Topological defects in the fully frustrated XY model and in ^{3}He-A films

1985 J. Phys. C: Solid State Phys. 18 2437

(http://iopscience.iop.org/0022-3719/18/12/008)

J. Phys. C: Solid State Phys. **18** (1985) 2437–2454. Printed in Great Britain

Topological defects in the fully frustrated *XY* model and in ^{3}He-A films

Thomas C Halsey†
IBM Thomas J Watson Research Center, Yorktown Heights, NY 10598, USA, and Department of Physics, Harvard University, Cambridge, MA 02138, USA

Received 7 August 1984

Abstract. The ground states of both the fully frustrated *XY* model in two dimensions and ^{3}He-A films have $Z_2 \times U(1)$ symmetries. Both line defects (domain walls) and point defects (vortices) can disrupt ordering in these systems at finite temperatures. The presence of line defects induces a breakdown of vortex charge quantisation in both of these systems. In the fully frustrated *XY* model on a square lattice, integral quantisation of charge is replaced by quarter-integral quantisation; in ^{3}He-A, quantisation of charge is completely destroyed. It follows that in both of these systems phase ordering cannot coexist with Z_2 disorder. The fully frustrated *XY* model is relevant to Josephson junction arrays in a transverse magnetic field, at a value of the field where half of a magnetic flux quantum threads each cell of the array.

1. Introduction

The last decade has seen enormous advances in our understanding of the statistical mechanics of two-dimensional systems. An appreciation of the importance of topological point defects in two-dimensional systems with continuous symmetries has stimulated much of this advance. Vortices in *XY* models and in superfluid ^{4}He films, and dislocations and disclinations in two-dimensional solids, are all predicted to play central roles in the destruction of ordering in these systems; many of these predictions have been verified experimentally (Nelson 1983).

Recently, some two-dimensional statistical mechanics problems have arisen in which the interaction of topological point defects with topological line defects or domain walls can be important to the physics. Two such systems are the subject of this study: the fully frustrated *XY* model of Villain, and superfluid films of ^{3}He-A.

The unfrustrated *XY* model can be expressed in terms of phases θ_i on the sites i of a lattice, with

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j). \qquad (1.1)$$

At low temperatures the Landau–Peierls fluctuations prevent this model from developing long-range order in the phases θ_i (Lifshitz and Pitaevski 1980, Nelson 1983).

† Current address: The James Franck Institute, The University of Chicago, 5640 Ellis Avenue, Chicago, IL 60637, USA.

However, at sufficiently low temperatures this model has a quasi-long-range-ordered phase in which the correlation function decays algebraically:

$$\langle \exp[\mathrm{i}(\theta(\boldsymbol{r}) - \theta(0))]\rangle \sim 1/|\boldsymbol{r}|^{\eta} \tag{1.2}$$

where the exponent η is dependent on the temperature (Nelson 1983).

At finite temperatures vortices may appear in the phase θ. In a continuum approximation these vortices can be characterised by the line integral of $\nabla\theta(\boldsymbol{r})$:

$$\oint \nabla\theta(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = 2\pi n \tag{1.3}$$

where n, an integer, is the total vortex charge enclosed by the circuit. At low temperatures these logarithmically interacting vortices only appear in bound pairs of total charge zero. For $T > T_{\mathrm{KT}}$, the temperature of the Kosterlitz–Thouless transition, these bound pairs dissociate; the resultant free vortex charge destroys the quasi-long-range ordering. At $T = T_{\mathrm{KT}}$, the exponent η possesses the universal value $\eta_c = \frac{1}{4}$ (Nelson 1983).

The system can be studied in the Villain approximation in which the cosine potential is replaced by a periodic gaussian potential. This allows a transformation to a Coulomb gas representation of the model (1.1):

$$H = -\pi\bar{J} \sum_{i,j} m_i G_{ij} m_j \tag{1.4}$$

with $\lim_{|i-j|\to\infty} G_{ij} = \log|\boldsymbol{i} - \boldsymbol{j}|$, and the m_i restricted to integral values (Villain 1975, Jose *et al* 1977, Nelson 1983).

Similar theories of defect-mediated phase transitions have been developed for ^{4}He films, two-dimensional solids and two-dimensional liquid crystal systems (Nelson 1983). The order parameters of all these systems possess continuous symmetries allowing topological point defects. However, there exist two-dimensional systems whose order parameters have both continuous and discrete symmetries. The simplest such case would be an order parameter with a $Z_2 \times U(1)$ symmetry. Any such system will not only possess topological point defects corresponding to integrally charged vortices in the U(1) phase but will also possess line defects or domain walls, separating regions in which the Z_2 components of the order parameter differ. In this case the behaviour of the U(1) phase degrees of freedom must be examined more carefully.

The ground state of the two-dimensional fully frustrated *XY* model of Villain has a $Z_2 \times U(1)$ symmetry (Villain 1977, Teitel and Jayaprakash 1983a). This is a model of phases θ_i on the sites of a square lattice interacting via a Hamiltonian:

$$H = -\sum_{\langle ij\rangle} J_{\langle ij\rangle} \cos(\theta_i - \theta_j). \tag{1.5}$$

The $J_{\langle ij\rangle}$ are chosen so that on any elementary plaquette of the lattice, three of the $J_{\langle ij\rangle}$ are given by $+J$ and one is given by $-J$. Because each plaquette is equally frustrated, there is no quenched disorder in this model. The ground state of this model has a broken Z_2 Ising-like symmetry stemming from a broken translational invariance in addition to the U(1) symmetry of uniform rotation of all phases θ_i. First introduced by Villain, this model has been studied numerically by Teitel and Jayaprakash (1983a). They observed in their simulations a phase transition at $kT \simeq 0.45\,J$; this transition does not appear to be of the Kosterlitz–Thouless kind. This model is identical to a recently proposed model of Josephson junction arrays in a transverse magnetic field, provided that the magnetic flux piercing one cell of the array is $\frac{1}{2}\Phi_0$, where Φ_0 is the magnetic flux quantum (Teitel and Jayaprakash 1983b, Halsey 1985).

Thin films of ^{3}He-A present another example of a system with both discrete and continuous symmetries. As originally pointed out by Stein and Cross (1979), this system also has independent Z_2 and U(1) symmetries. The Z_2 symmetry arises because the angular momentum $\boldsymbol{l}$ of the film must be oriented in one of the two directions normal to the film; the U(1) symmetry is associated with the phase of the ^{3}He-A order parameter.

While these are the only two systems considered explicitly in this study, we believe that our results will be relevant for a variety of systems, including the antiferromagnetic *XY* model on a triangular lattice, *XY* helimagnets and surface reconstruction on Mo(100) (Bak 1979, Garel and Doniach 1980, Natterman 1981, Lee *et al* 1984, Miyashita and Shiba 1983). The antiferromagnetic *XY* model on a triangular lattice is very similar to the fully frustrated *XY* model; its ground state also has a U(1) symmetry of global spin rotation and a discrete Z_2 symmetry corresponding to the broken translational invariance of the ground state. Mo(100) undergoes at low temperatures an incommensurate surface reconstruction; the wavevector of the reconstruction may lie in one of two perpendicular directions in the surface.

In both the fully frustrated *XY* model and in ^{3}He films, the presence of line defects associated with the discrete symmetry induces a breakdown in the integral quantisation of the vortex charges associated with the continuous symmetry. The presence of domain walls in the fully frustrated *XY* model causes the integral quantisation of vortex charge to be superseded by quantisation in units of $\pm\frac{1}{4}$. In ^{3}He-A, the quantisation of vortex charge is completely destroyed in the presence of domain walls.

These conclusions have important consequences for the finite-temperature behaviour of these systems. It is impossible for phase quasi-long-range ordering to coexist with Z_2 Ising-like disorder in both the fully frustrated *XY* model and in ^{3}He-A films, although the converse, phase disorder coexisting with Z_2 order, may still be allowed. However, we are not able to exclude the possibility that systems similar to the fully frustrated *XY* model may have phases in which Z_2 disorder does not preclude an exotic sort of phase quasi-long-range ordering.

This study is organised into six sections and one Appendix. Section 1 is the introduction. Section 2 discusses the fully frustrated *XY* model and its relationship with Josephson junction arrays, while § 3 studies domain walls in this model and the appearance of $\pm\frac{1}{4}$ charges. In § 4 we examine the implications of these results for the phase transition behaviour of this system. In § 5 domain walls and vorticity in ^{3}He-A are discussed; § 6 contains the conclusions. In an Appendix the general form of Hamiltonians for systems with the same symmetry properties as the fully frustrated *XY* model is examined, and possible types of ordering in these systems are discussed.

2. The fully frustrated *XY* model

The fully frustrated *XY* model can be realised physically by a Josephson junction array in a transverse magnetic field (Teitel and Jayaprakash 1983a, Halsey 1985). Consider such an array, a two-dimensional square lattice of Josephson junctions (see figure 1). The Hamiltonian will then be a sum of Josephson coupling energies:

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j - A_{ij}) \tag{2.1}$$

where the θ_i are the phases on the superconducting 'islands' separated by the junctions, and the A_{ij} are proportional to the line integrals of the vector potential across the

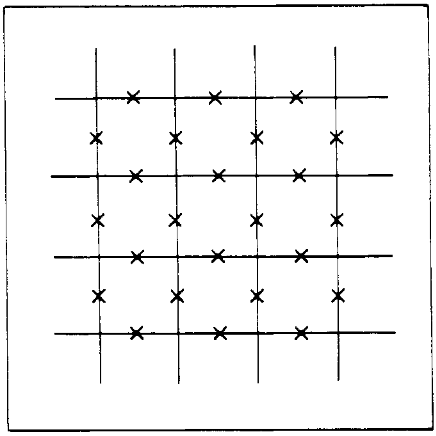

Figure 1. A schematic representation of a square lattice of Josephson junctions. The magnetic field is perpendicular to the plane of the paper.

junctions:

$$A_{ij} = \frac{2e}{hc} \int_i^j \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}. \tag{2.2}$$

We here assume that the capacitance of the junctions is sufficiently large so that we are justified in ignoring quantum fluctuations in the phases.

If we make the approximation that the transverse penetration depth of the array is infinite, then we regard the A_{ij} as being quenched in by an external magnetic field. The magnetic flux threading one plaquette of the array will be

$$\oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \frac{\Phi_0}{2\pi} \sum_p A_{ij} = f\Phi_0 \tag{2.3}$$

where Φ_0 is the magnetic flux quantum; $\Phi_0 = hc/2e$. If we consider only uniform transverse magnetic fields, then the dimensionless parameter f will be constant throughout the lattice.

In the Landau gauge, all A_{ij} on horizontal bonds will be zero, while the A_{ij} on the bonds of the mth vertical column will be $2\pi fm$. Thus, at $f = \frac{1}{2}$, alternating columns of bonds are ferromagnetic and antiferromagnetic, and all rows of bonds are ferromagnetic. Hence, each plaquette possesses one antiferromagnetic and three ferromagnetic bonds: this is the fully frustrated *XY* model.

The ground state of one such plaquette can be easily determined. The condition that a particular set of phases $\{\theta_i'\}$ be an extremum of (2.1) is that

$$\sum_{j'} \sin(\theta_i' - \theta_{j'}' - A_{ij'}) = 0 \tag{2.4}$$

where the sum is over the four nearest neighbours of the site i. The sine of the gauge-invariant phase difference $\theta_i - \theta_j - A_{ij}$ across a junction is proportional to the supercurrent flowing in that junction. Thus (2.4) is simply the requirement that supercurrent be conserved at every site in the array. The energy of a single plaquette will therefore be minimised in a state in which a constant (possibly zero) current circulates about the plaquette either in the clockwise or in the anticlockwise direction. If the gauge-invariant phase difference corresponding to this current is $\Delta\varphi$, then on the ferromagnetic bonds $\theta_i' - \theta_j' = \Delta\varphi$, while on the antiferromagnetic bond $\theta_i' - \theta_j' = \Delta\varphi - \pi$, so that $4\Delta\varphi = \pm\pi$ and $\Delta\varphi = \pm\pi/4$.

Since these gauge-invariant phase differences are opposite to one another, this one-plaquette solution may be extended to the entire lattice, as in figure 2(*a*). The ground state has a 'checkerboard' form, alternate plaquettes having currents circulating in a clockwise or an anticlockwise sense (see figure 2(*b*)). By inspection, this state conserves current at every site. The ground-state energy per site is given by $-2^{1/2}J$ (Villain 1977).

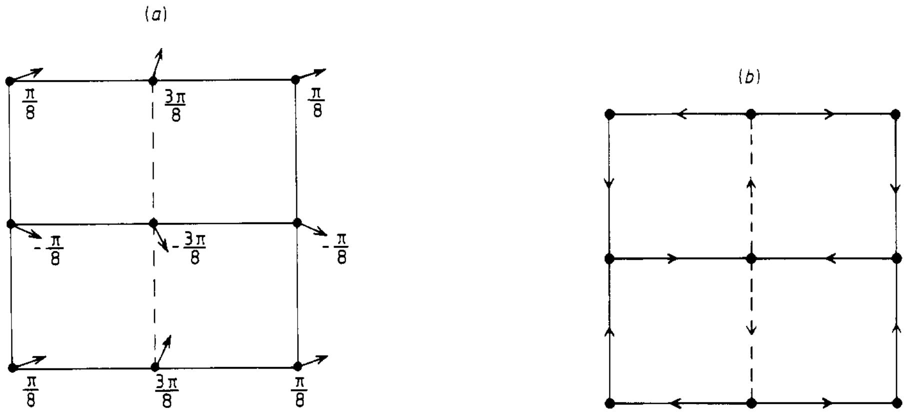

Figure 2. (*a*) The phase configuration in a 'checkerboard' ground state. The unit cell is 2 × 2. The broken lines represent antiferromagnetic bonds, the full lines ferromagnetic bonds. (*b*) Currents flowing in a 'checkerboard' ground state. The magnitudes of all currents are equal, so it is clear by inspection that current is conserved in this state.

The ground state of a continuum superconducting film in a magnetic field is a triangular lattice of vortices (Fisher 1980). At the centre of each vortex is a normal core. In an array it is no longer possible to identify vortices with their normal cores, as most of the array surface area is taken up by normal regions. It is possible to identify vortex-bearing plaquettes as those about which current circulates in a chosen sense. However, this mode of description ignores the symmetry between the two possible senses of current circulation, and will be avoided in this study.

In the Villain approximation the cosine potential in (2.1) is replaced by a periodic gaussian potential. The model in terms of the phases $\{\theta_i\}$ can be transformed into a model in terms of charges $\{m(\boldsymbol{r})\}$ on the sites of a square lattice (Teitel and Jayaprakash 1983a, Halsey 1985). The charges are restricted to integral values, and interact with one another and with a background field via the Hamiltonian

$$H = -\pi\tilde{J}\sum_{\boldsymbol{r},\boldsymbol{r}'}(m(\boldsymbol{r})+f)G(\boldsymbol{r},\boldsymbol{r}')(m(\boldsymbol{r}')+f) \tag{2.5}$$

with

$$\lim_{|\boldsymbol{r}-\boldsymbol{r}'|\to\infty} G(\boldsymbol{r},\boldsymbol{r}') = \log|\boldsymbol{r}-\boldsymbol{r}'|.$$

The configurations $\{m(\boldsymbol{r})\}$ are constrained by $\Sigma_{\boldsymbol{r}}(m(\boldsymbol{r})+f)=0$. Thus the density of charges in any state will be equal to f. It is easy to see that the ground state of this model

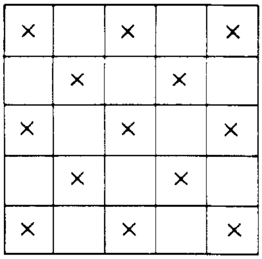

Figure 3. The 'checkerboard' ground state in the charge model: a $2^{1/2} \times 2^{1/2}$ superlattice. Crosses represent charges; the density of charges is $f = \frac{1}{2}$.

at $f = \frac{1}{2}$ also has a 'checkerboard' form, as in figure 3. This form is a square $2^{1/2} \times 2^{1/2}$ superlattice of charges. These charges should be distinguished from the vortices of the continuum system; they are only defined in the transformed model (2.5).

The ground state of the phase model (2.1) exhibits two broken symmetries (Villain 1977, Teitel and Jayaprakash 1983a). The first of these symmetries is the global U(1) symmetry of phase rotation. The phases in any ground state $\{\theta_i\}$ can be rotated into another ground state $\{\theta_i + \alpha\}$. Corresponding to this broken symmetry there will be Goldstone modes, which may be characterised by making the parameter α a function of position $\alpha(\boldsymbol{r})$; we imagine that $\alpha(\boldsymbol{r})$ is slowly varying on the length scale of the individual plaquettes. Clearly, quasi-long-range ordering in this phase field $\alpha(\boldsymbol{r})$ implies that the resistance of the array is zero, as in the zero-magnetic-field case (Lobb *et al* 1983, Halperin and Nelson 1979).

At finite temperatures, vortices may appear in this slowly varying part of the phase $\alpha(\boldsymbol{r})$. Thus, α given approximately by

$$\alpha(\boldsymbol{r}) \approx \tan^{-1}[(y - y_0)/(x - x_0)] \tag{2.6}$$

corresponds to a vortex of unit charge located at (x_0, y_0). In the presence of such a vortex there will be a net current circulating about the position of the vortex, in addition to the local circulating currents inherent in the ground state.

The second symmetry broken by the ground state of the phase model is a Z_2 Ising-like symmetry. This symmetry is associated with the sense of current flow about a particular plaquette; thus we call it a chiral symmetry. It can also be regarded as a broken translational invariance; in the charge model (2.5) this symmetry is a consequence of the existence of two possible $2^{1/2} \times 2^{1/2}$ superlattices. Indexing the (x, y) coordinates of the lattice by the integers (m, n), and writing $i_{m,n;m',n'} = \sin(\theta_{m,n} - \theta_{m',n'} - A_{m,n;m',n'})$, we have that $c(m, n)$, the total current circulating about the m, nth plaquette, is

$$c(m, n) = i_{m,n+1;m,n} + i_{m+1,n+1;m,n+1} + i_{m+1,n;m+1,n+1} + i_{m,n;m+1,n}. \tag{2.7}$$

A possible chiral order parameter is

$$\chi(m, n) = (-1)^{m+n} c(m, n). \tag{2.8}$$

In the Appendix we present a more systematic discussion of the order parameter of this system.

Because of this broken chiral symmetry, at finite temperatures line defects or domain walls can exist, separating regions in which $\chi(m, n)$ has different signs. However, the restoration of the chiral symmetry at finite temperature through the proliferation of these defects is not, by itself, sufficient to destroy superconductivity in the array.

To see this, imagine a persistent current established in an annular geometry, as in figure 4. Furthermore, imagine that there exists a path circumnavigating the annulus

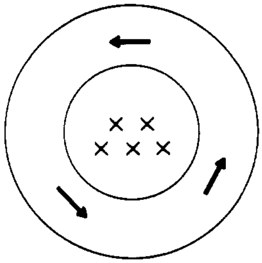

Figure 4. A current flowing around the annulus can be viewed as the topological consequence of the presence of a number of virtual vortices in the centre of the annulus.

such that the path always remains within one chiral domain; the amplitude of the chiral order parameter is supposed sufficiently large that it is meaningful to speak of the local phase $\alpha(\boldsymbol{r})$ of the plaquette ground state. This condition may not be consistent with the thermal destruction of chiral long-range order, which could conceivably involve the destruction of chiral ordering at even the shortest length scales. We presume, however, that immediately above the chiral disordering transition the domain walls will be sufficiently well localised that it will be possible to identify a local phase $\alpha(\boldsymbol{r})$.

The current flowing along this path about the annulus can be viewed as the consequence of the presence of a certain number of virtual vortices within the hole of the annulus. Only excitations that transport vortex charge from the interior of the annulus to its exterior are capable of degrading this supercurrent. Thus we must understand how the presence of domain walls can affect charge transport in the system in order to understand the resistive behaviour of such a phase.

We can obtain a different perspective on this question by considering the behaviour of the charge model. In a Z_2 disordered phase, the charges in this model will behave essentially as a 'liquid' at long wavelengths. We would expect such a 'liquid' to exhibit Debye screening, so interstitials and vacancies would be free to migrate across the system (Nelson 1983). The situation is, however, rather subtle, because in a true liquid the charges are free to occupy a continuum of possible positions, while in the charge model (2.5) the charges, even in a 'liquid' phase, are restricted to occupy the sites of a square lattice.

3. Domain walls

In the charge model (2.5) a domain wall between two chiral domains has the simple form shown in figure 5. These domain walls have a peculiar property: the corners of domain walls inevitably possess a net charge of $\pm\frac{1}{4}$. Average charges can be defined at the vertices of the lattice of figure 5 by averaging the charge of the four plaquettes neighbouring to

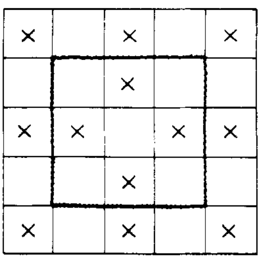

Figure 5. A domain wall between two chiral domains in the charge model.

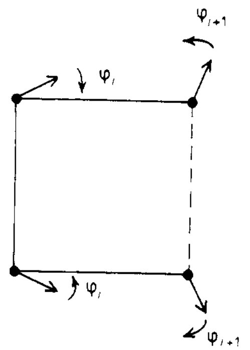

Figure 6. The definition of the phase deviations φ_i, φ_{i+1} used in (3.1), the determination of the structure of a domain wall in the phase model.

a vertex with the background field $f = \frac{1}{2}$. At vertices inside a chiral domain or on portions of walls oriented in either the horizontal or the vertical direction, this average will be zero. However, at the corners of domain walls there will be either one or three charges on the neighbouring plaquettes, so the average charge will be $\pm\frac{1}{4}$. The sign of the charge is determined by the position of the domain wall corner relative to the two $2^{1/2} \times 2^{1/2}$ sublattices.

In the phase model (2.1) the structure of a domain wall is somewhat more complicated. Restricting ourselves to a domain walls that are still local minima of (2.1), we may use the requirement of current conservation to determine the structure of the domain wall.

This requirement can be simply expressed in a difference equation. Consider an infinite domain wall oriented in parallel with the y axis, and define phase deviations φ_i from the chiral ground state in the manner shown in figure 6. Here i indexes the column. Then the difference equation satisfied by these deviations will be

$$2\sin(\pi/4 - 2\varphi_i) - \sin[\pi/4 + (\varphi_i + \varphi_{i-1})] - \sin[\pi/4 + (\varphi_i + \varphi_{i+1})] = 0. \tag{3.1}$$

In figure 7 we display two stable solutions to this equation. As $i \to \pm\infty$ these solutions match respectively onto the different chiral ground states. The excess energy of the configuration (i.e., the energy of the domain wall) may be determined numerically to be $0.343\,J$ per unit length of the wall. The phase of the chiral domain on one side of the wall is determined by the phase of the chiral domain on the other side of the wall and by the position of the domain wall. Thus, as shown in figure 7, the domain wall can be placed in either of two positions relative to the chiral structure, and the phases of the ground states on the right-hand side differ by π in the two cases.

This value for the domain wall energy can be used to make a preliminary estimate of the temperature at which long-range chiral ordering will be destroyed. The model (2.1) can be replaced by an effective model that is a function only of $\sigma_i = \mathrm{sgn}(\chi(i)) = \pm 1$, and excludes the phase degrees of freedom α. The Hamiltonian

$$H = -\tfrac{1}{2}(0.343 \times J) \sum_{\langle ij \rangle} \sigma_i \sigma_j \tag{3.2}$$

will correctly reproduce the energy of linear segments of domain wall. The chiral disordering temperature of this Hamiltonian is given by the well known formula for the two-dimensional Ising model (Onsager 1944)

$$kT_c = 0.343\,J/\log(1 + 2^{1/2}) = 0.389\,J. \tag{3.3}$$

The estimate can be compared with the Kosterlitz–Thouless estimate of the tem-

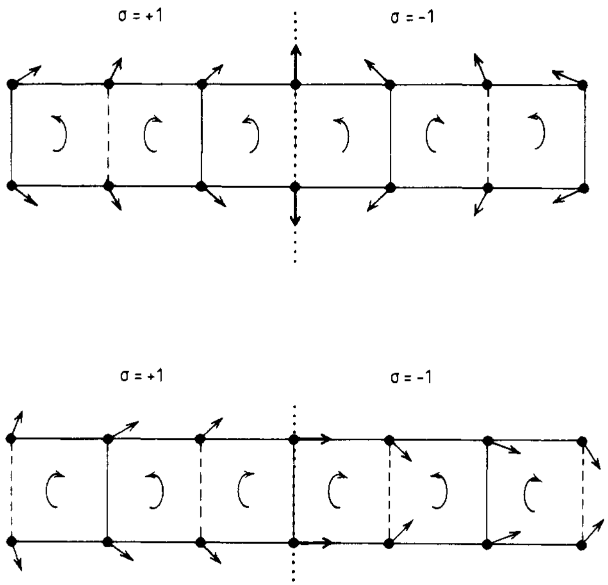

Figure 7. Two vertical domain walls (indicated by dotted lines) with degenerate energies. The phases of the right-hand chiral state for the two cases differ by π.

perature at which phase coherence in a system will be destroyed by the proliferation of free vortices. The bonds in a ground state all carry phase differences of $\pm\pi/4$. Thus an effective Hamiltonian for long-wavelength phase fluctuations in a single chiral ground state is

$$H = \tfrac{1}{2}J \cos\frac{\pi}{4} \int d^2r(\nabla\alpha(\boldsymbol{r}))^2. \tag{3.4}$$

The energy–entropy estimate for an isolated vortex now gives (Kosterlitz and Thouless 1973, Nelson 1983)

$$kT_{\mathrm{v}} = 2^{-3/2}\pi J = 1.11\, J \tag{3.5}$$

so at $kT_{\mathrm{v}} = 1.11\, J$ there will be an instability of the ground state towards the generation of free vortices.

Because $T_{\mathrm{c}} < T_{\mathrm{v}}$, the discussion in § 2, pointing out the subtlety of the question of whether or not chiral disorder leads to phase disorder, is quite apposite. However, the estimate for T_{c} is seriously flawed because it relies upon a domain wall energy derived for an infinite straight wall. This ignores the effect of domain wall corners.

The form of a horizontal domain wall may be obtained from the solution for a vertical wall by a combined rotation and gauge transformation. The two possible horizontal wall configurations are shown in figure 8. The phases of the ground states obtained above the wall differ by $\pm\pi/2$ from the phases obtained upon crossing a vertical domain wall.

Here there is an apparent contradiction. Any domain wall will possess both horizontal and vertical segments, yet the relative phases of the two chiral ground states appear to differ by $\pm\pi/2$ according to whether they are compared across a horizontal or a vertical segment of wall.

This apparent contradiction can be resolved by placing a vortex of charge $\pm\frac{1}{4}$ at each corner of a domain wall. The charge of $\pm\frac{1}{4}$ corresponds to a phase mismatch of $\pm\pi/2$.

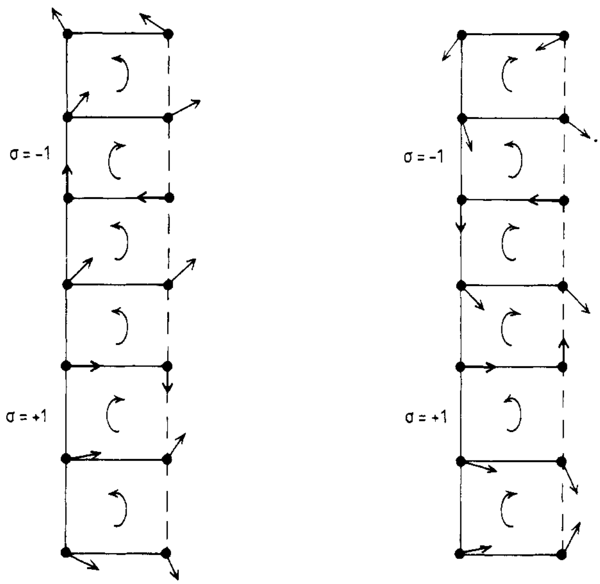

Figure 8. Two horizontal domain walls. Again the phases of the topmost chiral states differ by π from one another. However, they differ by $\pm\pi/2$ with respect to the right-hand states displayed in figure 7.

Thus if we define slowly varying phases $\alpha^+(\boldsymbol{r})$ and $\alpha^-(\boldsymbol{r})$, respectively, in the two chiral regions $\sigma = +1, -1$, then

$$\oint \left[\left(\frac{1+\sigma}{2} \right) \nabla \alpha^+(\boldsymbol{r}) + \left(\frac{1-\sigma}{2} \right) \nabla \alpha^-(\boldsymbol{r}) \right] \cdot \mathrm{d}\boldsymbol{r} = \tfrac{1}{4}(2\pi n) \tag{3.6}$$

where n is an integer. Clearly this discussion is most valid if the domain wall corners are well separated, so that the core regions of these vortices do not overlap.

4. Domain walls and phase ordering

These fractionally charged vortices will presumably have an important effect upon phase coherence in a chirally disordered system. However, they may also have a significant effect upon the chiral disordering transition itself.

The core energies of these vortices were not included in the estimate of the chiral transition temperature in § 3. Furthermore, in systems possessing quasi-long-range phase ordering, the possible configurations of domain walls will be strongly constrained. Phase ordering requires that all vortices, including those with fractional charge, appear in bound pairs with zero net charge. (In the presence of domain walls it is necessary to use a refined definition of phase ordering, as discussed in the Appendix.) As the sign of the charge of a domain wall is determined by its position, this restricts the types of domain wall that may occur.

The Peierls energy–entropy estimate for the transition temperature of systems with domain walls compares the energy of a domain wall of length L, in this case $0.343\,JL$, with the entropy of such a domain wall, regarded as a self-avoiding walk of length L (Peierls 1936, Domb 1974). The number of such walks on a square lattice is $N(L) \simeq$

$(2.639)^L$, so the Peierls estimate for kT_c is

$$kT_c \simeq 0.343\,J/\log(2.639) \tag{4.1}$$

in approximate agreement with (3.3).

If the fractional charges form bound pairs, however, they must at least alternate in sign along the wall. The number of walks satisfying this requirement is $N \simeq (2.639)^{L/2}$. This raises the estimate (4.1) of the chiral disordering temperature T_c by a factor of two. We are neglecting more complicated effects such as vortex core energies or the distortion of a domain wall induced by the pairing of charges.

The presence of these fractional charges will also affect the quasi-long-range phase ordering. Phase ordering can be related to the behaviour of the dielectric function for the vortex charges in the system (Nelson 1983, Rubenstein *et al* 1982). If the vortex charge density is given by $n(\boldsymbol{r})$, and its Fourier transform by $\hat{n}(\boldsymbol{q})$, this dielectric function is

$$\varepsilon^{-1}(\boldsymbol{q}) = 1 - (4\pi^2K/q^2)\langle|\hat{n}(\boldsymbol{q})|^2\rangle. \tag{4.2}$$

Here $K = (2^{1/2}J/kT)$ is the bare stiffness constant. A system can have phase ordering only if $\varepsilon^{-1}(\boldsymbol{q}\to 0) \neq 0$, so the vortices still interact logarithmically even at distant separations. Thus we need to understand the behaviour of

$$\varepsilon^{-1}(\boldsymbol{q}\to 0) = 1 + \pi^2K\int \mathrm{d}^2r\, r^2\langle n(\boldsymbol{r})n(0)\rangle \tag{4.3}$$

in the presence of domain walls and of fractionally charged vortices.

In the usual calculation of the dielectric function at low temperatures, $\langle n(\boldsymbol{r})n(0)\rangle$ is expanded in a power series in the fugacity of the integrally charged vortices, $y = \exp(-E_c/kT)$, where E_c is the core energy of a vortex. To the order y^2

$$\langle n(\boldsymbol{r})n(0)\rangle = -2y^2\exp(-2\pi K\log|\boldsymbol{r}|) \tag{4.4}$$

and

$$\varepsilon^{-1}(\boldsymbol{q}\to 0) = 1 - 4\pi^3K\int \mathrm{d}r\, r^{3-2\pi K}. \tag{4.5}$$

The integral diverges if $K \leqslant 2/\pi$, a signal of the vortex unbinding transition.

We can attempt a similar calculation for the $\pm\frac{1}{4}$ charges. The excitation of $\pm\frac{1}{4}$ charges necessarily involves the excitation of domain walls, which will carry many such charges. Thus it is not very meaningful literally to expand the dielectric function in powers of $y_{1/4}$, a fugacity for fractional charges. However, we can imagine a domain wall with all charges tightly bound in pairs except for two widely separated charges, as in figure 9. (Note that regions of a domain wall carrying only bound-charge pairs must have an anisotropic appearance, as in figure 9.) If the fugacity for these 'free' $\pm\frac{1}{4}$ charges is $y_{1/4}$, then their contribution to the dielectric function is

$$\pi^2K\int \mathrm{d}^2r\, r^2\langle n(\boldsymbol{r})n(0)\rangle = -\pi^2\frac{K}{16}y_{1/4}^2\int \mathrm{d}^2r\, r^{(2-2\pi K/16)}\exp(-\sigma(\boldsymbol{r})) + \mathrm{O}(y_{1/4}^4). \tag{4.6}$$

Here $\sigma(\boldsymbol{r})$ is proportional to the domain wall contribution to the free energy of the charge pair.

If $\sigma(\boldsymbol{r}) = 0$, this integral diverges if $K/16 \leqslant 2/\pi$. Thus, were it not for the free energy of the domain wall connecting the $\pm\frac{1}{4}$ charges, they would unbind at a temperature approximately $\frac{1}{16}$ of the integral vortex unbinding temperature. If fractional vortices are

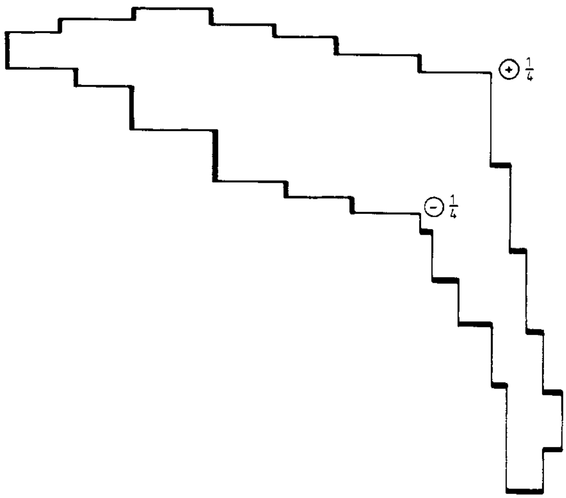

Figure 9. A domain wall with all charges bound except for one widely separated $\pm\frac{1}{4}$ charge dipole. The heavy lines represent closely bound $\pm\frac{1}{4}$ charge dipoles. Note the anisotropy of those segments of the domain wall carrying only bound charge.

free to proliferate, so are integral charges, because of the screening effect of the former.

If $\lim_{r\to\infty}\sigma(\boldsymbol{r}, T) = \infty$, then arbitrarily long domain walls will be suppressed. This is characteristic of the chirally ordered phase. If $\lim_{r\to\infty}\sigma(\boldsymbol{r}, T) \neq \infty$, then this single-domain-wall picture is not sensible, a characteristic of the chirally disordered phase. If $\lim_{r\to\infty}\sigma(\boldsymbol{r}, T) = 0$ and $T > \frac{1}{16}T_v$, the discussion above suggests that there will be free fractional vortices at (and above) this temperature and no quasi-long-range phase ordering.

Of course, $\sigma(\boldsymbol{r}, T)$ will depend strongly on whether or not the vortices along the intervening segments of wall are bound. If they are not bound, then no expansion in $y_{1/4}$ can be valid. It is plausible, however, that a divergence of the first term of such an expansion still signals an instability of bound vortex pairs.

Since our estimates suggest that the chiral disordering temperature is well above the temperature at which this instability occurs for $\pm\frac{1}{4}$-integral vortex pairs, the chiral disordering transition will correspond not merely to a proliferation of domain walls but to a proliferation of domain walls carrying free vortex charges. Thus there will be no phase ordering above this transition. The nature of the transition may be quite complex due to the interaction of phase disordering and chiral disordering effects.

5. Ordering in superfluid ^{3}He films

In the A phase of ^{3}He, the order parameter can be written in the form

$$d_{i\alpha} = d_i(\boldsymbol{\Delta}_1 + \mathrm{i}\boldsymbol{\Delta}_2)\alpha. \tag{5.1}$$

Here d_i is a spin vector and $|\boldsymbol{\Delta}_1| = |\boldsymbol{\Delta}_2| = \Delta$ (Leggett 1975). The orbital part of the angular momentum is proportional to $\boldsymbol{l} = \boldsymbol{\Delta}_1 \times \boldsymbol{\Delta}_2$. We shall ignore the spin part of the order parameter and concentrate instead upon the orbital part of the order parameter; spin–orbit coupling is only important at extremely low temperatures in ^{3}He (Leggett 1975).

At a surface, $\boldsymbol{l}$ must be perpendicular to the surface. Thus the order parameter of a ^{3}He-A film will have a $U(1) \times Z_2$ symmetry (Stein and Cross 1979). The Z_2 factor corresponds to the two choices for the direction of $\boldsymbol{l}$, parallel or antiparallel to $\hat{z}$, the normal to the film. The U(1) factor refers to proper rotations of the Δ vectors within the plane of the film. In this system there will be topological defects corresponding to each of the symmetry factors. At finite temperatures domain walls will be excited separating regions in which $\boldsymbol{l} \cdot \hat{z}$ has different signs; there will also be vortices in the $\boldsymbol{\Delta}$s. This situation is very similar to that for the fully frustrated *XY* model discussed in §§ 2–4.

Again we will concentrate upon the interaction between vortices and domain walls. A domain wall can be modelled by two parallel disclinations placed at the top and the bottom of the film thickness, as in figure 10 (Stein and Cross 1979). While this model should only be taken seriously if the film thickness d is greater than the coherence length ξ_0 of the ^{3}He Cooper pairs, it is qualitatively helpful even for $d \sim \xi_0$.

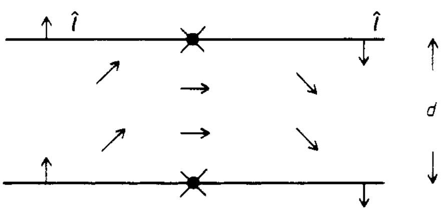

Figure 10. A domain wall in a ^{3}He-A film. The vector $\boldsymbol{l}$ rotates about an axis parallel to the domain wall. Filled circles indicate disclination lines, which are oriented perpendicularly to the plane of the paper.

The triad of vectors $\boldsymbol{\Delta}_1$, $\boldsymbol{\Delta}_2$, $\boldsymbol{l}$ rotates about some axis $\hat{m}$ in the plane of the film in moving across the domain wall. (In figure 10 this axis is chosen parallel to the domain wall.) In general, the energy of the domain wall will depend upon the angle between $\hat{m}$ and the local tangent to the domain wall. The hydrodynamics of this triad of vectors can be written in terms of the rotation matrix, $R_{\alpha\beta}(\boldsymbol{r})$, connecting the local orientation of the triad to some reference orientation (Cross 1975). The free-energy density as a function of $R_{\alpha\beta}(\boldsymbol{r})$ can be approximated by

$$\mathcal{F} = \tfrac{1}{2}K_1(\partial_\gamma R^{\mathrm{T}}_{\alpha\beta})(\partial_\gamma R_{\beta\alpha}) + \tfrac{1}{2}K_2(\partial_\gamma R^{\mathrm{T}}_{\alpha\gamma})(\partial_\beta R_{\beta\alpha}) + \tfrac{1}{2}K_3(\partial_\gamma R^{\mathrm{T}}_{\gamma\beta})(\partial_\alpha R_{\beta\alpha}). \tag{5.2}$$

It is straightforward to show that if $K_2 + K_3 > 0$, then it is energetically preferred for the rotation axis $\hat{m}$ to be parallel to the domain wall tangent, while if $K_2 + K_3 < 0$, it is energetically preferred for $\hat{m}$ to be perpendicular to the domain wall. In the following discussion we assume the former, but the results are general.

If the rotation axis $\hat{m}$ is always parallel to the domain wall, then even if the net vortex charge inside the wall is zero the wall itself will possess a vortex charge, as in figure 11. Thus a domain wall can serve as an extended vortex core. Small sections of the wall with large curvatures can have quite large vortex charge densities.

In general the axis $\hat{m}$ will not be oriented in the most energetically preferred direction. Parametrising the position along the domain wall by s, and writing the tangent angle as $\varphi(s)$ and the angle of $\hat{m}$ as $\theta(s)$, the free energy of the domain wall can be approximated as

$$F[\theta(s)] = \int \mathrm{d}s\,[K_1 + K_2 \sin^2(\theta(s) - \varphi(s))]. \tag{5.3}$$

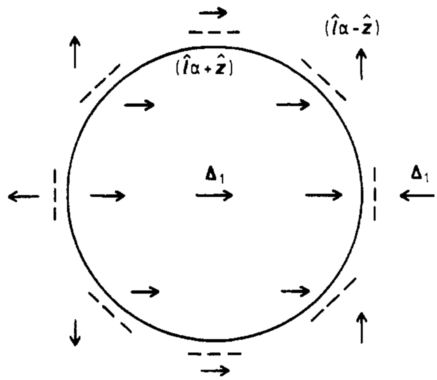

Figure 11. Broken lines indicate axes of reflection $\hat{m}$ of the vector $\boldsymbol{\Delta}_1$. These reflection axes are here chosen as tangents to the domain wall. In this case, although the vortex charge inside the domain wall is zero, the wall itself carries the charge +2.

The net vortex charge of the section of the wall between s_1 and s_2 will be

$$n(s_1, s_2) = \frac{1}{\pi}\int_{s1}^{s2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)\mathrm{d}s = \frac{1}{\pi}(\theta(s_2) - \theta(s_1)). \tag{5.4}$$

The total charge carried by a domain wall must be an integer.

Thus in the presence of domain walls local vorticity is not quantised. If we ignore the global constraints on the charge appearing along a domain wall, we can write an approximate Hamiltonian for the interaction of the vorticity degrees of freedom:

$$\frac{H}{kT} = -\pi K\int_a \mathrm{d}^2r\,\mathrm{d}^2r'\,n(\boldsymbol{r})\log|\boldsymbol{r}-\boldsymbol{r}|n(\boldsymbol{r}') + \frac{E_\mathrm{c}}{kT}\int_a \mathrm{d}^2r\,n^2(\boldsymbol{r}) \tag{5.5}$$

so we have

$$\langle|\hat{n}(\boldsymbol{q})|^2\rangle = (4\pi^2K/q^2 + E_\mathrm{c}a^2/kT)^{-1} \tag{5.6}$$

and using (4.2) we see that $\varepsilon^{-1}(\boldsymbol{q}\to 0) = 0$. Thus we conclude that no type of phase ordering in the $\boldsymbol{\Delta}$s is consistent with the proliferation of domain walls in this system.

6. Conclusion

The 'checkerboard' ground state determined for the fully frustrated *XY* model by (2.5) is actually the ground state for a wide class of models. The interaction energy of a general Josephson junction need not be cosinusoidal in the phase difference across the junction. Thus we may generalise the model (1.5) to

$$H = -J\sum_{\langle ij\rangle}\sum_{\nu=1}^{\infty}\lambda_\nu\cos^\nu(\theta_i - \theta_j - A_{ij}) \tag{6.1}$$

with

$$\sum_P A_{ij} = \pi.$$

The ground state will be of the checkerboard form if the current–phase relation $i(\varphi)$ has the standard one-maximum form over the interval $0 \leqslant \varphi \leqslant \pi$ (Baratoff *et al* 1970).

In the Appendix we show that the appearance of $\pm\frac{1}{4}$ charged vortices on the domain walls in the model (2.1) is a consequence of the symmetries of the 'checkerboard' state, and not of the particular form of the Hamiltonian. Thus much of the discussion in §§ 2–4 will apply to all of the models determined by (6.1) and (6.2).

However, in some of these models the sequence of inequalities $\frac{1}{16}T_v < T_c < T_v$ may not hold. There is no reason why T_c and T_v cannot be independent functions of the $\{\lambda_\nu\}$.

If $T_v < T_c$ (up to factors O(1)) for some class of models, then we expect that at low temperatures there will be a Kosterlitz–Thouless-type phase transition, an intermediate phase with chiral ordering but without phase ordering, and at a higher temperature a chiral disordering transition.

If $2T_c < \frac{1}{16}T_v$ (again up to factors O(1)), so that an instability towards domain walls carrying bound charge occurs at $T \simeq 2T_c$, the situation is rather subtle. One intriguing but rather unlikely possibility would be the appearance of a phase possessing no chiral ordering but in which the $\pm\frac{1}{4}$ charges remained bound, so that a remnant of phase ordering existed. At a still higher temperature the phase ordering would be destroyed. Another possibility would be a first-order transition destroying both chiral and phase ordering.

Acknowledgments

It is a pleasure to acknowledge the considerable encouragement and advice offered by C J Lobb and by J Toner. David R Nelson brought the array problem to my attention, and also offered valuable advice and criticism; he and C J Lobb also read the manuscript. I am grateful to M C Cross for pointing out to me the similarity between the array problem and ^{3}He films. I also benefited from discussions with P Bak, G M Grinstein, B I Halperin, C L Henley and R A Webb. At Harvard this work was supported by a National Science Foundation pre-doctoral fellowship, and by the NSF through the Harvard Materials Science Laboratory and through Grant No DMR-82-07431.

Appendix

The chirality order parameter $\chi(m, n)$ introduced in (2.8) does not include the phase degrees of freedom, and the phase order parameter $\alpha(\boldsymbol{r})$ introduced in § 2 neglects the chiral degrees of freedom. We can define an order parameter that includes both degrees of freedom, borrowing a method of describing states that are not translationally invariant that is used in the study of spin glasses (Henley 1983, Parisi 1983).

The Hamiltonian (2.1) can be written in a spin language, with $S_x^i = \cos\theta_i$, $S_y^i = \sin\theta_i$, and

$$H = -\sum_{\langle ij\rangle} J_{ij}\boldsymbol{S}^i \cdot \boldsymbol{S}^j. \tag{A1}$$

The J_{ij} are antiferromagnetic ($J_{ij} = -J$) on alternate columns of bonds, and ferromagnetic ($J_{ij} = J$) elsewhere.

Suppose $\{S_{g\alpha}^i\}$ are the values of these spins in a chiral ground state (see figure A1), α being a cartesian index. On each plaquette P it is possible to define a matrix $Q_{\alpha\beta}$ for any

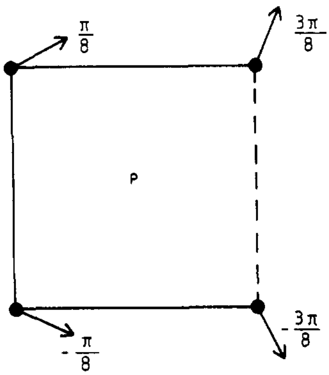

Figure A1. The reference ground state for the definition of the plaquette order parameter $Q_{\alpha\beta}$ in (A2).

configuration of spins $S^i_{s\alpha}$:

$$Q_{\alpha\beta} = \tfrac{1}{2}\sum_{i\in P} S^i_{s\alpha}S^i_{g\beta}. \tag{A2}$$

If the $\{S^i_{s\alpha}\}$ are uniform rotations of the chiral ground state spins, $S^i_{s\alpha} = R_{\alpha\beta}S^i_{g\beta}$, then

$$Q_{\alpha\beta} = \tfrac{1}{2}R_{\alpha\gamma}\sum_{i\in P} S^i_{g\gamma}S^i_{g\beta}. \tag{A3}$$

Since $\Sigma_{i\in P}S^i_{g\gamma}S^i_{g\beta} = 2\delta_{g\beta}$, in this case $Q_{\alpha\beta} = R_{\alpha\beta}$.

Similarly, if the $\{S^i_{s\alpha}\}$ are improper rotations of the ground-state spins,

$$S^i_{s\alpha} = \left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}R\right)_{\alpha\beta} S^i_{g\beta},$$

then

$$Q_{\alpha\beta} = \left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}R\right)_{\alpha\beta}.$$

In general,

$$Q_{\alpha\beta} = \begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}_{\alpha\beta}$$

can be written as

$$Q_{\alpha\beta} = Q_+R_{\alpha\beta}(\varphi_+) + Q_-\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}R(\varphi)\right)_{\alpha\beta} \tag{A4}$$

where

$$R(\varphi) = \begin{bmatrix}\cos(\varphi) & \sin(\varphi)\\ -\sin(\varphi) & \cos(\varphi)\end{bmatrix}.$$

Thus Q_+ multiplies a rotation matrix and Q_- an improper rotation matrix.

The state of the plaquette is now determined by two amplitudes and by two phases. These can be physically interpreted as the amplitudes of the two chiral ground states and their phase orientations. We can form new descriptions of the state of the plaquette:

$$\psi_+ = Q_+\exp(\mathrm{i}\varphi_+) \qquad \psi_- = Q_-\exp(\mathrm{i}\varphi_-) \tag{A5}$$

or of a coarse-graining of several plaquettes:

$$\psi_\pm(p) = \sum_{p'} w(p, p')\bar{\psi}_\pm(p') \tag{A6}$$

where $w(p, p')$ is some short-range kernel.

The free energy will now be function of $\psi_+(p)$, $\psi_-(p)$ and of their complex conjugates; we will use continuum versions $\psi_\pm(\boldsymbol{r})$. The terms appearing in the free energy will be restricted by the symmetries of the system (Lifshitz and Pitaevski 1980). Thus, for instance, under a global rotation of all spins in the system by an angle θ,

$$\psi_+ \to e^{i\theta}\psi_+ \qquad \psi_- \to e^{i\theta}\psi_- \tag{A7}$$

while under a global improper rotation of all spins in the system,

$$\psi_+ \to e^{i\theta}\psi_-^* \qquad \psi_- \to e^{i\theta}\psi_+^*. \tag{A8}$$

There are also symmetry operations that include the space group of the square lattice. These operations may include gauge transformations, to restore the choice of alternate vertical columns as the sites of antiferromagnetic bonds (Blankschtein *et al* 1984). Under a rotation of $\pi/2$ about a lattice site (including the appropriate gauge transformation),

$$\psi_+ \to e^{-i\pi/4}\psi_-^* \qquad \psi_- \to e^{i\pi/4}\psi_+^* \qquad r_i \to \varepsilon_{ij} r_j \tag{A9}$$

and under a translation by one lattice constant in the horizontal direction,

$$\psi_+ \to e^{i\pi/2}\psi_-^* \qquad \psi_- \to e^{-i\pi/2}\psi_+^*. \tag{A10}$$

The other space-group operations place no further constraints on the form of the Hamiltonian.

It is straightforward to determine the form of the free energy from the above. Initially keeping only the lowest-order-derivative terms and terms up to sixth order in $\psi_\pm$, we have

$$\begin{aligned}\mathscr{F} = c(|\nabla\psi_+|^2 + |\nabla\psi_-|^2) + r(|\psi_+|^2 + |\psi_-|^2) + u_4(|\psi_+|^4 + |\psi_-|^4) + v_4|\psi_+|^2|\psi_-|^2 \\ + u_6(|\psi_+|^6 + |\psi_-|^6) + v_6(|\psi_+|^2|\psi_-|^4 + |\psi_+|^4|\psi_-|^2).\end{aligned} \tag{A11}$$

To eighth order in $\psi_\pm$ there are two terms that couple the phase of ψ_+ with that of ψ_-,

$$\begin{aligned}\mathscr{F}_8^\varphi = g_8[(\psi_-\psi_+^*)^4 + (\psi_+\psi_-^*)^4] \\ + h_8\{[\partial_x(\psi_-\psi_+^* - \psi_-^*\psi_+)]^2 + [\partial_y(\psi_-\psi_+^* + \psi_-^*\psi_+)]^2\}.\end{aligned} \tag{A12}$$

These terms are responsible for the coupling of the phases of the two chiral ground states across a domain wall described in § 3.

The correlation functions reflecting the various possible types of ordering can be written in terms of the fields $\psi_\pm$. The chiral order parameter can be expressed as

$$\bar{\sigma}(\boldsymbol{r}) = \langle|\psi_+(\boldsymbol{r})|^2 - |\psi_-(\boldsymbol{r})|^2\rangle \tag{A13}$$

and the correlation function that reflects phase quasi-long-range ordering in a chirally ordered phase is

$$D(\boldsymbol{r} - \boldsymbol{r}) = \langle(\psi_+(\boldsymbol{r}) + \psi_-(\boldsymbol{r}))(\psi_+^*(\boldsymbol{r}) + \psi_-^*(\boldsymbol{r}))\rangle. \tag{A14}$$

In the presence of domain walls phases shift by 0, $\pm\pi/2$ or π in moving across domain walls. Thus the correlation function that reflects phase ordering in the presence of domain walls is

$$D^{1/4}(\boldsymbol{r}-\boldsymbol{r}) = \langle(\psi_+^4(\boldsymbol{r}) + \psi_-^4(\boldsymbol{r}))(\psi_+^{*4}(\boldsymbol{r}) + \psi_-^{*4}(\boldsymbol{r}))\rangle. \tag{A15}$$

References

Bak P 1979 *Solid State Commun.* **32** 581
Baratoff A, Blackburn J A and Schwartz B B 1970 *Phys. Rev. Lett.* **25** 1096
Blankschtein D, Ma M and Berker A N 1984 *Phys. Rev*. B **30** 1362
Cross M C 1975 *J. Low Temp. Phys.* **21** 525
Domb C 1974 *Phase Transitions and Critical Phenomena* vol 3, ed. C Domb and M S Green (New York: Academic Press)
Fisher D S 1980 *Phys. Rev.* B **22** 1190
Garel T and Doniach S 1980 *J. Phys. C: Solid State Phys.* **13** L887
Halperin B I and Nelson D R 1979 *J. Low Temp. Phys.* **36** 1165
Halsey T C 1985 *Phys. Rev.* B to be published
Henley C L 1983 *PhD Thesis* Harvard University
Jose J V, Kadanoff L P, Kirkpatrick S and Nelson D R 1977 *Phys. Rev.* B **16** 1217
Kosterlitz J M and Thouless D J 1973 *J. Phys. C: Solid State Phys.* **6** 1181
Lee D H, Joannopolous J D, Negele J W and Landau D P 1984 *Phys. Rev. Lett.* **52** 433
Leggett A J 1975 *Rev. Mod. Phys.* **47** 331
Lifshitz E M and Pitaevski L P 1980 *Statistical Physics* Part 1, (Oxford: Pergamon) pp 432–71
Lobb C J, Abraham D W and Tinkham M 1983 *Phys. Rev.* B **27** 150
Miyashita S and Shiba H 1983 *Tokyo University Preprint*
Natterman T 1981 *J. Phys. C: Solid State Phys.* **14** 2441
Nelson D R 1983 *Phase Transitions and Critical Phenomena* vol. 7, ed. C Domb and J Lebowitz (London: Academic Press)
Onsager L 1944 *Phys. Rev.* **65** 117
Parisi G 1983 *Phys. Rev. Lett.* **50** 1946
Peierls R E 1936 *Proc. Cambridge Phil. Soc.* **32** 477
Rubenstein M, Shraiman B and Nelson D R 1982 *Phys. Rev.* B **27** 1800
Stein D L and Cross M C 1979 *Phys. Rev. Lett.* **42** 504
Teitel S and Jayaprakash C 1983a *Phys. Rev.* B **27** 598
—— 1983b *Phys. Rev. Lett.* **51** 1999
Villain J 1975 *J. Physique* **36** 581
—— 1977 *J. Phys. C: Solid State Phys.* **10** 1717