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MULTIDIMENSIONAL PORTFOLIO OPTIMIZATION WITH PROPORTIONAL TRANSACTION COSTS

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We provide a computational study of the problem of optimally allocating wealth among multiple stocks and a bank account, to maximize the infinite horizon discounted utility of consumption. We consider the situation where the transfer of wealth from one asset to another involves transaction costs that are proportional to the amount of wealth transferred. Our model allows for correlation between the price processes, which in turn gives rise to interesting hedging strategies. This results in a stochastic control problem with both drift-rate and singular controls, which can be recast as a free boundary problem in partial differential equations. Adapting the finite element method and using an iterative procedure that converts the free boundary problem into a sequence of fixed boundary problems, we provide an efficient numerical method for solving this problem. We present computational results that describe the impact of volatility, risk aversion of the investor, level of transaction costs, and correlation among the risky assets on the structure of the optimal policy. Finally we suggest and quantify some heuristic approximations.

KEY WORDS: portfolio optimization, transaction costs, stochastic control, Hamilton–Jacobi–Bellman equation, free boundary problem

1. INTRODUCTION

This paper considers the optimal consumption–investment strategy of an investor who operates in a market containing one risk-free ("bank") asset and multiple risky assets ("stocks"). Price process of stocks are modeled as a multidimensional geometric Brownian motion. We allow for correlation between prices of various stocks. The investor is given an initial position in various assets. In time, he can choose to either consume money from the bank, buy stock with money in the bank, or add money to the bank by selling stock.

Transacting, that is, buying or selling stocks, incurs proportional transaction costs. The investors pays a proportion of the value transacted to a third party that enabled the transaction. This proportion may depend on the particular stock being transacted as well as on whether the transaction is a purchase or a sale. The investor obtains utility $u(c)$ by

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consuming c dollars from the bank. We take the utility function to be either the power utility, $u(c) = \frac{c^\gamma}{\gamma}$, $\gamma < 1$, $\gamma \neq 0$, or the log utility, $u(c) = \log(c)$, where γ is the relative risk aversion coefficient, a parameter that models the investor's risk preference. The investor's objective is to transact and consume so as to maximize the expected net present value of utility, namely, $\mathbf{E}\int_0^\infty e^{-\theta t}u(c_t)\,dt$, for some discount factor $\theta > 0$. The θ is also called the impatience factor, since it represents the impatience of the investor in consuming wealth. The investor is allowed to trade in continuous time and in infinitesimal quantities.

The solution to the portfolio optimization problem has two parts. First, the transaction policy is specified by the so-called *region of inaction*. When the proportions of the investor's wealth invested in each of the stocks lie within this region, the investor does not transact. He merely consumes from his bank account. When fluctuations in the price processes drive the proportions of wealth invested in the stocks to the boundary of the region of inaction, the investor transacts the minimal amount required to keep the proportions in the region of inaction. Second, the optimal amount to consume from the bank account is specified, using the solution to a nonlinear partial differential equation (PDE), known as the Hamilton–Jacobi–Bellman (HJB) equation. The key difficulty in obtaining this solution, either analytically or computationally, is that the domain over which the HJB equation must be solved is not exogenously specified. Rather, it is specified endogenously via conditions that determine the region of inaction. As a consequence, one is forced to solve a so-called *free boundary problem* involving the HJB equation, and this is far from trivial.

In this paper we provide a computational method to solve the free boundary problem arising from the portfolio optimization problem. Our method converts the free boundary problem into a sequence of fixed boundary problems with exogenously specified domains. Initially, one guesses a region of inaction (and thus a transaction policy) that contains the optimal region of inaction and computes the solution of the HJB equation over this region to obtain the optimal consumption. To compute the solution of the HJB equation, we use an iterative scheme that adapts the finite element method (FEM). In this scheme, we fix a consumption policy and solve for the corresponding value function by solving a linear elliptic PDE. This is done using a variant of FEM that is capable of handling the particular form of the boundary conditions (the so-called *implicit boundary conditions*) that arise. We then use the value function to determine an improved consumption policy and iterate this procedure until we determine the optimal consumption policy and thus the optimal value function for the given region of inaction, up to a numerical tolerance. We then use the value function again to update the region of inaction. Our update procedure is a variant of the scheme proposed in Kumar and Muthuraman (2004), which is based on the *principle of smooth pasting.* Our update procedure is designed to ensure that new region of inaction, and thus the new transaction policy, is an improvement over the previous one. The update procedure is also a conservative procedure in that it attempts to ensure that each of the regions of inaction generated contains the optimal region of inaction within it.

We apply this procedure to the two-stock and three-stock portfolio optimization problems. For the two-stock case we use the computational results to study the effect of changes of various problem parameters on the optimal policies. This helps build intuition about the structure of optimal policies. We also compare our computational results with the asymptotic results obtained by others. For the special case of portfolios with one stock and logarithmic utility, we state a theorem that provides theoretical guarantees on our computational method. The proof of this theorem is omitted here for brevity and can be found in Muthuraman (2005b). A lot more can be theoretically justified in the one-dimensional, case, using slightly modified update conditions and can be found in Muthuraman (2005a). Finally, we use our numerical results to suggest efficient heuristics for transacting and consuming from portfolios with many stocks and proportional transaction costs.

The problem described above has been well studied (a) when the portfolio has multiple stock with no transaction costs and (b) when the portfolio contains only one stock with transaction costs. In a seminal paper in 1969, Merton (1969) studied case (a). In this setting he obtained the optimal policy. The optimal policy proposed by Merton continuously transacts to hold fixed fractions of total wealth in various stocks and consumes a (different) fixed fraction of wealth. Merton's policy requires that an infinite number of transactions be made in any finite time interval. This suggests that in the presence of transaction costs, Merton's policy would no longer be optimal. With transaction costs, the investor would want to make fewer transactions. In particular, he would make transactions only if the fraction of his stock holding is "sufficiently" far away from Merton's optimal fraction, to warrant the transaction. Magill and Constantinides (1976) first considered case (b) and conjectured that the optimal policy would be characterized by an interval of inaction, such that the optimal policy would not transact when the fraction of wealth in stock lies in this interval. When the fraction lies outside the interval the optimal policy would be to buy or sell just enough to bring the fraction into the interval. Constantinides (1979) establishes the connectedness of the "no transactions" region and the form of the optimal policy for the discrete time version.

The problem with proportional transaction costs is now understood to be a singular stochastic control problem. Richard (1977) considered a problem with proportional and fixed costs of control in $\mathbf{R}^1$ and analyzed it by posing it as a stochastic control problem. Taksar, Klass, and Assaf (1988) considered a portfolio optimization problem under the stochastic control framework. They obtained optimal policies, for a model without consumption, that maximized asymptotic growth rate of portfolio. In 1990, Davis and Norman (1990) solved the Merton problem with proportional transaction costs for the one-stock case. They provided detailed characterization of the optimal policy and conditions under which the HJB equation has a smooth solution. They also provided a numerical method to calculate the optimal policy. Numerical methods for the one-stock problem can also be found in Tourin and Zariphopolou (1994, 1997). Later, Shreve and Soner (1994), in an exhaustive theoretical exposition, considered a relaxation of the Davis and Norman problem and used viscosity solution techniques to provide existence and uniqueness results and characterized the regularity of the value function. Leland (2000) noted that the value function and the region of inaction scale with $\lambda^{\frac{1}{3}}$. In a recent paper Janeček and Shreve (2004) have carried out an asymptotic analysis for the one-stock problem and have provided an asymptotic expansion of the value function in powers of $\lambda^{\frac{1}{3}}$, where λ, the cost of transaction, goes to zero. In Muthuraman (2005a), the author has extensively used the scheme proposed here to study the one-stock problem, compared it with asymptotic results of Janeček and Shreve (2004), and quantified various simplifications of the optimal policy, including that of the *simple policy* of Constantinides (1986) and the affine consumption that we discuss in this paper.

Duffie and Sun (1990) studied a model with a different transaction and information accrual structure and concluded that in their model an investor would choose to rebalance his portfolio at constant time intervals. Weiner (2000) has studied a model in which the stock price process has stochastic volatility. Pliska and selby (1994), Morton and Pliska (1995), and Atkinson, Pliska and Wilmott (1997) have studied a model with no consumption and with transaction costs that are of the management-fee type, that is, the investor pays a fixed fraction of his entire wealth when he wants to rebalance his portfolio.

Policies for optimizing portfolio with multiple stocks have received relatively less attention. Leland (2000) considers the optimal tracking of portfolio weights by minimizing a linear combination of expected transaction costs and tracking error. His model considers proportional transaction costs and does not involve consumption controls. He provides a relatively simple procedure for computing the optimal region of inaction with an explicit focus on implementation. His analysis also includes the effect of capital gains taxes on optimal strategies. Liu (2004) considers a problem very similar to ours. But he assumes no restrictions on borrowing.[1] This assumption, along with the assumption of independent asset returns and constant absolute risk aversion (CARA)-type exponential utility ($-e^{-\gamma c}$), results in the separability of optimal policies for transactions on each asset and consumption. As a result the multidimensional problem collapses to a set of one-dimensional problems which are characterized by ordinary differential equations (ODEs). We show in our paper that when there are restrictions on borrowing, like in the classical models considered by Merton (1969), Magill and Constantinides (1979), Davis and Norman (1990), and Shrever and Soner (1994), where strategies have to be self-financing, such separability is not possible. Therefore, one cannot decouple the problem into a set of ODEs and is forced to seek techniques that can solve the problem in multiple dimensions. However, his analysis considers a more general transactions cost structure with both fixed and proportional costs. Based on the form of the value function for the independent stock case, he conjectures a form for the correlated stock case, which is loosely consistent with our results.

Atkinson, Pliska, and Wilmott (1997) have been able to use asymptotic analysis around the Merton point to obtain optimal policies for multiple stocks for their model that assumes no consumption and considers the problem of optimizing long-term growth rate under a management fee structure. Akian, Menaldi, and Sulem (1996) consider the multiple-stock version of the Davis and Norman model when stock returns are uncorrelated. They restrict the risk aversion coefficient γ to lie in (0, 1). Under this set of utility functions they show existence and uniqueness results. They also use a numerical scheme to compute optimal policies for the two independent stock problems. As pointed out by Samuelson (1979), restricting the risk aversion coefficient to lie in (0, 1) leads to intolerably risky behavior Janeček and Shreve (2004). Ostrov and Goodman (2004) have recently obtained results on small transaction cost asymptotics for the case of multiple uncorrelated stocks. Recently, Lynch and Tan (2002) have worked on a discrete time model and have obtained results consistent with our results for the continuous time case.

Our focus is the multidimensional version of the classical Merton problem (Merton 1969) with proportional transaction costs. The problem of obtaining optimal transaction and consumption policies for this case with correlation has not been addressed satisfactorily in the existing literature. Complete theoretical analysis of this problem is difficult, and there is not much available beyond a few existence proofs (Akian, Menaldi, and Sulem 1996). Even computational results are unavailable, except in special cases (Akian, Menaldi, and Sulem 1996; Atkinson, Pliska, and Wilmott 1997). As a result, it is difficult to build intuition about the nature of optimal policies. In this paper, we attempt to fill this gap in the literature by providing a new computational scheme to solve such multistock portfolio optimization problems and use it to provide an exhaustive computational study of the case of two correlated stocks. By doing so, we not only provide intuition about

[1] However, he imposes some technical conditions to rule out arbitrages.

the behavior of the optimal policies, but also suggest suitable heuristics for portfolios with many stocks. Moreover the spirit behind the scheme introduced—converting a free boundary problem to a sequence of fixed boundary problems—we believe is much more powerful and can be adapted to various other models.

2. MODEL FORMULATION

We consider a market consisting of one risk-free and multiple risky investment opportunities. The risk-free investment, called the "bank," continuously pays an interest rate $r > 0$. The evolution of S_0, the value in the bank, can then be expressed as $S_0(t) = S_0(0)e^{rt}$ or,

$$dS_0(t) = r\,S_0(t)\,dt. \tag{2.1}$$

Let N be the number of available risky investments, called "stocks" hereafter. The N stocks have mean rates of returns $\alpha_1, \alpha_2, \ldots, \alpha_N$. We will take a standard N-dimensional Brownian motion $B = \{B(t) : t \geq 0\}$ on its standard filtered probability space $(\Omega, \mathcal{F}, \mathcal{P})$ as our source of uncertainty, where $\mathcal{F} = \{\mathcal{F}(t) : t \geq 0\}$ is a right-continuous filtration of σ-algebras on this space that represents the information revealed by the Brownian motion.

Let $S(t) = [S_1(t), S_2(t), \ldots, S_N(t)]'$ denote the vector of the values of N stocks at time t and let $\alpha = [\alpha_1, \ldots, \alpha_N]'$ denote the corresponding vector of mean rates of return. Then we have

$$dS(t) = \operatorname{diag}(S(t))\,[\alpha\,dt + \sigma\,dB(t)], \tag{2.2}$$

where $\operatorname{diag}(S(t))$ denotes the $N \times N$ matrix formed with the elements of $S(t)$ as its diagonal. The $N \times N$ positive definite matrix σ represents the covariance structure.

The investor is given an initial position of x dollars invested in the bank and $y = [y_1, y_2, \ldots, y_N]'$ dollars invested in the N stocks. He must choose a consumption and trading policy to maximize his objective. Consumption $c(.)$ occurs from money in the bank and only in nonnegative quantities. We assume that $c(.)$ is adapted to $\mathcal{F}_t$, that is, it is nonspeculative. We will also require that $c(.)$ be integrable for any finite t, that is,

$$\int_0^t c(s)\,ds < \infty \quad \forall t \geq 0. \tag{2.3}$$

To model the transaction controls we consider two $\mathcal{F}_t$-adapted processes $L(t)$ and $U(t)$ that are right continuous with left limits. $L(t)$ is a N-vector whose ith element represents the cumulative amount of money spent from the bank to buy stock i before incurring transaction costs. Similarly $U(t)$ is a N-vector whose ith component represents the cumulative amount of money obtained from selling stock i before incurring transaction costs. Thus $L(t)$ and $U(t)$ are nonnegative and nondecreasing processes. Buying and selling, stocks incur proportional transaction costs. Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]' \geq 0$ and $\mu = [\mu_1, \mu_2, \ldots, \mu_N]' \geq 0$ be the transaction cost for buying and selling, respectively. To be more precise, buying a unit worth of stock i will cost $(1 + \lambda_i)$ in cash from the bank, and selling a unit worth of stock i will result in $(1 - \mu_i)$ in cash that is added to the bank.

For the sake of readability in the rest of this paper, unless necessary, we will suppress the dependence on time t when denoting the processes $B(t)$, $S_0(t)$, $S(t)$, $c(t)$, $L(t)$, $U(t)$. With consumption and transaction, the controlled evolution o f S_0 and S can be described by the equations

$$dS_0 = (r S_0 - c)\, dt - (e + \lambda) \cdot dL + (e - \mu) \cdot dU \tag{2.4}$$

$$dS = \operatorname{diag}(S)\, [\alpha\, dt + \sigma\, dB] + dL - dU. \tag{2.5}$$

Here, the standard dot product is denoted by $\cdot$ and e denotes a vector of ones of appropriate length. Also (x, y) is the initial position that the investor starts with, that is, $S_0(0^-) = x$ and $S(0^-) = y$.

We define the solvency region as

$$\mathcal{S}_{\lambda,\mu} = \left\{ (x, y) \in (\mathbf{R}, \mathbf{R}^N) : x + \sum_{i=1}^{N} \min((1 + \lambda_i) y_i, (1 - \mu_i) y_i) \geq 0 \right\}.$$

The initial portfolio (x, y) and its future evolution are restricted to lie in $\mathcal{S}_{\lambda,\mu}$, which is the set of points from which the investor can conduct transactions to move to a point of nonnegative value in all assets.

That is, a consumption–transaction policy (c, L, U) is called *admissible* if S_0 and S, given by equations (2.4)–(2.5), lie in $\mathcal{S}_{\lambda,\mu}$ for all $t \geq 0$, that is,

$$\mathbf{P}[(S_0, S) \in \mathcal{S}_{\lambda,\mu} \quad \text{for all } t \geq 0] = 1. \tag{2.6}$$

Therefore, an admissible policy will ensure that bankruptcy does not occur in finite time. We will use $\mathcal{U}$ to denote the set of all admissible policies. The set $\mathcal{U}$ is clearly nonempty, since we can construct an admissible policy $(\tilde{c}, \tilde{L}, \tilde{U})$ from any policy (c, L, U) by terminating (c, L, U) at any arbitrary time when the state is still in $\mathcal{S}_{\lambda,\mu}$, and moving all wealth to the bank.

The utility that the investor obtains by consuming c dollars from the bank is given by the utility function $u(c)$. We will consider two common utility functions: the power utility function and the log utility function. They are given by

$$\text{Power utility:} \quad u(c) = \frac{c^\gamma}{\gamma} \quad \gamma \neq 0, \gamma < 1 \tag{2.7}$$

$$\text{Log utility:} \quad u(c) = \log(c). \tag{2.8}$$

Here γ is the relative risk aversion coefficient that describes the investors risk preference. These utility functions are very common in modeling investors risk preference and belong to a class of function called the *hyperbolic absolute risk aversion* functions (HARA). Let $\theta > 0$ be the discounting factor. Then, the investors objective is to choose an admissible consumption–transaction policy (c, L, U) so as to maximize

$$J_{x,y}(c, L, U) = \mathbf{E}_{x,y} \int_0^\infty e^{-\theta t} u(c)\, dt, \tag{2.9}$$

subject to (2.4)–(2.6). The value function v is defined as

$$v(x, y) = \sup_{(c,L,U) \in \mathcal{U}} J_{x,y}(c, L, U). \tag{2.10}$$

3. THE HJB EQUATION AND SCALING

The stochastic control problem defined in the previous section can be transformed into a PDE problem, using dynamic programing arguments and Itô's formula. The PDE problem characterizes the value function v (2.10).

We will use v_x to denote the partial differential of v with respect to x, v_i, to denote the partial differential with respect to y_i, and v_{ij} to denote $\frac{\partial^2 v}{\partial y_i \partial y_j}$. The value function v is expected to satisfy the so-called HJB equation

$$\max\left[\mathcal{L}v - \theta v + u(c), \quad \max_i(-(1+\lambda_i)v_x + v_i), \quad \max_i((1-\mu_i)v_x - v_i)\right] = 0 \tag{3.1}$$

where

$$\mathcal{L}v \equiv \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} y_i y_j v_{ij} + \sum_{i=1}^{N} \alpha_i y_i v_i + (rx - c)v_x \tag{3.2}$$

$$a = \sigma\sigma'$$

$$c = \begin{cases} v_x^{\frac{1}{\gamma-1}} & \text{when } u(c) = c^\gamma/\gamma \\ v_x^{-1} & \text{when } u(c) = \log(c). \end{cases} \tag{3.3}$$

Note that, since we are dealing with a infinite horizon discounted objective, it is natural that the optimal consumption is only a function of x and y and not a function of time t.

It is not clear whether a twice differentiable (C^2) solution exists for the HJB equation (3.1). When a C^2 solution does not exist, the second-order term in the HJB equation cannot be interpreted in a classical sense. Davis and Norman (1990) have shown under certain conditions that a C^2 solution exists for the one-stock problem. The theory of viscosity solutions provides a framework under which one can show that the value function is a unique viscosity solution to a HJB equation, without requiring that the value function be C^2. Existence and uniqueness proofs based on viscosity theory, for the single-stock problem and the multiple-stock problem with independent stock can be found in Shreve and Soner (1994) and Akian, Menaldi, and Sulem (1996), respectively. For the one-stock case, Shreve and Soner (1994) show that the value function is C^2 on $\mathcal{S}_{\lambda,\mu}\backslash\{(x, y): x = 0 \text{ or } y_i = 0 \text{ for some } i\}$. For the independent stock case the following theorem of Akian, Menaldi, and Sulem (1996) rigorously characterizes the value function v as a viscosity solution to the HJB equation.

THEOREM 3.1. *If the returns of stocks are independent and* $u(c) = \frac{c^\gamma}{\gamma}$, $\gamma \in (0, 1)$ *then* $v(x, y)$ *is the unique viscosity solution to the HJB equation (3.1).*

The focus of this paper is the computational scheme to solve (3.1) and hence we will assume that a solution exists for the independent stock case as well as the nonindependent case and also for the whole class of utility functions that we consider, $u(c) = \frac{c^\gamma}{\gamma}$, $\gamma < 1$, $\gamma \neq 0$ and $u(c) = \log(c)$.

Two properties of the value function are necessary for further analysis: concavity and *homothetic property*. Using concavity, we obtain the structure of optimal policies, and using the homothetic property we reduce the dimensionality of the problem and prepare it for further numerical analysis.

THEOREM 3.2. *If* $u(c) = c^\gamma/\gamma$ *or* $u(c) = \log(c)$ *then,*

1. *v is concave.*
2. *For $\rho > 0$,*

$$v(\rho x, \rho y) = \rho^\gamma v(x, y) \qquad \textit{when } u(c) = c^\gamma/\gamma$$

$$v(\rho x, \rho y) = \frac{1}{\theta}\log(\rho) + v(x, y) \quad \textit{when } u(c) = \log(c)$$

This is called the "homothetic property" of the value function.

The proof of 3.2.1 can be found in Akian, Menaldi, and Sulem (1996) and for the homothetic property refer to Davis and Norman (1990). The discrete time counterpart of Theorem 3.2 can be found in Constantinides (1979).

Now using the concavity of the value function we provide a heuristic characterization of the structure of the optimal transaction policy. Since a concave function is concave along each axis, v_i is a nondecreasing function of y_i. As we traverse the (x, y) space along the y_i-axis, v_i decreases from $v_i \geq (1 + \lambda_i)v_x$ to $v_i \leq (1 - \mu_i)v_x$. Recall that if $v_i \geq (1 + \lambda_i)v_x$ or $v_i \leq (1 - \mu_i)v_x$ we transact. Thus there are two boundaries, ϕ_{bi} and ϕ_{si}, such that when y_i is below ϕ_{bi} the optimal policy is to buy stock i, when above $\phi_{si}(x, y)$ to sell stock i, when in-between no transactions on stock i are made. To be precise, we define $\phi_{bi}, \phi_{si} : \mathbf{R}^{N+1} \to \mathbf{R}$ as $\phi_{bi}(x, y) = \{y_i : v_i(x, y) = (1 + \lambda_i)v_x(x, y)\}$ and $\phi_{si}(x, y) = \{y_i : v_i(x, y) = (1 - \mu_i)v_x(x, y)\}$.

Then the region of inaction, Ω can be expressed as

$$\Omega = \bigcap_{i=1}^{N} \{(x, y) \in \mathcal{S}_{\lambda,\mu} : \phi_{bi}(x, y) \leq y_i \leq \phi_{si}(x, y)\}. \tag{3.4}$$

It suffices to restrict our search for optimal transaction policies to those that are characterized by a region of inaction bounded by N buy boundaries $\{\phi_{bi}, i = 1, \ldots, N\}$ and N sell boundaries $\{\phi_{si}, i = 1, \ldots, N\}$.

Since v has the homothetic property it can be argued that the region of inaction Ω is a cone (Davis and Norman 1990). Therefore, we can reduce the dimensionality of the problem from $N + 1$ to N by defining a function $\psi(y)$,

$$\psi(y) = v(1 - e.y, y),$$

where e is the N-vector of ones. If we define wealth w as $w = x + e.y$, $\psi(y)$ is defined along the iso-wealth cut $w = 1$ of the (x, y) space and y represents the vector of the amounts of wealth in each stock when the total wealth is 1. Of course, we can find the value function $v(x, y)$ given $\psi(y)$ because of the homothetic property. Therefore, it is sufficient to compute $\psi(y)$. For $N = 2$, Figure 3.1 shows the region of inaction with the $w = 1$ cut, and Figure 3.2 shows the different regions along the $w = 1$ cut. It is possible that the investor's initial wealth proportions (x, y) fall outside the region of inaction Ω. Given the value function v within Ω, it is obvious that the investor would choose to rebalance to a position given by

$$\arg\max_{\tilde{x},\tilde{y}} [v(\tilde{x}, \tilde{y}) - J((x, y), (\tilde{x}, \tilde{y}))]$$

where $J((x, y), (\tilde{x}, \tilde{y}))$ is the cost incurred in transacting from (x, y) to $(\tilde{x}, \tilde{y})$. The regions outside the region of inaction Ω fall into two categories (Figure 3.2) indicated by I and II. In regions denoted by I, just enough of one asset is bought or sold to bring the fractions of wealth to Ω. In regions denoted by II, it is not possible to transact only one asset to reach Ω. Hence in II, two assets are transacted to reach a corner of Ω. The sequence in which the assets are transacted is irrelevant, since transactions are instantaneous.[2]

The solvency region, region of inaction, and the transaction boundaries in the $\mathbf{R}^{N+1}$ space were denoted by $\mathcal{S}_{\lambda,\mu}$, Ω, and ϕ, respectively. For notational simplicity we will use the same notations to denote the respective, on any cut (which is a $\mathbf{R}^N$ space). The following theorem is straightforward to verify.

[2] Though the transactions are indicated on the "wealth equals one" plane, all transactions change the total wealth. So, effectively one would be moving to a different wealth plane.

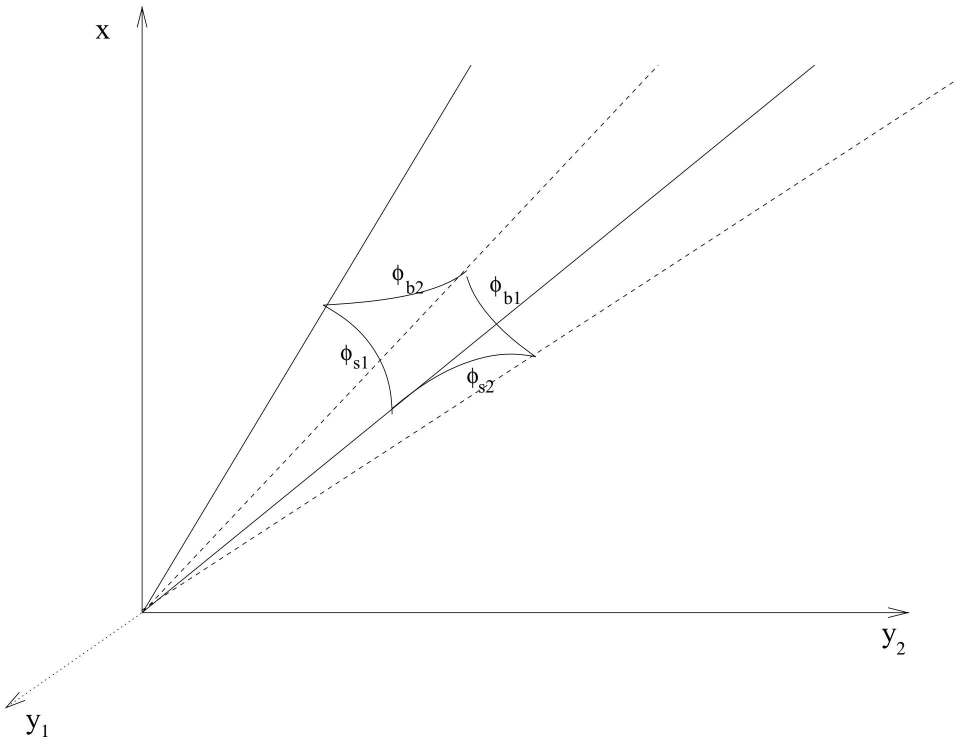

FIGURE 3.1. Illustrating homotheticity.

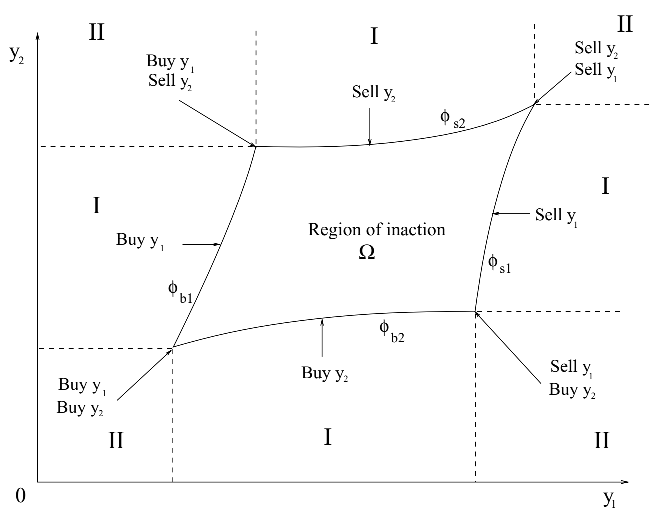

FIGURE 3.2. Regions of inaction and transactions.

THEOREM 3.3. *If* $v(x, y) : \mathbf{R} \times \mathbf{R}^N \Rightarrow \mathbf{R}$ *is concave in* $\mathcal{S}_{\lambda,\mu}$ *then* $\psi(y) : \mathbf{R}^N \Rightarrow \mathbf{R}$ *defined by*

$$\psi(y) = v(1 - e.y, y)$$

is concave in $\mathcal{S}_{\lambda,\mu}$.

In terms of $\psi(y)$, for the power utility case ($u(c) = c^\gamma/\gamma$), the HJB equation (3.1) becomes

$$\max\left[\max_c\left(\tilde{\mathcal{L}}\psi + \left(\frac{c^\gamma}{\gamma} - c(\gamma\psi - \sum_i y_i\psi_i)\right)\right), \tilde{\mathcal{B}}_j\psi, \tilde{\mathcal{S}}_j\psi\right] = 0. \tag{3.5}$$

Here c is only a function of y, since ψ is defined only on the $w = 1$ cut (for the sake of readability suppress $c(y)$'s dependence on y) and

$$\tilde{\mathcal{L}}\psi \equiv \sum_{i=1}^N\sum_{j=1}^N \eta_{ij}\psi_{ij} + \sum_{i=1}^N b_i\psi_i - \beta\psi, \tag{3.6}$$

$$\tilde{\mathcal{B}}_j\psi \equiv \psi_j\left(y_j + \frac{1}{\lambda_j}\right) - \left(\psi\gamma - \sum_{i\neq j}\psi_i y_i\right) \tag{3.7}$$

$$\tilde{\mathcal{S}}_j\psi \equiv \psi_j\left(y_j - \frac{1}{\mu_j}\right) - \left(\psi\gamma - \sum_{i\neq j}\psi_i y_i\right), \tag{3.8}$$

where

$$\begin{aligned}
\eta_{ij} &= \frac{y_i y_j}{2}\sum_k\sum_l a_{kl}(\delta_{ik} - y_k)(\delta_{jl} - y_l),\\
b_i &= \frac{1}{2}\sum_k\sum_l a_{kl}y_k y_l(\gamma - 1)(\delta_{ik} + \delta_{il} - 2y_i) + \sum_k y_k(\delta_{ik} - y_i)(\alpha_k - r)\\
\beta &= \theta - \gamma\left(r + \frac{1}{2}\sum_k\sum_l a_{kl}y_k y_l(\gamma - 1) + \sum_k(\alpha_k - r)y_k\right).
\end{aligned}$$

Here δ_{ik} represents the Kronecker delta function, with $\delta_{ik} = 1$ if and only if $i = k$ and $\delta_{ik} = 0$ otherwise. For the log utility case ($u(c) = \log(c)$), we have

$$\max\left[\max_c\left(\tilde{\mathcal{L}}\psi + \left(\log(c) - c\left(\frac{1}{\theta} - \sum_i y_i\psi_i\right)\right)\right), \tilde{\mathcal{B}}_j\psi, \tilde{\mathcal{S}}_j\psi\right] = 0 \tag{3.9}$$

with

$$\tilde{\mathcal{L}}\psi \equiv \sum_{i=1}^N\sum_{j=1}^N \eta_{ij}\psi_{ij} + \sum_{i=1}^N b_i\psi_i - \beta\psi + \upsilon, \tag{3.10}$$

$$\tilde{\mathcal{B}}_j\psi \equiv \psi_j\left(y_j + \frac{1}{\lambda_j}\right) - \left(\frac{1}{\theta} - \sum_{i\neq j}\psi_i y_i\right), \tag{3.11}$$

$$\tilde{\mathcal{S}}_j\psi \equiv \psi_j\left(y_j - \frac{1}{\mu_j}\right) - \left(\frac{1}{\theta} - \sum_{i\neq j}\psi_i y_i\right), \tag{3.12}$$

where

$$\eta_{ij} = \frac{y_i y_j}{2} \sum_k \sum_l a_{kl}(\delta_{ki} - y_k)(\delta_{lj} - y_l),$$

$$b_i = y_i \left(-\frac{1}{2} \sum_k \sum_l a_{kl}(\delta_{ki} y_l + \delta_{li} y_k - 2 y_k y_l) + \sum_k (\delta_{ki} - y_k)(\alpha_k - r) \right),$$

$$\beta = \theta$$

$$\upsilon = \frac{1}{\theta} \left(r - \frac{1}{2} \sum_k \sum_l a_{kl} y_k y_l + \sum_k y_k(\alpha_k - r) \right).$$

4. THE COMPUTATIONAL SCHEME

The basic computational procedure for solving the HJB equations for both the log utility and the power utility cases is the same. Only the specific equations used in the computations differ. Therefore we provide a detailed exposition of the log utility case only. Now consider the case when $u(c) = \log(c)$. As with v, using concavity of ψ we can argue that the solution to (3.9)–(3.12) must reduce to finding ψ and a region of inaction such that

$$\max_c \left[\tilde{\mathcal{L}}\psi + \left(\log(c) - c \left(\frac{1}{\theta} - \sum_i y_i \psi_i \right) \right) \right] = 0 \quad \text{in } \Omega^*, \text{ with} \tag{4.1}$$

$\tilde{\mathcal{B}}_j \psi = 0$ and $\tilde{\mathcal{S}}_j \psi = 0$ in the buy j region and the sell j region, respectively. We would also need that (3.9) holds in $\mathcal{S}_{\lambda,\mu}$. We will use a two-step procedure. We begin by choosing an arbitrary region of inaction, Ω^0. For the transaction policy corresponding to Ω^0, we calculate the optimal consumption $c^{(0)}$ and the associated value function $\psi^{(0)}$. In the next step we use a boundary update procedure that uses Ω^0 and $\psi^{(0)}$ to obtain a new region of inaction Ω^1 and thus a new transaction policy. We will repeat the procedure to get a sequence of regions of inaction $\Omega^0, \Omega^1, \Omega^2, \ldots$ and corresponding consumptions $c^{(0)}, c^{(1)}, c^{(2)}, \ldots$ and value functions $\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots$. In essence the procedure transforms the free boundary problem (3.9)–(3.12) into a sequence of fixed boundary problems that are easier to solve computationally. Step 1 solves the fixed boundary problem while Step 2 updates the region of inaction. Figure 4.1 summarizes the iterative procedure.

We assume that the arbitrarily chosen Ω^0 is large enough so that the optimal region of inaction, Ω^*, is a subset of Ω^0. However, if Ω^0 is not large enough we provide a condition (equation (4.11)) that will fail, in which case one can restart the iteration with a larger Ω^0.

4.1. Step 1: Solving for $c^{(n)}$, $\psi^{(n)}$ for a Given Ω^n

Given Ω^n, $c^{(n)}$ and $\psi^{(n)}$ must solve

$$\max_{c^{(n)}} \left[\tilde{\mathcal{L}}\psi^{(n)} + \left(\log\left(c^{(n)}\right) - c^{(n)} \left(\frac{1}{\theta} - \sum_i y_i \psi_i^{(n)} \right) \right) \right] = 0 \text{ in } \Omega^n \tag{4.2}$$

with boundary conditions

$$\tilde{\mathcal{B}}_j \psi^{(n)} = 0 \text{ at buy } j \text{ boundary,} \tag{4.3}$$

$$\tilde{\mathcal{S}}_j \psi^{(n)} = 0 \text{ at sell } j \text{ boundary.} \tag{4.4}$$

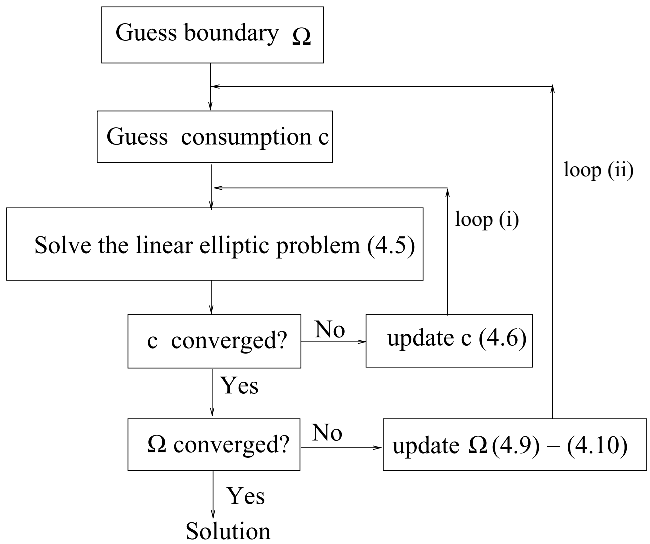

FIGURE 4.1. Summary of iteration procedure.

This is a nonlinear elliptic PDE with implicit boundary conditions. That is, equations (4.3)–(4.4) do not correspond to Dirichlet, Neumann, or even oblique derivative boundary conditions. But it is a fixed boundary problem. Implicit boundary can be represented as, $G(\psi, \nabla\psi) = 0$ for some function $G : (\mathbf{R}, \mathbf{R}^N) \to \mathbf{R}$. We will use an iterative scheme to solve (4.2)–(4.4) as follows. Given a consumption $c^{(n,m)}$

$$\tilde{\mathcal{L}}\psi^{(n,m)} + \left(\log\left(c^{(n,m)}\right) - c^{(n,m)}\left(\frac{1}{\theta} - \sum_i y_i\psi_i^{(n,m)}\right)\right) = 0 \text{ in } \Omega^n \tag{4.5}$$

is a linear elliptic equation and can be solved along with (4.3) and (4.4) to obtain $\psi^{(n,m)}$, where $\psi^{(n,m)}$ would be the value function, given a transaction policy Ω^n and a consumption $c^{(n,m)}$. Once we solve the linear problem we update our consumption with the first-order condition in equation (4.2), that is,

$$c^{(n,m+1)}(y) = \left[\gamma\psi^{(n,m)} - \sum_i \left(y_i\psi_i^{(n,m)}\right)\right]^{-1}. \tag{4.6}$$

In the representation $c^{(n,m)}$, n represents the iteration index of the boundary update sequence, while m represents the index of the consumption iteration. A good guess for the initial consumption $c^{(n,0)}$ would be a value less than the Merton consumption fraction, since with transaction costs one does not expect to consume more than the case when there are no transactions costs.

$$c^{(n,0)}(y) < \theta \quad \forall y \in \Omega^{(n)}. \tag{4.7}$$

We terminate the iterative procedure when

$$\sup_{y\in\Omega^{(n)}} \left|c^{(n,m+1)}(y) - c^{(n,m)}(y)\right| < \epsilon_c, \tag{4.8}$$

for some tolerance parameter ϵ. To implement this procedure we need to solve (4.5), along with (4.3) and (4.4), which constitutes a linear elliptic PDE with implicit boundary conditions. We solve this using a variant of the FEM where we modify the usual method to handle the implicit boundary conditions. Section 5 will discuss this in detail. We should indicate here that one need not necessarily use the FEM but can use any other numerical scheme, provided the scheme can address arbitrary geometries and implicit boundary conditions, in order to apply our method.

4.2. Step 2: The Boundary Update Procedure

The objective in Step 2 is to obtain the new region of inaction Ω^{n+1} from Ω^n and $\psi^{(n)}$. Our procedure updates the region of inaction Ω^n to Ω^{n+1} with ϕ_{bi}^{n+1}'s and ϕ_{si}^{n+1}'s given by

$$\begin{aligned}\phi_{bi}^{n+1} &= \min\left\{y^* > \phi_{bi}^n \mid y^* \text{ is the local maximizer of } \tilde{\mathcal{B}}_i\psi^{(n)} \text{ along the } y_i\text{-axis}\right\}\\ &= \min\left\{y^* > \phi_{bi}^n \;\middle|\; \exists\epsilon > 0 \text{ s.t. } \tilde{\mathcal{B}}_i\psi^{(n)}(y^*) = \max_{-\epsilon\le\tau\le\epsilon} \tilde{\mathcal{B}}_i\psi^{(n)}\big(y^* + \tau e_i\big)\right\},\end{aligned} \tag{4.9}$$

$$\begin{aligned}\phi_{si}^{n+1} &= \max\left\{y^* < \phi_{si}^n \mid y^* \text{ is the local maximizer of } \tilde{\mathcal{S}}_i\psi^{(n)} \text{ along the } y_i\text{-axis}\right\}\\ &= \max\left\{y^* < \phi_{si}^n \mid \exists\epsilon > 0 \text{ s.t. } \tilde{\mathcal{S}}_i\psi^{(n)}(y^*) = \max_{-\epsilon\le\tau\le\epsilon} \tilde{\mathcal{S}}_i\psi^{(n)}\big(y^* + \tau e_i\big)\right\}.\end{aligned} \tag{4.10}$$

Essentially, once ψ^n is known, each point on the old boundary $\phi_{bi}^n(\phi_{si}^n)$ is moved along the ith axis towards the interior of Ω^n, to the first point where $\mathcal{B}_i\psi^{(n)}(\mathcal{S}_i\psi^{(n)})$ is maximized.

If Ω^1 falls strictly within Ω^0, it guarantees that the arbitrarily chosen Ω^0 was large enough. This can be checked after computing $\psi^{(0)}$ using the condition

$$\begin{aligned}\tilde{\mathcal{B}}_i\psi^{(0)}\big|_{\phi_{bi}^0} &< \tilde{\mathcal{B}}_i\psi^{(0)}\big|_{\phi_{bi}^0+\tau e_i} \quad \forall i \text{ and}\\ \tilde{\mathcal{S}}_i\psi^{(0)}\big|_{\phi_{si}^0} &< \tilde{\mathcal{S}}_i\psi^{(0)}\big|_{\phi_{si}^0-\tau e_i} \quad \forall i,\end{aligned} \tag{4.11}$$

for some $\epsilon > 0$ and all $\tau \in (0, \epsilon)$. The above conditions simply say that it is necessary that the derivative of $\tilde{\mathcal{B}}_i\psi^{(0)}(\tilde{\mathcal{S}}_i\psi^{(0)})$ along the y_i-axis is nonnegative (nonpositive). If either of the above conditions fail, then it indicates that the arbitrarily chosen Ω^0 was not large enough. A restart of the procedure with a larger Ω^0 is required. A good way to choose a larger Ω^0 in such cases is to move each boundary half way between the old position and the boundary of the solvency region and check (4.11) again. Once $\Omega^1 \subset \Omega^0$ then subsequent Ω's are expected to be nested, that is, $\Omega^{n+1} \subset \Omega^n$. We provide heuristic justification for the update procedure below. Later, for the one-stock case with logarithmic utility we state a theorem that shows that this procedure gives policy improvement at each step and also gives a sequence of nested regions of inaction.

To relate to intuition, the heuristic explanation of the update procedure needs to be carried out in the $\mathbf{R}^{N+1}(x, y)$ space rather than in the $\mathbf{R}^N$ space along the $w = 1$ cut, which is a projection taken for numerical tractability. Consider $v^{(n)}$ that can be obtained from $\psi^{(n)}$. We will choose a particular i and consider updating ϕ_{si}^n to ϕ_{si}^{n+1}. Figure 4.2 shows the cross-section of the (x, y) space cut at $y_j =$ constant $\forall j \neq i$. Here the x- and y- axes indicate the wealth in bank and stock i, respectively. Now say the stochastic processes escapes Ω^n at point $p_1 \in \phi_{si}^n$. The transaction policy dictated by Ω^n is that we sell stock i instantaneously. This selling would move the process in the direction $((1-\mu_i), -1)$. Also at p_1, we have from the boundary conditions that $\nabla v.((1-\mu_i), -1) = (1-\mu_i)v_x - v_i = 0$. Consider a point $\tilde{p}$ very close to p_1 along the iso-wealth line, shown in figure. Say, at $\tilde{p}$, $(1-\mu_i)v_x - v_i > 0$, which indicates that pushing the processes in the

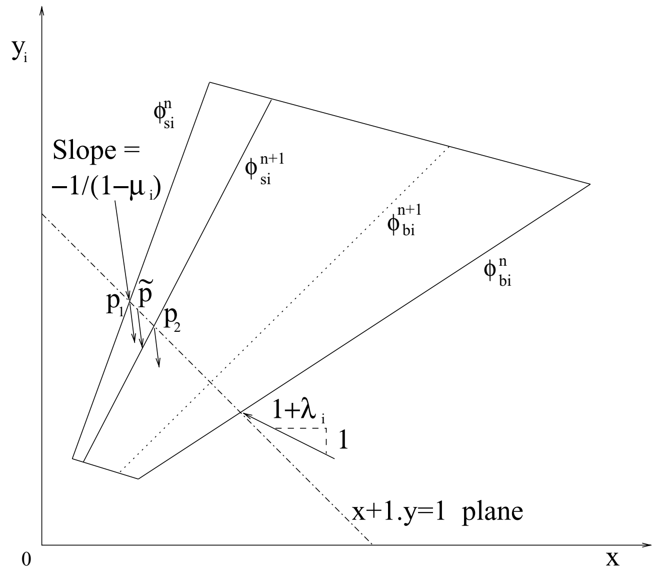

FIGURE 4.2. The boundary update procedure.

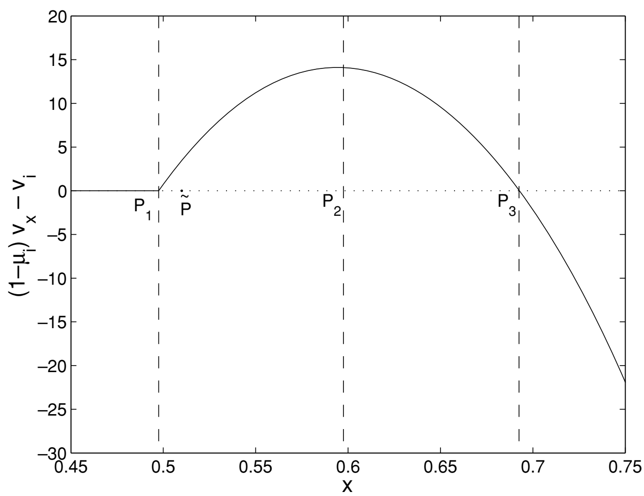

FIGURE 4.3. $(1-\mu_i)v_x - v_i$ along the transaction cut.

sell direction, that is, $((1-\mu_i), -1)$ increases the value function v. Since it is optimal to sell stock at $\tilde{p}$, a point arbitrarily close to p_1, it would be profitable to move the boundary to $\tilde{p}$. Extending the argument, it would be profitable to move the boundary to any point p_3 along the iso-wealth line, provided $(1-\mu_i)v_x - v_i > 0$ between p_1 and p_3. Figure 4.3 shows a typical plot of $(1-\mu_i)v_x - v_i$ along the iso-wealth cut. Moving the left boundary to any point between p_1 and p_3 is bound to yield an improvement in the value function. This suggests that the update procedure of (4.9)–(4.10) is a policy improvement procedure.

We still need to decide how far we need to move in from p_1. A consideration for where we wish to move is the desire for a procedure that yields a nested sequence of regions of inaction, that is, $\Omega^{n+1} \subset \Omega^n$. If we can choose such an update procedure, this would give us tremendous computational advantage because we no longer need to calculate the value function $\psi^{(n)}$ in the entire state space and can restrict our attention to calculating it only in the region of inaction Ω^n. Thus we would have to solve only one PDE rather than $2N + 1$ PDEs at each iteration. To ensure this we pick p_2, that is, the point at which $(1 - \mu_i)v_x - v_i$ is maximized, rather than p_3. This choice of boundary update is motivated by the so-called *principle of smooth pasting* and results in nested sequence of regions of inaction in closely related problems, cf. Kumar and Muthuraman (2004). That is, since the value function is linear outside the region of inaction, choosing a point where the directional derivative is maximized ensures that continuity of the second derivative in the value function did not change.

The criterion for terminating the iterations can be set in two ways—either in terms of the convergence of the region of inaction, that is, $\int \Omega^n \backslash \Omega^{n+1} ds < \epsilon_a$ or the convergence of the value function $\int_{\Omega^{n+1}} (\psi^{(n+1)} - \psi^{(n)})\, dy < \epsilon_\psi$. Implementation-wise, testing the convergence of the value function is easier, but making sure that the boundaries converge to within a tolerance value is necessary to conduct a discussion on the behavior and shapes of the region of inaction; so our implementation determines convergence in terms of convergence of the boundaries.

We state below a theorem without proof. The proof can be found in Muthuraman (2005b) and is omitted here for brevity. The theorem justifies that, for the one-stock case with log utility, each step of the computational scheme described above (a) results in a policy improvement and (b) yields a nested sequence of regions of inaction, provided the initial guess for the region of inaction contained the optimal region of inaction. It considers a function f_1 that solves equation (4.2) with boundary conditions (4.3) and (4.4) in Ω^1. It assumes that the optimal Ω^* is a subset of Ω^1, that is, transaction policy improvement is possible by stepping in (equations (4.12) and (4.13)). It first shows that the solution f_2 to the HJB equation on Ω^2 is such that $f_2 \geq f_1$. Also, Ω^2 is the new region of inaction that is obtained from Ω^1 and f_1, using the policy update procedure (equations (4.14) and (4.15)). Further it shows that $\Omega^* \subset \Omega^2$, which is equivalent to showing that further improvements are possible by stepping in (equations (4.17) and (4.18)). A lot more can be said and theoretically justified in the one-dimensional case, using slightly modified update conditions, the details of which can be found in Muthuraman (2005a).

THEOREM 4.1. *Let $N = 1$. Assume that the Merton point is less than 1, that is, $\frac{\alpha - r}{\sigma^2} < 1$ and $\theta \neq (r - \alpha) + \sigma^2$. Say, $f_1 \in C^2(\Omega^1)$ solves (4.2)–(4.4) in $\Omega^1 \equiv (\phi_b^1, \phi_s^1)$ and also that*

$$(\tilde{\mathcal{B}} f_1)' > 0 \quad \text{at } \phi_b^1 \quad \text{and} \tag{4.12}$$

$$(\tilde{\mathcal{S}} f_1)' < 0 \quad \text{at } \phi_s^1. \tag{4.13}$$

Define ϕ_b^2 and ϕ_b^2 as

$$\phi_b^2 = \inf\left\{ y > \phi_b^1 \,\middle|\, (\tilde{\mathcal{B}} f_1)' = 0 \right\} \quad \text{and} \tag{4.14}$$

$$\phi_s^2 = \sup\left\{ y < \phi_s^1 \,\middle|\, (\tilde{\mathcal{S}} f_1)' = 0 \right\} \tag{4.15}$$

if $f_2 \in C^2(\Omega^2)$ is such that it solves (4.2)–(4.4) in $\Omega^2 \equiv (\phi_b^2, \phi_s^2)$, then

(a)

$$f_2 > f_1 \quad \textit{in } \Omega^2 \tag{4.16}$$

(b)

$$(\tilde{B}f_2)' > 0 \quad \textit{at } \phi_b^2 \quad \textit{and} \tag{4.17}$$

$$(\tilde{S}f_2)' < 0 \quad \textit{at } \phi_s^2. \tag{4.18}$$

5. FINITE ELEMENTS AND IMPLICIT BOUNDARY CONDITIONS

Because of the presence of implicit boundary conditions the standard FEM, as found in books on this subject, is unsuitable to solve (4.5) with (4.3) and (4.4). In this section we introduce a variant of the standard FEM to accommodate implicit boundary conditions. For the sake of readability and completeness we provide a brief introduction to the FEM procedure while describing the variant.

Since we are not dealing with iterations in this section, we will represent $\psi^{(n,m)}$ only by ψ. We will also use $\partial\Omega$ to represent the boundaries of the region of inaction Ω. Since $c^{(n,m)}$ is known in equation (4.5), for some suitable functions of y: $\tilde{\eta}_{ij}$, $\tilde{b}_i$, $\tilde{\beta}$ a nd $\tilde{f}$, we can write (4.5) as

$$\sum_i \sum_j \tilde{\eta}_{ij}\psi_{ij} + \sum_i \tilde{b}_i \psi_i + \tilde{\beta}\psi = \tilde{f} \text{ in } \Omega. \tag{5.1}$$

The above form is general enough to accommodate the power utility case as well. Similarly for some suitable functions r and r_i, the boundary conditions (4.3)–(4.4) can be written as

$$\sum_i r_i \psi_i + r\psi = 0 \text{ on } \partial\Omega. \tag{5.2}$$

In the case of the log utility r would be zero, but we would have nonzero r for the power utility, hence we will deal with the above general form.

The statement of the problem, which is called the strong form is: *Find $\psi : \Omega \to \mathbf{R}$ such that equations* (5.1) *and* (5.2) *are satisfied.* The next step is to obtain the so-called weak form, which is the integral form of equation (5.1) and is called the weak form because the regularity of ψ that is required to satisfy (5.3) is less than that required to satisfy the strong form (5.1). Solving the weak form of (5.1) is *finding a $\psi \in H^1(\Omega)$ such that for any test function $\vartheta \in H^1(\Omega)$ the following holds*

$$\sum_i \sum_j \tilde{\eta}_{ij} \left(\int_{\partial\Omega} \psi_i \vartheta - \int_{\Omega} \psi_i \vartheta_j \right) + \sum_i \tilde{b}_i \int_{\Omega} \psi_i \vartheta + \tilde{\beta} \int_{\Omega} \psi\vartheta = \int_{\Omega} \tilde{f}\vartheta, \tag{5.3}$$

along with the necessary boundary conditions, going to be described shortly. Here $H^1(\Omega)$ is the Sobolev space Oden and Reddy (1978) of functions that have square integrable generalized first derivatives in Ω, that is, $f \in H^1(\Omega)$ provided

$$\int_{\Omega} f_{y_i}^2 dy_i < \infty \quad \forall i.$$

In the usual finite element procedure, the Neumann boundary conditions (i.e., ψ_i is known on $\partial\Omega$) are incorporated at this stage by substitution in the first term of (5.3).

Dirichlet boundary conditions (ψ is known on *partial* Ω) are incorporated by restricting that we find a ψ belonging to $H^1(\Omega)$ and satisfying the Dirichlet boundary condition for all $\vartheta \in H^1(\Omega)$. For our purposes, to account for implicit boundary conditions we modify the weak form to finding a Λ and $\psi \in H^1(\Omega)$ such that for any two test functions $\vartheta \in H^1(\Omega)$ and $\vartheta_\Lambda \in H^1(\Omega)$ the following holds:

$$\begin{aligned}\sum_i \sum_j \tilde{\eta}_{ij} \left(\int_{\partial\Omega} \psi_i \vartheta - \int_\Omega \psi_i \vartheta_j \right) + \sum_i \tilde{b}_i \int_\Omega \psi_i \vartheta + \tilde{\beta} \int_\Omega \psi \vartheta \\ + \int_{\partial\Omega} \left[\left(\sum_i r_i \psi_i + r\psi \right) \vartheta_\Lambda - \Lambda \vartheta \right] = \int_\Omega \tilde{f} \vartheta.\end{aligned} \tag{5.4}$$

In (5.4), Λ is an unknown (to be determined) Lagrange multiplier that is used to include the boundary conditions (5.2) into (5.3).

The following theorem shows that the solution to the strong form always is a solution of the weak form and a solution to the weak form is a solution of the strong form, provided it has sufficient smoothness.

THEOREM 5.1. *(i) If ψ (classically) solves* (5.1) *along with* (5.2), *then ψ solves* (5.4) *for any two test functions ϑ and ϑ_Λ. (ii) If $\psi \in C^2(\Omega)$ and solves equation* (5.4) *for any two test functions ϑ and ϑ_Λ, then ψ solves* (5.1) *along with* (5.2).

Proof.

(i) Multiplying (5.1) by ϑ, integrating over Ω, and integrating by parts yields

$$\sum_i \sum_j \tilde{\eta}_{ij} \left(\int_{\partial\Omega} \psi_i \vartheta - \int_\Omega \psi_i \vartheta_j \right) + \sum_i \tilde{b}_i \int_\Omega \psi_i \vartheta + \tilde{\beta} \int_\Omega \psi \vartheta = \int_\Omega \tilde{f} \vartheta.$$

Since ψ solves the above, ψ also solves

$$\sum_i \sum_j \tilde{\eta}_{ij} \left(\int_{\partial\Omega} \psi_i \vartheta - \int_\Omega \psi_i \vartheta_j \right) + \sum_i \tilde{b}_i \int_\Omega \psi_i \vartheta + \tilde{\beta} \int_\Omega \psi \vartheta - \int_{\partial\Omega} \Lambda \vartheta = \int_\Omega \tilde{f} \vartheta, \tag{5.5}$$

for some Lagrange multiplier Λ.

Now multiplying (5.2) by another arbitrary test function ϑ_Λ and integrating over $\partial\Omega$, we have

$$\int_{\partial\Omega} \left(\sum_i r_i \psi_i + r\psi \right) \vartheta_\Lambda = 0. \tag{5.6}$$

Adding (5.5) and (5.6) yields the result.

(ii) For any two arbitrary test functions ϑ and ϑ_Λ, ψ solves (5.4). Therefore, taking $\vartheta \equiv 0$, we have $\int_{\partial\Omega} (\sum_i r_i \psi_i + r\psi)\vartheta_\Lambda = 0$. Since the above is true for any ϑ_Λ, it implies that $\sum_i r_i \psi_i + r\psi = 0$ on $\partial\Omega$.

Now taking $\vartheta_\Lambda \equiv 0$, we have

(5.7)
$$\sum_i \sum_j \tilde{\eta}_{ij} \left(\int_{\partial\Omega} \psi_i \vartheta - \int_\Omega \psi_i \vartheta_j \right) + \sum_i \tilde{b}_i \int_\Omega \psi_i \vartheta + \tilde{\beta} \int_\Omega \psi \vartheta - \int_{\partial\Omega} \Lambda \vartheta = \int_\Omega \tilde{f} \vartheta.$$

Since ψ is twice differentiable in Ω, we have

$$\int_\Omega \left[\sum_i \sum_j \tilde{\eta}_{ij} \psi_{ij} + \sum_i \tilde{b}_i \psi_i + \tilde{\beta} \psi - \tilde{f} \right] \vartheta - \int_{\partial\Omega} \Lambda \vartheta = 0. \tag{5.8}$$

Now take $\vartheta = \hat{\vartheta}[\sum_i \sum_j \tilde{\eta}_{ij} \psi_{ij} + \sum_i \tilde{b}_i \psi_i + \tilde{\beta} \psi - \tilde{f}]$ for some function $\hat{\vartheta} = 0$ in $\partial\Omega$ and $\hat{\vartheta} > 0$ in Ω as our test function. Since $\psi \in C^2(\Omega)$, $\vartheta \in H^1(\Omega)$. We then have

$$\int_\Omega \left[\sum_i \sum_j \tilde{\eta}_{ij} \psi_{ij} + \sum_i \tilde{b}_i \psi_i + \tilde{\beta} \psi - \tilde{f} \right]^2 \hat{\vartheta} - 0 = 0. \tag{5.9}$$

Since $[\sum_i \sum_j \tilde{\eta}_{ij} \psi_{ij} + \sum_i \tilde{b}_i \psi_i + \tilde{\beta} \psi - \tilde{f}]^2 \geq 0$ and $\hat{\vartheta} > 0$ in Ω, the above implies that

$$\sum_i \sum_j \tilde{\eta}_{ij} \psi_{ij} + \sum_i \tilde{b}_i \psi_i + \tilde{\beta} \psi - \tilde{f} = 0 \text{ in } \Omega. \qquad \square$$

The FEM solves the weak form. As with any numerical scheme the first step is to divide the domain Ω into a collection of simple regions. This is called meshing and can be accomplished easily, using one of the many mesh generation routines that are widely available. We use meshes that have a larger mesh density in the neighborhood of the boundaries. Almost all mesh generation programs allow for control of mesh sizes and densities. An example of our mesh is shown in Figure 5.1. The vertices of the mesh elements are called nodes. Let M be the total number of nodes and M_p the number of nodes on the boundary. Once we have a mesh, the idea is to introduce an approximation $\hat{\psi}$ to our

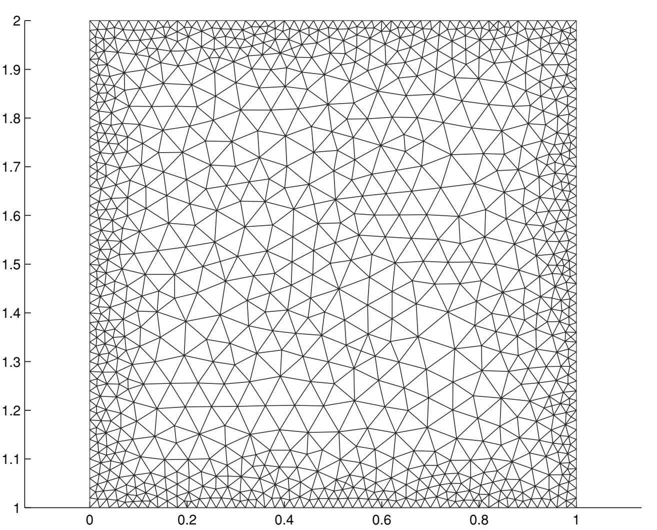

FIGURE 5.1. A typical mesh in two dimensions.

unknown ψ. Choosing a suitable set of basis functions $\varphi^{(1)}, \varphi^{(2)}, \dots, \varphi^{(M)}$, we introduce the approximation

$$\hat{\psi} = \sum_l p_l \varphi^{(l)}. \tag{5.10}$$

This transforms the problem of finding the function ψ to finding a $\hat{\psi}$, which is described by a finite number of unknowns p_l. By choosing $\varphi^{(l)}$ to be a function that equals 1 at node l and 0 at all other nodes, p_l will be the value of $\hat{\psi}$ at node l. Similarly we represent Λ by

$$\Lambda = \sum_{l\in\partial\Omega} p_l^{\Lambda} \varphi^{(l)}. \tag{5.11}$$

Note that the notation $\sum_l$ implies that l runs through all nodes, that is $l \in \{i : \text{Node } i \in \Omega\}$ and $\sum_{l\in\partial\Omega}$ implies that l runs through all boundary nodes only, that is $l \in \{i : \text{Node } i \textit{ in } \partial\Omega\}$. Many classes of basis functions are available and differ by the value they take within the element Hughes (1987). Since we seek new boundaries during our boundary iteration step, as functions of the derivative of the value function, we choose Hermite elements as our basis function. Functions that are specified as linear combinations of the Hermite basis functions can be made to have continuous derivatives on the vertices of the elements. This is essential for our work.

We now discretize the weak form by taking that test functions belong to the space of functions that are linear combinations of our basis functions $\varphi^{(l)}$. This is called the Galerkin method. Thus, representing the test functions by $\vartheta = \sum_k q_k \varphi^{(k)}$ and $\vartheta_\Lambda = \sum_{k\in\partial\Omega} q_k^{\Lambda} \varphi^{(k)}$, the weak form can be written as

$$\begin{aligned}\Bigg[\sum_i\sum_j \tilde{\eta}_{ij}\left(\int_{\partial\Omega}\psi_i\varphi^{(k)} - \int_\Omega \psi_i\varphi_j^{(k)}\right) + \sum_i \tilde{b}_i\int_\Omega \psi_i\varphi^{(k)} + \tilde{\beta}\int_\Omega \psi\varphi^{(k)} \\ - \int_{\partial\Omega}\Lambda\varphi^{(k)} - \int_\Omega \tilde{f}\varphi^{(k)}\Bigg] q_k + \left[\int_{\partial\Omega}\left(\sum_i r_i\psi_i + r\psi\right)\varphi^{(k)}\right] q_k^{\Lambda} = 0.\end{aligned} \tag{5.12}$$

Since the above holds for all test functions, each of the two terms in the square brackets above should be equal to zero. Approximating the ψ in the first term of (5.12) with $\hat{\psi}$ given by (5.10), and using equation (5.11) we have

$$\begin{aligned}\Bigg[\sum_i\sum_j \tilde{\eta}_{ij}\left(\int_{\partial\Omega}\varphi_i^{(l)}\varphi^{(k)} - \int_\Omega \varphi_i^{(l)}\varphi_j^{(k)}\right) + \sum_i \tilde{b}_i\int_\Omega \varphi_i^{(l)}\varphi^{(k)} + \tilde{\beta}\int_\Omega \varphi^{(l)}\varphi^{(k)}\Bigg] p_l \\ - \left[\int_{\partial\Omega}\varphi^{(l)}\varphi^{(k)}\right] p_l^{\Lambda} = \int_\Omega \tilde{f}\varphi^{(k)}.\end{aligned} \tag{5.13}$$

Equation (5.13) has M unknown p_l's and M_p unknown p_l^{Λ}'s. We can write (5.13) as

$$AP = F \tag{5.14}$$

where

$$A_{kl} = \begin{cases} \sum_i \sum_j \tilde{\eta}_{ij} \left(\int_{\partial\Omega} \varphi_i^{(l)} \varphi^{(k)} - \int_{\Omega} \varphi_i^{(l)} \varphi_j^{(k)} \right) \\ \quad + \sum_i \tilde{b}_i \int_{\Omega} \varphi_i^{(l)} \varphi^{(k)} + \tilde{\beta} \int_{\Omega} \varphi^{(l)} \varphi^{(k)} & \text{for } l \leq M \\ -\int_{\partial\Omega} \varphi^{(l)} \varphi^{(k)} & \text{for } M < l \leq M_p, \end{cases} \tag{5.15}$$

$$F_k = \int_{\Omega} \tilde{f} \varphi^{(k)} \tag{5.16}$$

$$P = \{p_1, p_2, \ldots, p_M, p_1^{\wedge}, \ldots, p_{M_p}^{\wedge}\}. \tag{5.17}$$

However, as specified A is $M \times (M + M_p)$ and is obviously not of rank $M + M_p$. We need to augment the system with additional constraints so as to uniquely specify P. These constraints come from the second term of (5.12).

$$\int_{\partial\Omega} \left(\sum_i r_i \varphi_i^{(l)} + r \varphi^{(l)} \right) \varphi^{(k)} p_l = 0,$$

which can be written as

$$RP = 0, \tag{5.18}$$

with

$$R_{kl} = \begin{cases} \int_{\partial\Omega} \left(\sum_i r_i \varphi_i^{(l)} + r \varphi^{(l)} \right) \varphi^{(k)} & \text{for } l \leq M \\ 0 & \text{for } M < l \leq M_p. \end{cases}$$

We combine (5.13) and (5.18) to form a combined $(M + M_p) \times (M + M_p)$ system

$$\begin{bmatrix} A \\ R \end{bmatrix} P = \begin{bmatrix} F \\ 0 \end{bmatrix}. \tag{5.19}$$

Thus, we solve (5.19) to obtain P. We report $\hat{\psi} = \sum p_l \varphi^{(l)}$ as the solution.

6. RESULTS AND DISCUSSIONS

The objectives of this section are threefold. First we illustrate the behavior of optimal policies. Second, we discuss why the observed behavior occurs, and third, we suggest some heuristic approximations to the optimal policy. We also compare our results to asymptotic results obtained by others Ostrov and Goodman (2004).

We begin our exploration of optimal policies for the two-stock problem by showing a typical boundary update sequence. Later, we consider the transaction policy dependence on various model parameters like transaction costs, risk aversion, volatility, and the covariance structure. Though the computational scheme would work in principle for problems in higher dimensions, the curse of dimensionality does exist. Solving, say, a 10-stock problem with this method would be impossible because of the enormous computational requirement. Hence, the only way to tackle the 10-stock problem would be to propose heuristic approximations for the optimal policy, from the observations made on the exact solution in lower dimensions. Once "good" approximations are discovered, one can use them to reduce the dimensionality of the search space. To that effect we propose and quantify some good heuristic approximations at the end of this section.

6.1. Discussion 1: Illustrating the Update Procedure

Now consider the two-stock problem. For concreteness, we choose the parameters as in Table 6.1. We begin our boundary update procedure with a guess of Ω, as the square, $[0.01, 0.5]^2$, that is, the initial guess for the transaction policy is not to trade when the fractions of wealth in either stock lie in the interval [0.01, 0.5]. We use the boundary update procedure and obtain various regions of inaction with each iteration. Figure 6.1 shows the sequence of boundaries generated. Convergence occurs in seven iterations for this case.

A contour plot of the value function associated with the "converged" region of inaction, that is, the optimal value function, is shown in Figure 6.2. For this set of parameters

TABLE 6.1
Table of Parameters

Parameter	Disc. 1	Disc. 2 and 3	Disc. 4	Disc. 5	Disc. 6	Disc. 7a	Disc. 7b
α_1, α_2	15%	15%, 12%	15%, 12%	12%	Varied,12%	12%	15%, 12%
σ_{11}, σ_{22}	0.4	0.4	Varied	0.4	0.3	$0.4 - \kappa$	$0.4 - \kappa$
σ_{12}, σ_{21}	0.1	0	0	0	0	κ	κ
r	7%	7%	7%	7%	7%	7%	7%
θ	10%	10%	10%	10%	10%	10%	10%
γ	−1	−1	−1	Varied	−1	−1	−1
λ_1, μ_1	1%	Varied	1%	1%	1%	1%	1%
λ_2, μ_2	1%	Varied	1%	1%	1%	1%	1%

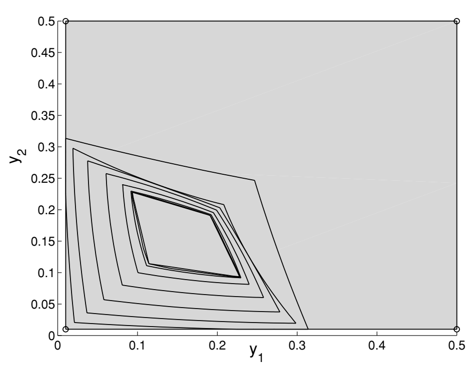

FIGURE 6.1. Sequence of boundaries generated (Discussion 1).

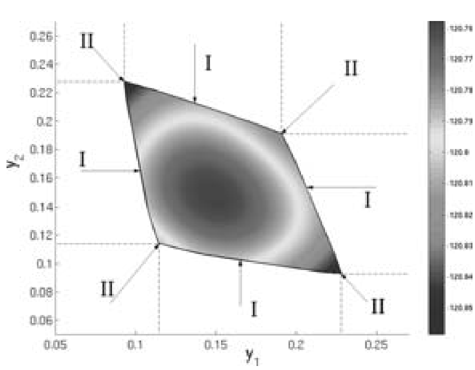

FIGURE 6.2. Contour plot of converged value function (Discussion 1).

Merton's optimal wealth allocation fraction is the point (0.16, 0.16). As our intuition suggests, we do find that the Merton point lies within the region of inaction. The Merton point is the point at which the investor would stay in the presence of zero transaction costs. In the presence of transaction costs, the investor would decrease the number of transactions, thereby decreasing the total costs of transaction, but at the same time the investor does not want to drift too far from the Merton point. So, he allows the process to wander in the region of inaction and conducts transactions only when the process tries to exit this region.

Figure 6.2 also shows the optimal action of the investor outside the region of inaction, for the correlated stock case. In regions marked I, only one asset needs to be transacted to reach Ω, while in regions marked II two assets have to be transacted to reach a corner of Ω.

6.2. Discussion 2: Impact of Transaction Costs (Uncorrelated Stocks)

Now we look at the dependence of the region of inaction on substantial changes in transaction costs beyond small cost asymptotics. Consider the case when both the stock returns are independent. Parameter choices are shown in Table 6.1.

Transaction costs for buying and selling stocks are kept equal. Figure 6.3 shows the optimal regions of inaction for this set of parameters for various values of transaction costs. First, observe that the regions of inaction are approximately rectangles, in all cases. This does suggest that the regions of inaction in the case of independent stocks are close to rectangles. This is consistent with previous suggestions (Akian, Menaldi, and Sulem 1996). Also note that since the expected rate of return from stock 1 is higher, as expected we not only put more money into stock 1 but also have a larger inhibition towards trading stock 1 as opposed to trading stock 2.

Finally, we consider the dependence of the region with respect to transaction levels. As the transaction cost increases it is obvious that the investor would prefer to make

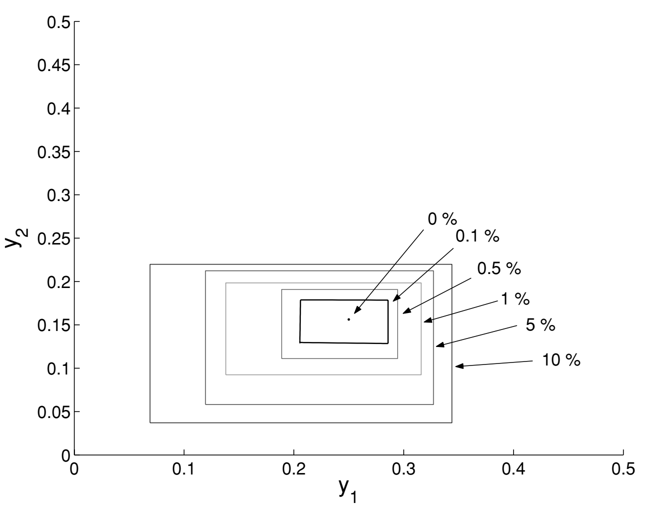

FIGURE 6.3. Impact of transaction costs (Discussion 2).

fewer transactions. This translates to a transaction policy that has a larger region of inaction. Figure 6.3 not only shows that the regions enlarge, but one can also note that the increase in dimensions of the region of inaction are sublinear for substantial change in transactions cost too. That is, changing the transaction cost from 0 to 1% changes the transaction policy much more than the change from 1 to 2%.

6.3. Discussion 3: Comparison with Asymptotics

Janeček and Shreve (2004) provide a asymptotic expansion for the value function around the Merton point in powers of the transaction cost parameter λ, for the one-stock case with power utility. The results obtained by Janeček and Shreve are compared in detail with the results obtained by this computational procedure in Muthuraman (2005a).

Ostrov and Goodman (2004) have carried out asymptotic analysis for the case of two independent stocks. From the first-order terms of the expansion in terms of $\lambda^{\frac{1}{3}}$, they conclude that the region of inaction is a rectangle, confirming with our observations for independent stock. The width of the region of inaction (rectangle) along the stock ith axis is given by Ostrov and Goodman (2004) as

$$\frac{\lambda^{\frac{1}{3}}}{1-\gamma}\left[12\frac{V_{ii}}{\sigma_i^2}\right]^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}) \tag{6.1}$$

where V_{ii} is the ith diagonal element of $V = v^T v$ and

$$v = \sigma^T\left(\frac{dd^T}{1-\gamma} - \mathrm{diag}(d)\right)$$

where $d = (\alpha - r)/\sigma\sigma^T$. Denoting the first term of (6.1) by $\tilde{\pi}^i$ and denoting by π^i the width of the region of inaction along the ith axis, as obtained by our computational method, we define $\pi_e^i = |\tilde{\pi}^i - \pi^i|$. Figure 6.4 shows a plot of $\log \pi_e^1$ and $\log \pi_e^2$ Vs $\log(\lambda)$. Again as in the one-dimensional case, if π^i was at least as close as or closer than $\tilde{\pi}^i$ to the exact width, then the difference between $\tilde{\pi}^i$ and π^i, that is π_e^i, would be $O(\lambda^{\frac{2}{3}})$. That is the slope would be at least $\frac{2}{3}$. As can be seen from the plots, the slopes are greater than $\frac{2}{3}$, confirming the asymptotic results of Ostrov and Goodman (2004). The parameters used are tabulated in Table 6.1.

6.4. Discussion 4: Impact of Volatility (Uncorrelated Stocks)

Now we study the impact of volatility on the regions of inaction. Parameter choices are shown in Table 6.1. As shown in Table 6.1, σ, which is our measure of volatility, is expressed in terms of σ_{11} and σ_{22} is varied, keeping $\sigma_{11} = \sigma_{22}$. Figure 6.5 shows the optimal regions of inaction for various values of σ. Since the stocks are independent, $\sigma_{21} = \sigma_{12} = 0$.

Since the transaction costs for stock 1 has doubled, as evident from the scaling of the axis, we do see that the regions of inaction have elongated along the stock 1 axis. Also as the variance increases, the regions of inaction not only move closer to the origin but also shrink in size. The move towards the origin is because the investor is risk averse and hence is less willing to keep wealth in the more variable stocks. The region of inaction

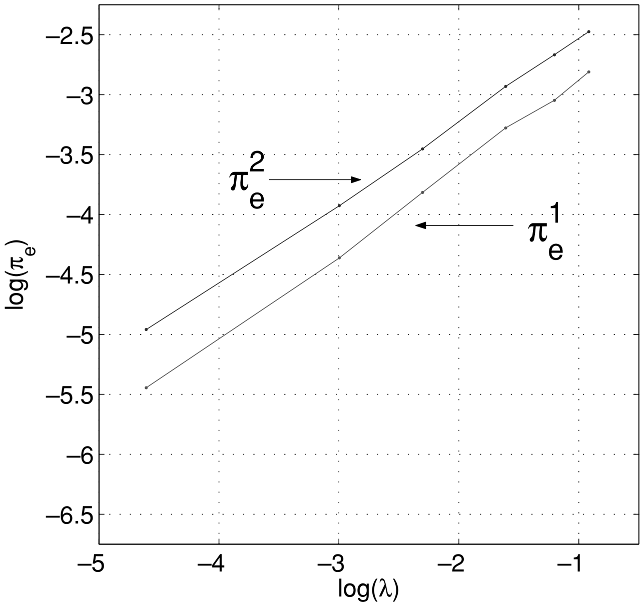

FIGURE 6.4. Asymptotic comparison in the two stock case (Discussion 3).

also starts to shrink because with higher variance the investor would be more willing to trade to hedge his risks and be closer to the Merton point.

6.5. Discussion 5: Impact of Risk Aversion (Uncorrelated Stocks)

As the investor becomes more risk averse he would put less money into the stock, and transact more. Here, as in the case of increasing volatility, the regions of inaction move closer to the origin and decrease in size. Figure 6.6 shows the regions of inaction for various values of γ. Parameter choices are shown in Table 6.1.

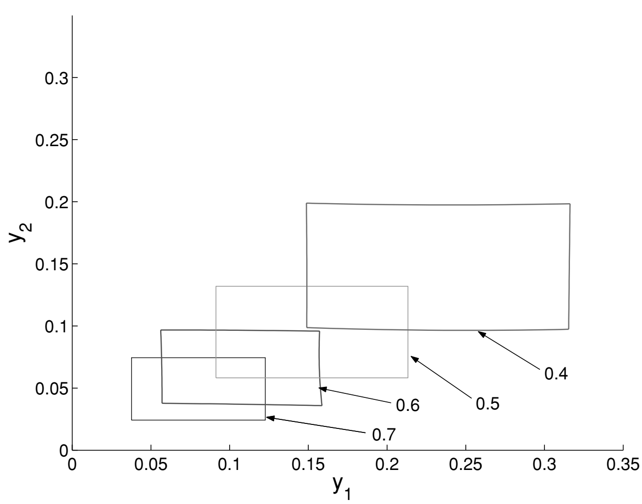

FIGURE 6.5. Impact of variance (Discussion 4).

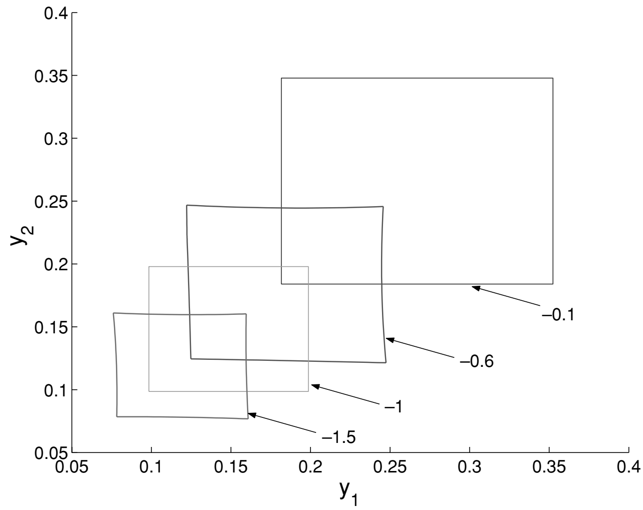

FIGURE 6.6. Impact of risk aversion (Discussion 5).

6.6. Discussion 6: Separability of Optimal Policies

As pointed out by Akian, Menaldi, and Sulem (1996), for the portfolio optimization problem with self-financing conditions, it is not possible to separate the optimal policies of each asset, even in independent stock case. In other words, a change in a particular asset's parameter can influence the buy/sell boundaries of other assets as well, when strategies are restricted to be self-financing. This dependence is evident even in the first-order asymptotic term obtained by Ostrov and Goodman (2004). This implies that the multistock problem has to be solved in multiple dimensions and cannot be decomposed to yield a set of one-dimensional problems like in Liu (2004).

Even away from the asymptotic regime, such dependence can be seen numerically as in Figure 6.7. The expected rate of growth of stock 1 (α_1) is varied and Ω's width along each

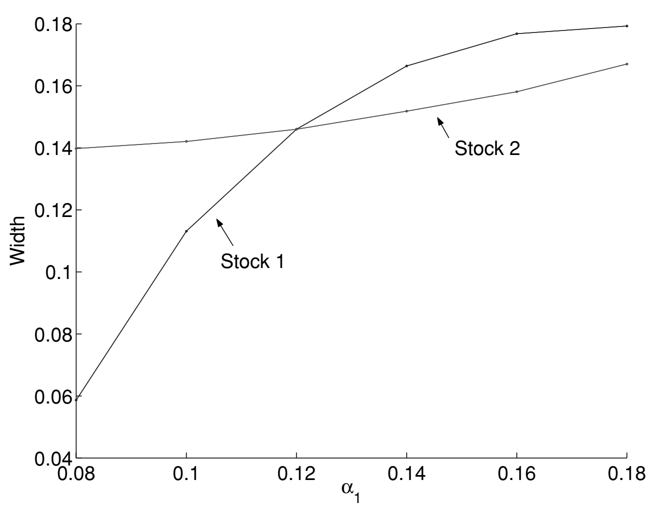

FIGURE 6.7. Width of the region of inaction along the stock 1 and 2 axis (Discussion 6).

stock axis is plotted for the parameter set given by Table 6.1 (Discussion 6). As can be seen, a change in α_1 does influence the width of Ω along the stock 2 axis despite keeping the parameters of stock 2 constant. Hence, the optimal policy cannot be obtained by optimizing along each dimension independently.

6.7. Discussion 7: Impact of Correlation

Figure 6.8 shows the impact of positive correlation on the shape of the region of inaction. To make meaningful inferences on the change of shape of the region of inaction, the covariance matrix σ should be changed in a manner so as to keep the Merton point fixed. For the power utility function the Merton point is given by the vector $\frac{(\sigma\sigma')^{-1}(\alpha - r)}{1-\gamma}$. If $\sigma \equiv \begin{pmatrix} \sigma_v & \sigma_c \\ \sigma_c & \sigma_v \end{pmatrix}$, then the Merton point is

$$\frac{1}{\left(\sigma_v^2 - \sigma_c^2\right)^2(1-\gamma)} \begin{pmatrix} \left(\sigma_v^2 + \sigma_c^2\right)(\alpha_1 - r) - 2\sigma_v\sigma_c(\alpha_2 - r) \\ \left(\sigma_v^2 + \sigma_c^2\right)(\alpha_2 - r) - 2\sigma_v\sigma_c(\alpha_1 - r) \end{pmatrix}.$$

If $\alpha_1 = \alpha_2 = \alpha$, say, then the above vector becomes $\frac{1}{(\sigma_v + \sigma_c)^2(1-\gamma)}\begin{pmatrix} (\alpha - r) \\ (\alpha - r) \end{pmatrix}$. The Merton point will not change as long as $\sigma_v + \sigma_c$ remains constant. Therefore, Figure 6.8 plots the region of inaction for $\sigma = \begin{pmatrix} 0.4-\kappa & \kappa \\ \kappa & 0.4-\kappa \end{pmatrix}$ for κ varying from 0 to 0.075. When $\kappa = 0.075$ the covariance term is around 23% of the variance term. The value of the other parameters are shown in Table 6.1 (Discussion 7a).

One could provide a heuristic explanation for why the shape of the region of inaction changes in the way it does in Figure 6.8. In particular the region of inaction shrinks along the (1, 1) direction (the main diagonal) and elongates along (1, − 1) with the increase in positive correlation, as measured by the parameter κ. With larger values of κ it is less likely that an increase in the value of stock 1 is accompanied by an decrease in the value of stock 2. Given a region of inaction, it is less likely that sample paths of the value

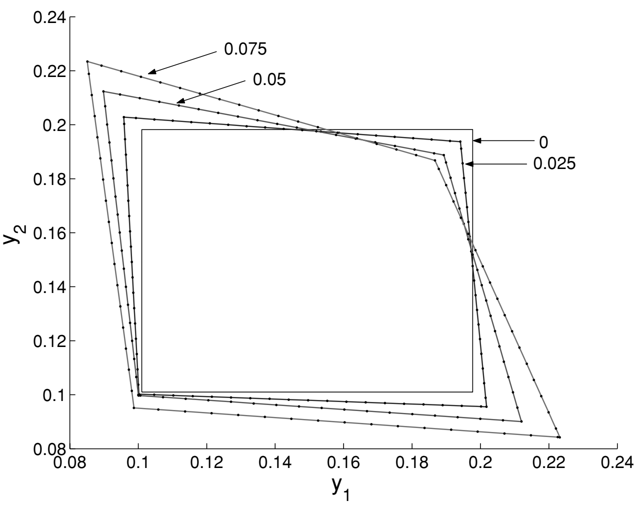

FIGURE 6.8. Impact of positive correlation. Stocks with equal expected returns for various values of κ (Discussion 7).

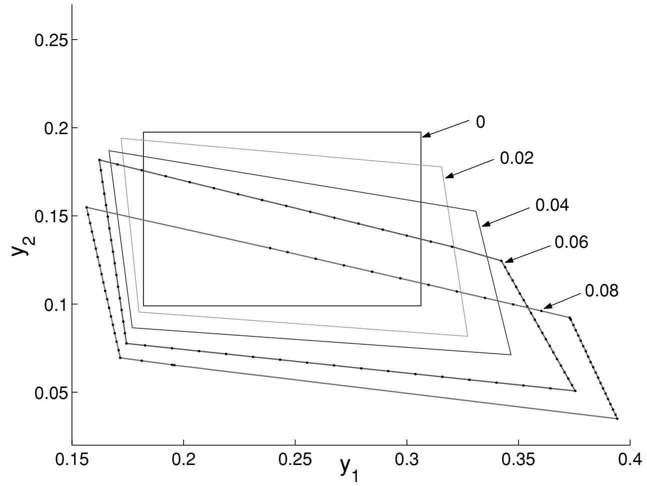

FIGURE 6.9. Impact of positive correlation. Stocks with different expected returns for various values of κ (Discussion 7).

processes will turn away from the main diagonal, and hence transactions are more likely to be inevitable along the main diagonal. Given this inevitability, one does not save much on transaction costs by giving the sample paths room to turn away from the boundary along the main diagonal. Therefore, a new region of inaction in which one transacts closer to the Merton point along the main diagonal will provide a better value function because it does not let the value function deteriorate as much before it intervenes. Of course, one can only shrink the region so much along the main diagonal before the transactions costs become prohibitive. The symmetric opposite reasoning explains the elongation along the (1, − 1) direction. Another explanation of this behavior is that in the case of positively correlated stock, one does not lose much by having more than the Merton value in one stock and less in the other, since one partially hedges the other. Therefore, one can tolerate more deviation from the Merton point along the (1, − 1) direction than along the (1, 1) direction.

Figure 6.9 shows the region of inaction for yet another set of parameters that are shown in Table 6.1 (Discussion 7b). Here, the expected rate of return for stock 1 (15 %) is more than that of stock 2 (12%). Since $\alpha_1 \neq \alpha_2$, the Merton point is not the same for all the regions of inaction shown. Here, κ is varied from 0 to 0.08. As correlation increases two phenomena can occur. The first is the case when the value function becomes infinite. To see this, consider the case when the two stocks are perfectly correlated. If the difference in their expected rates of return are such that they allow the construction of a riskless hedge with a sufficiently high growth in consumption then the value function can grow to infinity. The second is the case when the increase in correlation could possibly make the region of inaction cross one or more of the axes. Here, we run into a structural difficulty in solving the PDEs, since it loses ellipticity on the axes.

Figure 6.10 shows the case for the same set of parameters as in Table 6.1 (Discussion 7a), but with negative correlation. The three regions correspond to κ of 0, − 0.04, and −0.1, that is, 0%, − 9.09%, and −20% of the variance. An analog of the discussion in the case of positive correlation explains the behavior, namely the elongation along the (1, 1) diagonal and shrinkage along the (1, − 1) diagonal.

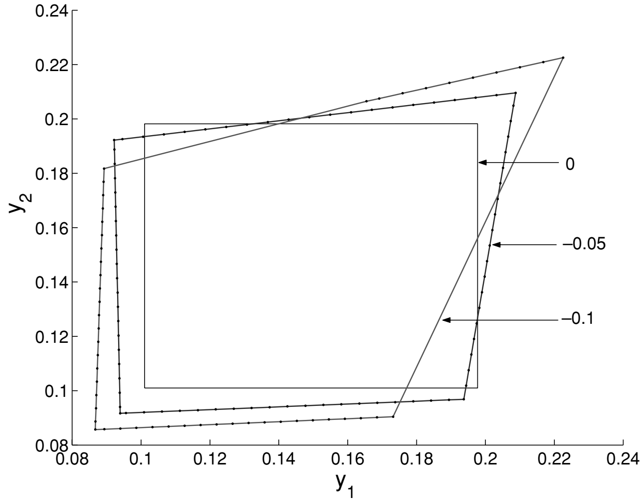

FIGURE 6.10. Impact of negative correlation. Stocks with equal expected returns for various values of κ (Discussion 7).

6.8. Convergence and Computational Complexity

In this subsection we show typical rates of convergence using numerical examples and also provide a discussion on the computational complexity of the scheme proposed.

As depicted in Figure 4.1, two iteration loops constitute the scheme, loop (i) and loop (ii). Initiated by a guess for the policy and consumption, loop (i) updates the consumption using equation (4.6) for the fixed policy (boundary), and loop (ii) updates the boundaries using (4.9)–(4.10). As mentioned in Section 4, we check for boundary convergence on the basis of the convergence of Ω^n's area to within a tolerance ϵ_b. The value of ϵ_b used for the results in this section is 10^{-6}. For the parameter set given in Table 6.1 (Discussions 2 and 3), Figure 6.11 plots the area of Ω^n versus n for various values of transaction costs.

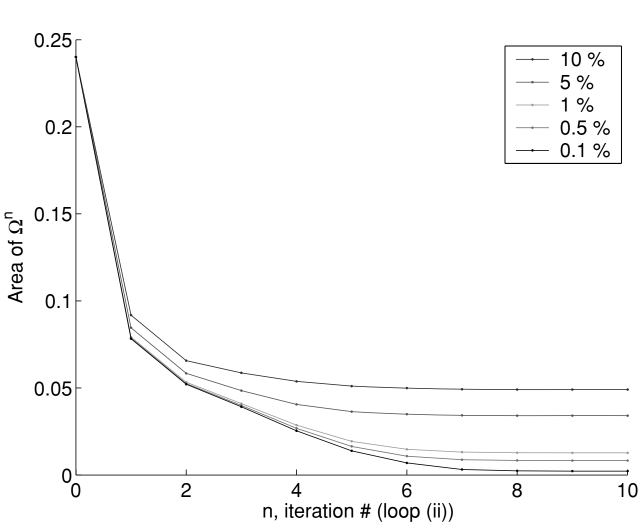

FIGURE 6.11. Convergence of Ω for various transaction costs.

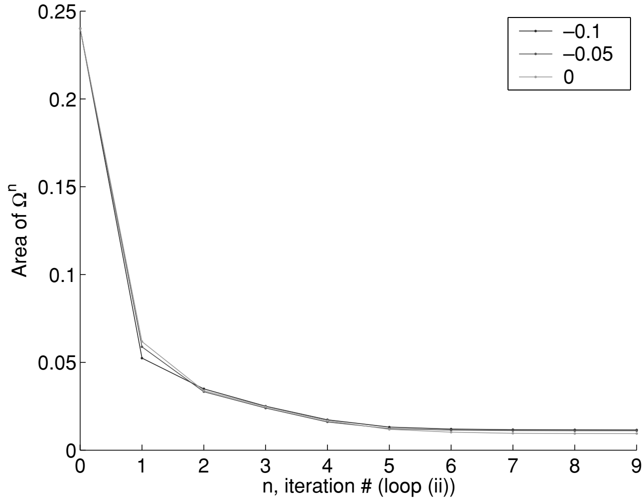

FIGURE 6.12. Convergence of Ω for various values of κ.

For all transaction values we start with the same initial guess for the region of inaction $[0.01, 0.5]^2$. Lower transaction costs take more iterations to converge mainly because the region of inaction is much smaller. But the growth in iterations is not that dramatic. For example, convergence is achieved in 8 iterations in the case of 10% transaction costs while it takes 10 iterations for the 0.1% case. The number of boundary iterations required for convergence does not seem to increase much with change in correlation. Figure 6.12 plots the area of Ω^n versus n for various values of κ with parameters given in Table 6.1 (Discussion 7a). The boundary converges in eight iterations for the independent case and the $\kappa = -0.05$ case, while it takes nine iterations for the $\kappa = -0.1$ case.

Figure 6.13 shows the consumption during both loops (i) and (ii) of the scheme. We start with 8% as the guess consumption. The line labeled "#1 of (ii)" traces the convergence

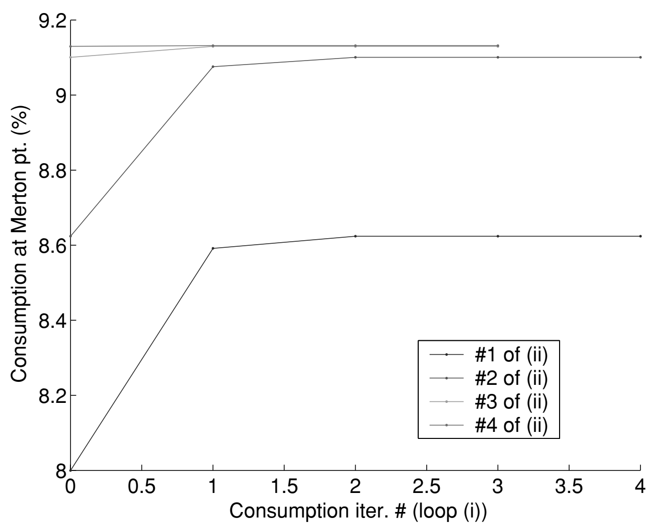

FIGURE 6.13. Convergence of consumption.

of consumption in the first iteration of loop (ii) and is plotted against the iteration number of loop (i). Convergence is checked using (4.8) with a tolerance value $\epsilon_c = 10^{-4}$. Once consumption converges within loop (i), the boundary is updated and the second iteration of loop (ii) is initiated. The initial consumption for iteration "#2 of (ii)" is set to the converged consumption in iteration "#1 of (ii)." Convergence of consumption is observed to be monotonic, provided the initial guess is lower than the optimal value. We do not however have a theoretical proof of monotonic convergence, but such monotonic convergence often occurs when solving elliptic nonlinear PDEs in fixed domains (Evans 1998). For almost all of the parameters that we tested for the two stock cases, convergence of the boundaries, that is, loop (ii) in Figure 4.1 occurs roughly in 5–10 iterations and the convergence of c, that is loop (i), takes 2–5 iterations.

Accurate estimates of the complexity involved in the updating conditions (4.6), (4.9), and (4.10), and the assembling of the matrices (A, R, and F) is not feasible because of the dependance on specific implementation. Moreover, many of the quantities involved in assembling can be precomputed and stored for use during runtime. The complexity of the scheme is dominated by the complexity involved in solving the linear system (5.19). Hence, a good estimate for the overall complexity is the complexity of solving (5.19) times the number of iterations.

Say we use an uniform rectangular mesh and H be the number of discretization points on each axis. Then, the total number of nodes in the region of inaction $M = H^N$ and the total number of boundary nodes is $M_p = 2NH^{N-1}$. Hence (5.19) is a $(H^N + 2NH^{N-1}) \times (H^N + 2NH^{N-1})$ system. Solution using direct methods like Gaussian elimination or LU decomposition (Golub and Loan 1996) brings the complexity to $O(H^{3N})$. If we represent k to be the total number of times (5.19) is solved until convergence, then an estimate of the complexity of the entire scheme is $O(kH^{3N})$. It is not clear how k depends on N, but our empirical observations on the one-, two- and three-dimensional cases suggest essentially linear dependence.

One can use faster iterative techniques (Golub and Loan 1996) like GMRES that access the matrix in (5.19) only by the matrix–vector products. In many cases the iterative method to solve the linear system will converge in a small number of iterations m. This will reduce the complexity to $O(mkH^{2N})$. By using fast multipole methods this can be reduced further to $O(mkH^N N \log H)$, but the costs of preprocessing increase significantly. However, the complexity always increases exponentially in dimension. In our computations as discussed in Section 5, we use nonuniform meshes like the one shown in Figure 5.1 and use LAPACK's linear solver to solve the linear system (5.19). The number of nodes used in all the two-dimensional cases presented is typically around 2,500.

6.9. Three-Stock Case

We test the computational scheme for the three-stock case for several reasons. First, it verifies that the procedure we have is genuinely an automated procedure, which does not rely on visualization. Second, it allows us to see if the insights we have in the two-stock case carry over to higher dimensions. Finally, it allows to test heuristics that we will propose shortly. Such heuristic approximations can be relied upon with greater confidence when their performance are quantified in the three-stock case as well. We will not extend every discussion in the two-stock case to three stocks. Rather, we illustrate here that this can be done by extending Discussion 1.

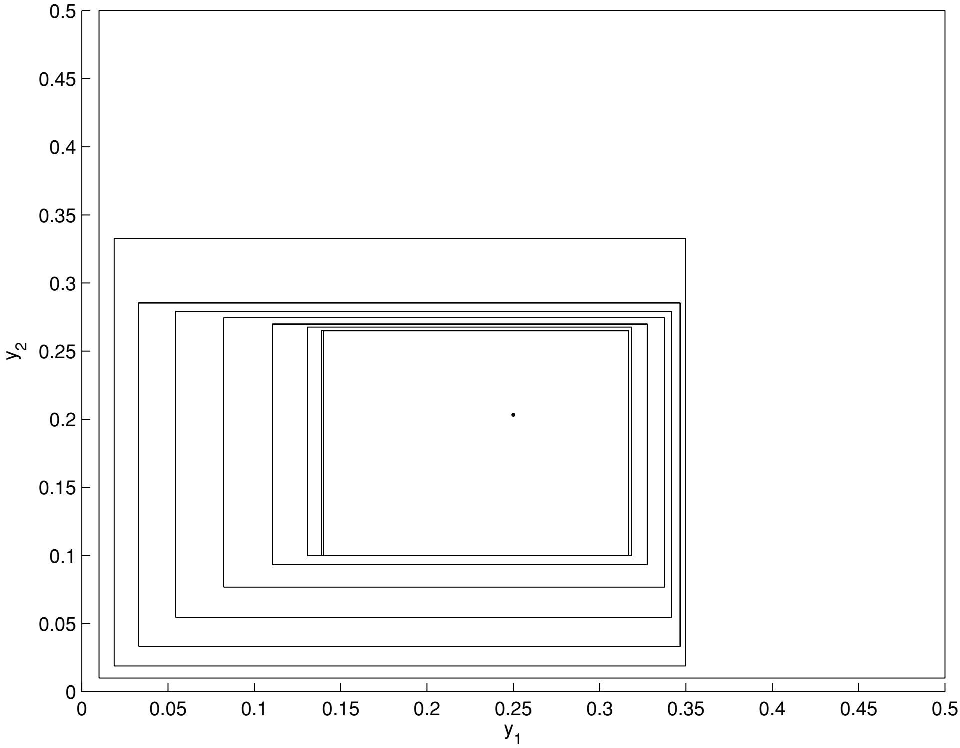

FIGURE 6.14. Sequence of boundaries, along the section $y_3 = 0.156$ (Discussion 1).

We consider three independent stocks described by the parameter set: $\alpha = [15\% \; 13.5\% \; 12\%]'$, $\sigma_i = 0.4$, $r = 7\%$, $\theta = 10\%$, $\gamma = -1$ and all transaction costs being 1%. The Merton point for this parameter set lies at (0.25, 0.203, 0.156). The regions of inaction for the three-stock case is bounded by six surfaces. Along the unit wealth cut, the region of inaction is a closed region set in three dimensions. It would be difficult to visualize the sequence of three-dimensional region in one plot. Therefore, we take two sections along the planes $y_3 = 0.156$ and $y_2 = 0.203$. These are planes that pass through the Merton point. The sections are shown in Figures 6.14 and 6.15. Here we start with an initial guess for Ω^0 as $[0.01, 0.5]^3$ and convergence occurs in 10 iterations.

6.10. Heuristic Approximations

As discussed in the introduction of this section, we seek to make "good" approximations based on the results obtained in two dimensions that not only perform well with regard to the value function obtained but are also easy to compute because they are specified by fewer parameters. We first look at the regions of inaction. When the stocks are independent, our observations suggest that the regions of inaction can be well approximated by rectangles or cuboids. For cases when the stocks are not independent it appears that the regions of inaction, though not rectangles any more, still appear to be well approximated by quadrilaterals in two dimensions and polyhedra in higher dimensions.

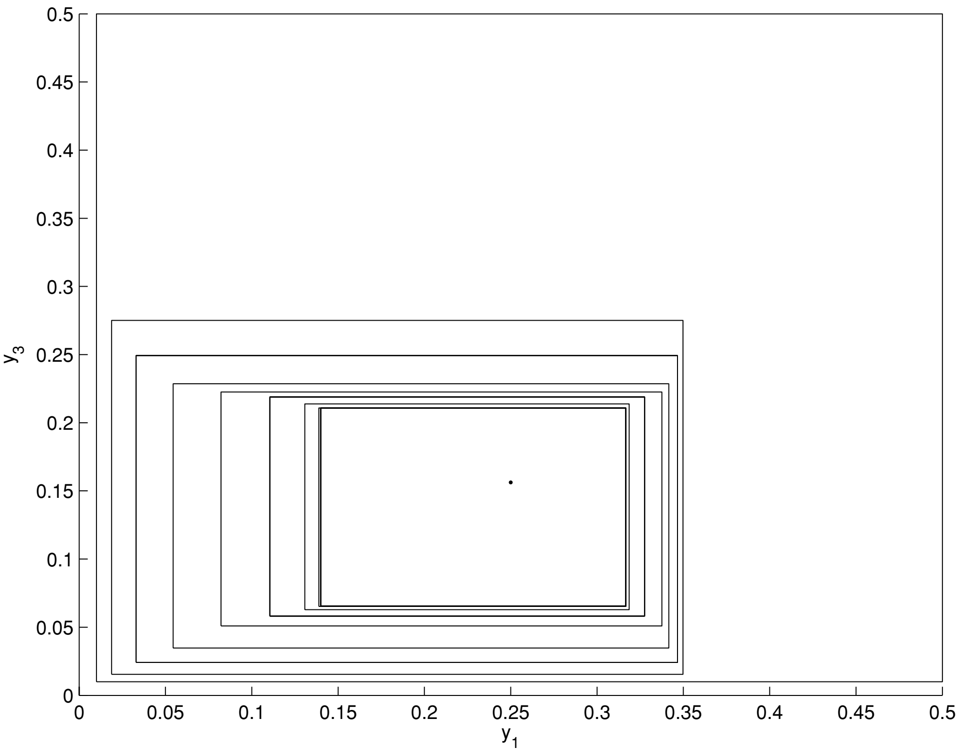

FIGURE 6.15. Sequence of boundaries, along the section $y_2 = 0.203$ (Discussion 1).

To quantify the "goodness" of such polygonal approximations we can compare the value functions obtained with and without the approximation. Since utility is just a numeraire, before we compare two value functions we need to normalize the utility function that defines the value function. The normalized utility function will derive a unit utility when one dollar is consumed. Normalization can be done by simply adding a constant to the utility function, as shown below. Defining the normalized utility function as $\bar{u}(c) = u(c) + K_u$, where $K_u = (1 - \frac{1}{\gamma})$ and 1 for the power utility and the log utility respectively, the normalized value function becomes

$$\bar{v}(x, y) = \sup_{(c,L,U)\in\mathcal{U}} \mathbf{E}_{x,y} \int_0^\infty e^{-\theta t}(u(c) + K_u)\, dt \tag{6.2}$$

$$= \sup_{(c,L,U)\in\mathcal{U}} \mathbf{E}_{x,y} \int_0^\infty e^{-\theta t} u(c)\, dt + \int_0^\infty e^{-\theta t} K_u\, dt \tag{6.3}$$

$$= v(x, y) + K_v, \tag{6.4}$$

where $K_v = \frac{\gamma - 1}{\gamma\lambda}$ for the power utility and $K_v = \frac{1}{\lambda}$ for the log utility.

For the case of N independent stocks, restricting our search for optimal policy to the space of regions of inaction bounded by hypercuboids reduces the problems to a search for $2\,N$ intercepts (each being a $N - 1$ dimensional vector). To see if this is a good approximation in two dimensions, we ran 60 experimental runs for 20 different values of transaction costs, variances and risk-aversion coefficients for the parameters in

Discussions 3, 4, and 5 respectively and found the best rectangular approximation of the region of inaction. These runs indicate that, for the two-stock case, restricting the search to rectangles decreases the normalized value function by $10^{-2}\%$ on average. This suggests that the regions of inaction might actually be rectangles when stocks are independent.

Similarly, we found the best quadrilateral approximator for 20 different values of transactions costs and risk-aversion, covariances, and other parameters as in Discussions 7a and 7b. We see a order $10^{-2}\%$ decrease in the value function for the case of nonindependent stocks when the search is restricted to quadrilateral regions of inaction, suggesting that quadrilateral regions are good approximations for two stocks. Further it suggests that $2N$ hyperpolygons are good approximations for the N-stock case.

Encouraged by the result in approximating the region of inaction by polyhedra, we evaluate the quality of this approximation in three dimensions. For the parameter set listed in Subsection 6.9, we calculated the value function with and without the approximation with various transaction costs. The transaction costs used in the test were $\lambda_i, \mu_i = \hat{\lambda}$ for $i = 1, 2, 3$ and $\hat{\lambda}$ taking values depicted in Figure 6.3. We observe that the decrease in value function due to approximating Ω by cuboids is still of order 10^{-2} %, confirming the "goodness" of the approximation. The enormous growth in the complexity of the scheme with dimensions forces us to limit our analysis to the three-stock case.

For the parameter case considered in Discussion 3, Figure 6.16 shows the optimal consumption. When no transaction costs are present Merton's consumption policy, for this parameter set, is to consume 9.14% of the wealth. Figure 6.16 is calculated using a 1% transaction cost. It is interesting to note that the consumption is roughly 9.10%, which does not differ by much from the consumption under zero transaction costs. So, in some sense, we would be justified in saying that in the presence of transaction costs, one stands to gain much more by finding the optimal region of inaction than by finding the optimal consumption. Figure 6.16 also suggests that an affine approximation for consumption might be a good approximation. An affine approximation for c of the form $c = K_1^c w + K_2^c$ reduces the search for c to a search for two scalar constants K_1^c and K_2^c. The next obvious question is, how much in value would we lose if we were to restrict consumption to be affine. It turns out again from the set of experimental parameters that on an average we

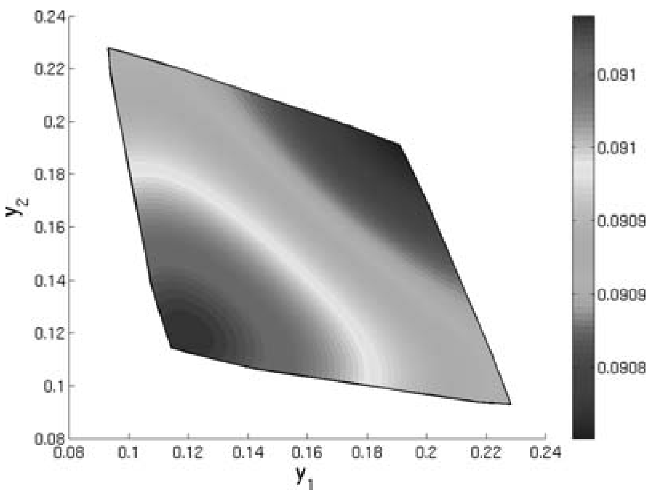

FIGURE 6.16. Contour plot of optimal consumption.

see a loss of 1% in value for 20 experiments done with two stocks. This does suggest that optimal consumption is not affine but approximating consumption by an affine function may be justified for the N-stock case.

7. CONCLUSIONS

In this work we have provided an efficient computational scheme that solves the portfolio optimization problem (Merton 1969, 1971) in the presence of transaction costs. We have stated a theorem that provides a theoretical basis for the computational scheme for the one-stock case. We have computed the shapes of the region of inaction for various parameter sets for the two-stock problem, including the correlated cases. Though the theory for multiple-stock case is incomplete, our results are the state of the art for the multiple correlated stock case. We have corroborated our results with asymptotics (Ostrov and Goodman 2004). We have also provided intuition for the structure of optimal policies in the two-stock case, using an exhaustive set of computational results. We have provided and evaluated some heuristics approximations that we hope will serve as the basis for building a tool for literal implementation.

This work opens up many directions for future research. Arguably the most exciting direction would be using approximations with proper justification and building a tool that can compute optimal policies for, say, a 10-stock problem. This would bring the literature in the area of portfolio optimization with transaction costs to a point where it can provide fund managers a tool that can be used for calibration and scientific decision making.

Regarding theory, the justification of the computational scheme in N dimensions needs to be addressed. Apart from the justification of the scheme in multiple dimensions, the exhaustive set of computational results in this paper open up a large set of potential theoretical results that could be established by future research.

The basic idea behind the procedure developed in this paper is in converting a free boundary problem to a sequence of fixed boundary problems and using an innovative update procedure. We do strongly believe that this idea is much more powerful and can be adapted to various other multidimensional models and problems where it is not possible to use techniques/transformations to reduce dimensionality to one.

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