Advanced Linear Algebra

Week 5

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Literature

For lectures (every week)

Advanced Vector Spaces, lecture notes by Henrik Schlichtkrull.

For exercises (every other week)

Exercises will be posted on the weekly sheet. Some of them will be outsourced to the book *Finite dimensional vector spaces* by P.R. Halmos.
Can be downloaded through SDU library.

Practical information

- Weekly note on ItsLearning.
- Written exam (date still unknown)
- 7-point grading scale.
- Censor: external.
- Exam will differ from MM562 and MM853 students. Otherwise the course is the same.

What is a vector space?

Old answer: let $\mathcal{F} \in \{\mathbb{R}, \mathbb{C}\}$. A vector space over $\mathcal{F}$ is a set V (of vectors) where you can

- add vectors together ($x + y \in V$ whenever $x, y \in V$),
- multiply vectors with elements in $\mathcal{F}$ (scalars): $\alpha x \in V$ whenever $x \in V$ and $\alpha \in \mathcal{F}$,

such that "everything is well-behaved".

Example

Suppose $f, g\colon \mathbb{R} \to \mathbb{R}$ are continuous. By basic calculus $f + g$ is also continuous. And if $\alpha \in \mathbb{R}$ then αf is also continuous. Let $C(\mathbb{R}, \mathbb{R})$ be the set of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. What we have (essentially) just argued above, is that $C(\mathbb{R}, \mathbb{R})$ is a vector space over $\mathbb{R}$.

Recall (from Algebra 1): A field (da: legeme) is a commutative ring with identity, such that every non-zero element is invertible.

More formally: a field is a set $\mathcal{F}$ with two binary operations $+, \cdot \colon \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ (addition and multiplication), such that

(a) $\mathcal{F}$ with $+$ is an abelian group with identity 0;

(b) $\mathcal{F} \setminus \{0\}$ with $\cdot$ is an abelian group with identity 1;

(c) (distributive law): $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ for all $\alpha, \beta, \gamma \in \mathcal{F}$.

Elements of a field $\mathcal{F}$ are called scalars.

Examples:

$\mathbb{Q}$ (rational numbers),

$\mathbb{R}$ (real numbers),

$\mathbb{C}$ (complex numbers),

$\mathbb{Z}_p$ for a prime $p \geq 2$ ($\mathbb{Z}$ mod p).

Definition

Let $\mathcal{F}$ be a field. A vector space over $\mathcal{F}$ (or $\mathcal{F}$-vector space) is a set V (in which elements are called vectors), with binary maps $+\colon V \times V \to V$ (addition) and $\cdot\colon \mathcal{F} \times V \to V$ (scalar multiplication) such that the following hold:

(a) V with $+$ is an abelian group with identity element 0 (the zero vector);

(b) (distributive law): $\alpha(x+y) = \alpha x + \alpha y$ and $(\alpha+\beta)x = \alpha x + \beta x$ for all $\alpha, \beta \in \mathcal{F}$ and $x, y \in V$;

(c) $\alpha(\beta x) = (\alpha\beta)x$ and $1_{\mathcal{F}} \cdot x = x$ for all $\alpha, \beta \in \mathcal{F}$ and $x \in V$.

Examples:

$\mathcal{F}$ is a vector space over $\mathcal{F}$.

$V = \mathcal{F}^n$ (n-tuples of elements in $\mathcal{F}$) is a vector space over $\mathcal{F}$.

$\{0\}$ is a vector space over any field (the trivial space)

$\mathbb{C}$ is a vector space over $\mathbb{R}$, and $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

Examples:

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$\{0\}$ is a vector space over any field (the trivial space)

$\mathbb{C}$ is a vector space over $\mathbb{R}$, and $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

Question: Which of the following is not true?

(a) $\mathbb{C}$ is a vector space over $\mathbb{Q}$;

(b) $\mathbb{R}$ is a vector space over $\mathbb{R}$;

(c) $\mathbb{Q}$ is a vector space over $\mathbb{C}$.

Answer: $\mathbb{Q}$ as not a vector space over $\mathbb{C}$.

In fact, if $\alpha \in \mathbb{C}$ and $q \in \mathbb{Q}$, then it is not always true that $\alpha q \in \mathbb{Q}$. E.g. $q = 1$ and $\alpha = i$.

Let V be a vector space over $\mathcal{F}$.

Definition

A subset $U \subseteq V$ is called a subspace if $0 \in U$ and if $x + y, \alpha x \in U$ whenever $x, y \in U$ and $\alpha \in \mathcal{F}$.

A subspace of V is itself a vector space over $\mathcal{F}$ (with the sum and scalar product it gets from V).

Definition

A linear combination is a finite sum $\sum_{i=1}^{n} \alpha_i x_i \in V$ where $x_1, \ldots, x_n \in V$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{F}$.
The vector $x = \sum_{i=1}^{n} \alpha_i x_i$ is called the sum of the linear combination.
If $S \subseteq V$ is a subset and $x_1, \ldots, x_n \in S$, we say that the linear combination comes from S.

In the definition we allow $n = 0$, which is the empty linear combination. Its sum is (by definition) the zero vector $0 \in V$.

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Definition

Let $S \subseteq V$ be a subset. The span of S is the set

$$\begin{aligned} \mathrm{Span} S &:= \{\text{linear combinations coming from } S\} \\ &= \{\textstyle\sum_{i=1}^{n} \alpha_i x_i \mid x_1, \ldots, x_n \in S, \alpha_1, \ldots, \alpha_n \in \mathcal{F}\}. \end{aligned}$$

By definition $\mathrm{Span}\emptyset = \{0\} \subseteq V$.
If $\mathrm{Span} S = V$ we say that S spans V.

Lemma (1.8)

Let $S \subseteq V$ be a subset, and let $U \subseteq V$ be a subspace. Then

(a) $\mathrm{Span} S$ *is a subspace of* V*;*

(b) *If* $S \subseteq U$ *then* $\mathrm{Span} S \subseteq U$.

In particular, $\mathrm{Span} S$ *is the smallest subspace of* V *containing* S.

Proof.

Omitted (it is straight forward from the definitions).

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Let $L \subseteq V$ be a subset.

Definition

L is said to be linearly dependent if $x \in \mathrm{Span}(L \setminus \{x\})$ for some $x \in L$.
L is called linearly independent if it is not linearly dependent, i.e. if $x \notin \mathrm{Span}(L \setminus \{x\})$ for every $x \in L$.

Lemma (1.10)

L is linearly dependent if and only if there exist distinct vectors $x_1, \ldots, x_n \in L$, and there exist scalars $\alpha_1, \ldots, \alpha_n \in \mathcal{F}$ with at least one $\alpha_k \neq 0$, such that $\sum_{i=1}^n \alpha_i x_i = 0$.

Proof.

"$\Leftarrow$": If $\sum_{i=1}^n \alpha_i x_i = 0$ and $\alpha_k \neq 0$, then
$x_k = -\sum_{i \neq k} \alpha_k^{-1} \alpha_i x_i$.
As the vectors $x_1, \ldots, x_n$ are distinct,
$x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in L \setminus \{x_k\}$, and thus
$x_k \in \mathrm{Span}(L \setminus \{x_k\})$. So L is linearly dependent.

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Lemma (1.10)

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Proof.

"$\Rightarrow$": Suppose L is linearly dependent and let $x \in L$ such that $x \in \mathrm{Span}(L \setminus \{x\})$.
There exist distinct $x_1, \ldots, x_n \in L \setminus \{x\}$ and $\beta_1, \ldots, \beta_n \in \mathcal{F}$ such that $x = \sum_{i=1}^{n} \beta_i x_i$.
Then $x, x_1, \ldots, x_n \in L$ are distinct(!) and $1_{\mathcal{F}} \cdot x - \sum_{i=1}^{n} \beta_i x_i = 0$.
Also, the scalar coefficient on x is $1_{\mathcal{F}}$ which is non-zero. $\square$

Lemma (1.11)

Let $L \subseteq V$ be linearly independent and let $x \in V \setminus L$. Then $L \cup \{x\}$ is linearly dependent if and only if $x \in \operatorname{Span} L$.

Proof.

"$\Leftarrow$": If $x \in \operatorname{Span} L$ then $L \cup \{x\}$ is linearly dependent by definition, since $(L \cup \{x\}) \setminus \{x\} = L$.
"$\Rightarrow$": If $L \cup \{x\}$ is linearly dependent, then by Lemma 1.10 there are distinct $x_1, \ldots, x_n \in L \cup \{x\}$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{F}$ at least one of which is non-zero, such that $\sum_{i=1}^{n} \alpha_i x_i = 0$.
If x was not one of the vectors $x_1, \ldots, x_n$, then it this would imply that L is linearly dependent (by Lemma 1.10). Assume WLOG that $x_1 = x$, and also that $\alpha_1 \neq 0$. Then

$$x = -\sum_{i=2}^{n} \alpha_1^{-1} \alpha_i x_i \in \operatorname{Span} L.$$

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Lemma (1.11)

Let $L \subseteq V$ be linearly independent and let $x \in V \setminus L$. Then $L \cup \{x\}$ is linearly dependent if and only if $x \in \mathrm{Span}\, L$.

Corollary

Let $L \subseteq V$ be linearly independent, and let $x \in V$ be an element such that $x \notin \mathrm{Span}\, L$. Then $L \cup \{x\}$ is linearly independent.

Spørgsmål: In the $\mathbb{R}$-vector space $\mathbb{R}^3$, let

$$L = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Is $L \cup \left\{ \begin{pmatrix} \sqrt{2} \\ 42 \\ 0 \end{pmatrix} \right\}$ linearly dependent or linearly independent?

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Definition

A basis for a vector space V is a subset $B \subseteq V$ for which

(a) $\mathrm{Span} B = V$;

(b) B is linearly independent.

Lemma (1.13)

A subset $B \subseteq V$ is a basis if and only if every vector $x \in V$ can obtained in a unique way as a linear combination from B.

Note that (a) if and only if every vector $x \in V$ can obtained as a linear combination from B (by definition of Span).
So for the proof, we may assume that $\mathrm{Span} B = V$.

What does uniqueness mean? if $B', B'' \subseteq B$ are finite subsets and $\alpha_v, \beta_w \in \mathcal{F}$ for $v \in B'$ and $w \in B''$ such that
$x = \sum_{v \in B'} \alpha_v v = \sum_{w \in B''} \beta_w w$, then
$\alpha_v = \beta_v$ for all $v \in B' \cap B''$ and all other coefficients α_v, β_v are zero.

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Lemma (1.13)

A subset $B \subseteq V$ is a basis if and only if every vector $x \in V$ can obtained in a unique way as a linear combination from B.

Proof.

Assuming $\mathrm{Span} B = V$ we will show B is linearly independent if and only if the uniqueness holds.
"$\Leftarrow$" We prove the contrapositive, so assume B is linearly dependent. By Lemma 1.10 there is a way of writing 0 as a linear combination in V where all coefficients are not zero, so uniqueness fails. $\square$

Lemma (1.13)

A subset $B \subseteq V$ is a basis if and only if every vector $x \in V$ can obtained in a unique way as a linear combination from B.

Proof.

"$\Rightarrow$" Assume B is linearly independent, and suppose that $x = \sum_{v \in B'} \alpha_v v = \sum_{w \in B''} \beta_w w$ (as before). By defining $\alpha_w = 0$ for $w \in B'' \setminus B'$ and $\beta_v = 0$ for $v \in B' \setminus B''$ and let $B''' = B' \cup B''$.

Then $0 = x - x = \sum_{v \in B'''} (\alpha_v - \beta_v) v$. As B is linearly dependent, it follows from (the contrapositive of) Lemma 1.10 that $\alpha_v - \beta_v = 0$ for all $v \in B'''$, equivalently $\alpha_v = \beta_v$ for $v \in B'''$. So uniqueness holds. $\square$

(Main) purpose of basis: Suppose $B \subseteq V$ is a basis.
Then every vector $x \in V$ is uniquely determined by a family $(\alpha_v)_{v \in B}$ in $\mathcal{F}$ where only finitely many α_v's are non-zero.
Namely $x = \sum_{v \in B} \alpha_v v$
(Even though this can be an infinite sum, it kind of makes sense since only finitely many terms of the sum are non-zero).

The α_v's are called the coordinates of x with respect to B.

When $B = \{x_1, \dots, x_n\}$ is a (finite) basis, this means that there is a bijection

$$V \to \mathcal{F}^n, \quad V \ni x = \sum_{i=1}^{n} \alpha_i x_i \mapsto (\alpha_i)_{i=1}^{n} \in \mathcal{F}^n$$

where $\alpha_i := \alpha_{x_i}$ is the coordinate of x wrt B.

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Consider the $\mathbb{R}$-vector space $\mathbb{R}^2$. Let $x_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. Then $B = \{x_1, x_2\}$ is a basis for $\mathbb{R}^2$.

Question: What are the coordinates of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ with respect to B?

(a) $\alpha_{x_1} = 2$ and $\alpha_{x_2} = 3$;
(b) $\alpha_{x_1} = 3$ and $\alpha_{x_2} = 2$;
(c) $\alpha_{x_1} = 1$ and $\alpha_{x_2} = 1$.

Answer: They should satisfy

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \alpha_{x_1} x_1 + \alpha_{x_2} x_2 = \alpha_{x_1} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \alpha_{x_2} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} \alpha_{x_1} 2 \\ \alpha_{x_2} 3 \end{pmatrix}.$$

So the correct answer is (c).

Lemma (1.15)

Let $L \subseteq S \subseteq V$ and assume that L is linearly independant and $\mathrm{Span} S = V$. Then L is a basis for V if and only if L is maximally linearly independent in S, i.e. $L \cup \{x\}$ is linearly dependent for every $x \in S \setminus L$.

Proof.

"$\Rightarrow$" If L is a basis, we have $\mathrm{Span} L = V$. Hence Any $x \in S \setminus L$ will be in $\mathrm{Span} L$, so $L \cup \{x\}$ is linearly dependent by Lemma 1.11.

"$\Leftarrow$": Assume L is maximally linearly independent in S. By Lemma 1.11, every $x \in S \setminus L$ is in $\mathrm{Span} L$. Hence $S \setminus L \subseteq \mathrm{Span} L$ which implies $S \subseteq \mathrm{Span} L$. Thus $V = \mathrm{Span} S \subseteq \mathrm{Span} L$, so L is a basis. $\square$

Theorem (1.16)

Let $L \subseteq S \subseteq V$ *where* L *is linearly independent and* $\mathrm{Span} S = V$. *Assume* S *is finite. Then there exists a basis* B *for* V *such that* $L \subseteq B \subseteq S$.

Proof.

There are only finitely many linearly independent sets B' such that $L \subseteq B' \subseteq S$. Any maximal one of these is a basis by Lemma 1.15. □

Corollary (1.17)

Any finitely spanned vector space has a basis.

Proof.

Take $L = \emptyset$ in Theorem 1.16. □

Remark: The assumption that S is finite is not necessary (the proof uses Zorn's lemma)! In particular, every vector space has a basis.

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Definition

Let V be a vector space over $\mathcal{F}$. We say that V is finite dimensional, $\dim V < \infty$, if V has a finite basis. Otherwise V is infinite dimensional, $\dim V = \infty$.

V is *n*-dimensional if it has a basis with n elements.

Question: What is the dimension of $\mathbb{C}$ as an $\mathbb{R}$-vector space?

(a) 1

(b) 2

(c) ∞.

Answer: $B = \{1, i\} \subseteq \mathbb{C}$ is a basis, so $\mathbb{C}$ is 2-dimensional.

Note: if you consider $\mathbb{C}$ as a $\mathbb{C}$-vector space, then $B = \{1\}$ is a basis, so it has dimension 1.

Fun fact: any n-dimensional $\mathcal{F}$-vector space is in bijection with $\mathcal{F}^n$. In the case $\mathcal{F} = \mathbb{Q}$, this implies that any finite dimensional $\mathbb{Q}$-vector space must be countable!

Hence considering $\mathbb{R}$ and $\mathbb{C}$ as vector spaces over $\mathbb{Q}$, these are ∞-dimensional.

Lemma (1.21)

Let $L \subseteq V$ be linearly independent, and let $S \subseteq V$ such that $\mathrm{Span} S = V$. For each $x \in L$ there exists $y \in S$ such that $y \notin \mathrm{Span}(L \setminus \{x\})$ and such that $(L \setminus \{x\}) \cup \{y\}$ is linearly independent.

Proof.

As L is linearly independent, $x \notin \mathrm{Span}(L \setminus \{x\})$. Hence $\mathrm{Span}(L \setminus \{x\})$ is a proper subspace of V.

If $S \subseteq \mathrm{Span}(L \setminus \{x\})$ then $x \in V = \mathrm{Span} S \subseteq \mathrm{Span}(L \setminus \{x\})$ – a contradiction.

Hence there is $y \in S$ such that $y \notin \mathrm{Span}(L \setminus \{x\})$.

By Lemma 1.11, $(L \setminus \{x\}) \cup \{y\}$ is linearly independent. □

Lemma (1.21)

Let $L \subseteq V$ be linearly independent, and let $S \subseteq V$ such that $\mathrm{Span} S = V$. For each $x \in L$ there exists $y \in S$ such that $y \notin \mathrm{Span}(L \setminus \{x\})$ and such that $(L \setminus \{x\}) \cup \{y\}$ is linearly independent.

Lemma (1.20)

Let $S \subseteq V$ such that $\mathrm{Span} S = V$, and let $k \in \mathbb{N}$. If there exists in V a linearly independent subset with k elements, then there exists in S a linearly independent subset with k elements.

Proof.

Let $L \subseteq V$ be linearly independent with $|L| = k$. By Lemma 1.21 we can exchange an element in L with one in S and obtain a linearly independent set with k elements. Doing this with all k elements in L, we end with $L' \subseteq S$ which is linearly independent and has k elements.

Theorem (1.22)

Let V be an n-dimensional vector space.

(1) *Every linearly independent subset of V has at most n elements and is contained in a basis;*

(2) *Every spanning subset of V has at least n elements and contains a basis;*

(3) *Every basis has exactly n elements.*

Theorem 1.16: Let $L \subseteq S \subseteq V$ where L is linearly independent and $\mathrm{Span} S = V$. Assume S is finite. Then there exists a basis B for V such that $L \subseteq B \subseteq S$.

Lemma 1.20: Let $S \subseteq V$ such that $\mathrm{Span} S = V$, and let $k \in \mathbb{N}$. If there exists in V a linearly independent subset with k elements, then there exists in S a linearly independent subset with k elements.

Theorem 1.22: Let V be an n-dimensional vector space.

(1) Every linearly independent subset of V has at most n elements and is contained in a basis;

Proof.

By assumption V has a basis B with n elements.

(1): Let $L \subseteq V$ be linearly independent. Using Lemma 1.20 with $S = B$, there is a subset of B with $k = |L|$ many elements. Hence $|L| \leq |B| = n$. By Theorem 1.16 (with $S = L \cup B$), L is contained in a basis.

Lemma 1.20: Let $S \subseteq V$ such that $\mathrm{Span} S = V$, and let $k \in \mathbb{N}$. If there exists in V a linearly independent subset with k elements, then there exists in S a linearly independent subset with k elements.

Theorem 1.22: Let V be an n-dimensional vector space.

(1) Every linearly independent subset of V has at most n elements and is contained in a basis;

(2) Every spanning subset of V has at least n elements and contains a basis;

Proof.

(2): Let $S \subseteq V$ be spanning. From Lemma 1.20 with $L = B$, it follows that S contains a linearly independent set with $|B| = n$ elements. By (1), this set is maximally linearly independent and hence a basis by Lemma 1.15. □

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Theorem 1.22: Let V be an n-dimensional vector space.

(1) Every linearly independent subset of V has at most n elements and is contained in a basis;

(2) Every spanning subset of V has at least n elements and contains a basis;

(3) Every basis has exactly n elements.

Proof.

(3): Any basis is linearly independent, so has at most n elements by (1).

Any basis is spanning, so has at least n elements by (2).

Hence any basis has exactly n elements. $\square$

Question Is

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ \pi \\ 42 \end{pmatrix} \right\}$$

a basis for the $\mathbb{R}$-vector space $\mathbb{R}^3$?

Answer: No.

By $\dim \mathbb{R}^3 = 3$, and by Theorem 1.22(3) any basis has exactly 3 elements.

Corollary (1.23)

Let $X \subseteq V$ be a subset with exactly $n = \dim V$ elements. If X is linearly independent, or if X spans V, then X is a basis.

Proof.

If X is linearly independent, Theorem 1.22(1) says that X is contained in a basis B with $n = |X|$ elements. Hence $X = B$ so X is a basis.

If $\mathrm{Span} X = V$, Theorem 1.22(2) says that X contains a basis B with n elements. Hence $X = B$, so X is a basis. $\square$

Lemma (1.24)

V is infinite-dimensional if and only if there exists an infinite linearly independent subset $L \subseteq V$.

Proof.

"$\Leftarrow$" Contrapositive: assume V is finite dimensional. Then any linearly independent subset is finite by Theorem 1.22(1).
"$\Rightarrow$" Assume V is infinite dimensional. By Lemma 1.15, any finite linearly independent subset is contained in a linearly independent subset with 1 more element.
Starting with $L_0 = \emptyset$, use this to construct $L_0 \subseteq L_1 \subseteq L_2 \subseteq \dots$ linearly independent with $|L_n| = n$. One checks (easy from the definition) that $\bigcup_{n \in \mathbb{N}} L_n$ is linearly independent and infinite. $\square$

Theorem (1.25)

Let V be a finite-dimensional vector space with $n = \dim V$, and let $U \subseteq V$ be a subspace. Then U is finite-dimensional with $\dim U \leq n$.
Moreover, any basis for U can be extended to a basis for V.

Proof.

Every linearly independent subset of U is also linearly independent in V – hence has at most n elements. Hence $\dim U < \infty$ by Lemma 1.24. Thus a basis for U has at most n elements, so $\dim U \leq n$.
By Theorem 1.22(1) any basis of U is linearly independent in V and is therefore contained in a basis for V. $\square$