Advanced Linear Algebra

Week 10

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Let V be an $\mathcal{F}$-vector space, and let $U, W \subseteq V$ be subspaces.

$$U + W := \mathrm{Span}(U \cup W) = \{u + w \in V \mid u \in U, w \in W\}$$

is the sum of U and W. This is a subspace of V.

Definition

The sum $U + W$ is direct if for every $v \in U + W$, there are *unique* $u \in U$ and $w \in W$ such that $v = u + w$.
In this case, we write $U \oplus W$.

Let $V = \mathbb{R}^2$, $U = \{\begin{pmatrix} \alpha \\ 0 \end{pmatrix} \mid \alpha \in \mathbb{R}\}$, $W = \{\begin{pmatrix} 0 \\ \beta \end{pmatrix} \mid \beta \in \mathbb{R}\}$.

Question: is $U + W$ a direct sum?

(1) Yes

(2) No

(3) Not enough information to determine this.

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Let $V = \mathbb{R}^2$, $U = \{\begin{pmatrix} \alpha \\ 0 \end{pmatrix} \mid \alpha \in \mathbb{R}\}$, $W = \{\begin{pmatrix} 0 \\ \beta \end{pmatrix} \mid \beta \in \mathbb{R}\}$.

Any $v \in \mathbb{R}^2$ is of the form

$$v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}}_{\in U} + \underbrace{\begin{pmatrix} 0 \\ \beta \end{pmatrix}}_{\in W}$$

and this decomposition is unique.
Hence $U + W = \mathbb{R}^2$ is direct, so $\mathbb{R}^2 = U \oplus W$.

Lemma (Example 5.2)

Suppose B is a basis for V, and write $B = C \cup D$ as a disjoint union. Let $U = \operatorname{Span} C$ and $W = \operatorname{Span} D$. Then $V = U \oplus W$.

Proof.

As $U + W = \operatorname{Span}(U \cup W) \supseteq \operatorname{Span}(C \cup D) = V$, it follows that $V = U + W$. So we should show that $U + W$ is direct.

Let $v \in U + W$. Write (uniquely)

$$v = \sum_{x \in B} \alpha_x x = \underbrace{\sum_{y \in C} \alpha_y y}_{\in U} + \underbrace{\sum_{z \in D} \alpha_z z}_{\in W}.$$

If $v = u + w$ with $u \in U$ and $w \in W$, write $u = \sum_{y \in C} \beta_y y$ and $w = \sum_{z \in D} \gamma_z z$. Then

$\sum_{y \in C} \alpha_y y + \sum_{z \in D} \alpha_z z = v = u + w = \sum_{y \in C} \beta_y y + \sum_{z \in D} \gamma_z z$

By uniqueness, $\alpha_y = \beta_y$ for $y \in C$ and $\alpha_z = \gamma_z$ for $z \in D$. Hence $u \in U$ and $w \in W$ are unique.

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Theorem (5.3)

Let V be a vector space, and $U, W \subseteq V$ be subspaces. TFAE:

(1) $V = U + W$ *and* $U \cap W = \{0\}$;

(2) $V = U \oplus W$.

Proof.

(1) $\Rightarrow$ (2): Let $v \in V$. As $V = U + W$, there exist $u \in U$ and $w \in W$ so that $v = u + w$. If $v = u' + w'$ with $u' \in U$ and $w' \in W$, then $u + w = v = u' + w'$ implies $u - u' = w' - w \in U \cap W = \{0\}$, so $u = u'$ and $w = w'$. Hence $U + W = U \oplus W$.

(2) $\Rightarrow$ (1): We have $V = U \oplus W = U + W$. Let $x \in U \cap W$. Then

$$0 = \underbrace{0}_{\in U} + \underbrace{0}_{\in W} = \underbrace{x}_{\in U} + \underbrace{(-x)}_{\in W}$$

By uniqueness, $x = 0$ so $U \cap W = \{0\}$.

Consider $\mathbb{R}^3$ with subspaces

$$U = \{ \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}$$

$$W = \{ \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}$$

Question: Is $\mathbb{R}^3 = U \oplus W$?

(1) Yes,

(2) No,

(3) Not enough information to determine this.

Consider $\mathbb{R}^3$ with subspaces

$$U = \{ \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}$$

$$V = \{ \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \}$$

Question: Is $\mathbb{R}^3 = U \oplus W$?

By Theorem 5.3 we want to check $\mathbb{R}^3 = U + W$ and $U \cap W = \{0\}$.

But $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in U \cap W$, so $U \cap W \neq \{0\}$. Hence $U + W$ is not a direct sum.

Theorem (5.4)

Assume $V = U \oplus W$. *Then* $\dim U < \infty$ *and* $\dim W < \infty$ *if and only if* $\dim V < \infty$, *and in this case*
$\dim V = \dim U + \dim W$.

Proof.

"If": If V is finite-dimensional then so are all its subspaces.
"Only if": If C and D are bases for U and W respectively, then $C \cap D = \emptyset$ since $U \cap W = \{0\}$. Hence $|C \cup D| = \dim U + \dim W$, so it remains to check that $C \cup D$ is a basis for V.
Note $\mathrm{Span}(C \cup D) = \mathrm{Span}(U \cup W) = U + W = V$.
Let $v \in V$ and write it as
$v = \sum_{x \in C} \alpha_x x + \sum_{y \in D} \alpha_y y = \sum_{x \in C} \beta_x x + \sum_{y \in D} \beta_y y$.
As $V = U \oplus W$ we have $\sum_{x \in C} \alpha_x x = \sum_{x \in C} \beta_x x$ and thus $\alpha_x = \beta_x$ for $x \in C$ (as C is a basis).
Similar for D, so $C \cup D$ is a basis.

Let $U \subseteq V$ be a subspace.

Definition

A subspace $W \subseteq V$ is called a complement to U, if $V = U \oplus W$.

Remark: Complements are not unique (unless $U = \{0\}$ or $U = V$)!

Lemma (5.6)

If $\dim V < \infty$ *then every subspace* $U \subseteq V$ *has a complement.*

Proof.

Let $C \subseteq U$ be a basis, and pick a basis $B \subseteq V$ such that $C \subseteq B$ (possiple by Theorem 1.25).
Let $W = \mathrm{Span}(B \setminus C)$. Then $V = U \oplus W$ by Example 5.2 (which we proved as a lemma). □

Remark: also true if $\dim V = \infty$, but much deeper to prove.

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Theorem (5.7)

Let $U, W \subseteq V$ be subspaces. Then $V = U \oplus W$ if and only if $\pi|_W \colon W \to V/U$ is an isomorphism, where $\pi \colon V \to V/U$ is the projection.

Proof.

We show: $\pi|_W$ is surjective if and only if $V = U + W$; and $\pi|_W$ is injective if and only if $U \cap W = \{0\}$.
This will imply the result from Theorem 5.3.
We have $V = U + W$
$\Leftrightarrow$ every $v \in V$ is of the form $u + w$ with $u \in U$ and $w \in W$
$\Leftrightarrow$ for every $v \in V$ there exists $w \in W$ so that $v - w \in U$
$\Leftrightarrow$ for every $v \in V$ there exists $w \in W$ such that $v + U = w + U$,
$\Leftrightarrow$ $\pi|_W$ is surjective.
Also, as $N(\pi) = U$, we have $N(\pi|_W) = U \cap W$ which is $\{0\}$ if and only if $\pi|_W$ is injective.

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Until now, it was required that U, W were subspaces of a bigger space V.

Hence $U + W$ and $U \oplus W$ are sometimes called the internal sum and internal direct sum respectively, since everything is happening inside V.

We can also form direct sums even if U, W do not belong to a common space V. We define the external direct sum

$$U \oplus W := U \times W = \{(u, w) \mid u \in U, w \in W\}$$

with the obvious vector space structure (see Lemma 5.8).

Lemma

If U, W are subspaces of V then there is a canonical isomorphism from the exterior direct sum $U \oplus W$ to the interior direct sum given by $(u, w) \mapsto u + w$.

Proof.

The map is linear, and clearly surjective. Injectivity follows from uniqueness of the sum $u + w$.

Let $V = U \oplus W$ (internally) where $U, W \subseteq V$ are subspaces. The map $E \colon V \to V$ given by $E(v) = u$ (where $v = u + w$ is the unique decomposition) is easily seen to be linear.
Warning: E depends not just on U, but also on W!

Definition

The map $E \in \mathrm{End}(V)$ is called the projection on U along W.

Let $V = \mathbb{R}^2$, $U = \{\begin{pmatrix} \alpha \\ 0 \end{pmatrix} \mid \alpha \in \mathbb{R}\}$, $W = \{\begin{pmatrix} 0 \\ \beta \end{pmatrix} \mid \beta \in \mathbb{R}\}$.
Recall that $\mathbb{R}^2 = U \oplus W$.

Question: E be the proj. on U along W. What is $E\begin{pmatrix} 2 \\ 3 \end{pmatrix}$?

(1) 2
(2) 5
(3) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
(4) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

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Let $V = \mathbb{R}^2$, $U = \{\begin{pmatrix} \alpha \\ 0 \end{pmatrix} \mid \alpha \in \mathbb{R}\}$, $W = \{\begin{pmatrix} 0 \\ \beta \end{pmatrix} \mid \beta \in \mathbb{R}\}$.
Recall that $\mathbb{R}^2 = U \oplus W$.

Question: E be the proj. on U along W. What is $E\begin{pmatrix} 2 \\ 3 \end{pmatrix}$?

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{\in U} + \underbrace{\begin{pmatrix} 0 \\ 3 \end{pmatrix}}_{\in W}.$$

So $E\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

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Let $V = U \oplus W$, and let $E \in \mathrm{End}(V)$ be the projection on U along W.
Consider the map $I - E \in \mathrm{End}(V)$.
For any $v \in V$ decompose it as $v = u + w$ (so $E(v) = u$).
Then

$$(I - E)(v) = \underbrace{v}_{u+w} - \underbrace{E(v)}_{u} = w$$

so $I - E$ is the projection on W along U.

Lemma (5.11)

If E is the projection on U along W then

(a) $U = R(E) = N(I - E)$;

(b) $W = N(E) = R(I - E)$.

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If E is the projection on U along W then

(a) $U = R(E) = N(I - E)$;

(b) $W = N(E) = R(I - E)$.

Proof.

Statements for $I - E$ follow by interchanging U and W, and E and $I - E$. So we only prove $U = R(E)$ and $W = N(E)$.

(a): By definition $R(E) \subseteq U$.

If $u \in U$, then $u = u + 0$ is the unique decomposition, so $Eu = u \in R(E)$. Hence $U \subseteq R(E)$, and thus $R(E) = U$.

(b): Let $w \in W$. Then $w = 0 + w$ is the unique decomposition, so $E(w) = 0$. Hence $w \in N(E)$, and thus $W \subseteq N(E)$.

Let $v \in N(E)$. Write $v = u + w$. Then $u = E(v) = 0$ so $v = w \in W$. Hence $N(E) \subseteq W$.

Definition

A linear map $E \in \mathrm{End}(V)$ is idempotent if $E^2 = E$.

Consider $V = \mathbb{R}^2$ and let $A, B \in \mathrm{End}(\mathbb{R}^2)$ be given by

$$A\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ 0 \end{pmatrix}, \qquad B\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.$$

Question: which is true?

(1) A and B are both idempotents

(2) A is an idempotent, B is *not* an idempotent

(3) A is *not* an idempotent, B is an idempotent

(4) Neither A nor B is an idempotent.

Definition

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Consider $V = \mathbb{R}^2$ and let $A, B \in \mathrm{End}(\mathbb{R}^2)$ be given by

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$$A^2\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A(A\begin{pmatrix} \alpha \\ \beta \end{pmatrix}) = A\begin{pmatrix} -\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

so $A \neq A^2$. Hence A is not an idempotent.

$$B^2\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = B\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = B\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

So $B^2 = B$ and thus B is an idempotent.

Recall: $E \in \mathrm{End}(V)$ is an idempotent if $E^2 = E$.

Theorem (5.13)

Let V be a vector space and $E \in \mathrm{End}(V)$. Then E is an idempotent if and only if E is the projection on U along W for some internal direct sum $V = U \oplus W$. Moreover, in this case, $U = R(E)$ and $W = R(I - E)$.

Proof.

"If": Suppose $V = U \oplus W$ and E is the projection on U along W. For $v \in V$ we have $E(v) \in U$, so $E(v) = E(v) + 0$ is the unique decomposition. Hence $E^2(v) = E(E(v)) = E(v)$, so $E^2 = E$. □

Recall: $E \in \mathrm{End}(V)$ is an idempotent if $E^2 = E$.

Theorem (5.13)

Let V be a vector space and $E \in \mathrm{End}(V)$. Then E is an idempotent if and only if E is the projection on U along W for some internal direct sum $V = U \oplus W$. Moreover, in this case, $U = R(E)$ and $W = R(I - E)$.

Proof.

"Only if": Suppose $E^2 = E$. Let $U = R(E)$ and $W = R(I - E)$. For any $v \in V$ we have $v = Ev + (I - E)v \in U + W$, so $V = U + W$.
For any $u \in U = R(E)$, there exists $v \in V$ so that $u = E(v)$.
Hence $Eu = E^2 v = Ev = u$. Hence if $x \in U \cap W$ then $Ex = x$. There also exists $v \in V$ so that $x = (I - E)v$. Hence

$$x = Ex = E(I - E)v = (E - E^2)v = (E - E)v = 0.$$

So $U \cap W = \{0\}$ and thus $V = U \oplus W$ (Thm 5.3).

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