1965: "There is a remote analogy between Hilbert's Lemma II and my Theorem 12.2 (*1940*, p. 54) for $\alpha = 0$. There is, however, this great difference that Hilbert considers only strictly constructive definitions and, moreover, transfinite iterations of the defining operations only up to constructive ordinals, while I admit, not only quantifiers in the definitions, but also iterations of the defining operations up to *any* ordinal number, no matter whether or how it can be defined. The term 'constructible set', in my proof, is justified only in a very weak sense and, in particular, only in the sense of 'relative to ordinal numbers', where the latter are subject to no conditions of constructivity. It was exactly by viewing the situation from this highly transfinite, set-theoretic point of view that in my approach the difficulties were overcome and a *relative* finitary consistency proof was obtained. Of course there is no need in this approach for anything like Hilbert's Lemma I. Hilbert probably hoped to prove it as a special case of a general theorem to the effect that transfinite modes of inference applied to a constructively correct system of axioms lead to no inconsistency."

Hilbert's paper gave an impulse to the study of the hierarchy of number-theoretic functions and to that of the various schemas for the recursive definitions of functions. In particular, Hilbert's work provides an approach to the problem of associating ordinals with number-theoretic functions defined by recursions (see *Kleene 1958* and *Péter 1951a*, *1953*).

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Weierstrass, through a critique elaborated with the sagacity of a master, created a firm foundation for mathematical analysis. By clarifying, among other notions, those of minimum, function, and derivative, he removed the remaining flaws from the calculus, cleansed it of all vague ideas concerning the infinitesimal, and conclusively overcame the difficulties that until then had their roots in the notion of infinitesimal. If today there is complete agreement and certitude in analysis whenever modes of inference are employed that rest upon the notions of irrational number and of limit in general, and if there is unanimity on all results concerning the most complicated questions in the theory of differential and integral equations, despite the boldest use of the most diverse combinations of superposition, juxtaposition, and nesting of limits, this is essentially due to the scientific activity of Weierstrass.

Nevertheless, the discussions about the foundations of analysis did not come to an end when Weierstrass provided a foundation for the infinitesimal calculus.

The reason for this is that the significance of the *infinite* for mathematics had not yet been completely clarified. To be sure, the infinitely small and the infinitely large were eliminated from analysis, as established by Weierstrass, through a reduction of the propositions about them to [[propositions about]] relations between finite magnitudes. But the infinite still appears in the infinite number sequences that define the real numbers, and, further, in the notion of the real number system, which we conceive to be an actually given totality, complete and closed.

The forms of logical inference in which this conception finds its expression—namely, those that we employ when, for example, we deal with *all* real numbers having a certain property or assert that *there exist* real numbers having a certain property—are called upon quite without restriction and are used again and again by Weierstrass precisely when he is establishing the foundations of analysis.

Thus the infinite, in a disguised form, was able to worm its way back into Weierstrass' theory and escape the sharp edge of his critique; therefore it is the *problem of the infinite* in the sense just indicated that still needs to be conclusively resolved. And just as the infinite, in the sense of the infinitely small and the infinitely large, could, in the case of the limiting processes of the infinitesimal calculus, be shown to be a mere way of speaking, so we must recognize that the infinite in the sense of the infinite totality (wherever we still come upon it in the modes of inference) is something merely apparent. And just as operations with the infinitely small were replaced by processes in the finite that have quite the same results and lead to quite the same elegant formal relations, so the modes of inference employing the infinite must be replaced generally by finite processes that have precisely the same results, that is, that permit us to carry out proofs along the same lines and to use the same methods of obtaining formulas and theorems.

That, then, is the purpose of my theory. Its aim is to endow mathematical method with the definitive reliability that the critical era of the infinitesimal calculus did not achieve; thus it shall bring to completion what Weierstrass, in providing a foundation for analysis, endeavored to do and toward which he took the first necessary and essential step.

But in clarifying the notion of the infinite we must still take into consideration a more general aspect of the question. If we pay close attention, we find that the literature of mathematics is replete with absurdities and inanities, which can usually be blamed on the infinite. So, for example, some stress the stipulation, as a kind of restrictive condition, that, if mathematics is to be rigorous, only a *finite* number of inferences is admissible in a proof—as if anyone had ever succeeded in carrying out an infinite number of them!

Even old objections that have long been regarded as settled reappear in a new guise. So in recent times we come upon statements like this: even if we could introduce a notion safely (that is, without generating contradictions) and if this were demonstrated, we would still not have established that we are justified in introducing the notion. Is this not precisely the same objection as the one formerly made against complex numbers, when it was said that one could not, to be sure, obtain a contradiction by means of them, but their introduction was nevertheless not justified, for, after all, imaginary magnitudes do not exist? No, if justifying a procedure means anything more than proving its consistency, it can only mean determining whether the procedure is successful in fulfilling its purpose. Indeed, success is necessary; here, too, it is the highest tribunal, to which everyone submits.

Another author seems to see contradictions, like ghosts, even when nothing has been asserted by anyone at all, namely, in the concrete world of perception ⟦Sinnenwelt⟧ itself, whose "consistent functioning" is regarded as a special assumption. I, for one, have always believed that only assertions and, insofar as they lead to assertions by means of inferences, assumptions could contradict each other, and the view that facts and events themselves could come to do so seems to me the perfect example of an inanity.

By these remarks I wanted to show only that the definitive clarification of the

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nature of the infinite has become necessary, not merely for the special interests of the individual sciences, but rather for the *honor of the human understanding* itself.

The infinite has always stirred the *emotions* of mankind more deeply than any other question; the infinite has stimulated and fertilized reason as few other *ideas* have; but also the infinite, more than any other *notion*, is in need of *clarification*.

If we now turn to this task, to the clarification of the nature of the infinite, we must ever so briefly call to mind the contentual significance that attaches to the infinite in reality; first we see what we can learn about this from physics.

The initial, naive impression that we have of natural events and of matter is one of uniformity, of continuity. If we have a piece of metal or a volume of liquid, the idea impresses itself upon us that it is divisible without limit, that any part of it, however small, would again have the same properties. But, wherever the methods of research in the physics of matter were refined sufficiently, limits to divisibility were reached that are not due to the inadequacy of our experiments but to the nature of the subject matter, so that we could in fact view the trend of modern science as an emancipation from the infinitely small and, instead of the old maxim "natura non facit saltus", now assert the opposite, "nature makes leaps".

As is well known, all matter is composed of small building blocks, *atoms*, which, when combined and connected, yield the entire multiplicity of macroscopic substances.

But physics did not stop at the atomic theory of matter. Toward the end of the last century the atomic theory of electricity, which at first seemed much stranger, took its place beside that theory. Whereas until that time electricity had been considered a fluid and had been the very model of an agent with a continuous effect, it too now proved to be made up of particles, namely, positive and negative *electrons*.

Besides matter and electricity there is in physics still something else that is real, for which the law of conservation also holds, namely, energy. Now not even energy, as we know today, permits of infinite division in an absolute and unrestricted way; Planck discovered that energy comes in *quanta*.

And the net result is, certainly, that we do not find anywhere in reality a homogeneous continuum that permits of continued division and hence would realize the infinite in the small. The infinite divisibility of a continuum is an operation that is present only in our thoughts; it is merely an idea, which is refuted by our observations of nature and by the experience gained in physics and chemistry.

We find the second place at which the question of infinity confronts us in nature when we consider the universe as a whole. Here we must investigate the vast expanse of the universe to see whether there is something infinitely large in it.

For a long time the opinion that the world is infinite was dominant; until the time of Kant and even afterward no one had entertained any doubt whatsoever about the infinitude of space.

Here again it is modern science, especially astronomy, that raises this question anew and seeks to decide it, not by the inadequate means of metaphysical speculation, but through reasons that are supported by experience and rest upon the application of the laws of nature. And weighty objections against infinity have appeared. *Euclidean* geometry necessarily leads to the assumption that space is infinite. Now, to be sure, Euclidean geometry, as a structure and a system of notions, is consistent in

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itself, but this does not imply that it applies to reality. Whether that is the case, only observation and experience can decide. In the attempt to prove the infinitude of space in a speculative way, moreover, obvious errors were committed. From the fact that outside of a region of space there always is still more space it follows only that space is unbounded but by no means that it is infinite. Unboundedness and finitude, however, do not exclude each other. In the geometry usually referred to as *elliptic*, mathematical research furnishes the natural model of a finite world. And the abandonment of Euclidean geometry is today no longer merely a purely mathematical or philosophical speculation; rather, we have come to abandon it also on account of other considerations, which originally had nothing at all to do with the question of the finitude of the world. Einstein showed that it was necessary to relinquish Euclidean geometry. On the basis of his theory of gravitation he attacks the cosmological questions and shows that a finite world is possible, and all the results discovered by astronomers are compatible also with the assumption of an elliptic world.

We have now ascertained in two directions, toward the infinitely small and toward the infinitely large, that reality is finite. Yet it could very well be the case that the infinite has a well-justified place *in our thinking* and plays the role of an indispensable notion. We shall examine what the situation in the science of mathematics is in this respect, and we shall first consult the purest and most naive child of the human intellect, the theory of numbers. Let us here select any formula from the rich multitude of elementary formulas, for example,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1).$$

Since in it we may substitute any integer for n, for example, $n = 2$ or $n = 5$, this formula contains *infinitely many* propositions, and this is obviously what is essential about it; that is why it constitutes the solution of an arithmetic problem and why its proof requires a genuine act of thought, whereas each of the specific numerical equations

$$1^2 + 2^2 = \tfrac{1}{6}\cdot 2\cdot 3\cdot 5$$

and

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \tfrac{1}{6}\cdot 5\cdot 6\cdot 11$$

can be verified by computation and hence is of no essential interest when considered by itself.

We come upon quite another, wholly different interpretation, or fundamental characterization, of the notion of infinity when we consider the method—so extremely important and fertile—of *ideal elements*. The method of ideal elements has an application already in the elementary geometry of the plane. There the points and straight lines of the plane are initially the only real, actually existing objects. The axiom of connection, among others, holds for them: for any two points there is always one and only one straight line that goes through them. From this it follows that two straight lines intersect each other in one point at most. The proposition that any two straight lines intersect each other in some point, however, does not hold; rather, the two lines can be parallel. But, as is well known, the introduction of ideal elements, namely, points at infinity and a line at infinity, renders the proposition according to which two straight lines always intersect each other in one and only one point universally valid.

The ideal elements "at infinity" have the advantage of making the system of the

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laws of connection as simple and perspicuous as is at all possible. As is well known, the symmetry between point and straight line then yields the duality principle of geometry, which is so fecund.

The ordinary *complex* magnitudes of algebra likewise are an instance of the use of ideal elements; they serve to simplify the theorems on the existence and number of the roots of an equation.

Just as in geometry infinitely many straight lines, namely, those that are parallel to one another, are used to define an ideal point, so in higher arithmetic certain systems of infinitely many numbers are combined into a *number ideal*, and indeed probably no use of the principle of ideal elements is a greater stroke of genius than this. When this procedure has been carried out generally within an algebraic field, we find in it again the simple and well-known laws of divisibility, just as they hold for the ordinary integers 1, 2, 3, 4, ... Here we have already entered the domain of higher arithmetic.

Now we come to analysis, the structure that in mathematical science is the most elaborate and has branched out more delicately than any other. You know what a dominant role the infinite plays there, how in a sense mathematical analysis is but a single symphony of the infinite.

The mighty advances made in the infinitesimal calculus rest for the most part upon operations with mathematical systems of infinitely many elements. Since, now, it was extremely tempting to identify the infinite with the "very large", there soon arose inconsistencies, the paradoxes of the infinitesimal calculus, as they are called, which in part were already known to the Sophists in antiquity. A fundamental advance was made when it was recognized that many propositions valid for the finite—for example, that the part is smaller than the whole, that a minimum or a maximum exists, that the order of terms or factors can be changed—may not be directly carried over to the infinite. At the beginning of my lecture I mentioned the fact that, chiefly through the sagacity of Weierstrass, these questions have been completely clarified, and today analysis has within its domain become an infallible guide and at the same time a practical instrument for the use of the infinite.

But analysis alone does not yet give us the deepest insight into the nature of the infinite. Rather, this is conveyed to us only by a discipline that is closer to the general philosophical way of thinking and was destined to place the entire complex of questions concerning the infinite in a new light. This discipline is set theory, whose creator was Georg Cantor. Here, however, we are concerned only with what was truly unique and original in Cantor's theory and constituted its real core, namely, his theory of *transfinite numbers*. This appears to me to be the most admirable flower of the mathematical intellect and in general one of the highest achievements of purely rational human activity. Now what is it all about?

If we wanted to characterize briefly the new conception of the infinite that Cantor introduced, we could no doubt say: in analysis we deal with the infinitely small or the infinitely large only as a limit notion—as something that is becoming, coming to be, being produced—that is, as we say, with the *potential infinite*. But this is not the real infinite itself. That we have when, for example, we consider the totality of the numbers 1, 2, 3, 4,... itself as a completed entity, or when we regard the points of a line segment as a totality of objects that is actually given and complete. This kind of infinite is called the *actual infinite*.

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Frege and Dedekind, two mathematicians who did highly meritorious work in the foundations of mathematics, already used—independently of each other—the actual infinite. Their specific aim was to make pure logic provide for arithmetic a foundation that would be independent of all intuition and experience as well as to derive arithmetic by means of logic alone. Dedekind even went so far as to refuse to draw upon intuition for the notion of finite number; instead, he strove to derive it by purely logical means, making essential use of the notion of infinite sets. It was Cantor, however, who systematically developed the notion of the actual infinite. If we look at the two examples of the infinite that we have mentioned, (1) $1, 2, 3, 4, \ldots$ and (2) the points of the line segment from 0 to 1, or, what is the same, the totality of real numbers between 0 and 1, then the idea that suggests itself most readily is to consider them purely from the point of view of cardinality, and when we do this we observe surprising facts that are familiar to every mathematician today. For, if we consider the set of all rational numbers, hence of all fractions $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \ldots, \frac{3}{7}, \ldots$, it turns out that, purely from the point of view of cardinality, this set is not larger than the set of integers; we say that the rational numbers can be denumerated in the ordinary way, or that they are denumerable. And the same still holds of the set of all numbers that can be obtained [[from the rational numbers]] by root extraction, and indeed of the set of all algebraic numbers. In our second example we have a similar situation: unexpectedly, the set of all points in a square or a cube is, purely from the point of view of cardinality, not larger than the set of points on the line segment from 0 to 1; indeed, the same still holds even of the set of all continuous functions. Someone who first learns of this might come to think that purely from the point of view of cardinality there is but a single infinite. No, the sets in our two examples, (1) and (2), are not "equivalent", as we say. Rather, the set in (2) cannot be denumerated; it is larger than the set in (1). Here Cantor's ideas take their distinctive turn. The points of the line segment cannot be denumerated in the ordinary way by means of $1, 2, 3, \ldots$! But, if we admit the actual infinite, we are not at all limited to this ordinary kind of denumeration or in any way compelled to leave off there. Rather, when we have counted $1, 2, 3, \ldots$, we can view the objects thus denumerated as an infinite set that has been completed in this definite order; if we denote the type of this ordering, as Cantor does, by ω, then the denumeration continues in a natural way with $\omega + 1, \omega + 2, \ldots$ to $\omega + \omega$, or $\omega \cdot 2$, and then again $\omega \cdot 2 + 1, \omega \cdot 2 + 2, \omega \cdot 2 + 3, \ldots, \omega \cdot 2 + \omega\ (= \omega \cdot 3)$, and further $\omega \cdot 2, \omega \cdot 3, \omega \cdot 4, \ldots, \omega \cdot \omega\ (= \omega^2), \omega^2 + 1, \ldots$, so that we finally obtain the following table:

$$
\begin{array}{l}
1, 2, 3, \ldots, \\
\omega, \omega + 1, \omega + 2, \ldots, \\
\omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots, \\
\omega \cdot 3, \omega \cdot 3 + 1, \omega \cdot 3 + 2, \ldots, \\
\omega^2, \omega^2 + 1, \ldots, \\
\omega^2 + \omega, \omega^2 + \omega \cdot 2, \omega^2 + \omega \cdot 3, \ldots, \\
\omega^2 \cdot 2, \ldots, \\
\omega^2 \cdot 2 + \omega, \ldots, \\
\omega^3, \ldots, \\
\omega^4, \ldots, \\
\omega^\omega, \ldots.
\end{array}
$$

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These are Cantor's first transfinite numbers, the numbers of the second number class, as he calls them.[1] Thus we come to them simply by a transnumeration [[Hinüberzählen]] beyond the ordinary denumerable infinite, that is, by an entirely natural and unambiguously determined, systematic continuation of ordinary counting as it takes place in the finite. Just as till now we merely counted the 1st, 2nd, 3rd, . . . object of a set, so we now also count the ωth, $(\omega + 1)$th, . . . , ω^{ω}th object.

Given this state of affairs, the question obviously immediately arises whether by means of this transfinite counting one could now actually enumerate the elements of sets that are not denumerable in the ordinary sense.

Now Cantor, in following these thoughts, developed the theory of transfinite numbers in a most successful way and created a complete calculus for them. So, finally, through the gigantic collaboration of Frege, Dedekind, and Cantor the infinite was enthroned and enjoyed the period of its greatest triumph. In the boldest flight the infinite had reached a dizzy pinnacle of success.

The reaction did not fail to set in; it took very dramatic forms. Events took quite the same turn as in the development of the infinitesimal calculus. In their joy over the new and rich results, mathematicians apparently had not examined critically enough whether the modes of inference employed were admissible; for, purely through the ways in which notions were formed and modes of inference used—ways that in time had become customary—contradictions appeared, sporadically at first, then ever more severely and ominously. They were the paradoxes of set theory, as they are called. In particular, a contradiction discovered by Zermelo and Russell had, when it became known, a downright catastrophic effect in the world of mathematics. Confronted with these paradoxes, Dedekind and Frege actually abandoned their standpoint and quit the field; for a long time Dedekind had reservations about permitting a new edition of his epoch-making booklet (*1888*), and Frege, too, was forced to recognize that the tendency of his book (*1893*, *1903*) was mistaken, as he confesses in an appendix. From the most diverse quarters extremely vehement attacks were directed against Cantor's theory itself. The reaction was so violent that the commonest and most fruitful notions and the very simplest and most important modes of inference in mathematics were threatened and their use was to be prohibited. There were, to be sure, defenders of the old; but the defensive measures were rather feeble, and moreover they were not put into effect at the right place in a unified front. Too many remedies were recommended for the paradoxes; the methods of clarification were too checkered.

Let us admit that the situation in which we presently find ourselves with respect to the paradoxes is in the long run intolerable. Just think: in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails?

But there is a completely satisfactory way of escaping the paradoxes without committing treason against our science. The considerations that lead us to discover this way and the goals toward which we want to advance are these:

(1) We shall carefully investigate those ways of forming notions and those modes

[1] [[Unlike Cantor, Hilbert takes the number classes to be cumulative; see his definition of the second number class below, p. 386.]]

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of inference that are fruitful; we shall nurse them, support them, and make them usable, wherever there is the slightest promise of success. No one shall be able to drive us from the paradise that Cantor created for us.

(2) It is necessary to make inferences everywhere as reliable as they are in ordinary elementary number theory, which no one questions and in which contradictions and paradoxes arise only through our carelessness.

Obviously we shall be able to reach these goals only if we succeed in completely clarifying *the nature of the infinite.*

We saw earlier that the infinite is not to be found anywhere in reality, no matter what experiences and observations or what kind of science we may adduce. Could it be, then, that thinking about objects is so unlike the events involving objects and that it proceeds so differently, so apart from all reality? Is it not clear, rather, that when we believed we had discovered that the infinite was in some sense real we were only allowing ourselves to be led to that belief by the circumstance that we so often actually encounter in reality such immeasurable dimensions in the large and in the small? And has the contentual logical inference ever deceived and abandoned us anywhere when we applied it to real objects or events? No, contentual logical inference is indispensable. It has deceived us only when we accepted arbitrary abstract notions, in particular those under which infinitely many objects are subsumed. What we did, then, was merely to use contentual inference in an illegitimate way; that is, we obviously did not respect necessary conditions for the use of contentual logical inference. And in recognizing that such conditions exist and must be respected we find ourselves in agreement with the philosophers, especially with Kant. Kant already taught—and indeed it is part and parcel of his doctrine—that mathematics has at its disposal a content secured independently of all logic and hence can never be provided with a foundation by means of logic alone; that is why the efforts of Frege and Dedekind were bound to fail. Rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation [[in der Vorstellung]], certain extralogical concrete objects that are intuitively [[anschaulich]] present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.

Let us call to mind the nature and methods of ordinary finitary number theory. It can certainly be developed through the construction of numbers by means solely of intuitive contentual considerations. But the science of mathematics is by no means exhausted by numerical equations and it cannot be reduced to these alone. One can claim, however, that it is an apparatus that must always yield correct numerical equations when applied to integers. But then we are obliged to investigate the structure of the apparatus sufficiently to make this fact apparent. And the only tool at our

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disposal in this investigation is the same as that used for the derivation of numerical equations in the construction of number theory itself, namely, a concern for concrete content, the finitist frame of mind. This scientific requirement can in fact be satisfied; that is, it is possible to obtain in a purely intuitive and finitary way, just like the truths of number theory, those insights that guarantee the reliability of the mathematical apparatus. Let us now consider number theory in more detail.

In number theory we have the numerals

$$1, 11, 111, 11111,$$

each numeral being perceptually recognizable by the fact that in it 1 is always again followed by 1 [[if it is followed by anything]]. These numerals, which are the object of our consideration, have no meaning at all in themselves. In elementary number theory, however, we already require, besides these signs, others that mean something and serve to convey information, for example, the sign[2] 2 as an abbreviation for the numeral 11, or the numeral 3 as an abbreviation for the numeral 111; further we use the signs $+$, $=$, $>$, and others, which serve to communicate assertions. So $2 + 3 = 3 + 2$ serves to communicate the fact that $2 + 3$ and $3 + 2$, when the abbreviations used are taken into account, are the same numeral, namely, the numeral 11111. Likewise, then, $3 > 2$ serves to communicate the fact that the sign 3 (that is, 111) extends beyond the sign 2 (that is, 11), or that the latter sign is a proper segment of the former.

When communicating, we also use letters, such as $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, for numerals. Accordingly, $\mathfrak{b} > \mathfrak{a}$ is the communication that the numeral $\mathfrak{b}$ extends beyond the numeral $\mathfrak{a}$. And likewise, from the present point of view, we would regard $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$ merely as the communication of the fact that the numeral $\mathfrak{a} + \mathfrak{b}$ is the same as $\mathfrak{b} + \mathfrak{a}$. Here, too, the contentual correctness of this communication can be proved by contentual inference, and we can go very far with this intuitive, contentual kind of treatment.

I should now like to show you a first example in which we go beyond the bounds of this contentual way of thinking. The largest prime number known up to now (39 digits) is

$$\mathfrak{p} = 170\ 141\ 183\ 460\ 469\ 231\ 731\ 687\ 303\ 715\ 884\ 105\ 727.$$

By means of Euclid's well-known procedure we can, completely within the framework of the attitude we have adopted, prove the theorem that between $\mathfrak{p} + 1$ and $\mathfrak{p}! + 1$ there certainly exists a new prime number. This proposition itself, moreover, is completely in conformity with our finitist attitude. For "there exists" here serves merely to abbreviate the proposition:

Certainly $\mathfrak{p} + 1$ or $\mathfrak{p} + 2$ or $\mathfrak{p} + 3$ or $\ldots$ or $\mathfrak{p}! + 1$ is a prime number.

But let us go on. Obviously, to say

There exists a prime number that (1) is $> \mathfrak{p}$ and (2) is at the same time $\leqq \mathfrak{p}! + 1$

[2] [[In the present paragraph and the next, "Zeichen" is uniformly translated as "sign" and "Zahlzeichen" as "numeral", although Hilbert on occasion uses the former as a short form for the latter; the *Zahlzeichen* are, of course, included among the *Zeichen*. Furthermore, when Hilbert uses "Abkürzung", here rendered as "abbreviation", he wavers between two meanings: pure abbreviation of a symbol and name of that symbol.]]

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would amount to the same thing, and this leads us to formulate a proposition that expresses only a part of Euclid's assertion, namely: there exists a prime number that is $>\mathfrak{p}$. So far as content is concerned, this is a much weaker assertion, stating only a part of Euclid's proposition; nevertheless, no matter how harmless the transition appears to be, there is a leap into the transfinite when this partial proposition, taken out of the context above, is stated as an independent assertion.

How can that be? We have here an existential proposition with "there exists". To be sure, we already had one in Euclid's theorem. But the latter, with its "there exists", was, as I have already said, merely another, shorter expression for

"$\mathfrak{p} + 1$ or $\mathfrak{p} + 2$ or $\mathfrak{p} + 3$ or ... or $\mathfrak{p}! + 1$ is a prime number",

just as, instead of saying: This piece of chalk is red or that piece of chalk is red or... or the piece of chalk over there is red, I say more briefly: Among these pieces of chalk there exists a red one. An assertion of this kind, that in a finite totality "there exists" an object having a certain property, is completely in conformity with our finitist attitude. On the other hand, the expression

"$\mathfrak{p} + 1$ or $\mathfrak{p} + 2$ or $\mathfrak{p} + 3$ or ... ad infinitum is a prime number"

is, as it were, an infinite logical product,[3] and such a passage to the infinite is no more permitted without special investigation and perhaps certain precautionary measures than the passage from a finite to an infinite product in analysis, and initially it has no meaning at all.

In general, from the finitist point of view an existential proposition of the form "There exists a number having this or that property" has meaning only as a *partial proposition*, that is, as part of a proposition that is more precisely determined but whose exact content is unessential for many applications.

Thus we here encounter the transfinite when from an existential proposition we extract a partial proposition that cannot be regarded as a disjunction.[4] In like manner we come upon a transfinite proposition when we negate a universal assertion, that is, one that extends to arbitrary numerals. So, for example, the proposition that, if $\mathfrak{a}$ is a numeral, we must always have

$$\mathfrak{a} + 1 = 1 + \mathfrak{a}$$

is from the finitist point of view *incapable of being negated.* This will become clear for us if we reflect upon the fact that [from this point of view] the proposition cannot be interpreted as a combination, formed by means of "and", of infinitely many numerical equations, but only as a hypothetical judgment that comes to assert something when a numeral is given.

From this it follows, in particular, that in the spirit of the finitist attitude we

[3] [[It is rather a logical sum, or disjunction. In the version published in *Grundlagen der Geometrie* (*1930a*) "logisches Produkt" is replaced by "Oder-Verknüpfung".]]

[4] [[The German text says: "Wir stoßen also hier auf das Transfinite durch Zerlegung einer existentialen Aussage, die sich nicht als eine Oder-Verknüpfung deuten läßt". In the version published in *Grundlagen der Geometrie* (*1930a*) this sentence has been amended thus: "Wir stoßen also hier auf das Transfinite durch Zerlegung einer existentialen Aussage in Teile, deren keiner sich als eine Oder-Verknüpfung deuten läßt". The English translation that we propose seems to be closer to what Hilbert's train of thought requires than either of these two German sentences.]]

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cannot make use of the alternative according to which an equation like the one above, in which an unspecified numeral occurs, is either satisfied by every numeral or refuted by a counterexample. For, being an application of the principle of excluded middle, this alternative essentially rests upon the assumption that the assertion of the validity of that equation is capable of being negated.

At all events we observe the following. In the domain of finitary propositions, in which we should, after all, remain, the logical relations that prevail are very imperspicuous, and this lack of perspicuity mounts unbearably if "all" and "there exists" occur combined or appear in nested propositions. In any case, those logical laws that man has always used since he began to think, the very ones that Aristotle taught, do not hold. Now one could attempt to determine the logical laws that are valid for the domain of finitary propositions; but this would not help us, since we just do not want to renounce the use of the simple laws of Aristotelian logic, and no one, though he speak with the tongues of angels, will keep people from negating arbitrary assertions, forming partial judgments, or using the principle of excluded middle. What, then, shall we do?

Let us remember that *we are mathematicians* and as such have already often found ourselves in a similar predicament, and let us recall how the method of ideal elements, that creation of genius, then allowed us to find an escape. I presented some shining examples of the use of this method at the beginning of my lecture. Just as $i = \sqrt{-1}$ was introduced so that the laws of algebra, those, for example, concerning the existence and number of the roots of an equation, could be preserved in their simplest form, just as ideal factors were introduced so that the simple laws of divisibility could be maintained even for algebraic integers (for example, we introduce an ideal common divisor for the numbers 2 and $1 + \sqrt{-5}$, while an actual one does not exist), so we must here *adjoin the ideal propositions to the finitary ones* in order to maintain the formally simple rules of ordinary Aristotelian logic. And it is strange that the modes of inference that Kronecker attacked so passionately are the exact counterpart of what, when it came to number theory, the same Kronecker admired so enthusiastically in Kummer's work and praised as the highest mathematical achievement.

How, then, do we come to the *ideal propositions*? It is a remarkable circumstance, and certainly a propitious and favorable one, that to enter the path that leads to them we need only continue in a natural and consistent way the development that the theory of the foundations of mathematics has already taken. Indeed, let us acknowledge that elementary mathematics already goes beyond the point of view of intuitive number theory. For the method of algebraic calculation with letters is not within the resources of contentual, intuitive number theory as we have hitherto conceived of it. This theory always uses formulas for communication only; letters stand for numerals, and the fact that two signs are identical is communicated by an equation. In algebra, on the other hand, we consider the expressions formed with letters to be independent objects in themselves, and the contentual propositions of number theory are formalized by means of them. Where we had propositions concerning numerals, we now have formulas, which themselves are concrete objects that in their turn are considered by our perceptual intuition, and the derivation of one formula from another in accordance with certain rules takes the place of the number-theoretic proof based on content.

Hence, as soon as we consider algebra, there is an increase in the number of finitary objects. Up to now these were only the numerals, such as 1, 11, . . . , 11111. They alone had been the objects of our contentual consideration. But in algebra mathematical practice already goes beyond that. Yes, even when a proposition, so long as it is combined with some indication as to its contentual interpretation, is still admissible from our finitist point of view, as, for example, the proposition that always

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a},$$

where $\mathfrak{a}$ and $\mathfrak{b}$ stand for specific numerals, we yet do not select this form of communication but rather take the formula

$$a + b = b + a.$$

This is no longer an immediate communication of something contentual at all, but a certain formal object, which is related to the original finitary propositions

$$2 + 3 = 3 + 2$$

and

$$5 + 7 = 7 + 5$$

by the fact that, if we substitute numerals, 2, 3, 5, and 7, for a and b in that formula (that is, if we employ a proof procedure, albeit a very simple one), we obtain these finitary particular propositions. Thus we arrive at the conception that a, b, $=$, and $+$, as well as the entire formula

$$a + b = b + \mathrm{a},$$

do not mean anything in themselves, any more than numerals do. But from that formula we can indeed derive others; to these we ascribe a meaning, by treating them as communications of finitary propositions. If we generalize this conception, mathematics becomes an inventory of formulas—first, formulas to which contentual communications of finitary propositions [hence, in the main, numerical equations and inequalities] correspond and, second, further formulas that mean nothing in themselves and are the *ideal objects of our theory.*

Now, what was our goal? In mathematics, we found, first, finitary propositions that contain only numerals, like

$$3 > 2, \quad 2 + 3 = 3 + 2, \quad 2 = 3, \quad \text{and} \quad 1 \neq 1,$$

which according to our finitist conception are immediately intuitive and directly intelligible. These are capable of being negated, and the result will be true or false; one can manipulate them at will, without any qualms, in all the ways that Aristotelian logic allows. The law of contradiction holds; that is, it is impossible for any one of these propositions and its negation to be simultaneously true. The principle of "excluded middle" holds; that is, of the two, a proposition and its negation, one is true. To say that a proposition is false is equivalent to saying that its negation is true. Besides these elementary propositions, which are of an entirely unproblematic character, we encountered finitary propositions of problematic character, for example, those that were not decomposable [[into partial propositions]]. Now, finally, we have introduced the ideal propositions to ensure that the customary laws of logic again

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hold one and all. But since the ideal propositions, namely, the formulas, insofar as they do not express finitary assertions, do not mean anything in themselves, the logical operations cannot be applied to them in a contentual way, as they are to the finitary propositions. Hence it is necessary to formalize the logical operations and also the mathematical proofs themselves; this requires a transcription of the logical relations into formulas, so that to the mathematical signs we must still adjoin some logical signs, say

$\&$,	$\vee$,	$\rightarrow$,	$\overline{}$,
and	or	implies	not

and use, besides the mathematical variables, $a, b, c, \ldots$, also logical variables, namely, variable propositions $A, B, C, \ldots$.

How can this be done? It is our good fortune to find the same preestablished harmony here that we observe so often in the history of science, a harmony that benefited Einstein when he found the general calculus of invariants fully developed for his theory of gravitation; we discover that considerable spadework has already been done: the *logical calculus* has been developed. To be sure, it was originally created in an entirely different context, and, accordingly, its signs were initially introduced for purposes of communication only; but we will be consistent in our course if we now divest the logical signs, too, of all meaning, just as we did the mathematical ones, and declare that the formulas of the logical calculus do not mean anything in themselves either, but are ideal propositions. In the logical calculus we possess a sign language that is capable of representing mathematical propositions in formulas and of expressing logical inference through formal processes. In a way that exactly corresponds to the transition from contentual number theory to formal algebra we regard the signs and operation symbols of the logical calculus as detached from their contentual meaning. In this way we now finally obtain, in place of the contentual mathematical science that is communicated by means of ordinary language, an inventory of formulas that are formed from mathematical and logical signs and follow each other according to definite rules. Certain of these formulas correspond to the mathematical axioms, and to contentual inference there correspond the rules according to which the formulas follow each other; hence contentual inference is replaced by manipulation of signs [[äußeres Handeln]] according to rules, and in this way the full transition from a naive to a formal treatment is now accomplished, on the one hand, for the axioms themselves, which originally were naively taken to be fundamental truths but in modern axiomatics had already for a long time been regarded as merely establishing certain interrelations between notions, and, on the other, for the logical calculus, which originally was to be only another language.

Let me still explain briefly just how a *mathematical proof* is formalized. As I said, certain formulas, which serve as building blocks for the formal edifice of mathematics, are called axioms. A mathematical proof is an array that must be given as such to our perceptual intuition; it consists of inferences according to the schema

$$\begin{array}{c}\mathfrak{S}\\ \underline{\mathfrak{S}\rightarrow\mathfrak{T}}\\ \mathfrak{T},\end{array}$$

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where each of the premisses, that is, the formulas $\mathfrak{S}$ and $\mathfrak{S} \rightarrow \mathfrak{T}$ in the array, either is an axiom or results from an axiom by substitution, or else coincides with the end formula of a previous inference or results from it by substitution. A formula is said to be provable if it is the end formula of a proof.

Through our program the choice of axioms for our proof theory is already indicated. Although the choice of axioms is to a certain extent arbitrary, they nevertheless fall into a number of qualitatively distinct groups, just as in geometry; we cite a few examples from each.[5]

I. Axioms of implication:

$$A \rightarrow (B \rightarrow A) \text{ (introduction of an assumption),}$$

$$(B \rightarrow C) \rightarrow \{(A \rightarrow B) \rightarrow (A \rightarrow C)\} \text{ (elimination of a proposition).}$$

II. Axioms of negation:

$$\{A \rightarrow (B \mathbin{\&} \overline{B})\} \rightarrow \overline{A} \text{ (principle of contradiction),}$$

$$\overline{\overline{A}} \rightarrow A \text{ (principle of double negation).}$$

[From the principle of contradiction the formula

$$(A \mathbin{\&} \overline{A}) \rightarrow B$$

follows, and from the principle of double negation the principle of excluded middle,

$$\{(A \rightarrow B) \mathbin{\&} (\overline{A} \rightarrow B)\} \rightarrow B,$$

follows.] These axioms of groups I and II are nothing but the axioms of the propositional calculus.

III. Transfinite axioms:

$(a)A(a) \rightarrow A(b)$ (inference from the universal to the particular, Aristotle's dictum),

$\overline{(a)}A(a) \rightarrow (Ea)\overline{A}(a)$ (if a predicate does not hold of all individuals, then there exists a counterexample),

$\overline{(Ea)}A(a) \rightarrow (a)\overline{A}(a)$ (if there is no individual for which a proposition holds, then the proposition is false for all a).

Here, moreover, we come upon a very remarkable circumstance, namely, that all of these transfinite axioms are derivable from a single axiom, one that also contains the core of one of the most attacked axioms in the literature of mathematics, namely, the axiom of choice:

$$A(a) \rightarrow A(\varepsilon(A)),$$

where ε is the transfinite logical choice function.

In addition there are the specifically mathematical axioms.

IV. Axioms of equality:

$$a = a,$$

$$a = b \rightarrow (A(a) \rightarrow A(b)),$$

and finally

[5] [[The full system of axioms is given below, pp. 465–469.]]

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V. Axioms of number:

$$a + 1 \neq 0$$

and

$$[\{A(0) \,\&\, (x)(A(x) \rightarrow A(x'))\} \rightarrow A(a)],$$

the axiom of mathematical induction.

In this way we are able to develop our proof theory and to construct the system of provable formulas, that is, the science of mathematics.

But in our joy over the fact that we have, in general, been so successful and that, in particular, we found ready-made that indispensable tool, the logical calculus, we must nevertheless not forget the essential prerequisite of our procedure. For there is a condition, a single but absolutely necessary one, to which the use of the method of ideal elements is subject, and that is the *proof of consistency*; for, extension by the addition of ideals is legitimate only if no contradiction is thereby brought about in the old, narrower domain, that is, if the relations that result for the old objects whenever the ideal objects are eliminated are valid in the old domain.

In the present situation, however, this problem of consistency is perfectly amenable to treatment. As we can immediately recognize, it reduces to the question of seeing that "$1 \neq 1$" cannot be obtained as an end formula from our axioms by the rules in force, hence that "$1 \neq 1$" is not a provable formula. And this is a task that fundamentally lies within the province of intuition just as much as does in contentual number theory the task, say, of proving the irrationality of $\sqrt{2}$, that is, of proving that it is impossible to find two numerals $\mathfrak{a}$ and $\mathfrak{b}$ satisfying the relation $\mathfrak{a}^2 = 2\mathfrak{b}^2$, a problem in which it must be shown that it is impossible to exhibit two numerals having a certain property. Correspondingly, the point for us is to show that it is impossible to exhibit a proof of a certain kind. But a formalized proof, like a numeral, is a concrete and surveyable object. It can be communicated from beginning to end. That the end formula has the required structure, namely "$1 \neq 1$", is also a property of the proof that can be concretely ascertained. The demonstration [[that "$1 \neq 1$" is not a provable formula]] can in fact be given, and this provides us with a justification for the introduction of our ideal propositions.

At the same time we experience the pleasant surprise that this gives us the solution also of a problem that became urgent long ago, namely, that of proving *the consistency of the arithmetic axioms*.[6] For the problem of proving consistency arises wherever the axiomatic method is used. After all, in selecting, interpreting, and manipulating the axioms and rules we do not want to have to rely on good faith and pure confidence alone. In geometry and the physical theories the consistency proof is successfully carried out by means of a reduction to the consistency of the arithmetic axioms. This method obviously fails in the case of arithmetic itself. By making this important final step possible through the method of ideal elements, our proof theory forms the necessary keystone in the edifice of axiomatic theory. And what we have experienced twice, first with the paradoxes of the infinitesimal calculus and then with the paradoxes of set theory, cannot happen a third time and will never happen again.

But our proof theory as it is sketched here is not only able to secure the foundations

[6] [[In his various papers on the foundations of mathematics Hilbert means by "arithmetic axioms" at times axioms for the real number system, at times axioms for number theory.]]

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of the science of mathematics; I believe, rather, that it also opens up a path that, if we follow it, will enable us to deal for the first time with general problems of a fundamental character that fall within the domain of mathematics but formerly could not even be approached.

Mathematics in a certain sense develops into a tribunal of arbitration, a supreme court that will decide questions of principle—and on such a concrete basis that universal agreement must be attainable and all assertions can be verified.

Even the assertions of the recent doctrine called "intuitionism", modest though they may be, can in my opinion obtain their certificate of justification only from this tribunal.

As an example of the way in which fundamental questions can be treated I would like to choose the thesis that every mathematical problem can be solved. We are all convinced of that. After all, one of the things that attract us most when we apply ourselves to a mathematical problem is precisely that within us we always hear the call: here is the problem, search for the solution; you can find it by pure thought, for in mathematics there is no *ignorabimus*. Now, to be sure, my proof theory cannot specify a general method for solving every mathematical problem; that does not exist. But the demonstration that the assumption of the solvability of every mathematical problem is consistent falls entirely within the scope of our theory.

I would still like to play a last trump. The final test of every new theory is its success in answering preexistent questions that the theory was not specifically created to answer. By their fruits ye shall know them—that applies also to theories. As soon as Cantor had discovered his first transfinite numbers, the numbers of the second number class as they are called, the question arose, as I have already mentioned, whether by means of this transfinite counting one could actually enumerate the elements of sets known in other contexts but not denumerable in the ordinary sense. The line segment was the first and foremost set of this kind to come under consideration. This question, whether the points of the line segment, that is, the real numbers, can be enumerated by means of the numbers of the table constructed above, is the famous problem of the continuum, which was formulated but not solved by Cantor. Some mathematicians believed that they could dispose of this problem by denying its existence. The considerations that follow show how mistaken this attitude is. The problem of the continuum is distinguished by its originality and inner beauty; in addition it is characterized by two features that raise it above other famous problems: its solution requires new ways, since the old methods fail in its case, and, besides, this solution is in itself of the greatest interest on account of the result to be determined.

The solution of the continuum problem can be carried out by means of the theory I have developed, and indeed the first and most important step toward this solution is precisely the demonstration that every mathematical problem can be solved. The answer turns out to be affirmative: the points of a line segment can be enumerated by means of the numbers of the second number class, that is, by mere transnumeration beyond the denumerable infinite, to express it in a popular way. I should like to call this assertion itself the continuum theorem and offer a brief intuitive presentation here of the fundamental idea of its proof.

Instead of the set of real numbers we consider—which is evidently the same here —the set of number-theoretic functions, that is, of those functions of an integral

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argument whose values are also integers. If we want to order the set of these functions in the way required by the problem of the continuum, we must consider how an individual function is generated. Now a function of one argument can be defined in such a way that the values of the function for some or even all values of the argument are made to depend upon whether certain well-defined mathematical problems have a solution, for example, whether certain diophantine problems are solvable, whether prime numbers having certain properties exist, or whether a given number, say $2^{\sqrt{2}}$, is irrational. In order to avoid the difficulty inherent in this we make use of precisely the assertion mentioned above, namely, that every well-posed mathematical problem is solvable. This assertion is a general lemma belonging to *metamathematics*, as I would like to call the contentual theory of formalized proofs. To the part of the lemma that is of relevance here I now give the following precise formulation:

LEMMA I. If a proof of a proposition contradicting the continuum theorem is given in a formalized version with the aid of functions defined by means of the transfinite symbol ε (axiom group III), then in this proof these functions can always be replaced by functions defined, without the use of the symbol ε, by means merely of ordinary and transfinite recursion, so that the transfinite appears only in the guise of the universal quantifier.

Further, I need to make a few stipulations in order to carry out my theory.

For *variable* ⟦*atomic*⟧ *propositions* (indeterminate formulas) we always use capital ⟦italic⟧ Latin letters, but for *constant* ⟦*atomic*⟧ *propositions* (specific formulas) capital Greek letters; for example,

$\mathsf{Z}(a)$: "a is an ordinary integer";

$\mathsf{N}(a)$: "a is a number of the second number class".

For *mathematical variables* we always use lower-case ⟦italic⟧ Latin letters, but for *constant mathematical objects* (specific functions) lower-case Greek letters.[7]

Concerning the procedure of *substitution* the following general conventions hold.

For propositional variables only other indeterminate or constant propositions (formulas) may be substituted.

Any array may be substituted for a mathematical variable; however, when a mathematical variable occurs in a formula, the constant proposition that specifies of what kind it is, together with the implication sign, must always precede; for example,

$$\mathsf{Z}(a) \rightarrow (\ldots a \ldots)$$

and

$$\mathsf{N}(a) \rightarrow (\ldots a \ldots).$$

[7] ⟦Hilbert's terms require some comments. Capital italic Latin letters, like A or B, stand for *propositional variables* (*Aussagenvariablen*) or predicate variables; when provided with parentheses and individual variables, they represent what Hilbert calls *variable Aussagen* (*variable propositions* in the translation); these "propositions" are in fact propositional functions. Capital Greek letters stand for predicate constants and, when provided with parentheses and individual variables, represent what Hilbert calls *individuelle Aussagen* or *Individualaussagen* (*constant propositions* in the translation); they are in fact propositional functions, and an example is $\mathsf{Z}(a)$, "a is a natural number". Lower-case italic Latin letters stand for what Hilbert calls *mathematische Variablen* (*mathematical variables* in the translation); they are individual variables ranging over the natural numbers, the numbers of the second number class, or the number-theoretic functions, as specified in each case. Lower-case Greek letters stand for *individuelle mathematische Gebilde* (*constant mathematical objects* in the translation); they are individual constants that stand for numbers or functions (*spezielle Funktionen*, or *specific functions* in the translation).⟧

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The effect of this convention is that only ordinary numbers or numbers of the second number class come into consideration after all as substituends, for example for a in $\mathsf{Z}(a)$ or $\mathsf{N}(a)$, respectively.

German capital as well as lower-case letters have *reference* and are used only to convey information.

It must still be remarked that by "*array*" we are to understand a perceptually given object composed of primitive signs.

In order to understand the idea of the proof of the continuum theorem we must above all gain a precise understanding of the notion of mathematical variable in its most general sense. Mathematical variables are of two kinds: (1) *primitive variables* [[*Grundvariablen*]] and (2) *variable-types* [[*Variablentypen*]].

(1) Now, while in all of arithmetic and analysis [[the variable that ranges over]] the ordinary integers suffices as sole primitive variable, with each of Cantor's transfinite number classes there is associated a *primitive variable* that ranges over precisely the ordinals of that class. Accordingly, to each primitive variable there corresponds a proposition that characterizes it as such; this proposition is implicitly characterized by the axioms, for example,

$$\begin{aligned}
&\mathsf{Z}(0),\\
&\mathsf{Z}(a) \rightarrow \mathsf{Z}(a+1),\\
&\{A(0)\ \&\ (a)(A(a) \rightarrow A(a+1))\} \rightarrow \{\mathsf{Z}(a) \rightarrow A(a)\}\\
&\qquad \text{(formula of ordinary induction)},\\
&\mathsf{N}(0),\\
&\mathsf{N}(a) \rightarrow \mathsf{N}(a+1),\\
&(n)\{\mathsf{Z}(n) \rightarrow \mathsf{N}(a(n))\} \rightarrow \mathsf{N}(\lim a(n)),
\end{aligned}$$

and, in addition, the formula of transfinite induction for the numbers of the second class.

With each kind of primitive variable there is associated one kind of recursion, by means of which we define functions whose argument is a primitive variable of that kind. The recursion associated with the number-theoretic variable is "ordinary recursion", by means of which a function of a number-theoretic variable n is defined when we indicate what value it has for $n = 0$ and how the value for $n + 1$ is obtained from that for n. The generalization of ordinary recursion is transfinite recursion; it rests upon the general principle that the value of the function for a value of the variable is determined by the preceding values of the function.

(2) From the primitive variables we derive further kinds of variables by applying logical connectives to the propositions associated with the primitive variables, for example, to Z and N. The variables thus defined are called *variable-types*, and the propositions defining them *type-propositions*; for these, new constant signs are introduced each time. Thus the formula

$$(a)\{\mathsf{Z}(a) \rightarrow \mathsf{Z}(f(a))\}$$

offers the simplest instance of a variable-type; for this formula defines the function variable f and, as type-proposition, will be denoted by $\Phi(f)$ ("being-a-function"). A further example is the formula

$$(f)\{\Phi(f) \rightarrow \mathsf{Z}(g(f))\};$$

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it defines the "being-a-function-of-a-function" $\Psi(g)$, where the argument g is the new function-of-a-function variable.[8]

To characterize the higher variable-types we must provide the type-propositions themselves with subscripts; a type-proposition thus provided with a subscript is defined by recursion, the place of equality ($=$) being taken by logical equivalence ($\sim$).

In all of arithmetic and analysis only the function variable, the function-of-a-function variable, and so on in finite iteration, are used as higher variables. A variable-type that goes beyond these simplest examples is offered by the variable g that assigns a numerical value $g(f_n)$ to a sequence f_n consisting of

a function f_1 of an integer, $\Phi(f_1)$,
a function f_2 of a function, $\Psi(f_2)$,
a function f_3 of a function-of-a-function,
and so forth.

The corresponding type-proposition $\Phi_\omega(g)$ is represented by means of the following equivalences:

$$\Phi_0(a) \sim \mathrm{Z}(a),$$
$$\Phi_{n+1}(f) \sim (b)\{\Phi_n(b) \rightarrow \mathrm{Z}(f(b))\},$$
$$\Phi_\omega(g) \sim \{(n)\Phi_n(f_n) \rightarrow \mathrm{Z}(g(f))\};$$

these at the same time offer an example of the definition of a type-proposition by means of recursion.

The variable-types can be classified according to their "*heights*". Among those of height 0 we include all number-theoretic constants and among those of height 1 all those functions whose arguments and values have the property of a primitive variable, for example, the property Z or the property N. A function whose argument and value have certain heights possesses a height greater by 1 than the greater, or possibly than both, of those two heights. A sequence of functions of various heights has the limit of those heights as its height.

After these preparations we return to our task and recall that, in order to prove the continuum theorem, it is essential to correlate those definitions of number-theoretic functions that are free from the symbol ε one-to-one with Cantor's numbers of the second number class, or at least to establish a correspondence between them in such a way that every number-theoretic function is associated with at least one number of the second number class.

Clearly, the elementary means that we have at our disposal for forming functions are *substitution* (that is, replacement of an argument by a new variable or function) and *recursion* (according to the schema of the derivation of the function value for $n + 1$ from that for n).

One might think that these two processes, substitution and recursion, would have to be supplemented with other elementary methods of definition, for example, definition of a function by specification of its values up to a certain argument value, beyond which the function is to be constant, and also definition by means of elementary processes obtained from arithmetic operations, such as that of the remainder in

[8] ⟦"Function-of-a-function" is used here in the sense of "functional", not "compound function".⟧

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division, say, or of the greatest common divisor of two numbers, or even the definition of a number as the least of finitely many given numbers.

It turns out, however, that any such definition can be represented as a special case of the use of substitutions and recursions. The method of search for the recursions required is in essence equivalent to that reflection by which one recognizes that the procedure used for the given definition is finitary.

Having ascertained this, we must now survey the results of the two operations of substitution and recursion. On account of the variety of ways in which we can pass from n to $n + 1$, however, we cannot, as it turns out, bring the recursions that are to be used into a standard form if we confine ourselves to operating with ordinary number-theoretic variables. This difficulty already becomes apparent in the following example.

Let us consider the functions $a + b$; from them we obtain by n-fold iteration and equating

$$a + a + \cdots + a = a \cdot n.$$

In the same way we pass from $a \cdot b$ to

$$a \cdot a \cdot \cdots \cdot a = a^n,$$

and, further, from a^b to

$$a^{(a^a)}, a^{(a^{(a^a)})}, \ldots.$$

Thus we successively obtain the functions

$$\begin{aligned} a + b &= \varphi_1(a, b), \\ a \cdot b &= \varphi_2(a, b), \\ a^b &= \varphi_3(a, b). \end{aligned}$$

$\varphi_4(a, b)$ is the bth term in the sequence

$$a, a^a, a^{(a^a)}, a^{(a^{(a^a)})}, \ldots.$$

In a corresponding way we arrive at $\varphi_5(a, b)$, $\varphi_6(a, b)$, and so on.

To be sure, we could now define $\varphi_n(a, b)$ for variable n by means of substitutions and recursions, but these recursions would not be ordinary, stepwise ones; rather, we would be led to a manifold simultaneous recursion, that is, a recursion on different variables at once, and a resolution of it into ordinary, stepwise recursions would be possible only if we make use of the notion of function variable; the function $\varphi_a(a, a)$ is an instance of a function, of the number-theoretic variable a, that cannot be defined by substitutions and ordinary, step-wise recursions alone if we admit only number-theoretic variables.[9] How we can define the function $\varphi_n(a, b)$ by using the function variable is shown by the following formulas:

$$\begin{aligned} \iota(f, a, 1) &= a, \\ \iota(f, a, n + 1) &= f(a, \iota(f, a, n)); \\ \varphi_1(a, b) &= a + b, \\ \varphi_{n+1}(a, b) &= \iota(\varphi_n, a, b). \end{aligned}$$

[9] This assertion was proved by W. Ackermann [[1928]].

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Here ι stands for a specific function of three arguments, of which the first is itself a function of two ordinary number-theoretic variables.

Another example of a more complicated recursion is this:

$$\varphi_0(a) = \mathfrak{a}(a),$$
$$\varphi_{n+1}(a) = \mathfrak{f}(a, n, \varphi_n(\varphi_n(n + a))),$$

where $\mathfrak{a}$ stands for a known expression containing one argument, and $\mathfrak{f}$ for a known expression containing three arguments. What is characteristic of this recursion is that here we cannot obtain a numerical value for $n + 1$ from one for n but must make use of the range of the function φ_n in determining φ_{n+1}.

The difficulties that come to light in these examples are overcome when we make use of variable-types; then the general recursion schema reads as follows:

$$\rho(\mathfrak{g}, \mathfrak{a}, 0) = \mathfrak{a},$$
$$\rho(\mathfrak{g}, \mathfrak{a}, n + 1) = \mathfrak{g}(\rho(\mathfrak{g}, \mathfrak{a}, n), n);$$

here $\mathfrak{a}$ is a given expression of arbitrary variable-type; $\mathfrak{g}$ likewise is a given expression, having two arguments, of which the first is of the same variable-type as $\mathfrak{a}$ and the second is a number; the additional condition that $\mathfrak{g}$ must satisfy is that its value again be of the same variable-type as $\mathfrak{a}$. Finally, ρ is the expression to be defined by the recursion; it depends on three arguments and, after the substitutions for $\mathfrak{g}$, $\mathfrak{a}$, and n have been made, is of the same variable-type as $\mathfrak{a}$; in addition, other arbitrary parameters are permitted to occur in $\mathfrak{a}$ and $\mathfrak{g}$, and consequently also in ρ.

From this general schema we obtain specific recursions through substitution. Thus we obtain the recursions of our examples by considering, in the first example, f and a as parameters and by representing, in the second, the transition from $\varphi_n(a)$ to $\varphi_{n+1}(a)$ as a transition, mediated by the function-of-a-function $\mathfrak{g}$, from a function φ_n to the function φ_{n+1}, so that a is not regarded as a parameter at all in the recursion. Compared with elementary recursion, the recursion used in our two examples is of a wider kind, since in one case we introduce a higher parameter that is not an ordinary integer and in the other choose a function for $\mathfrak{a}$ and a function-of-a-function for $\mathfrak{g}$.

The variable-types form the link that makes possible the correspondence between the functions of a number-theoretic variable and the numbers of the second number class. Indeed, we arrive at such a correspondence between the numbers of the second number class and certain variable-types if we compare the two generating processes for the numbers of the second number class, namely, the process of adding 1 and the limit process for denumerable sequences, with the way in which variable-types increase in height. Let us establish a correspondence between the process of adding 1 and the taking of a function (that is, substitution of a given variable-type into a function as argument) and between the limit process and the aggregating of the denumerable sequence associated with a variable-type into a new variable-type, and let us designate the variable-types that in this way come to correspond to the numbers of the second number class specifically as Z-*types*. Thus, when the Z-types are formed, we use, besides the logical operations, only ordinary (not transfinite) recursions, that is, just those necessary for the denumeration of a type sequence as a preparatory step for the

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limit process. If we order these Z-types according to their heights, we have a one-to-one correspondence by means of which the variable-types of a given height are associated with a number of the second number class.

But therewith we also arrive at a one-to-one correspondence between the numbers of the second number class and the functions defined by means of the Z-types. To make that clear the following considerations suffice. Whenever we begin with variable-types up to a given height only and then form functions solely by means of substitution and recursion, we obtain only denumerably many functions. We can even formalize this denumeration rigorously; in particular, we can do this by first constructing a recursion function ρ that subsumes all the recursions in question and, consequently, contains a parameter that exceeds the variable-types admitted up to that point. In the definition of ρ we apply the general recursion schema in a way that makes essential use of this higher variable-type. Now we order according to their heights the relevant specializations of the variable-types occurring in ρ and therewith obtain the various initial substitutions. These we arrange in an enumerated sequence. Once this enumeration has been established, the introduction of an ordering according to the number of substitutions to be performed yields the functions that were to be defined.

In the argument just described I essentially presupposed the theory of the numbers of the second number class. I introduced the numbers of the second number class simply as resulting from transnumeration beyond the denumerable infinite, and the constant proposition N, "to be a number of the second number class", was characterized afterwards by a listing of axioms. But these axioms furnish only the general framework of a theory. To provide a more precise foundation for it, it is necessary to determine how the process of transnumeration beyond the denumerable infinite is to be formalized. This is done by applying the process of transnumeration to a sequence; this sequence can be given only by means of an ordinary recursion, and for these recursions certain types are in turn necessary.

This circumstance seems to present a difficulty, but in fact it turns out that reflecting upon precisely this point enables us to make the correspondence between the numbers of the second number class and the functions of a number-theoretic variable a much closer one. For the variable-types that we need for the production of the numbers of the second number class are obtained by formal replacement of the sign Z by the sign N at one or more places in the defining type-propositions that we have up to this point. We shall call the variable-types that then result N-*types*; as is apparent, corresponding Z-types and N-types have the same height. Now we do not need to associate with a given number of the second number class all functions of the same height, but we can let the numbers of the second number class and the functions correspond to each other according to the heights of the variable-types required for their definition. Let us now come to a more precise formulation of this correspondence.

If in the Z-types we go up to a certain height only, the heights of the corresponding N-types are also bounded. From the numbers of the second number class that can be produced with these types we can, by means of an increasing sequence, obtain a greater number of the second number class, which is defined by means of a higher variable-type. If, on the other hand, we have N-types up to a certain height, the functions definable by means of the corresponding Z-types can also be denumerated —in the way described above, according to the number of substitutions. As is well known, we obtain from such an enumeration $\varphi(a, n)$ by means of Cantor's diagonal procedure (for example, by forming $\varphi(a, a) + 1$) a function that differs from all of the enumerated functions and therefore could not be defined by means of the variable-types previously admitted.

Thus we have made it possible to establish a one-to-one correspondence between the numbers of the second number class that are definable at the height in question, but not at any lower one, and the denumerably many functions that are definable at the same height, and in this way each function is associated with at least one number of the second number class.

With this, however, the proof of the continuum theorem is not yet complete; rather, it still requires essential supplementation. For in our entire investigation up to now we have, in order to establish the correspondence, made restrictive assumptions in two respects: first, our general recursion schema for ρ embodies only the case of ordinary recursion (the number-theoretic variable being the variable on which the recursion proceeds), and, second, we also restricted the variable-types to those that result from transnumeration beyond denumerated sequences. It is certain that, in general, transfinite recursions and, accordingly, higher variable-types are necessarily used in mathematical investigations, for example, for the formation of certain kinds of functions of real variables. But here in our problem, where it is a matter of forming functions of a number-theoretic variable, we do not in fact require such higher recursions and variable-types; for the following lemma holds.

Lemma II. In the formation of functions of a number-theoretic variable transfinite recursions are dispensable; in particular, not only does ordinary recursion (that is, the one that proceeds on a number-theoretic variable) suffice for the actual formation process of the functions, but also the substitutions call merely for those variable-types whose definition requires only ordinary recursion. Or, to express ourselves with greater precision and more in the spirit of our finitist attitude, if by adducing a higher recursion or a corresponding variable-type we have formed a function that has only an ordinary number-theoretic variable as argument, then this function can always be defined also by means of ordinary recursions and the exclusive use of Z-types.

The following typical example will make the meaning and scope of this lemma clear to us.

If we imagine that the correspondence between the functions of a number-theoretic argument and the numbers of the second number class has been formalized, then by this very fact we have a certain function $\zeta(a, n)$ that associates an ordinary number with an arbitrary number a of the second number class and the ordinary number n (for fixed a and variable n, $\zeta(a, n)$ represents precisely the function associated with a). If we now substitute for a a number α_n, of the second number class, that depends on n, the sequence ⟦of the α_n⟧ being defined by ordinary or even transfinite recursion, for example,

$$\alpha_{n+1} = \omega^{\alpha_n},$$

then $\zeta(\alpha_n, n)$ is a function of a number-theoretic variable n; and our Lemma II now asserts of this function that it can be defined through ordinary recursion by means of

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Z-types, whereas it is certainly impossible to define $\zeta(a, n)$ by these means, since, after all, the contrary assumption clearly leads to a contradiction.

I should like to note expressly once more that the presentation given here of the proof of the continuum theorem contains only the fundamental ideas; its complete realization would require, besides the proofs of the two lemmas, a recasting strictly faithful to the finitist attitude.

Finally let us recall our real subject and, so far as the infinite is concerned, draw the balance of all our reflections. The final result then is: nowhere is the infinite realized; it is neither present in nature nor admissible as a foundation in our rational thinking—a remarkable harmony between being and thought. We gain a conviction that runs counter to the earlier endeavors of Frege and Dedekind, the conviction that, if scientific knowledge is to be possible, certain intuitive conceptions [[Vorstellungen]] and insights are indispensable; logic alone does not suffice. The right to operate with the infinite can be secured only by means of the finite.

The role that remains to the infinite is, rather, merely that of an idea—if, in accordance with Kant's words, we understand by an idea a concept of reason that transcends all experience and through which the concrete is completed so as to form a totality—an idea, moreover, in which we may have unhesitating confidence within the framework furnished by the theory that I have sketched and advocated here.

In closing I should like to express my thanks to P. Bernays for his sympathetic collaboration and the valuable aid that he extended to me in questions both of matter and of form, especially in the proof of the continuum theorem.