Advanced Linear Algebra

Week 15

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Recall from now on $\mathcal{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$.

Definition

A map $\langle \cdot, \cdot \rangle \colon V \times V \to \mathcal{F}$ is an inner product on V if

(a) $x \mapsto \langle x, y \rangle$ is a linear map $V \to \mathcal{F}$ for every $y \in \mathcal{F}$;

(b) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in V$;

(c) $\langle x, x \rangle > 0$ for every non-zero $x \in V$.

A vector space equipped with an inner product is called an inner product space.

Recall that in general V is not equal to $U \oplus U^\perp$. However

Theorem (8.12)

Assume U is a finite dimensional subspace of V. Then $V = U \oplus U^\perp$.

Proof.

U is itself an inner product space, so by Theorem 8.7 it has an orthonormal basis $\{x_1, \ldots, x_r\}$.
For $y \in V$ let $u := \sum_{i=1}^r \langle y, x_i \rangle x_i \in U$. By Lemma 8.6(iii), $y - u \in U^\perp$.
Hence $y = u + (y - u) \in U + U^\perp$. Consequently $V = U + U^\perp$ and $U \cap U^\perp = \{0\}$, so $V = U \oplus U^\perp$. $\square$

Let V be an inner product space. For every $v \in V$ we obtain a linear map $x \mapsto \langle x, v \rangle$. In other words, $v \in V$ induces a linear functional $\Phi(v) \in V'$ given by

$$\Phi(v)(x) = \langle x, v \rangle, \qquad x \in V.$$

We thus get a map $\Phi\colon V \to V'$.
This map Φ is anti-linear (as inner products are anti-linear in the second variable), i.e. it is additive
$(\Phi(v_1 + v_2) = \Phi(v_1) + \Phi(v_2))$,
and $\Phi(\alpha v) = \overline{\alpha}\Phi(v)$.
In particular, if $\mathcal{F} = \mathbb{R}$, then Φ is linear.

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$v \in V$ induces a linear functional $\Phi(v) \in V'$ given by

$$\Phi(v)(x) = \langle x, v \rangle, \qquad x \in V.$$

We thus get a map $\Phi\colon V \to V'$.
Consider $\mathbb{C}^2$ with the standard inner product
$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1\overline{y}_1 + x_2\overline{y}_2.$
Consider $z \in (\mathbb{C}^2)'$ given by $z\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 3 - x_2 i$ for all $x_1, x_2 \in \mathbb{C}$.
Question: Which is true?

(1) $z = \Phi(3 + i)$
(2) $z = \Phi(3 - i)$
(3) $z = \Phi\begin{pmatrix} 3 \\ i \end{pmatrix}$
(4) $z = \Phi\begin{pmatrix} 3 \\ -i \end{pmatrix}$.

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Theorem (8.13)

Let $z \in V'$.

(a) *There exists at most one* $v \in V$ *such that* $z = \Phi(v)$, *i.e. such that* $z(x) = \langle x, v \rangle$ *for all* $v \in V$;

(b) *If* $\dim V < \infty$ *then such a vector* v *exists.*

Recall: for vector spaces U, V and $A \in \mathrm{Hom}(U, V)$, the adjoint $A' \in \mathrm{Hom}(V', U')$ is given by $A'(z) = z \circ A$.
We will give an alternative description which is almost the same. To make things confusing, this is also called the adjoint (yep...), but is denoted A^* instead of A'.
Main difference: $A \mapsto A^*$ is anti-linear, whereas $A \mapsto A'$ is linear.
However, $A^* \in \mathrm{Hom}(V, U)$ whereas $A' \in \mathrm{Hom}(V', U')$.

Definition

Let U, V be inner product spaces over $\mathcal{F}$ and let $A \in \mathrm{Hom}(U, V)$. We say that A has an adjoint if there exists $A^* \in \mathrm{Hom}(V, U)$ such that

$$\langle Ax, y\rangle = \langle x, A^* y\rangle, \qquad \forall x \in U, \forall y \in V.$$

When this is the case, A^* is called the adjoint of A.

Definition

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$$\langle Ax, y \rangle = \langle x, A^* y \rangle, \qquad \forall x \in U, \forall y \in V.$$

When this is the case, A^* is called the adjoint of A.

Consider $\mathbb{C}$ with the inner product $\langle x, y \rangle = x\overline{y}$.
Consider $A \in \mathrm{End}(\mathbb{C})$ given by $Ax = (2+i)x$.
Question: Which of the following is true?

(1) $A^*y = (2+i)y$;

(2) $A^*y = (2-i)y$;

(3) $A^*y = 2y$;

(4) A does not have an adjoint.

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$$\langle Ax, y\rangle = \langle x, A^* y\rangle, \qquad \forall x \in U, \forall y \in V.$$

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Consider $\mathbb{C}$ with the inner product $\langle x, y\rangle = x\bar{y}$.
Consider $A \in \mathrm{End}(\mathbb{C})$ given by $Ax = (2 + i)x$.

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Lemma (8.16)

If an adjoint exists, then it is unique. When $\dim U < \infty$ *then every* $A \in \mathrm{Hom}(U, V)$ *has an adjoint.*

The following can easily be checked:

$$(A+B)^* = A^*+B^*, \quad (\alpha A)^* = \overline{\alpha} A^* \quad (AB)^* = B^* A^* \quad A^{**} = A.$$

Lemma (8.17)

$N(A^*) = R(A)^\perp$.

We consider the case $U = V$ and $A \in \mathrm{End}(V)$.

Definition

$A \in \mathrm{End}(V)$ is called self-adjoint if

$$\langle x, Ay \rangle = \langle Ax, y \rangle, \qquad \forall x, y \in V.$$

In other words, A is its own adjoint.

Sometimes this property is called “symmetric” or “Hermitian” (instead of self-adjoint).

Lemma (8.19)

Suppose V is finite-dimensional and let $A \in \mathrm{End}(V)$. Let $\mathcal{B}$ be an orthonormal basis for V (Theorem 8.7), and let ${}_{\mathcal{B}}[A]_{\mathcal{B}}$ be the induced matrix. Then ${}_{\mathcal{B}}[A^*]_{\mathcal{B}}$ is the conjugate transpose of ${}_{\mathcal{B}}[A]_{\mathcal{B}}$.

Consider $\mathbb{C}^3$ with the standard inner product, and let $A \in \mathrm{End}(\mathbb{C}^3)$ be the linear map given by

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 & +ix_2 & \\ -ix_1 & & +(1+i)x_3 \\ & (1-i)x_2 & +ix_3 \end{pmatrix}$$

Question Which is true?

(1) A is self-adjoint

(2) A^* exists, but $A^* \neq A$

(3) A does not have an adjoint.

Consider $\mathbb{C}^3$ with the standard inner product, and let $A \in \mathrm{End}(\mathbb{C}^3)$ be the linear map given by

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 & +ix_2 & \\ -ix_1 & & +(1+i)x_3 \\ & (1-i)x_2 & +ix_3 \end{pmatrix}$$

Theorem (8.20)

Let $E \in \mathrm{End}(V)$ be an idempotent ($E^2 = E$). The following are equivalent:

(1) *E is an orthogonal projection;*

(2) *E is self-adjoint.*

Let U, V be inner product spaces. Then $A \in \mathrm{Hom}(U, V)$ is said to preserve inner products if

$$\langle Ax, Ay \rangle = \langle x, y \rangle \qquad \forall x, y \in U.$$

A linear isomorphism which preserves inner products is called a unitary isomorphism.

Lemma (8.23)

Assume $\dim U = \dim V < \infty$ *and let* $A \in \mathrm{Hom}(U, V)$ *be an isomorphism. The following are equivalent:*

(i) *A preserves inner products;*

(ii) *A carries orthonormal bases to orthonormal bases;*

(iii) $A^*A = I$;

(iv) *A is a unitary isomorphism.*

Proof.

Omitted.

Definition

A linear map $A \in \mathrm{End}(V)$ is called a unitary if it is a unitary isomorphism. When $\mathcal{F} = \mathbb{R}$ these are sometimes called "orthogonal maps".

When $\mathcal{F} = \mathbb{C}$ the set of unitaries in $\mathrm{End}(V)$ is denoted $\mathrm{U}(V)$, and is called the unitary group of V.

When $\mathcal{F} = \mathbb{R}$ the set of unitaries in $\mathrm{End}(V)$ is denoted $\mathrm{O}(V)$ and is called the orthogonal group of V.

Both $\mathrm{U}(V)$ and $\mathrm{O}(V)$ are subgroups of $\mathrm{GL}(V)$; the group of linear isomorphisms in $\mathrm{Hom}(V, V)$.