Advanced Linear Algebra Week 11

Jamie Gabe



Let X, Y, V be \mathcal{F} -vector spaces. A map $B: X \times Y \to V$ is bilinear if

(a) $x \mapsto B(x, y)$ is a linear map $X \to V$ for every $y \in Y$; and (b) $y \mapsto B(x, y)$ is a linear map $Y \to V$ for every $x \in X$.

What condition (a) says is

$$B(x_1 + x_2, y) = B(x_1, y) + B(x_2, y), \qquad B(\alpha x, y) = \alpha B(x, y).$$

Similarly, (b) says

$$B(x, y_1 + y_2) = B(x, y_1) + B(x, y_2), \qquad B(x, \beta y) = \beta B(x, y).$$

Note that bilinearity implies

$$B(\sum_{i} \alpha_{i} x_{i}, \sum_{i} \beta_{j} y_{j}) \stackrel{(a)}{=} \sum_{i} \alpha_{i} B(x_{i}, \sum_{i} \beta_{j} y_{j}) \stackrel{(b)}{=} \sum_{i, j} \alpha_{i} \beta_{j} B(x_{i}, y_{j}).$$



Recall: $B: X \times Y \rightarrow V$ is bilinear if

(a)
$$B(x_1+x_2, y) = B(x_1, y) + B(x_2, y), \quad B(\alpha x, y) = \alpha B(x, y).$$

(b)
$$B(x, y_1+y_2) = B(x, y_1)+B(x, y_2), \quad B(x, \beta y) = \beta B(x, y).$$

Consider $X, Y, V = M_2(\mathbb{R})$, and the multiplication map

$$B \colon M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to M_2(\mathbb{R}), \qquad B(A, B) = AB.$$

Question: Is B bilinear?

- (1) Yes;
- (2) No;
- (3) It depends on A and B.



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$$B(A_1 + A_2, B) = (A_1 + A_2)B \stackrel{\text{Dist r.}}{=} \underbrace{A_1B}_{B(A_1, B)} + \underbrace{A_2B}_{B(A_2, B)}$$

$$B(\alpha A, B) = (\alpha A)B = \alpha(AB) = \alpha B(A, B).$$

So (a) is satisfied. (b) is similar, so matrix multiplication is bilinear.

More generally (same argument), the multiplication map $M_{n,m}(\mathcal{F}) \times M_{m,k}(\mathcal{F}) \to M_{n,k}(\mathcal{F})$ is bilinear.

Definition

A bilinear map $X \times Y \to \mathcal{F}$ is called a bilinear form.

$$\operatorname{Bil}(X, Y) := \{ \text{bilinear forms } X \times Y \to \mathcal{F} \}.$$

When Y = X we write Bil(X) := Bil(X, X).

Recall $X' = \{ \text{linear maps } X \to \mathcal{F} \}.$

Example: Let $u \in X'$ and $v \in Y'$. Then

$$uv: X \times Y \to \mathcal{F}, \qquad (uv)(x,y) = u(x)v(y)$$

is a bilinear form

(since
$$(uv)(x_1 + x_2, y) = u(x_1 + x_2)v(y) = u(x_1)v(y) + u(x_2)v(y) = (uv)(x_1, y) + (uv)(x_2, y)$$
, etc.)



Example: Let $X = \mathcal{F}^m$ and $Y = \mathcal{F}^n$, and let $A \in M_{m,n}(\mathcal{F})$.

Then $B \colon \mathcal{F}^m \times \mathcal{F}^n \to \mathcal{F}$ given by

$$B(x,y) = x^t A y \in \mathcal{F} \quad (x^t A y \text{ is a } 1 \times 1\text{-matrix})$$

is a bilinear form. In fact,

$$B(x_1 + x_2, y) = (x_1 + x_2)^t A y = x_1^t A y + x_2^t A y$$
, etc.
If $A = (a_{i,j})_{i,j}$, $x = (\alpha_i)_{i=1}^n \in \mathcal{F}^m$ and $y = (\beta_i)_{i=1}^n \in \mathcal{F}^n$, then

$$B(x, y) = x^t A y = \sum_{i,j} \alpha_i a_{i,j} \beta_j.$$

Consider the map $B: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$B\left(\left(\begin{array}{c}\alpha_1\\\alpha_2\end{array}\right),\left(\begin{array}{c}\beta_1\\\beta_2\end{array}\right)\right)=\alpha_1\beta_1-\alpha_2\beta_1-\alpha_1\beta_2+\alpha_2\beta_2.$$

Question: Is B a bilinear form?

- (1) Yes
- (2) No
- (3) Depends on $\alpha_1, \alpha_2, \beta_1, \beta_2$.



Example: Let $X = \mathcal{F}^m$ and $Y = \mathcal{F}^n$, and let $A \in M_{m,n}(\mathcal{F})$.

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Consider the map $B: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$B(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \dot{\beta_1} \\ \beta_2 \end{pmatrix}) = \alpha_1 \beta_1 - \alpha_2 \beta_1 - \alpha_1 \beta_2 + \alpha_2 \beta_2.$$

If we define $A=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, then

$$B(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}) = \sum_{i,j} (-1)^{i+j} \alpha_i \beta_j = (\alpha_1 \ \alpha_2) A \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

so this map is a bilinear form by the above example.

PARTMENT OF MATHEMATICS O COMPUTER SCIENCE $\mathrm{Bil}(X,Y)=\{\mathrm{bilinear\ maps}\ X\times Y\to \mathcal{F}\}\ \mathrm{is\ an\ }\mathcal{F}\text{-vector}\ \mathrm{space\ in\ the\ usual\ way.}\ \mathrm{l.e.\ if}\ B_1,B_2\in\mathrm{Bil}(X,Y),\ \mathrm{then}$

$$(B_1 + B_2)(x, y) := B_1(x, y) + B_2(x, y)$$

and if $B \in Bil(X, Y)$ and $\alpha \in \mathcal{F}$, then

$$(\alpha B)(x,y) = \alpha B(x,y).$$

Suppose (x_1, \ldots, x_m) is an ordered basis for X, and (y_1, \ldots, y_n) is an ordered basis for Y. Let (u_1, \ldots, u_m) and (v_1, \ldots, v_n) be the ordered dual bases for X' and Y', i.e. $u_i(\alpha_1x_1 + \cdots + \alpha_mx_m) = \alpha_i$ etc. Define $w_{i,j} := u_iv_j \in \operatorname{Bil}(X, Y)$, i.e. $w_{i,j}(x,y) = u_i(x)v_i(y)$.



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Theorem (4.4)

The set of all $w_{i,j}$ is a basis for $\operatorname{Bil}(X,Y)$, and if $B \in \operatorname{Bil}(X,Y)$ then $B = \sum_{i,j} B(x_i,y_j)w_{i,j}$. In particular, dim $\operatorname{Bil}(X,Y) = mn$.

Proof.

If $x = \sum_{i} \alpha_{i} x_{i} \in X$ and $y = \sum_{j} \beta_{j} y_{j} \in Y$, then $w_{i,j}(x,y) = u_{i}(x) v_{j}(y) = \alpha_{i} \beta_{j}$. $B(x,y) = B(\sum_{i} \alpha_{i} x_{i} \sum_{j} \beta_{i} y_{j}) = \sum_{i} \alpha_{i} \beta_{i} B(x_{i},y_{i})$

 $B(x,y) = B(\sum_{i} \alpha_i x_i, \sum_{j} \beta_j y_j) = \sum_{i,j} \alpha_i \beta_j B(x_i, y_j) = \sum_{i,j} B(x_i, y_j) w_{i,j}(x, y).$

Hence $\operatorname{Span}\{w_{i,i}: i, j\} = \operatorname{Bil}(X, Y)$.

If $B = \sum_{i,j} \gamma_{i,j} w_{i,j}$ then $B(x_l, y_k) = \sum_{i,j} \gamma_{i,j} w_{i,j} (x_l, y_k) = \gamma_{l,k}$. Hence the coefficients $\gamma_{i,j}$ are unique, and thus

the $w_{i,j}$ form a basis for Bil(X, Y).

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Definition

Let X, Y be finite-dimensional \mathcal{F} -vector spaces. Then

$$X \otimes Y := Bil(X, Y)'$$

is called the tensor product of X and Y. If $x \in X$ and $y \in Y$ we obtain a linear functional

$$\mathbf{x} \otimes \mathbf{y} \colon \mathrm{Bil}(X,Y) \to \mathcal{F}, \qquad (\mathbf{x} \otimes \mathbf{y})\mathbf{w} = \mathbf{w}(\mathbf{x},\mathbf{y}).$$

Elements in $X \otimes Y$ of the form $x \otimes y$ are called elementary tensors (or pure tensors).

Lemma (5.15)

The map $X \times Y \to X \otimes Y$ given by $(x, y) \mapsto x \otimes y$ is bilinear. In other words,

(a)
$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$$
, $(\alpha x) \otimes y = \alpha(x \otimes y)$

(b)
$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$$
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$$(b) \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2, \quad x \otimes (\beta y) = \beta(x \otimes y).$$

Consider
$$X=Y=\mathbb{R}^2$$
 and let $e_1=\left(egin{array}{c}1\\0\end{array}
ight)$ and $e_2=\left(egin{array}{c}1\\0\end{array}
ight)$.

Question: What is
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
?

- $(1) e_1 \otimes e_1 + e_2 \otimes e_2$
- (2) $e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2$
- $(3) e_1 \otimes e_2 + e_2 \otimes e_1$
- (4) 0



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ight)=(e_1+e_2)\otimes(e_1+e_2)=e_1\otimes(e_1\otimes e_2)+e_2\otimes(e_1+e_2)$$

$$=e_1\otimes e_1+e_1\otimes e_2+e_2\otimes e_1+e_2\otimes e_2.$$



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(b)
$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2, \quad x \otimes (\beta y) = \beta(x \otimes y).$$

Proof.

Recall that $X \otimes Y = \text{Bil}(X, Y)'$ and $(x \otimes y)(w) = w(x, y)$. Hence

$$((x_1 + x_2) \otimes y)(w) = w(x_1 + x_2, y) = w(x_1, y) + w(x_2, y)$$

= $(x_1 \otimes y)(w) + (x_2 \otimes y)(w)$.

Hence $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$. The other parts are proved in a similar way.



Suppose $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ are bases for X and Y respectively.

Theorem (5.16)

The dimension of $X \otimes Y$ is mn, and the elementary tensors

$$x_i \otimes y_i \in X \otimes Y$$
, $i = 1, \ldots, m, j = 1, \ldots, n$

form a basis for $X \otimes Y$.

Proof.

By Theorem 4.4 we have a basis of $w_{i,j}$ of Bil(X, Y) where

$$w = \sum_{i,j} w(x_i, y_j) w_{i,j}, \quad \text{for } w \in \text{Bil}(X, Y).$$

So the (i,j)'th coordinate functional is $w \mapsto w(x_i,y_j)$, which is exactly $x_i \otimes y_j$! By Theorem 3.4 these form a basis for $\mathrm{Bil}(X,Y)' = X \otimes Y$.

Corollary (5.17)

For finite-dimensional vector spaces X and Y, one has

$$X \otimes Y = \operatorname{Span}\{x \otimes y \mid x \in X, y \in Y\}.$$

Not every element in $X \otimes Y$ is an elementary tensor! But the elementary tensors span all of $X \otimes Y$ by the above corollary.

What if X and/or Y is not finite-dimensional?

We can still define elementary tensors $x \otimes y \in \operatorname{Bil}(X, Y)'$ by $(x \otimes y)(w) = w(x, y)$.

In the infinite-dimensional case, $X \otimes Y$ can be defined as

$$X \otimes Y := \operatorname{Span}\{x \otimes y \mid x \in X, y \in Y\} \subseteq \operatorname{Bil}(X, Y)'.$$

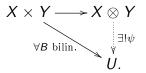
If dim $X = \infty$ or dim $Y = \infty$, then $X \otimes Y \neq \operatorname{Bil}(X, Y)'$ (it is a proper subspace).

Comment: This definition is very non-standard!



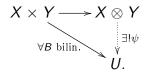
Tensor products have the following universal property: For every bilinear map $B\colon X\times Y\to V$ there exists a unique linear map $\psi\colon X\otimes Y\to V$ such that $B(x,y)=\psi(x\otimes y)$ for all $x\in X$ and $y\in Y$.

In diagrams, it is often written as



Here the map $X \times Y \to X \otimes Y$ is $(x, y) \mapsto x \otimes y$, and " \exists !" should be read as "exists(\exists) a unique(!)".

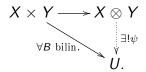




Proof.

Existence: Let $B_0: U' \to \operatorname{Bil}(X,Y)$ be the map $B_0(z) = z \circ B$. This is easily seen to be linear. Let $B'_0: \operatorname{Bil}(X,Y)' \to U''$ be the dual map of B_0 . Recall (Definition 3.21) that there is a canonical injective linear map $T: U \to U''$ given by T(u)(z) = z(u) for $z \in U'$, $u \in U$. Then for $z \in V'$, $x \in X$ and $y \in Y$ we have $(B'_0(x \otimes y))(z) = ((x \otimes y) \circ B_0)(z) = (x \otimes y)(z \circ B) =$ z(B(x,y)) = T(B(x,y))(z).So $B_0'(x \otimes y) = T(B(x, y))$ for all $x \in X$ and $y \in Y$. Hence $B'_0(X \otimes Y) \subset R(T) \cong U$. So there is a linear map $\psi \colon X \otimes Y \to U$ such that $\psi(x \otimes y) = B(x, y)$.





Proof.

Uniqueness: Suppose $\psi' \colon X \otimes Y \to U$ is linear and also satisfies $\psi'(x \otimes y) = B(x,y) = \psi(x \otimes y)$.

Since the elementary tensors $x \otimes y$ span all of $X \otimes Y$, and as ψ and ψ' are linear it follows that $\psi = \psi'$.

In the proof of existence, we used that $T: U \to U''$ is injective. We only proved (Theorem 3.23) that T is an isomorphism when U is finite-dimensional, but one can also prove that it is always injective. However, T is surjective (and thus an isomorphism) only when U is finite-dimensional. The proof that T is injective uses Zorn's lemma.

