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MM546/834 PDEs: theory, modeling and computing

Self–study and exercise problems, no 3

1. Given Poisson's equation $-\Delta u = f$, $x \in \Omega$ with Neumann boundary condition $\partial u/\partial n = 0$, $x \in \partial\Omega$ in its weak formulation
$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad , \quad \forall v \in H^1(\Omega) \quad .$$
   (a) Formulate and explain Galerkin's method for this problem.
   (b) Knowing that the problem is elliptic, show that the Galerkin solution is unique.
   (c) Derive and explain Galerkin orthogonality.

2. Given an element f in a vector space V with inner product $< \cdot, \cdot >$. Let $U \subset V$ be a subspace spanned by the basis functions $\{\varphi_j\}_{j=1}^N$. Show that the best approximation $f^* \in U$ to $f \in V$ is determined by the orthogonality condition $f - f^* \perp \varphi_j$ for all $j = 1, 2, \ldots, N$.
   Hint: Consider any other $g \in U$ and show that $\|f - f^*\| \leq \|f - g\|$.

3. Consider now the Poisson equation and it's variational formulation
$$a(u, v) = l(v) \quad \forall v \in V \quad .$$
   Argue that the approximation $u_\Delta \in V_\Delta \subset V$ by Galerkin's method is in fact the best possible approximation in V_Δ with respect to the energy norm.