IMADA
SDU

Spring 2021
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Advanced Linear Algebra (MM562/MM853)
Information sheet 7
Programme for week 18 and 19

Lectures.

- Week 18: Advanced Vector Spaces: p. 54 (Lemma 6.20), p. 60-61 (Jordan decomposition), p. 76-77 (diagonalization of normal endomorphisms).
- Week 19: Advanced Vector Spaces: p. 61 (Finishing Jordan decomposition), p. 76-77 (diagonalization of normal endomorphisms).

Exercises.

Exercises related to the lecture in week 18. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 & 0 & -2 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

as a matrix in $M_5(\mathbb{R}) = \mathrm{End}(\mathbb{R}^5)$. It has eigenvalues 1 and 2, and the generalised eigenspaces have dimensions: $\dim M_1 = 1$ and $\dim M_2 = 2$. Determine a Jordan decomposition for A as in Theorem 7.8.

Determine a similar Jordan decomposition when A is considered as an element in $\mathbb{M}_5(\mathbb{C}) = \mathrm{End}(\mathbb{C}^5)$.

Exercises related to the lecture in week 19.

1. Determine which of the endomorphism $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}) = \mathrm{End}(\mathbb{C}^2)$ is orthogonally diagonizable.
2. Show that if V is a finite-dimensional inner product space (over $\mathbb{R}$ or $\mathbb{C}$) and if $A \in \mathrm{End}(V)$ is an endomorphism for which there exists $x \in V$ such that $Ax \neq 0$ but $A^2x = 0$ then A cannot be normal.
   Hint: Use Lemma 9.7.
3. For each of the following endomorphisms, determine a basis in which the corresponding matrix is upper triangular.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -4 \\ 1 & 4 \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2) \quad \text{and} \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{pmatrix} \in \mathrm{End}(\mathbb{C}^3)$$

I consider them all doable by hand, but if you want to save a bit of time and use Maple or a similar programme for some of the matrix computations, feel free to do so.

4. Show that if $A \in \mathbb{M}_n(\mathbb{C})$ is a matrix and $p \in \mathbb{C}[x]$ is its characteristic polynomial then $p(A) = 0$ by following the steps below. This result is known as the Cayley-Hamilton Theorem.

   (a) Show that if $A = (a_{ij})_{ij}$ is upper triangular then $\det(A) = a_{11} \cdots a_{nn}$.

   (b) Now use the Jordan form of A to show that if $\lambda_1, \dots, \lambda_r$ are the different eigenvalues and their algebraic multiplicities are denoted $m_1, \dots, m_r$ respectively, then $p(x) = (\lambda_1 - x)^{m_1} \cdots (\lambda_r - x)^{m_r}$.

   (c) Deduce from Theorem 7.4 that $((A - \lambda_i I)|_{M_{\lambda_i}})^{m_i} = 0$.

   (d) Now conclude that $p(A)$ is zero on each M_{λ_i} and hence that $p(A) = 0$

5. If there are any exercises or topics from the previous weeks that you find tricky or have not gotten around to solving yet, I suggest that you spend some time on those, and perhaps discuss them with Max.