Advanced Linear Algebra

Week 7

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Recall from group theory: Let $(G, +)$ be an abelian group, and $H \subseteq G$ be a subgroup (automatically normal in the abelian case).
For each $x \in G$ define the coset

$$x + H = \{x + h \mid h \in H\} \subseteq G.$$

The quotient $G/H = \{x + H \mid x \in G\}$ is the set of all such cosets, and it is an abelian group with

$$(x + H) + (y + H) = (x + y) + H.$$

Recall that an $\mathcal{F}$-vector space V is an abelian group when equipped with $+$. And any subspace $U \subseteq V$ is a subgroup.
Form the quotient

$$V/U = \{x + U \mid x \in V\}$$

which is an $\mathcal{F}$-vector space with scalar multiplication

$$\alpha(x + U) := \alpha x + U.$$

Lemma (2.12)

Let V be an $\mathcal{F}$-vector space, and $U \subseteq V$ be a subspace. Then V/U (as before) is an $\mathcal{F}$-vector space. Moreover, the map $\pi\colon V \to V/U$ given by

$$\pi(x) = x + U, \qquad \textit{for } x \in V$$

is a surjective linear map with null-space $N(\pi) = U$.

Proof.

Omitted (this is straightforward). □

Definition

V/U is called the quotient space of V by U, and the linear surjection $\pi\colon V \to V/U$ is called the (canonical) projection.

In what follows, let $U \subseteq V$ be a subspace.

Theorem (2.14)

Let $C \subseteq U$ be a basis, and let $B \subseteq V$ be a basis such that $C \subseteq B$.
Then the projection π maps $B \setminus C$ bijectively onto a basis for V/U.
In particular, if $\dim V < \infty$, *then* $\dim(V/U) = \dim V - \dim U$.

Proof.

Let $W := \mathrm{Span}(B \setminus C)$. We will show that $\pi|_W \colon W \to V/U$ is a linear isomorphism.
Note that $V/U = \pi(V) = \pi(\mathrm{Span} B) = \mathrm{Span} \pi(B)$. Since $C \subseteq U = N(\pi)$, we have $\pi(C) = \{0\}$. Then $\pi(W) = \mathrm{Span} \pi(B \setminus C) = V/U$. So $\pi|_W$ is surjective. $\square$

Theorem (2.14)

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Proof.

$W = \mathrm{Span}(B \setminus C)$ and $\pi|_W \colon W \to V/U$ is surjective.
As B is linearly independent, so are C and $B \setminus C$. Hence C and $B \setminus C$ are bases for $U = \mathrm{Span}\, C$ and $W = \mathrm{Span}(B \setminus C)$ respectively. Any $x \in U \cap W$ can be written as a unique linear combination from B, C and $B \setminus C$. This implies that $x = 0$ is the only option, so $U \cap W = \{0\}$. Hence $N(\pi|_W) = N(\pi) \cap W = U \cap W = \{0\}$, so $\pi|_W$ is injective and thus bijective.
So $\pi|_W \colon W \to V/U$ is a linear isomorphism.
The rest follows easily.

Theorem (2.14)

Let $C \subseteq U$ be a basis, and let $B \subseteq V$ be a basis such that $C \subseteq B$. Then the projection π maps $B \setminus C$ bijectively onto a basis for V/U.

Consider the subspace $U = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \mid x \in \mathcal{F} \right\} \subseteq \mathcal{F}^2$.

Then $C = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is a basis for U, and

$B = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathcal{F}^2$.

Question: What can we conclude from Theorem 2.14?

(a) $\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} + U\}$ is a basis for $\mathcal{F}^2/U$;

(b) $\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + U\}$ is a basis for $\mathcal{F}^2/U$;

(c) Theorem 2.14 isn't applicable.

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Theorem (2.14)

Let $C \subseteq U$ be a basis, and let $B \subseteq V$ be a basis such that $C \subseteq B$. Then the projection π maps $B \setminus C$ bijectively onto a basis for V/U.

Consider the subspace $U = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} \mid x \in \mathcal{F} \right\} \subseteq \mathcal{F}^2$.

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$B = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathcal{F}^2$.

By Theorem 2.14, $B \setminus C = \{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$ maps bijectively onto a basis for $\mathcal{F}^2/U$, and thus $\pi(B \setminus C) = \{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} + U \}$ is a basis for $\mathcal{F}^2/U$.

Now, U, V are $\mathcal{F}$-vector spaces and $X \subseteq U$, $Y \subseteq V$ are subspaces.
Let $A \in \mathrm{Hom}(U, V)$ such that $A(X) \subseteq Y$.
If $u_1, u_2 \in U$ are such that $u_1 + X = u_2 + X$ then

$$u_1 - u_2 \in X \Rightarrow Au_1 - Au_2 \in Y \Rightarrow Au_1 + Y = Au_2 + Y.$$

Hence there is a quotient map $\overline{A}\colon U/X \to V/Y$ given by

$$\overline{A}(u + X) = Au + Y, \qquad u \in U.$$

It is easy to check that $\overline{A}$ is linear.
Special case (for $U = V$ and $X = Y$): Let $A \in \mathrm{End}(V)$. A subspace $X \subseteq V$ is A-invariant if $A(X) \subseteq X$.
In this case we get an induced $\overline{A} \in \mathrm{End}(V/X)$.

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Let $A \in \mathrm{Hom}(U, V)$ and consider $X = N(A) \subseteq U$ and $Y = \{0\} \subseteq V$. We get an induced $\overline{A} \colon U/N(A) \to V/\{0\} = V$ by

$$\overline{A}(u + N(A)) = Au.$$

Theorem (2.17 (a la first isomorphism theorem))

$\overline{A}$ defines an isomorphism $U/N(A) \to R(A)$.

Proof.

Clearly $R(A) = R(\overline{A})$, so $\overline{A}$ is surjects onto $R(A)$.
Also, $\overline{A}(u + N(A)) = Au = 0$ implies $u \in N(A)$, so $u + N(A) = 0$. Hence $N(\overline{A}) = \{0\}$.
So $\overline{A} \colon U/N(A) \to R(A)$ is an isomorphism. $\square$

Definition

For $A \in \mathrm{Hom}(U, V)$ we define

(a) $\mathrm{null}(A) = \dim N(A)$ called the nullity of A;

(b) $\mathrm{rank}(A) = \dim R(A)$ called the rank of A.

Corollary (2.19 (rank-nullity theorem))

If $\dim U < \infty$, *then*

$$\mathrm{rank}(A) + \mathrm{null}(A) = \dim U.$$

Proof.

As $U/N(A) \cong R(A)$ we have

$$\mathrm{rank}(A) = \dim(U/N(A)) = \dim U - \dim N(A) = \dim U - \mathrm{null}(A).$$

Consider the linear map $A\colon M_2(\mathbb{R}) \to \mathbb{R}$ of $\mathbb{R}$-vector spaces given by $A\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+b+c+d$.

Question What are the rank and nullity of A?

(a) $\mathrm{rank}(A) = 1$ and $\mathrm{null}(A) = 2$;

(b) $\mathrm{rank}(A) = 2$ and $\mathrm{null}(A) = 2$;

(c) $\mathrm{rank}(A) = 1$ and $\mathrm{null}(A) = 3$;

(d) $\mathrm{rank}(A) = 2$ and $\mathrm{null}(A) = 4$.

Answer: A is surjective so $\mathrm{rank}(A) = \dim \mathbb{R} = 1$. By rank-nullity we get

$$\mathrm{null}(A) = \dim M_2(\mathbb{R}) - \mathrm{rank}(A) = 4 - 1 = 3.$$

An $n \times m$-matrix over $\mathcal{F}$ is a matrix

$$[A] = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix} \text{ where all } \alpha_{ij} \text{ are in } \mathcal{F}.$$

$M_{n,m}(\mathcal{F})$ denotes the set of $n \times m$-matrices over $\mathcal{F}$.
Recall that $M_{n,m}(\mathcal{F}) = \mathrm{Hom}(\mathcal{F}^m, \mathcal{F}^n)$ in the following way: If $[A]$ is a matrix as above, we get a linear map $A\colon \mathcal{F}^m \to \mathcal{F}^n$ by

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^m \alpha_{1i}\beta_i \\ \vdots \\ \sum_{i=1}^m \alpha_{ni}\beta_i \end{pmatrix}.$$

If $A \in \mathrm{Hom}(\mathcal{F}^m, \mathcal{F}^n)$ we obtain the matrix coefficients α_{ij} by

$$Ae_j = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix} \in \mathcal{F}^n \text{ where } e_j \in \mathcal{F}^m \text{ is the vector with 1 in}$$

the j'th coordinate and zero everywhere else.

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If $A \in \mathrm{Hom}(\mathcal{F}^m, \mathcal{F}^n)$ we obtain the matrix coefficients α_{ij} by $Ae_j = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix} \in \mathcal{F}^n$ where $e_j \in \mathcal{F}^m$ is the vector with 1 in the j'th coordinate and zero everywhere else.

Note: this uses the standard (ordered) bases $\{e_1, \ldots, e_m\} \subseteq \mathcal{F}^m$ and(!) in order to extract α_{ij} from $Ae_j = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{pmatrix}$, we also use the standard (ordered) basis in $\mathcal{F}^n$ (to pick out the i'th coordinate).

Try to keep this idea in mind for the next slide!

Let U and V be finite-dimensional vector spaces with bases $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ respectively.
We also fix an ordering on the bases as indicated. E.g. we can talk about the first basis vector in U, in this case x_1, or the j'th basis vector, in this case x_j.
Let $A \in \mathrm{Hom}(U, V)$. We define coefficients α_{ij} as follows: Write

$$Ax_j = \alpha_{1j}y_1 + \alpha_{2j}y_2 + \cdots + \alpha_{ij}y_i + \cdots + \alpha_{nj}y_n.$$

Definition

If $A \in \mathrm{Hom}(U, V)$, then the matrix of A is $[A] \in M_{n,m}(\mathcal{F})$ with the coefficients α_{ij} as defined above (with respect to the fixed ordered bases).

Warning: This depends (very much!) on the bases and the ordering!

Let $A \in \mathrm{Hom}(U, V)$. We define coefficients α_{ij} as follows: Write

$$Ax_j = \alpha_{1j}y_1 + \alpha_{2j}y_2 + \cdots + \alpha_{ij}y_i + \cdots + \alpha_{nj}y_n.$$

Question: Consider $\mathbb{R}^2$ with ordered basis $\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right)$. Consider the linear map $A \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$A\begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \beta + \gamma \\ \beta + \gamma \end{pmatrix}.$$

What is the α_{11} coefficient in the matrix of A (with respect to the given ordered bases)?

(a) 0

(b) 1

(c) 2

Question: Consider $\mathbb{R}^2$ with ordered basis $\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right)$. Consider the linear map $A \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$A \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \beta + \gamma \\ \beta + \gamma \end{pmatrix}.$$

What is the α_{11} coefficient in the matrix of A (with respect to the given ordered bases)?

$$A \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \tfrac{2}{3} \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

So $\alpha_{11} = 1$. We also see that $\alpha_{21} = \frac{2}{3}$. Also

$$A \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \tfrac{3}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

so $\alpha_{12} = \frac{3}{2}$ and $\alpha_{22} = 1$. Hence $[A] = \begin{pmatrix} 1 & 3/2 \\ 2/3 & 1 \end{pmatrix}$.

Recall how to multiply matrices: We have a multiplication map

$$M_{p,n}(\mathcal{F}) \times M_{n,m}(\mathcal{F}) \to M_{p,m}(\mathcal{F})$$

given as follows: if B $\in M_{p,n}(\mathcal{F})$ has elements β_{ij}, and A $\in M_{n,m}(\mathcal{F})$ has elements α_{jk}, then the product BA $\in M_{p,m}(\mathcal{F})$ has elements γ_{ik} given by

$$\gamma_{ik} = \sum_{j=1}^{n} \beta_{ij}\alpha_{jk}.$$

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Let U, V have ordered bases $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_n)$ and $A \in \mathrm{Hom}(U, V)$ with induced matrix $[A] \in M_{n,m}(\mathcal{F})$ with elements α_{ij}. Let $u \in U$ and write $u = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_m x_m$. Then

$$Au = \beta_1 Ax_1 + \beta_2 Ax_2 + \cdots + \beta_j Ax_j + \cdots \beta_m Ax_m$$

And so $\beta_j Ax_j = \alpha_{1j}\beta_j y_1 + \cdots + \alpha_{nj}\beta_j y_n$. Hence

$$Au = (\sum_{j=1}^{m} \alpha_{1j}\beta_j)y_1 + \cdots + (\sum_{j=1}^{m} \alpha_{ij}\beta_j)y_i + \cdots + (\sum_{j=1}^{m} \alpha_{nj}\beta_j)y_n.$$

Compare with

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{m} \alpha_{1j}\beta_j \\ \vdots \\ \sum_{j=1}^{m} \alpha_{nj}\beta_j \end{pmatrix}$$

We have just argued for the following:

Lemma

Let U, V be finite-dimensional $\mathcal{F}$-vector spaces with ordered bases $(x_1, \dots, x_m)$, $(y_1, \dots, y_n)$, and $A \in \mathrm{Hom}(U, V)$ with induced matrix $[A] \in M_{n,m}(\mathcal{F})$.
Let $u \in U$ and let $[u] \in M_{m,1}(\mathcal{F})$ be its vector of coordinates in the basis $(x_1, \dots, x_m)$.
Then the matrix product $[A][u] \in M_{n,1}(\mathcal{F})$ is exactly the vector of coordinates for Au in the basis $(y_1, \dots, y_n)$.

Note: if we write $[Au] \in M_{n,1}(\mathcal{F})$ for the vector of coordinates of Au in the ordered basis $(y_1, \dots, y_n)$, the lemma states that

$$[A][u] = [Au].$$

Let U, V, W be vector spaces over $\mathcal{F}$, and let $A \in \mathrm{Hom}(U, V)$ and $B \in \mathrm{Hom}(V, W)$.
It is easy to check that the composition $B \circ A \colon U \to W$ is linear, and thus $B \circ A \in \mathrm{Hom}(U, W)$.
We call the composition the product of the linear maps A and B, which we write BA instead of $B \circ A$.
Suppose U, V, W all are finite dimensional and that we fix ordered bases for these. Arguing essentially the same way as for $[A][u] = [Au]$ before, one gets that

$$[B][A] = [BA].$$

Consider $\mathbb{R}^2$ with ordered basis $\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right)$. Consider the linear map $A \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$A \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \beta + \gamma \\ \beta + \gamma \end{pmatrix}.$$

Recall that $[A] = \begin{pmatrix} 1 & 3/2 \\ 2/3 & 1 \end{pmatrix}$.

Question: What is the first coordinate of $A \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ in the given basis?

(a) 0
(b) 1
(c) 2
(d) 2/3

Consider $\mathbb{R}^2$ with ordered basis $\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}\right)$. Consider the linear map $A \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$A \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \beta + \gamma \\ \beta + \gamma \end{pmatrix}.$$

Recall that $[A] = \begin{pmatrix} 1 & 3/2 \\ 2/3 & 1 \end{pmatrix}$.

Question: What is the first coordinate of $A \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ in the given basis?

We have $[\begin{pmatrix} 2 \\ 0 \end{pmatrix}] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so

$$[A][\begin{pmatrix} 2 \\ 0 \end{pmatrix}] = \begin{pmatrix} 1 & 3/2 \\ 2/3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}.$$

The first coordinate is 1.

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If $U = V = W$ then the product becomes a map

$$\mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V).$$

This map is bilinear in the sense that it is linear in each variable, i.e.

$$A \mapsto AB \text{ (fixed } B\text{)}, \qquad \textit{and} \qquad B \mapsto AB \text{ (fixed } A\text{)}$$

are linear maps $\mathrm{End}(V) \to \mathrm{End}(V)$.

Definition

An algebra over $\mathcal{F}$ is an $\mathcal{F}$-vector space $\mathcal{A}$ with a bilinear product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

Hence $\mathrm{End}(V)$ is an (associative) algebra (with unit).
This example includes $M_{n,n}(\mathcal{F}) = \mathrm{End}(\mathcal{F}^n)$.

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