IMADA | Spring 2021
SDU | Jamie Gabe

Advanced Linear Algebra (MM562/MM853)
Information sheet 2
Programme for week 7 and 8

Lectures.

- Week 7: Advanced Vector Spaces: last part of Lecture 2.
- Week 8: Advanced Vector Spaces: very last bits of Lecture 2 and the first part of Lecture 3.

Exercises.

Exercises related to the lecture in week 7.

1. Consider the matrices
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathbb{M}_3(\mathbb{R}).$$
Determine the rank and nullity of both matrices (considered as elements in $\mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^3)$).

2. Denote by $\mathcal{P}_n$ the subspace in $\mathbb{R}[x]$ consisting of polynomials of degree at most n and the differentiation operator $D\colon \mathcal{P}_n \to \mathcal{P}_n$ sending a polynomial to its derivative. From the exercises done in week 6, it follows that D is linear [convince yourself of this]. Determine the matrix $[D]$ of D in the standard basis $\{1, x, x^2, \ldots, x^n\}$.
MM853 students should be able to do this for general n, and MM562 students may, if they prefer, assume that $n = 4$.

3. Determine the rank and nullity of the operator D from the previous question.

4. Let U, V be vector spaces over and let $A \in \mathrm{Hom}(U, V)$. Show that if $U_0 \subseteq U$ is a subspace then $A(U_0)$ is a subspace in V.

5. Consider the real vector space $\mathbb{R}^3$ with its standard basis $B := \{(1,0,0),(0,1,0),(0,0,1)\}$.
   (a) Show that $B' := \{(1,0,0),(1,1,0),(1,1,1)\}$ is also a basis.
   (b) Consider the map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x_1, x_2, x_3) = (2x_1 - x_2, x_2 + x_3, 4x_3)$. Show that $T \in \mathrm{End}(\mathbb{R}^3)$ and determine the matrix $[T]_B$ and $[T]_{B'}$ in $\mathbb{M}_3(\mathbb{R})$ representing T in B and B', respectively. Also, determine the matrix representing $T^2 := T \circ T$ in the basis B.
   I did not cover page 19 in the lecture, so the exercise is intended to be done without these tools. See below for a slightly smarter way.

6. Let U, V, W be finite dimensional vector spaces equipped with fixed bases B_U, B_V and B_W. Let $A \in \mathrm{Hom}(U, V)$ and $B \in \mathrm{Hom}(V, W)$ be given and consider the composition $B \circ A \in \mathrm{Hom}(U, W)$. Show that the matrix $[B \circ A]$ representing $B \circ A$ equals the matrix product $[B] \cdot [A]$ of the two matrices representing A and B, respectively.
   Hint: one may start by considering the case $\dim(U) = \dim(V) = \dim(W) = 2$*, and then generalize from there (or simply choose to believe that it works in general).*

Exercises related to the lecture in week 8.

1. Consider again the bases B and B' for $\mathbb{R}^3$ introduced above as well as the operator $T \in \mathrm{End}(\mathbb{R}^3)$. Determine the transition matrix (a.k.a. basis change matrix) P from B' to B and use it to determine $[T]_{B'}$ from $[T]_B$.

2. Consider the complex vector space $\mathbb{M}_2(\mathbb{C})$ of 2×2 complex matrices and denote by Tr the usual trace (mapping a matrix to the sum of its diagonal elements). Show that $\mathrm{Tr} \in (\mathbb{M}_2(\mathbb{C}))'$ and determine a basis for the nullspace $N(\mathrm{Tr})$ and the quotient $\mathbb{M}_2(\mathbb{C})/N(\mathrm{Tr})$.

3. Halmos: §14, exercise 3 (in this exercise, $\mathcal{P} = \mathbb{C}[t]$ is the complex vector space of polynomials with complex coefficients)
   Just do a few and stop when you get bored and/or feel comfortable with the notion of a linear functional

4. Halmos: §14, exercise 5.

5. (if time permits) Consider again the bases B and B' for $\mathbb{R}^3$ defined above, and denote by $\{y_1', y_2', y_3'\}$ the dual basis of B'. Write y_1', y_2' and y_3' in terms of the basis B; i.e. as $y_1'(x_1, x_2, x_3) = \cdots$ and similarly for y_2' and y_3'.

An additional challenge for those so inclined.

- Show that any linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is automatically continuous with respect to the standard metric on $\mathbb{R}^n$ coming from the 2-norm.

- Next, you are supposed to show, by example, that the above 'automatic continuity result' is not true for infinite dimensional vector spaces. To this end, put $V := \mathbb{R}$ as a vector space over $\mathbb{Q}$ and note that $\sqrt{2}$ and 1 are linearly independent. Then apply the infinite dimensional version of Theorem 1.16 (see Remark 1.18) to extend $\{1, \sqrt{2}\}$ to a basis for V. Use this basis to define a discontinuous[1], $\mathbb{Q}$-linear map $T\colon V \to V$.

[1] with respect to the standard metric on $\mathbb{R}$