Advanced Linear Algebra

Week 8

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Let U be a finite-dimensional vector space with ordered basis $\mathcal{B} = (x_1, \dots, x_m)$.
Any $u \in U$ can be written uniquely as

$$u = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_m x_m.$$

This induces a vector ${}_{\mathcal{B}}[u] = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \in \mathcal{F}^m$.

Note that x_j is written

$$x_j = 0 \cdot x_1 + \cdots + 0 \cdot x_{j-1} + 1 \cdot x_j + 0 \cdot x_{j+1} + \cdots + 0 \cdot x_m$$

so ${}_{\mathcal{B}}[x_j]$ is the vector (often denoted e_j) in $\mathcal{F}^m$ with a 1 at the j'th coordinate, and 0's everywhere else.

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Let U and V be finite-dimensional vector spaces with ordered bases $\mathcal{B} = (x_1, \ldots, x_m)$ and $\mathcal{C} = (y_1, \ldots, y_n)$ respectively. We also fix an ordering on the bases as indicated. Let $A \in \mathrm{Hom}(U, V)$. We define coefficients α_{ij} as follows: Write

$$Ax_j = \alpha_{1j} y_1 + \alpha_{2j} y_2 + \cdots + \alpha_{ij} y_i + \cdots + \alpha_{nj} y_n.$$

Definition

If $A \in \mathrm{Hom}(U, V)$, then the matrix of A is ${}_{\mathcal{C}}[A]_{\mathcal{B}} = [A] \in M_{n,m}(\mathcal{F})$ with the coefficients α_{ij} as defined above (with respect to the fixed ordered bases).

Note: the j'th column in ${}_{\mathcal{C}}[A]_{\mathcal{B}}$ is exactly the vector

$$\begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix} = {}_{\mathcal{C}}[Ax_j].$$

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Let U, V have ordered bases $\mathcal{B} = (x_1, \ldots, x_m)$, $\mathcal{C} = (y_1, \ldots, y_n)$ and $A \in \mathrm{Hom}(U, V)$ with induced matrix ${}_{\mathcal{C}}[A]_{\mathcal{B}} \in M_{n,m}(\mathcal{F})$ with elements α_{ij}.
Let $u \in U$ and write $u = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_m x_m$ and ${}_{\mathcal{B}}[u] = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$ in $\mathcal{F}^m$ be the induced vector wrt the basis $\mathcal{B} = (x_1, \ldots, x_m)$.
Similarly ${}_{\mathcal{C}}[Au] \in \mathcal{F}^n$ is the induced vector wrt the basis $\mathcal{C} = (y_1, \ldots, y_n)$, then

$${}_{\mathcal{C}}[Au] = {}_{\mathcal{C}}[A]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[u].$$

Remember: the j'th column in ${}_{\mathcal{C}}[A]_{\mathcal{B}}$ is exactly the vector ${}_{\mathcal{C}}[Ax_j]$.
One can think of this as ${}_{\mathcal{C}}[A]_{\mathcal{B}} = ({}_{\mathcal{C}}[Ax_1] \; {}_{\mathcal{C}}[Ax_2] \; \ldots \; {}_{\mathcal{C}}[Ax_m])$ (interpreted in the right way).
Consider $\mathbb{C}$ as an $\mathbb{R}$-vector space with ordered basis $\mathcal{B} = (1, i)$.
Complex conjugation $A \in \mathrm{Hom}(\mathbb{C}, \mathbb{C})$ by

$$A(\alpha + \beta i) = \alpha - \beta i \qquad \textit{for} \quad \alpha, \beta \in \mathbb{R}.$$

Question: What is the matrix ${}_{\mathcal{B}}[A]_{\mathcal{B}}$?

(1) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$;

(2) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

(3) $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$;

(4) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

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Remember: the j'th column in ${}_{\mathcal{C}}[A]_{\mathcal{B}}$ is exactly the vector ${}_{\mathcal{C}}[Ax_j]$.
One can think of this as ${}_{\mathcal{C}}[A]_{\mathcal{B}} = ({}_{\mathcal{C}}[Ax_1] \ {}_{\mathcal{C}}[Ax_2] \ \ldots \ {}_{\mathcal{C}}[Ax_m])$
Consider $\mathbb{C}$ as an $\mathbb{R}$-vector space with ordered basis $\mathcal{B} = (1, i)$.
Complex conjugation $A \in \mathrm{Hom}(\mathbb{C}, \mathbb{C})$ by

$$A(\alpha + \beta i) = \alpha - \beta i \qquad \textit{for} \quad \alpha, \beta \in \mathbb{R}.$$

Question: What is the matrix ${}_{\mathcal{B}}[A]_{\mathcal{B}}$?
Here $x_1 = 1$ and $x_2 = i$.
First column: $Ax_1 = A(1) = 1 = 1 \cdot 1 + 0 \cdot i$.
So ${}_{\mathcal{B}}[Ax_1] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Second column: $Ax_2 = A(i) = -i = 0 \cdot 1 + (-1) \cdot i$.
So ${}_{\mathcal{B}}[Ax_2] = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.
Hence ${}_{\mathcal{B}}[A]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let U, V, W be finite dimensional $\mathcal{F}$-vector spaces, and let $A \in \mathrm{Hom}(U, V)$ and $B \in \mathrm{Hom}(V, W)$.
Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be ordered bases for U, V, W respectively.
Then

$$_{\mathcal{D}}[B]_{\mathcal{C}} \cdot {}_{\mathcal{C}}[A]_{\mathcal{B}} = {}_{\mathcal{D}}[BA]_{\mathcal{B}}.$$

This is checked just like $_{\mathcal{C}}[A]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[u] = {}_{\mathcal{C}}[Au]$ (which we proved last lecture).

Lemma (2.2)

$\mathrm{Hom}(U, V)$ *is a vector space over* $\mathcal{F}$ *defined by operations*

$$(A + B)(x) = A(x) + B(x), \qquad (\alpha A)(x) = \alpha A(x)$$

for $A, B \in \mathrm{Hom}(U, V)$, $\alpha \in \mathcal{F}$ *and* $x \in U$.
Moreover, the zero vector in $\mathrm{Hom}(U, V)$ *is the zero map.*

Proof.

Omitted (straight forward). $\square$

The special case $M_{n,m}(\mathcal{F})$ is a vector space where each matrix entry is "represents a dimension".
More formally, there is a canonical basis for $M_{n,m}(\mathcal{F})$: the matrices e_{ij} with a 1 in the (i,j)'th entry, and 0's everywhere else form such a basis.
2 × 2-case: in $M_{2,2}(\mathcal{F})$ the canonical (non-ordered) basis is

$$\{\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{e_{11}}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e_{12}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{e_{21}}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{e_{22}}\}.$$

In fact, any $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in M_{2,2}(\mathcal{F})$ is uniquely of the form

$$\alpha_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Suppose $\mathcal{B} = (x_1, \ldots, x_m)$ and $\mathcal{C} = (y_1, \ldots, y_n)$ are ordered bases of U, V respectively. Define $E_{ij} \in \mathrm{Hom}(U, V)$ to be the unique linear map (Lemma 2.3) satisfying

$$E_{ij}(x_k) = \begin{cases} y_i, & \text{if } k = j \\ 0, & \text{otherwise.} \end{cases}$$

The matrix ${}_{\mathcal{C}}[E_{ij}]_{\mathcal{B}} \in M_{n,m}(\mathcal{F})$ is precisely the $n \times m$-matrix e_{ij} with 1 in the (i, j)'th entry and 0 everywhere else.

Theorem (2.20)

The maps E_{ij} form a basis for $\mathrm{Hom}(U, V)$. In particular,

$$\dim \mathrm{Hom}(U, V) = \dim(U) \cdot \dim(V).$$

$$E_{ij}(x_k) = \begin{cases} y_i, & \text{if } k = j \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

For $A \in \mathrm{Hom}(U, V)$ and $k = 1, \ldots, m$ there are unique $\alpha_{ik} \in \mathcal{F}$ such that

$$Ax_k = \sum_{i=1}^{n} \alpha_{ik} y_i \in V.$$

Also

$$\sum_{i,j} \alpha_{ij} E_{ij}(x_k) = \sum_{i=1}^{n} \alpha_{ik} y_k = Ax_k$$

so $A = \sum_{i,j} \alpha_{ij} E_{ij}$ (since linear maps are uniquely determined by their values on $\{x_1, \ldots, x_m\}$).

So $\mathrm{Span}\{E_{ij} : i = 1, \ldots, n, j = 1, \ldots, m\} = \mathrm{Hom}(U, V)$.

Proof continued.

It is easy to see that different scalars α_{ij} yield different maps $\sum_{i,j} \alpha_{ij} E_{ij}$, so the set of E_{ij} is linearly independent. Hence also a basis for $\mathrm{Hom}(U, V)$.

Since there are $n \cdot m$ different maps E_{ij} for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the dimension is

$$\dim \mathrm{Hom}(U, V) = nm = \dim U \cdot \dim V.$$

$\square$

Consider $\mathbb{C}$ as an $\mathbb{R}$-vector space.

Question: What is the dimension of $\mathrm{Hom}(\mathbb{C}, \mathbb{C})$?

(1) 1

(2) 2

(3) 3

(4) 4

Answer: $\dim \mathbb{C} = 2$ so

$$\dim \mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \dim \mathbb{C} \cdot \dim \mathbb{C} = 4.$$

Note if we instead consider $\mathbb{C}$ as a $\mathbb{C}$-vector space, we have

$$\dim \mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \dim \mathbb{C} \cdot \dim \mathbb{C} = 1 \cdot 1 = 1.$$

Let V be an n-dimensional vector space, and let $\mathcal{B} = (x_1, \dots, x_n)$ and $\mathcal{C} = (y_1, \dots, y_n)$ be two ordered bases for V.

Consider the identity map I from V with basis $\mathcal{C}$ to V with basis $\mathcal{B}$.

Let ${}_{\mathcal{B}}[\mathrm{I}]_{\mathcal{C}} = \mathsf{P} \in M_n(\mathcal{F})$ be the induced matrix, i.e. we write

$$y_j = \sum_{i=1}^{n} p_{ij} x_j$$

and P is the matrix with entries p_{ij}.

P is called the transition matrix or change of basis matrix (from $\mathcal{C}$ to $\mathcal{B}$).

Note that

$${}_{\mathcal{C}}[\mathrm{I}]_{\mathcal{B}}\mathsf{P} = {}_{\mathcal{C}}[\mathrm{I}]_{\mathcal{B}\mathcal{B}}[\mathrm{I}]_{\mathcal{C}} = {}_{\mathcal{C}}[\mathrm{I} \cdot \mathrm{I}]_{\mathcal{C}} = {}_{\mathcal{C}}[\mathrm{I}]_{\mathcal{C}} = 1$$

where 1 is the identity matrix. pause Hence P is invertible.

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Lemma (2.25 (reformulated))

Let $u \in V$. Then $\mathrm{P}\,{}_{\mathcal{C}}[u] = {}_{\mathcal{B}}[u]$.

What does this say?

Suppose that we know the matrix P, and that we know the "old" $\mathcal{C}$-coordinates for $u \in V$ (so we know ${}_{\mathcal{C}}[u]$), i.e. we know $\gamma_1, \ldots, \gamma_n \in \mathcal{F}$ such that $u = \sum_{i=1}^{n} \gamma_i y_i$.
Then we can find the "new" $\mathcal{B}$-coordinates of u (in the form of ${}_{\mathcal{B}}[u]$) by computing $\mathrm{P}\,{}_{\mathcal{C}}[u]$.

Proof.

$$\mathrm{P}\,{}_{\mathcal{C}}[u] = {}_{\mathcal{B}}[\mathrm{I}]_{\mathcal{C}} \cdot {}_{\mathcal{C}}[u] = {}_{\mathcal{B}}[\mathrm{I}(u)] = {}_{\mathcal{B}}[u]$$

□

Theorem (2.26 (reformulated))

Let V be an n-dimensional vector space, and let $\mathcal{B} = (x_1, \ldots, x_n)$ and $\mathcal{C} = (y_1, \ldots, y_n)$ be two ordered bases for V. Construct the transition matrix $\mathsf{P} = {}_{\mathcal{B}}[\mathrm{I}]_{\mathcal{C}}$. Let $A \in \mathrm{End}(V)$. Then

$$ {}_{\mathcal{B}}[A]_{\mathcal{B}} = \mathsf{P}_{\mathcal{C}}[A]_{\mathcal{C}}\mathsf{P}^{-1}. $$

What does this say?

If we know P, P^{-1} and the matrix for A wrt $\mathcal{C}$, then we can compute the matrix for A wrt $\mathcal{B}$.

Proof.

We saw earlier that $\mathsf{P}^{-1} = {}_{\mathcal{C}}[\mathrm{I}]_{\mathcal{B}}$. Hence

$$ \mathsf{P}_{\mathcal{C}}[A]_{\mathcal{C}}\mathsf{P}^{-1} = {}_{\mathcal{B}}[\mathrm{I}]_{\mathcal{C}\mathcal{C}}[A]_{\mathcal{C}\mathcal{C}}[\mathrm{I}]_{\mathcal{B}} = {}_{\mathcal{B}}[\mathrm{I} \cdot A \cdot I]_{\mathcal{B}} = {}_{\mathcal{B}}[A]_{\mathcal{B}} $$

Theorem (2.26 (reformulated))

Let V be an n-dimensional vector space, and let $\mathcal{B} = (x_1, \dots, x_n)$ and $\mathcal{C} = (y_1, \dots, y_n)$ be two ordered bases for V. Construct the transition matrix $\mathsf{P} = {}_{\mathcal{B}}[\mathrm{I}]_{\mathcal{C}}$. Let $A \in \mathrm{End}(V)$. Then

$$ {}_{\mathcal{B}}[A]_{\mathcal{B}} = \mathsf{P}\,{}_{\mathcal{C}}[A]_{\mathcal{C}}\mathsf{P}^{-1}. $$

Note: that one also obtains

$$ {}_{\mathcal{C}}[A]_{\mathcal{C}} = \mathsf{P}^{-1}\,{}_{\mathcal{B}}[A]_{\mathcal{B}}\mathsf{P}. $$

(This is how Theorem 2.26 is stated in the notes).

Two $n \times n$-matrices A, B are called similar if there exists an invertible $n \times n$-matrix P such that $\mathsf{A} = \mathsf{P}\mathsf{B}\mathsf{P}^{-1}$.

Theorem 2.26 implies that if A and B are induced by the same endomorphism wrt different bases, then they are similar.

Consider $\mathbb{R}^2$ with the standard ordered basis
$\mathcal{B} = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ and the ordered basis
$\mathcal{C} = (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix})$.

Question: what is the transition matrix $\mathsf{P} = {}_{\mathcal{B}}[\mathrm{I}]_{\mathcal{C}}$?

(1) $\mathsf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

(2) $\mathsf{P} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$;

(3) $\mathsf{P} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

Answer: $\mathsf{P} = ({}_{\mathcal{B}}[\mathrm{I}(\begin{pmatrix} 1 \\ 1 \end{pmatrix})]\ {}_{\mathcal{B}}[\mathrm{I}(\begin{pmatrix} 2 \\ 3 \end{pmatrix})]) = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$.

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Let V be an $\mathcal{F}$-vector space.

Definition (3.1)

The vector space

$$V' := \mathrm{Hom}(V, \mathcal{F}) = \{y \colon V \to \mathcal{F} \mid y \text{ is linear}\}$$

is called the dual space of V.
The elements $y \in V'$ are called linear functionals (or linear forms).

Let $\mathcal{B} = \{x_i \mid i \in I\}$ be a basis for V parametrised by some index set I.
Recall that any $x \in V$ can uniquely be written as $x = \sum_{i \in I} \alpha_i x_i$ for scalars α_i (only finitely many α_i non-zero).

Definition

Let $i \in I$. The linear functional $y_i \in V'$ given by

$$y_i\Big(\sum_{j \in I} \alpha_j x_j\Big) = \alpha_i$$

is called the i'th coordinate functional.

Note: Depends on $\mathcal{B}$ and the index I.
Note that $y_i \colon V \to \mathcal{F}$ is the unique linear map given on the basis by

$$y_i(x_j) = \begin{cases} 1_{\mathcal{F}} & \text{if } i = j \\ 0_{\mathcal{F}} & \text{if } i \neq j. \end{cases}$$

Note also, that $x = \sum_{i \in I} \alpha_i x_i = \sum_{i \in I} y_i(x) x_i$.

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Definition

Let $i \in I$. The linear functional $y_i \in V'$ given by

$$y_i(\sum_{j \in I} \alpha_j x_j) = \alpha_i$$

is called the i'th coordinate functional.

Consider $\mathbb{C}$ as an $\mathbb{R}$-vector space with ordered basis $(1, i)$.
Let $y_1, y_2 \in \mathbb{C}'$ be the 1st and 2nd coordinate functionals.
Question: What is $y_2(2 + i)$?

(1) 1

(2) 2

(3) 3

(4) -1

Answer: we have $y_2(\alpha + \beta i) = \beta$, so $y_2(2 + 1 \cdot i) = 1$.
Similarly, $y_1(\alpha + \beta i) = \alpha$, so $y_1(2 + i) = 2$.

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$\mathcal{B} = \{x_i \mid i \in I\}$ is a basis parametrised by I.
Let $\mathcal{B}' = \{y_i \mid i \in I\} \subseteq V'$ be the coordinate functionals.

Theorem (3.4)

The set $\mathcal{B}'$ is linearly independent. If $\dim V < \infty$ then

$$y = \sum_{i \in I} y(x_i) y_i \qquad \textit{for all } y \in V'$$

and $\mathcal{B}'$ is a basis for the dual space V'.

Proof.

Linear independence: Suppose $i_1, \ldots, i_n \in I$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{F}$ such that $\sum_{j=1}^{n} \alpha_j y_{i_j} = 0$. Then for $k = 1, \ldots, n$ we have

$$0 = \sum_{j=1}^{n} \alpha_j y_{i_j}(x_{i_k}) = \alpha_k$$

so all the coefficients $\alpha_1, \ldots, \alpha_n$ are 0. Hence $\mathcal{B}'$ is linearly independent.

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Theorem (3.4)

The set $\mathcal{B}'$ is linearly independent. If $\dim V < \infty$ then

$$y = \sum_{i \in I} y(x_i) y_i \qquad \textit{for all } y \in V'$$

and $\mathcal{B}'$ is a basis for the dual space V'.

Proof.

Suppose $\dim V < \infty$. Recall that any $x \in V$ can be written as $x = \sum_{i \in I} y_i(x) x_i$. Let $y \in V'$. Then

$$y(x) = y(\sum_{i \in I} y_i(x) x_i) = \sum_{i \in I} y(x_i) y_i(x) = \left(\sum_{i \in I} y(x_i) y_i \right)(x)$$

for all $x \in V$. Hence $y = \sum_{i \in I} y(x_i) y_i$.
Since $y(x_i)$ are scalars, it follows that $\mathrm{Span}\mathcal{B}' = V'$, so $\mathcal{B}'$ is a basis.

Definition

When V is finite-dimensional with basis $\mathcal{B}$, then $\mathcal{B}'$ (of coordinate functionals) is called the dual basis og $\mathcal{B}$.

Note that the dual basis $\mathcal{B}'$ is a basis for V' (Theorem 3.4). If V is infinite-dimensional then $\mathcal{B}'$ can still be defined, but it is no longer a basis!

Corollary (3.5)

If V is a finite-dimensional vector space, then $\dim V' = \dim V$.
If V is an infinite dimensional vector space, then so is V'.

Consider the $\mathbb{R}$-vector space $\mathbb{C}^2$.

Question: What is the dimension of the dual space $(\mathbb{C}^2)'$?

(1) 1

(2) 2

(3) 3

(4) 4.

Answer: $\dim \mathbb{C}^2 = 4$, so $(\mathbb{C}^2)'$ has the same dimension.