Week 18 & 19 - Exercises

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Advanced Linear Algebra

Week 18 - Exercise 1

Exercises related to the lecture in week 18. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 & 0 & -2 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

as a matrix in $M_5(\mathbb{R}) = \mathrm{End}(\mathbb{R}^5)$. It has eigenvalues 1 and 2, and the generalised eigenspaces have dimensions: $\dim M_1 = 1$ and $\dim M_2 = 2$. Determine a Jordan decomposition for A as in Theorem 7.8.

Determine a similar Jordan decomposition when A is considered as an element in $\mathrm{M}_5(\mathbb{C}) = \mathrm{End}(\mathbb{C}^5)$.

The eigenspace for the eigenvalue 1 is $V_1 = \mathrm{span}\{(1,0,0,0,0)\}$. Since $\dim(M_1) = 1$, it follows that $M_1 = V_1$. We have $V_2 = \mathrm{span}\{(-1,1,0,0,0)\}$. Further,

$$(A-2I)^2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 8 \\ 0 & 0 & 0 & -1 & -8 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & -1 & -8 \\ 0 & 0 & 0 & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $M_2 = \mathrm{span}\{(-1,1,0,0,0),(0,0,1,0,0)\}$. Finally, set $R = \mathrm{span}\{(0,0,0,1,0)\},(0,0,0,0,1)\}$. Then we have the Jordan decomposition

$$\mathbb{R}^5 = M_1 \oplus M_2 \oplus R$$

Week 18 - Exercise 1 - continued

Exercises related to the lecture in week 18. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 & 0 & -2 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

as a matrix in $M_5(\mathbb{R}) = \mathrm{End}(\mathbb{R}^5)$. It has eigenvalues 1 and 2, and the generalised eigenspaces have dimensions: $\dim M_1 = 1$ and $\dim M_2 = 2$. Determine a Jordan decomposition for A as in Theorem 7.8.

Determine a similar Jordan decomposition when A is considered as an element in $\mathbb{M}_5(\mathbb{C}) = \mathrm{End}(\mathbb{C}^5)$.

If we consider A as an element in $\mathbb{M}_5(\mathbb{C})$ then the eigenvalues of A are

$$\sigma(A) = \{1, 2, i, -i\}$$

Then

$$\begin{aligned} M_1 &= V_1 = \mathrm{span}\{(1,0,0,0,0)\} \\ M_2 &= M_2 = \mathrm{span}\{(-1,1,0,0,0),(0,0,1,0,0)\} \\ M_i &= V_i = \mathrm{span}\{(1-i,-1+i,1-i,-1+i,1)\} \\ M_{-i} &= V_{-i} = \mathrm{span}\{(1+i,-1-i,1+i,-1-i,1)\} \end{aligned}$$

Then we have the Jordan decomposition

$$\mathbb{C}^5 = M_1 \oplus M_2 \oplus M_i \oplus M_{-i}$$

Week 19 - Exercise 1

1. Determine which of the endomorphism $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}) = \mathrm{End}(\mathbb{C}^2)$ is orthogonally diagonalizable.

Recall that $A \in \mathbb{M}_2(\mathbb{C})$ is orthogonally diagonalizable if and only if A is normal. Note that

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and that

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

Thus, the second matrix is orthogonally diagonalizable, but the first one is not.

Week 19 - Exercise 2

2. Show that if V is a finite-dimensional inner product space (over $\mathbb{R}$ or $\mathbb{C}$) and if $A \in \text{End}(V)$ is an endomorphism for which there exists $x \in V$ such that $Ax \neq 0$ but $A^2x = 0$ then A cannot be normal.
Hint: Use Lemma 9.7.

Assume for contradiction that A is normal. Then $N(A) = N(A^*)$ by Lemma 9.7. By assumption we have $Ax \in N(A)$, and consequently $A^*Ax = 0$. But then

$$0 \neq \|Ax\|^2 = \langle Ax, Ax \rangle = \langle A^*Ax, x \rangle = 0$$

which is a contradiction.

Week 19 - Exercise 3

3. For each of the following endomorphisms, determine a basis in which the corresponding matrix is upper triangular.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -4 \\ 1 & 4 \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2) \quad \text{and} \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{pmatrix} \in \mathrm{End}(\mathbb{C}^3)$$

Matrix A is already upper triangular so we can just choose the standard basis $\{(1,0),(0,1)\}$.
For matrix B the characteristic polynomial is $(1-\lambda)^2$. Thus, $\sigma(B) = \{1\}$. The eigenspace is $V_1 = \text{span}\{(0,1)\}$. Since $(B-1I)^2 = 0$, we conclude that B is upper triangular in the (ordered) basis $\{(0,1),(1,0)\}$.
For matrix C the characteristic equation is $-\lambda(4-\lambda)+4 = (\lambda-2)^2$. Thus, $\sigma(C) = \{2\}$ and $V_2 = \text{span}\{(-2,1)\}$. Since $(C-2I)^2 = 0$, we conclude that C is upper triangular in the basis $\{(-2,1),(1,0)\}$.
For matrix D the characteristic polynomial is $(3-\lambda)^3$. Thus $\sigma(D) = \{3\}$. Since $V_3 = \text{span}\{(0,1,1),(1,0,0)\}$ and $(D-3I)^2 = 0$, we conclude that D is upper triangular in the basis $\{(0,1,1),(1,0,0),(0,1,0)\}$.

Week 19 - Exercise 4

4. Show that if $A \in \mathbb{M}_n(\mathbb{C})$ is a matrix and $p \in \mathbb{C}[x]$ is its characteristic polynomial then $p(A) = 0$ by following the steps below. This result is known as the Cayley-Hamilton Theorem.

   (a) Show that if $A = (a_{ij})_{ij}$ is upper triangular then $\det(A) = a_{11} \cdots a_{nn}$.

   (b) Now use the Jordan form of A to show that if $\lambda_1, \ldots, \lambda_r$ are the different eigenvalues and their algebraic multiplicities are denoted $m_1, \ldots, m_r$ respectively, then $p(x) = (\lambda_1 - x)^{m_1} \cdots (\lambda_r - x)^{m_r}$.

   (c) Deduce from Theorem 7.4 that $((A - \lambda_i I)|_{M_{\lambda_i}})^{m_i} = 0$.

   (d) Now conclude that $p(A)$ is zero on each M_{λ_i} and hence that $p(A) = 0$

(a) First, we consider an upper triangular 2×2-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

Then

$$\det(A) = a_{11}a_{22} - 0 \cdot a_{12} = a_{11}a_{22}$$

By induction it then follows that $\det(A) = a_{11} \cdots a_{nn}$ for an $A \in \mathbb{M}_n(\mathbb{C})$.

Week 19 - Exercise 4 - continued

4. Show that if $A \in \mathbb{M}_n(\mathbb{C})$ is a matrix and $p \in \mathbb{C}[x]$ is its characteristic polynomial then $p(A) = 0$ by following the steps below. This result is known as the Cayley-Hamilton Theorem.

   (a) Show that if $A = (a_{ij})_{ij}$ is upper triangular then $\det(A) = a_{11} \cdots a_{nn}$.

   (b) Now use the Jordan form of A to show that if $\lambda_1, \ldots, \lambda_r$ are the different eigenvalues and their algebraic multiplicities are denoted $m_1, \ldots, m_r$ respectively, then $p(x) = (\lambda_1 - x)^{m_1} \cdots (\lambda_r - x)^{m_r}$.

   (c) Deduce from Theorem 7.4 that $((A - \lambda_i I)|_{M_{\lambda_i}})^{m_i} = 0$.

   (d) Now conclude that $p(A)$ is zero on each M_{λ_i} and hence that $p(A) = 0$

(b) There exist an upper triangular matrix U and an invertible matrix P such that $U = P^{-1}AP$ (the Jordan form of A) where the diagonal of U consists of m_i λ_i's for all $i \in \{1, \ldots, r\}$. Then U and A have the same characteristic polynomial, and the characteristic polynomial for U is by (a) given by

$$p(x) = (\lambda_1 - x)^{m_1} \cdots (\lambda_r - x)^{m_r}$$

(c) Since $A - \lambda_i I$ is nilpotent with its index of nilpotency not larger than m_i (by Theorem 7.4), it holds that $((A - \lambda_i I)|_{M_{\lambda_i}})^{m_i} = 0$.

(d) It follows that $p(A) = 0$ on each M_{λ_i}, and since $\mathbb{C}^n = M_1 \oplus \cdots \oplus M_r$, we conclude that $p(A) = 0$.