Advanced Linear Algebra

Week 16

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Question: which of the following matrices in $M_2(\mathbb{C})$ are self-adjoint? (Multiple choice)

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Last time: $U \in \mathrm{End}(V)$ is a unitary isomorphism if it is an isomorphism such that

$$\langle Ux, Uy\rangle = \langle x, y\rangle, \qquad \forall x, y \in V.$$

$U \in \mathrm{End}(V)$ is a unitary isomorphism if and only if it maps an ONB to an ONB.

If we consider the case $V = \mathcal{F}^n$ with the standard inner product, then $\mathrm{End}(\mathcal{F}^n) = M_n(\mathcal{F})$.

Then $U \in M_n(\mathcal{F})$ is a unitary $\Leftrightarrow Ue_1, Ue_2, \ldots, Ue_n$ is an ONB, $\Leftrightarrow$ the columns of U have length 1 and are pairwise orthogonal.

Question: Is $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_3(\mathcal{F})$ a unitary?

(1) "Yes" for both $\mathcal{F} = \mathbb{R}$ and $\mathcal{F} = \mathbb{C}$;

(2) "Yes" for $\mathcal{F} = \mathbb{R}$ and "No" for $\mathcal{F} = \mathbb{C}$,

(3) "No" for both $\mathcal{F} = \mathbb{R}$ and $\mathcal{F} = \mathbb{C}$.

Last time: $U \in \mathrm{End}(V)$ is a unitary isomorphism if it is an isomorphism such that

$$\langle Ux, Uy \rangle = \langle x, y \rangle, \qquad \forall x, y \in V.$$

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If we consider the case $V = \mathcal{F}^n$ with the standard inner product, then $\mathrm{End}(\mathcal{F}^n) = M_n(\mathcal{F})$.

Then $U \in M_n(\mathcal{F})$ is a unitary $\Leftrightarrow$ $Ue_1, Ue_2, \ldots, Ue_n$ is an ONB, $\Leftrightarrow$ the columns of U have length 1 and are pairwise orthogonal.

Question: Is $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in M_3(\mathcal{F})$ a unitary?

Recall: if $A \in \mathrm{End}(V)$, $\lambda \in \mathcal{F}$, then

$$V_\lambda := \{x \in V : Ax = \lambda x\}$$

is the eigenspace corresponding to λ.
If $V_\lambda \neq \{0\}$ then λ is an eigenvalue for A.
$\sigma(A)$ is the set of eigenvalues for A.
All non-zero $x \in V_\lambda$ are called eigenvectors corresponding to λ.

Definition (9.1)

Let V be a finite-dimensional inner product space, and $A \in \mathrm{End}(V)$. A is orthogonally diagonalisable if there exists an orthonormal basis (ONB) for V consisting of eigenvectors for A.
Equivalently,

$$V = \bigoplus_{\lambda \in \sigma(A)} V_\lambda$$

is an orthogonal direct sum.

Definition (9.1)

Let V be a finite-dimensional inner product space, and $A \in \mathrm{End}(V)$. A is orthogonally diagonalisable if there exists an orthonormal basis (ONB) for V consisting of eigenvectors for A.

Suppose $\mathcal{B} = (x_1, x_2, \ldots, x_n)$ is an ONB for V of eigenvectors for A.

Recall from last time: the (i,j)'th entry of ${}_{\mathcal{B}}[A]_{\mathcal{B}}$ is $\langle Ax_j, x_i\rangle$.

Let λ_j be the eigenvalue corresponding to x_j, so $Ax_j = \lambda_j x_j$.

Then

$$\langle Ax_j, x_i\rangle = \langle \lambda_j x_j, x_i\rangle = \lambda_j\langle x_j, x_i\rangle = \begin{cases} \lambda_j, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$${}_{\mathcal{B}}[A]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

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Suppose $\mathcal{B}$ and A are as before (so ${}_{\mathcal{B}}[A]_{\mathcal{B}}$ is diagonal). Let $\mathcal{C} = (y_1, \ldots, y_n)$ be an ONB. Then

$$
{}_{\mathcal{B}}[I]_{\mathcal{C}\,\mathcal{C}}[A]_{\mathcal{C}\,\mathcal{C}}[I]_{\mathcal{B}} = {}_{\mathcal{B}}[A]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.
$$

Since ${}_{\mathcal{C}}[I]_{\mathcal{B}}$ maps the ONB $\mathcal{B}$ to the ONB $\mathcal{C}$, it is a unitary.

Lemma (9.2)

Let V be a finite-dimensional inner product space with ONB $\mathcal{C}$. Then $A \in \mathrm{End}(V)$ is orthogonally diagonalisable if and only if there exists a unitary matrix $U \in M_n(\mathcal{F})$ such that $U^{-1}{}_{\mathcal{C}}[A]_{\mathcal{C}}U$ is a diagonal matrix.

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Lemma (9.3)

Let $A \in \mathrm{End}(V)$ *be self-adjoint. Then* $\sigma(A) \subseteq \mathbb{R}$ *and the eigenspaces of* A *are orthogonal to each other.*

Recall: a subspace $W \subseteq V$ is A-invariant if $AW \subseteq W$.

Lemma (9.4)

Suppose that $A \in \mathrm{End}(V)$ has an adjoint, and that $W \subseteq V$ is an A-invariant subspace. Then $W^\perp$ is an A^*-invariant subspace.

Recall Theorem 6.9: if V is a complex vector space, $0 < \dim V < \infty$, and $A \in \mathrm{End}(V)$, then there exists an eigenvector for A.

Theorem (9.5 (Spectral theorem - complex case))

Let V be a complex inner product space with $0 < \dim V < \infty$, and let $A \in \mathrm{End}(V)$ be self-adjoint. Then A is orthogonally diagonalisable.

It can be proved that $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is orthogonally diagonlisable in $M_2(\mathbb{C})$, but not in $M_2(\mathbb{R})$.
i is an eigenvalue:

$$A \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = \begin{pmatrix} i/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = i \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

and $-i$ is an eigenvalue:

$$A \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -i/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = (-i) \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}.$$

Moreover, the eigenvectors from above have length 1 and

$$\langle \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} \rangle = \tfrac{1}{\sqrt{2}} \cdot \overline{\tfrac{1}{\sqrt{2}}} + \tfrac{i}{\sqrt{2}} \cdot \overline{\tfrac{-i}{\sqrt{2}}} = \tfrac{1}{2} - \tfrac{1}{2} = 0.$$

So these eigenvectors for A form an ONB for $\mathbb{C}^2$.

Question: which of the following matrices in $M_2(\mathbb{C})$ are orthogonally diagonalisable? (Multiple choice)

$$B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

For the rest of the lecture: V is a real inner product space with $0 < \dim V < \infty$, and $A \in \mathrm{End}(V)$ is self-adjoint.

Lemma (9.10)

$\sigma(A^2) \subseteq [0, \infty)$.

Recall: a minimal polynomial for A is a non-zero $p(X) \in \mathbb{R}[X]$ such that $p(A) = 0$ and $p(X)$ has minimal degree amongst such polynomials.
Lemma 6.8: If $\dim V < \infty$ then A has a minimal polynomial $p(X)$ and $\sigma(A) = R(p)$ (roots).

Lemma (9.11)

For every real inner product space V with $0 < \dim V < \infty$ and self-adjoint $A \in \mathrm{End}(V)$, there exists an eigenvector for A.

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Theorem (9.12 (Spectral theorem - real case))

Let V be a real inner product space with $0 < \dim V < \infty$, and let $A \in \mathrm{End}(V)$ be self-adjoint. Then A is orthogonally diagonalisable.

Question: Is the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$:

(1) Orthogonally diagonalisable in $M_2(\mathbb{C})$ but not in $M_2(\mathbb{R})$;
(2) Orthogonally diagonalisable in both $M_2(\mathbb{C})$ and $M_2(\mathbb{R})$;
(3) Not orthogonally diagonalisable in either of $M_2(\mathbb{C})$ or $M_2(\mathbb{R})$

Final remarks: the spectral theorem in the complex case is actually more general: all normal $A \in \mathrm{End}(V)$ are orthogonally diagonalisable.
Normal means $AA^* = A^*A$ (this is trivially satisfied for self-adjoint A).
Moreover, this is "if and only if": A is orthogonally diagonalisable $\Leftrightarrow$ A is normal.
Note that $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is normal (since $A^*A = AA^* = I$), but it is not orthogonally diagonalisable in $M_2(\mathbb{R})$! Only in $M_2(\mathbb{C})$.