Advanced Linear Algebra

Week 17

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In this lecture we work with general vector spaces again, without an associated inner product!
Our fields $\mathcal{F}$ are also general fields (and not just $\mathbb{R}$ or $\mathbb{C}$).
Recall: if $A \in \mathrm{End}(V)$ then for $\lambda \in \mathcal{F}$

$$\begin{aligned} V_\lambda &:= \{x \in V : Ax = \lambda x\} \\ &= N(A - \lambda I). \end{aligned}$$

If $V_\lambda \neq \{0\}$ then λ is an eigenvalue, V_λ is the corresponding eigenspace, and all non-zero vectors in V_λ are the corresponding eigenvectors.

Definition (7.1)

Let $A \in \mathrm{End}(V)$ and $\lambda \in \mathcal{F}$. We call

$$M_\lambda := \{x \in V : \exists k > 0 \text{ such that } (A - \lambda I)^k x = 0\}$$

the generalised eigenspace corresponding to λ, and non-zero vectors in M_λ are called generalised eigenvectors.

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the generalised eigenspace corresponding to λ, and non-zero vectors in M_λ are called generalised eigenvectors.

Note: $M_\lambda = \bigcup_{k \in \mathbb{N}} N((A - \lambda I)^k)$.

In particular, $V_\lambda \subseteq M_\lambda$.

What about generalised eigenvalues?

$$\begin{aligned} V_\lambda &= \{x \in V : (A - \lambda I)x = 0\} \\ M_\lambda &= \{x \in V : \exists k > 0 \text{ such that } (A - \lambda I)^k x = 0\} \end{aligned}$$

Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2)$.

Note that $A^2 = 0$.

We have $N(A) = \mathrm{Span}\{e_1\}$, and thus 0 is an eigenvector for A.

Question: Which is true?

(1) $V_0 = M_0 = \mathrm{Span}\{e_1\}$;

(2) $V_0 = \{0\}$ and $M_0 = \mathrm{Span}\{e_1\}$;

(3) $V_0 = \mathrm{Span}\{e_1\}$ and $M_0 = \mathbb{R}^2$;

(4) $V_0 = M_0 = \mathbb{R}^2$.

$$
\begin{aligned}
V_\lambda &= \{x \in V : (A - \lambda I)x = 0\} \\
M_\lambda &= \{x \in V : \exists k > 0 \text{ such that } (A - \lambda I)^k x = 0\}
\end{aligned}
$$

Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2)$.

Note that $A^2 = 0$.

We have $N(A) = \mathrm{Span}\{e_1\}$, and thus 0 is an eigenvector for A.

Definition (7.1)

Let $A \in \mathrm{End}(V)$ and $\lambda \in \mathcal{F}$. We let

$$\begin{aligned} M_\lambda &:= \{x \in V : \exists k > 0 \text{ such that } (A - \lambda I)^k x = 0\} \\ &= \bigcup_{k \in \mathbb{N}} N((A - \lambda I)^k). \end{aligned}$$

We have $N(A - \lambda I) \subseteq N((A - \lambda)^2) \subseteq N((A - \lambda I)^3) \subseteq \dots$.
Note that if $x \in N((A - \lambda I)^k)$ then $y := (A - \lambda I)x$ is in $N((A - \lambda I)^{k-1}) \subseteq N((A - \lambda I)^k)$ since

$$(A - \lambda I)^{k-1} y = (A - \lambda I)^{k-1}(A - \lambda I)x = (A - \lambda I)^k x = 0.$$

Hence $(A - \lambda I)(N((A - \lambda I)^k)) \subseteq N((A - \lambda I)^k)$.
Consequently, if $x \in N((A - \lambda I)^k)$ then
$Ax = \lambda x + (A - \lambda I)x \in N((A - \lambda I)^k)$.
So each $N((A - \lambda I)^k)$ is an A-invariant subspace of V, i.e.
$A(N((A - \lambda I)^k)) \subseteq N((A - \lambda I)^k)$.
Hence M_λ is also A-invariant.

Note that since $N((A-\lambda I)^k)$ is $A-\lambda I$-invariant, $A-\lambda I$ induces an endomorphism on $N((A-\lambda I)^k)$.
This endomorphism has the property, that if you take its k-th power, you get the zero map.

Definition (7.3)

An $A \in \mathrm{End}(V)$ is called nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$.
In this case, the smallest such k is the index of nilpotency of A.

Question: Which is true for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R})$?

(1) A is nilpotent with index 1
(2) A is nilpotent with index 2
(3) A is nilpotent with index 3
(4) A is not nilpotent.

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If A is injective then so is A^k for all $k \in \mathbb{N}$.
Hence, if A is nilpotent then $N(A) \neq \{0\}$, and thus 0 is an eigenvalue of A. Moreover, it is the only eigenvalue: if $Ax = \lambda x$ for $x \neq 0$, then $\lambda^k x = A^k x = 0$ so $\lambda = 0$.
Conclusion: if A is nilpotent then $\sigma(A) = \{0\}$.

Consider $A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$.

A matrix in $M_n(\mathcal{F})$ is called upper triangular if the (i, j)'th entry is 0 whenever $i > j$. I.e. the matrix has the form

We say that a matrix is strictly upper triangular if it is upper triangular and all its diagonal entries are 0. I.e. the matrix has the form

Note: A is strictly upper triangular iff $Ae_j \in \mathrm{Span}\{e_1, \ldots, e_{j-1}\}$ for all j.

Theorem (7.4 (slightly different than lecture notes))

Suppose $\dim V < \infty$ *and let* $A \in \mathrm{End}(V)$. *TFAE:*

(a) *A is nilpotent;*

(b) *there exists a basis* $\{x_1, \ldots, x_n\}$ *for V such that* $Ax_1 = 0$ *and* $Ax_j \in \mathrm{Span}\{x_1, \ldots, x_{j-1}\}$ *for* $j = 2, \ldots, n$*;*

(c) *there exists an ordered basis $\mathcal{B}$ for V such that* ${}_{\mathcal{B}}[A]_{\mathcal{B}}$ *is strictly upper triangular.*

In particular, if A is nilpotent, then its index of nilpotency is at most $\dim V$.

Question: Consider $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2)$ which is lower triangular. Note that $A^2 = 0$. What can we conclude from the above theorem?

(1) A is not nilpotent, since A is lower triangular;

(2) A is both lower and upper triangular;

(3) there is a basis for $\mathbb{R}^2$ so that $[A]$ is strictly upper triangular;

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Theorem (7.4 (slightly different than lecture notes))

Let $\dim V = n < \infty$ *and* $A \in \mathrm{End}(V)$. *TFAE:*

(a) *A is nilpotent;*

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Theorem (7.5)

Let V be a finite-dimensional vector space and $A \in \mathrm{End}(V)$.

(a) *There exist unique A-invariant subspaces $N, R \subseteq V$ such that $V = N \oplus R$, $A|_N \in \mathrm{End}(N)$ is nilpotent, and $A|_R \in \mathrm{End}(R)$ is invertible;*

(b) *If $M \subseteq V$ is an A-invariant subspace for which $A|_M \in \mathrm{End}(M)$ is nilpotent, then $M \subseteq N$;*

(c) *If $S \subseteq V$ is an A-invariant subspace for which $A|_S \in \mathrm{End}(S)$ is invertible, then $S \subseteq R$.*

Example: if $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2)$, then $N = \mathrm{Span}\{e_1\}$ and $R = \mathrm{Span}\{e_2\}$ satisfy $\mathbb{R}^2 = N \oplus R$, $A|_N = 0$ and $A|_R = I$.

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Theorem (7.5)

Let V be a finite-dimensional vector space and $A \in \mathrm{End}(V)$.

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Theorem (7.5)

Let V be a finite-dimensional vector space and $A \in \mathrm{End}(V)$.

(b) *If $M \subseteq V$ is an A-invariant subspace for which $A|_M \in \mathrm{End}(M)$ is nilpotent, then $M \subseteq N$;*

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