Advanced Linear Algebra

Week 19

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Recall that for $A \in \mathrm{End}(V)$ and $\lambda \in \mathcal{F}$, the generalised eigenspace is

$$M_\lambda = \{x \in V : \exists k \in \mathbb{N} \text{ s.t. } (A - \lambda I)^k x = 0\} = \bigcup_{k \in \mathbb{N}} N((A - \lambda I)^k).$$

Note that when $\dim V < \infty$, then M_λ is the space N in Theorem 7.5 for the endomorphism $A - \lambda I$. So there is a unique $A - \lambda I$-invariant subspace R so that $V = M_\lambda \oplus R$, $(A - \lambda I)|_R$ is invertible and $(A - \lambda I)|_{M_\lambda}$ is nilpotent.

Theorem ((Main part of) 7.8 - Jordan decomposition)

Assume $\dim V < \infty$ *and let* $A \in \mathrm{End}(V)$*. Let* $\lambda_1, \ldots, \lambda_m$ *be the eigenvalues of* A*. There exists a unique* A*-invariant subspace* $R \subseteq V$ *such that*

$$V = M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m} \oplus R.$$

Moreover, $A|_R$ *has no eigenvectors.*

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Corollary 7.9

If $\mathcal{F}$ is algebraically closed then $V = M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_m}$.

We now go back to inner product spaces V.
Recall that $A \in \mathrm{End}(V)$ has an adjoint A^*, if $\langle Ax, y\rangle = \langle x, A^* y\rangle$ for all $x, y \in V$.
The adjoint corresponds to the conjugate transpose of a matrix.

Definition (9.6)

$A \in \mathrm{End}(V)$ is called normal if it has an adjoint, and $AA^* = A^* A$.

Lemma (9.7)

Let $A \in \mathrm{End}(V)$ be normal.

(1) $\|Ax\| = \|A^* x\|$ *for all* $x \in V$;

(2) $N(A) = N(A^*)$;

(3) *The* λ *eigenspace of* A *equals the* $\overline{\lambda}$ *eigenspace of* A^* *for each* $\lambda \in \mathcal{F}$;

(4) *All eigenspaces of* A *are orthogonal to each other.*

Normal: $A^*A = AA^*$.

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Recall Lemma 8.17: $N(A^*) \perp R(A)$. Hence $N(A^*) \cap R(A) = \{0\}$.

Lemma (9.8)

If $A \in \mathrm{End}(V)$ is both nilpotent and normal, then $A = 0$.

Recall that if $\dim V < \infty$ then $A \in \mathrm{End}(V)$ is orthogonally diagonalisable if $V = \bigoplus_{\lambda \in \sigma(A)} V_\lambda$ is an orthogonal direct sum.

Theorem (9.9, Spectral theorem for normal maps)

Let $\mathcal{F} = \mathbb{C}$ and $\dim V < \infty$. Then $A \in \mathrm{End}(V)$ is orthogonally diagonalisable if and only if A is normal.