Advanced Linear Algebra

Week 11

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Let X, Y, V be $\mathcal{F}$-vector spaces. A map $B\colon X \times Y \to V$ is **bilinear** if

(a) $x \mapsto B(x, y)$ is a linear map $X \to V$ for every $y \in Y$; and

(b) $y \mapsto B(x, y)$ is a linear map $Y \to V$ for every $x \in X$.

What condition (a) says is

$$B(x_1 + x_2, y) = B(x_1, y) + B(x_2, y), \qquad B(\alpha x, y) = \alpha B(x, y).$$

Similarly, (b) says

$$B(x, y_1 + y_2) = B(x, y_1) + B(x, y_2), \qquad B(x, \beta y) = \beta B(x, y).$$

Note that bilinearity implies

$$B\Big(\sum_i \alpha_i x_i, \sum_j \beta_j y_j\Big) \overset{(a)}{=} \sum_i \alpha_i B\Big(x_i, \sum_j \beta_j y_j\Big) \overset{(b)}{=} \sum_{i,j} \alpha_i \beta_j B(x_i, y_j).$$

Recall: $B\colon X\times Y\to V$ is bilinear if

$$(a)\quad B(x_1+x_2,y)=B(x_1,y)+B(x_2,y),\qquad B(\alpha x,y)=\alpha B(x,y).$$

$$(b)\quad B(x,y_1+y_2)=B(x,y_1)+B(x,y_2),\qquad B(x,\beta y)=\beta B(x,y).$$

Consider $X, Y, V = M_2(\mathbb{R})$, and the multiplication map

$$B\colon M_2(\mathbb{R})\times M_2(\mathbb{R})\to M_2(\mathbb{R}),\qquad B(\mathrm{A},\mathrm{B})=\mathrm{AB}.$$

Question: Is B bilinear?

(1) Yes;

(2) No;

(3) It depends on A and B.

Recall: $B\colon X \times Y \to V$ is bilinear if

$$(a) \quad B(x_1+x_2, y) = B(x_1, y)+B(x_2, y), \quad B(\alpha x, y) = \alpha B(x, y).$$

$$(b) \quad B(x, y_1+y_2) = B(x, y_1)+B(x, y_2), \quad B(x, \beta y) = \beta B(x, y).$$

Consider $X, Y, V = M_2(\mathbb{R})$, and the multiplication map

$$B\colon M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to M_2(\mathbb{R}), \qquad B(\mathsf{A}, \mathsf{B}) = \mathsf{AB}.$$

Question: Is B bilinear?

$$B(\mathsf{A}_1 + \mathsf{A}_2, \mathsf{B}) = (\mathsf{A}_1 + \mathsf{A}_2)\mathsf{B} \overset{\text{Distr.}}{=} \underbrace{\mathsf{A}_1\mathsf{B}}_{B(\mathsf{A}_1,\mathsf{B})} + \underbrace{\mathsf{A}_2\mathsf{B}}_{B(\mathsf{A}_2,\mathsf{B})}$$

$$B(\alpha\mathsf{A}, \mathsf{B}) = (\alpha\mathsf{A})\mathsf{B} = \alpha(\mathsf{AB}) = \alpha B(\mathsf{A}, \mathsf{B}).$$

So (a) is satisfied. (b) is similar, so matrix multiplication is bilinear.

More generally (same argument), the multiplication map $M_{n,m}(\mathcal{F}) \times M_{m,k}(\mathcal{F}) \to M_{n,k}(\mathcal{F})$ is bilinear.

Definition

A bilinear map $X \times Y \to \mathcal{F}$ is called a bilinear form.

$$\mathrm{Bil}(X, Y) := \{\text{bilinear forms } X \times Y \to \mathcal{F}\}.$$

When $Y = X$ we write $\mathrm{Bil}(X) := \mathrm{Bil}(X, X)$.

Recall $X' = \{\text{linear maps } X \to \mathcal{F}\}$.

Example: Let $u \in X'$ and $v \in Y'$. Then

$$uv \colon X \times Y \to \mathcal{F}, \qquad (uv)(x, y) = u(x)v(y)$$

is a bilinear form
(since $(uv)(x_1 + x_2, y) = u(x_1 + x_2)v(y) =$
$u(x_1)v(y) + u(x_2)v(y) = (uv)(x_1, y) + (uv)(x_2, y)$, etc.)

Example: Let $X = \mathcal{F}^m$ and $Y = \mathcal{F}^n$, and let $\mathsf{A} \in M_{m,n}(\mathcal{F})$.
Then $B\colon \mathcal{F}^m \times \mathcal{F}^n \to \mathcal{F}$ given by

$$B(x,y) = x^t \mathsf{A} y \quad \in \mathcal{F} \quad (x^t \mathsf{A} y \text{ is a } 1 \times 1\text{-matrix})$$

is a bilinear form. In fact,
$B(x_1 + x_2, y) = (x_1 + x_2)^t \mathsf{A} y = x_1^t \mathsf{A} y + x_2^t \mathsf{A} y$, etc.
If $\mathsf{A} = (a_{i,j})_{i,j}$, $x = (\alpha_i)_{i=1}^m \in \mathcal{F}^m$ and $y = (\beta_j)_{j=1}^n \in \mathcal{F}^n$, then

$$B(x,y) = x^t \mathsf{A} y = \sum_{i,j} \alpha_i a_{i,j} \beta_j.$$

Consider the map $B\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by
$$B\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}\right) = \alpha_1\beta_1 - \alpha_2\beta_1 - \alpha_1\beta_2 + \alpha_2\beta_2.$$
Question: Is B a bilinear form?

(1) Yes
(2) No
(3) Depends on $\alpha_1, \alpha_2, \beta_1, \beta_2$.

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Example: Let $X = \mathcal{F}^m$ and $Y = \mathcal{F}^n$, and let $\mathsf{A} \in M_{m,n}(\mathcal{F})$. Then $B\colon \mathcal{F}^m \times \mathcal{F}^n \to \mathcal{F}$ given by

$$B(x,y) = x^t \mathsf{A} y \quad \in \mathcal{F} \quad (x^t \mathsf{A} y \text{ is a } 1 \times 1\text{-matrix})$$

is a bilinear form.

If $\mathsf{A} = (a_{i,j})_{i,j}$, $x = (\alpha_i)_{i=1}^m \in \mathcal{F}^m$ and $y = (\beta_j)_{j=1}^n \in \mathcal{F}^n$, then

$$B(x,y) = x^t \mathsf{A} y = \sum_{i,j} \alpha_i a_{i,j} \beta_j.$$

Consider the map $B\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$B(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}) = \alpha_1\beta_1 - \alpha_2\beta_1 - \alpha_1\beta_2 + \alpha_2\beta_2.$$

If we define $\mathsf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, then

$$B(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}) = \sum_{i,j} (-1)^{i+j} \alpha_i \beta_j = (\alpha_1 \ \alpha_2) \mathsf{A} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

so this map is a bilinear form by the above example.

$\mathrm{Bil}(X, Y) = \{\text{bilinear maps } X \times Y \to \mathcal{F}\}$ is an $\mathcal{F}$-vector space in the usual way. I.e. if $B_1, B_2 \in \mathrm{Bil}(X, Y)$, then

$$(B_1 + B_2)(x, y) := B_1(x, y) + B_2(x, y)$$

and if $B \in \mathrm{Bil}(X, Y)$ and $\alpha \in \mathcal{F}$, then

$$(\alpha B)(x, y) = \alpha B(x, y).$$

Suppose $(x_1, \dots, x_m)$ is an ordered basis for X, and $(y_1, \dots, y_n)$ is an ordered basis for Y.
Let $(u_1, \dots, u_m)$ and $(v_1, \dots, v_n)$ be the ordered dual bases for X' and Y', i.e. $u_i(\alpha_1 x_1 + \cdots \alpha_m x_m) = \alpha_i$ etc.
Define $w_{i,j} := u_i v_j \in \mathrm{Bil}(X, Y)$, i.e.
$w_{i,j}(x, y) = u_i(x) v_j(y)$.

$(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ ordered bases for X, Y.
$(u_1, \dots, u_m)$ and $(v_1, \dots, v_n)$ ordered dual bases for X' and Y'.
Define $w_{i,j} := u_i v_j \in \mathrm{Bil}(X, Y)$, $w_{i,j}(x, y) = u_i(x) v_j(y)$.

Theorem (4.4)

The set of all $w_{i,j}$ is a basis for $\mathrm{Bil}(X, Y)$, and if $B \in \mathrm{Bil}(X, Y)$ then $B = \sum_{i,j} B(x_i, y_j) w_{i,j}$. In particular, $\dim \mathrm{Bil}(X, Y) = mn$.

Proof.

If $x = \sum_i \alpha_i x_i \in X$ and $y = \sum_j \beta_j y_j \in Y$, then
$w_{i,j}(x, y) = u_i(x) v_j(y) = \alpha_i \beta_j$.
$B(x, y) = B(\sum_i \alpha_i x_i, \sum_j \beta_j y_j) = \sum_{i,j} \alpha_i \beta_j B(x_i, y_j) = \sum_{i,j} B(x_i, y_j) w_{i,j}(x, y)$.
Hence $\mathrm{Span}\{w_{i,j} : i, j\} = \mathrm{Bil}(X, Y)$.
If $B = \sum_{i,j} \gamma_{i,j} w_{i,j}$ then $B(x_l, y_k) = \sum_{i,j} \gamma_{i,j} w_{i,j}(x_l, y_k) = \gamma_{l,k}$.
Hence the coefficients $\gamma_{i,j}$ are unique, and thus the $w_{i,j}$ form a basis for $\mathrm{Bil}(X, Y)$.

Definition

Let X, Y be finite-dimensional $\mathcal{F}$-vector spaces. Then

$$X \otimes Y := \mathrm{Bil}(X, Y)'$$

is called the tensor product of X and Y.

If $x \in X$ and $y \in Y$ we obtain a linear functional

$$x \otimes y \colon \mathrm{Bil}(X, Y) \to \mathcal{F}, \qquad (x \otimes y)w = w(x, y).$$

Elements in $X \otimes Y$ of the form $x \otimes y$ are called elementary tensors (or pure tensors).

Lemma (5.15)

The map $X \times Y \to X \otimes Y$ given by $(x, y) \mapsto x \otimes y$ is bilinear. In other words,

(*a*) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad (\alpha x) \otimes y = \alpha(x \otimes y)$

(*b*) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2, \quad x \otimes (\beta y) = \beta(x \otimes y)$

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(b) & x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2, & x \otimes (\beta y) = \beta(x \otimes y).
\end{array}
$$

Consider $X = Y = \mathbb{R}^2$ and let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Question: What is $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

(1) $e_1 \otimes e_1 + e_2 \otimes e_2$

(2) $e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2$

(3) $e_1 \otimes e_2 + e_2 \otimes e_1$

(4) 0

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$$\begin{array}{lll} (a) & (x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, & (\alpha x) \otimes y = \alpha(x \otimes y) \\ (b) & x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2, & x \otimes (\beta y) = \beta(x \otimes y). \end{array}$$

Consider $X = Y = \mathbb{R}^2$ and let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Question: What is $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (e_1 + e_2) \otimes (e_1 + e_2) = e_1 \otimes (e_1 \otimes e_2) + e_2 \otimes (e_1 + e_2)$$

$$= e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2.$$

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Lemma (5.15)

The map $X \times Y \to X \otimes Y$ given by $(x, y) \mapsto x \otimes y$ is bilinear. In other words,

(a) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad (\alpha x) \otimes y = \alpha(x \otimes y)$

(b) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2, \quad x \otimes (\beta y) = \beta(x \otimes y).$

Proof.

Recall that $X \otimes Y = \mathrm{Bil}(X, Y)'$ and $(x \otimes y)(w) = w(x, y)$. Hence

$$((x_1 + x_2) \otimes y)(w) = w(x_1 + x_2, y) = w(x_1, y) + w(x_2, y)$$

$$= (x_1 \otimes y)(w) + (x_2 \otimes y)(w).$$

Hence $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$.
The other parts are proved in a similar way.

Suppose $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ are bases for X and Y respectively.

Theorem (5.16)

The dimension of $X \otimes Y$ is mn, and the elementary tensors

$$x_i \otimes y_j \in X \otimes Y, \quad i = 1, \ldots, m, \, j = 1, \ldots, n$$

form a basis for $X \otimes Y$.

Proof.

By Theorem 4.4 we have a basis of $w_{i,j}$ of $\mathrm{Bil}(X, Y)$ where

$$w = \sum_{i,j} w(x_i, y_j) w_{i,j}, \qquad \text{for } w \in \mathrm{Bil}(X, Y).$$

So the (i, j)'th coordinate functional is $w \mapsto w(x_i, y_j)$, which is exactly $x_i \otimes y_j$! By Theorem 3.4 these form a basis for $\mathrm{Bil}(X, Y)' = X \otimes Y$.

Corollary (5.17)

For finite-dimensional vector spaces X and Y, one has

$$X \otimes Y = \mathrm{Span}\{x \otimes y \mid x \in X, y \in Y\}.$$

Not every element in $X \otimes Y$ is an elementary tensor!
But the elementary tensors span all of $X \otimes Y$ by the above corollary.
What if X and/or Y is not finite-dimensional?
We can still define elementary tensors $x \otimes y \in \mathrm{Bil}(X, Y)'$ by $(x \otimes y)(w) = w(x, y)$.
In the infinite-dimensional case, $X \otimes Y$ can be defined as

$$X \otimes Y := \mathrm{Span}\{x \otimes y \mid x \in X, y \in Y\} \subseteq \mathrm{Bil}(X, Y)'.$$

If $\dim X = \infty$ or $\dim Y = \infty$, then $X \otimes Y \neq \mathrm{Bil}(X, Y)'$ (it is a proper subspace).
Comment: This definition is very non-standard!

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Tensor products have the following universal property: For every bilinear map $B\colon X \times Y \to V$ there exists a unique linear map $\psi\colon X \otimes Y \to V$ such that $B(x, y) = \psi(x \otimes y)$ for all $x \in X$ and $y \in Y$.
In diagrams, it is often written as

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[dr, "\forall B \text{ bilin.}"'] & X \otimes Y \arrow[d, dotted, "\exists!\psi"] \\ & U. \end{tikzcd}$$

Here the map $X \times Y \to X \otimes Y$ is $(x, y) \mapsto x \otimes y$, and "$\exists$!" should be read as "exists($\exists$) a unique(!)".

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$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[dr, "\forall B \text{ bilin.}"'] & X \otimes Y \arrow[d, dotted, "\exists! \psi"] \\
& U.
\end{tikzcd}$$

Proof.

Existence: Let $B_0 \colon U' \to \mathrm{Bil}(X, Y)$ be the map $B_0(z) = z \circ B$. This is easily seen to be linear.

Let $B_0' \colon \mathrm{Bil}(X, Y)' \to U''$ be the dual map of B_0.

Recall (Definition 3.21) that there is a canonical injective linear map $T \colon U \to U''$ given by $T(u)(z) = z(u)$ for $z \in U'$, $u \in U$.

Then for $z \in V'$, $x \in X$ and $y \in Y$ we have

$(B_0'(x \otimes y))(z) = ((x \otimes y) \circ B_0)(z) = (x \otimes y)(z \circ B) = z(B(x, y)) = T(B(x, y))(z)$.

So $B_0'(x \otimes y) = T(B(x, y))$ for all $x \in X$ and $y \in Y$.

Hence $B_0'(X \otimes Y) \subseteq R(T) \cong U$. So there is a linear map $\psi \colon X \otimes Y \to U$ such that $\psi(x \otimes y) = B(x, y)$.

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[dr, "\forall B \text{ bilin.}"'] & X \otimes Y \arrow[d, dotted, "\exists! \psi"] \\ & U. \end{tikzcd}$$

Proof.

Uniqueness: Suppose $\psi' \colon X \otimes Y \to U$ is linear and also satisfies $\psi'(x \otimes y) = B(x, y) = \psi(x \otimes y)$.
Since the elementary tensors $x \otimes y$ span all of $X \otimes Y$, and as ψ and ψ' are linear it follows that $\psi = \psi'$. □

In the proof of existence, we used that $T \colon U \to U''$ is injective. We only proved (Theorem 3.23) that T is an isomorphism when U is finite-dimensional, but one can also prove that it is always injective. However, T is surjective (and thus an isomorphism) only when U is finite-dimensional.
The proof that T is injective uses Zorn's lemma.

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