Investigations in the foundations of set theory I

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(*1908a*)

This paper presents the first axiomatic set theory. Cantor's definition of set[a] had hardly more to do with the development of set theory than Euclid's definition of point with that of geometry. Dedekind, whom Zermelo considers one of the two creators of set theory, had explicitly stated (*1888*, § 1) a number of principles about sets (which he called "systems"), but his attempt had remained fragmentary and had been somewhat discredited by the nonmathematical way in which he justified the existence of an infinite set (*1888*, art. 66). In spite of the great advances that set theory was making, the very notion of set remained vague. The situation became critical after the appearance of the Burali-Forti paradox and intolerable after that of the Russell paradox, the latter involving the bare notions of set and element. One response to the challenge was Russell's theory of types (above, pp. 150–182). Another, coming at almost the same time, was Zermelo's axiomatization of set theory. The two responses are extremely different; the former is a far-reaching theory of great significance for logic and even ontology, while the latter is an immediate answer to the pressing needs of the working mathematician.

Zermelo's basic idea resembles, if anything, Russell's "theory of limitation of size" (*1905a*); both refuse to take as sets collections that are too "big", that of all "things" or that of all ordinals, for example.[b] Sets are not simply collections; they are objects satisfying certain axiomatic conditions. Zermelo's axioms are surprisingly few in number. The most original is perhaps Axiom III, the axiom of separation. In Peano and Russell sets were part of logic, being intimately connected with "conditions", or propositional functions. To express the relation, generally felt to exist, between a set and a stipulation asserted by a statement, Zermelo introduces the notion "definite property" and, together with it, the axiom of separation: a definite property separates a subset from an already given set. His definition of "definite property", however, invokes "the universally valid laws of logic"; since Zermelo pays no attention at all to the underlying logic, these laws are left unspecified, and the notion of definite property remains hazy. The flaw will be removed, in different manners, by Weyl, Fraenkel, Skolem, and von Neumann (see below, p. 285).

Zermelo was perhaps the first to see clearly that the existence of infinite sets has to be insured by a special axiom (Axiom VII, of infinity). The powerful new tool that he used in his proofs of the well-ordering theorem appears in two forms, as axiom of choice (Axiom VI) and as general principle of choice (Article 29). The latter, which does not assume that

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[a] See below, p. 200, footnote 1.

[b] Cantor already had a similar idea; see above, p. 114. See also *König 1905a*, above, p. 148, and *von Neumann 1925*, below, pp. 396–398.

the sets from which the choice is made are disjoint, is here derived from Axiom VI.

Zermelo does not have the Cartesian product; he makes do with the "connection set", the set of unordered pairs, and that renders his treatment of equivalence of sets (§ 2), for example, somewhat cumbersome.

Zermelo states his axioms, declares that he has been unable to prove their consistency, and shows that the usual derivations of a number of known paradoxes cannot be obtained from them. He then proves theorems about sets. The development goes as far as Cantor's theorem,[c] König's theorem, and the theorem of Article 36, which connects two notions of infinitude. A second paper, dealing with a theory of well-ordering and a set-theoretic definition of natural numbers, is announced at the end of the introduction but was never published. However, in a paper (*1908b*) prepared shortly after the one below, Zermelo briefly shows how the natural numbers can be defined in a theory of finite sets.

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[c] "Every set is of lower cardinality than the set of its subsets."

Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions "number", "order", and "function", taking them in their pristine, simple form, and to develop thereby the logical foundations of all of arithmetic and analysis; thus it constitutes an indispensable component of the science of mathematics. At present, however, the very existence of this discipline seems to be threatened by certain contradictions, or "antinomies", that can be derived from its principles—principles necessarily governing our thinking, it seems—and to which no entirely satisfactory solution has yet been found. In particular, in view of the "Russell antinomy" (*1903*, pp. 101–107 and 366–368) of the set of all sets that do not contain themselves as elements, it no longer seems admissible today to assign to an arbitrary logically definable notion a set, or class, as its extension. Cantor's original definition of a set (*1895*) as "a collection, gathered into a whole, of certain well-distinguished objects of our perception or our thought"[1] therefore certainly requires some restriction; it has not, however, been successfully replaced by one that is just as simple and does not give rise to such reservations. Under these circumstances there is at this point nothing left for us to do but to proceed in the opposite direction and, starting from set theory as it is historically given, to seek out the principles required for establishing the foundations of this mathematical discipline. In solving the problem we must, on the one hand, restrict these principles sufficiently to exclude all contradictions and, on the other, take them sufficiently wide to retain all that is valuable in this theory.

Now in the present paper I intend to show how the entire theory created by Cantor and Dedekind can be reduced to a few definitions and seven principles, or axioms, which appear to be mutually independent. The further, more philosophical, question about the origin of these principles and the extent to which they are valid will not be discussed here. I have not yet even been able to prove rigorously that my axioms

[1] [[Cantor's full definition reads: "Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) zu einem Ganzen" (*1895*, p. 481, or *1932*, p. 282).]]

are consistent, though this is certainly very essential; instead I have had to confine myself to pointing out now and then that the antinomies discovered so far vanish one and all if the principles here proposed are adopted as a basis. But I hope to have done at least some useful spadework hereby for subsequent investigations in such deeper problems.

The present paper contains the axioms and their most immediate consequences, as well as a theory of equivalence based upon these principles that avoids the formal use of cardinal numbers. A second paper, which will develop the theory of well-ordering together with its application to finite sets and the principles of arithmetic, is in preparation.[2]

§ 1. Fundamental definitions and axioms

1. Set theory is concerned with a *domain* $\mathfrak{B}$ of individuals, which we shall call simply *objects* and among which are the *sets*. If two symbols, a and b, denote the same object, we write $a = b$, otherwise $a \neq b$. We say of an object a that it "exists" if it belongs to the domain $\mathfrak{B}$; likewise we say of a class $\mathfrak{K}$ of objects that "there exist objects of the class $\mathfrak{K}$" if $\mathfrak{B}$ contains at least one individual of this class.

2. Certain *fundamental relations* of the form $a \varepsilon b$ obtain between the objects of the domain $\mathfrak{B}$. If for two objects a and b the relation $a \varepsilon b$ holds, we say "a is an *element* of the set b", "b contains a as an element", or "b possesses the element a". An object b may be called a *set* if and—with a single exception (Axiom II)—only if it contains another object, a, as an element.

3. If every element x of a set M is also an element of the set N, so that from $x \varepsilon M$ it always follows that $x \varepsilon N$, we say that M is a *subset* of N and we write $M \mathrel{\in\!\!\!-} N$.[3] We always have $M \mathrel{\in\!\!\!-} M$, and from $M \mathrel{\in\!\!\!-} N$ and $N \mathrel{\in\!\!\!-} R$ it always follows that $M \mathrel{\in\!\!\!-} R$. Two sets M and N are said to be *disjoint* if they possess no common element, or if no element of M is an element of N.

4. A question or assertion $\mathfrak{E}$ is said to be *definite* if the fundamental relations of the domain, by means of the axioms and the universally valid laws of logic, determine without arbitrariness whether it holds or not. Likewise a "propositional function" [["Klassenaussage"]] $\mathfrak{E}(x)$, in which the variable term x ranges over all individuals of a class $\mathfrak{K}$, is said to be definite if it is definite for *each single* individual x of the class $\mathfrak{K}$. Thus the question whether $a \varepsilon b$ or not is always definite, as is the question whether $M \mathrel{\in\!\!\!-} N$ or not.

The fundamental relations of our domain $\mathfrak{B}$, now, are subject to the following *axioms*, or *postulates*.

Axiom I. (Axiom of extensionality [[Axiom der Bestimmtheit]].) If every element of a set M is also an element of N and vice versa, if, therefore, both $M \mathrel{\in\!\!\!-} N$ and $N \mathrel{\in\!\!\!-} M$, then always $M = N$; or, more briefly: Every set is determined by its elements.

The set that contains only the elements $a, b, c, \ldots, r$ will often be denoted briefly by $\{a, b, c, \ldots, r\}$.

[2] [[This paper is apparently *1909*; see also *1908b*.]]

[3] This sign of inclusion was introduced by Schröder (*1890*). Peano and, following him, Russell, Whitehead, and others use the sign Ͻ instead.

Axiom II. (Axiom of elementary sets [[Axiom der Elementarmengen]].) There exists a (fictitious) set, the *null set*, 0, that contains no element at all. If a is any object of the domain, there exists a set $\{a\}$ containing a and only a as element; if a and b are any two objects of the domain, there always exists a set $\{a, b\}$ containing as elements a and b but no object x distinct from both.

5. According to Axiom I, the elementary sets $\{a\}$ and $\{a, b\}$ are always uniquely determined and there is only a single null set. The question whether $a = b$ or not is always definite (No. 4), since it is equivalent to the question whether or not $a \varepsilon \{b\}$.

6. The null set is a subset of every set $M: 0 \Subset M$; a subset of M that differs from both 0 and M is called a *part* [[*Teil*]][4] of M. The sets 0 and $\{a\}$ do not have parts.

Axiom III. (Axiom of separation [[Axiom der Aussonderung]].) Whenever the propositional function $\mathfrak{E}(x)$ is definite for all elements of a set M, M possesses a subset $M_{\mathfrak{E}}$ containing as elements precisely those elements x of M for which $\mathfrak{E}(x)$ is true.

By giving us a large measure of freedom in defining new sets, Axiom III in a sense furnishes a substitute for the general definition of set that was cited in the introduction and rejected as untenable. It differs from that definition in that it contains the following restrictions. In the first place, sets may never be *independently defined* by means of this axiom but must always be *separated* as subsets from sets already given; thus contradictory notions such as "the set of all sets" or "the set of all ordinal numbers", and with them the "ultrafinite paradoxes", to use Hessenberg's expression (*1906*, chap. 24), are excluded. In the second place, moreover, the defining criterion must always be definite in the sense of our definition in No. 4 (that is, for each single element x of M the fundamental relations of the domain must determine whether it holds or not), with the result that, from our point of view, all criteria such as "definable by means of a finite number of words", hence the "Richard antinomy" and the "paradox of finite denotation",[5] vanish. But it also follows that we must, prior to each application of our Axiom III, prove the criterion $\mathfrak{E}(x)$ in question to be definite, if we wish to be rigorous; in the considerations developed below this will indeed be proved whenever it is not altogether evident.

7. If $M_1 \Subset M$, then M always possesses another subset, $M - M_1$, the *complement* of M_1, which contains all those elements of M that are *not* elements of M_1. The complement of $M - M_1$ is M_1 again. If $M_1 = M$, its complement is the null set, 0; the complement of any part (No. 6) M_1 of M is again a part of M.

8. If M and N are any two sets, then according to Axiom III all those elements of M that are also elements of N are the elements of a subset D of M; D is also a subset of N and contains all elements *common* to M and N. This set D is called the *common component*, or *intersection*, of the sets M and N and is denoted by $[M, N]$. If $M = N$, then $[M, N] = M$; if $N = 0$ or if M and N are disjoint (No. 3), then $[M, N] = 0$.

9. Likewise, for several sets $M, N, R, \ldots$ there always exists an intersection $D = [M, N, R, \ldots]$. For, if T is any set whose elements are themselves sets, then according to Axiom III there corresponds to every object a a certain subset T_a of T that contains all those elements of T that contain a as an element. Thus it is definite

[4] [[Below, Zermelo also uses the expression "echter Teil" in the same sense. In the translation "part" has been used throughout for "nonempty proper subset".]]

[5] See *Hessenberg 1906*, chap. 23; on the other hand, see *König 1905a*.

for every a whether $T_a = T$, that is, whether a is a common element of all elements of T; if A is an arbitrary element of T, all elements a of A for which $T_a = T$ are the elements of a subset D of A that contains all these common elements. This set D is called the intersection associated with T and is denoted by $\mathfrak{D}T$. If the elements of T do not possess a common element, $\mathfrak{D}T = 0$, and this is always the case if, for example, an element of T is not a set or if it is the null set.

10. THEOREM. Every set M possesses at least one subset M_0 that is not an element of M.

Proof. It is definite for every element x of M whether $x \varepsilon x$ or not; the possibility that $x \varepsilon x$ is not in itself excluded by our axioms. If now M_0 is the subset of M that, in accordance with Axiom III, contains all those elements of M for which it is not the case that $x \varepsilon x$, then M_0 cannot be an element of M. For either $M_0 \varepsilon M_0$ or not. In the first case, M_0 would contain an element $x = M_0$ for which $x \varepsilon x$, and this would contradict the definition of M_0. Thus M_0 is surely not an element of M_0, and in consequence M_0, if it were an element of M, would also have to be an element of M_0, which was just excluded.

It follows from the theorem that not all objects x of the domain $\mathfrak{B}$ can be elements of one and the same set; that is, *the domain* $\mathfrak{B}$ *is not itself a set*, and this disposes of the Russell antinomy so far as we are concerned.

AXIOM IV. (Axiom of the power set [[Axiom der Potenzmenge]].) To every set T there corresponds another set $\mathfrak{U}T$, the *power set* of T, that contains as elements precisely all subsets of T.

AXIOM V. (Axiom of the union [[Axiom der Vereinigung]].) To every set T there corresponds a set $\mathfrak{S}T$, the *union* of T, that contains as elements precisely all elements of the elements of T.

11. If no element of T is a set different from 0, then, of course, $\mathfrak{S}T = 0$. If $T = \{M, N, R, \ldots\}$, where $M, N, R, \ldots$ all are sets, we also write $\mathfrak{S}T = M + N + R + \ldots$ and call $\mathfrak{S}T$ the *sum* of the sets $M, N, R, \ldots$, whether some of these sets $M, N, R, \ldots$ contain common elements or not. Always $M = M + 0 = M + M = M + M + \ldots$.

12. For the "addition" of sets that we have just defined, the commutative and associative laws hold:

$$M + N = N + M, \quad M + (N + R) = (M + N) + R.$$

Finally, for sums and intersections (No. 8) the distributive law also holds, in the two forms:

$$[M + N, R] = [M, R] + [N, R]$$

and

$$[M, N] + R = [M + R, N + R].$$

The proof is carried out by means of Axiom I and consists in a demonstration that every element of the set on the left is also an element of the set on the right, and conversely.[6]

[6] The complete theory of this logical addition and multiplication can be found in *Schröder 1890*.

13. *Introduction of the product.* If M is a set different from 0 and a is any one of its elements, then according to No. 5 it is definite whether $M = \{a\}$ or not. *It is therefore always definite whether a given set consists of a single element or not.*

Now let T be a set whose elements, $M, N, R, \ldots$, are various (mutually disjoint) sets, and let S_1 be any subset of its union $\mathfrak{S}T$. Then it is definite for every element M of T whether the intersection $[M, S_1]$ consists of a single element or not. Thus all those elements of T that have exactly one element in common with S_1 are the elements of a certain subset T_1 of T, and it is again definite whether $T_1 = T$ or not. All subsets S_1 of $\mathfrak{S}T$ that have exactly one element in common with each element of T then are, according to Axiom III, the elements of a set $P = \mathfrak{P}T$, which, according to Axioms III and IV, is a subset of $\mathfrak{U}\mathfrak{S}T$ and will be called the *connection set* [[*Verbindungsmenge*]] associated with T or the *product* of the sets $M, N, R, \ldots$. If $T = \{M, N\}$, or $T = \{M, N, R\}$, we write $\mathfrak{P}T = MN$, or $\mathfrak{P}T = MNR$, respectively, for short.

In order, now, to obtain the theorem that *the product of several sets can vanish* (that is, be equal to the null set) *only if a factor vanishes* we need a further axiom.

AXIOM VI. (Axiom of choice [[Axiom der Auswahl]].) If T is a set whose elements all are sets that are different from 0 and mutually disjoint, its union $\mathfrak{S}T$ includes at least one subset S_1 having one and only one element in common with each element of T.

We can also express this axiom by saying that it is always possible to *choose* a single element from each element $M, N, R, \ldots$ of T and to combine all the chosen elements, $m, n, r, \ldots$, into a set S_1.[7]

The preceding axioms suffice, as we shall see, for the derivation of all essential theorems of general set theory. But in order to secure the existence of infinite sets we still require the following axiom, which is essentially due to Dedekind.[8]

AXIOM VII. (Axiom of infinity [[Axiom des Unendlichen]].) There exists in the domain at least one set Z that contains the null set as an element and is so constituted that to each of its elements a there corresponds a further element of the form $\{a\}$, in other words, that with each of its elements a it also contains the corresponding set $\{a\}$ as an element.

14_{VII}.[9] If Z is an arbitrary set constituted as required by Axiom VII, it is definite for each of its subsets Z_1 whether it possesses the same property. For, if a is any element of Z_1, it is definite whether $\{a\}$, too, is an element of Z_1, and all elements a of Z_1 that satisfy this condition are the elements of a subset Z_1' for which it is definite whether $Z_1' = Z_1$ or not. Thus all subsets Z_1 having the property in question are the elements of a subset T of $\mathfrak{U}Z$, and the intersection (No. 9) $Z_0 = \mathfrak{D}T$ that corresponds to them is a set constituted in the same way. For, on the one hand, 0 is a common element of all elements Z_1 of T, and, on the other, if a is a common element of all of these Z_1, then $\{a\}$ is also common to all of them and is thus likewise an element of Z_0.

[7] For the justification of this axiom see my *1908*, where in § 2, pp. 111–128 [[above, pp. 186–198]], the relevant literature is discussed.

[8] *1888*, art. 66. The "proof" that Dedekind there attempts to give of this principle cannot be satisfactory, since it takes its departure from "the set of everything thinkable", whereas from our point of view the domain $\mathfrak{B}$ itself, according to No. 10, does *not* form a set.

[9] The subscript VI, or VII, on the number of a section indicates that explicit or implicit use has been made of Axiom VI, or VII, respectively, in establishing the theorem of that section.

Now if Z' is any other set constituted as required by the axiom, there corresponds to it a smallest subset Z'_0 having the same property, exactly as Z_0 corresponds to Z. But now the intersection $[Z_0, Z'_0]$, which is a common subset of Z and Z', must be constituted in the same way as Z and Z'; and just as, being a subset of Z, it must contain the component Z_0, so, as a subset of Z', it must contain the component Z'_0. According to Axiom I it then necessarily follows that $[Z_0, Z'_0] = Z_0 = Z'_0$ and that Z_0 thus is the *common component of all possible sets constituted like* Z, even though these need not be elements of a set. The set Z_0 contains the elements 0, $\{0\}$, $\{\{0\}\}$, and so forth, and it may be called the *number sequence*, because its elements can take the place of the numerals. It is the simplest example of a denumerably infinite set (below, No. 36).

§ 2. Theory of equivalence

From our point of view, the equivalence of two sets (*Cantor 1895*, p. 483) cannot be defined at first except for the case in which the sets are disjoint (No. 3); it is only afterward that the definition can be extended to the general case.

15. *Definition* A. Two disjoint sets M and N are said to be *immediately equivalent*, $M \sim N$, if their product MN (No. 13) possesses at least one subset Φ such that each element of $M + N$ occurs as an element in one and only one element $\{m, n\}$ of Φ. A subset Φ of MN thus constituted is called a *mapping of M onto N*; two elements m and n that occur together in one element of Φ are said to be "mapped onto each other"; they "correspond to each other", or one "is the image of the other".

16. If Φ is any subset of MN and therefore an element of $\mathfrak{U}(MN)$ and if x is any element of $M + N$, it is always definite (No. 4) whether the elements of Φ that contain x form a set consisting of a single element (No. 13). Thus it is also definite whether *all* elements x of $M + N$ possess this property, that is, whether Φ represents a mapping of M onto N or not. According to Axiom III, all of the mappings Φ therefore are the elements of a certain subset Ω of $\mathfrak{U}(MN)$, and it is definite whether Ω differs from 0 or not. *It is therefore always definite for two disjoint sets M and N whether they are equivalent or not.*

17. If two equivalent disjoint sets M and N are mapped onto each other by Φ, there also corresponds to each subset M_1 of M an equivalent subset N_1 of N under a mapping Φ_1, that is a subset of Φ.

For it is definite for every element $\{m, n\}$ of Φ whether $m \,\varepsilon\, M_1$ or not, and therefore all elements of Φ thus associated with M_1 are the elements of a subset Φ_1 of Φ. If we now denote by N_1 the intersection (No. 8) of $\mathfrak{S}\Phi_1$ with N, each element of $M_1 + N_1$ occurs as an element in only a single element of Φ_1, since otherwise it would occur more than once in Φ as well; and, according to No. 15, we in fact have $M_1 \sim N_1$.

18. If two disjoint sets M and N are disjoint from and equivalent to one and the same third set, R, or if $M \sim R$, $R \sim R'$, and $R' \sim N$, where each of these pairs of equivalent sets is assumed to be disjoint, then always also $M \sim N$.

Let the subset Φ of MR, the subset X of RR', and the subset Ψ of $R'N$ be three mappings (No. 15) that map M onto R, R onto R', and R' onto N, respectively. If then $\{m, n\}$ is any element of MN, it is definite whether there exist an element r of R and an element r' of R' such that $\{m, r\} \,\varepsilon\, \Phi$, $\{r, r'\} \,\varepsilon\, \mathrm{X}$, and $\{r', n\} \,\varepsilon\, \Psi$. All elements