Advanced Linear Algebra

Week 9

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Let V be an $\mathcal{F}$-vector space.

Definition (3.1)

The vector space

$$V' := \mathrm{Hom}(V, \mathcal{F}) = \{y \colon V \to \mathcal{F} \mid y \text{ is linear}\}$$

is called the dual space of V.
The elements $y \in V'$ are called linear functionals (or linear forms).

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Let $\mathcal{B} = \{x_i \mid i \in I\}$ be a basis for V parametrised by some index set I.

Definition

Let $i \in I$. The linear functional $y_i \in V'$ given by

$$y_i(\sum_{j \in I} \alpha_j x_j) = \alpha_i$$

is called the *i'th coordinate functional*.

So if $x \in V$ is written as $x = \sum_{i \in I} \alpha_i x_i$, then $y_i(x) = \alpha_i$.

$\mathcal{B} = \{x_i \mid i \in I\}$ is a basis parametrised by I.
Let $\mathcal{B}' = \{y_i \mid i \in I\} \subseteq V'$ be the coordinate functionals.

Theorem (3.4)

The set $\mathcal{B}'$ is linearly independent. If $\dim V < \infty$ then

$$y = \sum_{i \in I} y(x_i) y_i \qquad \textit{for all } y \in V'$$

and $\mathcal{B}'$ is a basis for the dual space V'.

Definition

When V is finite-dimensional with basis $\mathcal{B}$, then $\mathcal{B}'$ (of coordinate functionals) is called the dual basis og $\mathcal{B}$.

Note that the dual basis $\mathcal{B}'$ is a basis for V' (Theorem 3.4).

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Let $U \subseteq V$ be a subspace.

Definition

The subspace

$$U^\circ := \{y \in V' \mid y(x) = 0 \text{ for all } x \in U\}$$

is called the annihilator of U.

U° is a subspace.

Indeed, if $y^{(1)}, y^{(2)} \in U^\circ$ then for every $x \in U$ we have

$$(y^{(1)} + y^{(2)})(x) = \underbrace{y^{(1)}(x)}_{=0} + \underbrace{y^{(2)}(x)}_{=0} = 0$$

so $y^{(1)} + y^{(2)} \in U^\circ$. Similarly $\alpha y \in U^\circ$ whenever $y \in U^\circ$ and $\alpha \in \mathcal{F}$.

$$U^\circ := \{y \in V' \mid y(x) = 0 \text{ for all } x \in U\}$$

Let $V = \mathbb{R}^2$ with the standard ordered basis (e_1, e_2).
Let $y_1, y_2 \in (\mathbb{R}^2)'$ be the coordinate functionals, so

$$y_1 \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_1 \qquad y_2 \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_2$$

and $U = \mathrm{span}\{e_2\} = \{\begin{pmatrix} 0 \\ \alpha \end{pmatrix} \mid \alpha \in \mathbb{R}\}$.

Question: Which of the following is true?

(1) $y_1, y_2 \in U^\circ$;

(2) $y_1 \in U^\circ$ and $y_2 \notin U^\circ$;

(3) $y_1 \notin U^\circ$ and $y_2 \in U^\circ$;

(4) $y_1, y_2 \notin U^\circ$.

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and $U = \mathrm{span}\{e_2\} = \{\begin{pmatrix} 0 \\ \alpha \end{pmatrix} \mid \alpha \in \mathbb{R}\}$.
For an arbitrary $x = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \in U$, we have

$$y_1(x) = y_1 \begin{pmatrix} 0 \\ \alpha \end{pmatrix} = 0,$$

and since $x \in U$ was arbitrary, it follows that $y_1 \in U^\circ$.

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and $U = \mathrm{span}\{e_2\} = \{\begin{pmatrix} 0 \\ \alpha \end{pmatrix} \mid \alpha \in \mathbb{R}\}$.
For an arbitrary $x = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \in U$, we have

$$y_2(x) = y_2 \begin{pmatrix} 0 \\ \alpha \end{pmatrix} = \alpha,$$

so if we pick for instance $x = e_2$ (corresponding to $\alpha = 1$) then $y_2(x) = 1 \neq 0$. Hence $y_2 \notin U^\circ$.

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Let $U \subseteq V$ be a subspace, and let $\pi\colon V \to V/U$ be the projection.

Theorem (3.8)

There is an isomorphism $(V/U)' \to U^\circ$ given by $z \mapsto z \circ \pi$. In particular, if V is finite-dimensional, then

$$\dim U + \dim U^\circ = \dim V.$$

Remark: $(V/U)'$ consists of linear maps $z\colon V/U \to \mathcal{F}$
and $\pi\colon V \to V/U$ is linear.
So $z \circ \pi\colon V \to \mathcal{F}$ is a linear map, and hence $z \circ \pi \in V'$.

Proof.

Let $A\colon (V/U)' \to V'$ be the linear map $Az = z \circ \pi$.
Goal: to show $R(A) = U^\circ$ and $N(A) = \{0\}$. Then A induces an isomorphism $(V/U)' \to U^\circ$ by the first isomorphism theorem.

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$N(A) = \{0\}$: Suppose $Az = z \circ \pi = 0$. Goal: $z(x) = 0$ for all $x \in V/U$. Pick $x' \in V$ such that $\pi(x') = x$. Then $z(x) = z(\pi(x')) = (Az)(x') = 0$. Hence $z = 0$, so $N(A) = \{0\}$.

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Goal: to show $R(A) = U^\circ$ and $N(A) = \{0\}$. Then A induces an isomorphism $(V/U)' \to U^\circ$ by the first isomorphism theorem.
$R(A) \subseteq U^\circ$: Let $z \in (V/U)'$. If $x \in U$ then $\pi(x) = 0$, and hence $(Az)(x) = z(\pi(x)) = z(0) = 0$, so $Az \in U^\circ$ for all $z \in (V/U)'$. Hence $R(A) \subseteq U^\circ$.

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Proof.

Let $A\colon (V/U)' \to V'$ be the linear map $Az = z \circ \pi$.
Goal: to show $R(A) = U^\circ$ and $N(A) = \{0\}$. Then A induces an isomorphism $(V/U)' \to U^\circ$ by the first isomorphism theorem.
$U^\circ \subseteq R(A)$: Let $y \in U^\circ$. Define $\overline{y}\colon V/U \to \mathcal{F}$ by $\overline{y}(x+U) = y(x)$. Well-defined since $y(u) = 0$ for $u \in U$. Then $y = \overline{y} \circ \pi = A\overline{y}$, so $U^\circ \subseteq R(A)$.

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Proof.

Let $A\colon (V/U)' \to V'$ be the linear map $Az = z \circ \pi$.
By Theorem 2.14 we have $\dim V = \dim U + \dim V/U$.
By Corollary 3.5 we have $\dim V/U = \dim(V/U)'$.
As we just proved $(V/U)' \cong U^\circ$ we get $\dim(V/U)' = \dim U^\circ$.
Hence

$$\dim V = \dim U + \dim U^\circ.$$

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There is an isomorphism $(V/U)' \to U^\circ$ given by $z \mapsto z \circ \pi$. In particular, if V is finite-dimensional, then

$$\dim U + \dim U^\circ = \dim V.$$

Consider the $\mathcal{F}$-vector space $\mathcal{F}^n$ and let

$$U = \{ \begin{pmatrix} \alpha \\ \alpha \\ \vdots \\ \alpha \end{pmatrix} \mid \alpha \in \mathcal{F} \}.$$

Question: What is the dimension of the annihilator U°?

(1) 0
(2) 1
(3) $n - 1$
(4) n.

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$$U = \{ \begin{pmatrix} \alpha \\ \alpha \\ \vdots \\ \alpha \end{pmatrix} \mid \alpha \in \mathcal{F} \}.$$

We have $\dim \mathcal{F}^n = n$ and $\dim U = 1$. So

$$\dim U^\circ = \dim \mathcal{F}^n - \dim U = n - 1.$$

Let $U \subseteq V$ be a subspace, and assume $\dim V < \infty$.

Theorem (3.9, The Extension Theorem)

For every $z \in U'$ there exists $y \in V'$ such that $z(x) = y(x)$ for all $x \in U$. (In other words, $y|_U = z$).

Consider $\mathbb{R}^2$ and let $U = \{\begin{pmatrix} -\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R}\}$.

Consider $z \in U'$ given by $z\begin{pmatrix} -\alpha \\ \alpha \end{pmatrix} = 2\alpha$

and $y \in (\mathbb{R}^2)'$ given by $y\begin{pmatrix} \beta \\ \gamma \end{pmatrix} = 2\beta$.

Question: is $y|_U = z$?

(1) Yes,

(2) No,

(3) There is not enough information to determine this.

Let $U \subseteq V$ be a subspace, and assume $\dim V < \infty$.

Theorem (3.9, The Extension Theorem)

For every $z \in U'$ there exists $y \in V'$ such that $z(x) = y(x)$ for all $x \in U$. (In other words, $y|_U = z$).

Proof.

Let $J\colon V' \to U'$ be the restriction map $J(y) = y|_U$ for $y \in V'$, or equivalently, $J(y)\colon U \to \mathcal{F}$ is given by $J(y)(x) = y(x)$ for all $x \in U$.

It is easy to check that J is linear.

Goal: show that $R(J) = U'$.

Note that $N(J) = U^\circ$. By the rank–nullity theorem we have

$$\dim R(J) + \underbrace{\dim N(J)}_{\dim U^\circ} = \dim V' \overset{Cor 3.5}{=} \dim V \overset{Thm 3.8}{=} \dim U + \dim U^\circ$$

so $\dim R(J) = \dim U = \dim U'$. As $R(J) \subseteq U'$ is a subspace, it follows that $R(J) = U'$ (Corollary 1.23).

Lemma (3.10)

Suppose $\dim V < \infty$, *and let* $x \in V$ *be non-zero. Then* $y(x) \neq 0$ *for some* $y \in V'$.

Proof.

Let $U = \mathrm{span}\{x\} = \{\alpha x : \alpha \in \mathcal{F}\}$, which is a 1-dimensional subspace of V.
Define $z \in U'$ by $z(\alpha x) = \alpha$ for $\alpha \in \mathcal{F}$.
By the Extension Theorem we may find $y \in V'$ such that $y|_U = z$. Hence $y(x) = z(x) = 1$. □

Theorem (3.11, The Separation Theorem)

Let $\dim V < \infty$ *and* $U \subseteq V$ *be a subspace. For each* $x \in V \setminus U$ *there exists* $y \in V'$ *which annihilates* U *but not* x*, i.e.* $y(u) = 0$ *for all* $u \in U$ *and* $y(x) \neq 0$*.*
Alternatively (equivalently), this means

$$U = \{x \in V \mid y(x) = 0 \text{ for all } y \in U^\circ\}.$$

Proof.

Let $\pi \colon V \to V/U$ be the projection. As $x \notin U$ we have $\pi(x) \neq 0$.
By Lemma 3.10 there exists $z \in (V/U)'$ such that $z(\pi(x)) \neq 0$.
By Theorem 3.8 $y := z \circ \pi \in U^\circ$ (so y annihilates U), and $y(x) = z(\pi(x)) \neq 0$.

Let U, V be $\mathcal{F}$-vector spaces and $A \in \mathrm{Hom}(U, V)$.

Definition

The adjoint of A is the linear map $A' \colon V' \to U'$ given by

$$(A'y)(x) = y(A(x)), \qquad \text{for all } x \in U.$$

In other words, $A'y = y \circ A$.

Consider $A \colon \mathbb{R} \to \mathbb{R}^2$ given by $A\alpha = \begin{pmatrix} -\alpha \\ 3\alpha \end{pmatrix}$.

Let $y_1, y_2 \in (\mathbb{R}^2)'$ be the first and second coordinate functionals (wrt the standard ordered basis).

Question: Then $A'y_1 \in \mathbb{R}'$. What is $A'y_1(\alpha)$?

(1) 1

(2) 2

(3) 3

(4) -1.

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In other words, $A'y = y \circ A$.

Consider $A \colon \mathbb{R} \to \mathbb{R}^2$ given by $A\alpha = \begin{pmatrix} -\alpha \\ 3\alpha \end{pmatrix}$.

Let $y_1, y_2 \in (\mathbb{R}^2)'$ be the first and second coordinate functionals (wrt the standard ordered basis).

$$A'y_1(\alpha) = y_1(A(\alpha)) = y_1 \begin{pmatrix} -\alpha \\ 3\alpha \end{pmatrix} = -\alpha.$$

$$A'y_2(\alpha) = y_2(A(\alpha)) = y_2 \begin{pmatrix} -\alpha \\ 3\alpha \end{pmatrix} = 3\alpha.$$

Consider $A\colon \mathbb{R} \to \mathbb{R}^2$ given by $A\alpha = \begin{pmatrix} -\alpha \\ 3\alpha \end{pmatrix}$.

Let $y_1, y_2 \in (\mathbb{R}^2)'$ be the first and second coordinate functionals (wrt the standard ordered basis).

$$A' y_1(\alpha) = -\alpha \qquad A' y_2(\alpha) = 3\alpha.$$

The matrix for A is $[A] = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

$A'\colon (\mathbb{R}^2)' \to \mathbb{R}'$ has the matrix $[A'] = (-1 \;\; 3)$.

Lemma (3.13)

Let $A \in \mathrm{Hom}(U, V)$*. Then the adjoint* $A' \in \mathrm{Hom}(V', U')$ *and satisfies*

(1) $A \mapsto A'$ *is a linear map* $\mathrm{Hom}(U, V) \to \mathrm{Hom}(V', U')$*;*

(2) $(BA)' = A'B'$ *for* $A \in \mathrm{Hom}(U, V)$ *and* $B \in \mathrm{Hom}(V, W)$*;*

(3) $(I_V)' = I_{V'} \in \mathrm{End}(V')$.

Proof.

Straight-forward from definitions (omitted). □

Corollary (3.14)

If $A \in \mathrm{Hom}(U, V)$ *is an isomorphism then so is* $A' \in \mathrm{Hom}(V', U')$*, and* $(A')^{-1} = (A^{-1})'$.

Proof.

$$(A^{-1})'A' = (AA^{-1})' = (I_V)' = I_{V'}$$
$$A'(A^{-1})' = (A^{-1}A)' = (I_U)' = I_{U'}.$$

Let U, V be finite dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{C}$, let $m = \dim U$ and $n = \dim V$.
Let $A \in \mathrm{Hom}(U, V)$. Then ${}_{\mathcal{C}}[A]_{\mathcal{B}} \in M_{n,m}(\mathcal{F})$.
Then $A' \in \mathrm{Hom}(V', U')$ and therefore ${}_{\mathcal{B}'}[A']_{\mathcal{C}'} \in M_{m,n}(\mathcal{F})$.
Note: n and m swapped!

Lemma (3.15)

${}_{\mathcal{B}'}[A']_{\mathcal{C}'} = ({}_{\mathcal{C}}[A]_{\mathcal{B}})^T$ *(the transpose).*

Moral: the adjoint is really the transpose, but it does not require any bases (as opposed to the transpose of a matrix).

Proof.

Left as an exercise. (Good exercise for understanding the definitions). □

Let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Let $y_1, y_2 \in (\mathbb{R}^2)'$ be the standard coordinate functionals.

Question: What is $A'y_1$?

(1) $y_1 + 2y_2$

(2) $2y_1 + y_2$

(3) $y_1 + 3y_2$

(4) $3y_1 + y_2$.

Answer: $[A'] = [A]^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, and $[y_1] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Since

$$[A'y_1] = [A'][y_1] = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

we get $A'y_1 = y_1 + 2y_2$.

Definition

The vector space $V'' := (V')'$ is the double dual of V.

Suppose $x \in V$. Define a map $Tx \colon V' \to \mathcal{F}$ by $(Tx)(y) = y(x)$.

Lemma (3.20, 3.22)

There is a linear map $T \colon V \to V''$ given by $(Tx)(y) = y(x)$ for all $x \in V$ and $y \in V'$.

Proof.

Straight forward from definitions. $\square$

Definition

$T \colon V \to V''$ is called the natural correspondence from V to V''.

Theorem (3.23)

If $\dim V < \infty$ *then* $T\colon V \to V''$ *is an isomorphism.*

Proof.

First we show that $N(T) = \{0\}$, so let $x \in V$ such that $Tx = 0$.
Assume for contradiction that $x \neq 0$. By Corollary 3.10 there is $y \in V'$ such that $y(x) \neq 0$. Hence

$$(Tx)(y) = y(x) \neq 0$$

which contradicts that $Tx = 0$. So $x = 0$, and thus $N(T) = \{0\}$.
By Corollary 3.5 $\dim V = \dim V' = \dim V''$.
By Theorem 2.10 T is an isomorphism. $\square$

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If $A \in \mathrm{Hom}(U, V)$, then $A' \in \mathrm{Hom}(V', U')$.
Taking the adjoint of A', we get $A'' \in \mathrm{Hom}(U'', V'')$.
If $T_U \colon U \to U''$ and $T_V \colon V \to V''$ are the natural correspondences, we have the diagram

$$\begin{tikzcd}
U \arrow[r, "A"] \arrow[d, "T_U"'] & V \arrow[d, "T_V"] \\
U'' \arrow[r, "A''"'] & V''.
\end{tikzcd}$$

Lemma (3.24)

$$A'' \circ T_U = T_V \circ A.$$

In other words: the diagram above is commutative.

Proof.

Follows from unraveling the definitions.