Advanced Linear Algebra

Week 6

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Recall: Let $\mathcal{F}$ be a field (elements: scalars). A vector space over $\mathcal{F}$ (or an $\mathcal{F}$-vector space) is a set V (elements: vectors) where you can

- add vectors together ($x + y \in V$ whenever $x, y \in V$),
- multiply vectors with scalars: $\alpha x \in V$ whenever $x \in V$ and $\alpha \in \mathcal{F}$,

such that "everything is well-behaved".
A linear combination coming from $S \subseteq V$ is a finite sum $\sum_{i=1}^{n} \alpha_i x_i \in V$ where $x_1, \ldots, x_n \in S$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{F}$.
The span of S is the subspace

$$\begin{aligned} \mathrm{Span} S &:= \{\text{linear combinations coming from } S\} \\ &= \{\textstyle\sum_{i=1}^{n} \alpha_i x_i \mid x_1, \ldots, x_n \in S, \alpha_1, \ldots, \alpha_n \in \mathcal{F}\}. \end{aligned}$$

$L \subseteq V$ is said to be linearly dependent if $x \in \mathrm{Span}(L \setminus \{x\})$ for some $x \in L$.
Otherwise L is linearly independent, i.e. if $x \notin \mathrm{Span}(L \setminus \{x\})$ for every $x \in L$.
A basis for a vector space V is a subset $B \subseteq V$ s.t.

(a) $\mathrm{Span} B = V$;

(b) B is linearly independent.

If $B \subseteq V$ is a basis, then for $x \in V$ there is a unique family $(\alpha_v)_{v \in B}$ of scalars with only finitely many non-zero, such that

$$x = \sum_{v \in B} \alpha_v v.$$

We say that V is finite dimensional, $\dim V < \infty$, if V has a finite basis. Otherwise V is infinite dimensional, $\dim V = \infty$.
V is n-dimensional if it has a basis with n elements.

Theorem (1.22)

Let V be an n-dimensional vector space.

1. *Every linearly independent subset of V has at most n elements and is contained in a basis;*
2. *Every spanning subset of V has at least n elements and contains a basis;*
3. *Every basis has exactly n elements.*

Lemma (1.24)

V is infinite-dimensional if and only if there exists an infinite linearly independent subset $L \subseteq V$.

Proof.

"$\Leftarrow$" Contrapositive: assume V is finite dimensional. Then any linearly independent subset is finite by Theorem 1.22(1).
"$\Rightarrow$" Assume V is infinite dimensional. By Lemma 1.15, any finite linearly independent subset is contained in a linearly independent subset with 1 more element.
Starting with $L_0 = \emptyset$, use this to construct $L_0 \subseteq L_1 \subseteq L_2 \subseteq \dots$ linearly independent with $|L_n| = n$. One checks (easy from the definition) that $\bigcup_{n\in\mathbb{N}} L_n$ is linearly independent and infinite. $\square$

Theorem (1.25)

Let V be a finite-dimensional vector space with $n = \dim V$, and let $U \subseteq V$ be a subspace. Then U is finite-dimensional with $\dim U \leq n$.
Moreover, any basis for U can be extended to a basis for V.

Proof.

Every linearly independent subset of U is also linearly independent in V – hence has at most n elements. Hence $\dim U < \infty$ by Lemma 1.24. Thus a basis for U has at most n elements, so $\dim U \leq n$.
By Theorem 1.22(1) any basis of U is linearly independent in V and is therefore contained in a basis for V. $\square$

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From now on, U and V are vector spaces over a fixed field $\mathcal{F}$.

Definition

A linear map (or homomorphism) from U to V is a map A such that

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$$

for all $x, y \in U$ and $\alpha, \beta \in \mathcal{F}$.
$\mathrm{Hom}(U, V)$ is the set of linear maps from U to V.
$\mathrm{End}(U) := \mathrm{Hom}(U, U)$ and $A \in \mathrm{End}(U)$ are called endomorphisms.

Example

A linear map $\mathcal{F}^m \to \mathcal{F}^n$ is given by an $n \times m$-matrix with coefficients in $\mathcal{F}$.
So $\mathrm{Hom}(\mathcal{F}^m, \mathcal{F}^n) = M_{n,m}(\mathcal{F})$ is the set of $n \times m$-matrices over $\mathcal{F}$.
Similarly $\mathrm{End}(\mathcal{F}^n) = M_n(\mathcal{F})$ are the square $n \times n$-matrix.

Lemma (2.3)

Let B be a basis for U, and let $f\colon B \to V$ be a map (no linearity assumed!). Then there is a unique linear map $A\colon U \to V$ which extends f, i.e. $A(x) = f(x)$ for all $x \in B$.

Proof.

Any linear map $A\colon U \to V$ satisfies

$$A(\sum_i \alpha_i x_i) = \sum_i \alpha_i A(x_i)$$

for all linear combinations $\sum_i \alpha_i x_i$ from B (i.e. $x_i \in B$). Hence a linear map is uniquely determined by its values on B.

Any $x \in U$ is uniquely a linear combination $x = \sum_{v \in B} \alpha_v v$. It is straight forward to check that

$$A(\sum_{v \in B} \alpha_v v) = \sum_{v \in B} \alpha_v f(v)$$

defines a (necessarily unique) linear map which extends f.

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Question: Consider $\mathbb{R}^3$ as an $\mathbb{R}$-vector space, with the basis $B = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Let $f: B \to \mathbb{R}$ be given by

$$f(v) = 1, \qquad v \in B$$

and let $A: \mathbb{R}^3 \to \mathbb{R}$ be the unique linear extension of f. What is

$$A \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}?$$

(a) 0
(b) 1
(c) 2
(d) 3.

Question: Consider $\mathbb{R}^3$ as an $\mathbb{R}$-vector space, with the basis $B = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Let $f\colon B \to \mathbb{R}$ be given by

$$f(v) = 1, \qquad v \in B$$

and let $A\colon \mathbb{R}^3 \to \mathbb{R}$ be the unique linear extension of f. Then

$$\begin{aligned} A\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} &= A\left(1 \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \\ &= 1 \cdot f\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot f\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + 1 \cdot f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= 1 + 1 + 1 = 3. \end{aligned}$$

Definition

Let $A \in \mathrm{Hom}(U, V)$.

(a) $N(A) := \{x \in U \mid Ax = 0\}$ is the null-space (or kernel) of A;

(b) $R(A) := \{Ax \mid x \in U\}$ is the range (or image) og A.

$N(A)$ is a subspace of U.
$R(A)$ is a subspace of V.

Lemma (2.5)

(a) *A is injective if and only if $N(A) = \{0\}$;*

(b) *A is surjective if and only if $R(A) = V$.*

(a): "$\Rightarrow$" if A is injective and $x \in N(A)$, then $Ax = 0 = A0$, so $x = 0$. Hence $N(A) \subseteq \{0\}$ (and $\{0\} \subseteq N(A)$ is trivial).
"$\Leftarrow$": If $N(A) = \{0\}$, let $x_1, x_2 \in U$ such that $Ax_1 = Ax_2$.
Then $A(x_1 - x_2) = Ax_1 - Ax_2 = 0$ so $x_1 - x_2 \in N(A)$.
Hence $x_1 - x_2 = 0$ so A is injective.

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Let $A \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ be given by

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 0 \end{pmatrix}$$

Question: What is the null-space of A?

(a) $N(A) = \{0\}$;

(b) $N(A) = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \mid x \in \mathbb{R} \right\}$;

(c) $N(A) = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$;

(d) $N(A) = \mathbb{R}^2$.

Answer: We have

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x+y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x = -y.$$

So (b) is the answer.

Definition

A bijective linear map $U \to V$ is called a linear isomorphism. If such a map exists, we say that U and V are isomorphic.

Lemma (2.7)

Let $A \in \mathrm{Hom}(U, V)$. Then

(1) *Let $S \subseteq U$ be spanning. Then A is surjective $\Leftrightarrow$ $A(S)$ spans V;*

(2) *Let $L \subseteq U$ be linearly independent. Then A is injective $\Rightarrow$ $A|_{\mathrm{Span}(L)}$ is injective $\Leftrightarrow$ $A|_L$ is injective and $A(L)$ is linearly independent;*

(3) *Let $B \subseteq U$ be a basis. Then A is bijective $\Leftrightarrow$ $A|_B$ is injective and $A(B)$ is a basis for V.*

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Lemma (2.7)

Let $A \in \mathrm{Hom}(U, V)$. *Then*

(1) *Let* $S \subseteq U$ *be spanning. Then* A *is surjective* $\Leftrightarrow$ $A(S)$ *spans* V;

Proof.

(1): Check that $A(\mathrm{Span} S) = \mathrm{Span} A(S)$ for every $S \subseteq U$. When $\mathrm{Span} S = U$ then it implies that $A(U) = \mathrm{Span} A(S)$, so (1) follows. $\square$

Lemma (2.7)

Let $A \in \mathrm{Hom}(U, V)$. Then

(2) *Let $L \subseteq U$ be linearly independent. Then A is injective $\Rightarrow$ $A|_{\mathrm{Span}(L)}$ is injective $\Leftrightarrow$ $A|_L$ is injective and $A(L)$ is linearly independent;*

Proof.

(2): The first implication is trivial.
Assume $A|_{\mathrm{Span}(L)}$ is injective. Clearly $A|_L$ is injective. For $x \in L$ we have $x \notin \mathrm{Span}(L \setminus \{x\})$ (by definition of linear independence). By injectivity of $A|_{\mathrm{Span}(L)}$ we have

$$Ax \notin A(\mathrm{Span}(L \setminus \{x\})) = \mathrm{Span}(A(L \setminus \{x\})),$$

and by injectivity $A(L \setminus \{x\}) = A(L) \setminus \{Ax\}$. So $A(L)$ is linearly independent.

Lemma (2.7)

Let $A \in \mathrm{Hom}(U, V)$. *Then*

(2) *Let* $L \subseteq U$ *be linearly independent. Then* A *is injective* $\Rightarrow$ $A|_{\mathrm{Span}(L)}$ *is injective* $\Leftrightarrow$ $A|_L$ *is injective and* $A(L)$ *is linearly independent;*

Proof.

(2): Conversely, assume $A|_L$ is injective and $A(L)$ is linearly independent.

By Lemma 2.5(1) it suffices to show that if $x = \sum_i \alpha_i x_i \in \mathrm{Span}(L)$ and $Ax = 0$ then $x = 0$. Since $0 = Ax = \sum_i \alpha_i A x_i$ and since $A(L)$ is linearly independent, we get that all α_i are zero. Hence $x = 0$.

Lemma (2.7)

Let $A \in \mathrm{Hom}(U, V)$*. Then*

(1) *Let* $S \subseteq U$ *be spanning. Then* A *is surjective* $\Leftrightarrow$ $A(S)$ *spans* V*;*

(2) *Let* $L \subseteq U$ *be linearly independent. Then* A *is injective* $\Rightarrow$ $A|_{\mathrm{Span}(L)}$ *is injective* $\Leftrightarrow$ $A|_L$ *is injective and* $A(L)$ *is linearly independent;*

(3) *Let* $B \subseteq U$ *be a basis. Then* A *is bijective* $\Leftrightarrow$ $A|_B$ *is injective and* $A(B)$ *is a basis for* V*.*

Proof.

(3): This follows from (1) and (2). □

Corollary (2.8)

If U and V are finite-dimensional vector spaces over $\mathcal{F}$, then $\dim U = \dim V$ *if and only if U and V are isomorphic.*

Proof.

"$\Leftarrow$": If $A\colon U \to V$ is a linear isomorphism, then it maps a basis in U bijectively onto a basis in V (Lemma 2.7(3)). Hence a basis in U has the same number of elements as a basis in V, so $\dim U = \dim V$.

"$\Rightarrow$": Suppose $\dim U = \dim V$ and let $B \subseteq U$ and $C \subseteq V$ be bases. As $|B| = |C|$ there exists a bijection $f: B \to C$. Extend f to a linear map $A\colon U \to V$ (Lemma 2.3). Then $A|_B$ is injective and $A(B) = C$ is a basis for V. Hence A is a linear isomorphism. $\square$

Question Consider $M_2(\mathbb{R})$ - the $\mathbb{R}$-vector space of 2×2-matrices with coefficients in $\mathbb{R}$. Is $M_2(\mathbb{R})$ isomorphic with $\mathbb{C}$ (considered as an $\mathbb{R}$-vector space)?

(a) Yes;

(b) No.

Answer No.

$\mathbb{C}$ is 2-dimensional and $M_2(\mathbb{R})$ is 4-dimensional.

However $M_2(\mathbb{R})$, $\mathbb{C}^2$ and $\mathbb{R}^4$ are all 4-dimensional, hence isomorphic.

If $A \in \mathrm{Hom}(U, V)$ is bijective, then $A^{-1}\colon V \to U$ is linear (straightforward). So $A^{-1} \in \mathrm{Hom}(V, U)$ (whenever A^{-1} is defined!).

In the following, I_U and I_V are the identity maps on U and V respectively.

Lemma (2.9)

If $AT = I_V$ and $SA = I_U$ for some $S, T \in \mathrm{Hom}(V, U)$, then A is an isomorphism and $S = T = A^{-1}$.

Proof.

$S = S(AT) = (SA)T = T$. Recall that A^{-1} is the unique map such that $A^{-1}A = I_U$ and $AA^{-1} = I_V$, so $S = T = A^{-1}$. □

Theorem (2.10)

Assume that U and V both have the finite dimension n. If $A \in \mathrm{Hom}(U, V)$ is surjective or injective, then it is bijective.

Proof.

Let $B \subseteq U$ be a basis. Then $|B| = n$. By Lemma 2.7(3) it suffices to show that $A|_B$ is injective (i.e. $A(B)$ has n elements), and that $A(B)$ is a basis for V.

Suppose A is surjective. By Lemma 2.7(1) $A(B)$ spans V. By Theorem 1.22(2), $|A(B)| \geq n$. As $|B| = n$ we have $|A(B)| \leq n$ and thus $|A(B)| = n$. By Corollary 1.23, $A(B)$ is a basis.

Suppose A is injective. Then $|A(B)| = |B| = n$. By Lemma 2.7(2), $A(B)$ is linearly independent. By Corollary 1.23, $A(B)$ is a basis. $\square$

Corollary (2.11)

Suppose $\dim U = \dim V < \infty$ *and let* $T \in \mathrm{Hom}(V, U)$. *If* $AT = I_V$ *or* $TA = I_U$, *then* A *is an isomorphism and* $A^{-1} = T$.

Proof.

If $AT = I_V$ then A is surjective, and if $TA = I_U$ then A is injective. In either case, Theorem 2.10 implies that A is bijective. This implies that $A^{-1} = T$. □

When $U = V$ we let

$$\mathrm{GL}(V) = \{A \in \mathrm{End}(V) \mid A \text{ is bijective}\}.$$

This is then general linear group, which is a group with identity I_V.

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Suppose that the n-system of linear equations with n variables

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ & & & & \vdots & & & & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & 0 \end{array}$$

only has the unique solution $x_1 = x_2 = \cdots = x_n = 0$.
Question: Does the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 1 \\ & & & & \vdots & & & & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & 1 \end{array}$$

have a solution?

(a) Yes, there is a solution
(b) No, there is not a solution
(c) There is not enough information to determine this.

Answer: Let $A \in \mathrm{Hom}(\mathcal{F}^n, \mathcal{F}^n)$ be the linear map

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n a_{1i} x_i \\ \sum_{i=1}^n a_{2i} x_i \\ \vdots \\ \sum_{i=1}^n a_{ni} x_i \end{pmatrix}.$$

Uniqueness of the solution $x_1 = x_2 = \cdots = x_n = 0$ means that $N(A) = \{0\}$. Hence A is injective and thus also surjective (Theorem 2.10). So $R(A) = \mathcal{F}^n$, which implies that there exists $\alpha_1, \ldots, \alpha_n \in \mathcal{F}$ so that

$$\begin{pmatrix} \sum_{i=1}^n a_{1i} \alpha_i \\ \sum_{i=1}^n a_{2i} \alpha_i \\ \vdots \\ \sum_{i=1}^n a_{ni} \alpha_i \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

So yes, $x_1 = \alpha_1, \cdots, x_n = \alpha_n$ solves the system.

Recall from group theory: Let $(G, +)$ be an abelian group, and $H \subseteq G$ be a subgroup (automatically normal in the abelian case).
For each $x \in G$ define the coset

$$x + H = \{x + h \mid h \in H\} \subseteq G.$$

The quotient $G/H = \{x + H \mid x \in G\}$ is the set of all such cosets, and it is an abelian group with

$$(x + H) + (y + H) = (x + y) + H.$$

Recall that an $\mathcal{F}$-vector space V is an abelian group when equipped with $+$. And any subspace $U \subseteq V$ is a subgroup.
Form the quotient

$$V/U = \{x + U \mid x \in V\}$$

which is an $\mathcal{F}$-vector space with scalar multiplication

$$\alpha(x + U) := \alpha x + U.$$

Lemma (2.12)

Let V be an $\mathcal{F}$-vector space, and $U \subseteq V$ be a subspace. Then V/U (as before) is an $\mathcal{F}$-vector space. Moreover, the map $\pi\colon V \to V/U$ given by

$$\pi(x) = x + U, \qquad \textit{for } x \in V$$

is a surjective linear map with null-space $N(\pi) = U$.

Proof.

Omitted (this is straightforward). $\square$

Definition

V/U is called the quotient space of V by U, and the linear surjection $\pi\colon V \to V/U$ is called the (canonical) projection.

Consider the subspace $U = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \mid x \in \mathcal{F} \right\} \subseteq \mathcal{F}^2$. Let's describe $\mathcal{F}^2/U$.

The elements in $\mathcal{F}^2/U$ are $\begin{pmatrix} x \\ y \end{pmatrix} + U$. Note that

$$\begin{pmatrix} x \\ y \end{pmatrix} + U = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y \\ -y \end{pmatrix} + U = \begin{pmatrix} x+y \\ 0 \end{pmatrix} + U.$$

This implies that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + U$ spans $\mathcal{F}^2/U$. Any non-zero one-point set is linearly independent, and hence $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + U \right\}$ is a basis for $\mathcal{F}^2/U$.

Question: What is the dimension of $\mathcal{F}^2/U$?

(a) $\dim(\mathcal{F}^2/U) = 0$

(b) $\dim(\mathcal{F}^2/U) = 1$

(c) $\dim(\mathcal{F}^2/U) = 2$