IMADA
SDU

Spring 2021
Jamie Gabe

Advanced Linear Algebra (MM562/MM853)
Information sheet 4
Programme for week 11 and 12

Lectures.

- Week 11: Advanced Vector Spaces: Page 29-30 and 42-44.
- Week 12: Advanced Vector Spaces: first half of Section 6.

Exercises.

Exercises related to the lecture in week 11.

1. Check, if you have not already done so, that the dot-product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is bilinear.
2. Work through Example 4.2 (2) in the lecture notes. If you prefer, you may assume that $n = m = 2$. What is the connection between this example and the dot-product exercise above?
3. (Halmos, §25, exercise 1) Let $\{e_1, e_2\}$ and $\{e_1, e_2, e_3\}$ denote the standard basis for $\mathbb{R}^2$ and $\mathbb{R}^3$, and consider $x = (1, 1) \in \mathbb{R}^2$ and $y = (1, 1, 1) \in \mathbb{R}^3$. Determine the coordinates of $x \otimes y \in \mathbb{R}^2 \otimes \mathbb{R}^3$ with respect to the basis $\{e_i \otimes e_j \mid i = 1, 2; j = 1, 2, 3\}$.
4. (Function spaces) Let X be a finite set[1] and consider the space $\mathcal{F}(X)$ consisting of all functions $f \colon X \to \mathbb{C}$.

   (a) If you have not already done so, check that $\mathcal{F}(X)$ is a vector space with respect to the coordinate-wise operations $(f + g)(x) := f(x) + g(x)$ and $(\alpha \cdot f)(x) = \alpha f(x)$.

   (b) For a fixed $x_0 \in X$ consider the *Dirac mass* at x_0; i.e. the function given by

   $$\delta_{x_0}(x) := \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$$

   Show that $\{\delta_x \mid x \in X\}$ is a basis for $\mathcal{F}(X)$.

   (c) Let Y be another finite set and show that $\mathcal{F}(X) \otimes \mathcal{F}(Y)$ is isomorphic to $\mathcal{F}(X \times Y)$. *Hint:* consider the natural bases for the two spaces in question and build a map that maps one to the other and then extend by linearity.

[1] So, here X just denotes a set even though it often in this course denotes a vector space – don't let this confuse you.

5. (Slice maps) Let U, V be finite dimensional vector spaces over a common field $\mathcal{F}$ and let $f \in U'$ be given. Show that there exists a linear map $T\colon U \otimes V \to V$ with the property that $T(u \otimes v) = f(u)v$ for all $u \in U$ and $v \in V$. The map T is often denoted $f \otimes I$ and is called the *slice map* associated with f

6. (A non-commutativity result) Let V be a finite dimensional vector space and consider the tensor product $V \otimes V$. For $x, y \in V \setminus \{0\}$, show that if x and y are linearly independent then $x \otimes y \neq y \otimes x$.
   Hint: use Theorem 3.11 together with the exercise on slice maps.

7. (A commutativity result) Let U, V be finite dimensional vector spaces over a common field $\mathbb{F}$. Show that $U \otimes V \simeq V \otimes U$ by providing an explicit isomorphism.

Exercises related to the lecture in week 12.

1. If you have not done so already, determine a basis for $\mathbb{R}^2$ consisting of eigenvectors for $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

2. Consider the trivial example vector space $V = \{0\}$, which I made a point out of excluding in the lecture. What is $\mathrm{End}(V)$? What is the minimal polynomial for the (there is only one) operator in $\mathrm{End}(V)$? And why does this operator not have any eigenvectors?

3. Consider the endomorphisms

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in \mathbb{M}_2(\mathbb{R}) = \mathrm{End}(\mathbb{R}^2) \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathbb{M}_3(\mathbb{R}) = \mathrm{End}(\mathbb{R}^3),$$

   and determine their minimal polynomials.

4. Drawing on your experience from the previous exercise, determine the minimal polynomial of an arbitrary diagonal matrix?

An additional challenge for those so inclined. Let $\bar{\mathbb{Q}}$ denote the set of all complex numbers λ such that there exists a $p \in \mathbb{Q}[x]$ with $p(\lambda) = 0$. The set $\bar{\mathbb{Q}}$ is called the set of algebraic numbers.

(a) Show that one could equally well replace $\mathbb{Q}[x]$ with $\mathbb{Z}[x]$ in the definition of $\bar{\mathbb{Q}}$.

(b) Show that $\bar{\mathbb{Q}}$ is countable. You have thus proven the existence of uncountable many non-algebraic numbers, without pointing out a single one! However, it can be proven that the natural suspects, like π and e, are not algebraic.

(c) Show that if $\lambda \in \mathbb{C}$ is an eigenvalue of a matrix in $\mathbb{M}_n(\mathbb{Q})$ for some $n \in \mathbb{N}$ then λ is algebraic (i.e. in $\bar{\mathbb{Q}}$).

(d) Show that the converse is also true. This is more tricky, and you may want to look op the so-called *companion matrix* on wikipedia for this part. Or, you can simply assume it to be true and move on.

(e) Show that $\bar{\mathbb{Q}}$ is a field, by showing the following:

- If $\alpha \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of a matrix with rational entries, then this matrix can be chosen invertible and α^{-1} is an eigenvalue of the inverse matrix.
- Show that if $\alpha, \beta \in \bar{\mathbb{Q}}$ are eigenvalues of A and B with eigenvectors x and y, respectively, then $\alpha\beta$ and $\alpha + \beta$ are eigenvalues for $A \otimes B$ and $A \otimes I + I \otimes B$ with eigenvector $x \otimes y$ in both cases. (contemplate why one may always assume that the matrices A and B have the same size, so that $A \otimes I + I \otimes B$ makes sense)