Week 9 & 10 - Exercises

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Advanced Linear Algebra

Week 9 - Exercise 1

1. Consider the map $A \in \mathrm{End}(\mathbb{R}^3)$ given by (multiplication with) the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the standard basis for $\mathbb{R}^3$. Determine rank and nullity of both A and its adjoint A'.

Let $(x, y, z) \in \mathbb{R}^3$ we have

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + y \\ 0 \end{pmatrix}$$

Thus,

$$\begin{aligned} R(A) &= \{(s + 2t, 2s + t, 0) \mid s, t \in \mathbb{R}\} \\ &= \{s(1, 2, 0) + t(2, 1, 0) \mid s, t \in \mathbb{R}\} \\ &= \mathrm{span}\{(1, 2, 0), (2, 1, 0)\} \end{aligned}$$

We conclude that

$$\mathrm{rank}(A) = \mathrm{rank}(A') = 2$$
$$\mathrm{null}(A) = \mathrm{null}(A') = 3 - 2 = 1$$

Week 9 - Exercise 2

2. Consider the subspace $U := \text{span}((1,0,0)) \leq \mathbb{R}^3$ and determine a basis for the annihilator U°.

First, we note that

$$3 = \dim(U) + \dim(U^\circ) = 1 + \dim(U^\circ) \implies \dim(U^\circ) = 2$$

We have

$$\begin{aligned} U^\circ &= \{y \in (\mathbb{R}^3)' \mid \forall x \in U : y(x) = 0\} \\ &= \{y \in (\mathbb{R}^3)' \mid y(1,0,0) = 0\} \end{aligned}$$

We denote by $\{y_1, y_2, y_3\}$ the dual basis for $(\mathbb{R}^3)'$ by with respect to the standard basis for $\mathbb{R}^3$. Since $y_2, y_3 \in U^\circ$ and y_2, y_3 are linearly independent and $\dim(U^\circ) = 2$, we conclude that $\{y_2, y_3\}$ is a basis for U°.

Week 9 - Exercise 3

Corollary 3.18. *The column rank and the row rank coincide for every matrix* $\mathbf{A}$ *over an arbitrary field* $\mathcal{F}$.

Observe that for $A \in \mathbb{M}_{mn}(\mathcal{F})$ and $(x_1, \ldots, x_n) \in \mathcal{F}^n$ we have

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Thus, the column space for A agrees with the range of A and consequently rank(A) is equal to the column rank of A. By combining Lemma 3.15 with Theorem 3.17, we see that also the column rank of A^T agrees with rank(A). But the column rank of A^T is equal to the row rank of A. Thus, the column and rank and row rank for A coincide.

Week 9 - Exercise 4

4. Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $U \leq V$ be a subspace. What is the relationship between U and its double annihilator $U^{\circ\circ}$ under the natural identification of V with V''.

Recall that the natural identification of V with V'' is given by

$$T(x)(y) = y(x)$$

We claim that

$$T|_U \colon U \to U^{\circ\circ}$$

is an isomorphism. First, note that $T|_U$ is well-defined since if $x \in U$ and $y \in U^\circ$ then

$$T(x)(y) = y(x) = 0$$

and thus $T(x) \in U^{\circ\circ}$. Now,

$$\begin{aligned} \dim(U^{\circ\circ}) &= \dim(V') - \dim(U^\circ) \\ &= \dim(V) - (\dim(V) - \dim(U)) = \dim(U) \end{aligned}$$

Clearly, $T|_U$ is injective as T is injective, and since $\dim(U) = \dim(U^{\circ\circ})$, it follows that $T|_U$ is an isomorphism.

Week 9 - Exercise 5

5. Let V be a finite dimensional vector space over $\mathbb{F}$ and consider an endomorphism $A \in \operatorname{End}(V)$. Let $U \leq V$ be a subspace which is invariant under A, in the sense that $A(U) \subseteq U$. Show that U° is invariant under the dual map A'.

We are asked to show that $A'(U^\circ) \subseteq U^\circ$ or equivalently that

$$A'(y)(x) = y(A(x)) = 0$$

for all $y \in U^\circ$ and $x \in U$. But this is true since $A(x) \in U$ which follows from the assumption $A(U) \subseteq U$.

Week 9 - Exercise 6

6. (Halmos, §17, Exercise 2) Show that the vectors $x_1 := (1,1,1), x_2 := (1,1,-1)$ and $x_3 := (1,-1,-1)$ form a basis for $\mathbb{C}^3$. Denote by y_1, y_2, y_3 the dual basis and determine $y_1(0,1,0)$, $y_2(0,1,0)$ and $y_3(0,1,0)$

We have

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, x_1, x_2, x_3 are linearly independent and consequently they form a basis for $\mathbb{C}^3$. Now,

$$(0,1,0) = 0 \cdot (1,1,1) + \frac{1}{2} \cdot (1,1,-1) + \left(-\frac{1}{2}\right) \cdot (1,-1,-1)$$

Hence,

$$y_1(0,1,0) = 0, \qquad y_2(0,1,0) = \frac{1}{2}, \qquad y_3(0,1,0) = -\frac{1}{2}$$

Week 9 - Exercise 7

7. (Halmos, §17, Exercise 3) Let V be an n-dimensional vector space over a field $\mathbb{F}$ and let $y \in V'$ be given. Argue that $\{x \in V \mid y(x) = 0\}$ is a subspace in V; what is its dimension?

Note that

$$\{x \in V \mid y(x) = 0\} = N(y)$$

Thus,

$$\dim N(y) = \text{null}(y) = \dim(V) - \text{rank}(y) = n - \text{rank}(y)$$

If $y(x) = 0$ for all $x \in V$ then $\text{rank}(y) = 0$ and $\dim N(y) = n$. Else $\text{rank}(y) = \dim(\mathbb{F}) = 1$ and $\dim N(y) = n - 1$.

Week 9 - Exercise 8

8. Prove Lemma 3.15: Let U and V be finite-dimensional $\mathcal{F}$-vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{C}$ respectively. Let $\mathcal{B}'$ and $\mathcal{C}'$ be the induced bases of U' and V' respectively. For any $A \in \mathrm{Hom}(U, V)$ show that

$$_{\mathcal{B}'}[A']_{\mathcal{C}'} = ({}_{\mathcal{C}}[A]_{\mathcal{B}})^T$$

(Hint: compute the (i,j)'th entry of $_{\mathcal{B}'}[A']_{\mathcal{C}'}$ and the (j,i)'th entry of $_{\mathcal{C}}[A]_{\mathcal{B}}$ individually (using the definitions), and show that they agree)

We write $\mathcal{B} = \{u_1, \ldots, u_m\}$ and $\mathcal{C} = \{x_1, \ldots, x_n\}$. Further, we write $\mathcal{B}' = \{z_1, \ldots, z_m\}$ and $\mathcal{C}' = \{y_1, \ldots, y_n\}$ for the dual bases. We consider the entries α_{ij} of $_{\mathcal{C}}[A]_{\mathcal{B}}$ which are determined from

$$Au_j = \sum_{i=1}^{n} \alpha_{ij} x_i$$

Similarly, the entries β_{kl} of $_{\mathcal{B}'}[A']_{\mathcal{C}'}$ are determined from

$$A' y_l = \sum_{k=1}^{m} \beta_{kl} z_k$$

Then

$$\beta_{jl} = \sum_{k=1}^{m} \beta_{kl} z_k(u_j) = A'(y_l)(u_j) = y(Au_j) = \sum_{i=1}^{n} \alpha_{ij} y_l(x_i) = \alpha_{lj}$$

which proves the statement.

Week 10 - Exercise 1 (Halmos §20, exercise 1)

1. Suppose that x, y, u, and v are vectors in $\mathbb{C}^4$; let $\mathfrak{M}$ and $\mathfrak{N}$ be the subspaces of $\mathbb{C}^4$ spanned by $\{x, y\}$ and $\{u, v\}$ respectively. In which of the following cases is it true that $\mathbb{C}^4 = \mathfrak{M} \oplus \mathfrak{N}$?

(a) $x = (1, 1, 0, 0)$, $y = (1, 0, 1, 0)$
$u = (0, 1, 0, 1)$, $v = (0, 0, 1, 1)$.

(b) $x = (-1, 1, 1, 0)$, $y = (0, 1, -1, 1)$
$u = (1, 0, 0, 0)$, $v = (0, 0, 0, 1)$.

(c) $x = (1, 0, 0, 1)$, $y = (0, 1, 1, 0)$
$u = (1, 0, 1, 0)$, $v = (0, 1, 0, 1)$.

We write $M = \operatorname{span}\{x, y\}$ and $N = \operatorname{span}\{u, v\}$. We claim that $M \oplus N = \mathbb{C}^4$ if and only if x, y, u, v are linearly independent.

Indeed, if x, y, u, v are linearly independent and $\alpha_1 x + \alpha_2 y = \beta_1 u + \beta_2 v \in M \cap N$ then $\alpha_1 x + \alpha_2 y - \beta_1 u - \beta_2 v = 0$ and thus $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ by linearly independence so that $M \cap N = \{0\}$. Consequently, the sum $M + N$ is direct and $\dim(M \oplus N) = \dim(M) + \dim(N) = 4$ which shows that $M \oplus N = \mathbb{C}^4$. Conversely, if $M \oplus N = \mathbb{C}^4$ and $\alpha_1 x + \alpha_2 y + \beta_1 u + \beta_2 v = 0$ then $\alpha_1 x + \alpha_2 y = -\beta_1 u - \beta_2 v \in M \cap N = \{0\}$ so that $\alpha_1 x + \alpha_2 y = 0$ and $\beta_1 u + \beta_2 v = 0$. But $\dim(M) + \dim(N) = 4$ so that $\dim(M) = \dim(N) = 2$ and thus x, y and u, v are pairwise linearly independent. Hence, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$.

Week 10 - Exercise 1 (Halmos §20, exercise 1 - continued)

1. Suppose that x, y, u, and v are vectors in $\mathbb{C}^4$; let $\mathfrak{M}$ and $\mathfrak{N}$ be the subspaces of $\mathbb{C}^4$ spanned by $\{x, y\}$ and $\{u, v\}$ respectively. In which of the following cases is it true that $\mathbb{C}^4 = \mathfrak{M} \oplus \mathfrak{N}$?

(a) $x = (1, 1, 0, 0)$, $y = (1, 0, 1, 0)$
$u = (0, 1, 0, 1)$, $v = (0, 0, 1, 1)$.

(b) $x = (-1, 1, 1, 0)$, $y = (0, 1, -1, 1)$
$u = (1, 0, 0, 0)$, $v = (0, 0, 0, 1)$.

(c) $x = (1, 0, 0, 1)$, $y = (0, 1, 1, 0)$
$u = (1, 0, 1, 0)$, $v = (0, 1, 0, 1)$.

We write $M = \text{span}\{x, y\}$ and $N = \text{span}\{u, v\}$. We have shown that $M \oplus N = \mathbb{C}^4$ if and only if x, y, u, v are linearly independent. Thus, we need to determine whether the four vectors are linearly independent.

(a) We have $v = u + y - x$ and thus $M \oplus N \neq \mathbb{C}^4$.

(b) We have

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, $M \oplus N = \mathbb{C}^4$.

(c) We have $v = x + y - u$ and thus $M \oplus N \neq \mathbb{C}^4$.

Week 10 - Exercise 1 (Halmos §20, exercise 3)

3. Construct three subspaces $\mathfrak{M}$, $\mathfrak{N}_1$, and $\mathfrak{N}_2$ of a vector space $\mathfrak{V}$ so that $\mathfrak{M} \oplus \mathfrak{N}_1 = \mathfrak{M} \oplus \mathfrak{N}_2 = \mathfrak{V}$ but $\mathfrak{N}_1 \neq \mathfrak{N}_2$. (Note that this means that there is no cancellation law for direct sums.) What is the geometric picture corresponding to this situation?

One geometric picture is that if we take three lines in $\mathbb{R}^2$ only intersecting at $(0,0)$ then the lines will pairwise span $\mathbb{R}^2$, but the lines are of course not equal. E.g., for

$$M = \text{span}\{(1,0)\}, \quad N_1 = \text{span}\{(1,1)\}, \quad N_2 = \text{span}\{(1,-1)\}$$

we have

$$M \oplus N_1 = \mathbb{R}^2 = M \oplus N_2$$

but $N_1 \neq N_2$.

Week 10 - Exercise 2

2. Let V be a vector space over a field $\mathbb{F}$ and let $U, W \leq V$ be subspaces with $V = U \oplus W$. Check that the projection onto U along W is indeed in $\mathrm{End}(V)$.

We consider the map $P_U : V \to V$ given by $P_U(u + w) = u$. Let $u_1 + w_1, u_2 + w_2 \in U \oplus W$ and $\alpha \in \mathbb{F}$. Then

$$\begin{aligned} P_U\left((u_1 + w_2) + (u_2 + w_2)\right) &= P_U\left((u_1 + u_2) + (w_1 + w_2)\right) \\ &= u_1 + u_2 \\ &= P_U(u_1 + w_1) + P_U(u_2 + w_2) \end{aligned}$$

and

$$\begin{aligned} P_U\left(\alpha(u_1 + w_1)\right) &= P_U\left(\alpha u_1 + \alpha w_1\right) \\ &= \alpha u_1 \\ &= \alpha P_U(u_1 + w_1) \end{aligned}$$

Thus, $P_U \in \mathrm{End}(V)$.

Week 10 - Exercise 3

3. Consider the real vector space $V := C(\mathbb{R}, \mathbb{R})$. Show that $U := \{f \in C(\mathbb{R}, \mathbb{R}) \mid f(0) = 0\}$ is a subspace and determine a complement (i.e. a subspace W such that $V = U \oplus W$. Moreover, write down formulas for the projections of V onto U along W and vice versa.

Clearly, the zero function is in U. If $f, g \in U$ and $\alpha \in \mathbb{R}$ then $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$ so that $f + g \in U$ and $(\alpha f)(0) = \alpha f(0) = 0$ so that $\alpha f \in U$. Hence, U is a subspace of V.

Now, consider the constant function $\mathbb{1} \in V$ defined by $\mathbb{1}(x) = 1$ for all $x \in \mathbb{R}$. We set $W = \operatorname{span}\{\mathbb{1}\}$. Since $\mathbb{1}(0) = 1 \neq 0$ we have $U \cap W = \{0\}$, and thus the sum $U + W$ is direct. Further, for $f \in V$ we can write

$$f = f - f(0)\mathbb{1} + f(0)\mathbb{1}$$

where $f(0)\mathbb{1} \in W$ and $(f - f(0)\mathbb{1})(0) = f(0) - f(0) = 0$ so that $f - f(0)\mathbb{1} \in U$. Hence, $V = U \oplus W$. Further,

$$P_U(f) = f - f(0)\mathbb{1}, \qquad P_W(f) = f(0)\mathbb{1}$$

Week 10 - Exercise 4

4. Let $E \in \mathbb{M}_n(\mathbb{R})$ be an idempotent. What are the possible eigenvalues for E?[1] Find examples in $\mathbb{M}_2(\mathbb{R})$ showing that the potential eigenvalues you found can be realised.

Let $\lambda \in \mathbb{R}$ be an eigenvalue for E. Then there is an $x \in \mathbb{R}^n \setminus \{0\}$ such that $E(x) = \lambda x$. Since E is an idempotent we have

$$\lambda x = E(x) = E(E(x)) = E(\lambda x) = \lambda E(x) = \lambda^2 x$$

Thus, $(\lambda^2 - \lambda)x = 0$ which implies that $\lambda^2 = \lambda$ since $x \neq 0$. It follows that $\lambda \in \{0, 1\}$. The matrix

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is an idempotent with eigenvalue 0. The matrix

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is an idempotent with eigenvalue 1.