Week 11 & 12 - Exercises

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Advanced Linear Algebra

Week 11 - Exercise 1

1. Check, if you have not already done so, that the dot-product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is bilinear.

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n), z = (z_1, \dots, z_n) \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.
Then

$$\begin{aligned} x \cdot (\alpha y + \beta z) &= (x_1, \dots, x_n) \cdot (\alpha y_1 + \beta z_1, \dots, \alpha y_n + \beta z_n) \\ &= \alpha x_1 y_1 + \beta x_1 z_1 + \cdots + \alpha x_n y_n + \beta x_n z_n \\ &= \alpha(x_1 y_1 + \cdots + x_n y_n) + \beta(x_1 z_1 + \cdots + x_n z_n) \\ &= \alpha(x \cdot y) + \beta(x \cdot z) \end{aligned}$$

Linearity in the first argument is proved similarly.

Week 11 - Exercise 2

2. Work through Example 4.2 (2) in the lecture notes. If you prefer, you may assume that $n = m = 2$. What is the connection between this example and the dot-product exercise above?

   (2) Let $X = \mathcal{F}^m$ and $Y = \mathcal{F}^n$, and consider vectors in $x \in X$ and $y \in Y$ as columns. Let $\mathbf{B}$ be an $m \times n$ matrix with elements $b_{ij} \in \mathcal{F}$. Then

   $$B(x, y) := x^t\,\mathbf{B}\,y = \sum_{i,j} \alpha_i b_{ij} \beta_j$$

   defines a bilinear form on $X \times Y$. Here (α_i) and (β_j) are the coordinates of x and y.

Let $x = (x_1, \dots, x_m) \in \mathcal{F}^m$ and $y = (y_1, \dots, y_n), z = (z_1, \dots, z_n) \in \mathcal{F}^n$ and $\alpha, \beta \in \mathcal{F}$. Then

$$\begin{aligned} B(x, \alpha y + \beta z) &= \sum_{i,j} x_i b_{ij}(\alpha y_j + \beta z_j) = \sum_{i,j} x_i b_{ij} \alpha y_j + \sum_{i,j} x_i b_{ij} \beta z_j \\ &= \alpha \sum_{i,j} x_i b_{ij} y_j + \beta \sum_{i,j} x_i b_{ij} z_j = \alpha B(x, y) + \beta B(x, z) \end{aligned}$$

Linearity in the first argument is proved similarly. If $m = n$ and $b_{ij} = 1$ for $i = j$ and $b_{ij} = 0$ for $i \neq j$ (i.e if B is the identity matrix) then the bilinear form B reduces to the dot-product from the previous exercise.

Week 11 - Exercise 3

3. (Halmos, §25, exercise 1) Let $\{e_1, e_2\}$ and $\{e_1, e_2, e_3\}$ denote the standard basis for $\mathbb{R}^2$ and $\mathbb{R}^3$, and consider $x = (1,1) \in \mathbb{R}^2$ and $y = (1,1,1) \in \mathbb{R}^3$. Determine the coordinates of $x \otimes y \in \mathbb{R}^2 \otimes \mathbb{R}^3$ with respect to the basis $\{e_i \otimes e_j \mid i = 1,2; j = 1,2,3\}$.

We have

$$\begin{aligned} x \otimes y &= (e_1 + e_2) \otimes (e_1 + e_2 + e_3) \\ &= e_1 \otimes e_1 + e_1 \otimes e_2 + e_1 \otimes e_3 + e_2 \otimes e_1 + e_2 \otimes e_2 + e_2 \otimes e_3 \end{aligned}$$

Thus, $x \otimes y$ has the coordinates $(1,1,1,1,1,1)$ with respect to the given basis.

Week 11 - Exercise 4

4. (Function spaces) Let X be a finite set[1] and consider the space $\mathcal{F}(X)$ consisting of all functions $f\colon X \to \mathbb{C}$.

(a) If you have not already done so, check that $\mathcal{F}(X)$ is a vector space with respect to the coordinate-wise operations $(f+g)(x) := f(x) + g(x)$ and $(\alpha \cdot f)(x) = \alpha f(x)$.

(b) For a fixed $x_0 \in X$ consider the *Dirac mass* at x_0; i.e. the function given by

$$\delta_{x_0}(x) := \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$$

Show that $\{\delta_x \mid x \in X\}$ is a basis for $\mathcal{F}(X)$.

(c) Let Y be another finite set and show that $\mathcal{F}(X) \otimes \mathcal{F}(Y)$ is isomorphic to $\mathcal{F}(X \times Y)$. *Hint:* consider the natural bases for the two spaces in question and build a map that maps one to the other and then extend by linearity.

(a) This follows from the fact that $\mathbb{C}$ is a vector space.

(b) We write $X = \{x_1, \dots, x_n\}$. Given an $f \in \mathcal{F}(X)$ we have

$$f = f(x_1)\delta_{x_1} + \cdots + f(x_n)\delta_{x_n}$$

Thus, $\mathcal{F}(X) = \operatorname{span}\{\delta_x : x \in X\}$. Now, let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and assume that

$$\alpha_1\delta_{x_1} + \cdots + \alpha_n\delta_{x_n} = 0$$

Then for all $i \in \{1, \dots, n\}$ we have

$$\alpha_i = \alpha_1\delta_{x_1}(x_i) + \cdots + \alpha_i\delta_{x_i}(x_i) + \cdots + \alpha_n\delta_{x_n}(x_i) = 0$$

Thus, $\{\delta_x : x \in X\}$ is linearly independent and consequently a basis for $\mathcal{F}(X)$.

Week 11 - Exercise 4

4. (Function spaces) Let X be a finite set[1] and consider the space $\mathcal{F}(X)$ consisting of all functions $f\colon X \to \mathbb{C}$.

   (a) If you have not already done so, check that $\mathcal{F}(X)$ is a vector space with respect to the coordinate-wise operations $(f+g)(x) := f(x) + g(x)$ and $(\alpha \cdot f)(x) = \alpha f(x)$.

   (b) For a fixed $x_0 \in X$ consider the *Dirac mass* at x_0; i.e. the function given by

   $$\delta_{x_0}(x) := \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$$

   Show that $\{\delta_x \mid x \in X\}$ is a basis for $\mathcal{F}(X)$.

   (c) Let Y be another finite set and show that $\mathcal{F}(X) \otimes \mathcal{F}(Y)$ is isomorphic to $\mathcal{F}(X \times Y)$. *Hint:* consider the natural bases for the two spaces in question and build a map that maps one to the other and then extend by linearity.

(c) From (b) we know that $\{\delta_{(x,y)} : x \in X, y \in Y\}$ is a basis for $\mathcal{F}(X \times Y)$ and that $\{\delta_x \otimes \delta_y : x \in X, y \in Y\}$ is a basis for $\mathcal{F}(X) \otimes \mathcal{F}(Y)$. We define the map $\varphi\colon \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \times Y)$ given on the basis by $\varphi(\delta_x \otimes \delta_y) = \delta_{(x,y)}$ and extend by linearity. Since φ is a bijection from the basis of $\mathcal{F}(X) \otimes \mathcal{F}(Y)$ to the basis of $\mathcal{F}(X \times Y)$, it follows from Lemma 2.7 that φ is an isomorphism.

Week 11 - Exercise 5

5. (Slice maps) Let U, V be finite dimensional vector spaces over a common field $\mathcal{F}$ and let $f \in U'$ be given. Show that there exists a linear map $T \colon U \otimes V \to V$ with the property that $T(u \otimes v) = f(u)v$ for all $u \in U$ and $v \in V$. The map T is often denoted $f \otimes I$ and is called the *slice map* associated with f

We fix bases $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_m\}$ for U and V respectively. Then $\{u_i \otimes v_j : i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$ is a basis for $U \otimes V$. We define $T \colon U \otimes V \to V$ given on the basis by $T(u_i \otimes v_j) = f(u_i)v_j$ and extend by linearity. To see that this map T has the desired property, we let $u = \sum_{i=1}^{n} \alpha_i u_i \in U$ and $v = \sum_{j=1}^{m} \beta_j v_j \in V$. Then

$$
\begin{aligned}
T(u \otimes v) &= T\left(\sum_{i=1}^{n} \alpha_i u_i \otimes \sum_{j=1}^{m} \beta_j v_j\right) = T\left(\sum_{i=1}^{n}\sum_{j=1}^{m} (\alpha_i \beta_j)(u_i \otimes v_j)\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \beta_j T(u_i \otimes v_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \beta_j f(u_i) v_j \\
&= f\left(\sum_{j=1}^{n} \alpha_i u_i\right) \sum_{j=1}^{m} \beta_j v_j = f(u)v
\end{aligned}
$$

Week 11 - Exercise 6

6. (A non-commutativity result) Let V be a finite dimensional vector space and consider the tensor product $V \otimes V$. For $x, y \in V \setminus \{0\}$, show that if x and y are linearly independent then $x \otimes y \neq y \otimes x$.
Hint: use Theorem 3.11 together with the exercise on slice maps.

We define the subspace $U := \text{span}\{x\}$ of V. Since x, y are linearly independent, we have $y \notin U$. Then by Theorem 3.11, there is an $f \in V'$ such that $f(y) \neq 0$ and $f|_U = 0$. Now, we consider the slice map T associated with f from Exercise 5. Then

$$T(x \otimes y) = f(x)y = 0$$

and

$$T(y \otimes x) = f(y)x \neq 0$$

Hence, $x \otimes y \neq y \otimes x$.

Week 11 - Exercise 7

7. (A commutativity result) Let U, V be finite dimensional vector spaces over a common field $\mathbb{F}$. Show that $U \otimes V \simeq V \otimes U$ by providing an explicit isomorphism.

We fix bases $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ for U and V respectively. Then $\{u_i \otimes v_j : i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$ and $\{v_j \otimes u_i : i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$ are bases for $U \otimes V$ and $V \otimes U$ respectively. We define $\varphi \colon U \otimes V \to V \otimes U$ given on the basis elements by $\varphi(u_i \otimes v_j) = v_j \otimes u_i$ and extend by linearity. Since φ is a bijection from the basis of $U \otimes V$ to the basis of $V \otimes U$, it follows that φ is an isomorphism.

Week 12 - Exercise 1

1. If you have not done so already, determine a basis for $\mathbb{R}^2$ consisting of eigenvectors for $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

We determine the eigenvalues by solving the characteristic equation:

$$0 = \begin{vmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = \lambda(\lambda - 2)$$

Thus, the eigenvalues are 0 and 2. Observe that

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, the standard basis vectors for $\mathbb{R}^2$ are eigenvectors for the matrix.

Week 12 - Exercise 2

2. Consider the trivial example vector space $V = \{0\}$, which I made a point out of excluding in the lecture. What is $\mathrm{End}(V)$? What is the minimal polynomial for the (there is only one) operator in $\mathrm{End}(V)$? And why does this operator not have any eigenvectors?

Note that $\mathrm{End}(V)$ only contains the identity operator $I \colon V \to V$ given by $I(0) = 0$. Now, if $p(X) = \alpha$ is a constant polynomial then $p(I) = \alpha I = 0$. Thus, the minimal polynomials for I are the constant polynomials. The operator I cannot have any eigenvectors as eigenvectors by definition are non-zero.

Week 12 - Exercise 3

3. Consider the endomorphisms

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2) \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathrm{M}_3(\mathbb{R}) = \mathrm{End}(\mathbb{R}^3),$$

and determine their minimal polynomials.

We consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and let $p(X) = \alpha_0 + \alpha_1 X + \cdots + \alpha_n X^n \in \mathbb{R}[X]$. We have

$$A^n = \begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix}$$

and consequently

$$\begin{aligned} p(A) &= \alpha_0 I + \alpha_1 A + \cdots + \alpha_n A^n \\ &= \begin{pmatrix} \alpha_0 + \alpha_1 1 + \alpha_2 1^2 + \cdots + \alpha_n 1^n & 0 \\ 0 & \alpha_0 + \alpha_1 2 + \alpha_2 2^2 + \cdots + \alpha_n 2^n \end{pmatrix} \end{aligned}$$

Thus, $p(A) = 0$ if and only if 1 and 2 are roots of $p(X)$ if and only if $p(X) = (X-1)(X-2)q(X)$. It follows the minimal polynomials for A are of the form $C(X-1)(X-2)$. Note that repeated diagonal entries do not change this argument. Thus, B has the same minimal polynomials as A.

Week 12 - Exercise 4

4. Drawing on your experience from the previous exercise, determine the minimal polynomial of an arbitrary diagonal matrix?

Let $A \in \mathbb{M}_n(\mathbb{R})$ be a diagonal matrix and let $\lambda_1, \ldots, \lambda_m$ for an $m \leq n$ be the *distinct* diagonal entries of A. Then the minimal polynomials of A are of the form

$$p(X) = C(X - \lambda_1)(X - \lambda_2) \cdots (X - \lambda_m)$$