Week 5 & 6 - Exercises

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Advanced Linear Algebra

Week 5 - Exercise 1

- Show that, in a vector space $(V, +, \cdot)$ over a field $\mathbb{F}$, for all $x \in V$ and $\alpha \in \mathbb{F}$ one has $\alpha \cdot x = 0$ iff $\alpha = 0_{\mathbb{F}}$ or $x = 0_V$. Moreover, show that $(-1_{\mathbb{F}}) \cdot x = -x$. Here $0_{\mathbb{F}}$, $1_{\mathbb{F}}$ denote the additive and multiplicative neutral elements in $\mathbb{F}$ and 0_V the zero vector in V.

Assume $\alpha \cdot x = 0$. We prove that $\alpha = 0$ or $x = 0$ by showing that $\alpha \neq 0$ implies $x = 0$. So suppose $\alpha \neq 0$. Then

$$\begin{aligned}\alpha \cdot x = 0 &\Longrightarrow \alpha \cdot x = \alpha \cdot x + \alpha \cdot x = \alpha \cdot (x + x)\\ &\Longrightarrow x = \alpha^{-1} \cdot \alpha \cdot x = \alpha^{-1} \cdot \alpha \cdot (x + x) = x + x \Longrightarrow x = 0\end{aligned}$$

Conversely, assume that $\alpha = 0$ or $x = 0$. If $\alpha = 0$ then

$$\alpha \cdot x = (\alpha + \alpha) \cdot x = \alpha \cdot x + \alpha \cdot x \Longrightarrow \alpha \cdot x = 0$$

A similar argument proves that $\alpha \cdot x = 0$ if $x = 0$.

Week 5 - Exercise 1 continued

- Show that, in a vector space $(V, +, \cdot)$ over a field $\mathbb{F}$, for all $x \in V$ and $\alpha \in \mathbb{F}$ one has $\alpha \cdot x = 0$ iff $\alpha = 0_{\mathbb{F}}$ or $x = 0_V$. Moreover, show that $(-1_{\mathbb{F}}) \cdot x = -x$. Here $0_{\mathbb{F}}$, $1_{\mathbb{F}}$ denote the additive and multiplicative neutral elements in $\mathbb{F}$ and 0_V the zero vector in V.

We have

$$(-1) \cdot x + x = (-1) \cdot x + 1 \cdot x = (-1 + 1) \cdot x = 0 \cdot x = 0$$
$$\implies (-1) \cdot x = -x$$

Week 5 - Exercise 2

- Let $p \in \mathbb{N}$ be a prime. Show that if V is a vector space over $\mathbb{Z}_p$, then $\underbrace{x + x + \cdots + x}_{p \text{ times}} = 0$ for every $x \in V$.

We have

$$\underbrace{x + x + \cdots + x}_{p \text{ times}} = \underbrace{1 \cdot x + 1 \cdot x + \cdots + 1 \cdot x}_{p \text{ times}}$$
$$= (\underbrace{1 + 1 + \cdots + 1}_{p \text{ times}}) \cdot x = p \cdot x = 0$$

since $p = 0$ in $\mathbb{Z}_p$.

Week 5 - Exercise 3 (Halmos §1, exercise 5)

5. Let $Q(\sqrt{2})$ be the set of all real numbers of the form $\alpha + \beta\sqrt{2}$, where α and β are rational.
(a) Is $Q(\sqrt{2})$ a field?
(b) What if α and β are required to be integers?

We consider the set

$$\mathbb{Q}(\sqrt{2}) = \{\alpha + \beta\sqrt{2} \mid \alpha, \beta \in \mathbb{Q}\}$$

(a) For all $\alpha, \beta \in \mathbb{Q}$ not both 0 we have

$$\frac{1}{\alpha + \beta\sqrt{2}} = \frac{\alpha - \beta\sqrt{2}}{(\alpha + \beta\sqrt{2})(\alpha - \beta\sqrt{2})} = \frac{\alpha - \beta\sqrt{2}}{\alpha^2 - 2\beta^2}$$
$$= \frac{\alpha}{\alpha^2 - 2\beta^2} - \frac{\beta}{\alpha^2 - 2\beta^2}\sqrt{2} \in \mathbb{Q}(\sqrt{2})$$

Thus, $\mathbb{Q}(\sqrt{2})$ is a field.

(b) $\mathbb{Z}(\sqrt{2})$ is not a field since $2 - \sqrt{2}$ (for example) does not have a multiplicative inverse.

Week 5 - Exercise 4 (Halmos §4, exercise 4)

4. Sometimes a subset of a vector space is itself a vector space (with respect to the linear operations already given). Consider, for example, the vector space $\mathcal{C}^3$ and the subsets $\mathcal{V}$ of $\mathcal{C}^3$ consisting of those vectors (ξ_1, ξ_2, ξ_3) for which

(a) ξ_1 is real,
(b) $\xi_1 = 0$,
(c) either $\xi_1 = 0$ or $\xi_2 = 0$,
(d) $\xi_1 + \xi_2 = 0$,
(e) $\xi_1 + \xi_2 = 1$.

In which of these cases is $\mathcal{V}$ a vector space?

We denote the subsets of $\mathbb{C}^3$ by U_j for $j = a, \dots, e$.

(a) We have $(1,0,0) \in U_a$, but $i \cdot (1,0,0) = (i,0,0) \notin U_a$. Thus, U_a is not closed under scalar multiplication.

(b) We have $(0,0,0) \in U_b$. If $(0,\xi_2,\xi_3),(0,\eta_2,\eta_3) \in U_b$ and $\alpha \in \mathbb{C}$ we have

$$\alpha \cdot (0,\xi_2,\xi_3) = (0,\alpha\xi_2,\alpha\xi_3) \in U_b$$
$$(0,\xi_2,\xi_3) + (0,\eta_2,\eta_3) = (0,\xi_2+\eta_2,\xi_3+\eta_3) \in U_b$$

Hence, U_b is a subspace.

Week 5 - Exercise 4 (Halmos §4, exercise 4) - continued

4. Sometimes a subset of a vector space is itself a vector space (with respect to the linear operations already given). Consider, for example, the vector space $\mathcal{C}^3$ and the subsets $\mathcal{V}$ of $\mathcal{C}^3$ consisting of those vectors (ξ_1, ξ_2, ξ_3) for which

(a) ξ_1 is real,
(b) $\xi_1 = 0$,
(c) either $\xi_1 = 0$ or $\xi_2 = 0$,
(d) $\xi_1 + \xi_2 = 0$,
(e) $\xi_1 + \xi_2 = 1$.

In which of these cases is $\mathcal{V}$ a vector space?

(c) We have $(1,0,0) \in U_c$ and $(0,1,0) \in U_c$, but $(1,0,0) + (0,1,0) = (1,1,0) \notin U_c$. Thus, U_c is not closed under addition.

(d) We have $(0,0,0) \in U_d$. Let $(\xi_1, \xi_2, \xi_3), (\eta_1, \eta_2, \eta_3) \in U_d$ and $\alpha \in \mathbb{C}$. Then $\xi_1 + \xi_2 = 0$ and $\eta_1 + \eta_2 = 0$. We have $\alpha\xi_1 + \alpha\xi_2 = \alpha(\xi_1 + \xi_2) = 0$ and $\xi_1 + \eta_1 + \xi_2 + \eta_2 = 0$. Thus,

$$\alpha \cdot (\xi_1, \xi_2, \xi_3) = (\alpha\xi_1, \alpha\xi_2, \alpha\xi_3) \in U_d$$

$$(\xi_1, \xi_2, \xi_3) + (\eta_1, \eta_2, \eta_3) = (\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3) \in U_d$$

Hence, U_d is a subspace.

(e) We have $(0,0,0) \notin U_e$. Further, $(1,0,0) \in U_e$ and $(0,1,0) \in U_e$, but $(1,0,0) + (0,1,0) = (1,1,0) \notin U_e$ and $2 \cdot (1,0,0) = (2,0,0) \notin U_e$. Thus, U_e is neither closed under addition nor scalar multiplication.

Week 5 - Exercise 5 (Halmos §7, exercise 1)

1. (a) Prove that the four vectors

$$x = (1, 0, 0),$$
$$y = (0, 1, 0),$$
$$z = (0, 0, 1),$$
$$u = (1, 1, 1),$$

in $\mathbb{C}^3$ form a linearly dependent set, but any three of them are linearly independent.

(b) If the vectors x, y, z, and u in $\mathcal{P}$ are defined by $x(t) = 1$, $y(t) = t$, $z(t) = t^2$, and $u(t) = 1 + t + t^2$, prove that x, y, z, and u are linearly dependent, but any three of them are linearly independent.

Since $u = x + y + z$, the four vectors form a linearly dependent set. To see why x, y, u are linearly independent, we assume that

$$(0, 0, 0) = \alpha_1 x + \alpha_2 y + \alpha_3 u = \alpha_1(1, 0, 0) + \alpha_2(0, 1, 0) + \alpha_3(1, 1, 1)$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. This yields the following set of equations:

$$\alpha_1 + \alpha_3 = 0, \qquad \alpha_2 + \alpha_3 = 0, \qquad \alpha_3 = 0$$

Hence, $\alpha_1 = \alpha_2 = \alpha_3 = 0$ which shows that x, y, u are linearly independent. Similarly, one shows that any three of x, y, z, u are linearly independent. Since $1 \mapsto (1, 0, 0)$, $t \mapsto (0, 1, 0)$, $t^2 \mapsto (0, 0, 1)$ yields an isomorphism from $\mathcal{P}_2$ to $\mathbb{C}^3$, exactly the same arguments apply in (b).

Week 5 - Exercise 6 (Halmos §7, exercise 2)

2. Prove that if $\mathfrak{R}$ is considered as a rational vector space (see § 3, (8)), then a necessary and sufficient condition that the vectors 1 and ξ in $\mathfrak{R}$ be linearly independent is that the real number ξ be irrational.

We consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Let $\xi \in \mathbb{R}$. We prove the claim by contraposition.
Assume that 1 and ξ are linear dependent. This means that we can write ξ as a linear combination of 1, i.e. there is an $\alpha \in \mathbb{Q}$ such that $\xi = \alpha \cdot 1 = \alpha$. Thus, $\xi \in \mathbb{Q}$.
Assume that $\xi \in \mathbb{Q}$. Since $\xi = \xi \cdot 1$, we have written ξ as a linear combination of 1, and we conclude that 1 and ξ are linearly dependent.

Week 5 - Exercise 7 (Halmos §7, exercise 5)

5. (a) The vectors (ξ_1, ξ_2) and (η_1, η_2) in $\mathbb{C}^2$ are linearly dependent if and only if $\xi_1\eta_2 = \xi_2\eta_1$.
(b) Find a similar necessary and sufficient condition for the linear dependence of two vectors in $\mathbb{C}^3$. Do the same for three vectors in $\mathbb{C}^3$.
(c) Is there a set of three linearly independent vectors in $\mathbb{C}^2$?

(a) Let $(\xi_1, \xi_2), (\eta_1, \eta_2) \in \mathbb{C}^2$. Assume without loss of generality that $\xi_1 \neq 0 \neq \xi_2$. Suppose $\xi_1\eta_2 = \xi_2\eta_1$. Set $\alpha = \frac{\eta_1}{\xi_1} = \frac{\eta_2}{\xi_2}$. Then $(\eta_1, \eta_2) = \alpha(\xi_1, \xi_2)$ which shows that (ξ_1, ξ_2) and (η_1, η_2) are linearly dependent.
Suppose that (ξ_1, ξ_2) and (η_1, η_2) are linearly dependent. This means that there is an $\alpha \in \mathbb{C}$ such that $(\eta_1, \eta_2) = \alpha(\xi_1, \xi_2)$ or equivalently $\eta_1 = \alpha\xi_1$ and $\eta_2 = \alpha\xi_2$. Hence, $\alpha = \frac{\eta_1}{\xi_1} = \frac{\eta_2}{\xi_2}$ from which $\xi_1\eta_2 = \xi_2\eta_1$ follows.

(b) Note that $\xi_1\eta_2 = \xi_2\eta_1$ is equivalent to the condition that

$$\det \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} = 0$$

The generalisation to $\mathbb{C}^3$ is that $(\xi_1, \xi_2, \xi_3), (\eta_1, \eta_2, \eta_3), (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3$ are linearly dependent if and only if

$$\det \begin{pmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \\ \xi_3 & \eta_2 & \zeta_3 \end{pmatrix} = 0$$

(c) No, see Theorem 1.22.

Week 5 - Exercise 8

- (From the Exam 2020): Argue that the set $B' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ constitutes a basis for $\mathbb{R}^2$.

Since $\dim(\mathbb{R}^2) = 2$ and $|B'| = 2$, it suffices to show that B' is linearly independent (see Corollary 1.23). To show linear independence, we assume that

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$. This yields the following set of equations:

$$\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ -\alpha_2 &= 0 \end{aligned}$$

From the second equation we see that $\alpha_2 = 0$, and the first equation then implies that $\alpha_1 = 0$. Thus, B' is linearly independent, and B' constitutes a basis for $\mathbb{R}^2$.

Week 6 - Exercise 1

- Consider the vector space $\mathbb{C}^5$ and denote by $P \colon \mathbb{C}^5 \to \mathbb{C}^5$ the map $P(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0, 0)$. Show that P is linear and determine its kernel, range, nullity and rank. Moreover, show that $\mathbb{C}^5 / \ker(P) \simeq \mathbb{C}^2$.

Let $(x_1, x_2, x_3, x_4, x_5)$, $(y_1, y_2, y_3, y_4, y_5) \in \mathbb{C}^5$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} P\left((x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)\right) &= P(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + y_5) \\ &= (x_1 + y_1, x_2 + y_2, 0, 0, 0) \\ &= (x_1, x_2, 0, 0, 0) + (y_1, y_2, 0, 0, 0) \\ &= P(x_1, x_2, x_3, x_4, x_5) + P(y_1, y_2, y_3, y_4, y_5) \end{aligned}$$

and

$$\begin{aligned} P\left(\alpha(x_1, x_2, x_3, x_4, x_5)\right) &= P(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5) \\ &= (\alpha x_1, \alpha x_2, 0, 0, 0) \\ &= \alpha(x_1, x_2, 0, 0, 0) \\ &= \alpha P(x_1, x_2, x_3, x_4, x_5) \end{aligned}$$

Thus, P is linear. We have

$$\begin{aligned} N(P) &= \operatorname{span}\{(0, 0, 1, 0, 0); (0, 0, 0, 1, 0); (0, 0, 0, 0, 1)\}, \\ R(P) &= \operatorname{span}\{(1, 0, 0, 0, 0); (0, 1, 0, 0, 0)\} \end{aligned}$$

$$\operatorname{null}(P) = \dim N(P) = 3, \qquad \operatorname{rank}(P) = \dim R(P) = 2$$

Week 6 - Exercise 1 - continued

- Consider the vector space $\mathbb{C}^5$ and denote by $P\colon \mathbb{C}^5 \to \mathbb{C}^5$ the map $P(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0, 0)$. Show that P is linear and determine its kernel, range, nullity and rank. Moreover, show that $\mathbb{C}^5/\ker(P) \simeq \mathbb{C}^2$.

We define the map $A\colon \mathbb{C}^5/N(P) \to \mathbb{C}^2$ by $A((x_1, x_2, x_3, x_4, x_5) + N(P)) = (x_1, x_2)$. Then A is well-defined. Indeed, if $(x_1, x_2, x_3, x_4, x_5) + N(P) = (y_1, y_2, y_3, y_4, y_5) + N(P)$ then $(x_1 - y_1, x_2 - y_2, x_3 - y_3, x_4 - y_4, x_5 - y_5) \in N(P)$ so that

$$(0,0,0,0,0) = P(x_1 - y_1, x_2 - y_2, x_3 - y_3, x_4 - y_4, x_5 - y_5) = (x_1 - y_1, x_2 - y_2, 0, 0, 0)$$

Hence, $x_1 = y_1$ and $x_2 = y_2$ and thus $(x_1, x_2) = (y_1, y_2)$. It is straightforward to check that A is linear (more or less same calculations as for the map P). To see injectivity, we assume that $A((x_1, x_2, x_3, x_4, x_5) + N(P)) = (0, 0)$ for some $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5$. Then $(x_1, x_2) = (0, 0)$ and thus

$$(x_1, x_2, x_3, x_4, x_5) + N(P) = (0, 0, x_3, x_4, x_5) + N(P) = (0, 0, 0, 0, 0) + N(P)$$

This shows that $N(A) = \{(0,0,0,0,0) + N(P)\}$ or equivalently that A is injective. The surjectivity of A is clear. Hence, A is an isomorphism and $\mathbb{C}^5/N(P) \simeq \mathbb{C}^2$. Alternatively, one may argue that

$$\dim \mathbb{C}^5/N(P) = \dim \mathbb{C}^5 - \dim N(P) = 5 - 3 = 2$$

by Theorem 2.14, and thus $\mathbb{C}^5/N(P) \simeq \mathbb{C}^2$ follows from the fact that $\dim \mathbb{C}^5/N(P) = \dim \mathbb{C}^2$, see Corollary 2.8.

Week 6 - Exercise 2 (Halmos §22, exercise 1)

1. Consider the quotient spaces obtained by reducing the space $\mathcal{P}$ of polynomials modulo various subspaces. If $\mathfrak{M} = \mathcal{P}_n$, is $\mathcal{P}/\mathfrak{M}$ finite-dimensional? What if $\mathfrak{M}$ is the subspace consisting of all even polynomials? What if $\mathfrak{M}$ is the subspace consisting of all polynomials divisible by x_n (where $x_n(t) = t^n$)?

We consider the vector space

$$\mathbb{R}[x] = \{a_0 + a_1 x + \cdots + a_m x^m \mid m \in \mathbb{N}_0, a_i \in \mathbb{R} \text{ for } i = 0, \ldots, m\}$$

and the subspaces

$$\begin{aligned} \mathcal{P}_n &= \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R} \text{ for } i = 1, \ldots, m\} \\ \mathcal{P}_{\text{even}} &= \{a_0 + a_1 x^2 + \cdots + a_m x^{2m} \mid m \in \mathbb{N}_0, a_i \in \mathbb{R} \text{ for } i = 0, \ldots, m\} \\ \mathcal{P}_{x^n} &= \{x^n(a_0 + a_1 x + \cdots + a_m x^m) \mid m \in \mathbb{N}_0, a_i \in \mathbb{R} \text{ for } i = 0, \ldots, m\} \\ &= \{a_0 x^n + a_1 x^{n+1} + \cdots + a_m x^{n+m} \mid m \in \mathbb{N}_0, a_i \in \mathbb{R} \text{ for } i = 0, \ldots, m\} \end{aligned}$$

Then we have the quotient spaces $\mathbb{R}[x]/\mathcal{P}_n$, $\mathbb{R}[x]/\mathcal{P}_{\text{even}}$ and $\mathbb{R}[x]/\mathcal{P}_{x^n}$.

Week 6 - Exercise 2 (Halmos §22, exercise 1) - continued

Recall that if V is a vector space and U is a subspace of V then we have the projection $\pi\colon V \to V/U$ given by $\pi(x) = x + U$ with the following property:

Theorem 2.14. *Let C be a basis for U and B a basis for V with $C \subseteq B$. Then the projection π maps the complement $B \setminus C$ bijectively onto a basis for V/U. In particular, if $\dim V$ is finite then $\dim(V/U) = \dim V - \dim U$.*

The vector spaces have the following bases:

$$B(\mathbb{R}[x]) = \{1, x, x^2, \dots\}, \qquad B(\mathcal{P}_n) = \{1, x^2, \dots x^n\}$$
$$B(\mathcal{P}_{\text{even}}) = \{1, x^2, x^4, \dots\}, \qquad B(\mathcal{P}_{x^n}) = \{x^n, x^{n+1}, x^{n+2}, \dots\}$$

Thus, it follows from Theorem 2.14 that we have the following bases for the quotient spaces:

$$B(\mathbb{R}[x]/\mathcal{P}_n) = \{x^{n+1} + \mathcal{P}_n, x^{n+2} + \mathcal{P}_n, \dots\}$$
$$B(\mathbb{R}[x]/\mathcal{P}_{\text{even}}) = \{x + \mathcal{P}_{\text{even}}, x^3 + \mathcal{P}_{\text{even}}, \dots\}$$
$$B(\mathbb{R}[x]/\mathcal{P}_{x^n}) = \{1 + \mathcal{P}_{x^n}, x + \mathcal{P}_{x^n}, \dots, x^{n-1} + \mathcal{P}_{x^n}\}$$

Hence, $\mathbb{R}[x]/\mathcal{P}_{x^n}$ is finite-dimensional whereas $\mathbb{R}[x]/\mathcal{P}_n$ and $\mathbb{R}[x]/\mathcal{P}_{x^n}$ are infinite-dimensional.

Week 6 - Exercise 3

- Let U, V be vector spaces over a field $\mathbb{F}$ and let $A \in \mathrm{Hom}(U, V)$ be an isomorphism. Show that the set theoretical inverse map $A^{-1}\colon V \to U$ is linear and hence an isomorphism as well.

Let $v, w \in V$ and $\alpha \in \mathbb{F}$. Then

$$A\left(A^{-1}(v) + A^{-1}(w)\right) = AA^{-1}(v) + AA^{-1}(w) = v + w$$

Applying A^{-1} to both sides yields $A^{-1}(v + w) = A^{-1}(v) + A^{-1}(w)$. Similarly,

$$A\left(\alpha A^{-1}(v)\right) = \alpha AA^{-1}(v) = \alpha v \implies A^{-1}(\alpha v) = \alpha A^{-1}(v)$$

Thus, A^{-1} is linear.

Week 6 - Exercise 4

- Recall that $C(\mathbb{R},\mathbb{R})$ denotes the vector space (over $\mathbb{R}$) of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

  (a) Show that $C^1(\mathbb{R},\mathbb{R}) := \{f \in C(\mathbb{R},\mathbb{R}) \mid f \text{ continuously differentiable}\}$ is a subspace of $C(\mathbb{R},\mathbb{R})$.

  (b) Show that the map $f \mapsto f'$ defines a linear map $D\colon C^1(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ and determine its kernel and range.

  (c) Contemplate why the result in (b) does not contradict Theorem 2.10.

Theorem 2.10. *Assume that U and V both have the finite dimension n. If A is surjective or injective, then it is bijective.*

(a) We have $0 \in C^1(\mathbb{R},\mathbb{R})$. If $f, g \in C^1(\mathbb{R},\mathbb{R})$ and $\alpha \in \mathbb{R}$ we know from calculus that $f+g$ and αf are also differentiable, and if f' and g' are continuous then so is $(f+g)' = f' + g'$ and $(\alpha f)' = \alpha f'$. Thus, $C^1(\mathbb{R},\mathbb{R})$ is a subspace of $C(\mathbb{R},\mathbb{R})$.

(b) If $f, g \in C^1(\mathbb{R},\mathbb{R})$ and $\alpha \in \mathbb{R}$ we have $(f+g)' = f' + g'$ and $(\alpha f)' = \alpha f'$. Thus, the map $f \mapsto f'$ is linear. The kernel consists of all constant functions from $\mathbb{R}$ to $\mathbb{R}$ and the range is $C(\mathbb{R},\mathbb{R})$, i.e. the map is surjective. To see this, let $g \in C(\mathbb{R},\mathbb{R})$ be given. Then we know from the Fundamental Theorem of Calculus that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \int_{-\infty}^{x} g(t)dt$ satisfies $f \in C^1(\mathbb{R},\mathbb{R})$ and $f' = g$.

(c) The map $f \mapsto f'$ is surjective and not injective (the kernel is nontrivial). This does not contradict Theorem 2.10 since it only applies for finite-dimensional vector spaces, and $C(\mathbb{R},\mathbb{R})$ and $C^1(\mathbb{R},\mathbb{R})$ are infinite-dimensional.