IMADA
SDU

Spring 2021
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Advanced Linear Algebra (MM562/MM853)
Information sheet 3
Programme for week 9 and 10

Lectures.

- Week 9: Advanced Vector Spaces: last part of Lecture 3.
- Week 10: Advanced Vector Spaces: first part of Lecture 5.

Exercises.

Exercises related to the lecture in week 9.

1. Consider the map $A \in \mathrm{End}(\mathbb{R}^3)$ given by (multiplication with) the matrix
$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
in the standard basis for $\mathbb{R}^3$. Determine rank and nullity of both A and its adjoint A'.
2. Consider the subspace $U := \mathrm{span}((1, 0, 0)) \leq \mathbb{R}^3$ and determine a basis for the annihilator U°.
3. Present a proof of Corollary 3.18.
4. Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $U \leq V$ be a subspace. What is the relationship between U and its double annihilator $U^{\circ\circ}$ under the natural identification of V with V''.
5. Let V be a finite dimensional vector space over $\mathbb{F}$ and consider an endomorphism $A \in \mathrm{End}(V)$. Let $U \leq V$ be a subspace which is invariant under A, in the sense that $A(U) \subseteq U$. Show that U° is invariant under the dual map A'.
6. (Halmos, §17, Exercise 2) Show that the vectors $x_1 := (1, 1, 1), x_2 := (1, 1, -1)$ and $x_3 := (1, -1, -1)$ form a basis for $\mathbb{C}^3$. Denote by y_1, y_2, y_3 the dual basis and determine $y_1(0, 1, 0)$, $y_2(0, 1, 0)$ and $y_3(0, 1, 0)$
7. (Halmos, §17, Exercise 3) Let V be an n-dimensional vector space over a field $\mathbb{F}$ and let $y \in V'$ be given. Argue that $\{x \in V \mid y(x) = 0\}$ is a subspace in V; what is its dimension?

8. Prove Lemma 3.15: Let U and V be finite-dimensional $\mathcal{F}$-vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{C}$ respectively. Let $\mathcal{B}'$ and $\mathcal{C}'$ be the induced bases of U' and V' respectively. For any $A \in \mathrm{Hom}(U, V)$ show that

$$_{\mathcal{B}'}[A']_{\mathcal{C}'} = ({}_{\mathcal{C}}[A]_{\mathcal{B}})^T$$

(Hint: compute the (i,j)'th entry of $_{\mathcal{B}'}[A']_{\mathcal{C}'}$ and the (j,i)'th entry of $_{\mathcal{C}}[A]_{\mathcal{B}}$ individually (using the definitions), and show that they agree)

Exercises related to the lecture in week 10.

1. Halmos, §20: Exercise 1, 3. Recall that, in Halmos' book, the symbol $\mathcal{C}$ denotes the complex numbers.

2. Let V be a vector space over a field $\mathbb{F}$ and let $U, W \leq V$ be subspaces with $V = U \oplus W$. Check that the projection onto U along W is indeed in $\mathrm{End}(V)$.

3. Consider the real vector space $V := C(\mathbb{R}, \mathbb{R})$. Show that $U := \{f \in C(\mathbb{R}, \mathbb{R}) \mid f(0) = 0\}$ is a subspace and determine a complement (i.e. a subspace W such that $V = U \oplus W$. Moreover, write down formulas for the projections of V onto U along W and vice versa.

4. Let $E \in \mathbb{M}_n(\mathbb{R})$ be an idempotent. What are the possible eigenvalues for E?[1] Find examples in $\mathbb{M}_2(\mathbb{R})$ showing that the potential eigenvalues you found can be realised.

[1]Recall that an *eigenvalue* for E is a $\lambda \in \mathbb{R}$ such that there exists a non-zero $x \in \mathbb{R}^n$ so that $Ex = \lambda x$. The vector x is called an *eigenvector* for λ.