Week 16 & 17 - Exercises

Max Holst Mikkelsen

Advanced Linear Algebra

Week 16 - Exercise 1

1. Which of the following endomorphisms are self-adjoint?

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{R}), \qquad \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}), \qquad \begin{pmatrix} 1 & 2+i \\ 2+i & 3 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$$

For those of the endomorphisms above that are selfadjoint, find an orthonormal basis in which they are diagonal.

We set

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2+i \\ 2+i & 3 \end{pmatrix}$$

Then

$$A^* = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}, \quad C^* = \begin{pmatrix} 1 & 2-i \\ 2-i & 3 \end{pmatrix}$$

Thus, A and B are self-adjoint.

Week 16 - Exercise 1 - continued

1. Which of the following endomorphisms are self-adjoint?

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}), \qquad \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}), \qquad \begin{pmatrix} 1 & 2+i \\ 2+i & 3 \end{pmatrix} \in \mathrm{M}_2(\mathbb{C})$$

For those of the endomorphisms above that are selfadjoint, find an orthonormal basis in which they are diagonal.

To find the eigenvalues of A, we solve the characteristic equation:

$$0 = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = (\lambda - 1)\lambda - 4 = \lambda^2 - \lambda - 4$$

Thus, the eigenvalues are

$$\lambda_1 = \frac{1}{2} - \frac{\sqrt{17}}{2}, \qquad \lambda_2 = \frac{1}{2} + \frac{\sqrt{17}}{2}$$

Then

$$\begin{pmatrix} 1-\lambda_1 & 2 \\ 2 & -\lambda_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{17}}{2} & 2 \\ 2 & -\frac{1}{2} + \frac{\sqrt{17}}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{4} + \frac{\sqrt{17}}{4} \\ 0 & 0 \end{pmatrix}$$

Hence, we have the eigenspace $V_{\lambda_1} = \mathrm{span}\{(\frac{1}{4} - \frac{\sqrt{17}}{4}, 1)\}$. Similarly, one finds that $V_{\lambda_2} = \mathrm{span}\{(\frac{1}{4} + \frac{\sqrt{17}}{4}, 1)\}$. By normalizing these two eigenvectors, we have found an orthonormal basis in which A is diagonal.

A similar procesdure yields that the eigenvalues for B are $\lambda_1 = \frac{1}{2} - \frac{\sqrt{5}}{2}$ and $\lambda_2 = \frac{1}{2} + \frac{\sqrt{5}}{2}$ with eigenspaces $V_{\lambda_1} = \mathrm{span}\{(\frac{i(1-\sqrt{5})}{2}, 1)\}$ and $V_{\lambda_2} = \mathrm{span}\{(\frac{i(1+\sqrt{5})}{2}, 1)\}$.

Week 16 - Exercise 2

2. Let V be a finite dimensional inner product space and assume that $A \in \mathrm{End}(V)$ is orthogonally diagonizable. Show that A is *normal*; i.e. that $AA^* = A^*A$.

Since A is orthogonally diagonalizable, there exists by Lemma 9.2 a unitary matrix P and a diagonal matrix D such that $[A] = PDP^{-1}$. Then

$$\begin{aligned}
[AA^*] = [A][A^*] = [A][A]^* &= PDP^{-1}(PDP^{-1})^* \\
&= PDP^{-1}(P^{-1})^* D^* P^* = PDP^{-1}PD^*P^{-1} \\
&= PDD^*P^{-1} = PD^*DP^{-1}
\end{aligned}$$

A similarly argument shows that $[A^*A] = PD^*DP^{-1}$. Consequently, $AA^* = A^*A$.

Week 16 - Exercise 3

3. Let V be a finite dimensional inner product space and let $B = \{x_1, \dots, x_n\}$ be an orthonormal basis (ONB) for V. Show that the coordinate isomorphism $\kappa_B \colon V \to \mathbb{F}^n$ is a unitary with respect to the usual dot-product on $\mathbb{F}^n$. Recall that κ_B is given by

$$\kappa_B\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

By Lemma 8.23, it suffices to show that κ_B preserves inner products. Let $x = \sum_{i=1}^{n} \alpha_i x_i$, $y = \sum_{j=1}^{n} \beta_j x_j \in V$. Then

$$\begin{aligned}
\langle x, y\rangle &= \langle \sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{n} \beta_j x_j \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \overline{\beta_j} \langle x_i, x_j \rangle \\
&= \sum_{i=1}^{n} \alpha_i \overline{\beta_i} \\
&= \langle (\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n \rangle \\
&= \langle \kappa_B(x), \kappa_B(y) \rangle
\end{aligned}$$

Week 16 - Exercise 4

4. Let V be a finite dimensional inner product space and let $A \in \mathrm{End}(V)$ be an endomorphism which is both unitary and selfadjoint. Show that $\sigma(A) \subseteq \{1, -1\}$ and that V decomposes as $V = V_- \oplus V_+$ such that $A(x) = x$ when $x \in V_+$ and $A(x) = -x$ when $x \in V_-$.

Let $\lambda \in \sigma(A)$. Take $x \in V_\lambda \setminus \{0\}$. This means that $Ax = \lambda x$. Then

$$\|x\|^2 = \langle x, x \rangle = \langle Ax, Ax \rangle = \langle A^2 x, x \rangle = \lambda^2 \langle x, x \rangle = \lambda^2 \|x\|^2$$

Since $\|x\| \neq 0$, it follows that $\lambda^2 = 1$ and consequently $\lambda = \pm 1$. Denoting by V_- and V_+ the eigenspaces for -1 and 1 respectively, we have by the spectral theorem that

$$V = V_- \oplus V_+$$

By definition, the eigenspaces have the property that $Ax = x$ when $x \in V_+$ and $Ax = -x$ when $x \in V_-$.

Week 16 - Exercise 5

5. Let V be a finite dimensional inner product space and let $E \in \mathrm{End}(V)$ be an orthogonal projection. Show by hand (i.e. without using the spectral theorem) that E is orthogonally diagonizable and determine its diagonal form.

By definition of an orthogonal projection we have

$$V = R(E) \oplus N(E)$$

Now, if $\lambda \in \sigma(E)$ and $x \in V_\lambda \setminus \{0\}$, we have

$$\lambda x = Ex = E^2 x = \lambda^2 x$$

Thus $\lambda^2 = \lambda$ and consequently $\sigma(E) \subseteq \{0, 1\}$. Clearly $V_0 = N(E)$. Moreover $V_1 = R(E)$. Indeed, if $x \in V_1$ then $x = Ex \in R(E)$ and conversely if $x \in R(E)$ then $Ey = x$ for some $y \in V$ and consequently $Ex = E^2 y = Ey = x$ so that $x \in V_1$. Thus, we have that E is orthogonally diagonalizable with the diagonal form

$$V = V_0 \oplus V_1$$

Week 16 - Exercise 6

6. Consider the inner product space $C([0,1],\mathbb{R})$ with its usual inner product $\langle f\,,g\rangle = \int_0^1 f(x)g(x)dx$, and consider the space $\mathcal{P}_n$ as a subspace of $C([0,1],\mathbb{R})$; i.e., consider a polynomial as the corresponding function on $[0,1]$. In this way $\mathcal{P}_n$ becomes an inner product space and we can consider the differentiation operator $D\colon \mathcal{P}_n \to \mathcal{P}_n$. Show that D is not selfadjoint.

Observe that D has only one eigenvalue 0 for which the eigenspace is $V_0 = \text{span}\{1\}$ where 1 denotes the function on $[0,1]$ which is contantly equal to 1. If D were self-adjoint we would have by the spectral theorem that

$$\mathcal{P}_n = V_0$$

which is a contradiction.

Week 17 - Exercise 1

1. Consider $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2)$. In both cases, determine the decomposition $N \oplus R$ given by Theorem 7.5.

We set

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

From the proof of theorem 7.5 we see that

$$N = \cup_{k \in \mathbb{N}} N(A^k), \quad R = \cap_{k \in \mathbb{N}} R(A^k)$$

First, we consider A. Since A is invertible, we have $N = \{0\}$ and $R = \mathbb{R}^2$.
Next, we consider B. For a given $k \in \mathbb{N}$ we have

$$B^k = \begin{pmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{pmatrix}$$

Hence, $N = N_k = \mathrm{span}\{(1, -1)\}$ and $R = R_k = \mathrm{span}\{(1, 1)\}$.

Week 17 - Exercise 2

2. Consider the matrix $\begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix} \in \mathbb{M}_3(\mathbb{C}) = \mathrm{End}(\mathbb{C}^3)$ and determine its generalized eigenspaces.

We denote the matrix by A. Then $\sigma(A) = \{1\}$ (obtained by solving the characteristic equation). Now,

$$A - I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$$

Consequently,

$$(A - I)^2 = 0$$

which shows that

$$M_1 = \{x \in \mathbb{C}^3 \mid \exists k > 0 : (A - I)^k x = 0\} = \mathbb{C}^3$$

Week 17 - Exercise 3

3. Assume that V is a non-trivial vector space (i.e. $V \neq \{0\}$). Show that a nilpotent endomorphism $A \in \mathrm{End}(V)$ must have $\sigma(A) = \{0\}$.

First, we show that $0 \in \sigma(A)$. Let $x \in V \setminus \{0\}$. If $Ax = 0$ then clearly $0 \in \sigma(A)$. Else there is a smallest $k \in \mathbb{N}, k \geq 2$ such that $0 = A^k x = AA^{k-1}x$. Hence, also $0 \in \sigma(A)$ in this case.
Conversely, let $\lambda \in \sigma(A)$ and let $x \in V_\lambda \setminus \{0\}$. Then for some $k \in \mathbb{N}$ we have

$$0 = A^k x = \lambda^k x$$

and consequently $\lambda = 0$.

Week 17 - Exercise 4

4. Consider the real vector space $V = C^\infty(\mathbb{R}, \mathbb{R})$ and the differentiation endomorphism $D \in \mathrm{End}(V)$; i.e. $D(f) = f'$. Determine the *generalized eigenspace* M_0 for D corresponding to the eigenvalue 0.

We say in a previous exercise that $\sigma(D) = \{0\}$. We show that $M_0 = \mathcal{P}$ where $\mathcal{P}$ is the subspace of polynomials. First, let $f \in \mathcal{P}$. Then $f(x) = \sum_{n=1}^{k} \alpha_n x^n$ and consequently $D^{k+1} f = 0$. Thus, $f \in M_0$. Conversely, let $f \in M_0$. This means that there is a $k \in \mathbb{N}$ such that $D^k f = 0$. But then we know from calculus that $f(x) = \sum_{n=0}^{k-1} \alpha_n x^n$. Consequently, $f \in \mathcal{P}$.

Week 17 - Exercise 5

5. Show that the spectral theorem for normal maps (Theorem 9.9) is *not* true over $\mathbb{R}$. Consider for instance the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2)$

Note that A is normal, but A has no eigenvalues. Indeed, the characteristic equation

$$0 = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

has no (real) solutions. Thus, A cannot be orthogonally diagonable.

Week 17 - Exercise 6

6. Let V be a vector space over a field $\mathbb{F}$ and let $A \in \operatorname{End}(V)$, $\lambda \in \mathbb{F}$ and $p \in \mathbb{F}[x]$ be given. Show that if $v \in V_\lambda$ is an eigenvector then $p(A)v = p(\lambda)v$.

We $A^n v = \lambda^n v$ for all $n \in \mathbb{N}$. Thus, if $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{F}[x]$ we obtain

$$p(A)v = (a_0 I + a_1 A + \cdots + a_n A^n)v = (a_0 + a_1\lambda + \cdots + a_n\lambda^n)v = p(\lambda)v$$