IMADA
SDU

Spring 2021

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Advanced Linear Algebra (MM562/MM853)
Information sheet 5
Programme for week 14 and 15

Important! Lectures and exercises on April 14 are swapped!
So lectures are 8.15-10.00 and exercises are 10.15-12.00.

Lectures.

- Week 14: Advanced Vector Spaces: Section 8, page 65-69.
- Week 15: Advanced Vector Spaces: Section 8, page 70-73 (except for 8.21 and 8.22 that we skip for now).

Exercises.

Exercises related to the lecture in week 14.

1. Consider $\mathbb{R}^2$ endowed with the usual dot-product as inner product. Give a geometric interpretation of what it means for two vectors to be orthogonal. Also, what does Lemma 8.3 have to do with the usual Pythagoras theorem about right angled triangles?
2. If you have not done so already, try out the Gram-Schmidt orthonormalization algorithm on a set of vectors. For instance $\{(2,0,0),(1,1,0),(1,1,1)\} \subset \mathbb{R}^3$.
3. Let V be an inner product space and let $X \subseteq V$ be a subset. Show that $(\mathrm{span}(X))^\perp = X^\perp$
4. Let V be an inner product space. Show that $d(x,y) := \|x-y\|$ defines a metric on V.
5. If time permits, understand how Theorem 8.11 says that the orthogonal projection of $y \in V$ onto U is the vector in U closest to y with respect to the metric from the previous question.
6. Consider the space $C([0,1],\mathbb{R})$ of continuous functions on $[0,1]$. Show that
$$\langle f,g\rangle = \int_0^1 f(x)g(x)dx,$$
defines an inner product on $C([0,1],\mathbb{R})$.
7. Let $1_{[0,1]}$ denote the function on $[0,1]$ which is constantly equal to one. Show that $E(f) = (\int_0^1 f(x)dx) \cdot 1_{[0,1]}$ is the orthogonal projection onto $\mathrm{span}(1_{[0,1]})$.

Exercises related to the lecture in week 15. I know that you don't have time to go through all of these problems, as you don't have time to prepare them between the lecture and the exercises. I will likely post some of them again for the following week exercise session.

1. Consider the map $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $A(x_1, x_2) = (2x_1, x_1 - x_2)$. Determine the adjoint A^* with respect to the ordinary inner product on $\mathbb{R}^2$.
2. Let V be an inner product space and consider the map $\Phi\colon V \to V'$ given by $\Phi(v) = \langle -\,, v\rangle$ that we studied in the lecture. Show that this is indeed an antilinear map as I claimed.
3. Let U and V be finite dimensional inner product spaces and let $A \in \operatorname{Hom}(U, V)$ be given. Show that the map A^* that we constructed in the lecture is actually linear, so that have defined an element in $\operatorname{Hom}(V, U)$.
4. Consider an inner product space V over $\mathbb{C}$ and let $\lambda \in \mathbb{C}$ have $|\lambda| = 1$. Show that the map $U\colon V \to V$ given by $U(v) = \lambda v$ is a unitary and determine its adjoint.
5. Let V be an inner product space and $U\colon V \to V$ a unitary. Show that U is an isometry with respect to the metric d studied in the exercise above (i.e. prove that $d(Ux, Uy) = d(x, y)$ for all $x, y \in V$).
6. Consider the linear map $A\colon \mathbb{C}^2 \to \mathbb{C}^3$ given by $A(z_1, z_2) = (z_1, z_2, 0)$. Determine A^* and show that A satisfies the relation $A^*A = I_{\mathbb{C}^2}$ but not the relation $AA^* = I_{\mathbb{C}^3}$, thus showing the the assumption in Lemma 8.23 is necessary.
7. Prove Lemma 8.23.
8. If you have not already done so, read and understand the proof of Lemma 8.19.

An additional challenge for those so inclined. Show that Theorem 8.12 does not hold true without the assumption of finite-dimensionality.
Hint: One could use $V := \ell^2(\mathbb{N}) := \left\{(x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}, \sum_{n=1}^{\infty} |x_n|^2\right\}$ with inner product given by

$$\langle (x_n)_n, (y_n)_n \rangle := \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

If you have not seen this inner product space before, you may either try to prove that the above formula does indeed provide $\ell^2(\mathbb{N})$ with an inner product, or you can simply choose to go ahead and accept this as a fact without proof. Now find a subspace $U \leq \ell^2(\mathbb{N})$ such that $\ell^2(\mathbb{N}) \neq U \oplus U^\perp$.