Simplex Method - 2 Duality Theory.

The essence of duality theory.

Every linear programming problem has been associated with another linear programming problem. The original problem is called "primal" while the other is called its dual. In general either problem can be considered the primal with the remaining one its dual. If the primal is solved it is equivalent to solving its dual.

Definition of the dual problem

Let the primal problem be,

$$\text{Max} \quad z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$$
$$x_1, x_2, \dots x_n \ge 0$$

Dual : The dual problem is defined as

$$\text{Min} \quad z' = b_1w_1 + b_2w_2 + \dots + b_mw_m$$

subject to

$$a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m \ge c_1$$
$$a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m \ge c_2$$
$$\vdots$$
$$a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m \ge c_n$$
$$w_1, w_2, \dots w_m \ge 0$$

where $w_1, w_2 \dots w_m$ are called dual variables.

Characteristics of the Dual problem.

Duality in linear programming has the following characteristics:

1) Dual of the dual LP is primal.
2) If either the primal or dual of the problem has the optimal solution, then the other one will also have the same.
3) If any of the two problem has an infeasible solution then the value of the objective function on the other is unbounded.
4) The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual.

~~8) If any of the~~

5) If either the primal or the dual has an unbounded objective function value then the solution to the other problem is infeasible.
6) If the primal has a feasible solution but the dual does not have, then the primal will not have finite optimal solution & vice versa.

Formulation of Dual problems

i) Change the objective function of maximization in the primal into minimization in the dual and vice versa.

ii) The number of variables in the primal will be the number of constraint in the dual and vice versa

iii) The cost coefficients $C_1, C_2 \ldots C_n$ in the objective function of the primal will be the RHS constant of the constraint in the dual and vice versa.

iv) In forming the constraints for the dual, we consider the transpose of the body matrix of the primal problem.

v) The variables in both problems are non negative

vi) If the variable in the primal is unrestricted in sign, then the corresponding constraints in the dual will be an equation and vice versa.

Problems:

1) Write the dual for the following primal LP problem

$$\text{Max} \quad z = x_1 + 2x_2 + x_3$$

Subject to

$$2x_1 + x_2 + x_3 \leq 2$$
$$-2x_1 + x_2 - 5x_3 \geq -6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0$$

$\Rightarrow$ Since the problem is not in canonical form we interchange the inequality of the second constraint

$$\text{Max} \quad z = x_1 + 2x_2 + x_3$$

Subject to

$$2x_1 + x_2 - x_3 \leq 2$$
$$2x_1 - x_2 + 5x_3 \leq 6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0$$

Dual: Let w_1, w_2, w_3 be the dual variables

$$z' = 2w_1 + 6w_2 + 6w_3$$

Subject to
$$2w_1 + 2w_2 + 4w_3 \geq 1$$
$$+w_1 - w_2 + w_3 \geq 2$$
$$-w_1 + 5w_2 + w_3 \geq 1$$
where $w_1, w_2, w_3 \geq 0$

2) Find the dual of the following LPP

max $\cancel{z = 3x_1 +}$ $z = 3x_1 - x_2 + x_3$

Subject to
$$4x_1 - x_2 \leq 8$$
$$8x_1 + x_2 + 3x_3 \geq 12$$
$$5x_1 - 6x_3 \leq 13$$
$$x_1, x_2, x_3 \geq 0$$

⇒ Interchange the inequality of the second constraint.

MAX $z = 3x_1 - x_2 + x_3$

$$4x_1 - x_2 + 0x_3 \leq 8$$
$$-8x_1 - x_2 - 3x_3 \leq -12$$
$$5x_1 + 0x_2 - 6x_3 \leq 13$$
$$x_1, x_2, x_3 \geq 0$$

Dual : $z' = 8w_1 - 12w_2 + 13w_3$
$$4w_1 - 8w_2 + 5w_3 \geq 3$$
$$-w_1 - w_2 \geq -1$$
$$-3w_2 + 6w_3 \geq 1$$
$$w_1, w_2, w_3 \geq 0$$

3) Write the dual of the following LPP

$$\max z = 40x_1 + 35x_2$$

S.T $\quad 2x_1 + 3x_2 \le 60$

$4x_1 + 3x_2 \le 96$

$x_1, x_2 \ge 0$

$\Rightarrow$ Dual : $\min z = 60w_1 + 96w_2$

$2w_1 + 4w_2 \ge 40$

$3w_1 + 3w_2 \ge 35$

$w_1, w_2 \ge 0$

Dual of the above dual

$$\max z = 40x_1 + 35x_2$$

S.T $\quad 2x_1 + 3x_2 \le 60$

$4x_1 + 3x_2 \le 96$

$x_1, x_2 \ge 0$

4) Write the dual of the following LPP

$$\max z = 3x_1 + 4x_2 + 7x_3$$

S.T $\quad x_1 + x_2 + x_3 \le 10$

$4x_1 - x_2 - x_3 \ge 15$

$x_1 + x_2 + x_3 = 7$

$x_1, x_2 \ge 0$

$\Rightarrow$ Convert the second constraint to standard form

$-4x_1 + x_2 + x_3 \le -15$

The third constraint can be expressed as a pair of inequalities.

~~x1 + x2 + y3~~

$x_1 + x_2 + x_3 \le 7$

$-x_1 - x_2 - x_3 \le -7$

Let $y_3 = y_3' - y_3''$

Dual : $z' = 10y_1 + 15y_2 + 7(y_3' - y_3'')$

$$y_1 - 4y_2 + (y_3' - y_3'') \geq 3$$
$$y_1 - y_2 + (y_3' - y_3'') \geq 4$$
$$y_1 + y_2 + (y_3' - y_3'') \geq 7$$

$\therefore$ $z' = 10y_1 + 15y_2 + 7y_3$

$$y_1 - 4y_2 + y_3 \geq 3$$
$$y_1 - y_2 + y_3 \geq 4$$
$$y_1 + y_2 + y_3 \geq 7.$$

The Dual Simplex Method.

The algorithm is designed to solve a class of LP models efficiently. It is used to solve problems which start dual feasible. i.e., whose primal is optimal but infeasible. In this method the solution starts better than optimum but infeasible and remains infeasible until the true optimum is reached at which the solution becomes feasible.

Application of Dual simplex method.

1. Parametric programming.
2. Integer programming algorithms
3. Some non linear programing algorithms
4. It eliminates the introduction of artificial variables in the LP problems.

Dual Simplex Algorithm.

Step 1: Convert the problem into maximization problem if it is initially in the minimization form.

Step 2: Convert $\geq$ type constraints, if any, into $\leq$ type by multiplying both sides of such constraints by -1.

Step 3: Convert the inequality constraints into equalities by addition of slack variables and obtain the initial solution. Express this in the form of a table.

Step 4: Compute $c_j - z_j$ for every coloumn. Three cases arise:

a) If all $c_j - z_j$ are either negative or zero and all b_i are non negative, the solution obtained above is the optimal basic feasible solution.

b) If all $c_j - z_j$ are either negative or zero and at least one b_i is negative, then proceed to step 5

c) If any $c_j - z_j$ is positive, the method fails.

Step 5: Select the row that contains the most negative b_i. This row is called the key row or the pivot row. The corresponding basic variable leaves basis. This is called dual feasibility condition.

Step 6: Look at the elements of the key row.

a) If all the elements are non negative, the problem does not have a feasible solution.

b) If atleast one element is negative, find the ratio of the corresponding elements of $c_j - z_j$ row to these elements. Ignore the ratio's associated with positive or zero elements of the key row. Choose the smallest of these ratio's. The corresponding coloumn is the key column and the associated variable is the entering variable. This is called Dual optimality condition. Mark the key element or the pivot element.

Step 7: Make the key element unity. Perform the row operation as in the regular simplex method and repeat iterations until either an optimal feasible solution is obtained in a finite number of steps or there is an indication of the non existence of the feasible solution.

Problems:

1) Solve the dual simplex method for the following LPP

min $\quad z = 2x_1 + 2x_2 + 4x_3$

S.T $\quad 2x_1 + 3x_2 + 5x_3 \geq 2$

$\quad 3x_1 + x_2 + 7x_3 \leq 3$

$\quad x_1 + 4x_2 + 6x_3 \leq 5$

$\quad x_1, x_2, x_3 \geq 0$

$\Rightarrow$ Step 1: The given problem is converted to minimization

$z = -2x_1 - 2x_2 - 4x_3$

Step 2: The constraint of type $\geq$ is converted to $\leq$ type

$-2x_1 - 3x_2 - 5x_3 \leq -2$

Step 3: Add slack variable to convert the given problem to standard form.

$$z = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$$

$$-2x_1 - 3x_2 - 5x_3 + s_1 = -2$$

$$3x_1 + x_2 + 7x_3 + s_2 = 3$$

$$x_1 + 4x_2 + 6x_3 + s_3 = 5$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

	c_j	-2	-2	-4	0	0	0	
c_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	-2	$\boxed{-3}$	-5	1	0	0	-2 ←
0	s_2	3	1	7	0	1	0	3
0	s_3	1	4	6	0	0	1	5
$z_j = \sum c_B \cdot x_{ij}$		0	0	0	0	0	0	0
$c_j - z_j$		-2	-2 ↑	-4	0	0	0	

Step 4: Compute $c_j - z_j$ where $z_j = \sum c_B a_{ij}$. As all $c_j - z_j$ are either negative or zero and b_1 is negative the solution is optimal but infeasible.

Step 5: As $b_1 = -2$, the first row is the key row and s_1 is the outgoing variable.

Step 6: Find the ratio of elements of $c_j - z_j$ row to the elements of the key row. Neglect the ratio corresponding to positive or zero elements of key row.

$$\frac{-2}{-2} = 1, \quad \frac{-2}{-3} = \frac{2}{3}, \quad \frac{-4}{-5} = \frac{4}{5}$$

Since $\frac{2}{3}$ is the smallest ratio, 'x_2' column is the key column, x_2 is the incoming variable and -3 is the key element.

Step 7 : Replace s_1 by x_2. Apply corresponding row operation

$R_1 = R_1/-3$

R1: $\frac{2}{3}$ 1 $5/3$ $-1/3$ 0 0 $2/3$

$R_2 = R_2 - R_1$: 3 1 7 0 1 0 3

$-$ $2/3$ 1 $5/3$ $-1/3$ 0 0 $2/3$

R2: $7/3$ 0 $16/3$ $1/3$ 1 0 $7/3$

$R_3 = (R_1 \times 4) - R_3$

$\underline{4}$ $\frac{8}{3}$ 4 $\frac{20}{3}$ $\frac{-4}{3}$ 0 0 $\frac{8}{3}$

$-$ 1 4 6 0 0 1 5

R3: $-5/3$ 0 $-2/3$ $4/3$ 0 -1 $7/3$

	C_j	-2	-2	-4	0	0	0	
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b
-2	x_2	$2/3$	1	$5/3$	$-1/3$	0	0	$2/3$
0	s_2	$7/3$	0	$16/3$	$1/3$	1	0	$7/3$
0	s_3	$-5/3$	0	$-2/3$	$4/3$	0	1	$7/3$
$Z_j = \sum C_B a_{ij}$		$-4/3$	-2	$-10/3$	$2/3$	0	0	$-4/3$
$C_j - Z_j$		$-2/3$	0	$-2/3$	$-2/3$	0	0	

optimal basic feasible solution.

As all $c_j - z_j$ are negative or zero and all b_i are positive, the given solution is optimal

$x_1 = 0$, $x_2 = 2/3$ $x_3 = 0$

max $Z = \cancel{-2 \times (0 \times -2) \times \frac{2}{3}}$

max : $(-2 \times 0) - (2 \times \frac{2}{3}) - (4 \times 0) = -4/3$

or min $z = 4/3$.

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2) Use dual simplex method to

maximize $z = -3x_1 - 2x_2$

ST $x_1 + x_2 \geq 1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \geq 10$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

=) Step 1: The given problem is maximization

Step 2: Convert the constraint of $\geq$ type to $\leq$ type.

$-x_1 - x_2 \leq -1$

$x_1 + x_2 \leq 7$

$-x_1 - 2x_2 \leq -10$

$x_2 \leq 3$

Step 3: Add slack variable to express the given problem in standard form.

$z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

$-x_1 - x_2 + s_1 = -1$

$x_1 + x_2 + s_2 = 7$

$-x_1 - 2x_2 + s_3 = -10$

$x_2 + s_4 = 3$

	C_j	-3	-2	0	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	$\boxed{-2}$	0	0	1	0	-10 ←
0	s_4	0	1	0	0	0	1	3
$Z_j = \Sigma C_B \cdot x_{ij}$		0	0	0	0	0	0	0
$C_j - Z_j$		-3	-2 ↑	0	0	0	0	

Step 4: Compute $C_j - Z_j$ where $Z_j = \Sigma C_B \cdot x_{ij}$. As all $C_j - Z_j$ are either negative or zero and b_1 and b_3 are negative the solution is optimal but infeasible. We proceed to step 5.

Step 5: $b_3 = -10$ is the key row and s_3 is the outgoing variable

Step 6: Find the ratio of elements of $C_j - Z_j$ row to the elements of key row.

$\frac{-3}{-1} = 3$ $\quad \frac{-2}{-2} = 1$

x_2 column is the key column and (-2) is the key element. s_3 is replaced by x_2.

$R3 = R3 / -2$

$R3: \frac{1}{2} \quad 1 \quad 0 \quad 0 \quad -1/2 \quad 0 \quad 5$

$R1 = R1 + R3$

-1	-1	1	0	0	0	-1
1/2	1	0	0	-1/2	0	5

$R1: -1/2 \quad 0 \quad 1 \quad 0 \quad -1/2 \quad 0 \quad 4$

$R2 = R2 - R3$

1	1	0	1	0	0	7
1/2	1	0	0	-1/2	0	5

$R2: 1/2 \quad 0 \quad 0 \quad 1 \quad 1/2 \quad 0 \quad 2$

$R4 = \cancel{R3}/\cancel{R4} \quad R4 - R3$

1/2	1	0	0	-1/2	0	5
0	1	0	0	0	1	3
-1/2	0	0	0	1/2	1	-2

	C_j	-3	-2	0	0	0	0		
C_B	Basis	$x1$	$x2$	$s1$	$s2$	$s3$	$s4$	b	
0	$s1$	-1/2	0	1	0	-1/2	0	4	
0	$s2$	1/2	0	0	1	1/2	0	2	
-2	$x2$	1/2	1	0	0	-1/2	0	5	
0	$s4$	$\boxed{-1/2}$	0	0	0	1/2	1	-2	←
$z_j = \Sigma C_B \cdot x_{ij}$		-1	-2	0	0	1	0	-10	
$C_j - z_j$		-2	0	0	0	-1	0		
$\frac{C_j - z_j}{a_{ij}}$		4 ↑	$\cancel{0}$ -	$\cancel{0}$ -	$\cancel{0}$ -	-	-		

Replace $s4$ and $x1$

$R4 = (R4)/(-1/2)$

$= (-1/2 \quad 0 \quad 0 \quad 0 \quad 1/2 \quad 1 \quad -2)/(-1/2)$

$= 1 \quad 0 \quad 0 \quad 0 \quad -1 \quad -2 \quad 4$

$R3 = (R4 \times (-1/2)) + R3$

	$\frac{-1}{2}$	0	0	0	1/2	1	-2
+	$\frac{1}{2}$	1	0	0	-1/2	0	5
R3 :	0	1	0	0	0	1	3

$R2 = (R4 \times (-1/2)) + R2$

	$\frac{-1}{2}$	0	0	0	1/2	1	-2
+	1/2	0	0	1	1/2	0	2
R2	0	0	0	1	1	1	0

$R1 = (R4 \times (-1/2)) - R1$

	$\frac{-1}{2}$	0	0	0	1/2	1	-2
-	-1/2	0	1	0	-1/2	0	4
R1 =	0	0	1	0	-1	-1	6

	c_j	-3	-2	0	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	0	0	1	0	-1	-1	6
0	s_2	0	0	0	1	1	1	0
-2	x_2	0	1	0	0	0	1	3
-3	x_1	1	0	0	0	-1	-2	4
$z_j = \sum c_B \cdot a_{ij}$		-3	-2	0	0	3	4	-18
$c_j - z_j$		0	0	0	0	-3	-4	

The table gives optimal feasible solution

$x_1 = 4$, $x_2 = 3$

$Z_{max} = -(3 \times 4) - (2 \times 3)$

$= \underline{\underline{-18}}$