concentrating on the work of mathematicians, but they also show us more clearly that mathematics is an integral part of intellectual history. Its isolation and that of its historians is a part of their self-construction and self-understanding. It can be otherwise and, if so, certainly fruitful for a meaningful historiography that is not only presentistic and antiquarian but also futuristic.

NOTES

1. 'It is remarkable too that at the very period in history when significant steps where taken to release geometry from its Euclidean shackles, a similar movement was taking place, quite independently, to rescue algebra from arithmetic' (Dubbey 1977, p. 302; see also Nový 1973, Chap. 6).
2. 'Semiology' is a term coined by Saussure (1916) for the study of signs in their social use and meaning (see also Kristeva 1977).
3. My translation, cited according to Imre Toth, who gives the most convincing interpretation of the rise of non-Euclidean geometry (Toth 1980).

4

Conceptual revolutions and the history of mathematics: two studies in the growth of knowledge (1984)*

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> In most sciences one generation tears down what another has built, and what one has established another undoes. In mathematics alone each generation builds a new storey to the old structure.
>
> Hermann Hankel

> Je le vois, mais je ne le crois pas.
>
> Georg Cantor

> Transformation, by presenting each anterior concept, theory, law, or principle as the *occasion* of an innovation, focuses attention on the *cause*, the possible reason why only one of the many scientists to whom the scientific idea was known produced the transformation in question.
>
> I. Bernard Cohen

It has often been argued that revolutions do not occur in the history of mathematics and that, unlike the other sciences, mathematics accumulates positive knowledge without revolutionizing or rejecting its past.[1] But there are certain critical moments, even in mathematics, that suggest that revolutions do occur—that new orders are brought about and eventually serve to supplant an older mathematics. Although there are many important examples of such innovation in the history of mathematics, two are particularly instructive: the discovery by the ancient Greeks of incommensurable magnitudes, and the creation of transfinite set theory by Georg Cantor in the nineteenth century. Both examples are as different in character as they are separated in time, and yet each provides a clear instance of a major transformation in mathematical thought. The Greeks' discovery of incommensurable magnitudes brought about changes that were no less significant than the revolutionary transformation mathematics experienced in the twentieth century as a result of Georg

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Cantor's set theory. Taking each of these as marking important transitional periods in mathematics, this essay is an attempt to investigate the character of such transformations.

Recently there has been considerable interest in the growth of mathematics, the nature of that growth, and its relation to the development of knowledge generally. In autumn 1974, at the fiftieth anniversary meeting of the History of Science Society, an entire session was devoted to the historiography of mathematics and to the relationship between the growth of mathematical knowledge and the patterns described in Thomas S. Kuhn's book *The structure of scientific revolutions* (1962, second edition; enlarged, 1970*a*). Naturally, the question of revolutions arose, and with it the problem of whether revolutions occur at all in the history of mathematics. When invited to consider the example of Cantorian set theory, I took the opportunity to suggest that revolutions did indeed occur in mathematics, although the example of transfinite set theory seemed to imply that Cantor's revolutionary work did not fit the framework of Professor Kuhn's model of anomaly–crisis–revolution.[2] Nor is there, perhaps, any reason to expect that a purely logicodeductive discipline like mathematics should undergo the same sort of transformations, or revolutions, as the natural sciences.

Similar interest in the nature of mathematical knowledge and its growth was evidenced at the Workshop on the Evolution of Modern Mathematics held at the American Academy of Arts and Sciences in Boston, 7–9 August 1974. Of all the participants at the workshop, no one questioned the phenomenon of revolutions in mathematics so directly as did Professor Michael Crowe of the University of Notre Dame. In a short paper prepared for the workshop and subsequently published in *Historia Mathematica*, he concluded emphatically with his tenth 'law' that 'revolutions never occur in mathematics'.[3] My intention here, however, is to argue that revolutions can and *do* occur in the history of mathematics, and that the Greeks' discovery of incommensurable magnitudes and Georg Cantor's creation of transfinite set theory are especially appropriate examples of such revolutionary transformations.

4.1. REVOLUTIONS AND THE HISTORY OF MATHEMATICS

Whether one can discern revolutions in any discipline depends upon what one means by the term 'revolution'. In insisting that revolutions never occur in mathematics, Professor Crowe explains that his reason for asserting this 'law' depends on his own definition of revolutions. As he puts it, 'My denial of their existence is based on a somewhat restricted definition of "revolution" which in my view entails the specification that a previously accepted entity *within* mathematics proper be rejected' (Crowe 1975, p. 470). Having said this, however, he is willing to admit that non-Euclidean geometry, for example, 'did lead to a revolutionary change in views as to the nature of mathematics, but not within mathematics itself' (Crowe 1975, p. 470).

Certainly one can question the definition Professor Crowe adopts for 'revolution'. It is unnecessarily restrictive, and in the case of mathematics it defines revolutions in such a way that they are inherently impossible within his conceptual framework. Nevertheless, revolutionary moments have been identified, not only by historians but by mathematicians as well. Rather than dictate the meaning of revolution, there is no reason not to allow its use in legitimately describing certain penetrating changes in the evolution of mathematics. However, before challenging further the assertion that revolutions never occur in the history of mathematics, it will be helpful to consider briefly the meaning of revolution as a historical concept. Here we are fortunate in having a recent study by Professor Cohen to guide us. In fact, what follows is a very brief résumé of results owing largely to Professor Cohen's research on the subject of revolutions.[4]

The concept of revolution first made its appearance with reference to scientific and political events in the eighteenth century, although with considerable confusion and ambiguity as to the meaning of the term in such contexts. In general, the word was regarded in the eighteenth century as indicating a breach of continuity, a change of great magnitude, even though the old astronomical sense of revolution as a cyclical phenomenon persisted as well. But, following the French Revolution, the new meaning gained currency, and thereafter revolution commonly came to imply a radical change or departure from traditional or acceptable modes of thought. Revolutions, then, may be visualized as a series of discontinuities of such magnitude as to constitute definite breaks with the past. After such episodes, one might say that there is no returning to an older order.

Bernard de Fontenelle may well have been the first author to apply the word 'revolution' to the history of mathematics, specifically to its evolution in the seventeenth century. In his *Éléments de la géométrie de l'infini* (1727), he was thinking of the infinitesimal calculus of Newton and Leibniz.[5] What Fontenelle perceived was a change of so great an order as to have altered completely the state of mathematics. In fact, Fontenelle went so far as to pinpoint the date at which this revolution had gathered such force that its effect was unmistakable. In his eulogy of the mathematician Rolle, published in the *Histoire de l'Académie Royale des Sciences* of 1719, Fontenelle referred to the work of the Marquis de l'Hôpital, his *Analyse des infiniment petits* (first published in 1696, with later editions in 1715, 1720, and 1768), as follows:

> In those days the book of the Marquis de l'Hôpital had appeared, and almost all the mathematicians began to turn to the side of the new geometry of the infinite, until then hardly known at all. The surpassing universality of its methods, the elegant brevity of its demonstrations, the finesse and directness of the most difficult solutions, its singular

and unprecedented novelty, it all embellishes the spirit and has created, in the world of geometry, an unmistakable revolution.[6]

Clearly this revolution was qualitative, as all revolutions must be. It was a revolution that Fontenelle perceived in terms of character and magnitude, without invoking any displacement principle—any rejection of earlier mathematics—before the revolutionary nature of the new geometry of the infinite could be proclaimed. For Fontenelle, Euclid's geometry had been surpassed in a radical way by the new geometry in the form of the calculus, and this was undeniably revolutionary.

Traditionally, then, revolutions have been those episodes of history in which the authority of an older, accepted system has been undermined and a new, better authority appears in its stead. Such revolutions represent breaches in continuity, and are of such degree, as Fontenelle says, that they are unmistakable even to the casual observer. Fontenelle has aided us, in fact, by emphasizing the discovery of the calculus as one such event—and he even takes the work of l'Hôpital as the identifying marker, much as Newton's *Principia* of 1687 marked the scientific revolution in physics or the Glorious Revolution of the following year marked England's political revolution from the Stuart monarchy. The monarchy, we know, persisted, but under very different terms.

In much the same sense, revolutions have occurred in mathematics. However, because of the special nature of mathematics, it is not always the case that an older order is refuted or turned out. Although it may persist, the old order nevertheless does so under different terms, in radically altered or expanded contexts. Moreover, it is often clear that the new ideas would never have been permitted within a strictly construed interpretation of the old mathematics, even if the new mathematics finds it possible to accommodate the old discoveries in a compatible or consistent fashion. Often, many of the theorems and discoveries of the older mathematics are relegated to a significantly lesser position as a result of a conceptual revolution that brings an entirely new theory or mathematical discipline to the fore. This was certainly how Fontenelle regarded the calculus. Similarly, it is also possible to interpret the discovery of incommensurable magnitudes in Antiquity as the occasion for the first great transformation in mathematics, namely, its transformation from a mathematics of discrete numbers and their ratios to a new theory of proportions as presented in Book V of Euclid's *Elements*.

4.2. THE PYTHAGOREAN DISCOVERY OF INCOMMENSURABLE MAGNITUDES

Aristotle reports the Pythagorean doctrine that all things were numbers, and surmises that this view doubtless originated in several sorts of empirical

observation.[7] For example, in terms of Pythagorean music theory the study of harmony had revealed the striking mathematical constancies of proportionality. When the ratios of string lengths or flute columns were compared, the harmonics produced by other, but proportionally similar lengths, were the same. The Pythagoreans also knew that any triangle with sides of length 3, 4, 5, whatever unit might be taken, was a *right* triangle. This too supported their belief that ratios of whole numbers reflected certain invariant and universal properties. In addition, Pythagorean astronomy linked such terrestrial harmonies with the motions of the planets, where the numerical harmony, or cyclic regularity of the daily, monthly, or yearly revolutions, was as striking as the musical harmonies the planets were believed to create as they moved in their eternal cycles. All these invariants gave substance to the Pythagorean doctrine that numbers—the whole numbers—and their ratios were responsible for the hidden structure of all nature. As Aristotle comments:

> The so-called Pythagoreans, having begun to do mathematical research and having made great progress in it, were led by these studies to assume that the principles used in mathematics apply to all existing things . . . they were more than ever disposed to say that the elements of all existing things are found in numbers.[8]

But what were these numbers? For the early Pythagoreans, Aristotle indicates that they were apparently something like physical 'monads'. In the *Metaphysics*, for example, one passage offers the following elaboration: '[The Pythagoreans] compose all heaven of numbers (ἐξ ἀϱιθμῶν), not of numbers in the purely arithmetical sense, though, but assuming that monads have size.'[9]

Thus the Pythagoreans apparrently came to regard the numbers themselves as providing the structure and form of the material universe, their ratios determining the shapes and harmonies of all symmetrical things. The Pythagoreans gave the word λόγοι to the groups of numbers determining the character of a given object, and later the meaning of this word was extended, as we shall see, from that of 'word' to 'ratio'.[10]

This sort of arithmology found its realization in the Pythagoreans' quest to associate numbers with all things, and to determine the internal properties, ratios, and relations between numbers themselves. Thus the number of stones needed to outline the figure of a man or a horse was taken by the Pythagorean Eurytus as the 'number' for man or horse.[11] The essence of such things was expressed by a particular number. Moreover, some Pythagoreans sought to establish the number for justice, or for marriage. Others distinguished numbers that were perfect (the tetractys, for example, $1+2+3+4=10$), amicable, or friendly. Figurate numbers, including pentagonal and solid numbers, were also subjects of great interest.[12] It is against this background of Pythagorean numerology, in which the λόγος of all things was thought to be an invariant principle of the universe, expressible in terms of whole numbers and their ratios, that the discovery of incommensurable magnitudes must be viewed. The Pythagoreans' arithmology would doubtless have provided sufficient incentive for their search for the hidden numbers, the prevailing logos governing the most important objects of their mysticism, for example the pentagon or the golden section. It is also possible that the discovery was made in less rarefied contexts, through study of the simplest of right triangles, the isosceles right triangle.

Exactly when incommensurable magnitudes were first discovered is not particularly relevant for the argument here.[13] Similarly, the details of the initial discovery are also of secondary importance, and we can dispense with the dilemma of whether the discovery was first made in the context that Aristotle reports it, by studying the ratio of the length of a square's edge with its diagonal, or whether, as has been argued by K. von Fritz (1945) and by S. Heller (1958), that Hippasus found incommensurability in considering the construction of the regular pentagon.[14] What concerns us is the discovery and its subsequent effect. Philosophically, it would certainly have represented a crisis for the Pythagoreans.[15] Having been tempted by the seductive harmony of generalization, some Pythagoreans had carried too far their universal principle that all things were numbers. The complete generalization was inadmissible, and this realization was a major blow to Pythagorean thought, if not to Greek mathematics. In fact, a scholium to Book X of Euclid's *Elements* reflects the gravity of the discovery of incommensurable magnitudes in the well-known fable of the shipwreck and the drowning of Hippasus:

> It is well known that the man who first made public the theory of irrationals perished in a shipwreck in order that the inexpressible and unimaginable [Καὶ ἄλογον Καὶ ἀνεὶδεον] should ever remain veiled . . . and so the guilty man, who fortuitously touched on and revealed this aspect of living things, was taken to the place where he began and there is forever beaten by the waves.[16]

What deserves attention here are the words 'inexpressible' and 'unimaginable'. It is difficult, if not impossible, for us to appreciate how hard it must have been to conceive of something one could not determine or name—the inconceivable—and this was exactly the name given to the diagonal: ἄλογον. This reflects the double meaning of the word *logos* as *word*, as the 'utterable' or 'nameable', and now the irrational, the *alogon*, as the 'unspeakable', the 'unnameable'. In this context, it is easy to understand the commentary: 'Such fear had these men of the theory of irrationals, for it was literally the discovery of the "unthinkable".'[17]

Ultimately, however, the Greeks regarded the discovery not as a crisis but as a great advance. Whether or not discovery of incommensurable magnitudes precipitated a crisis in Greek mathematics, and, if so, whether it affected only the foundations of mathematics rather than the mathematics itself, the significant issue concerns the *response* mathematicians were forced to make once the existence of incommensurable magnitudes had been divulged and was a matter of general knowledge.[18]

What ultimate effect did this discovery have on the content and nature of Greek mathematics? Above all, the theories of proportion advanced by Theaetetus and Eudoxus in the early fourth century BC (390–350 BC) served to reverse the emphasis of earlier mathematics. Consider, for example, the statement of Archytas (an early Pythagorean and teacher of Eudoxus), who was emphatic that arithmetic was superior to geometry for supplying satisfactory proofs.[19] After the discovery of incommensurable magnitudes, such a statement would be virtually impossible to justify. In fact, the opposite was closer to the truth, as the subsequent development of Greek geometric algebra demonstrates.

Basically, the transformation from a simple theory of commensurable proportions (where geometry and arithmetic might be regarded as coextensive) to a new theory embracing incommensurable magnitudes (for which arithmetic was inadequate) centres on the contributions of Theaetetus and Eudoxus. However, we know from Plato's *Theaetetus* that a major step toward the better understanding of the irrational was taken by Theaetetus's teacher, Theodorus, who established the incommensurability of certain magnitudes up to (but not including) $\sqrt{17}$ by means of geometric constructions. Although Theodorus's achievements were limited owing to his lack of a sufficiently developed arithmetic theory, some historians have argued that he began to develop a metric geometry capable of handling arithmetic properties in much the form of propositions in Book II of Euclid's *Elements*.[20]

Following his teacher Theodorus, Theaetetus became interested in the general properties of incommensurables and produced the classification that so impressed Socrates in Plato's dialogue (*Theaetetus*, 147C–148B). Also, Theaetetus realized that, to treat incommensurables successfully, geometry had to embody more of the results of arithmetic theory, and so he sought to translate necessary algebraic results into geometric terms. Here he focused on the arithmetic properties of relative primes, using the process of determining greatest common factors by means of successive subtraction, or *anthyphairesis*.[21] This enabled Theaetetus to reformulate the theory of proportion to include certain incommensurable magnitudes that he classified as the *medial*, *binomial*, and *apotome*, and these were enough for the results in which he was interested. But Theaetetus apparently was not inspired to study the new theory of proportion itself—something his premature death certainly precluded.

Eudoxus, however, realized that the methods Theaetetus had brought to geometry from arithmetic for the purpose of studying incommensurables could actually provide the basis for an even more comprehensive theory of proportion. In studying the construction of the regular pentagon, dodecahedron, and icosahedron, Eudoxus seems to have realized that these, like segments divided into mean and extreme ratio, involved incommensurable magnitudes that were not included in the three classes treated by Theaetetus (Knorr 1975, pp. 286–8). Because of his interest in a formal, more comprehensive theory of proportions, he transformed Theaetetus's methods involving *anthyphairesis* by focusing on the theory of proportion itself and producing in large measure the theorems elaborated in Book V of Euclid's *Elements*, where the concept of equal multiples made it possible to develop a theory of proportion that was generally applicable to incommensurables. The advantages of the new Eudoxan theory were considerable, and comparison with Theaetetus's anthyphairetic approach made clear the differences. Aristotle, in fact, contrasted the two on several occasions, and noted the superiority of Eudoxus's formulation explicitly.[22]

Having produced a comprehensive theory of proportion, however, Eudoxus and his followers, perhaps chief among them Hermotimus of Colophon, were also interested in providing a systematic development of the new theory that eventually provided the basic framework for Euclid's Book V of the *Elements*, a book a scholiast tentatively attributes to Eudoxus.[23] In dealing with incommensurable magnitudes, 'unfamiliar and troublesome' concepts as Morris Kline (1972, p. 50) has described them, the need to formulate axioms and to deduce consequences one by one so that no mistakes might be made was of special importance. This emphasis, in fact, reflects Plato's interest in the dialectic certainty of mathematics and was epitomized in the great Euclidean synthesis, which sought to bring the full rigour of axiomatic argumentation to geometry. It was in this spirit that Eudoxus undertook to provide the precise logical basis for the incommensurable ratios, and in so doing, gave great momentum to the logical, axiomatic, *a priori* 'revolution' identified by Kant (1781–7) as the great transformation wrought upon mathematics by the Greeks (see aso Cohen 1976, pp. 283–4).

In concluding this brief summary of Greek mathematics and the transformation caused by the discovery of incommensurable magnitudes, several aspects of that transformation deserve particular emphasis. Primarily, two things were unacceptable after the discovery of incommensurables: (1) the Pythagorean interpretation of ratio, and (2) the coming into play of proofs they had given concerning commensurable magnitudes. A new theory was needed to accommodate irrational magnitudes—and this was provided by Theaetetus and Eudoxus. The less dramatic transformation of the definition of the number concept was a lengthier process, but over the course of centuries it eventually led to admission of irrational *numbers* as being as acceptable ontologically as natural numbers or fractions.[24]

Wholly apart from the slower, more subtle transformation of the number concept, however, was the dramatic, much quicker transformation of the character of Greek mathematics itself. Because Pythagorean arithmetic could

not accommodate irrational magnitudes, geometric algebra (cumbersome though it was) developed in its stead. In the process, Greek mathematics was directly transformed into something more powerful, more general, more complete. Central to this transformation were auxiliary elements that reflected the transformation under way. A new interpretation of mathematics must have discarded as untenable the older Pythagorean doctrine that all things were number—there were now clearly things that did not have numbers in the Pythagorean sense of the word—and consequently their view of number was correspondingly inadequate. The older concept of number was severely limited, and in the realization of this inadequacy and the creation of a remedy to solve it came the revolution. New proofs replaced old ones.[25] Soon a new theory of proportion emerged, and as a result, after Eudoxus, no one could look at mathematics and think that it was the same as it had been for the Pythagoreans. Nor was it possible to assert that Eudoxus had merely added something to a theory that previously was perfectly all right. The lesson of the irrational was that everything was *not* all right. As a result of the new theory of proportion, the methods and content of Greek mathematics were vastly different, and comparison of Book V of Euclid with the Pythagorean books VII–IX (perhaps reflecting directly earlier arithmetics from the previous century) reveals the deep transformation that Eudoxus and his theory of proportion brought to Greek mathematics.[26] The old methods were supplanted, and eventually, although the same words, 'number' or 'proportion', might continue in use, their meaning, scope, and content would not be the same.

In fact, the transformation in conceptualization from irrational magnitudes to irrational numbers represented a revolution of its own in the number concept, although this was not a transformation accomplished by the Greeks. Nor was it an upheaval of a few years, as are most political revolutions, but a basic, fundamental change. Even if the evolution was relatively slow, this does not alter the ultimate effect of the transformation. The old concept of number, although the word was retained, was gone, and in its place, numbers included irrationals as well.

This transformation of the concept of number, however, entailed more than just extending the old concept of number by adding on the irrationals—the entire concept of number was inherently changed, transmuted as it were, from a world-view in which integers alone were numbers, to a view of number that was eventually related to the completeness of the entire system of real numbers.

In much the same way, Georg Cantor's creation of transfinite numbers in the nineteenth century transformed mathematics by enlarging its domain from finite to infinite numbers. Above all, the conceptual step from transfinite sets to transfinite numbers represents a shift that was in many ways the same as the shift from irrational magnitudes to irrational numbers. From the concrete to the abstract, the transformation in both cases revolutionized mathematics.

4.3. GEORG CANTOR'S DEVELOPMENT OF TRANSFINITE SET THEORY

Born in St Petersburg (Leningrad) in 1845, Georg Cantor left Russia for Germany with his parents in 1856.[27] Following study at the *Gymnasium* in Wiesbaden, private schools in Frankfurt-am-Main and the Realschule in Darmstadt, he entered a *Höhere Gewerbeschule* (Trade School), also in Darmstadt, from which he graduated in 1862 with the endorsement that he was a 'very gifted and highly industrious pupil' (Fraenkel 1930, p. 192). But his interests in mathematics prompted him to go on to university, and with his parents' blessing he began his advanced studies in the autumn of that same year at the *Polytechnicum* in Zürich. Unfortunately, his first year there was interrupted early in 1863 by the sudden death of his father, although within the year he resumed his studies, at the university in Berlin. There he studied mathematics, physics, and philosophy, and was greatly influenced by three of the greatest mathematicians of the day: Kummer, Weierstrass, and Kronecker.

After the summer term of 1886, which he spent in Göttingen, Cantor returned to the University of Berlin from which he graduated in December with the distinction 'Magna cum laude' (Fraenkel 1930, p. 194). Following three years of local teaching and study as a member of the prestigious Schellbach seminar for teachers, Cantor left Berlin for Halle in 1869 to accept an appointment as a *Privatdozent* in the Department of Mathematics. There he came under the influence of one of his senior colleagues, Eduard Heine, who was just completing a study of trigonometric series. Heine urged Cantor to turn his talents to a particularly interesting but extremely difficult problem: that of establishing the uniqueness of the representations of arbitrary functions by means of trigonometric series.[28] Within the next three years Cantor published five papers on the subject. The most important of these was the last, published in 1872, in which he presented a remarkably general and innovative solution to the representation problem.

With impressive skill Cantor was able to show that any function represented by a trigonometric series was not only uniquely represented, but that in the interval of representation an infinite number of points could be excepted provided only that the set of exceptional points be distributed in a specific way.[29] The condition was limited to sets Cantor described as point sets of the *first species* (Dauben 1979, pp. 41–2). Given a set P, the collection of all limit points p in P defined its first derived set, P'. Similarly, P'' represented the second derived set of P, and contained all limit points of P'. Proceeding analogously, for any set P Cantor was able to generate an entire sequence of derived sets $P', P'', \ldots$. P was described as a point set of the first species if, for some index n, $P^n = \varnothing$.

As outlined in the paper of 1872, Cantor's elementery set-theoretic concepts could not break away into a new autonomy of their own. Though he had the basic idea of the transfinite numbers in the sequence of derived sets P', $P'', \ldots, P^{\infty}, P^{\infty+1}, \ldots$, the basis for any articulate conceptual differentiation between P^n and P^{∞} was lacking. As yet, Cantor had no precise basis for defining the first transfinite number ∞ following all finite natural numbers n.[30] A general framework within which to establish the meaning and utility of the transfinite numbers was lacking. The only guide Cantor could offer was the vague condition that $P^n \neq \varnothing$ for all n, which separated sets of the first species from those of the second. Cantor could not begin to make meaningful progress until he had realized that there were further distinctions yet to be made in orders of magnitude between discrete and continuous sets. Until the close of 1873, Cantor did not even suspect the possibility of such differences.

In order to argue his uniqueness theorem of 1872, Cantor discovered that he needed to present a careful analysis of limit points and the elementary properties of derived sets, as well as a rigorous theory of irrational numbers.[31] It was the problem of carefully and precisely defining the irrational numbers that forced Cantor to face the topological complexities of the real line and to consider seriously the structure of derived sets of the first species.

After the success of his paper of 1872, it was a natural step to search for properties that would distinguish the continuum of real numbers from other infinite sets like the totality of rational or algebraic numbers. What Cantor soon established was something most mathematicians had assumed, but which no one had been able to formulate precisely: that there were more real numbers than natural, rational, or algebraic numbers (Cantor 1874). Cantor's discovery that the real numbers were non-denumerable was not in itself revolutionary, but it made possible the invention of new concepts and a radically new theory of the infinite. When coupled with the idea of one-to-one correspondences, it was possible to distinguish mathematically for the first time between different magnitudes, or powers, of infinity. In 1874 he was only able to identify denumerable and non-denumerable sets. But as his thinking advanced, he was eventually able to detach his theory from the specific examples of point sets, and in 1883 he was ready to publish his *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, in which he presented a completely general theory of transfinite numbers.[32] It was in the *Grundlagen* that Cantor introduced the entire hierarchy of infinite number classes in terms of the order types of well-ordered sets. More than twelve years later, in his last major publication, the *Beiträge* of 1895 and 1897, he formulated the most radical and powerful of his new ideas, the entire succession of his transfinite cardinal numbers:[33]

$$\aleph_0, \aleph_1, \ldots .$$

Cantor's introduction of the actual infinite in the form of transfinite numbers was a radical departure from traditional mathematical practice, even dogma. This was especially true because mathematicians, philosophers, and theologians in general had repudiated the concept since the time of Aristotle.[34] Philosophers and mathematicians rejected completed infinities largely because of their alleged logical inconsistency. Theologians represented another tradition of opposition to the actual infinite, regarding it as a direct challenge to the unique and absolute infinite nature of God. Mathematicians, like philosophers, had been wary of the actual infinite because of the difficulties and paradoxes it seemed inevitably to introduce into the framework of mathematics. Gauss, in most authoritative terms expressed his opposition to the use of such infinities in mathematics in a celebrated letter to Heinrich Schumacher:

> But concerning your proof, I protest above all against the use of an infinite quantity [*Grösse*] as a *completed* one, which in mathematics is never allowed. The infinite is only a *façon de parler*, in which one properly speaks of limits.[35]

Cantor believed, on the contrary, that on the basis of rigorous, mathematical distinctions between the potential and the actual infinite, there was no reason to hold the old objections and that it was possible to overcome the objections of mathematicians like Gauss, philosophers like Aristotle, and theologians like Thomas Aquinas, and to do so in terms even they would find impossible to reject. In the process, Cantor was led to consider not only the epistemological problems his new transfinite numbers raised, but to formulate as well an accompanying metaphysics. In fact, he argued convincingly that the idea of the actual infinite was implicitly part of any view of the potential infinite and that the only reason mathematicians had avoided using the actual infinite was because they were unable to see how the well-known paradoxes of the infinite, celebrated from Zeno to Bolzano, could be understood and avoided. He argued that once the self-consistency of his transfinite numbers was recognized, they could not be refused a place alongside the other accepted but once disputed members of the mathematical family, including irrational and complex numbers (Cantor 1883, p. 182). In creating transfinite set theory, Cantor was making a significant contribution to the constellation of mathematical ideas.

Of central concern to Cantor's entire defence of transfinite set theory was the nature of mathematics and the question of what criteria determined the acceptability of mathematical concepts and arguments. He reinforced his support of transfinite set theory with a simple analysis of the familiar and accepted positive integers. Insofar as they were regarded as well defined in the mind, distinct and different from all other components of thought, they served in a connectional or relational sense, he said, to modify the substance of thought itself (Cantor 1883, p. 181). Cantor described this reality that the whole numbers consequently assumed as their intrasubjective or immanent reality. In contradistinction to the reality numbers could assume strictly in terms of mind, however, was the reality they could assume in terms of body, manifest in objects of the physical world. Cantor explained further that this second sort of reality arose from the use of numbers as expressions or images of processes in the world of natural phenomena. This aspect of the integers, be they finite or infinite, Cantor described as their transubjective or transient reality.[36]

Cantor specifically claimed the reality of both the physical and ideal aspects of his approach to the number concept. The dual realities, in fact, were always found in a joined sense, in so far as a concept possessing an immanent reality always possessed a transient reality as well. Cantor believed that to determine the connections between the two kinds of reality was one of the most difficult problems of metaphysics.

In emphasizing the intrasubjective nature of mathematics, Cantor concluded that it was possible to study only the immanent realities, without having to confirm or conform to any subjective content. As noted earlier, this set mathematics apart from all other sciences and gave it an independence from the physical world that provided great freedom for mathematicians in the creation of mathematical concepts. It was on these grounds that Cantor offered his now-famous dictum that the essence of mathematics is its freedom. As he put it in the *Grundlagen* (Cantor 1883, p. 182):

> Because of this extraordinary position which distinguishes mathematics from all other sciences, and which produces an explanation for the relatively free-and-easy way of pursuing it, it especially deserves the name of *free mathematics*, a designation which I, if I had the choice, would prefer to the now customary 'pure' mathematics.

Cantor was asserting the freedom within mathematics to allow the creation and application of new ideas on the basis of intellectual consistency alone. Mathematics was therefore absolutely free in its development and bound only to the requirement that its concepts permit no internal contradictions, but that they follow in definite relation to previously given definitions, axioms, and theorems. Mathematics, Cantor believed, was the one science that was justified in releasing itself from any metaphysical fetters. Its freedom, insisted Cantor, was its essence.

The detachment of mathematics from the constraints of an imposed structure embedded in the natural world frees it from the metaphysical problems inherent in any attempt to understand the ultimate status of the physical and life sciences. Mathematicians do not face the preoccupation of scientists who must try to make theory conform with some sort of given, external reality against which those theories may be tested, articulated, improved, revised, or rejected.[37] Mathematicians, if they worry at all, need do so only in terms of the internal consistency of their work. This effectively eliminates the possibility of later discrepancies. Thus the grounds do not seem present within mathematics for generating anomaly and crisis, or for displacing earlier theory with some incompatible new theory.

One important consequence, in fact, of the insistence on self-consistency within mathematics is that its advance is necessarily cumulative. New theories cannot displace the old, just as the calculus did not displace geometry. Though revolutionary, the calculus was not an incompatible advance requiring subsequent generations to reject Euclid; nor did Cantor's transfinite mathematics require displacement and rejection of previously established work in analysis, or in any other part of mathematics.

Advances in mathematics, therefore, are generally compatible and consistent with previously established theory; they do not confront and challenge the correctness or validity of earlier achievements and theory, but augment, articulate, and generalize what has been accepted before. Cantor's work managed to transform or to influence large parts of modern mathematics without requiring the displacement or rejection of previous mathematics.

4.4. REVOLUTIONARY ADVANCE IN MATHEMATICS

Does this mean, then, that mathematics, because it represents a form of knowledge in which progress is genuinely cumulative, cannot experience periods of legitimate revolution? Surely not. To say that mathematics grows by the successive accumulation of knowledge, rather than by the displacement of discredited past theory by new theory, is not the same as to deny revolutionary advance. Cantor's proof of the non-denumerability of the real numbers, for example, led to the creation of the transfinite numbers. This was conceptually impossible within the bounds of traditional mathematics, yet in no way did it contradict or compromise finite mathematics. Cantor's work did not displace, but it *did* augment the capacity of previous theory in a way that was revolutionary, that would otherwise have been impossible. It was revolutionary in breaking the bonds and limitations of earlier analysis, just as imaginary and complex numbers carried mathematics to new levels of generality and made solutions possible that would otherwise have been impossible to formulate. Moreover, the extensive revision due to transfinite set theory of large parts of mathematics, involving the rewriting of textbooks and precipitating debates over foundations, are all results of what Thomas Kuhn has diagnosed as companions to revolutions.[38] And all these are reflected in the historical development of Cantorian set theory.

4.5. THE NATURE OF SCIENTIFIC RESOLUTION

I have deliberately juxtaposed the words 'revolution' and 'resolution' in order to emphasize what I take to be the nature of scientific advance reflected in the development of the history of mathematics—be it the Greek discovery of

incommensurables and the concomitant creation of a theory of proportion to accommodate them, or Cantor's profound discovery of the non-denumerability of the real numbers and his subsequent creation of transfinite numbers and the development of a general, transfinite set theory. Because mathematics is restricted only by the limits imposed by consistency, the inherent structure of logic determines the structure of mathematical evolution. I have already suggested the way in which that evolution is necessarily cumulative. As theory develops, it provides more complete, more powerful, more comprehensive problem-solutions, sometimes yielding entirely new and revolutionary theories in the process. But the fundamental character of such advance is embodied in the idea of resolution. Like the microscopist, moving from lower to higher levels of resolution, successive generations of mathematicians can claim to understand more, with a greater stockpile of results and increasingly refined techniques at their disposal. As mathematics becomes increasingly articulated, the process of resolution brings the areas of research and subjects for problem-solving into greater focus, until solutions are obtained or new approaches developed to extend the boundaries of mathematical knowledge. Discoveries accumulate, and some inevitably lead to revolutionary new theories uniting entire branches of study, producing new points of view, sometimes wholly new disciplines that would have been impossible to produce within the bounds of previous theory.

This is as true of the discovery of incommensurable magnitudes as it is of the advent of irritational, imaginary, and transfinite numbers, of the invention of the calculus, or the discovery of non-Euclidean geometries. None of these involved crisis or the rejection of earlier mathematics, although each represented a response to the failures and limitations of prevailing theory. New discoveries, particularly those of revolutionary import like those discussed here, provide new modes of thought within which more powerful and general results are possible than ever before. As Hermann Hankel (1871, p. 25) once wrote, 'In mathematics alone each generation builds a new storey to the old structure.' This is the most obvious sense in which I mean that the nature of scientific advance can be understood directly, in terms of the logic of argument and mathematics, as one of increasingly powerful resolution.

4.6. RESISTANCE TO CHANGE

One last feature of the evolution of mathematics may help to corroborate further the fact that it does experience revolutionary transformations, for resistance to new discoveries may be taken as a strong measure of their revolutionary quality. One form of this resistance was reflected in the Greeks' inability to conceive of anything as number except the integers—although eventually this prejudice was overcome, just as Cantor eventually overcame even his own discomfort with the actual infinite to support his transfinite numbers. Perhaps there is no better indication of the revolutionary quality of a new advance in mathematics than the extent to which it meets with opposition. The revolution, then, consists as much in overcoming establishment opposition as it does in the visionary quality of the new ideas themselves.

From the examples we have investigated here, it seems clear that mathematics may be revolutionized by the discovery of something entirely new and completely unexpected within the bounds of previous theory. Discovery of incommensurable magnitudes and the eventual creation of irrational numbers, the imaginary numbers, the calculus, non-Euclidean geometry, transfinite numbers, the paradoxes of set theory, even Gödel's incompleteness proof, are all revolutionary—they have all changed the content of mathematics and the ways in which mathematics is regarded. They have each done more than simply add to mathematics—they have each transformed it. In each case the old mathematics is no longer what it seemed to be, perhaps no longer even of much interest when compared with the new and revolutionary ideas that supplant it.

NOTES

1. The most adamant statement that mathematics does not experience revolutions may be found in M. J. Crowe (1975, pp. 15–20, esp. p. 19). The literature on the subject, however, is vast. Of authors who have claimed that mathematics grows by accumulation of results, without rejecting any of its past, the following sample is indicative: H. Hankel (1871, p. 25); G. D. Birkhoff (1934, esp. p. 302; 1950, p. 557); C. Truesdell (1968, foreword)—'While "imagination, fancy, and invention" are the soul of mathematical research, in mathematics there has never yet been a revolution.'
2. J. W. Dauben, Set theory and the nature of scientific resolution. (MS) for the Colloquium History of Mathematics and Recent Philosophies of Science (at the semicentennial meeting of the History of Science Society, Burndy Library, Norwalk, Conn., 27 October 1974).
3. Crowe's ten 'laws' (Crowe 1975, p. 16); see also M. J. Crowe (1967*b*, pp. 105–26, esp. pp. 123–4).
4. I. B. Cohen (1976*a*). More recently, Professor Cohen has also developed this material in a number of articles (see also Cohen 1980, esp. pp. 39–49).
5. Bernard de Fontenelle (1727); refer in particular to the preface, which is also reprinted in Fontenelle (1792, Vol. VI, p. 43).
6. Bernard de Fontenelle (1719, esp. p. 98). See also Fontenelle (1792, Vol. VII, p. 67).
7. For details of the background to Greek mathematics, and in particular to the history of incommensurability, see the recent works by W. R. Knorr (1975) and H. J. Waschkies (1977). I am especially indebted to Wilbur Knorr for his comments on an early draft of this chapter. Our discussion of the many difficulties in dealing with pre-Socratic material has been of great help to me in clarifying many murky or puzzling aspects of the history of the theory of incommensurable magnitudes and early Greek geometry.

8. Aristotle, *Metaphysics*, 985b23–986a3. Similarly, 1090a20–25. See the more direct interpretation that 'things are numbers' and variations at 1080b16–21; 1083b11, 18.
9. Aristotle, *Metaphysics*, 1080b16–20; see also *De caelo*, 300a16–19. The whole question of Pythagorean number theory and its character has been vigorously debated. For a general introduction that is careful to underscore the problems in reconstructing what the Pythagoreans may have believed, see J. A. Philip (1966). Harold Cherniss (1951, esp. p. 336) has described the Pythagorean point of view as more 'a materialization of number than a mathematization of nature'. The source for number atomism in Pythagorean mathematics comes from Ecphantus of Syracuse, and as W. Knorr (1975, p. 43) notes, this provides the basis for a thesis long in fashion via P. Tannery and F. M. Cornford, but which seems more recently to have fallen into disrepute. Yet I believe a form of "number-atomism" may be accepted as having been a doctrine of some Pythagoreans.' In a review of J. E. Raven's *Pythagoreans and Eleatics*, Gregory Vlastos (1953, p. 32) argued vigorously that 'number-atomism was not regarded by the tradition stemming from Theophrastus as an original feature of Pythagoreanism'. He carries this further by arguing that number-atomism was surely not a feature of Pythagorean musical formulae, 'nor could there be any question of number-atomism in the extensions of this theory to medicine, moral, or psychological concepts'. Fortunately, the question of number-atomism is not crucial to the issues presented here. Whether the early Pythagoreans, or only some later Pythagoreans like Ecphantus, adopted a view of number as material monads, the significant feature of Pythagorean arithmetic for the present purposes was its emphasis on *ratio*, and its belief that all things could be expressed through ratios of whole numbers.
10. H. Vogt (1909–10, 1913–14) was among the first to attempt the reconstruction of the development of a theory of proportion in response to the discovery of incommensurable magnitudes through transformations in terminology. Later Kurt von Fritz developed a similar approach in his articles on 'Theodoros' and 'Theaitetos' in *Paulys Real-Encyclopädie der classischen Altertumswissenschaft* (second series, Metzlersche Verlagsbuchhandlung, 1934), pp. 1811–31, 1351–72, respectively. See also Fritz (1945).
11. Aristotle, *Metaphysics*, 1092b10. Aristotle reports that Eurytus decided the number of man or horse, for example, 'by imitating the figures of living things with pebbles'. For commentaries on this passage by Alexander (*Metaphysics*, 827, 9) and Theophrastus (*Metaphysics*, 6a19), see G. S. Kirk and J. E. Raven (1957, p. 314). Wilbur Knorr (1975, p. 45) maintains that Eurytus's approach was an attempt to modify Pythagorean number-atomism in response to discovery of incommensurables.
12. For representative passages in Aristotle, *Metaphysics*, turn to 985b23–31, 986a2–8. See as well the discussion in Kirk and Raven (1957, pp. 236–62, esp. pp. 248–50). It should be noted that some writers mimimize the significance of the Pythagoreans in the history of mathematics and science. See, for example, W. A. Heidel (1940, p. 31): 'The role of the Pythagoreans must appear to have been much

exaggerated.' Even more emphatic is the view of W. Burkert (1972, p. 482) 'The tradition of Pythagoras as a philosopher and scientist is, from the historical view, a mistake . . . Thus, after all, there lived on, in the image of Pythagoras, the great Wizard whom even an advanced age, though it be unwilling to admit the fact, cannot entirely dismiss.' As for the Pythagorean concept of a 'perfect number', it must be remembered that their definition differed from that now standard in mathematics. For the Pythagoreans, the number 10 was perfect because it was the sum of the first four integers, $1+2+3+4=10$. Only after Aristotle did the sense of 'perfect numbers', as used by Euclid, make its appearance. Then, as now, a perfect number is equal to the sum of it divisors. Consequently, $6=1+2+3$ and $28=1+2+4+7+14$ are both perfect numbers, but 10 is not, since $10 \neq 1+2+5$. For further information see Burkert (1972, p. 431).

13. This, too, is a question that has received much discussion but little agreement in literature on the subject. For the most recent study of the problem, W. Knorr (1975, pp. 36–49, esp. p. 40) presents numerous arguments to establish the discovery within a twenty-year span from 430 to 410 BC.
14. For Aristotle's discussion of the incommensurability of the side and diagonal of a square, see *Prior analytics*, 41–29. W. Knorr (1975, pp. 22–8, esp. p. 23) discusses this proof and its version in Euclid's Book X of the *Elements* at length, noting that 'arguing for the antiquity of this version of the proof is its application of the even and the odd'. Arguing for the discovery of incommensurability by Pythagoreans studying the method of *anthyphairesis*, discussed later (see n. 21), are Kurt von Fritz (1945, p. 46) and S. Heller (1956, 1958). See also the discussion in W. Knorr (1975, pp. 29–36).
15. Although much debate has centred on the advisability of referring to the discovery as a 'crisis', as did H. Hasse and H. Scholz (1928), an important distinction must be made between the effect of the discovery of incommensurability upon mathematics as opposed to Pythgorean arithmology and its close connection with their cosmology or arithmological philosophy. For non-Pythagoreans and mathematicians in general, the ancient literature never mentions a 'crisis' but refers instead to the discovery as an advance, or even as a great 'wonder'. This is precisely the attitude of Aristotle (*Metaphysics*, 983a13–20): 'As we said, all men begin wondering that a thing should be so; the subject may be, for example, the automata in a peepshow, the solstices, or the incommensurability of the diagonal. For it must seem a matter for wonder, to all who have not studied the case, that there should be anything that cannot be measured by any measure, however small.' For Pythagorean arithmology, on the other hand, the discovery must have posed a major problem, and in this context its effect can be accurately described as representing a 'crisis.'

    G. E. L. Owen (1957–8, p. 214) is even more emphatic in asserting that 'discovery of incommensurables was a real crisis in mathematics'. For arguments that there was no such crisis, however, see K. Reidemeister (1949, p. 30) and H. Freudenthal (1966). Burkert (1972, p. 462) comes to similar conclusions.
16. Scholium to Euclid, *Elementa*, X, I, in *Opera omnia* (ed. J. L. Heiberg, Teubner, 1888), p. 417. For other accounts of the drowning episode, see Iamblichus *De vita Pythagorica liber*, XXXIV, 247, and XVIII, 88 (ed. Ludwig Deubner, Teubner, 1937), pp. 132 and 52, respectively, and Iamblichus, *De communi mathematica scientia liber*, XXV (ed. Nicola Festa, Teubner, 1891). pp. 76–8. Burkert (1972, p. 455) writes that 'the tradition of secrecy, betrayal, and divine punishment provided the occasion for the reconstruction of a veritable melodrama in intellectual history'. Pappus, however, viewed the story of the drowning as a 'parable', *The commentary of Pappus on Book X of Euclid's Elements*, Book I, Section 2 (ed. G. Junge and W. Thomson, Harvard University Press, 1930; reprinted by Johnson Reprint Corp., 1968), p. 64: the story was 'most probably a parable by which they sought to express their conviction that firstly, it is better to conceal (or veil) every surd, or irrational, or inconceivable in the universe, and, secondly, that the soul which by error or heedlessness discovers or reveals anything of this nature which is in it or in this world, wanders [thereafter] hither and thither on the sea of non-identity (i.e. lacking all similarity of quality or accident), immersed in the stream of the coming-to-be and the passing-away, where there is no standard of measurement.'
17. Scholium to Euclid, *Elementa*, X, I. For discussion of this passage, see Moritz Cantor (1894, Vol. 1, p. 175). As Burkert (1972, p. 461) has pointed out, later commentators like Plutarch and Pappus might have been especially tempted to seize on the *double entendre* made possible by the multiple connotations of the word ἄρρητος as irrational and unspeakable: 'In Plutarch it is clear that the word ἄρρητος, set in quotation marks, as it were, by λεγόμγναι, is to be understood in a double sense. The "ineffable because irrational" is at the same time the "unspeakable because secret" . . . The fascination of the ἀρρήτου lies in the pretense to indicate the fundamental limitations of human expression, which are at the same time transcended by the initiate . . . This exciting double sense of the word ἄρρητος is what makes the story of the discovery and betrayal of the irrational an *exemplum* for Plutarch, and even more for Pappus, who is probably following some Platonic source.' For additional discussion of these terminological transformations, refer to K. von Fritz (1939, p. 69; 1955, pp. 13–103, esp. pp. 80–7), as well as to the articles by von Fritz and Vogt cited in n. 10. It should also be added that Mugler, in defining ἄρρητος, writes that 'son sens étymologique étant «indicible, inexprimable»; il était synonyme, à l'origine, de ἄλογος au sens primitif' (*Dictionnaire*, p. 83).
18. The position adopted by Michael Crowe (1975, p. 19), for one, is that 'revolutions may occur in mathematical nomenclature, symbolism, metamathematics, methodology, and perhaps even in the historiography of mathematics', but *not* within mathematics itself.
19. Archytas, *Fragment* B4 (Fragmente der Gespräche) in H. Diels, *Die Fragmente der Vorsokratiker*, Vol. I (Weidmannsche, 1922), p. 337: 'Und die Arithmetik hat . . . einen recht betrachtlichen Vorrang . . . besonders aber auch vor der Geometrie, da sie deutlicher als diese was sie will behandeln kann . . . ⟨Denn die Geometrie beweist, wo die anderen Künste im Stiche lassen,⟩ und wo die Geometrie wiederum versagt, bringt die Arithmetik sowohl Beweise zustande wie auch die *Darlegung* der Formem [Prinzipien?], wenn es überhaupt irgend eine wissenschaftliche Behandlung der Formen gibt.
20. W. Knorr (1975, pp. 170–210, esp. pp. 199, 220–1) 'The early study of

incommensurabilty: Theodorus'. Here the recent research of D. Fowler is also relevant, above all his pair of articles, (Fowler 1980, 1982). I am happy to acknowledge a very stimulating correspondence with David Fowler covering a range of subjects including incommensurability, *anthyphairesis*, and Greek theories of ratio and proportion in general. Although our correspondence on these matters came after this essay was already in the press, I am grateful for his very careful reading of my original paper, and his subsequent comments, only a few of which it has been possible to incorporate here. Readers should also note in particular D. Fowler (1979, 1981).

21. O. Becker (1933), in analysing the concept of ανθυφαίρεσις, reconstructed a pre-Eudoxan theory of proportion. For a detailed discussion of *anthyphairesis*, see W. Knorr (1975, pp. 29–36), '*Anthyphairesis* and the side and diameter', and H. Waschkies (1977, pp. 77–100), 'Die anthyphairetische Proportionentheorie'. Mugler, *Dictionnaire*, p. 61, connects ἀνθυφαιρεῖν, the process of reciprocal subtraction, with study of the irrational magnitudes and the older, archaic term, '*probablement d'origine pythagoricienne*, ἀνταναίρεσις', p. 65. See as well the commentary on Theaetetus's demonstration and *anthyphairesis* by François Lasserre (1964, pp. 68–9).

22. Aristotle, *Posterior analytics*, 74a17–30, refers to the new, more general techniques of proof (ὃ καθόλου ὑποτίθεται ὑπαρχειν). Moreover, Scholia 1 and 3 to Book V of the *Elements* comment on the generality of the results obtained there. See Euclid, *Opera omnia* (ed. Heiberg), Vol. V, pp. 280 and 282, respectively. In fact, the differences between the earliest theory of proportion, generally regarded as authentically Pythagorean and set forth in Book VII of Euclid's *Elements*, and Eudoxus's powerful more general theory as represented in Euclid Book V, may be seen in a comparison of several parallel definitions. For example:

 Book VII, Definition 3: Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείξονος ὅταν καταμετρῇ τὸν μείξονα.

 Book V, Definition 1: Μέρος ἐστὶ μεγέθος μεγέθους τὸ ἔλασσον τοῦ μείζονος; ὅταν καταμετρῇ τὸ μεῖζον.

 Book VII, Definition 5: Πολλαπλάσις δὲ ὁ μείξων τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος.

 Book V, Definition 2: Πολλαπλάσιον δὲ τὸ μεῖξον τοῦ ἐλάττονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάττονος.

 Waschkies (1977, p. 19) also underscores the significance of the term μέγεθος for magnitude in Book V by noting that it became a technical term in geometry directly as a result of Eudoxus's influence.

23. Scholium 1 to Book V of Euclid's *Elements* in *Opera omnia* (ed. Heiberg), Vol. V, p. 280. As W. Knorr (1975, p. 274) notes, 'The fundamental conception of proportion in *Elements* V, if not the completion of the entire theory, is due to Eudoxus.'

24. It should be stressed, however, that the Greeks never attained such a general concept of number. For them, ἀριθμοι, or numbers, were always defined, as in Euclid VII, Definition 2, as a sum of *units*. There were no rational or irrational *numbers*, only ratios of whole numbers and proportions defined as equal ratios (van der Waerden 1961, p. 125). Despite the conjectures of some historians (see, e.g., Heath 1921, Vol. I, p. 327), the Greeks *never* had the concept of real numbers, Dedekind cuts, or even the set of rational numbers. For details, see F. Beckmann (1967, esp. pp. 21, 37–41). Knorr (1975, pp. 9–10) stresses that ἀριθμός (=number) and λόγος (=ratio) were *never* equated in the ancient tradition.

25. Aristotle takes Theorem V, 16, on the *ennalax* property of proportions, as epitomizing the great transformation in proof techniques and capabilities brought about by Eudoxus's theory (see n. 22). On a simpler level, Book V duplicates propositions from Book II, where they were originally established for line segments only. Book V, of course, establishes similar theorems for all magnitudes in general. One may also compare, for example, specific propositions like the *di' isou* theorem for proportions, *Elements* V, 22, with the earlier version, VII, 14, where a different method was originally used employing the special properties of integers as opposed to magnitudes. Recently, Wilbur Knorr (1975, p. 304) has argued that in Theorems X, 9–10, Euclid saw the unsuitability of the original pre-Eudoxan proofs of these propositions, and therefore gave them a new, if not very skilful version suitable to post-Eudoxan theory.

26. By directly comparing the proofs of various Euclidean propositions in their pre- and post-Eudoxan forms, it is possible to make clear their comparative 'advantages and limitations', as Knorr (1975, Appendix B, pp. 332–44) does in drawing direct comparisons where possible between theorems in Book V and their counterparts in Book VII.

 As Zeuthen (1910) observed, it is precisely at Theorem VII, 19, that the relation between Book V and Book VII is directly established, for in VII, 19, Euclid shows that the definition of proportion used in Book V is equivalent to definition VII, 20 when applied to numbers. It therefore follows that all theorems on proportion in Book V may be applied to any of the theorems dealing with proportions between numbers alone in Book VII. Zeuthen (1910, p. 412) states that 'l'importance logique du No. 19 consiste précisément en ce qu'on y établit que la définition d'une proportion donné dans le V^e livre a, si on l'applique à des nombres entiers, tout à fait la même portée que la définition donnée au VII^e livre'.

 Wholly apart from the significance of Eudoxus's theory of proportion for the development of the Euclidean *Elements*, Kurt von Fritz (1945, p. 264) has pointed out that Eudoxus was 'the author of the method of exhaustion, of the theorem that the volume of a cone is one-third of the volume of a cylinder with the same base and altitude, and undoubtedly of other stereometric theorems which must have been used in the proof of that proposition. All this would have been impossible without the new definition of proportion invented by Eudoxus.' Similarly, Wilbur Knorr (1975, p. 306) has noted that 'the renovation of proportion theory (Book V) was used to improve the foundations of geometry (Books VI and XI) and with the "method of exhaustion" to effect the measurements in Book XIII'.

27. For the details of Cantor's biography and the origins of transfinite set theory, sketched here only in the broadest outline, consult A. Fraenkel (1930). For more recent studies, refer to H. Meschkowski (1967), I. Grattan-Guinness (1971), J. Dauben (1979), and 'The development of Cantorian set theory', Chap. 5 in

I. Grattan-Guinness 1980, pp. 1181–219). I am grateful to Esther Phillips for her comments on an earlier version of this paper. Conversations with her on the subject of revolutions in mathematics have also greatly benefited the analysis that follows.

28. See E. Heine (1870, esp. p. 353). As Cantor noted in a footnote to his first paper on the subject, 'Zu den folgenden Arbeiten bin ich durch Herrn *Heine* angeregt worden. Derselbe hat die Güte gehabt, mich mit seinen Untersuchungen über trigonometrische Reihen frühzeitig bekannt zu machen' (Cantor 1870, p. 130).
29. G. Cantor (1872). For a discussion of the significance of this paper in the context of Cantor's early work, consult J. Dauben (1971) and 'The origins of Cantorian set theory', Chap. 2 in Dauben (1979, pp. 30–46).
30. For a fuller discussion of Cantor's early conceptualization of derived sets and the distinction between sets of the first and second species, see J. Dauben (1974).
31. It should be noted that Richard Dedekind's famous theory of 'cuts' used to define the real numbers was also published in the same year (Dedekind 1872). See also P. E. B. Jourdain (1910) and J. Cavaillès (1962, esp. pp. 35–44).
32. G. Cantor (1883), translated, in part, into French as 'Fondements d'une théorie générale des ensembles', *Acta Mathematica*, **2** (1883), pp. 381–408. There is also an English translation by U. Parpart, 'Foundations of the theory of manifold's', *The Campaigner* (The Theoretical Journal of the National Caucus of Labor Committees), **9**, (January and February), pp. 69–96. The reader should be warned, however, that in addition to missing the distinction between *reellen* and *realen Zahlen* in translating the *Grundlagen*, Parpart also fails to distinguish between *Zahlen* and *Anzahlen*, translating both as 'number' throughout without making clear the differences crucial to Cantor's introduction of the transfinite numbers. For fuller discussion of the significance of such terminological aspects of the *Grundlagen*, see J. Dauben (1979, pp. 125–8).
33. G. Cantor (1895–7). Part I was translated into Italian by F. Gerbaldi 'Contribuzione al fondamento della teoria degli inseimi transfinite', *Rivista di Matematica*, (**5**) (1985), pp. 129–62. Both parts were translated into French by F. Marotte *Sur les fondements de la théorie des ensembles transfinis* (Paris: Hermann, 1899), and into English by P. E. B. Jourdain *Contributions to the founding of the theory of transfinite numbers* (Open Court, 1915). For discussion of Cantor's terminology, and the remarkable fact that he only introduced the transfinite alephs in 1893, although he had introduced the ω for transfinite ordinal numbers in 1883, see J. Dauben (1979, pp. 179–81).
34. See in particular the discussion by Cantor (1833, Sections 4–8, reprinted in *Gesammelte Abhandlungen*, pp. 173–83). The following analysis presents, in its major outline, the views Cantor held on these matters.
35. Gauss wrote to Schumacher from Göttingen on 12 July 1831. See letter 396 (Gauss's letter 177) in *Briefwechsel zwischen K. F. Gauss und H. C. Schumacher* (ed. C. A. F. Peters, Esch, 1860), Vol. II, p. 269.
36. See Cantor's explanation of immanent and transient realities (Cantor 1883, Section 8, reprinted in *Gesammelte Abhandlungen*, pp. 181–3).
37. This is exactly Cantor's point in Section 8 of the *Grundlagen* (1883), where he stresses that the natural sciences are always concerned with the 'fit with facts', while mathematics need not be concerned with the conditions of natural phenomena as an ultimate arbiter of the truth or success of a given theory. In the natural sciences, however, historians and philosophers of science have been especially interested in the nature of the connections between observation, experiment, and theory. Among many works that might be cited, that of Thomas Kuhn is perhaps the best known and will suffice here to give some sense of the connections that set the sciences in general apart from mathematics: 'The decision to reject one paradigm is always simultaneously the decision to accept another, and the judgment leading to that decision involves the comparison of both paradigms with nature and with each other' (Kuhn 1962, p. 77). It was precisely its independence from nature that gave mathematics, in Cantor's view, its 'freedom' as characterized in the passage quoted on p. 61 (Cantor 1883, p. 182).
38. See 'The invisibility of revolutions', Chap. 11 in T. S. Kuhn (1962, 135–42, esp. p. 136).

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