Non-linear parabolic pdes in mathematical finance (Black Scholes)

Anders Kinch Møller

Relaxing the assumptions of the BS-model results in a non-linear BS equation.
For example we could look at the volatility σ and the drift μ depending on time t or the transaction cost not being 0. In the case of transaction cost, we would need to relax the hedging condition. This can be done by trading at discrete times, as we see in the Leland's model.

stochastic volatility (Heston model)
stochastic interest rates
increase number of underlying assets
look at different payoff functions (like options on the maximum or average)

Hamilton-Jacobi-Bellman equations
allowing for jumps in the development of the assets (integro PDE's)

A pde is said to be parabolic if $B^2 - 4AC = 0$
considering stability we find a more telling constraint on the size of δt. Heat conduction is said to be stable if the u_j^n do not grow in magnitude with n.
In general, the PDEs of relevance are of theconvection-di usion-reactiontype innspace variables and one time variable.
Look at a generalized model, we relax the assumptions of the BS-model: we now model a multi=asset environment with dividend D. We can also include transaction cost, relaxing the friction-less assumption.

Let's consider the general parabolic equation:

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u - \frac{\partial u}{\partial t} = f(x,t) \quad \text{with}$$

$a_i j(x,t)\alpha_i\alpha_j > 0 \quad \text{if} \quad \sum_{j=1}^{n} \alpha^2{}_j > 0.$

This equation is the general equation describing the behaviour of many derivative types. For example if x is a point in a $n = 1$-dimensional space, the equation reduces to the BS-equation. We can generalize this equation to include more general kinds of options. We can generalize it to the multivariate case.

$$ \tag{4} $$

We want to find a numerical scheme which produce accurate results.
In order to reduce the number of solutions for the pde, we define the initial condition and boundary conditions. These are:

- First boundary value problem (Dirchlet problem)
- Second boundary value problem (Neumann, robin problems).
- Cauchy problem.

Assumptions:

1. The underlying stock $S(t)$ follows a geometric brownian motion.

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dW && \text{(geometric brownian motion)} \\
V &= V(T-t, S_t; r, \sigma, K) \\
dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2 \text{By Itos lemma}
\end{aligned}
$$