Week 7 & 8 - Exercises

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Advanced Linear Algebra

Week 7 - Exercise 1

Consider the matrices

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \in \mathbb{M}_3(\mathbb{R}).$$

Determine the rank and nullity of both matrices (considered as elements in $\mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^3)$).

For $(x, y, z) \in \mathbb{R}^3$ we have

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y+z\\x+y+z\\x+y+z\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\\y\\x\end{pmatrix}$$

Thus, if we denote the matrices by A and B respectively, we have

$$R(A) = \mathrm{span}\{(1,1,1)\}$$
$$R(B) = \mathbb{R}^3$$

Hence,

$$\mathrm{rank}(A) = \dim R(A) = 1, \quad \mathrm{null}(A) = \dim \mathbb{R}^3 - \mathrm{rank}(A) = 3 - 1 = 2$$
$$\mathrm{rank}(B) = \dim R(B) = 3, \quad \mathrm{null}(B) = 3 - 3 = 0$$

Week 7 - Exercise 2

Denote by $\mathcal{P}_n$ the subspace in $\mathbb{R}[x]$ consisting of polynomials of degree at most n and the differentiation operator $D\colon \mathcal{P}_n \to \mathcal{P}_n$ sending a polynomial to its derivative. From the exercises done in week 6, it follows that D is linear [convince yourself of this]. Determine the matrix $[D]$ of D in the standard basis $\{1, x, x^2, \ldots, x^n\}$.
MM853 students should be able to do this for general n, and MM562 students may, if they prefer, assume that $n = 4$.

We have

$$\begin{aligned}
D(1) &= 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + \cdots + 0\cdot x^{n-1} + 0\cdot x^n \\
D(x) &= 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 + \cdots + 0\cdot x^{n-1} + 0\cdot x^n \\
&\vdots \\
D(x^n) &= nx^{n-1} = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + \cdots + n\cdot x^{n-1} + 0\cdot x^n
\end{aligned}$$

Thus,

$$[D] = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & \ldots & n \\
0 & 0 & \ldots & \ldots & \ldots & 0
\end{pmatrix}$$

Week 7 - Exercise 3

Determine the rank and nullity of the operator D from the previous question.

We have

$$N(D) = \text{span}\{1\}$$

Thus,

$$\text{null}(D) = \dim N(D) = 1$$

$$\text{rank}(D) = \dim \mathcal{P}_n - \text{null}(D) = n + 1 - 1 = n$$

Week 7 - Exercise 4

Let U, V be vector spaces over and let $A \in \mathrm{Hom}(U, V)$. Show that if $U_0 \subseteq U$ is a subspace then $A(U_0)$ is a subspace in V.

We have $0 \in U_0$ and thus $0 = A(0) \in A(U_0)$. Now, let $A(u), A(v) \in A(U_0)$ and $\alpha \in \mathbb{F}$. Then $u + v \in U_0$ and thus

$$A(u) + A(v) = A(u + v) \in A(U_0)$$

Further, $\alpha u \in U_0$ and thus

$$\alpha A(u) = A(\alpha u) \in A(U_0)$$

Hence, $A(U_0)$ is a subspace of V.

Week 7 - Exercise 5

Consider the real vector space $\mathbb{R}^3$ with its standard basis $B := \{(1,0,0),(0,1,0),(0,0,1)\}$.

(a) Show that $B' := \{(1,0,0),(1,1,0),(1,1,1)\}$ is also a basis.

(b) Consider the map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x_1,x_2,x_3) = (2x_1 - x_2, x_2 + x_3, 4x_3)$. Show that $T \in \mathrm{End}(\mathbb{R}^3)$ and determine the matrix $[T]_B$ and $[T]_{B'}$ in $\mathbb{M}_3(\mathbb{R})$ representing T in B and B', respectively. Also, determine the matrix representing $T^2 := T \circ T$ in the basis B.
I did not cover page 19 in the lecture, so the exercise is intended to be done without these tools. See below for a slightly smarter way.

(a) Since $\dim \mathbb{R}^3 = 3$ and $|B'| = 3$, it suffices to show that B' is linearly independent. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and assume that

$$\alpha_1(1,0,0) + \alpha_2(1,1,0) + \alpha_3(1,1,1) = (0,0,0)$$

This corresponds to the following set of equations:

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ \alpha_2 + \alpha_3 &= 0 \\ \alpha_3 &= 0 \end{aligned}$$

From the third equation we have $\alpha_3 = 0$. Then the second equation yields $\alpha_2 = 0$. Inserting in the first equation, we obtain $\alpha_1 = 0$. Hence, B' is linearly independent, and B' is a basis for $\mathbb{R}^3$.

Week 7 - Exercise 5 - continued

Consider the real vector space $\mathbb{R}^3$ with its standard basis $B := \{(1,0,0),(0,1,0),(0,0,1)\}$.

(a) Show that $B' := \{(1,0,0),(1,1,0),(1,1,1)\}$ is also a basis.

(b) Consider the map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x_1,x_2,x_3) = (2x_1 - x_2, x_2 + x_3, 4x_3)$. Show that $T \in \mathrm{End}(\mathbb{R}^3)$ and determine the matrix $[T]_B$ and $[T]_{B'}$ in $\mathrm{M}_3(\mathbb{R})$ representing T in B and B', respectively. Also, determine the matrix representing $T^2 := T \circ T$ in the basis B.
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(b) We have

$$\begin{aligned}
T(1,0,0) &= (2,0,0) = 2\cdot(1,0,0) + 0\cdot(0,1,0) + 0\cdot(0,0,1)\\
T(0,1,0) &= (-1,1,0) = -1\cdot(1,0,0) + 1\cdot(0,1,0) + 0\cdot(0,0,1)\\
T(0,0,1) &= (0,1,4) = 0\cdot(1,0,0) + 1\cdot(0,1,0) + 4\cdot(0,0,1)
\end{aligned}$$

Thus,

$$[T]_B = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

Week 7 - Exercise 5 - continued

Consider the real vector space $\mathbb{R}^3$ with its standard basis $B := \{(1,0,0),(0,1,0),(0,0,1)\}$.

(a) Show that $B' := \{(1,0,0),(1,1,0),(1,1,1)\}$ is also a basis.

(b) Consider the map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x_1,x_2,x_3) = (2x_1 - x_2, x_2 + x_3, 4x_3)$. Show that $T \in \mathrm{End}(\mathbb{R}^3)$ and determine the matrix $[T]_B$ and $[T]_{B'}$ in $\mathbb{M}_3(\mathbb{R})$ representing T in B and B', respectively. Also, determine the matrix representing $T^2 := T \circ T$ in the basis B.
I did not cover page 19 in the lecture, so the exercise is intended to be done without these tools. See below for a slightly smarter way.

(b) We have

$$\begin{aligned} T(T(1,0,0)) &= T(2,0,0) = (4,0,0) \\ T(T(0,1,0)) &= T(-1,1,0) = (-3,1,0) \\ T(T(0,0,1)) &= T(0,1,4) = (-1,5,16) \end{aligned}$$

Thus,

$$[T^2]_B = \begin{pmatrix} 4 & -3 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 16 \end{pmatrix}$$

Note that

$$[T]_B^2 = [T^2]_B$$

Week 7 - Exercise 5 - continued

Consider the real vector space $\mathbb{R}^3$ with its standard basis $B := \{(1,0,0),(0,1,0),(0,0,1)\}$.

(a) Show that $B' := \{(1,0,0),(1,1,0),(1,1,1)\}$ is also a basis.

(b) Consider the map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x_1,x_2,x_3) = (2x_1 - x_2, x_2 + x_3, 4x_3)$. Show that $T \in \mathrm{End}(\mathbb{R}^3)$ and determine the matrix $[T]_B$ and $[T]_{B'}$ in $\mathbb{M}_3(\mathbb{R})$ representing T in B and B', respectively. Also, determine the matrix representing $T^2 := T \circ T$ in the basis B.
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(b) We have

$$
\begin{aligned}
T(1,0,0) &= (2,0,0) = 2\cdot(1,0,0) + 0\cdot(1,1,0) + 0\cdot(1,1,1)\\
T(1,1,0) &= (1,1,0) = 0\cdot(1,0,0) + 1\cdot(1,1,0) + 0\cdot(1,1,1)\\
T(1,1,1) &= (1,2,4) = -1\cdot(1,0,0) + (-2)\cdot(1,1,0) + 4\cdot(1,1,1)
\end{aligned}
$$

Thus,

$$
[T]_{B'} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{pmatrix}
$$

Week 7 - Exercise 6

Let U, V, W be finite dimensional vector spaces equipped with fixed bases B_U, B_V and B_W. Let $A \in \mathrm{Hom}(U,V)$ and $B \in \mathrm{Hom}(V,W)$ be given and consider the composition $B \circ A \in \mathrm{Hom}(U,W)$. Show that the matrix $[B \circ A]$ representing $B \circ A$ equals the matrix product $[B] \cdot [A]$ of the two matrices representing A and B, respectively.
Hint: one may start by considering the case $\dim(U) = \dim(V) = \dim(W) = 2$, *and then generalize from there (or simply choose to believe that it works in general).*

Let us consider $\dim(U) = \dim(V) = \dim(W) = 2$. We denote the basis vectors by $B_U = \{u_1, u_2\}$, $B_V = \{v_1, v_2\}$, $B_W = \{w_1, w_2\}$ and the entries of the matrices by

$$[A] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad [B] = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Now,

$$\begin{aligned} A(u_1) &= a_{11} v_1 + a_{21} v_2, & A(u_2) &= a_{12} v_1 + a_{22} v_2 \\ B(v_1) &= b_{11} w_1 + b_{21} w_2, & B(v_2) &= b_{12} w_1 + b_{22} w_2 \end{aligned}$$

Then

$$\begin{aligned} B(A(u_1)) &= a_{11} B(v_1) + a_{21} B(v_2) \\ &= a_{11}(b_{11} w_1 + b_{21} w_2) + a_{21}(b_{12} w_1 + b_{22} w_2) \\ &= (a_{11} b_{11} + a_{21} b_{12}) w_1 + (a_{11} b_{21} + a_{21} b_{22}) w_2 \end{aligned}$$

Week 7 - Exercise 6 - continued

$$[A] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad [B] = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\begin{aligned} A(u_1) &= a_{11}v_1 + a_{21}v_2, \qquad & A(u_2) &= a_{12}v_1 + a_{22}v_2 \\ B(v_1) &= b_{11}w_1 + b_{21}w_2, \qquad & B(v_2) &= b_{12}w_1 + b_{22}w_2 \end{aligned}$$

Then

$$\begin{aligned} B(A(u_1)) &= a_{11}B(v_1) + a_{21}B(v_2) \\ &= a_{11}(b_{11}w_1 + b_{21}w_2) + a_{21}(b_{12}w_1 + b_{22}w_2) \\ &= (a_{11}b_{11} + a_{21}b_{12})w_1 + (a_{11}b_{21} + a_{21}b_{22})w_2 \end{aligned}$$

and

$$\begin{aligned} B(A(u_2)) &= a_{12}B(v_1) + a_{22}B(v_2) \\ &= a_{12}(b_{11}w_1 + b_{21}w_2) + a_{22}(b_{12}w_1 + b_{22}w_2) \\ &= (a_{12}b_{11} + a_{22}b_{12})w_1 + (a_{12}b_{21} + a_{22}b_{22})w_2 \end{aligned}$$

Thus,

$$[BA] = \begin{pmatrix} a_{11}b_{11} + a_{21}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \end{pmatrix} = [B] \cdot [A]$$

Week 8 - Exercise 1

Consider again the bases B and B' for $\mathbb{R}^3$ introduced above as well as the operator $T \in \mathrm{End}(\mathbb{R}^3)$. Determine the transition matrix (a.k.a. basis change matrix) P from B' to B and use it to determine $[T]_{B'}$ from $[T]_B$.

$$B := \{(1,0,0),(0,1,0),(0,0,1)\}. \quad B' := \{(1,0,0),(1,1,0),(1,1,1)\}$$

Since

$$\begin{aligned}
(1,0,0) &= 1 \cdot (1,0,0) + 0 \cdot (0,1,0) \cdot 0 \cdot (0,0,1)\\
(1,1,0) &= 1 \cdot (1,0,0) + 1 \cdot (0,1,0) \cdot 0 \cdot (0,0,1)\\
(1,1,1) &= 1 \cdot (1,0,0) + 1 \cdot (0,1,0) \cdot 1 \cdot (0,0,1)
\end{aligned}$$

we have

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$[T]_{B'} = P^{-1}[T]_B P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

Week 8 - Exercise 2

Consider the complex vector space $\mathbb{M}_2(\mathbb{C})$ of 2×2 complex matrices and denote by Tr the usual trace (mapping a matrix to the sum of its diagonal elements). Show that $\mathrm{Tr} \in (\mathbb{M}_2(\mathbb{C}))'$ and determine a basis for the nullspace $N(\mathrm{Tr})$ and the quotient $\mathbb{M}_2(\mathbb{C})/N(\mathrm{Tr})$.

Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$ and let $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}
\mathrm{Tr}\left(\lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right)
&= \mathrm{Tr}\begin{pmatrix} \lambda a_{11} + b_{11} & \lambda a_{12} + b_{12} \\ \lambda a_{21} + b_{21} & \lambda a_{22} + b_{22} \end{pmatrix} \\
&= \lambda a_{11} + b_{11} + \lambda a_{22} + b_{22} \\
&= \lambda (a_{11} + a_{22}) + (b_{11} + b_{22}) \\
&= \lambda \,\mathrm{Tr}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \mathrm{Tr}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}
\end{aligned}$$

Thus, $\mathrm{Tr} \in (\mathbb{M}_2(\mathbb{C}))'$.

Week 8 - Exercise 2 - continued

Consider the complex vector space $\mathbb{M}_2(\mathbb{C})$ of 2×2 complex matrices and denote by Tr the usual trace (mapping a matrix to the sum of its diagonal elements). Show that $\mathrm{Tr} \in (\mathbb{M}_2(\mathbb{C}))'$ and determine a basis for the nullspace $N(\mathrm{Tr})$ and the quotient $\mathbb{M}_2(\mathbb{C})/N(\mathrm{Tr})$.

We have

$$\begin{aligned} N(\mathrm{Tr}) &= \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}) : a_{11} = -a_{22} \right\} \\ &= \operatorname{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \end{aligned}$$

Note that a basis for $\mathbb{M}_2(\mathbb{C})$ is given by

$$B(\mathbb{M}_2(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

Thus, it follows from Theorem 2.14 that a basis for the quotient $\mathbb{M}_2(\mathbb{C})/N(\mathrm{Tr})$ is given by

$$B(\mathbb{M}_2(\mathbb{C})/N(\mathrm{Tr})) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + N(\mathrm{Tr}) \right\}$$

Week 8 - Exercise 3 (Halmos §14, exercise 3)

3. Suppose that for each x in $\mathcal{P}$ the function y is defined by

(a) $y(x) = \int_{-1}^{+2} x(t)\, dt$,

(b) $y(x) = \int_0^2 (x(t))^2\, dt$,

(c) $y(x) = \int_0^1 t^2 x(t)\, dt$,

(d) $y(x) = \int_0^1 x(t^2)\, dt$,

(e) $y(x) = \frac{dx}{dt}$,

(f) $y(x) = \left.\frac{d^2x}{dt^2}\right|_{t=1}$.

In which of these cases is y a linear functional?

(a) Let $x_1, x_2 \in \mathbb{R}[t]$ and $\lambda \in \mathbb{R}$. Then

$$y(\lambda x_1 + x_2) = \int_{-1}^{2} (\lambda x_1 + x_2)(t)\, dt = \int_{-1}^{2} (\lambda x_1(t) + x_2(t))\, dt$$

$$= \lambda \int_{-1}^{2} x_1(t)\, dt + \int_{-1}^{2} x_2(t)\, dt = \lambda y(x_1) + y(x_2)$$

Thus, y is a linear functional.

(b) y is not a linear functional since for example,

$$y(2 \cdot 1) = \int_0^2 2^2\, dt = 4 \int_0^2 dt \neq 2 \int_0^2 dt = 2y(1)$$

(c) Let $x_1, x_2 \in \mathbb{R}[t]$ and $\lambda \in \mathbb{R}$. Then

$$y(\lambda x_1 + x_2) = \int_0^1 t^2 \left((\lambda x_1 + x_2)(t)\right) = \lambda \int_0^1 t^2 x_1(t)\, dt + \int_0^1 t^2 x_2(t)\, dt = \lambda y(x_1) + y(x_2)$$

Thus, y is a linear functional.

Week 8 - Exercise 3 (Halmos §14, exercise 3) - continued

3. Suppose that for each x in $\mathcal{P}$ the function y is defined by

(a) $y(x) = \int_{-1}^{+2} x(t)\, dt$,

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(d) $y(x) = \int_0^1 x(t^2)\, dt$,

(e) $y(x) = \frac{dx}{dt}$,

(f) $y(x) = \frac{d^2x}{dt^2}\Big|_{t=1}$.

In which of these cases is y a linear functional?

(d) Let $x_1, x_2 \in \mathbb{R}[t]$ and $\lambda \in \mathbb{R}$. Then

$$y(\lambda x_1 + x_2) = \int_0^1 (\lambda x_1 + x_2)(t^2)\, dt = \lambda \int_0^1 x_1(t^2)\, dt + \int_0^1 x_2(t^2)\, dt = \lambda y(x_1) + y(x_2)$$

Thus, y is a linear functional.

(e) y is not a linear functional since it is not even a map into the scalars ($\frac{dx}{dt}$ is not a complex number).

(f) Let $x_1, x_2 \in \mathbb{R}[t]$ and $\lambda \in \mathbb{R}$. Then

$$y(\lambda x_1 + x_2) = \frac{d^2(\lambda x_1 + x_2)}{dt^2}\Big|_{t=1} = \lambda \frac{d^2 x_1}{dt^2}\Big|_{t=1} + \frac{d^2 x_2}{dt^2}\Big|_{t=1} = \lambda y(x_1) + y(x_2)$$

Thus, y is a linear functional.

Week 8 - Exercise 4 (Halmos §14, exercise 5)

5. If y is a non-zero linear functional on a vector space $\mathcal{V}$, and if α is an arbitrary scalar, does there necessarily exist a vector x in $\mathcal{V}$ such that $[x, y] = \alpha$?

Note that we are asked to show that y is surjective. From Week 7 - Exercise 4 we know that $R(y)$ is a subspace of $\mathbb{F}$. Hence $\dim R(y) \leq \dim \mathbb{F} = 1$, and since y is non-zero, it then follows that $\dim R(y) = 1 = \dim \mathbb{F}$. Thus, $R(y) = \mathbb{F}$ which means that y is surjective.

Week 8 - Exercise 5

5. (if time permits) Consider again the bases B and B' for $\mathbb{R}^3$ defined above, and denote by $\{y_1', y_2', y_3'\}$ the dual basis of B'. Write y_1', y_2' and y_3' in terms of the basis B; i.e. as $y_1'(x_1, x_2, x_3) = \cdots$ and similarly for y_2' and y_3'.

$$B := \{(1,0,0), (0,1,0), (0,0,1)\}. \quad B' := \{(1,0,0), (1,1,0), (1,1,1)\}$$

For $(x_1, x_2, x_3) \in \mathbb{R}^3$ we note that

$$(x_1, x_2, x_3) = (x_1 - x_2) \cdot (1,0,0) + (x_2 - x_3) \cdot (1,1,0) + x_3 \cdot (1,1,1)$$

Thus,

$$\begin{aligned} y_1'(x_1, x_2, x_3) &= x_1 - x_2 \\ y_2'(x_1, x_2, x_3) &= x_2 - x_3 \\ y_3'(x_1, x_2, x_3) &= x_3 \end{aligned}$$