathrm{d}W(t),$$

thus

$$\mathbb{E}\, Y(T) = \mathbb{E}\, Y(0) = \tilde{V}(0, \tilde{S}(0)).$$

The fair price of the option is thus given by

$$\tilde{V}(0, \tilde{S}(0)) = \mathbb{E}\, e^{-\int_0^T r(\tau)\mathrm{d}\tau} \tilde{V}\Big(T, X(T)\Big) = \mathbb{E}\, e^{-\int_0^T r(\tau)\mathrm{d}\tau} f(X(T)), \tag{3.4}$$

where the SDE (3.3) for X is obtained from SDE (3.1) for Y by replacing all drift rates μ_i by the risk-free interest rate r.

Remark. (3.3) *is the risk-neutral version of* (3.1) *(in a risk-neutral world, all assets would have the same expected rate of return, as investors would not require a higher rate of return for holding risky assets).*

Thus, alternative to solving PDE (3.2), we can also determine the expectation (3.4) of the solution of (3.3). Therefore, in this course we will consider both SDE and PDE methods.

Exercise 8. *Solve [6, Reality check 9 "The Black–Scholes formula", pages 464–465] nr. (1–2).*

Exercise 9. *Solve [6, Reality check 9 "The Black–Scholes formula", pages 464–465] nr. (3–6).*

Numerical challenges

- efficient approximation of expectations $\mathbb{E}\, X(T)$ of SDE solutions
  - variance reduction
  - minimizing computational complexity
  - ensuring positivity of solutions
  - SDEs with jumps
- maximize expectations over stopping times,

$$\sup_{t \in \mathcal{T}} \mathbb{E}\left(e^{-\int_0^t r(s)ds} f(X(t))\right)$$

- numerical approximation of PDE solutions by FEM
  - parabolic PDEs of Black–Scholes type
  - Integro-PDEs
  - linear complementary problems

$$\Big(\partial_t V + \tfrac{\sigma^2 x^2}{2} V_{xx} + rxV_x - rV\Big)\Big(V(t,S) - (K-S)^+\Big) = 0,$$
$$\partial_t V + \tfrac{\sigma^2 x^2}{2} V_{xx} + rxV_x - rV \leq 0,$$
$$V(t,S) - (K-S)^+ \geq 0$$

4 Monte Carlo simulation of SDEs

Exercise 10 (Discretized Brownian Paths). *Simulate a one-dimensional Wiener process $W(t)$ on the time interval $[0,1]$ with time step size $1/500$. Evaluate the function $u(t, W(t)) = e^{t+\frac{1}{2}W(t)}$ along the paths, and plot the result for five individual paths as well as the mean value in one diagram. Choose the number of paths adaptively such that the sample variance of the mean value at the end point is not larger than 10^{-3}.*

Exercise 11. *Assume that a sample is separated into two sets with sample means $\hat{x}_{1,M_1}$ and $\hat{x}_{2,M_2}$ and sum of squares of differences from the current mean $S_{i,M_i} = \sum_{j=1}^{M_i}(x_{i,j} - \hat{x}_{i,M_i})^2$, $i \in \{1,2\}$, and assume that for the sample sizes it holds $M_1 \geq M_2$. Prove that then the mean and sum of differences of the mean of the whole sample are given by*

$$\hat{x}_{M_1+M_2} = \hat{x}_{1,M_1} + \delta \frac{M_2}{M_1+M_2}, S = S_{1,M_1} + S_{2,M_2} + \delta^2 \frac{M_1 M_2}{M_1+M_2}$$

where $\delta = (\hat{x}_{2,M_2} - \hat{x}_{1,M_1})$.

Computational complexity of MC simulation for SDEs
SDE: $\mathrm{d}X(t) = g_0(X(t))\mathrm{d}t + g_1(X(t))\mathrm{d}W(t), \qquad X(t_0) = X_0$
Discrete approximation and Monte Carlo simulation:

$$\mathbb{E}\, f(X) \approx \mathbb{E}\, f(Y^h) \approx \hat{f}_M(Y^h) = \frac{1}{M}\sum_{i=1}^{M} f(Y^h(\omega_i))$$

Approximation with weak order p:

$$\begin{aligned} MSE(\hat{f}_M(Y^h)) :=& \mathbb{E}\left[\left(\hat{f}_M(Y^h) - \mathbb{E}\, f(X)\right)^2\right] \overset{!}{<} \epsilon^2 \\ =& \mathbb{E}\left[\hat{f}_M(Y^h)^2\right] + \left(\mathbb{E}\, f(X)\right)^2 - 2\,\mathbb{E}\,\hat{f}_M(Y^h)\,\mathbb{E}\, f(X) \\ & - \left(\mathbb{E}\,\hat{f}_M(Y^h)\right)^2 + \left(\mathbb{E}\,\hat{f}_M(Y^h)\right)^2 \\ =& \underbrace{\mathrm{Var}(\hat{f}_M(Y^h))}_{=\frac{1}{M}\mathrm{Var}\, f(Y^h)} + \underbrace{\left(\mathbb{E}\, f(Y^h) - \mathbb{E}\, f(X)\right)^2}_{\approx C h^{2p}} \overset{!}{<} \epsilon^2 \end{aligned}$$

$\Rightarrow M \sim \frac{1}{\epsilon^2}$ simulations with each $\frac{T-t_0}{h} \sim \frac{1}{\epsilon^{\frac{1}{p}}}$ steps $\Rightarrow$ Computational complexity $\mathcal{O}(\epsilon^{-\left(2+\frac{1}{p}\right)})$. Thus, for $p=1$ we have $\mathcal{O}(\epsilon^{-(3)})$, for $p=2$ we have $\mathcal{O}(\epsilon^{-(2.5)})$, and only in the limit $p \to \infty$ we obtain $\mathcal{O}(\epsilon^{-(2)})$.
In practice, we will aim at having both variance and bias bounded by $\frac{\varepsilon^2}{2}$.

Exercise 12. *For a stock price following the SDE*

$$\mathrm{d}S = \mu(t,s)S\mathrm{d}t + \sigma S\mathrm{d}W, \qquad 0 < t < 1$$

with $\sigma = 0.2$ and initial value $S(0) = 1$ determine, using the Euler-Maruyama scheme, the fair value of a European call option with expiration time 1 and strike price 1, assuming the riskless interest rate r to be constantly 5 %, such that the root mean square error ε equals 10^{-3}.

Exercise 13. *Repeat exercise 12 with additional, smaller values of ε and plot the dependency of the computational costs on ε.*

Exercise 14. *Repeat exercises 12 and 13 for the Milstein method and Platen's explicit order 2 weak scheme, and discuss the results.*

To reduce the computational effort, we will aim at reducing $\operatorname{Var} f(Y^h)$.

Variance reduction: control variates
Aim: Given some random variable V, reduce $\operatorname{Var} V$
Idea: Find a second random variable Z with known expectation $\mathbb{E}\, Z$, and consider for $b \in \mathbb{R}$

$$V(b) = V - b(Z - \mathbb{E}\, Z).$$

For the sample mean

$$\hat{V}_M(b) = \hat{V}_M - b(\hat{Z}_M - \mathbb{E}\, Z) = \frac{1}{M}\sum_{i=1}^{M}\left(V(\omega_i) - b\left(Z(\omega_i) - \mathbb{E}\, Z\right)\right)$$

it holds that it is unbiased, $\mathbb{E}\, \hat{V}_M(b) = \mathbb{E}\, V$. How to find b and Z such that $\operatorname{Var} V(b)$ is considerably smaller than $\operatorname{Var} V$? It holds

$$\begin{aligned}\operatorname{Var} \hat{V}_M(b) &= \frac{1}{M}\operatorname{Var}\Big(V - b(Z - \mathbb{E}\, Z)\Big)\\ &= \operatorname{Var} \hat{V}_M + \frac{1}{M}b^2 \operatorname{Var} Z - \frac{2b}{M}\operatorname{Cov}(V,Z)\end{aligned}$$

$\Longrightarrow$ optimal parameter: $b^\star = \frac{\operatorname{Cov}(V,Z)}{\operatorname{Var} Z}$

$$\Longrightarrow \frac{\operatorname{Var} \hat{V}_M(b^\star)}{\operatorname{Var} \hat{V}_M} = 1 - \underbrace{\frac{\operatorname{Cov}(V,Z)^2}{\operatorname{Var} Z \operatorname{Var} V}}_{=:\rho_{V,Z}^2}$$

$\Longrightarrow$ Z should be chosen such that it is as much as possible correlated or anticorrelated to V
Estimate for $b^\star$:

$$\hat{b}_M^\star = \frac{\sum_{i=1}^{M}(V(\omega_i) - \hat{V}_M)(Z(\omega_i) - \hat{Z}_M)}{\sum_{i=1}^{M}(Z(\omega_i) - \hat{Z}_M)^2}$$

Example 4.1 (Underlying assets, [4, Example 4.1.1]). *When pricing a derivative security with payoff $f(X(T))$, where X follows the risk-neutral dynamic* (3.3)*, we know that*

$$\mathbb{E}\, X_i(t) = e^{\int_0^t r(s)\mathrm{d}s}\tilde{S}_i(0).$$

Exercise 15. *Estimate the correlation between $S(T)$ and $(S(T)-K)^+$ for $K \in \{40, 45, 50, 55, 60, 65, 70\}$, where S with $S(0) = 50$ follows*

$$\mathrm{d}S(t) = r\mathrm{d}t + \sigma \mathrm{d}W(t)$$

with $\sigma = 30\%$, $r = 5\%$, and $T = 0.25$, i. e., generate [4, Table 4.1].

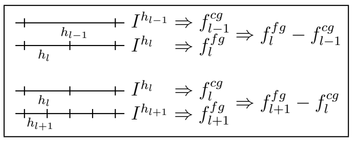
Exercise 16. *Repeat exercise 15 for the parameters used in exercise 12. Repeat then exercises 12 to 14, now by using additionally the underlying asset as control variate to reduce the variance. Compare and discuss the results.*

Variance reduction: Multilevel Monte Carlo simulation

Idea (Giles '08,[1]): Consider hierarchy of uniform grids I^{h_l} with $h_l = (T - t_0)/2^l$, $l = 0, \dots, L$. Let $f_l = f(Y^{h_l})$. Then

$\mathbb{E}(f_L^{fg}) = \mathbb{E}(f_0^{fg}) + \sum_{l=1}^{L} \mathbb{E}(f_l^{fg} - f_{l-1}^{cg})$

with $\mathbb{E}(f_l^{fg}) = \mathbb{E}(f_l^{cg})$.



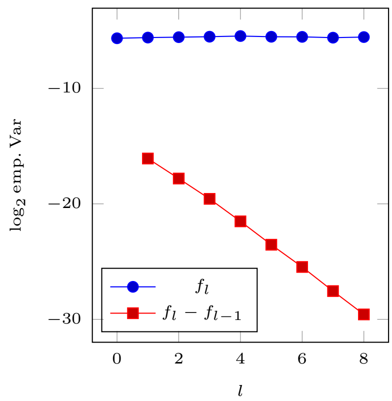
Approximation by

$$\hat{f}_L = \sum_{i_0=1}^{M_0} \frac{f_0^{fg}(\omega_{0,i_0})}{M_0} + \sum_{l=1}^{L}\sum_{i_l=1}^{M_l} \frac{(f_l^{fg} - f_{l-1}^{cg})(\omega_{l,i_l})}{M_l}$$

$$\Rightarrow \mathbb{E}(\hat{f}_L) = \mathbb{E}(f_L^{fg}),\ \mathrm{Var}(\hat{f}_L) = \frac{1}{M_0}\mathrm{Var}(f_0^{fg}) + \sum_{l=1}^{L} \frac{1}{M_l}\mathrm{Var}(f_l^{fg} - f_{l-1}^{cg})$$

Numerical example: European Lipschitz call, Milstein method

$dX(t) = 0.05X(t)\mathrm{d}t + 0.2X(t)\mathrm{d}W(t)$, $X(0) = 1$, $K = 1$, $T = 1$



$$\mathrm{Var}(f_l - f_{l-1}) = \mathcal{O}(h_l^2)$$

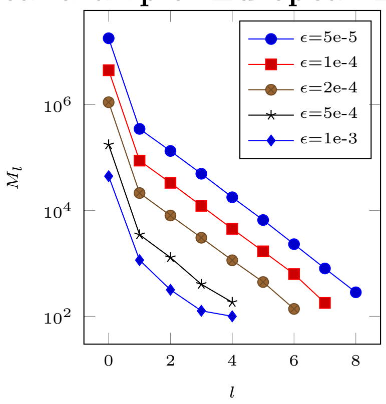
Exercise 17. *Given L, h_l and V_l, $l = 0, \ldots, L$, find the simulation numbers M_l, $l = 0, \ldots, L$ which minimize the computational costs*

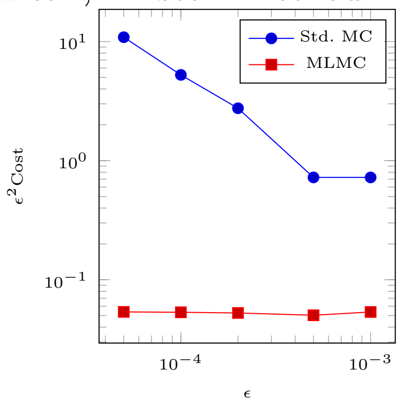
$$C = \sum_{l=0}^{L} \frac{M_l}{h_l}$$

under the condition that $\sum_{l=0}^{L} \frac{V_l}{M_l} < \frac{\epsilon^2}{2}$.

Exercise 18. *Repeat exercises 13 and 14, now applying the multilevel Monte Carlo method, and compare the results with the ones you obtained in exercises 13, 14 and 16.*

Numerical example: European Lipschitz call, Milstein method





$M_l = \mathcal{O}(h_l^{3/2})$ gain: ≈ 200

Computational complexity

Theorem 4.1 (Giles '08). *Assume that*

- *the computational complexity of evaluating $f_l^{fg} - f_{l-1}^{cg}$ is bounded by $\mathcal{O}(h_l^{-1})$,* ✓
- $\mathbb{E}(f_l^{cg} - f(X)) = \mathbb{E}(f_l^{fg} - f(X)) = \mathcal{O}(h_l^p)$ *with* $p \geq \frac{1}{2}$, ✓
- $\mathrm{Var}(f_l^{fg} - f_{l-1}^{cg}) = \mathcal{O}(h_l^q)$ *with* $q \geq 1$.

Then there exist values L and M_l for which the multilevel estimator $\hat{f}_L$ fulfills

$$MSE = \mathbb{E}(\hat{f}_L - \mathbb{E}\, f(X))^2 < \epsilon^2$$

with computational complexity $\begin{cases} \mathcal{O}(\epsilon^{-2}) & : \; q > 1 \\ \mathcal{O}(\epsilon^{-2}(\log \epsilon)^2) & : \; q = 1 \end{cases}$.

Proof.

$$\begin{aligned} V_l \le C_0 h_l^q \qquad &\implies M_l = \frac{2}{\epsilon^2}\sqrt{V_l h_l}\sum_{i=0}^{L}\sqrt{\frac{V_i}{h_i}} \le \frac{2}{\epsilon^2} C_1 h_l^{\frac{q+1}{2}} \sum_{i=0}^{L} h_i^{\frac{q-1}{2}} \\ &\implies C = \sum_{l=0}^{L}\frac{M_l}{h_l} \le \frac{2}{\epsilon^2} C_1 \left(\sum_{i=0}^{L} h_i^{\frac{q-1}{2}}\right)^2 . \end{aligned}$$

□

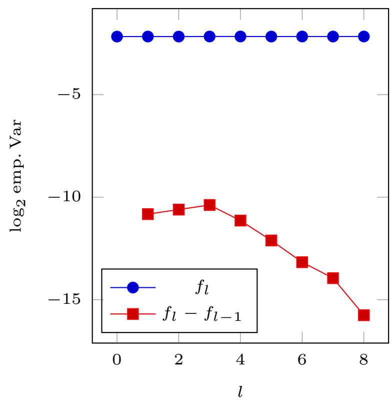
Lipschitz case
Goal: Prove $\mathrm{Var}(f_l^{fg} - f_{l-1}^{cg}) = \mathcal{O}(h_l^q)$ with $q > 1$. $(\star)$
Assume $f_l^{fg} = f_l^{cg} = f(Y^{h_l}(T))$ and f is Lipschitz with constant L_f (e. g. European call with $f(Y) = (Y(T) - K)^+$).

$$\begin{aligned} &\mathrm{Var}(f_l - f_{l-1}) = \mathbb{E}(f_l - f_{l-1})^2 - (\mathbb{E}(f_l - f_{l-1}))^2 \\ &\le \mathbb{E}(f_l - f_{l-1})^2 \le L_f^2\, \mathbb{E}((Y^{h_l}(T) - Y^{h_{l-1}}(T))^2) \\ &\le 2L_f^2 \left(\mathbb{E}((Y^{h_l}(T) - X(T))^2) + \mathbb{E}((Y^{h_{l-1}}(T) - X(T))^2)\right) \\ &= \mathcal{O}(h_l^{2p}) \end{aligned}$$

for approximations Y^h *mean square convergent* of order p
$\Rightarrow$ $(\star)$ fulfilled if $p > \frac{1}{2}$ $\implies$ Where possible, the Milstein method should be used

Digital call, Milstein method
$f_D(X) = \chi(X(T) > K)$
Standard numerical discretisation: Simulate $Y(T)$,

$$f_D(Y) = \chi(Y(T) > K) \quad \Rightarrow \quad \mathrm{Var}(f_l - f_{l-1}) = \mathcal{O}(h_l)$$



Exercise 19. *Repeat exercises 3 to 5, 13, 14 and 18, now for a digital call.*

Digital call with smoothed payoff
$f_D(X) = \chi(X(T) > K)$ Given $Y^{h_l}(T-h_l)$, approximate motion thereafter as Brownian motion with constant drift $a = g_0(Y^{h_l}(T-h_l))$ and constant diffusion $b = g_1(Y^{h_l}(T-h_l))$.

$$\begin{aligned}\to f_l^{fg} =& P(Y^{h_l}(T) > K | Y^{h_l}(T-h_l)) \\ =& P(Y^{h_l}(T-h_l) + a h_l + b\Delta_l W > K | Y^{h_l}(T-h_l)) \\ =& \Phi\left(\frac{Y^{h_l}(T-h_l) + a h_l - K}{|b|\sqrt{h_l}}\right)\end{aligned}$$

(see e. g. Glasserman '04)

Exercise 20. *Derive the corresponding expression for the coarse grid.*

Digital call: Sketch of proof

1. $\mathbb{E}(f_l^{fg}) = \mathbb{E}(f_l^{cg})$, $\Phi\left(\frac{Y^{h_l}(T-h_l)+a' h_l - K}{|b|\sqrt{h_l}}\right) = \mathbb{E}\left(\Phi\left(\frac{Y^{h_l}(T-h_l)+a h_l + b\Delta_{l+1}W - K}{|b|\sqrt{h_l/2}}\right)\middle| Y^{h_l}(T-h_l)\right)$.

   ✓

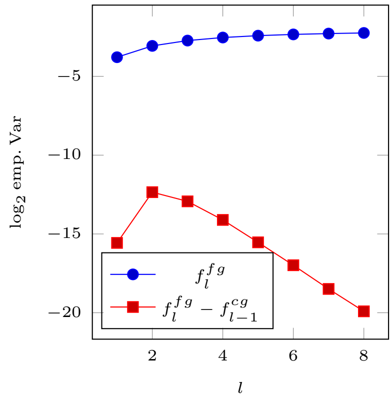
2. $\mathrm{Var}(f_l^{fg} - f_{l-1}^{cg}) = o(h_l^{3/2-\delta})$ for any $\delta > 0$

   - Define the set of "extreme" Brownian paths E_l: For $\gamma = \delta/14$, there exists n such that for $t_n \in I^{h_l}$
     - $\max\{|X(t_n)|, |Y_n^{h_l}|, |Y_n^{h_{l-1}}|\} > h_l^{-\gamma}$, or
     - $\max\{|X(t_n) - Y_n^{h_l}|, |X(t_n) - Y_n^{h_{l-1}}|, |Y_n^h - Y_n^{h_{l-1}}|\} > h_l^{1-\gamma}$, or
     - $|W(t_{n+1}) - W(t_n)| > h_l^{1/2-\gamma}$, or
     - $|X(T) - K| > h_l^{1/2-3\gamma}$.
   - $\mathrm{Var}(f_l^{fg} - f_{l-1}^{cg}) \leq \mathbb{E}((f_l^{fg} - f_{l-1}^{cg})^2 \mathbb{1}_{E_l}) + \mathbb{E}((f_l^{fg} - f_{l-1}^{cg})^2 \mathbb{1}_{E_l^c})$
   - $\mathbb{E}((f_l^{fg} - f_{l-1}^{cg})^2 \mathbb{1}_{E_l}) = o(h_l^r)$ for all $r > 0$
   - $\mathbb{E}((f_l^{fg} - f_{l-1}^{cg})^2 \mathbb{1}_{E_l^c}) = \mathcal{O}(h_l^{3/2-13\gamma})$

For a complete proof, see [2, Section 3.8].

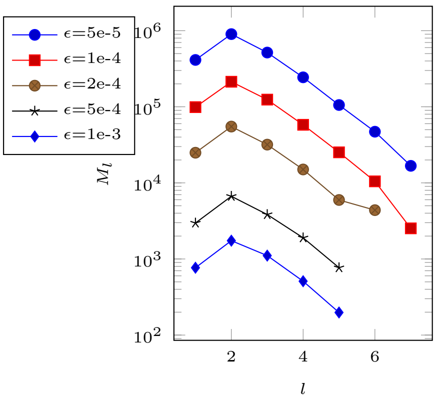
Exercise 21. *Repeat exercise 19, now using the smoothed payoff, and discuss the differences.*

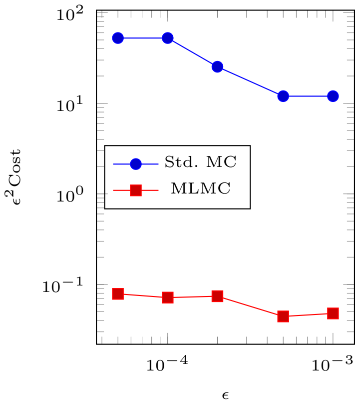
Numerical example: Digital call, Milstein method, smoothed payoff
$dX(t) = 0.05X(t)\mathrm{d}t + 0.2X(t)\mathrm{d}W(t)$, $X(0) = 1$, $K = 1$, $T = 1$



$$\mathrm{Var}(f_l - f_{l-1}) = \mathcal{O}(h_l^{3/2})$$

Numerical example: Digital call, Milstein method, smoothed payoff





$M_l = \mathcal{O}(h_l^{5/4})$ gain: ≈ 650

Exercise 22. *The smoothed payoff used so far has been based on an Euler-Maruyama approximation in the last step. Repeat exercise 21, now using a smoothed payoff based on a Milstein approximation in the last step, and discuss the differences.*

So far, we used a linear SDE (geometric Brownian motion) to model the underlyings. Now we will look at the Heston model as representative for systems of SDEs.

Heston model

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sqrt{\nu(t)}S(t)\mathrm{d}W_1(t), \qquad S(t_0) = S_0$$

$$\mathrm{d}\nu(t) = \kappa\Big(\vartheta - \nu(t)\Big)\mathrm{d}t + \xi\sqrt{\nu(t)}\mathrm{d}W_2(t), \qquad\qquad \nu(t_0) = \nu_0$$

with $\mathbb{E}[\mathrm{d}W_1(t)\mathrm{d}W_2(t)] = \rho\mathrm{d}t$.

Exercise 23. *Reformulate the Heston model such that the occurring Wiener processes are stochastically independent.*

The first difficulty is to ensure that the numerical approximation of $\nu(t)$ remains positive.

CIR (Cox-Ingersoll-Ross) process

$$\mathrm{d}\nu(t) = \kappa\Big(\vartheta - \nu(t)\Big)\mathrm{d}t + \xi\sqrt{\nu(t)}\mathrm{d}W(t), \qquad\qquad \nu(t_0) = \nu_0$$

with $2\kappa\vartheta \geq \xi^2$. One can prove: If $\nu_0 > 0$, then $\nu(t)$ is strictly positive.
How to find a suitable numerical approximation?
Idea: apply Lamperti transformation

$$X(t) = F(\nu(t)) \text{ with } F(z) = \int_0^z \frac{1}{\sqrt{y}}\mathrm{d}y = 2\sqrt{z}$$

Exercise 24. *Derive the SDE for X.*

The resulting SDE is stiff. We apply therefore the backward Euler-Maruyama method and obtain

$$Y_{k+1} = Y_k + \kappa\Big((\vartheta - \frac{\xi^2}{4\kappa})\frac{2}{Y_{k+1}} - \frac{Y_{k+1}}{2}\Big)\Delta_k t + \xi\Delta_k W. \tag{4.1}$$

Exercise 25. *Derive an explicit representation for Y_{k+1} given Y_k.*

Exercise 26. *For $\vartheta = \frac{1}{8}$, $\kappa = 2$, $\xi = 0.5$, $T = 1$ determine numerically the strong order of approximation of the Lamperti backward Euler-Maruyama method applied to the CIR process by performing simulations with step sizes 2^{-i}, $i = 8, \ldots, 11$, and comparing to a reference solution generated with step size 2^{-15}. Can you explain the result?*

Now we are able to price options based on the Heston stochastic volatility model.

Exercise 27. *For the Heston model with $\nu_0 = 1$, $S_0 = 1$, $T = 1$, $\kappa = 0.5$, $\vartheta = 0.9$, $\xi = 0.05$ and riskless interest rate $r = 0.05$, determine the fair price of a European call option with strike price 1, using the backward Euler-Maruyama method, both with standard and with Multilevel Monte Carlo simulation, such that the root mean-square error is smaller than some prescribed bound ε, for different values of ε, and plot and discuss the dependency of the computational costs on ε.*

To improve the results, we would like to use Milstein's method due to its higher strong order. For an SDE

$$X(t) = X(t_0) + \int_{t_0}^{t} f(s, X(s))\mathrm{d}s + \int_{t_0}^{t} g(s, X(s))\mathrm{d}W(s)$$

with m-dimensional Wiener process W a family of Milstein methods is given by [5]

$$\begin{aligned} Y_{n+1} = Y_n &+ \Big((1-\alpha)f(t_n, Y_n) + \alpha f(t_{n+1}, Y_{n+1})\Big)\Delta_n t + \sum_{j=1}^{m} g_j(t_n, Y_n)\Delta_n W_j \\ &+ \frac{1}{2}\sum_{j,k=1}^{m} g_j'(t_n, Y_n) g_k(t_n, Y_n)\Big((\Delta_n W_j)(\Delta_n W_k) - A_{n,jk} - \delta_{j,k}\Delta_n t\Big) \end{aligned} \tag{4.2}$$

where

$$A_{n,jk} = \int_{t_n}^{t_{n+1}} \Big(W_j(t) - W_j(t_n)\Big)\mathrm{d}W_k(t) - \int_{t_n}^{t_{n+1}} \Big(W_k(t) - W_k(t_n)\Big)\mathrm{d}W_j(t)$$

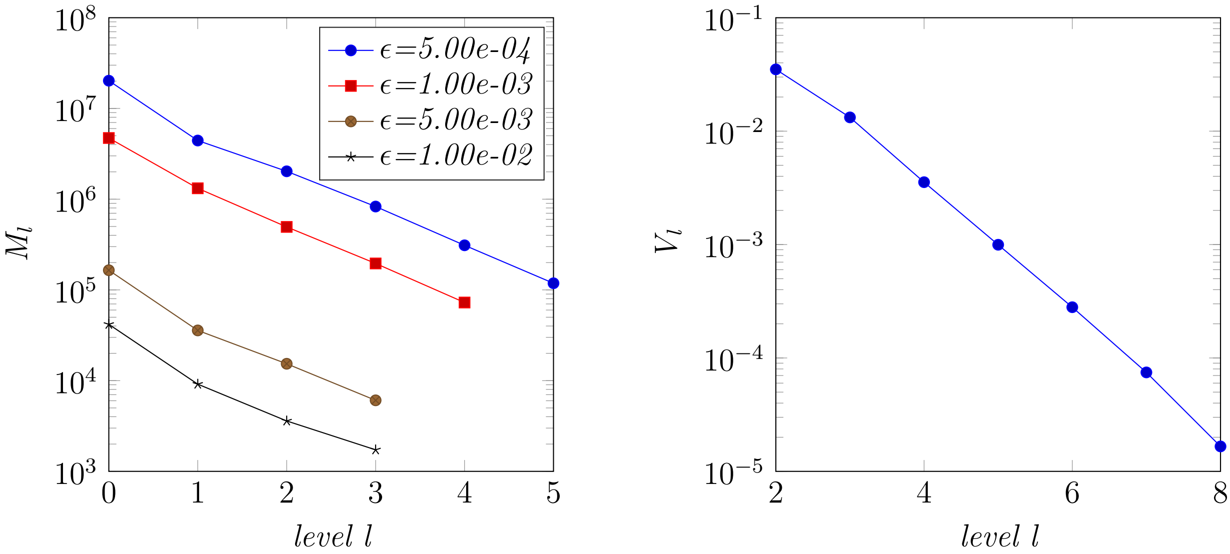
and $\delta_{j,k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$ denotes the Kronecker[15] symbol.Problem: efficient simulation of the Lévy areas $A_{n,jk}$.
Idea ([3]): Put $A_{n,jk} = 0$ ($\Longrightarrow$ strong convergence order only 0.5), but obtain still fast variance reduction in the MLMC method by using antithetic pairs of fine level simulations, so defining $f_{l-1}^{cg} = f\Big(Y^{h_{l-1}}(T)\Big)$, but

$$f_l^{fg} = \frac{1}{2}\left(f\Big(Y^{h_l}(T)\Big) + f\Big(Y^{h_l,a}(T)\Big)\right),$$

where $Y^{h_l,a}$ is calculated by swapping the fine path increments of the Brownian motion, leaving the ones of the coarse path unchanged.

Exercise 28. *Repeat exercise 27, now applying the semi-implicit Milstein method (i. e.* $\alpha = 1$ *in* (4.2)*) with neglected Lévy areas, with and without antithetic treatment in the MLMC method, and compare the results.*

[15] Leopold Kronecker, *Liegnitz 7.12.1823; †Berlin 29.12.1891, German mathematician, professor at the University of Berlin



Estimated price rounded to three digits: 0.395

Merton's jump-diffusion model

$$\mathrm{d}S(t) = \mu S(t-)\mathrm{d}t + \sigma S(t-)\mathrm{d}W(t) + S(t-)\mathrm{d}J(t)$$

with $S(t-) = \lim_{s \to t-} S(s)$, $S(t) = \lim_{s \to t+} S(s)$, and

$$J(t) = \sum_{j=1}^{N(t)} (Y_j - 1)$$

where Y_j are random variables and N is a counting process, i. e. there are random arrival times

$$0 < \tau_1 < \tau_2 < \dots$$

and

$$N(t) = \sup\{n : \ \tau_n \leq t\}.$$

It follows that

$$S(\tau_j) - S(\tau_j-) = S(\tau_j-)[J(\tau_j) - J(\tau_j-)] = S(\tau_j-)(Y_j - 1),$$

hence

$$S(\tau_j) = S(\tau_j-)Y_j$$

and consequently

$$S(t) = S(t_0)e^{(\mu - \frac{\sigma^2}{2})(t-t_0) + \sigma(W(t) - W(t_0))} \prod_{j=1}^{N(t)} Y_j.$$

If $N(t)$ is a Poisson process with rate λ, we have

$$P(\tau_{j+1} - \tau_j \leq t) = 1 - e^{-\lambda t},$$

and for Y_j i.i.d and independent of N and W, J is called a compound Poisson process.

Exercise 29. *Simulate 1000000 paths of a compound Poisson process $J(t)$ on the time interval $[0, 1]$ with rate $\lambda = 1$ and the jump magnitudes Y_i fulfilling $\ln Y_i \sim N(0.1, 0.2)$. Plot five individual paths as well as the mean.*

Let $J(t)$ be a compound Poisson process. Then $\mathbb{E}\, J(t) = \lambda m t$ with $m = \mathbb{E}\, Y_j - 1$, and the jump-diffusion SDE under the risk-neutral measure is thus given by

$$\mathrm{d}S(t) = (r - \lambda m)S(t-)\mathrm{d}t + \sigma S(t-)\mathrm{d}W(t) + S(t-)\mathrm{d}J(t).$$

Jump-adapted simulation: For each path simulation, the set of jump times $J = \{\tau_1, \tau_2, \ldots, \tau_m\}$ within the time interval $[0, T]$ is simulated and combined with a set of equidistant time steps to form

$$I^h = J \cup \{i\frac{T}{N} : i = 0, \ldots, N\}.$$

Within the n-th time step,

1. Simulate, starting at S_n, one step of a one step method with step size $h_n = t_{n+1} - t_n$, leading to $S_{n+1}^- = A(t_n, S_n, h_n; \xi)$,
2. $S_{n+1} = \begin{cases} S_{n+1}^- + S_{n+1}^-(Y_i - 1) & \text{if } \{t_{n+1}\} \cap J = \{\tau_i\}, \\ S_{n+1}^- & \text{otherwise.} \end{cases}$

Exercise 30. *For Merton's jump diffusion model with $S_0 = 100$, $\sigma = 0.2$, $T = 1$, jump intensity $\lambda = 1$, jump magnitudes Y_i fulfilling $\ln Y_i \sim N(0.1, 0.2)$, and riskless interest rate $r = 0.05$, determine the fair price of a European call option with strike price 100, both with standard and with Multilevel Monte Carlo simulation, such that the root mean-square error is smaller than some prescribed bound ε, for different values of ε, and plot and discuss the dependency of the computational costs on ε*

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