IMADA
SDU

Spring 2021
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Advanced Linear Algebra (MM562/MM853)
Information sheet 6
Programme for week 16 and 17

Lectures.

- Week 16: Advanced Vector Spaces: 75, 76 (first half), 78 (the Spectral Theorem for self-adjoint endomorphisms)
- Week 17: Advanced Vector Spaces: 57-59 (Generalized eigenspaces and nilpotence)

Exercises. If you didn't have time to go through all exercises last week, then you can choose to look at some of those as well.

Exercises related to the lecture in week 16.

1. Which of the following endomorphisms are self-adjoint?
$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{R}), \qquad \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}), \qquad \begin{pmatrix} 1 & 2+i \\ 2+i & 3 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$$
For those of the endomorphisms above that are selfadjoint, find an orthonormal basis in which they are diagonal.

2. Let V be a finite dimensional inner product space and assume that $A \in \mathrm{End}(V)$ is orthogonally diagonizable. Show that A is *normal*; i.e. that $AA^* = A^*A$.

3. Let V be a finite dimensional inner product space and let $B = \{x_1, \dots, x_n\}$ be an orthonormal basis (ONB) for V. Show that the coordinate isomorphism $\kappa_B \colon V \to \mathbb{F}^n$ is a unitary with respect to the usual dot-product on $\mathbb{F}^n$. Recall that κ_B is given by
$$\kappa_B\left(\sum_{i=1}^n \alpha_i x_i\right) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

4. Let V be a finite dimensional inner product space and let $A \in \mathrm{End}(V)$ be an endomorphism which is both unitary and selfadjoint. Show that $\sigma(A) \subseteq \{1, -1\}$ and that V decomposes as $V = V_- \oplus V_+$ such that $A(x) = x$ when $x \in V_+$ and $A(x) = -x$ when $x \in V_-$.

5. Let V be a finite dimensional inner product space and let $E \in \mathrm{End}(V)$ be an orthogonal projection. Show by hand (i.e. without using the spectral theorem) that E is orthogonally diagonizable and determine its diagonal form.

6. Consider the inner product space $C([0,1],\mathbb{R})$ with its usual inner product $\langle f\,,\,g\rangle = \int_0^1 f(x)g(x)dx$, and consider the space $\mathcal{P}_n$ as a subspace of $C([0,1],\mathbb{R})$; i.e., consider a polynomial as the corresponding function on $[0,1]$. In this way $\mathcal{P}_n$ becomes an inner product space and we can consider the differentiation operator $D\colon \mathcal{P}_n \to \mathcal{P}_n$. Show that D is not selfadjoint.

Exercises related to the lecture in week 17.

1. Consider $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2)$. In both cases, determine the decomposition $N \oplus R$ given by Theorem 7.5.
2. Consider the matrix $\begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix} \in \mathbb{M}_3(\mathbb{C}) = \mathrm{End}(\mathbb{C}^3)$ and determine its generalized eigenspaces.
3. Assume that V is a non-trivial vector space (i.e. $V \neq \{0\}$). Show that a nilpotent endomorphism $A \in \mathrm{End}(V)$ must have $\sigma(A) = \{0\}$.
4. Consider the real vector space $V = C^\infty(\mathbb{R},\mathbb{R})$ and the differentiation endomorphism $D \in \mathrm{End}(V)$; i.e. $D(f) = f'$. Determine the *generalized eigenspace* M_0 for D corresponding to the eigenvalue 0.
5. Show that the spectral theorem for normal maps (Theorem 9.9) is *not* true over $\mathbb{R}$. Consider for instance the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2)$
6. Let V be a vector space over a field $\mathbb{F}$ and let $A \in \mathrm{End}(V)$, $\lambda \in \mathbb{F}$ and $p \in \mathbb{F}[x]$ be given. Show that if $v \in V_\lambda$ is an eigenvector then $p(A)v = p(\lambda)v$.